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PhD thesis

Towards a formalized theory of solid modules

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Til Skorra

Abstract

This thesis is concerned with the formalization of condensed mathematics. It consists of three papers, each of which is a substantial step towards a formalization of the theory of solid abelian groups in the Lean theorem prover. The first paper presents a formalization of Nöbeling's theorem, a technical result which is independent of condensed mathematics proper, but important when setting up the foundations of the solid theory. The second paper describes the most general categorical framework into which condensed sets fit, and how the theory is formalized using this framework. Important examples of condensed sets are given by discrete sets, and the third paper proves the equivalence of several conditions on a condensed set which characterize it as discrete. All the results in the papers have been formalized in Lean and integrated into its mathematical library, mathlib.

Resumé

Denne afhandling handler om formalisering af kondenseret matematik. Den består af tre artikler, som hver især udgør et vigtigt skridt imod en formalisering af teorien om solide Abelske grupper i bevisassistenten Lean. Den første artikel præsenterer en formalisering af Nöbelings sætning, et teknisk resultat der er uafhængigt af kondenseret matematik, men som er vigtigt for at etablere den solide teori. Den anden artikel beskriver den mest generelle kategoriteoretiske ramme, i hvilken kondenserede mængder passer, og hvordan teorien formaliseres inden for denne ramme. Diskrete mængder er vigtige eksempler på kondenserede mængder, og den tredje artikel beviser ækvivalensen af flere betingelser på en kondenseret mængde, der karakteriserer egenskaben at være diskret. Alle resultaterne i disse artikler er formaliserede i Lean og integrerede i den matematiske database mathlib.

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By getting involved in the formalization of mathematics, I was introduced to the Lean community, of which I am proud to call myself a member. I am grateful to everyone involved in maintaining mathlib, the Lean language itself, and other libraries. I want to particularly thank the organizers and participants of the 2023 Banff workshop on formalization of cohomology theories. It provided me with an excellent opportunity to present my earliest work in this area, and to meet for the first time many of the people that share my interests.

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Introduction

The subject of this thesis, condensed mathematics, is a theory which is designed as a generalization of topology.[1] It is better suited for the use of algebraic techniques than the classical theory of topological spaces.

The majority of results presented here, indeed all the results in the three papers [Asg24b, ABK+24, Asg24a] the thesis consists of, which we will link to in this introduction as paper 1, paper 2, and paper 3 respectively, have been formalized by the author in an interactive theorem prover. Formalizing mathematics can be summarized as the process of writing it in a programming language which allows the computer to check correctness (this is explained in more detail in §1.2).

In this introduction, we start in §1 by giving an overview of the background material, of the structure of the thesis, and of the path taken when completing this thesis. In §2, §3, §4, and §5 we give summaries of the papers and reports that appear in the thesis. Finally, in §6, we make a few remarks on possible future work building on this material.

1. Overview and background

1.1. Condensed mathematics. The theory of condensed mathematics, introduced independently by Clausen-Scholze [Sch19b, Sch19a, CS20, CS22, CS23] and Barwick-Haine [BH19], provides a framework for mixing algebra and geometry. More precisely, it allows for the use of algebraic techniques in a setting where the objects are topological in nature. As an example, one might attempt to do homological algebra on topological abelian groups. The issue with this is that the category of topological abelian groups is not an abelian category, as shown by the fact that the identity map $\mathbb{R}^\delta \to \mathbb{R}$, where the source is $\mathbb{R}$ with the discrete topology and the target is $\mathbb{R}$ with the usual topology, is a monomorphism and an epimorphism in this category, but not an isomorphism. Such attempts have therefore previously relied on somewhat complicated ad-hoc constructions, which have been successful, see e.g. [HS07, Sch99], but have their limitations. For example, it is possible to define a version of the bounded derived category of the category of locally compact abelian groups, but not an unbounded version.

The proposed solution of condensed mathematics is to replace the category of topological abelian groups with that of *condensed abelian groups*. This is an abelian category which satisfies the same Grothendieck axioms as the category of abelian groups, and thus provides a very convenient setting for doing homological algebra.

We define a condensed object in a category $\mathcal{C}$ as a $\mathcal{C}$-valued sheaf on the category of small compact Hausdorff spaces, with respect to the *coherent topology* (see paper 2 for the precise definition of this Grothendieck topology). A more concrete description is given as follows:

(1) X preserves finite products: in other words, for every finite family $(T_i)_{i \in I}$ of compact Hausdorff spaces, the natural map

$$X\Big(\coprod_{i\in I} T_i\Big) \longrightarrow \prod_{i\in I} X(T_i)$$

is a bijection.

(2) For every continuous surjection $\pi\colon S \to T$ of compact Hausdorff spaces, the diagram

$$X(T) \xrightarrow{X(\pi)} X(S) \rightrightarrows X(S \times_T S)$$

is an equalizer (the two parallel morphisms are the ones induced by the projections in the pullback).

We then take a condensed set to be a sheaf of *large*[2] sets on this site, and the same for condensed groups, rings, modules, etc. The fact that condensed sets are defined as sheaves of large sets on a large Grothendieck site means that the category of condensed sets satisfies all the axioms of a topos except for the existence of a small set of generators (instead, it has a large set of generators). For all practical purposes, it can be considered a topos.

Condensed abelian groups, and condensed modules over a ring more generally, form an abelian category (this follows from general theorems about categories of sheaves valued in abelian categories). In addition, they satisfy the same of Grothendieck's AB axioms as the category of abelian groups. These are AB3 (colimits exist), AB3* (limits exist), AB4 (direct sums are exact), AB4* (products are exact), AB5 (filtered colimits are exact), and AB6 (an arbitrary product of filtered colimits can be rewritten as a filtered colimit of products in the expected way). Condensed R-modules over a commutative ring R moreover admit a closed symmetric monoidal structure. The

[1]More precisely, it strictly generalizes the theory of *compactly generated* topological spaces, via a fully faithful functor.

[2]This is to ensure that sheafification is possible.

tensor product is given by the sheafification of the objectwise tensor product, and is in general badly behaved. It can be *completed* in a certain sense, which gives a much more useful notion, especially in non-archimedean contexts. This is the *solid tensor product*, given as the *solidification* (defined below) of the condensed tensor product.

We say that a condensed abelian group M is *solid* if it satisfies the following condition:

For every profinite set $S = \varprojlim_i S_i$*, the canonical map of condensed abelian groups* $\mathbb{Z}[S] \to \varprojlim_i \mathbb{Z}[S_i]$ *induces a bijection of hom sets*

$$\mathrm{Hom}\left(\varprojlim_i \mathbb{Z}[S_i], M\right) \to \mathrm{Hom}\left(\mathbb{Z}[S], M\right).$$

The full subcategory $\mathsf{SolidAb}$ of solid abelian groups in condensed abelian groups is an abelian category, closed under limits, colimits and extension. The inclusion functor

$$\mathsf{SolidAb} \to \mathsf{CondAb}$$

admits a left adjoint called *solidification*, denoted by $M \mapsto M^{\blacksquare}$. For every profinite set $S = \varprojlim_i S_i$, there exists a set J and isomorphisms

$$\mathbb{Z}[S]^{\blacksquare} \cong \varprojlim_i \mathbb{Z}[S_i] \cong \prod_{j \in J} \mathbb{Z}.$$

The objects $\prod_{j \in J} \mathbb{Z}$ are compact projective generators of $\mathsf{SolidAb}$. Proving this isomorphism relies on a technical result which is otherwise independent of condensed mathematics. This is Nöbeling's theorem, formalized in paper 1.

The notion of being solid can be defined for condensed modules over a ring R, but the analogous definition to the one above is not the most useful one in full generality. However, it is the correct notion when R is a finitely generated $\mathbb{Z}$-algebra. This gives a definition of solid $\mathbb{Z}[X]$-modules, and for general R, being solid can then be defined as the condition that for every $r \in R$ and every ring homomorphism $\mathbb{Z}[X] \to R$ such that X maps to r, the underlying $\mathbb{Z}[X]$-module is solid.[3] One can go a lot further in generality and develop a theory of *analytic ring structures* (see [Sch19b, Lecture 7]) on condensed rings (or rather, higher variants thereof — the general theory of analytic rings is best phrased in the language of ∞-categories, and is not ready for formalization yet). The theory of solid abelian groups is the study of one specific analytic ring structure on the condensed ring $\mathbb{Z}$ and can be seen as a prerequisite for analogous analytic ring structures on other condensed rings. For example, the theory of *solid spectra*, which we develop later in this thesis, is the study of an analytic ring structure on the condensed $\mathbb{E}_\infty$-ring $\mathbb{S}$. We will see that proofs of many results in that theory can be reduced to the case of solid abelian groups.

1.2. Mathlib and formalized mathematics. The phrase "formalized mathematics" refers to mathematics (definitions, theorems and proofs) written in a formal language, adhering to strict rules of logic. Most traditional mathematicians supposedly work in set-theoretic foundations, in the sense that their informal proofs could in principle be translated to a formal language associated to a theory of sets such as ZFC. This "in principle" has mostly been enough for the mathematical community thus far, without anyone ever performing this translation. However, the practice of translating informal mathematics into *a* formal language is gaining popularity, and the languages that turn out to be the most convenient for this are based on type theory. For most mathematics, it does not matter whether type-theoretic or set-theoretic foundations are used. The upshot of using type theory is that computers understand it and are very good at the process of *type checking*.

The Lean theorem prover [dMU21] is a programming language based on a version of *dependent type theory* which is equivalent to ZFC together with a rather mild assumption about existence of some inaccessible cardinals (see [Car19, Corollary 6.8]). It is the proof assistant that I have used to formalize condensed mathematics. There is an open-source library of formalized pure mathematics written in Lean, called mathlib [mC20]. It adheres to a number of important design principles, in terms of mathematical cohesion and generality, which make it increasingly useful as an encyclopaedic source of known mathematics, even for traditional mathematicians who are not necessarily aiming to formalize their work. These design principles and the type theory of Lean are described in more detail in §2 of paper 2 and the sources cited therein.

There are numerous benefits to formalizing mathematics, besides the obvious one of a better guarantee of correctness than the one provided by a human's careful reading. One problem with informal mathematics is that it is written with one specific audience in mind. It is a lot of work to get to the level of knowledge required to understand the papers in a given field, which are mostly written for experts. Formalizing mathematics forces us to be precise about the details of the argument. Moreover, it allows learners to delve into the argument in as much detail as they desire.[4] This becomes increasingly important as the field of mathematics continues to grow, making it more difficult to get to the level of understanding required to do research. A different aspect of the same phenomenon is the prospect of preserving knowledge that might otherwise get lost. Massot points

[3] This definition can be extracted from [Sch19b, Footnote 15, page 53].

[4] Patrick Massot has written about the possibility of using formalization to produce mathematical texts with "varying levels of detail". See for example [Mas21] and the talk [Mas23].

out an example of this kind from geometric topology in [Mas21, §2]. One could imagine similar examples from homotopy theory which has, over the last decade or so, been switching to the language of ∞-categories. Much of the classical theory has never been explicitly phrased using the modern language. Preserving the connection between the two through formalization would be a great service to future generations.

There are various resources available to learn *using* LEAN, for example [AM, AdMKU, Buz]. Furthermore, we refer the uninitiated reader of this thesis to the excellent guide to *reading* LEAN code found in [Baa24, Chapter 2].

1.3. Formalized condensed mathematics. The first formalization of condensed mathematics was a project called the *Liquid Tensor Experiment* (LTE, see [CT+22, Sch22][5]). It was a highly successful endeavour of the LEAN community following a challenge from Peter Scholze to formalize a fundamental result in the theory of *liquid vector spaces* [Sch19a, Theorem 9.1].

My work on formalizing condensed mathematics started out when LTE was about to finish, and at that time I relied heavily on the machinery developed there. Since then, much of my work has been devoted to leading the task of getting the theory into MATHLIB. This has meant a complete revision of the basic definitions, to conform with the generality standards of MATHLIB (see paper 2 for a detailed account of this). All the formalized results described in this thesis, and more results about condensed mathematics, are in MATHLIB.

1.4. Structure of thesis. The three papers in this thesis (1, 2, 3) describe steps towards a formalization of the theory of solid abelian groups. Paper 1 (summarized in §2) is about my formalization of Nöbeling's theorem, a technical result necessary to establish the solid theory, as mentioned in §1.1. Paper 2 (summarized in §3) is about the foundations of formalized condensed mathematics and how it is set up in MATHLIB. The requirement of high generality in MATHLIB led to generalizations of results in category theory, about the coherent, regular and extensive Grothendieck topologies. Paper 3 (summarized in §4) sets out the foundations of discrete condensed objects, a surprisingly subtle notion. It proves the equivalence of three conditions on a condensed set or module which characterize it as discrete. One of these conditions is very useful when setting up the theory of solid abelian groups. This thesis also contains a report (summarized in §5) on work I did in late 2021 and early 2022, before I started formalizing mathematics. This report is titled *Solid K-theory of p-adic Banach algebras* and contains partial progress towards understanding a condensed version of algebraic K-theory of some p-complete rings.

1.5. History of this project. I started out my PhD in 2021 planning to work on traditional mathematical problems and having limited knowledge of what formalized mathematics was. At first, I worked on problems related to condensed algebraic K-theory, and in particular the notion of *solidified K-theory* of p-complete rings. A dissatisfaction with the state of the fundamentals of the theory in the literature led me to think about the foundations of solid spectra. My work on solid K-theory of p-complete rings and the foundations of solid spectra is described in detail in the project report appearing last in this thesis. Later, the tendency to want detailed understanding and secure foundations of the theory led me into formalization. My goal from the start has been to formalize the theory of solid abelian groups, and eventually that of solid spectra and solid modules (where "module" may be taken in the most general, ∞-categorical sense possible — these latter two are long-term goals which require a lot more foundational work in formalization of ∞-category theory). I was quickly able to formalize a definition of solid abelian groups, but getting the basic theory off the ground requires many more ingredients. One of them is the discreteness characterization described in Paper 3, and this was the first big formalization project I worked on. Initially, I proved this characterization using the formalization of condensed sets developed in LTE, but after going on a big detour to get the basic theory of condensed sets into MATHLIB (which is mostly described in Paper 2), I was able to greatly improve on that work and integrate also the discreteness characterization into MATHLIB. This is the subject of Paper 3. The work described in Paper 1 is, as mentioned, an important step towards formalizing the solid theory, but independent of condensed mathematics proper, and could hence be formalized directly in MATHLIB before any of the condensed theory was there.

2. Nöbeling's theorem

In this section, we summarize paper 1 by giving a proof of Nöbeling's theorem, which is a shortened informal version of the formalized proof explained in the paper. This proof idea is sketched in [Sch19b, Proof of Theorem 5.4] as a translation of the argument in [Fuc70, Theorem 97.2] to our context. Throughout the proof, we try to indicate any omission, and refer to the paper for details.

THEOREM 1. *(Nöbeling). Let S be a profinite set. Then the abelian group $C(S, \mathbb{Z})$ of continuous (equivalently, locally constant) maps from S to $\mathbb{Z}$ is a free abelian group.*

PROOF. We can assume that S is a closed subset of a topological space of the form $\prod_{i \in I}\{0, 1\}$ where I is some set and each $\{0, 1\}$ has the discrete topology (for example, take I to be the set of clopen subsets of S). Throughout this proof, we will work with a fixed set I, and prove the result for every closed subset S of

[5] See also [CT24] for an interesting account of the methodology used in LTE in the context of the prospect of using interactive theorem provers to help tame complexity in pure mathematics research.

$\prod_{i\in I}\{0,1\}$. Pick a well-ordering on the set I. Let $e_{S,i} : S \to \{0,1\} \to \mathbb{Z}$ be the i-th projection. Order the formal products $e_{S,i_1}\cdots e_{S,i_r}$, with $i_1 > \cdots > i_r$, lexicographically. Denote by $P(S)$ the set of all such products and by $E(S)$ the set of such products which cannot be written as a linear combination of smaller products. The set $E(S)$ is our proposed basis of $C(S,\mathbb{Z})$.

For a subset J of I, denote by

$$\pi_J : \prod_{i\in I}\{0,1\} \to \prod_{i\in I}\{0,1\}$$

the map which is the projection map on J and zero outside J. Given a subset $S \subseteq \prod_{i\in I}\{0,1\}$, we denote by S_J the image $\pi_J(S)$. We obtain induced injective $\mathbb{Z}$-linear maps

$$\pi_J^* : C(S_J,\mathbb{Z}) \to C(S,\mathbb{Z})$$

by precomposition. Many of the details of the proof (which are omitted in [Sch19b, Proof of Theorem 5.4], but done explicitly in our formal version) are concerned with proving various compatibilities for these maps and the sets $E(S)$. These are generally easy to prove, but nevertheless very important.

We start by proving that $E(S)$ spans the whole $\mathbb{Z}$-module $C(S,\mathbb{Z})$. Since the set $P(S)$ is well-ordered (this is not entirely obvious but fairly straightforward to prove), it suffices to show that $P(S)$ spans $C(S,\mathbb{Z})$. We can write S as a limit of the diagram with transition maps π_J^*, where J runs over all finite subsets of I (proof omitted). Given a continuous map $f : S \to \mathbb{Z}$, by a general fact about profinite sets, there exists a finite set J of I and a map $g : S_J \to \mathbb{Z}$ such that $f = \pi_J^*(g)$. The map π_J^* takes elements of $P(S_J)$ to the corresponding elements of $P(S)$ provided that all the indices are in J (proof omitted). Therefore, it suffices to show that $P(S_J)$ spans $C(S_J,\mathbb{Z})$. Since $g \in C(S_J,\mathbb{Z})$ can be written as a linear combination of maps of the form $f_x : S_J \to \mathbb{Z}$, where $x \in S_J$ and $f_x(y) = \delta_{xy}$ is the Kronecker delta for every $y \in S_J$. Each of these can be written as a product of maps of the form $e_{S_J,j}$ and $(1 - e_{S_J,j})$, where $j \in J$, and hence a linear combination of elements in $P(S_J)$.

The difficult part is to prove that $E(S)$ is linearly independent. The proof is by induction over ordinals. More precisely, we regard I as an ordinal and every element of I as a smaller ordinal. The statement we prove is the following predicate on an arbitrary ordinal $\mu \le I$:

For all closed subsets S of $\prod_{i\in I}\{0,1\}$ such that for all $x \in S$ and $i \in I$, $x_i = 1$ implies $i < \mu$, $E(S)$ is linearly independent in $C(S,\mathbb{Z})$.

For an ordinal λ, we use the notation $\pi_\lambda = \pi_{\{i\in I | i<\lambda\}}$ and $S_\lambda := S_{\{i\in I|i<\lambda\}}$.

We proceed by induction over the ordinal μ, split into three cases; $\mu = 0$, μ is a limit ordinal and μ is a successor.

In the base case $\mu = 0$, S is empty or a singleton, and the proof is straightforward.

Suppose that μ is a limit ordinal. Then $S = S_\mu$ and the inductive hypothesis tells us that $E(S_{\mu'})$ is linearly independent for all $\mu' < \mu$. The technical result needed here is that

$$E(S_\mu) = \bigcup_{\mu'<\mu} \pi_{\mu'}^*\left(E(S_{\mu'})\right)$$

(proof omitted). The family of subsets in the union above is directed, and each of them is linearly independent by the inductive hypothesis and injectivity of $\pi_{\mu'}^*$. By a general theorem in linear algebra, it follows that the union is linearly independent as well.

For the successor case, we suppose that the predicate holds for μ and proceed to prove it for $\mu + 1$. We do this by constructing an exact sequence

$$0 \longrightarrow C(S_\mu,\mathbb{Z}) \xrightarrow{\pi_\mu^*} C(S,\mathbb{Z}) \xrightarrow{g} C(S',\mathbb{Z})$$

where S' meets the conditions to satisfy the inductive hypothesis and writing $E(S)$ as a disjoint union of $E(S_\mu)$ and $E'(S)$, where $E'(S)$ denotes the subset of those products of $E(S)$ which start with $e_{S,\mu}$. Then we show that g has the property that for every element of $E(S)$, it removes the first factor in the product, and the resulting product is in $E(S')$. Then the result follows by general linear algebra.

We describe here how the set S' and the map g is defined and omit the proofs. Let

$$S_0 = \{x \in S | x_\mu = 0\},$$
$$S_1 = \{x \in S | x_\mu = 1\},$$

and

$$S' = S_0 \cap \pi_\mu(S_1).$$

This set clearly satisfies the inductive hypothesis. Let

$$g_0 : S' \to S$$

denote the inclusion map, and let

$$g_1 : S' \to S$$

denote the map which swaps the μ-th coordinate to 1. Then we define $g : C(S,\mathbb{Z}) \to C(S',\mathbb{Z})$ as $g_1^* - g_0^*$, where * denotes precomposition. □

The paper contains a much more detailed proof and a discussion about the formalization process and implementation details.

3. The categorical foundations

We now summarize paper 2, joint with R. Brasca, N. Kuhn, F. A. E. Nuccio, and A. Topaz, which sets up a very general categorical framework, which, when specialized appropriately, gives the theory of condensed sets.

Recall that a condensed set is defined as a sheaf of sets on the site of compact Hausdorff spaces where the covers are given by finite, jointly surjective families of morphisms. It has a more explicit description as a presheaf of sets $X : \mathsf{CompHaus}^{\mathrm{op}} \to \mathsf{Set}$, satisfying the two conditions:

(1) X preserves finite products: in other words, for every finite family $(T_i)_{i \in I}$ of compact Hausdorff spaces, the natural map
$$X\Big(\coprod_{i \in I} T_i\Big) \longrightarrow \prod_{i \in I} X(T_i)$$
is a bijection.

(2) For every continuous surjection $\pi\colon S \to T$ of compact Hausdorff spaces, the diagram
$$X(T) \xrightarrow{X(\pi)} X(S) \rightrightarrows X(S \times_T S)$$
is an equalizer (the two parallel morphisms being induced by the projections in the pullback).

A condensed set can equivalently be defined as a sheaf on either of the subsites $\mathsf{Profinite}$ (consisting of the totally disconnected spaces) or $\mathsf{Stonean}$ (consisting of the extremally disconnected spaces, i.e. those in which the closure of every open set is open) of $\mathsf{CompHaus}$. On $\mathsf{Profinite}$, the same exact explicit description as described above for $\mathsf{CompHaus}$ applies. On $\mathsf{Stonean}$, the equalizer condition is irrelevant, and the sheaves are simply those presheaves which take finite disjoint unions to the corresponding finite products.

This definition, the more explicit characterization, and the equivalence between the three sheaf categories all fit into a much more general categorical framework. The Grothendieck topology which defines condensed sets is called the *coherent topology* and can be defined on any category which satisfies a condition called being *precoherent*.[6] These categories generalize the well-known class of coherent categories. The study of the coherent topology on coherent categories is classical, and includes the sites $\mathsf{CompHaus}$ and $\mathsf{Profinite}$, but not sites which do not have pullbacks, like $\mathsf{Stonean}$, which is why the generalization was required.

Being precoherent is a property of the category postulating how so-called effective epimorphic families of morphisms should behave. So before describing precoherence more precisely, we need to discuss effective epimorphisms. Usually in the literature, a morphism $f : X \to Y$ is said to be an *effective epimorphism* if it exhibits Y as the coequalizer of the two projection maps $X \times_Y X \to X$ from the kernel pair. In other words, the definition requires the kernel pair to exist. There is a definition that does not require the kernel pair to exist, and can briefly be described as saying that f exhibits Y as the joint coequalizer of all pairs of maps to X which f coequalizes (we refer to the paper for details). The upshot is that if the kernel pair exists, then the two definitions agree. Thus, we (and MATHLIB) refer to the more general term as *effective epimorphism.*

Similarly, one can define what it means for a family of morphisms $(f_i : X_i \to Y)_{i \in I}$ to be *effective epimorphic.* The easiest way to define this is to assume that the coproduct of the X_i exists and say that the family $(f_i)_{i \in I}$ is effective epimorphic if the induced map $f : \coprod_i X_i \to Y$ is an effective epimorphism. This again puts restrictions on the category $\mathcal{C}$ in the form of requiring the coproduct to exist. Instead, one can give a similar definition to the one for a single morphism, which does not require the existence of any limits of colimits. Under mild assumptions on the category, this is equivalent to the condition on the induced morphism from the coproduct. Again, we refer to the paper for details.

We can now define three predicates on any category $\mathcal{C}$, which each allows us to define a Grothendieck topology on $\mathcal{C}$. The first is the property of being *precoherent*, which says that for every finite, effective epimorphic family $(f_i : X_i \to Y)_{i \in I}$, and every morphism $h : Z \to Y$, there exists a finite set J, an effective epimorphic family $(g_j : W_j \to Z)$, a map $a : J \to I$, and for every $j \in J$ a morphism $h_j : W_j \to X_{a(j)}$, such that for all $j \in J$, $h \circ g_j = f_{a(j)} \circ h_j$. In other words, every finite effective epimorphic family can be "pulled back" along any morphism. The second is the property of being *preregular*, which is analogous to precoherent, but with effective epimorphic families replaced by single effective epimorphisms. The third is the property of being *finitary extensive.* We give a precise definition in the paper, but vaguely speaking it says that finite coproducts exist, and pullbacks along their coprojections exist and interact well with them.

On a category which is precoherent, preregular, or finitary extensive, one can define a Grothendieck topology called the *coherent*, *regular*, and *extensive* topology, respectively. We refer to the paper for the precise definitions of these topologies. They have the following properties:

- A presheaf is a sheaf for the coherent topology if and only if it satisfies the sheaf condition with respect to finite effective epimorphic families.

[6]The term *precoherent* was invented for the purpose of formalization and is indeed the minimal condition required to define the coherent topology.

- A presheaf is a sheaf for the regular topology if and only if it satisfies the sheaf condition with respect to single effective epimorphisms (if the category has pullbacks, then this is equivalent to condition (2) in the explicit description of a condensed set above, with the phrase "continuous surjection" replaced by "effective epimorphism").
- A presheaf is a sheaf for the extensive topology if and only if it preserves finite products (in other words it satisfies condition (1) in the explicit description of a condensed set above).

Now we can state some of the general categorical results we proved in the paper, and describe how they relate to condensed sets. First, the properties of being preregular and finitary extensive together imply precoherence:

Proposition 2. *Let $\mathcal{C}$ be a category. If $\mathcal{C}$ is preregular and finitary extensive, then $\mathcal{C}$ is precoherent.*

Second, the relationship between sheaves for these topologies (this follows from the fact that together, the regular and extensive topologies *generate* the coherent topology, in the appropriate sense):

Proposition 3. *Let $\mathcal{C}$ be a preregular and finitary extensive category and let F be a presheaf on $\mathcal{C}$. F is a sheaf for the coherent topology if and only if it is a sheaf for the regular and extensive topologies.*

In light of Proposition 3, proving the explicit description of condensed sets at the beginning of this section becomes a matter of showing that in the category of compact Hausdorff spaces and in the category of profinite sets, the effective epimorphisms are precisely the continuous surjections.

Proposition 4. *Let $\mathcal{C}$ be a preregular and finitary extensive category, suppose that every object of $\mathcal{C}$ is projective, and let F be a presheaf on $\mathcal{C}$. Then F is a sheaf for the coherent topology if and only if it preserves finite products.*

In light of Proposition 4, proving the explicit description of condensed sets as finite-product preserving presheaves on Stonean becomes a matter of showing that in that category, the effective epimorphisms are precisely the continuous surjections, and that every object of Stonean is projective (in fact, the category Stonean is equivalent to the full subcategory of projective objects in CompHaus).

Finally, the equivalence of sheaf categories is given by the following general result:

Proposition 5. *Let $\mathcal{C}$ be a category and let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor into a precoherent category $\mathcal{D}$ such that*

- *F preserves and reflects finite effective epimorphic families.*
- *For every object Y of $\mathcal{D}$, there exists an object X of $\mathcal{C}$ and an effective epimorphism $F(X) \to Y$.*

Then the following holds:

(1) $\mathcal{C}$ is precoherent.
(2) Let G be a sheaf for the coherent topology on $\mathcal{D}$. The presheaf $G \circ F^{\mathrm{op}}$ is a sheaf for the coherent topology on $\mathcal{C}$.
(3) Precomposition with F induces an equivalence between the categories of sheaves for the coherent topology on $\mathcal{C}$ and on $\mathcal{D}$.

Showing that $\mathcal{C} :=$ Stonean and $\mathcal{D} :=$ CompHaus satisfy the hypotheses of Proposition 5 amounts to showing that CompHaus has enough projectives. The object of Stonean surjecting onto a given compact Hausdorff space S is given by the Stone-Čech compactification of the underlying set of S with the discrete topology.

4. Discrete condensed objects

Paper 3 studies one aspect of the analogy between condensed sets and topological spaces; the important notion of *discreteness*. In this section we give an overview of the main results of that paper.

Condensed sets generalize a large class of topological spaces. It is important to be able to study the notion of discreteness in this setting, which becomes a bit more subtle than for topological spaces. A topological space X is discrete if and only if for every topological space Y, every map $X \to Y$ is continuous. In other words, there is an adjoint pair of functors

$$\mathsf{Set} \underset{U}{\overset{\delta}{\rightleftarrows}} \mathsf{Top} \quad (\delta \dashv U)$$

where δ is the functor which equips a set with the discrete topology and U is the forgetful functor mapping a topological space to its underlying set.

Given a condensed set X, we may think of $X(*)$ as its "underlying set". It is not difficult to show that there is an adjoint pair of functors

$$\mathsf{Set} \underset{U}{\overset{\underline{(-)}}{\rightleftarrows}} \mathsf{CondSet} \quad (\underline{(-)} \dashv U)$$

where U is the functor which takes X to its underlying set $X(*)$ and $\underline{Y}$ is the constant sheaf at a set Y. This makes $\underline{(-)}$ a good candidate for the functor which "equips a set with the discrete condensed structure". The problem is that this functor is defined somewhat abstractly, and thus there is a priori no concrete description of its sections.

A discrete topological space X also satisfies the property that every continuous map to it is locally constant. In fact, if a compactly generated[7] topological space X has the property that every continuous map from a compact Hausdorff space to X is locally constant, then X is discrete. Carrying this characterization of discrete topological spaces over to the setting of condensed sets, we define another functor $L : \mathsf{Set} \to \mathsf{CondSet}$ as follows: Given a set X, $L(X)$ is the sheaf which takes a compact Hausdorff space S to the set of all locally constant maps $S \to X$. It takes some work to show that L is left adjoint to U, the underlying set functor. This gives the expected isomorphism $L \cong \underline{(-)}$ and hence the desired explicit description of the sections of a discrete condensed set.

It is fairly easy to see that a discrete condensed set X satisfies the condition that for every profinite set $S = \varprojlim_i S_i$, the canonical map $\varinjlim_i X(S_i) \to X(S)$ is a bijection. This is because locally constant maps from profinite sets factor through one of the discrete quotients S_i. The other direction is true as well, but more difficult to prove (see the paper for details).

All the above gives the following theorem, which gives three equivalent conditions on a condensed set which characterize it as discrete:

Theorem 6. *The following conditions on a condensed set X are equivalent:*

(1) X is in the essential image of the functor $\underline{(-)}$.
(2) X is in the essential image of the functor L.
(3) For every profinite set $S = \varprojlim_i S_i$, the canonical map $\varinjlim_i X(S_i) \to X(S)$ is a bijection.

The condensed set X being discrete is also equivalent to the counit of the adjunction $\underline{(-)} \dashv U$ (and equivalently that of $L \dashv U$) being an isomorphism at X. This follows from full faithfulness of these functors $\mathsf{Set} \to \mathsf{CondSet}$ (which is not a priori obvious for $\underline{(-)}$, but easy to prove for L).

The paper also contains the important result that a condensed module over a ring R is discrete if and only if its underlying condensed set is discrete. This easily gives the same characterization of discrete condensed R-modules as for condensed sets.

5. Solid K-theory

We now summarize the project report appearing last in this thesis. It is about partial progress towards computing solidified K-theory

An important invariant of rings is given by *algebraic K-theory*. This associates to every ring R a spectrum $K(R)$. Many important rings, such as $\mathbb{R}, \mathbb{Q}_p, \mathbb{Z}_p$, etc. are naturally equipped with a topology. The spectrum $K(R)$ does not carry any information about this topology. To resolve this, one can promote the functor K to a functor from *condensed rings* to *condensed spectra*. We denote this functor by K, and the usual algebraic K-theory functor by K^δ (δ for "discrete").

We have so far only mentioned condensed objects in 1-categories. To define condensed algebraic K-theory, we need to extend this to ∞-categories. A condensed object in an ∞-category $\mathcal{C}$ is a hypercomplete $\mathcal{C}$-valued sheaf on the coherent site of compact Hausdorff spaces. A more detailed explanation of this definition is given in the report. The characterization as a finite-product preserving presheaf on $\mathsf{Stonean}$ still holds in this setting. We may then take $\mathcal{C}$ to be the ∞-category of spectra, and define condensed algebraic K-theory of a condensed ring R as the presheaf defined by

$$K(R)(T) = K^\delta(R(T))$$

for every $T \in \mathsf{Stonean}$. By general properties of K^δ, this satisfies the sheaf condition of preserving finite products. To define $K(R)$ on a compact Hausdorff space or a profinite set, we need to sheafify.

Condensed spectra, like condensed abelian groups, admit a notion of being *solid*, defined similarly (we establish this notion in detail in the report). This is useful in a non-archimedean setting. Consider a p-complete ring $R \simeq \varprojlim_{n\in\mathbb{N}} R/p^n$ and let $K(R)$ denote the condensed K-theory spectrum of R. There is a canonical map

$$K(R) \to \varprojlim_{n\in\mathbb{N}} K^\delta(R/p^n)$$

The full condensed K-theory spectrum is ill-behaved in the same way as the tensor product of condensed abelian groups. It can be tamed by solidification; the general theory shows that the target of the map above is solid, and thus we can produce a map

$$K^{\blacksquare}(R) \to \varprojlim_{n\in\mathbb{N}} K^\delta(R/p^n).$$

It is natural to ask whether this map is an equivalence. This is only known in very simple cases, and this thesis does not answer the general question. However, we study the lower homotopy groups of the source for two examples; $R = \mathcal{O}_{\mathbb{C}_p}$ and $R = \mathbb{Z}[T]_p^\wedge$. These are examples of condensed rings R which arise as rings of integers of fully multiplicative p-adic Banach algebras, such that R/p is discrete.

[7]This is the category of topological spaces which embeds fully faithfully into condensed sets.

We show that $K_0(R) \simeq \mathbb{Z}$ and $K_1(R) \simeq R^\times$ as condensed abelian groups for both these examples. They are both solid, and in general this implies that $(K_2(R))^■ \simeq \pi_2(K^■(R))$. Moreover, we show that $(K_2(R))^■$, as a condensed abelian group, admits a Steinberg presentation as a quotient of the tensor product $R^\times \otimes^■ R^\times$.

6. Summary and future work

The papers in this thesis describe important steps towards a formalization of the theory of solid abelian groups in the way the theory was defined in [Sch19b, Lectures 5-6]. Moreover, the project report on solidified K-theory establishes the general theory of solid spectra in an analogous way.

A new approach, based on *light condensed sets*, to the solid theory (and the rest of condensed mathematics) was introduced in [CS23]. In many ways that will be easier to formalize. This is work in progress which is well underway. The closed symmetric monoidal structure on sheaf categories plays a bigger role when developing the solid theory in this setting, and that is currently under development in MATHLIB in the proper generality of monoidal structures on localized categories. Although perhaps not reflected entirely by the contents of this thesis, I have formalized roughly the same amount of the foundations of condensed mathematics in the light setting as in the classical setting. Getting the solid theory off the ground using these foundations will be well within reach as soon as the closed symmetric monoidal structure is there.

A more ambitious goal is to formalize the theory of solid *spectra* in a similar way, for example, to the approach in the project report. This requires the language of ∞-categories, which is still in its infancy in LEAN. Emily Riehl has recently launched a project (see [R+24]) to formalize the general theory of ∞-categories in LEAN. This is one big necessary step towards a formalization of solid spectra. The next step is then to formalize the explicit examples of ∞-categories of anima, then spectra, and then condensed anima and condensed spectra. One needs a good way for these explicit examples to interact with the general theory. MATHLIB has good infrastructure for that in the setting of 1-categories. To be able to work with ∞-categories in LEAN in the same way as practitioners of, say, homotopy theory do in informal mathematics, we should try to mimic this infrastructure as much as possible. This is as yet an unsolved problem, which will be exciting to work on in the future.

Solid abelian groups and their higher analogues provide a useful setting for doing nonarchimedean geometry. One goal of MATHLIB is to support formalization of modern research mathematics. It varies by field how close we are to achieving that goal, but in the case of papers like [AM24, Aok24, JC22, JC23], we are not yet at the point of formalizing the statements. The goals described above are important prerequisites for this.

7. Thesis statement

This thesis consists of the following papers and manuscripts:

- Paper 1 is the published version of:
  - **Dagur Asgeirsson**. *Towards Solid Abelian Groups: A Formal Proof of Nöbeling's Theorem.* In: 15th International Conference on Interactive Theorem Proving (ITP 2024). Edited by Yves Bertot, Temur Kutsia, and Michael Norrish. Vol. 309. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2024, 6:1-6:17. DOI: `https://doi.org/10.4230/LIPIcs.ITP.2024.6`.
- Paper 2 is the accepted version, to appear in the Journal of Symbolic Logic, of:
  - **Dagur Asgeirsson**, Riccardo Brasca, Nikolas Kuhn, Filippo Alberto Edoardo Nuccio Mortarino Majno di Capriglio, and Adam Topaz. *Categorical Foundations of Formalized Condensed Mathematics.* 2024. DOI: `https://doi.org/10.48550/arXiv.2407.12840`.

  A co-author statement detailing my contributions to the paper has been submitted along with this thesis.
- Paper 3 is the preprint version of:
  - **Dagur Asgeirsson**. *A Formal Characterization of Discrete Condensed Objects.* 2024. DOI: `https://doi.org/10.48550/arXiv.2410.17847`.
- The project report, *Solid K-theory of p-adic Banach algebras*, is appearing for the first time in this thesis. However, parts of it are slightly modified versions of two notes that had previously appeared on my website. More precisely:
  - §2.1 is based on: **Dagur Asgeirsson**. *Discrete condensed objects.* 2022. URL: `https://dagur.sites.ku.dk/files/2022/03/discrete.pdf`.
  - §2.2 is based on: **Dagur Asgeirsson**. *Solid mathematics.* 2022. URL: `https://dagur.sites.ku.dk/files/2022/02/solid.pdf`.

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Paper 1 — Towards solid abelian groups: A formal proof of Nöbeling's theorem

This chapter contains the paper:

Dagur Asgeirsson. *Towards solid abelian groups: a formal proof of Nöbeling's theorem*. 2024. [Asg24b]

This is the version published in the proceedings of the 15th International Conference on Interactive Theorem Proving (ITP 2024).

Towards Solid Abelian Groups: A Formal Proof of Nöbeling's Theorem

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Abstract

Condensed mathematics, developed by Clausen and Scholze over the last few years, is a new way of studying the interplay between algebra and geometry. It replaces the concept of a topological space by a more sophisticated but better-behaved idea, namely that of a condensed set. Central to the theory are solid abelian groups and liquid vector spaces, analogues of complete topological groups.

Nöbeling's theorem, a surprising result from the 1960s about the structure of the abelian group of continuous maps from a profinite space to the integers, is a crucial ingredient in the theory of solid abelian groups; without it one cannot give any nonzero examples of solid abelian groups. We discuss a recently completed formalisation of this result in the Lean theorem prover, and give a more detailed proof than those previously available in the literature. The proof is somewhat unusual in that it requires induction over ordinals – a technique which has not previously been used to a great extent in formalised mathematics.

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Keywords and phrases Condensed mathematics, Nöbeling's theorem, Lean, Mathlib, Interactive theorem proving

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Supplementary Material *Software*: `https://github.com/leanprover-community/mathlib4/blob/ba9f2e5baab51310883778e1ea3b48772581521c/Mathlib/Topology/Category/Profinite/Nobeling.lean` archived at `swh:1:cnt:2fb2985994d43409a52761d0e853d37deeabdc74`

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1 Introduction

Nöbeling's theorem says that the abelian group $C(S, \mathbb{Z})$ of continuous maps from a profinite space S to the integers, is a free abelian group. In fact, the original statement [14, Satz 1] is the more general result that bounded maps from any set to the integers form a free abelian group, but this special case has recently been applied [16, Theorem 5.4] in the new field of condensed mathematics (see also [15, 5, 3]).

We report on a recently completed formalisation of this theorem using the Lean 4 theorem prover [12], building on its *Mathlib* library of formalised mathematics (which was recently ported from Lean 3, see [11, 10]). The proof uses the well-ordering principle and a tricky induction over ordinals. This is the first use of the induction principle for ordinals in Mathlib outside the directory containing the theory of ordinals. Often, one can replace such transfinite constructions by appeals to Zorn's lemma. The author is not aware of any proof of Nöbeling's theorem that does this, or otherwise avoids induction over ordinals[1].

When formalising a nontrivial proof, one inevitably makes an effort to organise the argument carefully. One purpose of this paper is to give a well-organised and detailed proof of Nöbeling's theorem, written in conventional mathematical language, which is essentially a by-product of the formalisation effort. This will hopefully be a more accessible proof than those that already exist in the literature; the one in [14] is in German, while the proofs of the result in [7, 16] are the same argument as the one presented here, but in significantly less detail. This is the content of section 4; some mathematical prerequisites are found in section 3.

In section 2 we give more details about the connection to condensed mathematics and in sections 5 and 6 we discuss the formalisation process and the integration into Mathlib.

Throughout the text, we use the symbol "↗" for external links, usually directly to the source code for the corresponding theorems and definitions in Mathlib. In order for the links to stay usable, they are all to a fixed commit to the master branch (the most recent one at the time of writing).

Mathlib is a growing library of mathematics formalised in Lean. All material is maintained continuously by a team of experts. There is a big emphasis on unity, meaning that there is *one* official definition of every concept, and it is the job of contributors to provide proofs that alternative definitions are equivalent. All the code in this project has been integrated into Mathlib; a process that took quite some time, as high standards are demanded of code that enters the library. However, it is an important part of formalisation to get the code into Mathlib, because doing so means that it stays usable to others in the future.

2 Motivation

Condensed mathematics [16, 15, 5] is a new theory developed by Clausen and Scholze (and independently by Barwick and Haine, who called the theory *pyknotic sets* [3]). It has the purpose of generalising topology in a way that gives better categorical properties, which is desirable e.g. when the objects have both a topological and an algebraic structure. Condensed objects[2] can be described as sheaves on a certain site of profinite spaces. A topological abelian group A can be regarded as a condensed abelian group with S-valued points $C(S, A)$ for profinite spaces S. Discrete abelian groups such as $\mathbb{Z}$ are important examples of topological abelian groups. There is a useful characterisation of discrete condensed sets (which leads to the same characterisation for more general condensed objects such as condensed abelian groups), which has been formalised in Lean 3 by the author in [1].

The discreteness characterisation can be stated somewhat informally as follows: A condensed set X is discrete if and only if for every profinite space $S = \varprojlim_i S_i$ (written as a cofiltered limit of finite discrete spaces), the natural map

$$\varinjlim_i X(S_i) \to X(S)$$

is an isomorphism.

[1] The proof does not use any ordinal arithmetic. However, it crucially uses the principle of induction over ordinals with a case split between limit ordinals and successor ordinals.

[2] This notion was first formalised in the *Liquid Tensor Experiment* [6, 17], see section 6 for a more detailed discussion.

There is a notion of completeness of condensed abelian groups, called being *solid* [16, Definition 5.1]. For the convenience of the reader, we give the informal definition here in Definition 1. First, we need to recall two facts about condensed abelian groups:

- The category of condensed abelian groups has all limits.
- The forgetful functor from condensed abelian groups to condensed sets has a left adjoint, denoted by $\mathbb{Z}[-]$ (adopted from the analogous relationship between the category of sets and the category of abelian groups).

▶ **Definition 1.** *Let $S = \varprojlim_i S_i$ be a profinite space and define a condensed abelian group as follows:*

$$\mathbb{Z}[S]^{\blacksquare} := \varprojlim_i \mathbb{Z}[S_i]$$

There is a natural map $\mathbb{Z}[S] \to \mathbb{Z}[S]^{\blacksquare}$, and we say that a condensed abelian group A is solid *if for every profinite space S and every morphism $f : \mathbb{Z}[S] \to A$ of condensed abelian groups, there is a unique morphism $g : \mathbb{Z}[S]^{\blacksquare} \to A$ making the obvious triangle commute*[3]*.*

Using the discreteness characterisation and Nöbeling's theorem, one can prove that for every profinite space S, there is a set I and an isomorphism of condensed abelian groups

$$\mathbb{Z}[S]^{\blacksquare} \cong \prod_{i \in I} \mathbb{Z}.$$

This structural result is essential to developing the theory of solid abelian groups. Without it one cannot even prove the existence of a nontrivial solid abelian group.

Since the proof of Nöbeling's theorem has nothing to do with condensed mathematics, people studying the theory might be tempted to skip the proof and use Nöbeling's theorem as a black box. Now that it has been formalised, they can do this with a better conscience. On the other hand, people interested in understanding the proof might want to turn to sections 3 and 4 of this paper for a more detailed account.

3 Preliminaries

For ease of reference, we collect in this section some prerequisites for the proof of Nöbeling's theorem. Most of them were already in Mathlib.

3.1 Order theory

▶ **Definition 2.** *Let I and X be sets and let r be a binary relation on X. An I-indexed family (x_i) in X is* directed *if for all $i, j \in I$, there exists $k \in I$ such that $r(x_i, x_k)$ and $r(x_j, x_k)$.*

▶ **Lemma 3.** *A monotone map on a poset with a join operation (i.e. a least upper bound of two elements) is directed.*

▶ Remark 4. Taking the union of two sets is an example of a join operation.

▶ **Definition 5.** *A category $\mathcal{C}$ is* filtered *if it satisfies the following three conditions*

(i) *$\mathcal{C}$ is nonempty.*

[3] This definition has also been formalised by the author in Lean 3 in [2]

(ii) *For all objects X, Y, there exists an object Z and morphisms $f : X \to Z$ and $g : Y \to Z$.*
(iii) *For all objects X, Y and all morphisms $f, g : X \to Y$, there exists an object Z and a morphism $h : Y \to Z$ such that $h \circ f = h \circ g$.*

A category is cofiltered *if the opposite category is filtered.*

▶ Remark 6. A poset is filtered if and only if it is nonempty and directed.

▶ Remark 7. The poset of finite subsets of a given set is filtered.

3.2 Linear Independence

▶ **Lemma 8.** *If (X_i) is a family of linearly independent subsets of a module over a ring R, which is directed with respect to the subset relation, then its union is linearly independent.*

▶ **Lemma 9.** *Suppose we have a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \xrightarrow{f} & M & \xrightarrow{g} & P \\ & & \uparrow v & & \uparrow u & & \uparrow w \\ & & I & \hookrightarrow & I \sqcup J & \hookleftarrow & J \end{array}$$

where N, M, P are modules over a ring R, the top row is exact, and the bottom maps are the inclusion maps. If v and w are linearly independent, then u is linearly independent.

3.3 Cantor's intersection theorem

▶ **Theorem 10.** Cantor's intersection theorem. *If $(Z_i)_{i \in I}$ is a nonempty family of nonempty, closed and compact subsets of a topological space X, which is directed with respect to the* superset *relation $(V, W) \mapsto V \supseteq W$, then the intersection $\bigcap_{i \in I} Z_i$ is nonempty.*

▶ Remark 11. Cantor's intersection theorem is often stated only for the special case of decreasing nested sequences of nonempty compact, closed subsets. The generalisation above can be proved by slightly modifying the standard proof of that special case.

3.4 Cofiltered limits of profinite spaces

▶ **Definition 12.** *A* profinite space *is a totally disconnected compact Hausdorff space.*

▶ **Lemma 13.** *Every profinite space has a basis of clopen subsets.*

▶ **Lemma 14.** *Every profinite space is totally separated, i.e. any two distinct points can be separated by clopen neighbourhoods.*

▶ Remark 15. A topological space is profinite if and only if it can be written as a cofiltered limit of finite discrete spaces. See section 6 for a further discussion.

▶ **Lemma 16.** *Any continuous map from a cofiltered limit of profinite spaces to a discrete space factors through one of the components.*

▶ Remark 17. In particular, a continuous map from a profinite space

$$S = \varprojlim_i S_i$$

to a discrete space factors through one of the finite quotients S_i.

4 The theorem

This section is devoted to proving

▶ **Theorem 18.** *(Nöbeling's theorem). Let S be a profinite space. Then the abelian group $C(S,\mathbb{Z})$ of continuous maps from S to $\mathbb{Z}$ is free.*

We can immediately reduce this to proving Lemma 19 below as follows: Let I denote the set of clopen subsets of S. Then the map

$$S \to \prod_{i\in I}\{0,1\}$$

whose i-th projection is given by the indicator function of the clopen subset i is a closed embedding.

▶ **Lemma 19.** *Let I be a set and let S be a closed subset of $\prod_{i\in I}\{0,1\}$. Then $C(S,\mathbb{Z})$ is a free abelian group.*

To prove Lemma 19, we need to construct a basis of $C(S,\mathbb{Z})$. Our proposed basis is defined as follows:

- Choose a well-ordering on I.
- Let $e_{S,i} \in C(S,\mathbb{Z})$ denote the composition

  $$S \hookrightarrow \prod_{i\in I}\{0,1\} \xrightarrow{p_i} \{0,1\} \hookrightarrow \mathbb{Z}$$

  where p_i denotes the i-th projection map, and the other two maps are the obvious inclusions.
- Let P denote the set of finite, strictly decreasing sequences in I. Order these lexicographically.
- Let $\mathrm{ev}_S : P \to C(S,\mathbb{Z})$ denote the map

  $$(i_1,\cdots,i_r) \mapsto e_{S,i_1}\cdots e_{S,i_r}.$$

- For $p \in P$, let $\Sigma_S(p)$ denote the span in $C(S,\mathbb{Z})$ of the set

  $$\mathrm{ev}_S\left(\{q \in P \mid q < p\}\right).$$

- Let $E(S)$ denote the subset of P consisting of those elements whose evaluation cannot be written as a linear combination of evaluations of smaller elements of P, i.e.

  $$E(S) := \{p \in P \mid \ \mathrm{ev}_S(p) \notin \Sigma_S(p)\}.$$

In Subsection 4.2 we prove that the set $\mathrm{ev}_S\left(E(S)\right)$ spans $C(S,\mathbb{Z})$, and in Subsection 4.3 we prove that the family

$$\mathrm{ev}_S : E(S) \to C(S,\mathbb{Z})$$

is linearly independent, concluding the proof of Nöbeling's theorem. Subsection 4.1 defines some notation which will be convenient for bookkeeping in the subsequent proof.

4.1 Notation and generalities

For a subset J of I we denote by

$$\pi_J : \prod_{i\in I}\{0,1\} \to \prod_{i\in I}\{0,1\}$$

the map whose i-th projection is p_i if $i \in J$, and 0 otherwise. These maps are continuous, and since source and target are compact Hausdorff spaces, they are also closed. Given a subset $S \subseteq \prod_{i\in I}\{0,1\}$, we let

$$S_J := \pi_J(S).$$

We can regard I with its well-ordering as an ordinal. Then I is the set of all strictly smaller ordinals. Given an ordinal μ, we let

$$\pi_\mu := \pi_{\{i\in I \mid i<\mu\}}$$

and

$$S_\mu := S_{\{i\in I \mid i<\mu\}}.$$

These maps induce injective $\mathbb{Z}$-linear maps

$$\pi_J^* : C(S_J, \mathbb{Z}) \to C(S, \mathbb{Z})$$

by precomposition.

Recall that we have defined P as the set of finite, strictly decreasing sequences in I, ordered lexicographically. We will use this notation throughout the proof of Nöbeling's theorem.

▶ **Lemma 20.** *For $p \in P$ and $x \in S$, we have*

$$\mathrm{ev}_S(p)(x) = \begin{cases} 1 & \text{if } \forall i \in p,\ x_i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Obvious. ◀

▶ **Lemma 21.** *Let J be a subset of I and let $p \in P$ be such that $i \in p$ implies $i \in J$. Then $\pi_J^*(\mathrm{ev}_{S_J}(p)) = \mathrm{ev}_S(p)$.*

Proof. Since $i \in p$ implies $i \in J$, we have

$$x_i = \pi_J^*(x)_i$$

for all $x \in S$ and $i \in p$. The result now follows from Lemma 20. ◀

▶ Remark 22. The hypothesis in Lemma 21 holds in particular if $p \in E(S_J)$. Indeed, suppose $i \in p$, then if $i \notin J$, we have $\mathrm{ev}_{S_J}(p) = 0$.

▶ **Lemma 23.** *If μ', μ are ordinals satisfying $\mu' < \mu$, then $E(S_{\mu'}) \subseteq E(S_\mu)$.*

Proof. Let $p \in E(S_{\mu'})$. Then every entry of p is $< \mu'$, and it suffices to show that if

$$\mathrm{ev}_{S_\mu}(p) = \pi_{\mu'}^*(\mathrm{ev}_{S_{\mu'}}(p))$$

is in the span of

$$\mathrm{ev}_{S_\mu}(\{q \in P \mid q < p\}) = \pi_{\mu'}^*\Big(\mathrm{ev}_{S_{\mu'}}(\{q \in P \mid q < p\})\Big)$$

then $\mathrm{ev}_{S_{\mu'}}(p)$ is in the span of $\mathrm{ev}_{S_{\mu'}}(\{q \in P \mid q < p\})$. This follows by injectivity of $\pi_{\mu'}^*$. ◀

4.2 Span

The following series of lemmas proves that $\mathrm{ev}_S(E(S))$ spans $C(S, \mathbb{Z})$.

▶ **Lemma 24.** *The set P is well-ordered.*

Proof sketch. Suppose not. Take a strictly decreasing sequence (p_n) in P. Let a_n denote first term of p_n. Then (a_n) is a decreasing sequence in I and hence eventually constant. Denote its limit by a. Let $q_n = p_n \setminus a_n$. Then there exists an N such that $(q_n)_{n \geq N}$ is a strictly decreasing sequence in P and we can repeat the process of taking the indices of the first factors, get a decreasing sequence in I whose limit is strictly smaller than a. Continuing this way, we get a strictly decreasing sequence in I, a contradiction. ◀

▶ Remark 25. The proof sketch of Lemma 24 above is ill-suited for formalisation. Kim Morrison gave a formalised proof, following similar ideas to those above, which used close to 300 lines of code. A few days later, Junyan Xu found a proof that was ten times shorter, directly using the inductive datatype `WellFounded`. This is the only result whose proof indicated in this paper differs significantly from the one used in the formalisation.

▶ **Lemma 26.** *If $\mathrm{ev}_S(P)$ spans $C(S, \mathbb{Z})$, then $\mathrm{ev}_S(E(S))$ spans $C(S, \mathbb{Z})$.*

Proof. It suffices to show that $\mathrm{ev}_S(P)$ is contained in the span of $\mathrm{ev}_S(E(S))$. Suppose it is not, and let p be the smallest element of P whose evaluation is not in the span of $\mathrm{ev}_S(E(S))$ (this p exists by Lemma 24). Write $\mathrm{ev}_S(p)$ as a linear combination of evaluations of strictly smaller elements of P. By minimality of p, each term of the linear combination is in the span of $\mathrm{ev}_S(E(S))$, implying that p is as well, a contradiction. ◀

▶ **Lemma 27.** *Let F denote the contravariant functor from the (filtered) poset of finite subsets of I to the category of profinite spaces, which sends J to S_J. Then S is homeomorphic to the limit of F.*

Proof sketch. Since S is compact and the limit is Hausdorff, it suffices to show that the natural map from S to the limit of F induced by the projection maps $\pi_J : S \to S_J$ is bijective.

For injectivity, let $a, b \in S$ such that $\pi_J(a) = \pi_J(b)$ for all finite subsets J of I. For all $i \in I$ we have $a_i = \pi_{\{i\}}(a) = \pi_{\{i\}}(b) = b_i$, hence $a = b$.

For surjectivity, let $b \in \lim F$. Denote by

$$f_J : \lim F \to S_J$$

the projection maps. We need to construct an element a of C such that $\pi_J(a) = f_J(b)$ for all J. In other words, we need to show that the intersection

$$\bigcap_J \pi_J^{-1}\{f_J(b)\},$$

where J runs over all finite subsets of I, is nonempty. By Cantor's intersection theorem 10, it suffices to show that this family is directed (all the fibres are closed by continuity of the π_J, and closed subsets of a compact Hausdorff space are compact). To show that it is directed, it suffices to show that for $J \subseteq K$, we have

$$\pi_K^{-1}\{f_K(b)\} \subseteq \pi_J^{-1}\{f_J(b)\}$$

(by Lemma 3). This follows easily because the transition maps in the limits are just restrictions of the π_J. ◀

▶ **Lemma 28.** *Let J be a finite subset of I. Then $\mathrm{ev}_{S_J}(E(S_J))$ spans $C(S_J, \mathbb{Z})$.*

Proof. By lemma 26, it suffices to show that $\mathrm{ev}_{S_J}(P)$ spans. For $x \in S_J$, denote by f_x the map $S_J \to \mathbb{Z}$ given by the Kronecker delta $f_x(y) = \delta_{xy}$.

Since S_J is finite, the set of continuous maps is actually the set of all maps, and the maps f_x span $C(S_J, \mathbb{Z})$.

Now let $j_1 > \cdots > j_r$ be a decreasing enumeration of the elements of J. Let $x \in S_J$ and let e_i denote e_{S_J, j_i} if $x_{j_i} = 1$ and $(1 - e_{S_J, j_i})$ if $x_{j_i} = 0$. Then

$$f_{j_i} = \prod_{i=1}^{r} e_i$$

is in the span of $P(S_J)$, as desired. ◀

▶ **Lemma 29.** *P spans $C(S, \mathbb{Z})$.*

Proof. Let $f \in C(S, \mathbb{Z})$. Then by Lemmas 27 and 16, there is a $g \in C(S_J, \mathbb{Z})$ such that $f = \pi_J^*(g)$. Writing this g as a linear combination of elements of $E(S_J)$, by Lemma 21 we see that f is a linear combination of elements of P as desired. ◀

4.3 Linear independence

▶ **Notation 30.** *Regard I with its well-ordering as an ordinal. Let Q denote the following predicate on an ordinal $\mu \leq I$:*

For all closed subsets S of $\prod_{i \in I}\{0, 1\}$, such that for all $x \in S$ and $i \in I$, $x_i = 1$ implies $i < \mu$, $E(S)$ is linearly independent in $C(S, \mathbb{Z})$.

We want to prove the statement $Q(I)$. We prove by induction on ordinals that $Q(\mu)$ holds for all ordinals $\mu \leq I$.

▶ **Lemma 31.** *The base case of the induction, $Q(0)$, holds.*

Proof. In this case, S is empty or a singleton. If S is empty, the result is trivial. Suppose S is a singleton. We want to show that $E(S)$ consists of only the empty list, which evaluates to 1 and is linearly independent in $C(S, \mathbb{Z}) \cong \mathbb{Z}$. Let $p \in P$ and suppose p is nonempty. Then it is strictly larger than the empty list. But the evaluation of the empty list is 1, which spans $C(S, \mathbb{Z}) \cong \mathbb{Z}$, and thus $\mathrm{ev}_S(p)$ is in the span of strictly smaller products, i.e. not in $E(S)$. ◀

4.3.1 Limit case

Let μ be a limit ordinal, S a closed subset such that for all $x \in S$ and $i \in I$, $x_i = 1$ implies $i < \mu$. In other words, $S = S_\mu$. Suppose $Q(\mu')$ holds for all $\mu' < \mu$. Then in particular $E(S_{\mu'})$ is linearly independent

▶ **Lemma 32.** *Let $\mu' < \mu$ and $p \in P$ whose entries are all $< \mu'$. Then*

$$\pi_{\mu'}^*\left(\mathrm{ev}_{S_{\mu'}}(\{q \in P \mid q < p\})\right) = \mathrm{ev}_{S_\mu}(\{q \in P \mid q < p\}).$$

Proof. If $q < p$, then every element of q is also $< \mu'$. Thus, by lemma 21,

$$\pi_{\mu'}^*\left(\mathrm{ev}_{S_{\mu'}}(q)\right) = \mathrm{ev}_{S_\mu}(q),$$

as desired. ◀

▶ **Lemma 33.**

$$E(S_\mu) = \bigcup_{\mu' < \mu} E(S_{\mu'})$$

Proof. The inclusion from right to left follows from Lemma 23, so we only need to show that if $p \in E(S_\mu)$ then there exists $\mu' < \mu$ such that $p \in E(S_{\mu'})$. Take μ' to be the supremum of the set $\{i + 1 \mid i \in p\}$. Then $\mu' = 0$ if p is empty, and of the form $i + 1$ for an ordinal $i < \mu$ if p is nonempty. In either case, $\mu' < \mu$.

Since every $i \in p$ satisfies $i < \mu' < \mu$, we have

$$\mathrm{ev}_{S_\mu}(p) = \pi^*_{\mu'}(\mathrm{ev}_{S_{\mu'}}(p))$$

and

$$\mathrm{ev}_{S_\mu}(\{q \in P \mid q < p\}) = \pi^*_{\mu'}\left(\mathrm{ev}_{S_{\mu'}}(\{q \in P \mid q < p\})\right)$$

so if $\mathrm{ev}_{S_{\mu'}}(p)$ is in the span of

$$\mathrm{ev}_{S_{\mu'}}(\{q \in P \mid q < p\}),$$

then $\mathrm{ev}_{S_\mu}(p)$ is in the span of

$$\mathrm{ev}_{S_\mu}(\{q \in P \mid q < p\}),$$

contradicting the fact that $p \in E(S_\mu)$. ◀

▶ **Lemma 34.**

$$\mathrm{ev}_{S_\mu}(E(S_\mu)) = \bigcup_{\mu' < \mu} \pi^*_{\mu'}\left(\mathrm{ev}_{S_{\mu'}}(E(S_{\mu'}))\right)$$

Proof. This follows from a combination of Lemmas 21 and 33. ◀

The family of subsets in the union in Lemma 34 is directed with respect to the subset relation (this follows from Lemmas 3, 23, and 21). The sets $\mathrm{ev}_{S_{\mu'}}(E(S_{\mu'}))$ are all linearly independent by the inductive hypothesis, and by injectivity of π^*_μ, their images under that map are as well. Thus, by Lemma 8, the union is linearly independent, and we are done.

4.3.2 Successor case

Let μ be an ordinal, S a closed subset such that for all $x \in S$ and $i \in I$, $x_i = 1$ implies $i < \mu+1$. In other words, $S = S_{\mu+1}$. Suppose $Q(\mu)$ holds. Then in particular $\mathrm{ev}_{S_\mu} : E(S_\mu) \to C(S_\mu, \mathbb{Z})$ is linearly independent.

To prove the inductive step in the successor case, we construct a closed subset S' of $\prod_{i \in I}\{0,1\}$ such that for all $x \in S'$, $x_i = 1$ implies $i < \mu$, and a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C(S_\mu,\mathbb{Z}) & \xrightarrow{\pi^*_\mu} & C(S,\mathbb{Z}) & \xrightarrow{g} & C(S',\mathbb{Z}) \\ & & {\scriptstyle \mathrm{ev}_{S_\mu}}\uparrow & & {\scriptstyle \mathrm{ev}_S}\uparrow & & \\ & & E(S_\mu) & \hookrightarrow & E(S) & \hookleftarrow & E'(S) \end{array} \tag{1}$$

where the top row is exact and $E'(S)$ is the subset of $E(S)$ consisting if those p with $\mu \in p$ (note that p necessarily starts with μ). For $p \in P$, we denote by $p^t \in P$ the sequence obtained by removing the first element of p (t stands for *tail*). The linear map g has the property that $g(\mathrm{ev}_S(p)) = \mathrm{ev}_{S'}(p^t)$ and $p^t \in E(S')$. Given such a construction, the successor step in the induction follows from lemma 9.

▶ **Construction 35.** *Let*

$$S_0 = \{x \in S \mid x_\mu = 0\},$$

$$S_1 = \{x \in S \mid x_\mu = 1\},$$

and

$$S' = S_0 \cap \pi_\mu(S_1).$$

Then S' satisfies the inductive hypothesis.

▶ **Construction 36.** *Let $g_0 : S' \to S$ denote the inclusion map, and let $g_1 : S' \to S$ denote the map that swaps the μ-th coordinate to 1 (since $S' \subseteq \pi_\mu(S_1)$, this map lands in S). These maps are both continuous, and we obtain a linear map*

$$g_1^* - g_0^* : C(S, \mathbb{Z}) \to C(S', \mathbb{Z}),$$

which we denote by g.

▶ **Lemma 37.** *The top row in diagram (1) is exact.*

Proof. We already know that π_μ^* is injective. Also, since $\pi_\mu \circ g_1 = \pi_\mu \circ g_0$, we have

$$g \circ \pi_\mu^* = 0.$$

Now suppose we have

$$f \in C(S, \mathbb{Z}) \text{ with } g(f) = 0.$$

We want to find an

$$f_\mu \in C(S_\mu, \mathbb{Z}) \text{ with } f_\mu \circ \pi_\mu = f.$$

Denote by

$$\pi'_\mu : \pi_\mu(S_1) \to S_1$$

the map that swaps the μ-th coordinate to 1. Since $g(f) = 0$, we have

$$f \circ g_1 = f \circ g_0$$

and hence the two continuous maps $f_{|S_0}$ and $f_{|S_1} \circ \pi'_\mu$ agree on the intersection

$$S' = S_0 \cap \pi_\mu(S_1)$$

Together, they define the desired continuous map f_μ on all of $S_0 \cup \pi_\mu(S_1) = S_\mu$. ◀

▶ **Lemma 38.** *If $p \in P$ starts with μ, then $g(\mathrm{ev}_S(p)) = \mathrm{ev}_{S'}(p^t)$.*

Proof. This follows from considering all the cases given by Lemma 20. We omit the proof here and refer to the Lean proof linked above. ◀

▶ Remark 39. If $p \in E(S)$ and $\mu \in p$, then p satisfies the hypotheses of Lemma 38.

▶ **Lemma 40.** *If $p \in E(S)$ and $\mu \in p$, then $p^t \in E(S')$.*

Proof. Contraposing the statement, it suffices to show that if

$$\mathrm{ev}_{S'}\left(p^t\right) \in \mathrm{Span}\left(\mathrm{ev}_{S'}\left(\{q \mid q < p^t\}\right)\right),$$

then

$$\mathrm{ev}_S(p) \in \mathrm{Span}\left(\mathrm{ev}_S\left(\{(q) \mid q < p\}\right)\right).$$

Given a $q \in P$ such that $i \in q$ implies $i < \mu$, we denote by $q^\mu \in P$ the sequence obtained by adding μ at the front. Write

$$g(\mathrm{ev}_S(p)) = \mathrm{ev}_{S'}(p^t) = \sum_{q<p^t} n_q \,\mathrm{ev}_{S'}(q) = \sum_{q<p^t} n_q g(\mathrm{ev}_S(q^\mu)).$$

Then by Lemma 37, there exists an $n \in C(S_\mu, \mathbb{Z})$ such that

$$\mathrm{ev}_S(p) = \pi_\mu^*(n) + \sum_{q<p^t} n_q(\mathrm{ev}_S(q^\mu)).$$

Now it suffices to show that each of the two terms in the sum above is in the span of $\{\mathrm{ev}_S(q) \mid q < p\}$. The latter term is because $q < p^t$ implies $q^\mu < p$. The former term is because we can write n as a linear combination indexed by $E(S_\mu)$, and for $q \in E(S_\mu)$ we have $\pi_\mu^*\left(\mathrm{ev}_{S_\mu}(q)\right) = \mathrm{ev}_S(q)$ and $\mu \notin q$ so $q < p$. ◀

▶ **Lemma 41.** *The set $E(S)$ is the disjoint union of $E(S_\mu)$ and*

$$E'(S) = \{p \in E(S) \mid \mu \in p\}.$$

Proof. We already know by Lemma 23 that $E(S_\mu) \subseteq E(S)$. Also, as noted in Remark 22, if $p \in E(S_\mu)$ then all elements of p are $< \mu$ and hence $p \notin E'(S)$. Now it suffices to show that if $p \in E(S) \setminus E'(S)$, then $p \in E(S_\mu)$.

Since $p \in E(S) \setminus E'(S)$, every $i \in p$ satisfies $i < \mu$. We have

$$\mathrm{ev}_S(p) = \pi_\mu^*(\mathrm{ev}_{S_\mu}(p))$$

and

$$\mathrm{ev}_S\left(\{q \in P \mid q < p\}\right) = \pi_\mu^*\left(\mathrm{ev}_{S_\mu}\left(\{q \in P \mid q < p\}\right)\right)$$

so if $\mathrm{ev}_{S_\mu}(p)$ is in the span of

$$\mathrm{ev}_{S_\mu}\left(\{q \in P \mid q < p\}\right),$$

then $\mathrm{ev}_S(p)$ is in the span of

$$\mathrm{ev}_S\left(\{q \in P \mid q < p\}\right),$$

contradicting the fact that $p \in E(S)$. ◀

The above lemmas prove all the claims made at the beginning of this section, concluding the inductive proof.

5 The formalisation

First a note on terminology: in the mathematical exposition of the proof in section 4, we have talked about continuous maps from S to $\mathbb{Z}$. Since $\mathbb{Z}$ is discrete, these are the same as the locally constant maps. The statement we have formalised is Listing 1.

```
instance LocallyConstant.freeOfProfinite (S : Profinite.{u}) :
    Module.Free ℤ (LocallyConstant S ℤ)
```

Listing 1 Nöbeling's theorem

which says that the $\mathbb{Z}$-module of locally constant maps from S to $\mathbb{Z}$ is free. When talking about locally constant maps, one does not have to specify a topology on the target, which is slightly more convenient when working in a proof assistant.

The actual proof is about closed subsets of the product $\prod_{i \in I}\{0,1\}$, which is of course the same thing as the space of functions $I \to \{0,1\}$. We implement it as the type `I → Bool`, where `Bool` is the type with two elements called `true` and `false`. This is the canonical choice for a two-element discrete topological space in Mathlib.

5.1 The implementation of P and $E(S)$

We implemented the set P as the type `Products I` defined as

```
def Products (I : Type*) [LinearOrder I] := {l : List I // l.Chain' (·>·)}
```

The predicate `l.Chain' (·>·)` means that adjacent elements of the list `l` are related by ">". We define the evaluation ev_S of products as

```
def Products.eval (S : Set (I → Bool)) (l : Products I) :
    LocallyConstant S ℤ := (l.val.map (e S)).prod
```

where `l.val.map (e S)` is the list of $e_{S,i}$ for i in the list `l.val`, and `List.prod` is the product of the elements of a list.

We define a predicate on `Products`

```
def Products.isGood (S : Set (I → Bool)) (l : Products I) : Prop :=
  l.eval S ∉ Submodule.span ℤ ((Products.eval S) '' {m | m < l})
```

and then the set $E(S)$ becomes

```
def GoodProducts (S : Set (I → Bool)) : Set (Products I) :=
  {l : Products I | l.isGood S}
```

It is slightly painful to prove completely trivial lemmas like 20 and its corollary 21 in Lean. Indeed, these results are not mentioned in the proof of [16, Theorem 5.4]. Although trivial, they are used often in the proof of the theorem and hence very important to making the proof work. Reading an informal proof of this theorem, one might never realise that these trivialities are used. This is an example of a useful by-product of formalisation; more clarity of exposition.

5.2 Ordinal induction

When formalising an inductive proof of any kind, one has to be very precise about what statement one wants to prove by induction. This is almost never the case in traditional mathematics texts. For example, the proof of [16, Theorem 5.4] claims to be proving by induction that $E(S)$ is a basis of $C(S,\mathbb{Z})$, not just that it is linearly independent. Furthermore, the set I is not fixed throughout the inductive proof which makes it somewhat unclear what the inductive hypothesis actually says. Working inside the topological space $\prod_{i\in I}\{0,1\}$ for a fixed set I throughout the proof was convenient in the successor step. This avoided problems that are solved by abuse of notation in informal texts, such as regarding a set as the same thing as its image under a continuous embedding.

The statement of the induction principle for ordinals in Mathlib is the following[4]:

```
def Ordinal.limitRecOn {Q : Ordinal → Sort _} (o : Ordinal)
    (H₁ : Q 0)
    (H₂ : ∀ o, Q o → Q (succ o))
    (H₃ : ∀ o, IsLimit o → (∀ o' < o, Q o')→ Q o) :
    Q o
```

In our setting, given a map `Q : Ordinal → Prop` (in other words, a *predicate on ordinals*)[5], we can prove $Q(\mu)$ for any ordinal μ if three things hold:

- The zero case: $Q(0)$ holds.
- The successor case: for all ordinals λ, $Q(\lambda)$ implies $Q(\lambda+1)$.
- The limit case: for every limit ordinal λ, if $Q(\lambda')$ holds for every $\lambda' < \lambda$, then $Q(\lambda)$ holds.

Finding the correct predicate Q on ordinals was essential to the success of this project:

```
def Q (I : Type*) [LinearOrder I] [IsWellOrder I (·<·)] (o : Ordinal) : Prop :=
  o ≤ Ordinal.type (·<· : I → I → Prop) →
    (∀ (S : Set (I → Bool)), IsClosed S → contained S o →
      LinearIndependent ℤ (GoodProducts.eval S))
```

The inequality

```
o ≤ Ordinal.type (·<· : I → I → Prop)
```

means that $o \leq I$ when I is considered as an ordinal, and the proposition `contained S o` is defined as

```
def contained {I : Type*} [LinearOrder I] [IsWellOrder I (·<·)]
    (S : Set (I → Bool)) (o : Ordinal) : Prop :=
  ∀ f, f ∈ S → ∀ (i : I), f i = true → ord I i < o
```

and `ord I i` is an abbreviation for

```
Ordinal.typein (·<· : I → I → Prop) i
```

i.e. the element $i \in I$ considered as an ordinal. The conclusion

```
LinearIndependent ℤ (GoodProducts.eval S)
```

means that the map $\mathrm{ev}_S : E(S) \to C(S,\mathbb{Z})$ is linearly independent.

As is often the case, this is quite an involved statement that we are proving by induction, and when writing informally, mathematicians wouldn't bother to specify the map `Q : Ordinal → Prop` explicitly.

[4] We have altered the notation slightly to match the notation in this paper.
[5] `Prop` is `Sort 0`

5.3 Piecewise defined locally constant maps

In the proof of Lemma 37, we defined a locally constant map $S_\mu \to \mathbb{Z}$ by giving locally constant maps from S_0 and $\pi_\mu(S_1)$ that agreed on the intersection, and noting that this gives a locally constant map from the union which is equal to S_μ. To do this in Lean, the following definition was added to Mathlib:

```
def LocallyConstant.piecewise {X Z : Type*} [TopologicalSpace X] {C₁ C₂ : Set X}
    (h₁ : IsClosed C₁) (h₂ : IsClosed C₂) (h : C₁ ∪ C₂ = Set.univ)
    (f : LocallyConstant C₁ Z) (g : LocallyConstant C₂ Z)
    (hfg : ∀ (x : X) (hx : x ∈ C₁ ∩ C₂), f ⟨x, hx.1⟩ = g ⟨x, hx.2⟩)
    [∀ j, Decidable (j ∈ C₁)] : LocallyConstant X Z where
  toFun i := if hi : i ∈ C₁ then f ⟨i, hi⟩
    else g ⟨i, (compl_subset_iff_union.mpr h) hi⟩
  isLocallyConstant := omitted
```

It says that given locally constant maps f and g defined respectively on closed subsets C_1 and C_2 which together cover the space X, such that f and g agree on $C_1 \cap C_2$, we get a locally constant map defined on all of X. This seems like exactly what we need in the above-mentioned proof. However, there is a subtlety, in that because of how the rest of the inductive proof is structured, we want the sets S_0, $\pi_\mu(S_1)$ and S_μ all to be considered as subsets of the underlying topological space $\prod_{i \in I}\{0,1\}$. To use `LocallyConstant.piecewise`, we would have to consider S_μ as the underlying topological space and S_0 and $\pi_\mu(S_1)$ as subsets of it. This is possible and is what was done initially, but a cleaner solution is to define a variant of `LocallyConstant.piecewise`:

```
def LocallyConstant.piecewise' {X Z : Type*} [TopologicalSpace X]
    {C₀ C₁ C₂ : Set X}
    (h₀ : C₀ ⊆ C₁ ∪ C₂) (h₁ : IsClosed C₁) (h₂ : IsClosed C₂)
    (f₁ : LocallyConstant C₁ Z) (f₂ : LocallyConstant C₂ Z)
    [DecidablePred (· ∈ C₁)]
    (hf : ∀ x (hx : x ∈ C₁ ∩ C₂), f₁ ⟨x, hx.1⟩  = f₂ ⟨x, hx.2⟩) :
    LocallyConstant C₀ Z
```

which satisfies the equations

```
lemma LocallyConstant.piecewise'_apply_left {X Z : Type*} [TopologicalSpace X]
    {C₀ C₁ C₂ : Set X} (h₀ : C₀ ⊆ C₁ ∪ C₂)
    (h₁ : IsClosed C₁) (h₂ : IsClosed C₂)
    (f₁ : LocallyConstant C₁ Z) (f₂ : LocallyConstant C₂ Z)
    [DecidablePred (· ∈ C₁)]
    (hf : ∀ x (hx : x ∈ C₁ ∩ C₂), f₁ ⟨x, hx.1⟩ = f₂ ⟨x, hx.2⟩)
    (x : C₀) (hx : x.val ∈ C₁) :
    piecewise' h₀ h₁ h₂ f₁ f₂ hf x = f₁ ⟨x.val, hx⟩
```

and

```
lemma LocallyConstant.piecewise'_apply_right {X Z : Type*} [TopologicalSpace X]
    {C₀ C₁ C₂ : Set X} (h₀ : C₀ ⊆ C₁ ∪ C₂)
    (h₁ : IsClosed C₁) (h₂ : IsClosed C₂)
    (f₁ : LocallyConstant C₁ Z) (f₂ : LocallyConstant C₂ Z)
    [DecidablePred (· ∈ C₁)]
    (hf : ∀ x (hx : x ∈ C₁ ∩ C₂), f₁ ⟨x, hx.1⟩ = f₂ ⟨x, hx.2⟩)
    (x : C₀) (hx : x.val ∈ C₂) :
    piecewise' h₀ h₁ h₂ f₁ f₂ hf x = f₂ ⟨x.val, hx⟩
```

Here C_0, C_1, and C_2 are subsets of the same underlying topological space X; C_1 and C_2 are closed sets covering C_0, and f_1 and f_2 are locally constant maps defined on C_1 and C_2 respectively, such that f_1 and f_2 agree on the intersection. This fits the application perfectly and shortened the proof of Lemma 37 considerably. Subtleties like this come up frequently, and can stall the formalisation process, especially when formalising general topology. When formalising Gleason's theorem (another result in general topology relevant to condensed mathematics, see [16, Definition 2.4] and [8]), similar subtleties arose about changing the "underlying topological space" to a subset of the previous underlying topological space.

The phenomenon that it is sometimes more convenient to formalise the definition of an object rather as a subobject of some bigger object is, of course, well known. It was noted in the context of group theory by Gonthier et al. during the formalisation of the odd order theorem, see [9, Section 3.3].

5.4 Reflections on the proof

The informal proof in [16] is about half a page; 21 lines of text. Depending on how one counts (i.e. what parts of the code count as part of the proof and not just prerequisites), the formalised proof is somewhere between 1500 and 3000 lines of Lean code. A big part of the difference is because of omissions in the proof in [16].

A more fair comparison would be with the entirety of section 4 in this paper, which is an account of all the mathematical contents of the formalised proof. Still, there is quite a big difference, which is mostly explained by the pedantry of proof assistants, as discussed in subsections 5.1, 5.2, and 5.3.

5.5 Mathlib integration

As discussed in recent papers by Nash [13] and Best et al. [4], when formalising mathematics in Lean, it is desirable to develop as much as possible directly against Mathlib. Otherwise, the code risks going stale and unusable, while if integrated into Mathlib it becomes part of a library that is continuously maintained.

The development of this project took place on a branch of Mathlib, all code being written in new files. This was a good workflow to get the formalisation done as quickly as possible, because if new code is put in the "correct places" immediately, one has to rebuild part of Mathlib to be able to use that code in other places, which can be a slow process if changes are made deep in the import hierarchy.

The proof of Nöbeling's theorem described in this paper has now been fully integrated into Mathlib. An unusually large portion of the code was of no independent interest, which resulted in a pull request adding one huge file, which Johan Commelin and Kevin Buzzard kindly reviewed in great detail, improving both the style and performance of the code.

6 Towards condensed mathematics in Mathlib

The history of condensed mathematics in Lean started with the *Liquid Tensor Experiment* (LTE) [6, 17]. This is an example of a formalisation project that was in some sense too big to be integrated into Mathlib. Nevertheless, it was a big success in that it demonstrated the capabilities of Lean and its community by fully formalising the complicated proof of a highly nontrivial theorem about so-called liquid modules. Moreover, it provided a setting in which to experiment with condensed mathematics and find the best way to do homological algebra in Lean. As mentioned above, the goal of LTE was to formalise one specialised theorem.

This is somewhat orthogonal to the goal of Mathlib which is to build a coherent, unified library of formalised mathematics. It is thus understandable that the contributors of LTE chose to focus on completing the task at hand instead of spending time on moving some parts of the code to Mathlib. Now that both LTE and the port of Mathlib to Lean 4 have been completed, we are seeing some important parts of LTE being integrated into Mathlib.

The definition of a condensed object was recently added to Mathlib. During a masterclass on formalisation of condensed mathematics organised in Copenhagen in June 2023, participants collaborated, under the guidance of Kevin Buzzard and Adam Topaz, on formalising as much condensed mathematics as possible in one week (all development took place in Lean 4 and the goal was to write material for Mathlib). The code can be found in the masterclass GitHub repository and much of it has already made it into Mathlib.

Profinite spaces form a rich category of topological spaces and there is more work other than Nöbeling's theorem to be done in Mathlib. Being the building blocks of condensed sets, it is important to develop a good API for profinite spaces in Mathlib. There, profinite spaces are defined as totally disconnected compact Hausdorff spaces. It is proved that every profinite space can be expressed as a cofiltered limit (more precisely, over the poset of its discrete quotients). It is also proved that the category of profinite spaces has all limits and that the forgetful functor to topological spaces preserves them. From this we can extract the following useful theorem:

▶ **Theorem 42.** *A topological space is profinite if and only if it can be written as a cofiltered limit of finite discrete spaces.*

The story about profinite spaces as limits does not end there, though. Sometimes it is not enough to know just that some profinite space *can* be written as *a* limit, but rather that there is a specific limit formula for it. Lemma 27 gives one specific way of writing a compact subset of a product as a cofiltered limit, which can be useful. Another example can be extracted from [1]. This is the fact that the identity functor on the category of profinite spaces is right Kan extended from the inclusion functor from finite sets to profinite spaces along itself. This gives another limit formula for profinite spaces, coming from the limit formula for right Kan extensions, and is useful when formalising the definition of solid abelian groups [2].

It can also be useful to regard the category of profinite spaces as the pro-category of the category of finite sets. The definition of pro-categories and this equivalence of categories would make for a nice formalisation project and be a welcome contribution to Mathlib.

7 Conclusion and future work

By formalising Nöbeling's theorem, we have illustrated that the induction principle for ordinals in Mathlib can be used to prove nontrivial theorems outside the theory ordinals themselves. Another contribution is the detailed proof given in section 4, and of course as mentioned before, it is an important step for the formalisation of condensed mathematics to continue.

A natural next step in the formalisation of the theory of solid abelian groups is to port the code in [1, 2] to Lean 4 and get it into Mathlib. Then one can put together the discreteness characterisation and Nöbeling's theorem to prove the structural results about $\mathbb{Z}[S]^{\blacksquare}$, which would lead us one step closer to an example of a nontrivial solid abelian group in Mathlib.

More broadly, it is important to continue moving as much as possible of the existing Lean code about condensed mathematics (from the LTE and the Copenhagen masterclass) into Mathlib.

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Paper 2 — Categorical foundations of formalized condensed mathematics

This chapter contains the paper:

Dagur Asgeirsson, Riccardo Brasca, Nikolas Kuhn, Filippo Alberto Edoardo Nuccio Mortarino Majno Di Capriglio, and Adam Topaz. *Categorical foundations of formalized condensed mathematics*. 2024. [ABK+24]

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CATEGORICAL FOUNDATIONS OF FORMALIZED CONDENSED MATHEMATICS

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Abstract. Condensed mathematics, developed by Clausen and Scholze over the last few years, proposes a generalization of topology with better categorical properties. It replaces the concept of a topological space by that of a condensed set, which can be defined as a sheaf for the coherent topology on a certain category of compact Hausdorff spaces. In this case, the sheaf condition has a fairly simple explicit description, which arises from studying the relationship between the coherent, regular and extensive topologies. In this paper, we establish this relationship under minimal assumptions on the category, going beyond the case of compact Hausdorff spaces. Along the way, we also provide a characterization of sheaves and covering sieves for these categories. All results in this paper have been fully formalized in the Lean proof assistant.

1. Introduction

The main goal of condensed mathematics (see e.g. [14, 15, 7]) is to provide a better framework to study the interplay between algebra and geometry. To do this, one has to generalize the notion of a topological space to obtain better categorical properties; the category of condensed sets achieves this remarkably well. A condensed set is defined as a sheaf for the so-called *coherent topology* on the category of compact Hausdorff spaces. The category of condensed sets contains a very large class of topological spaces as a full subcategory. In addition, it almost forms a topos[1], and the category of condensed abelian groups is a particularly well-behaved abelian category.

The formalization of the theory of condensed sets started with the *Liquid Tensor Experiment*, see [8, 16]. In that work the authors formalized the definition and various properties of the category of condensed abelian groups and of liquid vector spaces, including the main result [14, Theorem 9.1], using the Lean proof assistant. In §2 we will offer a brief outline both of Lean and of its main mathematical library mathlib.

Even if the achievements of the *Liquid Tensor Experiment* are spectacular, most of the work is not suitable to be integrated into a large mathematical library like mathlib. Indeed, a lot of results in the Liquid Tensor Experiment were stated and proven in an *ad-hoc* way and are not applicable in other contexts. This approach contradicts many of the design decisions prevalent throughout mathlib, which we briefly discuss in §2.2.

The main goal of our work is to formalize the foundations of the theory of condensed sets in an organic way, being as general as possible in all the various prerequisites. Indeed, the present work has already been incorporated in the mathlib library. Besides correctness, which is checked by Lean, this ensures that the results are stated in a way that is compatible with the rest of the library and that they can be used by others.

The goal of this paper is to prove, in the most general setting, results relating the coherent, regular and extensive topologies on a category, as well as characterizations of their sheaves. While the results we discuss in this paper are known to some experts as part of the folklore, we provide both a detailed exposition, while simultaneously minimizing various assumptions. The more general approach we take in this paper was motivated primarily by the formalization of these results.

Throughout the text, we use the symbol ↗ for external links. Almost every mathematical statement and definition will be accompanied by such a link directly to the source code for the corresponding statement in mathlib. The only exceptions are results that we use in the informal proof but not in the formal one. In particular, all relevant results are completely formalized in mathlib. In order for the links to stay usable, they are all to a fixed commit to the master branch (the most recent one at the time of writing).

[1] There are some set-theoretic issues that prevent it from satisfying all the axioms of a topos; these can be resolved in various ways and, for all practical purposes, the category of condensed sets can be regarded as a topos.

Here is a brief outline of the paper. In §2 we give a brief overview of the LEAN proof assistant and its mathematical library MATHLIB, explaining the general philosophy behind the library and the main design decisions that have been taken, focusing on the aspects that are most relevant to the present work. In §3, we review the theory of sheaves for Grothendieck topologies as it is formalized in MATHLIB: this section is standard, but we think it is a good idea to fix the notation and the terminology, as the literature is not always consistent. In §4, we introduce the notions of strict, regular and effective epimorphism. We prove in Proposition 4.12 that the effective epimorphisms in the category $\mathcal{C}$ of topological spaces are the quotient maps and Proposition 4.13 characterizes effective epimorphisms in $\mathcal{C}$ as the continuous surjections. Strict, regular, and effective epimorphisms are then used in §5 to define the regular (*resp.* extensive, coherent) topology on a category satisfying the technical condition of being preregular (*resp.* finitary extensive, precoherent). We prove in Proposition 5.8 that a preregular and finitary extensive category is precoherent and in Proposition 5.9 that the coherent topology is generated by the union of the regular and extensive topologies. In §6, we study sheaves on these three topologies: first of all we prove in Propositions 6.1, 6.6, and 6.8 that the three topologies are subcanonical. We then give in Propositions 6.4, 6.5, 6.13, and 6.14 various conditions for a presheaf to be a sheaf (characterizing sheaves in terms of the preservation of finite products and equalizers). We then give in Proposition 6.15 a condition for a functor[2] to induce an equivalence between the categories of sheaves for certain topologies. In §7 we apply our general categorical framework to the theory of condensed sets, proving our main theorems, that we now summarize.

Consider the following three categories, each containing the next as a full subcategory, and whose morphisms are continuous maps:

- $\mathsf{CompHaus}$: the category of compact Hausdorff spaces ↗.
- $\mathsf{Profinite}$: the category of *profinite* spaces, that we define, following MATHLIB, as totally disconnected compact Hausdorff spaces. This category is equivalent to the pro-category of the category of finite sets (this last statement has not yet been formalized; see [3, Section 6] for a more detailed discussion of the state of the category $\mathsf{Profinite}$ in MATHLIB) ↗.
- $\mathsf{Stonean}$: the category of *Stonean* spaces, whose objects are extremally disconnected compact Hausdorff spaces ↗. The condition of being extremally disconnected means that the closure of every open set is open. These spaces are precisely the projective objects in $\mathsf{CompHaus}$ (see [10, Theorem 2.5] and ↗). It is easy to see that Stonean spaces are totally disconnected, so we have a fully faithful inclusion $\mathsf{Stonean} \subseteq \mathsf{Profinite}$ ↗.

Let $\mathcal{C}$ be any of these categories. We prove in Proposition 5.8 and Proposition 7.1 that the categories $\mathcal{C}$ fit into the general framework we describe in this paper. As a consequence, we recover the following two key results (stated here as Theorem 7.4 and 7.7) which have appeared early on in the theory of condensed mathematics [15, Definition 1.2 and Proposition 2.7].

Theorem. *We have the following characterizations of sheaves on $\mathcal{C}$.*

- *When $\mathcal{C}$ is $\mathsf{CompHaus}$ or $\mathsf{Profinite}$, a presheaf $X\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ is a sheaf for the coherent topology on $\mathcal{C}$ if and only if it satisfies the following two conditions:*
  1) *X preserves finite products: in other words, for every finite family (T_i) of objects of $\mathcal{C}$, the natural map*
  $$X\Big(\coprod_i T_i\Big) \longrightarrow \prod_i X(T_i)$$
  is a bijection.
  2) *For every surjection $\pi\colon S \to T$ in $\mathcal{C}$, the diagram*
  $$X(T) \xrightarrow{X(\pi)} X(S) \rightrightarrows X(S \times_T S)$$
  is an equalizer (the two parallel morphisms being induced by the projections in the pullback).

[2]In this work we follow the convention that all functors are, by definition, covariant; we refer to contravariant functors as *presheaves*.

- *A presheaf* $X\colon \mathsf{Stonean}^{\mathrm{op}} \to \mathsf{Set}$ *is a sheaf for the coherent topology on* Stonean *if and only if it preserves finite products: in other words, for every finite family* (T_i) *of object of* $\mathcal{C}$*, the natural map*

$$X\Big(\coprod_i T_i\Big) \longrightarrow \prod_i X(T_i)$$

is a bijection.

Theorem. *The inclusion functors* Profinite → CompHaus *and* Stonean → CompHaus *induce equivalences of categories between the categories of sheaves for the coherent topology on* CompHaus*,* Profinite*, and* Stonean*.*

Recall that a condensed set is defined as a sheaf for the coherent topology on CompHaus. Thanks to the second theorem, the category of condensed sets is equivalent to the category of sheaves for the coherent topology on Profinite or Stonean.

In fact these theorems hold for very general target categories other than that of sets, they certainly hold for the category of modules over a ring, for example. Regarding condensed objects simply as product-preserving presheaves on Stonean allows us to perform many constructions "objectwise" on Stonean. For example, limits and filtered colimits of condensed sets are given objectwise on Stonean; in the setting of condensed abelian groups or modules, the situation is even better — *all* colimits are computed objectwise on Stonean. Furthermore, epimorphisms of condensed objects in a sufficiently nice concrete category are simply those morphisms $X \to Y$ which satisfy the property that the induced map $X(S) \to Y(S)$ is surjective for every object S of Stonean. These two facts are essential in proving that condensed abelian groups form an abelian category which satisfies all the same of Grothendieck's AB axioms as the category of abelian groups. This result has not yet made it into MATHLIB, but is well within reach.

2. Mathlib

The results we describe in this paper have all been formalized using the LEAN interactive theorem prover, and incorporated into its open-source formalized mathematical library MATHLIB [13]. The LEAN community maintains MATHLIB as a large monolith with a number of overarching design decisions, which must be taken into account in all mathematical contributions to it. This section explains the particulars of MATHLIB that played a key motivating role in the presentation results we discuss in this paper. While we do not provide an introduction to the LEAN theorem prover itself, we refer the reader to [9] for a comprehensive discussion.

2.1. **Mathematical cohesion.** One of the key design decisions made in MATHLIB is that it strives to be a *cohesive* library. This point of view manifests concretely in a few ways. Most notably, it often means that mathematical concepts usually have one "official" definition in MATHLIB, and various related definitions and lemmas are built around such official definitions (this collection of ancillary results is often referred to as "the API") allowing users to work with them effectively. The importance of this approach cannot be understated when it comes to formalization of advanced mathematics.

MATHLIB allows formalizers to efficiently use the constructions from the library, even when their work lies at the intersection of several subjects, which condensed mathematics certainly does. To take a small example, the definition of a condensed set mentions the category of compact Hausdorff spaces, and one frequently has to use both the topological properties of the objects of this category and the more abstract properties of the category itself. The cohesive nature of MATHLIB ensures that the interplay between these two aspects of compact Hausdorff spaces runs smoothly. This is in contrast with the alternative approach where there are separate libraries for different areas of mathematics, which can potentially be problematic should the same concept appear in two different libraries following different conventions, since results from one library would not be directly compatible with results in the other

2.2. **The "right" generality.** A related and equally important design decision in MATHLIB is that mathematical contributions should be developed in the "right" level of generality. Although the utility of this approach is clear — a more general result applies in more contexts — it is often more convenient for mathematicians to work in the correct level of generality *for their current project.* However, when making contributions to MATHLIB, formalizers are encouraged to keep in mind the cohesive and interconnected nature of the library, since it is often impossible to know how an initial contribution may be used in the future, and in what context.

Nevertheless, it is important to mention that it is usually difficult to find the right level of generality for MATHLIB at first. It often happens that preexisting code in MATHLIB is *refactored* to bring it closer to MATHLIB prescribed ideals. In fact, such a refactoring process often occurs in conjunction or in parallel with API development as discussed above.

2.3. **Contribution Process.** Ensuring that the design decisions of MATHLIB are maintained requires significant experience with the library. In practice, this means that contributions must pass a process resembling peer review, whereby "pull requests" are opened for potential contributions, which are then reviewed by a team of reviewers and maintainers before being incorporated into the library.

2.4. **Condensed Mathematics.** Having discussed some of the key design decisions of MATHLIB, and how these relate to contributions of formalized mathematics within the library, it should come as no surprise that the development of condensed mathematics in MATHLIB follows the same lines. The goal of this paper is to describe the mathematics behind the foundations of condensed mathematics in a way which is suitable for inclusion in MATHLIB. In fact, the general categorical approach we outline in this paper was originally *motivated* by the goal of finding the "right" level of generality appropriate for its inclusion in MATHLIB.

2.5. **Size issues.** Condensed mathematics is known to raise subtle set-theoretic issues, see [15, Remark 1.3]. These can be solved in different ways, one is explained in [15, Appendix to Lecture II] and another in [4, 1.2–1.4], the latter being closer to the approach used in MATHLIB. One advantage of formalizing the theory is to guarantee that all these problems are solved in a precise way. Roughly speaking, the idea is to use Grothendieck's universes. These are more or less built into the axiomatic framework of LEAN, which is a version of dependent type theory relying on the calculus of inductive constructions. For a more detailed explanation of the foundations of LEAN, we refer the reader to [6].

The basic objects of the theory are *terms* and *types*. Every term has a type, and a type can be regarded as a collection of elements, which are the terms of that type. In this way, types replace sets in their everyday use in mathematics as "collections of elements". The notation `a : A` is used to signify that `a` is a term of the type `A`. To avoid an analogue of Russell's paradox known as Girard's paradox, LEAN uses a *hierarchy of universes* indexed by the natural numbers

```
Type = Type 0
Type 0 : Type 1
Type 1 : Type 2
...
```

MATHLIB's definition of a category has two universe parameters u and v. The definition consists of a "set of objects" `(C : Type u)`, and for every pair of objects `X Y : C`, of a "set of morphisms" `(X ⟶ Y : Type v)`. Throughout this paper, we will use the word "set" informally in this way, letting LEAN take care of making sure that the "set" in question has a high enough universe level. For a concrete example, see Definition 3.11 where we mention the *top sieve* on an object X in a category $\mathcal{C}$. This is supposed to be the "set of all morphisms in $\mathcal{C}$ with target X". When $\mathcal{C}$ is a large category, this is not a set in the sense of set-theoretic foundations, but as explained above, our use of the word "set" is not abusive in this case.

MATHLIB's axioms are known to be equivalent to Zermelo–Fraenkel set theory plus the axiom of choice and the existence of n inaccessible cardinals for all $n \in \mathbb{N}$, see [6, Corollary 6.8]. In particular, the existence of the hierarchy of universes (and their precise behavior with respect to various constructions) is provable in ZFC using a relatively weak assumption about large cardinals.

3. Preliminaries

3.1. **Coverages.** There are various ways to formulate the notion of a site and Grothendieck topology on a category $\mathcal{C}$, which allows us to define the notion of a *sheaf* on $\mathcal{C}$. In order to fix the terminology, we start this section by recalling some basic definitions and results in this area. The terminology we describe here matches the terminology used in the corresponding definitions that can be found in MATHLIB.

Fix a category $\mathcal{C}$ throughout this section.

Definition 3.1. Let X be an object of $\mathcal{C}$. A *presieve* S on X is a set of morphisms with target X. If $f \in S$ is a morphism, we will use the notation $\operatorname{dom} f$ for the domain of f.

Remark 3.2. It is sometimes convenient to consider an *indexed* family of morphisms $(f_i : X_i \to X)_{i\in I}$, indexed by some set I. Of course, any such family yields a presieve S on X which contains only the morphisms f_i for $i \in I$. Conversely, any presieve can be considered as a family indexed by its elements.

The notion of an indexed family of morphisms over X is not exactly equivalent to that of a presieve over X, as an indexed family may have duplicates while a presieve cannot. However, it is sometimes convenient to use indexed families as opposed to presieves, and we will allow ourselves to freely go back and forth as discussed above.

Definition 3.3. Let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf on $\mathcal{C}$ and let S be a presieve on an object X of $\mathcal{C}$. A *family of elements* for S is a collection $(x_f)_{f\in S}$ where $x_f \in F(\operatorname{dom} f)$ for all $f \in S$. We say that such a family of elements $(x_f)_f$ is *compatible* provided that for all commutative squares in $\mathcal{C}$ of the form

$$\begin{array}{ccc} Y & \xrightarrow{\;g\;} & \operatorname{dom} f \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{dom} f' & \xrightarrow[\;f'\;]{} & X, \end{array}$$

with $f, f' \in S$, one has $F(g)(x_f) = F(g')(x_{f'})$. We say that $x \in F(X)$ is an *amalgamation* for $(x_f)_{f\in S}$ if $F(f)(x) = x_f$ for all $f \in S$.

Definition 3.4. We say that a presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ *is a sheaf for the presieve* S if for every compatible family of elements for S there exists a unique amalgamation.

Remark 3.5. If a presieve S on X is constructed out of an indexed family $(f_i : X_i \to X)_{i\in I}$ such that for all $i, j \in I$, the pullback $X_i \times_X X_j$ exists, one can rephrase the sheaf condition for the presieve as saying that the diagram

$$F(X) \longrightarrow \prod_{i\in I} F(X_i) \rightrightarrows \prod_{i,j\in I} F(X_i \times_X X_j)$$

is an equalizer, where the map on the left is given by the collection $\big(F(f_i)\big)_{i\in I}$ and the two parallel maps are induced by the projections in the pullbacks.

Definition 3.6. A *coverage* on $\mathcal{C}$ is the datum of a set of presieves on each object X of $\mathcal{C}$, called *covering presieves*, satisfying the following property: For every morphism $f\colon X \to Y$ in $\mathcal{C}$ and every covering presieve S on Y, there exists a covering presieve T on X such that for each $g \in T$, the composition $f \circ g$ factors through some morphism $h \in S$.

Definition 3.7. A *sieve* S on an object X of $\mathcal{C}$ is a presieve on X which is *downwards closed* in the sense that for each $f \in S$ and every g that is composable with f, we have that $f \circ g \in S$. The sieve $\langle R\rangle$ *generated* by a presieve R is the sieve consisting of all morphisms that factor through a morphism of R; this is the smallest sieve containing R. We also call $\langle R\rangle$ the *sieve associated to* R.

Remark 3.8. A sieve S on X can be regarded as a full subcategory of the overcategory $\mathcal{C}_{/X}$, and thus it comes equipped with a forgetful functor $S \to \mathcal{C}$. The sieve S induces a cocone over this functor, whose cocone point is X, and whose coprojections are the morphisms in S. This cocone will be used later.

Proposition 3.9. *Let X be an object in $\mathcal{C}$ and let S be a presieve on X. A presheaf F is a sheaf for S if and only if it is a sheaf for $\langle S\rangle$.*

Proof. See [11, Lemma C.2.1.3] or MATHLIB. □

Definition 3.10. The *pullback* of a sieve $S = (g_i\colon Y_i \to Y)_{i\in I}$ on Y along a morphism $f\colon X \to Y$ is the sieve on X consisting of all morphisms $g\colon Y_i \to X$ (for $i \in I$) such that $f \circ g \in S$. It is denoted f^*S.

Definition 3.11. A *Grothendieck topology* on $\mathcal{C}$ is the datum of a set of sieves on each object X of $\mathcal{C}$, called *covering sieves* satisfying the following properties:

GT-1) The *top sieve* — consisting of all morphisms in $\mathcal{C}$ with target X — is a covering sieve on X.

GT-2) For every covering sieve S on Y and every morphism $f\colon X \to Y$, the pullback f^*S is a covering sieve on X.

GT-3) Given a covering sieve S on Y, suppose another sieve R on Y satisfies the property that for every $f : X \to Y \in S$, f^*R is a covering sieve on X. Then R is also a covering sieve on Y.

Lemma 3.12. *Let $\mathcal{T}$ be a Grothendieck topology on $\mathcal{C}$, let X be an object of $\mathcal{C}$, and S and R be two sieves on X such that S is contained in R (meaning that every morphism in S is in R). If S is a covering sieve for $\mathcal{T}$, then R is a covering sieve as well.*

Proof. By axiom GT-3), it suffices to show that for every $f : Y \to X$ in S, f^*R is a covering sieve of Y. By axiom GT-1) it suffices to show that f^*R contains every morphism to Y. So let $g: Z \to Y$ be a morphism. Since $f \circ g$ is in S, it is in R, meaning that g is in f^*R, as desired. □

Definition 3.13. The coverage associated to a Grothendieck topology $\mathcal{T}$ is the coverage whose covering presieves are those whose associated sieve is a covering sieve in $\mathcal{T}$. The Grothendieck topology generated by a coverage $\mathcal{S}$ is the intersection of all Grothendieck topologies whose associated coverage contains $\mathcal{S}$.

Another definition of the Grothendieck topology $\mathcal{T}$ generated by a coverage can be given in terms of a *saturation* process. To define this, we start by ordering the collections of sieves on an object X by objectwise inclusion; given a coverage $\mathcal{S}$, its *saturation* is the smallest family $\big(C(X)\big)_{X\in\mathcal{C}}$ of collections of sieves satisfying:

Sat-1) For every object X, the top sieve on X is in $C(X)$.
Sat-2) For every object X and every covering presieve S on X in $\mathcal{S}$, we have $\langle S\rangle \in C(X)$.
Sat-3) For every object X and every pair S, R of sieves on X such that $S \in C(X)$ and such that for each $f \in S$ the pullback f^*R belongs to $C(Y)$, we have that R lies in $C(X)$.

In terms of the dependent type theory underlying Lean, requiring that this be "the smallest family" with a certain property is particularly handy, as it can be formalized in terms of *inductive types*, a notion that lies at the very core of the foundational set-up of Lean and therefore whose implementation and development is remarkably well integrated. This inductive construction is the one that is currently implemented in mathlib as follows :

```
inductive saturate (K : Coverage C) : (X : C) → Sieve X → Prop where
  | of (X : C) (S : Presieve X) (hS : S ∈ K X) : saturate K X (Sieve.generate S)
  | top (X : C) : saturate K X Top
  | transitive (X : C) (R S : Sieve X) :
    saturate K X R →
    (∀ ⦃Y : C⦄ ⦃f : Y ⟶ X⦄, R f → saturate K Y (S.pullback f)) →
    saturate K X S
```

To prove that the saturation of $\mathcal{S}$ is in fact a Grothendieck topology, axioms GT-1) and GT-3) follow at once from the defining properties Sat-1) and Sat-3) of the saturation. Verifying property GT-2) requires a bit more work and is achieved by applying the principle of induction on this inductive type. The formalization of this property is :

```
def toGrothendieck (K : Coverage C) : GrothendieckTopology C where
  sieves := saturate K
  top_mem' := .top
  pullback_stable' := by ... --the inductive proof mentioned above
  transitive' X S hS R hR := .transitive _ _ _ hS hR
```

It follows quite easily that the definition through saturations coincides with the one in Definition 3.13, an equivalence whose proof is formalized in the theorem :

```
theorem toGrothendieck_eq_sInf (K : Coverage C) : toGrothendieck _ K =
    sInf {J | K ≤ ofGrothendieck _ J } := by ...
```

Definition 3.14. Let $\mathcal{T}$ be a Grothendieck topology on $\mathcal{C}$. A presheaf $F: \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ is a *sheaf* for $\mathcal{T}$ if it is a sheaf for every covering sieve.

Proposition 3.15. *If a Grothendieck topology $\mathcal{T}$ is generated by a coverage, then a presheaf is a sheaf if and only if it is a sheaf for every covering presieve in the coverage.*

Proof. A proof can be found in [11, Proposition C.2.1.9]. The proof that appears in MATHLIB uses induction based on the inductive definition of the Grothendieck topology generated by a coverage discussed above. If one uses Definition 3.13 instead, a proof can be obtained by using the equivalence of this definition with the inductive construction. □

4. Effective epimorphisms

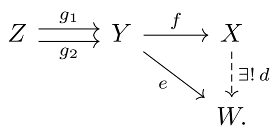
In the literature, there are three related conditions on a morphism, designed to capture the property of surjectivity better than the standard notion of an epimorphism. These are called *strict*, *regular* and *effective* epimorphisms respectively; each property implies the previous one. However, each property requires more assumptions on the underlying category than the previous one, and when the assumptions to define *effective epimorphism* hold, then strict implies effective. So, in a sense, these conditions are all equivalent. This is why it was decided to use the name *effective* in MATHLIB for the most generally applicable notion, usually called *strict*. For a more precise explanation of this justification of terminology, see the text following Definition 4.6.

In the category of topological spaces and the category of compact Hausdorff spaces, the effective epimorphisms are precisely the quotient maps. In the latter, the quotient maps are simply the continuous surjections, so the properties of being surjective, an epimorphism and an effective epimorphism all coincide (see Propositions 4.12 and 4.13).

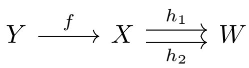
Definition 4.1. A morphism $f: X \to B$ in a category $\mathcal{C}$ is a *regular epimorphism* if it exhibits B as a coequalizer of some pair of morphisms $g_1, g_2: Z \to X$.

Remark 4.2. If a regular epimorphism $f: X \to B$ has a kernel pair (meaning that the pullback $X \times_B X$ exists), then B is the coequalizer of the two projections $X \times_B X \to X$.

Definition 4.3. A morphism $f: Y \to X$ in a category $\mathcal{C}$ is an *effective epimorphism* if it satisfies the following condition: for every morphism e that coequalizes every pair of parallel morphisms which f coequalizes, there exists a unique morphism d such that $d \circ f = e$:



Remark 4.4. It is easy to check that if $f: Y \to X$ is an effective epimorphism, then it is an epimorphism. Indeed, given a diagram



such that $h_1 \circ f = h_2 \circ f$, observe that $h_1 \circ f$ equalizes every pair of morphisms $g_1, g_2: Z \to Y$ equalized by f. In particular, there is a unique map $d: X \to W$ such that $d \circ f = h_1 \circ f$, and since h_1 and h_2 both satisfy this property, we deduce $h_1 = h_2$.

In MATHLIB, the notion of effective epimorphism is implemented in two steps. First, we define a structure `EffectiveEpiStruct` that contains the data required to be an effective epimorphism:

```
structure EffectiveEpiStruct {X Y : C} (f : Y ⟶ X) where
  desc : ∀ {W : C} (e : Y ⟶ W),
    (∀ {Z : C} (g₁ g₂ : Z ⟶ Y), g₁ ≫ f = g₂ ≫ f → g₁ ≫ e = g₂ ≫ e) → (X ⟶ W)
  fac : ∀ {W : C} (e : Y ⟶ W)
    (h : ∀ {Z : C} (g₁ g₂ : Z ⟶ Y), g₁ ≫ f = g₂ ≫ f → g₁ ≫ e = g₂ ≫ e),
    f ≫ desc e h = e
  uniq : ∀ {W : C} (e : Y ⟶ W)
    (h : ∀ {Z : C} (g₁ g₂ : Z ⟶ Y), g₁ ≫ f = g₂ ≫ f → g₁ ≫ e = g₂ ≫ e)
    (m : X ⟶ W), f ≫ m = e → m = desc e h
```

The field `desc` provides, given a morphism $e\colon Y \to W$ which coequalizes every morphism that f coequalizes, the morphism $d\colon X \to W$; the field `fac` is a proof that $d \circ f = e$; and the field `uniq` is a proof that d is unique.

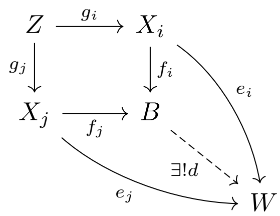
We then define a class `EffectiveEpi`, which is a proposition saying that the type of `EffectiveEpiStruct`'s associated to f is nonempty[3]:

```
class EffectiveEpi {X Y : C} (f : Y ⟶ X) : Prop where
  effectiveEpi : Nonempty (EffectiveEpiStruct f)
```

Definition 4.5. Given a family of morphisms $f = (f_i\colon X_i \to B)_{i\in I}$ and a pair of morphisms $g_{j_1}\colon Z \to X_{j_1}$ and $g_{j_2}\colon Z \to X_{j_2}$, we say that the family *coequalizes* g_{j_1} *and* g_{j_2} if $f_{j_1} \circ g_{j_1} = f_{j_2} \circ g_{j_2}$.

Definition 4.6. A family of morphisms $(f_i\colon X_i \to B)_{i\in I}$ in a category $\mathcal{C}$ is *effective epimorphic* if it satisfies the following universal property:

Given any family $(e_i\colon X_i \to W)_{i\in I}$ coequalizing every pair of morphisms $g_i\colon Z \to X_i$, $g_j\colon Z \to X_j$ which f coequalizes, there exists a unique morphism d such that for all i, $d \circ f_i = e_i$:



The notion of effective epimorphic family is formalized in a similar two-step process where we first define

```
structure EffectiveEpiFamilyStruct {B : C} {α : Type*}
    (X : α → C) (π : (a : α) → (X a ⟶ B)) where
  desc : ∀ {W} (e : (a : α) → (X a ⟶ W)),
          (∀ {Z : C} (a₁ a₂ : α) (g₁ : Z ⟶ X a₁) (g₂ : Z ⟶ X a₂),
            g₁ ≫ π _ = g₂ ≫ π _ → g₁ ≫ e _ = g₂ ≫ e _) → (B ⟶ W)
  fac : ∀ {W} (e : (a : α) → (X a ⟶ W))
          (h : ∀ {Z : C} (a₁ a₂ : α) (g₁ : Z ⟶ X a₁) (g₂ : Z ⟶ X a₂),
            g₁ ≫ π _ = g₂ ≫ π _ → g₁ ≫ e _ = g₂ ≫ e _)
              (a : α), π a ≫ desc e h = e a
  uniq : ∀ {W} (e : (a : α) → (X a ⟶ W))
          (h : ∀ {Z : C} (a₁ a₂ : α) (g₁ : Z ⟶ X a₁) (g₂ : Z ⟶ X a₂),
            g₁ ≫ π _ = g₂ ≫ π _ → g₁ ≫ e _ = g₂ ≫ e _)
              (m : B ⟶ W), (∀ (a : α), π a ≫ m = e a) → m = desc e h
```

and then

```
class EffectiveEpiFamily {B : C} {α : Type*} (X : α → C)
    (π : (a : α) → (X a ⟶ B)) : Prop where
  effectiveEpiFamily : Nonempty (EffectiveEpiFamilyStruct X π)
```

Definitions 4.3 and 4.6 work in *any* category; the morphism in question is not required to have a kernel pair. It is easy to see that if f is a regular epimorphism, then it is an effective epimorphism. Conversely, if an effective epimorphism f has a kernel pair, then it is a regular epimorphism (see). This justifies the use of the terminology "effective epimorphism";

[3]The fact that `EffectiveEpi` is a class allows Lean to use *typeclass inference* to infer that a morphism is effective epimorphic in some cases: for example, in `CompHaus`, given a morphism `f` with an `[Epi f]` instance, Lean can automatically infer an instance `EffectiveEpi f`. Moreover, the internal axiomatic of Lean guarantees that two terms of a proposition are definitionally equal: in particular, two *proofs* of non-emptiness of `EffectiveEpiStruct f` automatically coincide, whereas producing explicit witnesses might lead to different outcomes, and that would often be troublesome.

We give some characterizations of effective epimorphic families. For an object W of $\mathcal{C}$, let h_W denote the representable presheaf $h_W(X) = \mathrm{Hom}_{\mathcal{C}}(X, W)$.

Lemma 4.7. *Let $(f_i\colon X_i \to B)$ be a family of morphisms in $\mathcal{C}$. Let S be the sieve generated by the set $(f_i)_{i\in I}$, regarded as a presieve. Then the following are equivalent:*

- *(i) The family $(f_i)_i$ is effective epimorphic.*
- *(ii) For every object W of $\mathcal{C}$, the presheaf h_W is a sheaf for S.*
- *(iii) The cocone in $\mathcal{C}$ corresponding to the sieve S (described in Remark 3.8) is colimiting.*

Proof. (i) $\iff$ (ii): First of all, observe that (ii) is equivalent to h_W being a sheaf for $(f_i)_i$. Moreover, the data of a compatible family (in the sense of Definition 3.3) for $(f_i)_i$ is a family $(x_i\colon X_i \to W)_i$ that coequalizes every pair of morphisms that $(f_i)_i$ coequalizes and an amalgamation for it is the morphism denoted d in Definition 4.6. The equivalence between (i) and (ii) follows.

(ii) $\implies$ (iii): Suppose we have another cocone on the same functor, with cocone point W and coprojections $x_f : X \to W$ for any $f : X \to B$ contained in S. We will now prove that this is precisely the data of a compatible family for S. Indeed, if $f\colon X \to B$ and $f'\colon X' \to B$ are in S, and the square

$$\begin{array}{ccc} Y & \xrightarrow{g'} & X' \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ X & \xrightarrow[f]{} & B \end{array}$$

commutes, then $f\circ g = f'\circ g' \in S$ because of the downwards closed property of sieves. We have coprojections $x_f\colon X \to W$, $x_{f'}\colon X' \to W$ and $x_{f\circ g} = x_{f'\circ g'}\colon Y \to W$ of the cocone with cocone point W, which satisfy

$$x_{f'} \circ g' = x_{f'\circ g'} = x_{f\circ g} = x_f \circ g$$

which is what we wanted. The unique amalgamation given by (ii) gives the unique cocone morphism required to satisfy the universal property of the colimit.

(iii) $\implies$ (i): Given a family $(e_i\colon X_i \to W)$ that coequalizes any pair of morphisms $g_i\colon Z \to X_i$, $g_j\colon Z \to X_j$ that is coequalized by f, we obtain a cone over S with cone point W as follows: recall that S is generated by the $(f_i)_i$, and thus the morphisms in S are precisely those which factor through f_i for some i. Thus, for each morphism $g\colon Y \to B$ in S, we may write $g = f_i \circ h$ for some i, and set $w_g := e_i \circ h$ — this is well-defined by the assumption on $(e_i)_i$ We get the desired map $d\colon B \to W$ by the universal property of colimits. ☐

Lemma 4.8. *Let $(\pi_i\colon X_i \to B)_{i\in I}$ be an effective epimorphic family in $\mathcal{C}$, such that the coproduct of $(X_i)_i$ exists. The map*

$$\pi\colon \coprod_i X_i \longrightarrow B$$

induced by $(\pi_i)_i$ is an effective epimorphism.

Proof. Let $\iota_i\colon X_i \to \coprod_i X_i$ denote the coprojections of the coproduct. Let $e\colon \coprod_i X_i \to W$ be a morphism which coequalizes every pair of morphisms that π coequalizes. It is clear that the family $(e\circ\iota_i)_{i\in I}$ coequalizes every pair $g_i\colon Z \to X_i$, $g_j\colon Z \to X_j$ that $(\pi_i)_{i\in I}$ coequalizes. It is easy to see that the morphism $d\colon B \to W$ obtained from the universal property of the effective epimorphic family gives the universal property of effective epimorphisms for π. ☐

Lemma 4.9. *Let $(\pi_i\colon X_i \to B)_{i\in I}$ be a family of morphisms in $\mathcal{C}$. Suppose that*

1) *All coproducts and pullbacks appearing in 2) exist.*
2) *For every object Z and every morphism*

$$g\colon Z \longrightarrow \coprod_i X_i,$$

the induced map

$$i(g) := \coprod_i Z \times_{\coprod_i X_i} X_i \longrightarrow Z$$

is an epimorphism.

3) The map

$$\pi\colon \coprod_i X_i \longrightarrow B$$

induced by $(\pi_i)_i$ is an effective epimorphism.

Then $(\pi_i)_i$ is an effective epimorphic family.

Proof. Let $(e_i\colon X_i \to Z)_{i\in I}$ be a family which coequalizes every pair of morphisms $g_i\colon Z \to X_i$, $g_j\colon Z \to X_j$ which $(\pi_i)_i$ coequalizes. We need to show that there exists a unique $d\colon B \to Z$ such that for all such g_i, g_j, we have $d \circ g_i = d \circ g_j$. To obtain this, we will apply the property that π is an effective epimorphism to the induced morphism $e\colon \coprod_i X_i \to Z$. To be able to do this, we need to check that e coequalizes every pair of morphisms which π coequalizes.

Let $f_1, f_2\colon Z \to \coprod_i X_i$ be given and suppose that $\pi \circ f_1 = \pi \circ f_2$. We want to show that $e \circ f_1 = e \circ f_2$. Applying the fact that $i(f_1)$ is an epimorphism, it suffices to prove that

$$e \circ f_1 \circ i(f_1) = e \circ f_2 \circ i(f_1).$$

This identity can be checked on each component of the coproduct $\coprod_i Z \times_{\coprod_i X_i} X_i$. In other words, we need to show that for every $a \in I$,

$$e \circ f_1 \circ i(f_1) \circ \iota_a = e \circ f_2 \circ i(f_1) \circ \iota_a,$$

where

$$\iota_a\colon Z \times_{\coprod_i X_i} X_a \longrightarrow \coprod_i Z \times_{\coprod_i X_i} X_i$$

denotes the coprojection. One easily checks that

$$i(f_1) \circ \iota_a\colon Z \times_{\coprod_i X_i} X_a \longrightarrow Z$$

is simply the first projection map in the pullback, which we denote by p_1. We thus need to show that

$$e \circ f_1 \circ p_1 = e \circ f_2 \circ p_1.$$

The left-hand side simplifies to $e_a \circ p_2$, where

$$p_2\colon Z \times_{\coprod_i X_i} X_a \longrightarrow X_a$$

denotes the second projection in the pullback.

Now it again suffices to prove the equality after precomposition with the epimorphism $i(f_2 \circ p_1)$, i.e. to show that

$$e_a \circ p_2 \circ i(f_2 \circ p_1) = e \circ f_2 \circ p_1 \circ i(f_2 \circ p_1).$$

Again we can check this equality on the components of the coproduct $\coprod_b \left(Z \times_{\coprod_i X_i} X_a\right) \times_{\coprod_i X_i} X_b$, and similarly to above, this reduces to showing that for every $b \in I$,

$$e_a \circ g_a = e_b \circ g_b,$$

where

$$g_a\colon \left(Z \times_{\coprod_i X_i} X_a\right) \times_{\coprod_i X_i} X_b \longrightarrow X_a$$

is the first projection followed by the second projection, and

$$g_b\colon \left(Z \times_{\coprod_i X_i} X_a\right) \times_{\coprod_i X_i} X_b \longrightarrow X_b$$

is the second projection. Doing the same manipulation on the equality $\pi \circ f_1 = \pi \circ f_2$, we see that g_a, g_b is a pair of morphisms that the family $(\pi_i)_i$ coequalizes. By assumption, the family $(e_i)_i$ coequalizes it as well. This means that $e \circ f_1 = e \circ f_2$ and we obtain the unique $d\colon B \to Z$ we wanted. □

Propositions 4.12 and 4.13 provide an explicit description of effective epimorphisms in the categories of topological spaces, compact Hausdorff spaces, profinite spaces, and Stonean spaces. Both results ultimately rely on the observation that epimorphisms in these four categories are surjective, and we start with this result:

Lemma 4.10. *Let $\mathcal{C}$ be any of the categories* Top*,* CompHaus*,* Profinite *or* Stonean*. Then epimorphisms in $\mathcal{C}$ are surjective (continuous) maps.*

Proof. Note first that one direction is clear, because a surjective morphism in any concrete category is an epimorphism. Now let $f\colon Y \to X$ be a morphism in $\mathcal{C}$.

When $\mathcal{C} = \mathsf{Top}$ the result is very well known: suppose f is an epimorphism and consider the diagram

$$Y \xrightarrow{f} X \underset{e_1}{\overset{\chi}{\rightrightarrows}} \{0,1\}^\flat$$

where $\{0,1\}^\flat$ denotes the set $\{0,1\}$ endowed with the indiscrete topology, where χ is the characteristic function of $\mathrm{im}(f)$ and where e_1 is the constant map with image 1. Clearly, $\chi \circ f = e_1 \circ f$ and when f is an epimorphism this implies that $\chi = e_1$, which is the statement $\mathrm{im}(f) = X$.

When $\mathcal{C} = \mathsf{CompHaus}$, the above proof breaks down because $\{0,1\}^\flat$ is not in $\mathcal{C}$. But since spaces in $\mathcal{C}$ are normal, we can argue as follows: the subspace $\mathrm{im}(f) \subseteq X$ is compact, hence closed. Suppose that f is not surjective, and let $x \notin \mathrm{im}(f)$: by Urysohn's lemma, there is a continuous function $\theta\colon X \to [0,1]$ such that $\theta(x) = 0$ and $\theta(\mathrm{im}(f)) = 1$. Denote by $e_1\colon X \to [0,1]$ the constant function with image 1: then $e_1 \neq \theta$ and yet $f \circ \theta = f \circ e_1$ showing that f is not an epimorphism.

When $\mathcal{C} = \mathsf{Profinite}$ or $\mathcal{C} = \mathsf{Stonean}$ the above proof breaks down, because the unit interval is not in $\mathcal{C}$. But the argument for Top can be adapted by replacing the indiscrete space $\{0,1\}^\flat$ with the *discrete* space $\{0,1\}^\delta$, which is in $\mathcal{C}$. First, observe that, given any topological space Z and a clopen $U \subseteq Z$, the characteristic function χ_U is continuous. Moreover, since every object in $\mathcal{C}$ is totally disconnected, its topology admits a basis of open neighbourhoods that are clopen sets. Now suppose f is not surjective, and let $x \notin \mathrm{im}(f)$. Since — as before — $\mathrm{im}(f)$ is closed, there exists an open neighbourhood V of x contained in the complement $\mathrm{im}(f)^c$ and we can find a clopen neighbourhood $U \subseteq V$ such that $x \in U$ and $U \cap \mathrm{im}(f) = \emptyset$. Consider the diagram in $\mathcal{C}$

$$Y \xrightarrow{f} X \underset{e_0}{\overset{\chi_U}{\rightrightarrows}} \{0,1\}^\delta$$

where e_0 is the constant function with value 0. Now $\chi_U \neq e_0$, as can be seen by evaluating them on x, yet $\chi_U \circ f = e_0 \circ f$ since $U \cap \mathrm{im}(f) = \emptyset$. This shows that f is not an epimorphism. ☐

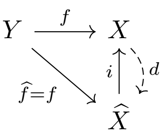
Lemma 4.11. *Let $\mathcal{C}$ be a full subcategory of* Top *and let $f\colon Y \to X$ be a morphism in $\mathcal{C}$ which is a quotient map. Then f is an effective epimorphism in $\mathcal{C}$.*

Proof. Suppose that $e\colon Y \to Z$ equalizes every morphism that f equalizes. This means that for every pair of points $y_1, y_2 \in Y$ such that $f(y_1) = f(y_2)$, we have $e(y_1) = e(y_2)$, as can be seen by considering the parallel morphisms $e_{y_1}, e_{y_2}\colon Y \to Y$ sending everything to y_1 and to y_2, respectively. The universal property of the quotient topology on X provides the existence of a unique continuous $d\colon X \to Z$ such that $d \circ f = e$, showing that f is an effective epimorphism. ☐

Proposition 4.12. *The effective epimorphisms in* Top *are the quotient maps.*

Proof. A quotient map is an effective epimorphism in Top by Lemma 4.11.

In the other direction, let $f\colon Y \to X$ be an effective epimorphism in Top. By Remark 4.4 and Lemma 4.10, f is surjective and we are simply left to prove that in this situation X is endowed with the quotient topology, namely the final topology induced by f. Denote by $\widehat{X}$ the space whose underlying set coincides with X, but endowed with the final topology induced by f, so that the identity map $i\colon \widehat{X} \to X$ is continuous. In the diagram



the morphism $\widehat{f}$ equalizes every pair of morphisms equalized by f, so there exists a unique *continuous* map $d\colon X \to \widehat{X}$ making the diagram commute. It follows that d is induced by the identity, showing that X is homeomorphic to $\widehat{X}$, as required. ☐

Proposition 4.13. *The effective epimorphisms in* $\mathsf{CompHaus}, \mathsf{Profinite}$ *and in* $\mathsf{Stonean}$ *are the (continuous) surjections.*

Proof. Let $\mathcal{C}$ be any of the categories $\mathsf{CompHaus}, \mathsf{Profinite}$ or $\mathsf{Stonean}$ and let $f : Y \to X$ be an effective epimorphism in $\mathcal{C}$. Combining Remark 4.4 and Lemma 4.10, yields that f is a continuous surjection.

In the other direction, consider a continuous surjection $f : Y \to X$ in $\mathcal{C}$. Since the objects of $\mathcal{C}$ are compact Hausdorff spaces, f is also a closed map and hence a quotient map, and thus an effective epimorphism by Lemma 4.11. ☐

5. Three Grothendieck topologies

5.1. The regular topology.

Definition 5.1. A category $\mathcal{C}$ is *preregular* if the collection of presieves consisting of single effective epimorphisms forms a coverage. In other words, if for every effective epimorphism $g : Z \to Y$ and every morphism $f : X \to Y$, there exists an effective epimorphism $h : W \to X$ and a morphism $i : W \to Z$ such that the following diagram commutes:

$$\begin{array}{ccc} W & \xrightarrow{i} & Z \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

In this case, we call this coverage the *regular coverage* on $\mathcal{C}$, and the Grothendieck topology generated by this coverage is called the *regular topology* on $\mathcal{C}$.

In MATHLIB, we define a predicate `Preregular` on categories:

```
class Preregular : Prop where
  exists_fac : ∀ {X Y Z : C} (f : X ⟶ Y) (g : Z ⟶ Y) [EffectiveEpi g],
    (∃ (W : C) (h : W ⟶ X) (_ : EffectiveEpi h) (i : W ⟶ Z), i ≫ g = h ≫ f)
```

Then the definition of the regular topology follows :

```
def regularCoverage [Preregular C] : Coverage C where
  covering B := { S | ∃ (X : C) (f : X ⟶ B), S = Presieve.ofArrows (fun (_ : Unit) ↦
    X)
    (fun (_ : Unit) ↦ f) ∧ EffectiveEpi f }
  pullback := by ...

def regularTopology [Preregular C] : GrothendieckTopology C :=
  Coverage.toGrothendieck _ <| regularCoverage C
```

5.2. The extensive topology.

Definition 5.2. A category $\mathcal{C}$ is *finitary extensive* if it satisfies the following properties:

1) $\mathcal{C}$ has finite coproducts.
2) $\mathcal{C}$ has pullbacks along coprojections of finite coproducts.
3) Every commutative diagram

$$\begin{array}{ccccc} Z_1 & \longrightarrow & Z & \longleftarrow & Z_2 \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X \coprod Y & \longleftarrow & Y \end{array}$$

consists of two pullback squares if and only if the top row is a coproduct diagram.

Remark 5.3. Our definition of finitary extensive category is precisely [5, Definition 2.1 and Proposition 2.2].

MATHLIB already had the predicate `FinitaryExtensive` on categories:

```
class FinitaryExtensive (C : Type u) [Category.{v} C] : Prop where
  [hasFiniteCoproducts : HasFiniteCoproducts C]
  [hasPullbacksOfInclusions : HasPullbacksOfInclusions C]
  van_kampen' : ∀ {X Y : C} (c : BinaryCofan X Y), IsColimit c → IsVanKampenColimit c
```

The field `van_kampen'` is condition 3) in Definition 5.2.

Proposition 5.4. *Let $\mathcal{C}$ be a finitary extensive category. The collection of finite families $(X_i \to X)_{i\in I}$ exhibiting X as a coproduct of the family $(X_i)_{i\in I}$, forms a coverage.*

Proof. The axioms of a finitary extensive category ensure that the required property holds, namely that given a morphism $f\colon X \to Y$ and a finite family of morphisms $(g_i\colon Y_i \to Y)_{i\in I}$, the family $(X \times_Y Y_i \to X)_{i\in I}$ exhibits X as a coproduct of the family $(X \times_Y Y_i)_{i\in I}$. This has been formalized in MATHLIB, but it appears *ibid.* as a definition: this is because the proof that the collection is a coverage is part of the definition in question. □

Definition 5.5. Let $\mathcal{C}$ be a finitary extensive category. The coverage defined in Proposition 5.4 is called the *extensive coverage* on $\mathcal{C}$, and the Grothendieck topology generated by this coverage is called the *extensive topology* on $\mathcal{C}$.

In MATHLIB, we define the extensive topology as follows:

```
def extensiveCoverage [FinitaryPreExtensive C] : Coverage C where
  covering B := { S | ∃ (α : Type) (_ : Finite α) (X : α → C) (π : (a : α) → (X a ⟶
    B)), S = Presieve.ofArrows X π ∧ IsIso (Sigma.desc π) }
  pullback := by ...

def extensiveTopology [FinitaryPreExtensive C] : GrothendieckTopology C :=
  Coverage.toGrothendieck _ <| extensiveCoverage C
```

Note that the definition of the extensive coverage and extensive topology only requires an assumption `[FinitaryPreExtensive C]`. This condition is slightly weaker than `FinitaryExtensive`, but the difference is unimportant. For the characterization of sheaves for the extensive topology, the stronger condition is indeed required.

5.3. The coherent topology.

Definition 5.6. A category $\mathcal{C}$ is *precoherent* if the collection of finite effective epimorphic families forms a coverage. In other words, if for any finite effective epimorphic family $(\pi_i\colon X_i \to B)_{i\in I}$ and any morphism $f\colon B' \to B$, there exists a finite effective epimorphic family $(\pi'_j\colon X'_j \to B')_{j\in I'}$, such that for each $j \in I'$, the composition $f \circ \pi'_j$ factors through π_i for some $i \in I$. In this case, we call this coverage the *coherent coverage* on $\mathcal{C}$, and the Grothendieck topology generated by this coverage is called the *coherent topology* on $\mathcal{C}$.

In MATHLIB, we define a predicate `Precoherent` on categories:

```
class Precoherent : Prop where
  pullback {B₁ B₂ : C} (f : B₂ ⟶ B₁) :
    ∀ (α : Type) [Finite α] (X₁ : α → C) (π₁ : (a : α) → (X₁ a ⟶ B₁)),
      EffectiveEpiFamily X₁ π₁ →
    ∃ (β : Type) (_ : Finite β) (X₂ : β → C) (π₂ : (b : β) → (X₂ b ⟶ B₂)),
      EffectiveEpiFamily X₂ π₂ ∧
      ∃ (i : β → α) (ι : (b :  β) → (X₂ b ⟶ X₁ (i b))),
      ∀ (b : β), ι b ≫ π₁ _ = π₂ _ ≫ f
```

Then the definition of the coherent topology follows:

```
def coherentCoverage [Precoherent C] : Coverage C where
  covering B := { S | ∃ (α : Type) (_ : Finite α) (X : α → C) (π : (a : α) → (X a ⟶
    B)),
```

```
    S = Presieve.ofArrows X π ∧ EffectiveEpiFamily X π }
  pullback := by ...

def coherentTopology [Precoherent C] : GrothendieckTopology C :=
  Coverage.toGrothendieck _ <| coherentCoverage C
```

Remark 5.7. The notion of a precoherent category naturally arose through the formalization process, and was forced upon us by the "MATHLIB philosophy" where definitions are often phrased in the most general way (see §2.2). Indeed, the condition that $\mathcal{C}$ is a precoherent category is precisely the minimal axiom needed to ensure that what we call the *coherent coverage* above is indeed a coverage. A similar approach was taken to define the notion of a *preregular* category. For example, we do not require the existence of pullbacks required in the definition of *regular* and *coherent* categories as in [11, A1.3] and [11, A1.4] respectively.

Due to our weaker assumptions, several of our results about the regular and coherent topology strengthen existing standard results. For example, [11, Example C.2.1.12 (d)] states that the coherent topology on a *coherent* category is subcanonical, which we extend in Proposition 6.8 below to *precoherent* categories. The analogous statement for the regular topology on a regular category can be found in [12, Corollary B.3.6], and is extended to preregular categories in Proposition 6.1 below. In Proposition 6.10 (respectively Lemma 6.2), we explicitly characterize the covering sieves in the coherent (respectively regular) topology on a precoherent (respectively preregular) category. Under stronger assumptions on the category, this result can be found in [12, Definition B.5.1 and Proposition B.5.2] (respectively in [11, C.2.1.12 (c)]).

5.4. **The coherent topology on a regular extensive category.**

Proposition 5.8. *Let $\mathcal{C}$ be a category which is preregular and finitary extensive. Then $\mathcal{C}$ is precoherent.*

Proof. Since $\mathcal{C}$ is finitary extensive, Lemmas 4.8 and 4.9 imply that finite effective epimorphic families in $\mathcal{C}$ are precisely those which induce an effective epimorphism on the coproduct.

Let $(f_i \colon X_i \to X)_{i \in I}$ be a finite effective epimorphic family and let $g \colon Y \to X$ be a morphism. Since the morphism $\coprod_i X_i \to X$ is an effective epimorphism, the fact that $\mathcal{C}$ is preregular ensures the existence of a diagram

$$\begin{array}{ccc} Z & \xrightarrow{e} & \coprod_i X_i \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & X \end{array}$$

in which $h \colon Z \to Y$ is an effective epimorphism.

Now, the fact that $\mathcal{C}$ is extensive ensures that the family $(Z \times_{\coprod_i X_i} X_i \to Z)_{i \in I}$ exhibits Z as a coproduct in the sense that the canonical map

$$\coprod_i Z \times_{\coprod_i X_i} X_i \longrightarrow Z$$

is an isomorphism. Therefore, the composition

$$\coprod_i Z \times_{\coprod_i X_i} X_i \longrightarrow Y$$

is an effective epimorphism, and therefore the family $(Z \times_{\coprod_i X_i} X_i \to Y)_{i \in I}$ works as the desired effective epimorphic family. □

It is obvious that the union of two coverages is a coverage. This allows us to state:

Proposition 5.9. *Let $\mathcal{C}$ be a category which is preregular and finitary extensive. The union of the regular and extensive coverages generates the coherent topology.*

Proof. Denote by $\mathcal{T}$ the topology generated by the union of the regular and extensive coverages. Note that the regular and extensive coverages are both contained in the coherent coverage, hence $\mathcal{T}$ is contained in the coherent topology, so it suffices to show that the coherent topology is contained in $\mathcal{T}$.

Let X be an object of $\mathcal{C}$ and let S be a covering sieve on X for the coherent topology: in other words, S is generated by a finite effective epimorphic family $(f_i\colon X_i \to X)_{i\in I}$. We want to show that S is a $\mathcal{T}$-covering sieve. Denote by

$$f\colon \coprod_{i\in I} X_i \longrightarrow X$$

the map induced by the f_i and for each $j \in I$, let

$$\iota_j\colon X_j \longrightarrow \coprod_{i\in I} X_i$$

be the coprojection. For each j,

$$f \circ \iota_j = f_j \in S, \text{ so } \iota_j \in f^*S.$$

Therefore, the sieve T generated by the family $(\iota_i)_i$ is contained in f^*S. Since the presieve generated by the family $(\iota_i)_i$ is a covering presieve of the coproduct in the extensive coverage, T is a $\mathcal{T}$-covering sieve and hence by Lemma 3.12, f^*S is a $\mathcal{T}$-covering sieve of $\coprod_i X_i$. By Lemma 4.8, f is an effective epimorphism, and hence the sieve S_f generated by the singleton presieve $\{f\}$ is a $\mathcal{T}$-covering sieve. Now by axiom GT-3) for Grothendieck topologies, it suffices to show that g^*S is a $\mathcal{T}$-covering sieve for every g in S_f. Given such a $g = f \circ h$, we have $g^*S = h^*(f^*S)$ which is a $\mathcal{T}$-covering sieve because f^*S is. □

6. Sheaves

6.1. **Regular sheaves.** Let $\mathcal{C}$ be a preregular category (see Definition 5.1).

Proposition 6.1. *The regular topology on $\mathcal{C}$ is subcanonical*[4].

Proof. We need to show that each presheaf of the form $h_W = \mathrm{Hom}(-, W)$ with W an object of $\mathcal{C}$ is a sheaf. By Proposition 3.15, it is enough to check that h_W is a sheaf for each family consisting of a single effective epimorphism. Noting that a singleton family is effective epimorphic if and only if it consists of an effective epimorphism, this is now clear from Lemma 4.7. □

Lemma 6.2. *A sieve in $\mathcal{C}$ is a covering sieve for the regular topology if and only if it contains an effective epimorphism.*

Proof. The proof is a simpler version of the proof of Proposition 6.10 below. The reader can easily take that proof and replace effective epimorphic families by effective epimorphisms, thereby filling in this proof (the key is to prove that effective epimorphisms in preregular categories are stable under composition). □

Lemma 6.3. *Suppose $\mathcal{C}$ has kernel pairs of effective epimorphisms. Then a presheaf F on $\mathcal{C}$ is a sheaf for the regular topology if and only if for every effective epimorphism $\pi\colon X \to B$, the diagram*

$$F(B) \xrightarrow{F(\pi)} F(X) \rightrightarrows F(X \times_B X) \tag{EqCond}$$

is an equalizer (the two parallel morphisms being given by the projections in the pullback).

Proof. This follows from the fact that a presheaf is a sheaf for the regular topology if and only if it is a sheaf for every family consisting of a single effective epimorphism, and the characterization (discussed in Remark 3.5) of the sheaf condition in the case where the relevant pullbacks exist. □

Proposition 6.4. *Suppose every object in $\mathcal{C}$ is projective*[5]. *Then every presheaf on $\mathcal{C}$ is a sheaf for the regular topology.*

Proof. Since every object is projective, every sieve generated by an epimorphism is the top sieve, for which every presheaf is a sheaf. □

[4] A Grothendieck topology is called *subcanonical* if every representable presheaf is a sheaf. By *representable*, we mean a presheaf of the form $\mathrm{Hom}(-, W)$ for some object W of $\mathcal{C}$.

[5] An object P is *projective* if every morphism out of P lifts along every epimorphism with the same target.

6.2. **Extensive sheaves.** Let $\mathcal{C}$ be a finitary extensive category (see Definition 5.2).

Proposition 6.5. *A presheaf on $\mathcal{C}$ is a sheaf with respect to the extensive topology if and only if it preserves finite products.*

Proof. This is proved in [12, Proposition B.4.5] (there, the extensive topology is defined only for categories with pullbacks, but the proof remains valid in our setting since only pullbacks along coprojections of finite coproducts are used). Our formalization follows the same ideas used *ibid.* □

Proposition 6.6. *The extensive topology on $\mathcal{C}$ is subcanonical.*

Proof. Since $\mathrm{Hom}(-, W)$ preserves limits, this follows from Proposition 6.5 □

Proposition 6.7. *Let X be an object of $\mathcal{C}$ and S a sieve on X. Then S is a covering sieve for the extensive topology on $\mathcal{C}$ if and only if it contains a family of morphisms $(f_i : X_i \to X)_{i \in I}$ which exhibit X as a coproduct of the X_i.*

Proof. The proof is a simpler version of the proof of Proposition 6.10 below. The reader can easily take that proof and replace effective epimorphic families by families of morphisms exhibiting the target as a coproduct, thereby filling in this proof. □

6.3. **Coherent sheaves.**

Proposition 6.8. *Let $\mathcal{C}$ be a precoherent category (see Definition 5.6). The coherent topology on $\mathcal{C}$ is subcanonical.*

Proof. We need to show that each presheaf of the form $h_W = \mathrm{Hom}(-, W)$ with W an object of $\mathcal{C}$ is a sheaf. By Proposition 3.15, it is enough to check that h_W is a sheaf for each finite effective epimorphic family, and this follows from Lemma 4.7. □

Remark 6.9. If $\mathcal{C}$ is finitary extensive and preregular (and hence precoherent), then Proposition 6.8 implies Proposition 6.6 and Proposition 6.1, because the coherent topology is finer than the extensive and regular one. On the other hand, being precoherent might not in general imply being finitary extensive or preregular (for example, when $\mathcal{C}$ does not have finite coproducts) and this is why we proved Proposition 6.6 and Proposition 6.1 separately.

Proposition 6.10. *Let $\mathcal{C}$ be a precoherent category. A sieve in $\mathcal{C}$ is a covering sieve for the coherent topology if and only if it contains a finite effective epimorphic family.*

Before proving Proposition 6.10 we provide some preliminary results.

Lemma 6.11. *If a sieve S contains a finite effective epimorphic family, then S is a covering sieve for the coherent topology.*

Proof. Let $(\pi_i : X_i \to X)_{i \in I}$ be a finite effective epimorphic family contained in S. By definition, the sieve S_0 generated by the family $(\pi_i)_{i \in I}$ is a covering sieve for the coherent topology, and since S contains the family $(\pi_i)_{i \in I}$, it contains S_0. Lemma 3.12 yields the conclusion. □

Lemma 6.12. *Assume that $\mathcal{C}$ is precoherent and that $(\pi_i : X_i \to B)_{i \in I}$ is a finite effective epimorphic family, and suppose that for each $i \in I$, we are given a finite effective epimorphic family $(\pi_{i,j} : Y_{i,j} \to X_i)_{j \in J_i}$. Then the induced collection $(\varpi_{i,j} = \pi_i \circ \pi_{i,j} : Y_{i,j} \to B)_{i \in I, j \in J_i}$ is an effective epimorphic family.*

Proof. By Lemma 4.7, a family is effective epimorphic if and only if for each object W the presheaf h_W is a sheaf for the sieve generated by this family. Thus, since the coherent topology is subcanonical by Proposition 6.8, it is enough to show that the sieve S generated by the family $(\varpi_{i,j})_{i \in I, j \in J_i}$ is a covering sieve for the coherent topology.

By and GT-3) of Definition 3.11, it is enough to check that f^*S is a covering sieve for every map f in the sieve generated by $(\pi_i)_{i \in I}$ (which is a covering sieve by Lemma 6.11). In fact, by GT-2), it is enough to check that each π_i^*S is a covering sieve. Since π_i^*S contains the finite effective epimorphic family $(\pi_{i,j})_{j \in I_j}$, it is a covering sieve for the coherent topology by Lemma 6.11. □

Proof of Proposition 6.10. Let $\mathcal{T}$ denote the collection of sieves in $\mathcal{C}$ that contain a finite effective epimorphic family. By Lemma 6.11, we know that $\mathcal{T}$ is contained in the coherent topology. Our goal is to show that they are equal, so it remains to show that $\mathcal{T}$ contains the coherent topology. By definition, the coherent topology is the smallest Grothendieck topology whose associated coverage contains the coherent coverage. Therefore, it suffices to show that

a) the collection $\mathcal{T}$ forms a Grothendieck topology and
b) the coverage associated to $\mathcal{T}$ contains the coherent coverage.

Once a) is established, point b) is immediate from the definitions of $\mathcal{T}$ and of the associated coverage (Definition 3.13). It remains to show a) by checking the conditions of Definition 3.11. Condition GT-1) is immediate, since for every object X of $\mathcal{C}$, the identity on X forms a finite effective epimorphic family. Condition GT-2) is a consequence of the precoherence assumption: Let $f\colon X \to Y$ be a morphism and let S be a sieve on Y that is contained in $\mathcal{T}$, i.e. that contains a finite effective epimorphic family $(\pi_i\colon Y_i \to Y)_{i\in I}$. Then the condition of being precoherent (see Definition 5.6) provides an effective epimorphic family $(\pi'_j\colon : X_j \to X)_{j\in I'}$ that is contained in the pullback sieve f^*S. Finally, we address GT-3). Let S, R be sieves on Y with $S \in \mathcal{T}$ such that for every $f\colon X \to Y \in S$, the pullback sieve f^*R is in $\mathcal{T}$. Then we have a finite effective epimorphic family $(f_i\colon X_i \to Y)_{i\in I}$ contained in S, and for each $i \in I$, a finite effective epimorphic family $(g_{i,j}\colon X_{i,j} \to X_i)_{j\in J_i}$ contained in f_i^*R. By Lemma 6.12, the finite family $(f_i \circ g_{i,j}\colon X_{i,j} \to Y)_{i\in I, j\in J_i}$ is effective epimorphic. By Definition 3.10 of the pullback sieve, the composition $f_i \circ g_{i,j}$ factors through some morphism in R, hence lies in R for each pair (i, j). Thus the whole family $(f_i \circ g_{i,j})_{i\in I, j\in J_i}$ is contained in R, showing that $R \in \mathcal{T}$. This finishes the proof of Condition GT-3). □

Proposition 6.13. *Let $\mathcal{C}$ be a preregular, finitary extensive category with pullbacks of kernel pairs. A presheaf on $\mathcal{C}$ is a sheaf for the coherent topology if and only if it satisfies the equalizer condition* (EqCond) *of Lemma 6.3, and preserves finite products.*

Proof. It is easy to see that a presheaf is a sheaf for the topology generated by a union of coverages if and only if it is a sheaf for every covering presieve of both coverages. The result now follows by combining Proposition 5.9 with Lemma 6.3 and Proposition 6.5. □

Proposition 6.14. *Let $\mathcal{C}$ be a preregular, finitary extensive category in which every object is projective. A presheaf on $\mathcal{C}$ is a sheaf for the coherent topology if and only if it preserves finite products.*

Proof. As in the proof of Proposition 6.13, the result follows by combining Proposition 6.4 with Proposition 6.5. □

Proposition 6.15. *Let $\mathcal{C}$ be a category and let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor into a precoherent category $\mathcal{D}$ such that*

1) F preserves and reflects finite effective epimorphic families.
2) For every object Y of $\mathcal{D}$, there exists an object X of $\mathcal{C}$ and an effective epimorphism $F(X) \to Y$.

Then the following holds:

a) $\mathcal{C}$ is precoherent.
b) Let G be a sheaf for the coherent topology on $\mathcal{D}$. The presheaf $G \circ F^{\mathrm{op}}$ is a sheaf for the coherent topology on $\mathcal{C}$.
c) Precomposition with F induces an equivalence between the categories of sheaves for the coherent topology on $\mathcal{C}$ and on $\mathcal{D}$.

Before proving Proposition 6.15, we need to fix some terminology and state some preliminary results. These preliminaries were already in MATHLIB, and we simply state them here without proof. The results can be extracted from [1, Exposé III], but the approach *ibid.* differs slightly from the one in MATHLIB.

Definition 6.16. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, both endowed with a Grothendieck topology, and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Fix an object X in $\mathcal{C}$ and an object Y in $\mathcal{D}$.

a) Given a sieve S on X, the *functor-pushforward of S along F* is the sieve F_*S on $F(X)$ consisting of those morphisms $f\colon Y \to F(X)$ that factor through $F(g)$ for some morphism $g\colon Z \to X$ in S.
b) Given a sieve S on $F(X)$, the *functor-pullback of S along F* is the sieve F^*S on X consisting of those arrows f such that $F(f)$ belongs to S.

c) The F-*image sieve* is the sieve S_Y^F on Y consisting of those morphisms to Y that factor through an object of the form $F(X)$ for some X in $\mathcal{C}$.

We omit the verification that Definition 6.16 is actually defining sieves (one needs to check that they are downwards closed). This verification was formalized in MATHLIB, and each point of Definition 6.16 contains the corresponding external link.

Definition 6.17. In the same setting of Definition 6.16, denote by $\mathcal{T}$ the topology on $\mathcal{C}$ and by $\mathcal{T}'$ that on $\mathcal{D}$.

a) We say that F is *continuous* if for every $\mathcal{T}'$-sheaf P on $\mathcal{D}$, the presheaf $P \circ F^{\mathrm{op}}$ on $\mathcal{C}$ is a $\mathcal{T}$-sheaf. In particular, if F is continuous it induces a functor

$$F^* \colon \mathrm{Sh}_{\mathcal{T}'}(\mathcal{D}) \longrightarrow \mathrm{Sh}_{\mathcal{T}}(\mathcal{C}).$$

b) We say that F is *cocontinuous* if for every object U of $\mathcal{C}$ and every $\mathcal{T}'$-covering sieve S on $F(U)$, the functor-pullback F^*S is a $\mathcal{T}$-covering sieve of U.

c) We say that F is *cover-dense* if for every object Y of $\mathcal{D}$, the F-image sieve S_Y^F is a $\mathcal{T}'$-covering sieve.

Remark 6.18. Observe that in point c) of Definition 6.17 the topology $\mathcal{T}$ on $\mathcal{C}$ plays no role. Hence, to speak of cover-dense functors one only needs a Grothendieck topology on the target.

Proposition 6.19. *In the setting of Definition 6.17, suppose that F is continuous and cocontinuous. Then we have an adjunction $F^* \dashv F_*$. If F is also fully faithful and cover-dense, then this adjunction is an adjoint equivalence of categories.*

Definition 6.20. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $\mathcal{T}'$ be a Grothendieck topology on $\mathcal{D}$ and let $F \colon \mathcal{C} \to \mathcal{D}$ be a fully faithful cover-dense functor. Define a Grothendieck topology $\mathcal{T}$ on $\mathcal{C}$ as follows: we declare that a sieve S on an object X in $\mathcal{C}$ is a $\mathcal{T}$-covering sieve if and only if the functor-pushforward sieve F_*S is a $\mathcal{T}'$-covering sieve of $F(X)$ (see for a proof of the fact that this indeed defines a Grothendieck topology). This is called the *topology induced by F*.

Lemma 6.21. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $\mathcal{T}'$ a Grothendieck topology on $\mathcal{D}$ and let $F \colon \mathcal{C} \to \mathcal{D}$ be a fully faithful cover-dense functor. Equip $\mathcal{C}$ with the induced topology. Then F is continuous and cocontinuous.*

Proof of Proposition 6.15. To show that $\mathcal{C}$ is precoherent, let $(\pi_i \colon X_i \to B)_i$ be a finite effective epimorphic family in $\mathcal{C}$ and let $f \colon B' \to B$ be any morphism. The family $F(\pi_i)$ is finite effective epimorphic (in $\mathcal{D}$) by condition 1): then, the hypothesis that $\mathcal{D}$ is precoherent, applied to the morphism $F(f) \colon F(B') \to F(B)$, provides a finite effective epimorphic family $\varpi_j \colon Y_j \to F(B')$ whose components factor through some of the $F(\pi_i)$. By condition 2) there exist objects $(X'_j)_j$ in $\mathcal{C}$ together with effective epimorphisms $\varphi_j \colon F(X'_j) \to Y_j$, that combine into an effective epimorphic family $F(X'_j) \to F(B')$ thanks to Lemma 6.12; moreover, the morphisms in this family are of the form $F(\pi'_j)$ for suitable $\pi'_j \colon X'_j \to B'$ because F is fully faithful. Applying again condition 1), this family $(\pi'_j)_j$ is finite effective epimorphic; that, for each j, the morphism π'_j factors through some of the π_i follows from the equivalent statement for the components of ϖ_j, combined once more with the full faithfulness of F. This establishes point a).

We claim that the topology on $\mathcal{C}$ induced by F is the coherent topology. It suffices to show that given an object X of $\mathcal{C}$, a sieve S on X is a covering for the induced topology if and only if it contains a finite effective epimorphic family. Suppose first that S contains a finite effective epimorphic family $(\pi_i \colon Y_i \to X)_i$. By condition 1), the family $(F(\pi_i))_i$ is finite effective epimorphic, and is clearly contained in F_*S. Hence F_*S is a covering sieve of $F(X)$ with respect to the coherent topology on $\mathcal{D}$ by Proposition 6.10, which means that S is a covering sieve with for the induced topology (see Definition 6.20). For the other direction, suppose that S is a covering sieve for the induced topology: as for the first implication, this is equivalent to the condition that F_*S contains a finite effective epimorphic family $(\pi_i \colon Z_i \to F(X))_i$. Condition 2) ensures that for every i, there is an effective epimorphism of the form $f_i \colon F(Y_i) \to Z_i$; moreover, since $\mathcal{C}$ is precoherent, we can apply Lemma 6.12 to obtain that the family $(\pi_i \circ f_i \colon F(Y_i) \to F(X))_i$ is effective epimorphic. Since F is full and reflects finite effective epimorphic families by condition 1), this family can be pulled back to a finite effective epimorphic family $(Y_i \to X)_i$ contained in $F^*(F_*S)$. We conclude thanks to Proposition 6.19.

Endowing $\mathcal{D}$ with the coherent topology, point b) is now immediate from Lemma 6.21 (see Definition 6.17).

To finish the proof, we pass to point c), again endowing $\mathcal{D}$ with the coherent topology. By Proposition 6.19, it suffices to prove that F is cover-dense, continuous and cocontinuous. By Lemma 6.21 and the above discussion, it suffices to prove that F is cover-dense. By Proposition 6.10, it suffices to show that for every object Y of $\mathcal{D}$, the F-image sieve S_Y^F contains an effective epimorphism. Condition 2) ensures the existence of an object X in $\mathcal{C}$ and of an effective epimorphism $F(X) \to Y$, that, by definition of the F-image sieve, belongs to S_Y^F. □

Remark 6.22. A finite-coproduct preserving functor between finitary extensive categories preserves (*resp.* reflects) finite effective epimorphic families if and only if it preserves (*resp.* reflects) effective epimorphisms. This is because finite effective epimorphic families in extensive categories are precisely those that induce effective epimorphisms on the coproduct (see Lemmas 4.8 and 4.9). This observation yields variants (see for instance ☐) of Proposition 6.15 in the case where the target is preregular and finitary extensive and the functor preserves certain pullbacks and coproducts, or when the target category is already finitary extensive.

7. Condensed mathematics

We can now introduce condensed sets and prove the main theorems from our general categorical results. We begin with the following result:

Proposition 7.1. *The categories* CompHaus, Profinite *and* Stonean *are preregular and finitary extensive.*

Proof. ☐ ☐ ☐ ☐ ☐ ☐ Let $\mathcal{C}$ denote any of the categories CompHaus, Profinite or Stonean. Note that the effective epimorphisms in $\mathcal{C}$ are precisely the continuous surjections (Proposition 4.13). These also coincide with the epimorphisms, by Lemma 4.10. Given the explicit description of pullbacks in Profinite and CompHaus, we immediately obtain that effective epimorphisms can be pulled back, and therefore Profinite and CompHaus are preregular. To see that Stonean is preregular, we use the fact that every object in Stonean is projective, and hence every epimorphism can be pulled back to the identity.

We also need to show that $\mathcal{C}$ is finitary extensive. In mathlib it was already proved that the category of all topological spaces is finitary extensive, and that given a functor $F\colon \mathcal{C} \to \mathcal{D}$ to a finitary extensive category which preserves and reflects finite coproducts, preserves pullbacks along coprojections in finite coproducts and reflects pullbacks, if $\mathcal{C}$ has finite coproducts and pullbacks along coprojections, then $\mathcal{C}$ is finitary extensive. To see that $\mathcal{C}$ together with its inclusion functor to the category of topological spaces has these properties, the only point which needs clarification is the case for pullbacks in Stonean. We know that Stonean does not have all pullbacks, but in the very special case of coprojections in finite coproducts, it does. Indeed, these are clopen embeddings, in which case the pullback is identified with the preimage of the image of the embedding. □

Definition 7.2. ☐ A *condensed set* is a sheaf for the coherent topology on CompHaus. (We refer the reader to Definition 5.6 for the definition of coherent topology.)

Remark 7.3. Thanks to Theorem 7.7 below, a condensed set can be defined as a sheaf for the coherent topology on Profinite or Stonean.

Theorem 7.4. ☐ ☐

a) *When $\mathcal{C}$ is* CompHaus *or* Profinite, *a presheaf $X\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ is a sheaf for the coherent topology on $\mathcal{C}$ if and only if it satisfies the following two conditions:*
   1) *X preserves finite products: in other words, for every finite family (T_i) of object of $\mathcal{C}$, the natural map*
   $$X\Big(\coprod_i T_i\Big) \longrightarrow \prod_i X(T_i)$$
   is a bijection.
   2) *For every surjection $\pi\colon S \to T$ in $\mathcal{C}$, the diagram*
   $$X(T) \xrightarrow{X(\pi)} X(S) \rightrightarrows X(S \times_T S)$$
   is an equalizer (the two parallel morphisms being given by the projections in the pullback).

b) *A presheaf* $X\colon \mathsf{Stonean}^{\mathrm{op}} \to \mathsf{Set}$ *is a sheaf for the coherent topology on* Stonean *if and only if it preserves finite products: in other words, for every finite family* (T_i) *of object of* $\mathcal{C}$*, the natural map*

$$X\Big(\coprod_i T_i\Big) \longrightarrow \prod_i X(T_i)$$

is a bijection.

Proof. In the case when $\mathcal{C}$ is CompHaus or Profinite, it has all pullbacks and we obtain the characterization from Proposition 6.13. In the case of Stonean, since every object is projective, we obtain the characterization from Proposition 6.14. □

Remark 7.5. A detailed proof of Theorem 7.4 is given in [2, Theorems 1.2.17 and 1.2.18].

Remark 7.6. A condition similar to the one in point a) of Theorem 7.4 above holds true when $\mathcal{C}$ is Stonean as well, except that condition a)-1) must be modified slightly (for example, using 1-hypercovers) due to the fact that Stonean does not have pullbacks. The content of b) is that this analogous condition turns out to be vacuously true in Stonean.

Theorem 7.7. *The inclusion functors* Profinite $\to$ CompHaus *and* Stonean $\to$ CompHaus *induce equivalences of categories between the categories of sheaves for the coherent topology on* CompHaus, Profinite, *and* Stonean.

Proof. We are going to apply Proposition 6.15. We spell out the case of Stonean here, the other one being similar. It is clear that the inclusion functor preserves and reflects effective epimorphisms (by the characterization of effective epimorphisms as continuous surjections). Verifying the other condition in the theorem amounts to proving that CompHaus has enough projectives. Given a compact Hausdorff space S, denote by S^δ the set S equipped with the discrete topology. Then the Stone–Čech compactification βS^δ is a projective object with a continuous surjection $\beta S^\delta \to S$. □

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Paper 3 — A formal characterization of discrete condensed objects

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A FORMAL CHARACTERIZATION OF DISCRETE CONDENSED OBJECTS

DAGUR ASGEIRSSON

Abstract. Condensed mathematics, developed by Clausen and Scholze over the last few years, proposes a generalization of topology with better categorical properties. It replaces the concept of a topological space by that of a condensed set, which can be defined as a sheaf on a certain site of compact Hausdorff spaces. Since condensed sets are supposed to be a generalization of topological spaces, one would like to be able to study the notion of discreteness. There are various ways to define what it means for a condensed set to be discrete. In this paper we describe them, and prove that they are equivalent. The results have been fully formalized in the Lean proof assistant.

1. Introduction

Condensed mathematics (originally defined in [13], see also [1, 4, 6, 7, 8, 12]) is a new framework which is suitable for applying algebraic techniques, such as homological algebra, in a setting where the objects of study are of a topological nature. In this framework, topological spaces are replaced by so-called *condensed sets*. The goal of this paper is to explore one aspect of the connection between condensed sets and topological spaces — the important example of discrete spaces. In condensed mathematics, the notion of discreteness becomes surprisingly subtle.

All results presented in this paper, except for an application described in §1.2, have been formalized in the Lean theorem prover [9], and integrated into its mathematical library mathlib [11]. There are subtle set-theoretic issues that arise in the foundations of condensed mathematics, to which the type theory of Lean provides a satisfactory solution, as explained in §2.2 (a short overview of the design principles of Lean and mathlib, as relevant to condensed mathematics, is given in [4, §2]). Another approach to these issues is to avoid the problem by switching to the theory of *light condensed objects* (see the recent lecture series [8]). I have also formalized the foundations of light condensed mathematics. This material is in mathlib, and the formalization there provides a possibility to unify the two as described in §2.2. The material in §4 was formalized in such a unified way (see §4.4 for more details).

One of the main contributions of the present work is that it describes and completely resolves a big subtlety in the theory of discrete condensed sets, which was discovered during the formalization process. Discrete condensed sets are most naturally defined as constant sheaves. It turns out to be quite difficult to prove that a constant sheaf on the defining site of condensed sets (and light condensed sets) is in fact given by locally constant maps — the analogue of a result that is well known in the case of sheaves on a topological space. This result is important because without it, there is no chance of an explicit description of the sections of a discrete condensed set. More broadly, it is my hope that this paper further improves the state of the literature on condensed mathematics, which prior to [4] consisted mostly of online lecture notes [13, 12, 7] and videos [8].

We provide several equivalent conditions on a condensed set that characterize it as discrete (Theorem A, Theorem B, Theorem 7.1). Moreover, we prove that the characterization of discrete condensed sets carries over nicely to condensed modules over a ring (Theorem C, Theorem 7.2). All the results hold in both the original setting of condensed objects, and that of light condensed objects.

Throughout the text, we use the symbol ↗ for external links. Every mathematical declaration will be accompanied by such a link to the corresponding formal statement in mathlib. All the links are to the same commit to the master branch, ensuring that the links stay usable.

1.1. **Main results.** We give the definition of a condensed set and some related terminology, before stating the main results of this paper.

Recall that a *condensed set* is a sheaf for the coherent topology on the category of compact Hausdorff spaces[1]. More concretely, it is a presheaf $X : \mathsf{CompHaus}^{\mathrm{op}} \to \mathsf{Set}$, satisfying the two properties :

1) X preserves finite products: in other words, for every finite family (T_i) of compact Hausdorff spaces, the natural map
$$X\Big(\coprod_i T_i\Big) \longrightarrow \prod_i X(T_i)$$
is a bijection.
2) For every continuous surjection $\pi\colon S \to T$ of compact Hausdorff spaces, the diagram
$$X(T) \xrightarrow{X(\pi)} X(S) \rightrightarrows X(S \times_T S)$$
is an equalizer (the two parallel morphisms being induced by the projections in the pullback).

Furthermore, there is a functor from topological spaces to condensed sets , which takes a topological space X to the sheaf $\mathsf{Cont}(-, X)$, which maps a compact Hausdorff space S to the set of continuous maps $S \to X$. This functor is fully faithful when restricted to *compactly generated spaces* (these include all reasonable topological spaces most mathematicians care about). Continuous maps from a point to a topological space X identify with the underlying set X, which is why we define the *underlying set* of a condensed set X to be the set $X(*)$. Even though condensed sets do not form a concrete category, we will sometimes call the functor $X \mapsto X(*)$ the *forgetful functor*. By analogy with topological spaces, we would like to be able to study *discrete condensed sets*. But what does it mean for a condensed set to be discrete?

Let's take a look at discreteness in topology. A discrete topological space X is one in which all subsets are open. It is characterized by the property that every map $X \to Y$, where Y is any topological space, is continuous. In other words, we have an adjoint pair of functors
$$\mathsf{Set} \underset{U}{\overset{\delta}{\rightleftarrows}} \mathsf{Top} \quad (\delta \dashv U)$$
where U denotes the forgetful functor and δ the functor which equips a set with the discrete topology. This suggests that we should try to define a functor $\mathsf{Set} \to \mathsf{CondSet}$ which "equips a set with a discrete condensed structure", which is left adjoint to the underlying set functor $\mathsf{CondSet} \to \mathsf{Set}$.

An obvious candidate for this functor $\mathsf{Set} \to \mathsf{CondSet}$ is
$$\underline{(-)} : \mathsf{Set} \to \mathsf{CondSet}$$
where $\underline{X}$ denotes the condensed set given by the constant sheaf at X. It is almost by definition left adjoint to the underlying set functor $\mathsf{CondSet} \to \mathsf{Set}$.

Another candidate comes from the functor from topological spaces to condensed sets that we already have, which takes a topological space X to the sheaf of continuous maps to X. Precomposing this functor with the functor which equips a set X with the discrete topology, we obtain a functor $\mathsf{Set} \to \mathsf{CondSet}$. Rephrasing the above, this is the functor which takes a set X to the sheaf of locally constant maps to X . We denote this functor by
$$L : \mathsf{Set} \to \mathsf{CondSet}$$
$$X \mapsto \mathsf{LocConst}(-, X)$$

Each of these functors has its advantages and disadvantages. The constant sheaf functor has nice abstract properties and is obviously a left adjoint to the underlying set functor, while the functor L has better definitional properties given by its concrete description. Our first objective is to construct a natural isomorphism $L \cong \underline{(-)}$. We do this by constructing an adjunction $L \dashv U$ where $U : \mathsf{CondSet} \to \mathsf{Set}$ is the underlying set functor mapping a condensed set X to $X(*)$.

We can now give the official definition of a discrete condensed set.

[1]This is the MATHLIB definition. The original definition uses the category of profinite sets as the defining site. These two definitions are equivalent, as formalized in MATHLIB and described in [4].

Definition 1.1. A condensed set X is *discrete* if it is in the essential image of the functor

$$\underline{(-)} : \mathsf{Set} \to \mathsf{CondSet},$$

i.e. if there exists some set X' and an isomorphism

$$X \cong \underline{X'}.$$

The discussion preceding Definition 1.1 immediately gives our first main theorem:

Theorem A. *A condensed set is discrete if and only if it is in the essential image of the functor*

$$L : \mathsf{Set} \to \mathsf{CondSet},$$

i.e. if there exists some set X' and an isomorphism of condensed sets

$$X \cong \mathsf{LocConst}(-, X')$$

A condensed set is completely determined by its values on profinite sets (see [4, Theorem 7.7]). Profinite sets are those topological spaces which can be written as a cofiltered limit of finite discrete sets. A particular limit formula for a profinite set S is given by writing it as the limit of its discrete quotients, described in more detail in §5. When we write $S = \varprojlim_i S_i$, we mean this formula. We can now state the second of our main theorems:

Theorem B. *A condensed set X is discrete if and only if for every profinite set $S = \varprojlim_i S_i$, the canonical map $X(S) \to \varinjlim_i X(S_i)$ is an isomorphism.*

The definition of a discrete condensed set can be extended to condensed R-modules in an obvious way, leading to the third and final main theorem:

Theorem C. *Let R be a ring. A condensed R-module is discrete (i.e. constant as a sheaf) if and only if its underlying condensed set is discrete.*

1.2. **Application.** The colimit characterization of discrete condensed sets given by Theorem B (or more precisely, its analogue for condensed abelian groups, which follows from Theorems B and C together) is used when setting up the theory of *solid abelian groups*. The results presented in this subsection have not yet been formalized, but a formalization is well within reach and is currently only blocked by the fact that the closed symmetric monoidal structure on sheaf categories has not been completely formalized yet (this is work in progress by the author and Joël Riou).

Solid abelian groups are condensed analogues of complete topological groups. We say that a condensed abelian group A is *solid* if for every profinite set $T = \varprojlim_i T_i$, the natural map

$$\mathbb{Z}[T] \to \varprojlim_i Z[T_i]$$

induces a bijection

$$\mathrm{Hom}\left(\varprojlim_i \mathbb{Z}[T_i], A\right) \to \mathrm{Hom}\left(\mathbb{Z}[T], A\right).$$

We denote by $\mathbb{Z}[T]^{\blacksquare}$ the condensed abelian group $\varprojlim_i Z[T_i]$. These are compact projective generators of the category of solid abelian groups. To get the theory off the ground, one needs to prove that this category has some nice properties (described by [13, Theorem 5.8]), which relies on a structural result [13, Corollary 5.5] about these building blocks. That result says that for every profinite set T, there is a set I and an isomorphism

$$\mathbb{Z}[T]^{\blacksquare} \cong \prod_{i \in I} \mathbb{Z}.$$

To prove this, one provides an isomorphism

$$\mathbb{Z}[T]^{\blacksquare} \cong \underline{\mathrm{Hom}}\left(\underline{\mathrm{Hom}}\left(\mathbb{Z}[T], \mathbb{Z}\right), \mathbb{Z}\right)$$

and then proves that the internal hom $\underline{\mathrm{Hom}}\left(\mathbb{Z}[T], \mathbb{Z}\right)$ is a discrete condensed abelian group, using the colimit characterization[2]. This means that it is simply given by the abelian group of continuous maps $C(T, \mathbb{Z})$, and

[2] This isomorphism, and the proof that the internal hom is discrete, is explained in an unpublished note by the author, see [2, 2.4-2.7]

thus Nöbeling's theorem [13, Theorem 5.4], which was formalized by the author in [3], applies. Nöbeling's theorem gives an isomorphism

$$\underline{\mathrm{Hom}}\,(\mathbb{Z}[T], \mathbb{Z}) \cong \bigoplus_{i \in I} \mathbb{Z},$$

and the desired isomorphism

$$\mathbb{Z}[T]^{\blacksquare} \cong \underline{\mathrm{Hom}}\left(\bigoplus_{i \in I} \mathbb{Z}, \mathbb{Z}\right) \cong \prod_{i \in I} \mathbb{Z}$$

follows.

1.3. **Terminology and notation.** Throughout the paper, we use the same terminology as MATHLIB. In particular, for an overview of the terminology surrounding Grothendieck topologies and sheaf theory that MATHLIB (and this paper) uses, we refer to [4, §3.1]. We will need the notion of initial and final functors. Sometimes the latter is referred to as *cofinal*, which is unfortunate because one might think that *cofinal* refers to the dual of *final*. We explain precisely what we mean by these in §2.1.2.

1.4. **Outline.** We prove Theorem A in §4, Theorem B in §5, and Theorem C in §6. The general categorical setup to prove Theorem C is in §3. We gather a few generalities in category theory which are perhaps not completely standard in §2.1. We discuss some set theoretic issues which are important to consider when setting up the formalized theory of condensed mathematics in §2.2. MATHLIB has an unusual approach to sheafification; this is described in §2.3. Finally, we tie together the main results as Theorems 7.1 and 7.2 in §7, giving a characterization of discrete condensed sets and discrete condensed modules over any ring.

2. Preliminaries

2.1. Generalities in category theory.

2.1.1. *Adjunctions.* We prove two useful results about adjunctions. The results in this subsection were formalized by the author as part of the prerequisites for this project. Let $\mathcal{C}, \mathcal{D}$ be categories and

$$\mathcal{C} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}$$

a pair of adjoint functors.

Proposition 2.1. *Suppose that L is fully faithful and let X be an object of $\mathcal{D}$. Then X is in the essential image of L if and only if the counit induces an isomorphism $L(R(X)) \to X$.*

Proof. The backward direction is obvious. For the forward direction, pick an isomorphism of the form $X \cong L(Y)$. Then the unit gives a one-sided inverse to the counit $L(R(L(Y))) \to L(Y)$ by one of the triangle identities. Since the left adjoint is fully faithful, the unit is an isomorphism and hence a two-sided inverse. We conclude that the counit induces an isomorphism $L(R(L(Y))) \to L(Y)$, and since the counit is a natural transformation, and there is an isomorphism $X \cong L(Y)$, this means it also induces an isomorphism $L(R(X)) \to X$. □

Proposition 2.2. *If there exists a natural isomorphism $R \circ L \cong \mathrm{Id}$, then the unit $\mathrm{Id} \to R \circ L$ is an isomorphism.*

Proof. Let $i : R \circ L \to \mathrm{Id}$ be an isomorphism. Let (Id, η, μ) denote the monad obtained by transporting the monad on $R \circ L$ induced by the adjunction, along the isomorphism i. The inverse of the unit of the adjunction is then given by $\mu \circ i$. The fact that this is an inverse follows from the coherence conditions of the monad and the fact that any monad on the identity functor is commutative. □

For a more detailed proof of Proposition 2.2, we refer to MATHLIB. This same proof idea is described in [10, Lemma A.1.1.1]

Corollary 2.3. *If there exists a natural isomorphism $R \circ L \cong \mathrm{Id}$, then L is fully faithful.*

2.1.2. *Initial and final functors.* The results in this subsection were not formalized by the author and were already in MATHLIB before the start of this project. The purpose of stating them here is to clarify the terminology. We omit all proofs and refer to the LEAN code for details.

Definition 2.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

- F is *initial* if the comma category $F/\mathcal{D}$ (whose objects are morphisms of the form $F(X) \to Y$ with $X \in \mathcal{C}$ and $Y \in \mathcal{D}$) is connected.
- F is *final* if the comma category $\mathcal{D}/F$ (whose objects are morphisms of the form $Y \to F(X)$ with $X \in \mathcal{C}$ and $Y \in \mathcal{D}$) is connected.

Proposition 2.5. *A functor is initial if and only if its opposite is final, and vice versa.*

Proposition 2.6. *Let $\mathcal{E}$ be a category and $G : \mathcal{D} \to \mathcal{E}$ be a functor. Then*

- *If F is initial, then G has a limit if and only if $G \circ F$ has a limit. In this case, the canonical map*
$$\lim G \to \lim(G \circ F)$$
is an isomorphism.
- *If F is final, then G has a colimit if and only if $G \circ F$ has a colimit. In this case, the canonical map*
$$\operatorname{colim}(G \circ F) \to \operatorname{colim} G$$
is an isomorphism.

2.2. **Set-theoretic issues in condensed mathematics.** In sheaf theory, one sometimes needs to be careful about the size of categories involved. For example, the category of sheaves of sets on a large site cannot strictly speaking have sheafification. This is because the process of sheafification involves taking a colimit which is the same size as the defining site.

This issue arises in condensed mathematics, and can be solved in various ways. Originally it was done in [13] by altering the definition of a condensed set slightly. Instead of defining it as a sheaf on the whole site of compact Hausdorff spaces, one defines a κ-condensed set to be a sheaf on the essentially small site of κ-small compact Hausdorff spaces, where κ is a certain type of cut-off cardinal. Then one takes a colimit of these categories of κ-condensed sets and defines a condensed set to be an object in this colimit in the category of categories.

The way this issue is solved in [6] is to make the sheaves take values in a larger category of sets. So a condensed set is defined as a sheaf of large sets on the site of all small compact Hausdorff spaces. This is the approach we take in MATHLIB as well, by taking advantage of the universe hierarchy built in to the type theory of LEAN (for a short explanation of the universe hierarchy, see [4, §2.5]; for more details, see [5, Chapter 2]). A `u`-condensed set is defined as a sheaf on the site `CompHaus.{u}` of all compact Hausdorff spaces in the universe `u`, valued in the category `Type (u+1)` of types in the universe `u+1`:

```
def Condensed (C : Type w) [Category.{v} C] :=
  Sheaf (coherentTopology CompHaus.{u}) C

abbrev CondensedSet := Condensed.{u} (Type (u+1))
```

The definition of a condensed object in a general category `C` has three universe parameters `u`, `v`, and `w` where the latter two are the size of the sets of morphisms and objects of `C` respectively. This is a recurring theme in the category theory library in MATHLIB. A *small category* in this context is category whose objects and morphisms live in the same universe `u`. More precisely, the type `C` of objects is a term of `Type u` and for each pair of objects `X Y : C`, the type `X ⟶ Y` of morphisms between them is also a term of `Type u`. In a *large category* the objects form a type of size `u+1` and the morphisms between two objects a type of size `u`.

Many theorems in category theory hold for categories regardless of whether they are small or large. Others require modifications to depending on whether some categories in the statement are small or large. MATHLIB strives to avoid code duplication as much as possible, and therefore, theorems in category theory often have assumptions of the form

```
variable (C : Type u) [Category.{v} C]
```

This means that the objects and morphisms of `C` live in completely arbitrary universes, which specializes directly to both small and large categories.

In an informal mathematics text, one might see a definition which would start as follows:

Let $(\mathcal{C}, J)$ be a small site and let $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf. The *sheafification* $\hat{F}$ of F is a sheaf of sets for J on $\mathcal{C}$ defined as follows . . . .

And it would then go on to describe the values of this sheaf as a colimit in Set, which exists because $\mathcal{C}$ was assumed to be small.

In MATHLIB, the approach taken in this situation is instead to depend on

```
variable {C : Type u} [Category.{v} C] (J : GrothendieckTopology C)
    (F : Cᵒᵖ ⥤ Type (max u v))
```

This specializes directly to the case of a small defining site (`[Category.{u} C]`), because in that case the target category of the presheaf becomes `Type u`. But it also allows for a more general construction, namely sheafifying presheaves on a large site (`{C : Type (u+1)} [Category.{u} C]`), by making sure they take values in a category of large enough sets (`Type (u+1)`). This way, sheafification is possible in the setting of `CondensedSet` in MATHLIB.

The size issues in condensed mathematics described above can be avoided by modifying the theory slightly. Instead of the category of compact Hausdorff spaces or that of profinite sets, we can take the category of *light profinite sets* as our defining site. This category consists of those profinite sets that can be written as a sequential limit $\varprojlim_{n \in \mathbb{N}} S_n$ of finite discrete sets. Equivalently, it is the category of all second countable, totally disconnected compact Hausdorff spaces (this is the MATHLIB definition). This category is essentially small, so we can define a *light condensed set* as a sheaf of (small) sets on the coherent site of light profinite sets. In MATHLIB:

```
def LightCondensed (C : Type w) [Category.{v} C] :=
  Sheaf (coherentTopology LightProfinite.{u}) C

abbrev LightCondSet := LightCondensed.{u} (Type u)
```

A common generalization of condensed and light condensed sets is a term of the type

```
Sheaf (coherentTopology (CompHausLike.{u} P)) (Type (max u w))
```

Here, `P` is a predicate on topological spaces, `CompHausLike.{u} P` denotes the category of compact Hausdorff spaces in the universe `u`, satisfying `P`, and `w` is an auxiliary universe variable. In MATHLIB, the categories `CompHaus` and `LightProfinite` are defined as abbreviations for `CompHausLike P` for the relevant predicates `P`. This is a recent design by the author with the purpose of being able to prove results about categories such as `CompHaus` and `LightProfinite` simultaneously.

All constructions in this paper carry over to the setting of light condensed objects. In the formalization, this is done either in each case separately or by arguing directly about the common generalization described in the previous paragraph. However, for clarity of exposition, we focus in this paper on the case of condensed sets, and let the reader themselves either fill in the details for the translation to light condensed sets, or look at the LEAN code.

2.3. **Sheafification in mathlib.** MATHLIB has a somewhat unusual approach to sheafification. In this section we define some terminology which may be non-standard in the literature, but follows MATHLIB conventions. Many of these constructions were formalized in part by the author, but Construction 2.11 and the material related to preserving sheafification was mostly formalized by Joël Riou.

Definition 2.7. Let $\mathcal{C}$ be a category equipped with a Grothendieck topology J and let $\mathcal{A}$ be a category.

- We say that $\mathcal{A}$ *has weak sheafification with respect to* J if the inclusion functor $\mathsf{Sh}_J(\mathcal{C}, \mathcal{A}) \to \mathsf{PSh}(\mathcal{C}, \mathcal{A})$ has a left adjoint (called the *sheafification functor*).
- We say that $\mathcal{A}$ *has sheafification with respect to* J if it has weak sheafification and the sheafification functor preserves finite limits.

Previously, MATHLIB only contained definitions of sheafification for sheaves valued in concrete categories with a long list of assumptions about existence of certain limits and colimits as well as the forgetful functor preserving these. Moreover, it only worked for presheaves of large sets (or other large concrete categories satisfying the required properties) defined on a large site, or presheaves of small sets defined on a small site, not for presheaves of small sets defined on a large, essentially small site. The type classes `HasWeakSheafify` and `HasSheafify` were added to MATHLIB by the author while adding the possibility to sheafify presheaves of small sets defined on a large, essentially small site. This streamlined many files which previously contained said long lists of assumptions, which now instead only contain an assumption of the form `HasWeakSheafify` or `HasSheafify`.

Definition 2.8. Let $\mathcal{C}$ be a category equipped with a Grothendieck topology J, let $\mathcal{A}$ and $\mathcal{B}$ be categories and $U : \mathcal{A} \to \mathcal{B}$ a functor. We say that U *has sheaf composition with respect to* J if for every J-sheaf F with values in $\mathcal{A}$, the presheaf $U \circ F$ is a sheaf. This yields a functor $\mathsf{Sh}_J(\mathcal{C}, \mathcal{A}) \to \mathsf{Sh}_J(\mathcal{C}, \mathcal{B})$ which we call the *sheaf composition functor associated to* U.

Proposition 2.9. *Let $\mathcal{C}, J, \mathcal{A}, \mathcal{B}$ and U be as in Definition 2.8. Suppose that U has sheaf composition with respect to J. Denote by F_U the sheaf composition functor associated to U. Then the following holds:*

(1) If U is faithful, then F_U is faithful.
(2) If U is full and faithful, then F_U is full.
(3) If U reflects isomorphisms, then F_U reflects isomorphisms.

Definition 2.10. Let $\mathcal{C}$ be a category equipped with a Grothendieck topology J, let $\mathcal{A}$ and $\mathcal{B}$ be categories and $U : \mathcal{A} \to \mathcal{B}$ a functor. Suppose that A and B have weak sheafification with respect to J. We say that U *preserves sheafification* if for every morphism $\phi : P_1 \to P_2$ of $\mathcal{A}$-valued presheaves which becomes an isomorphism after sheafification, the whiskering $U\phi$ also becomes an isomorphism after sheafification.[3]

Construction 2.11. Let $\mathcal{C}, J, \mathcal{A}, \mathcal{B}$, and U be like in Definition 2.10. Suppose that U has sheaf composition with respect to J. Denote by G_1 and G_2 the sheafification functors associated to $\mathcal{A}$ and $\mathcal{B}$ respectively. Consider the following square of functors:

$$\begin{array}{ccc} \mathsf{PSh}(\mathcal{C}, \mathcal{A}) & \xrightarrow{U\circ} & \mathsf{PSh}(\mathcal{C}, \mathcal{B}) \\ {\scriptstyle G_1}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle G_2} \\ \mathsf{Sh}_J(\mathcal{C}, \mathcal{A}) & \xrightarrow[F_U]{} & \mathsf{Sh}_J(\mathcal{C}, \mathcal{B}) \end{array}$$

where F_U is the sheaf composition functor associated to U. We construct a natural transformation α between the two composite functors as indicated in the diagram. Its components consist of natural transformations of functors $(U \circ P)^{sh} \to U \circ P^{sh}$ for each presheaf $P : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$, where $(-)^{sh}$ denotes sheafification. By adjunction, this is the same as giving a natural transformation $U \circ P \to U \circ P^{sh}$, which is obtained from whiskering U with the unit of the sheafification adjunction applied to P. We omit the proof of naturality, and refer to the LEAN code for details.

Proposition 2.12. *Let $\mathcal{C}, J, \mathcal{A}, \mathcal{B}$, and U be like in Construction 2.11. Suppose in addition that U preserves sheafification. Then the natural transformation α in Construction 2.11 is an isomorphism.*

Proof. This follows from the fact that U preserves sheafification, we refer to the LEAN code for details. □

Proposition 2.13. *Let $\mathcal{C}$ be a large category equipped with a Grothendieck topology J.*

(1) The category of large sets has sheafification with respect to J.
(2) Let R be a ring. The category of large R-modules has sheafification with respect to J

Suppose now that $\mathcal{C}$ is essentially small.

(3) The category of small sets has sheafification with respect to J.
(4) Let R be a ring. The category of small R-modules has sheafification with respect to J

Proof. This follows from the fact that these categories have limits and their forgetful functors preserve limits and filtered colimits. □

Proposition 2.14. *Let $\mathcal{C}$ be a large category equipped with a Grothendieck topology J.*

[3] In MATHLIB, this definition is stated slightly differently without assuming that the categories have weak sheafification. In our application, the categories do have sheafification and our definition is equivalent.

- *Let R be a ring. The forgetful functor from the category of large R-modules to the category of large sets has sheaf composition with respect to J.*
- *If $\mathcal{C}$ is essentially small and R is a ring, then the forgetful functor from the category of small R-modules to the category of small sets has sheaf composition with respect to J.*

Proof. This follows from the fact that this forgetful functor preserves limits. □

Proposition 2.15. *Let $\mathcal{C}$ be a large category equipped with a Grothendieck topology J.*

- *Let R be a ring. The forgetful functor from the category of large R-modules to the category of large sets preserves sheafification.*
- *If $\mathcal{C}$ is essentially small and R is a ring, then the forgetful functor from the category of small R-modules to the category of small sets preserves sheafification.*

Proof. This follows from the fact that this forgetful functor preserves limits and filtered colimits, and reflects isomorphisms. □

Propositions 2.13, 2.14, and 2.15 are not stated explicitly in this way in MATHLIB. Instead, there are much more general instances for sheaves valued in concrete categories whose forgetful functor preserves certain limits and colimits. For the convenience of the reader, we link to files on a branch of MATHLIB (one file for each Proposition), which show that LEAN automatically synthesizes these instances.

Proposition 2.13 shows that in the setting of condensed- and light condensed sets and modules, we can sheafify. Proposition 2.14 shows that there is a "forgetful functor" $\mathsf{CondMod}_R \to \mathsf{CondSet}$ which takes a condensed R-module to its "underlying" condensed set (and the analogue for light condensed objects). Proposition 2.15 shows that this functor fits into a diagram like the one in Construction 2.11, which commutes up to isomorphism, as the bottom row.

3. Constant sheaves

Notation 3.1. Let $\mathcal{C}$ be a category equipped with a Grothendieck topology J. Let $\mathcal{A}$ be a category which has weak sheafification with respect to J and let X be an object of $\mathcal{A}$. We denote by $\underline{X}$ the constant sheaf at X, i.e. the sheafification of the constant presheaf $\mathsf{cst}_X : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}$, $Y \mapsto X$.

Construction 3.2. *The constant sheaf adjunction.* Let $\mathcal{C}$ be a category equipped with a Grothendieck topology J. Let $\mathcal{A}$ be a category which has weak sheafification with respect to J. Suppose that $\mathcal{C}$ has a terminal object $*$ and let ev_* denote the functor which maps a sheaf or presheaf F to its evaluation $F(*)$. We construct an adjunction

$$\underline{(-)} \dashv \mathsf{ev}_*$$

We do this by constructing an adjunction $\mathsf{cst}_{-} \dashv \mathsf{ev}_*$ and composing it with the sheafification adjunction. The unit of the desired adjunction comes from the fact that the constant presheaf followed by evaluation is simply the identity. To define the counit, we need to give a natural transformation from the functor given by evaluation at the point followed by the constant presheaf functor, to the identity functor. Its component at a presheaf F corresponds to a natural transformation from the constant presheaf at $F(*)$ to F, and the component of that natural transformation at an object X is just a map $F(*) \to F(X)$, which is induced by the unique map $X \to *$. Naturality is easy.

Definition 3.3. Let $\mathcal{C}$ be a category equipped with a Grothendieck topology J. Let $\mathcal{A}$ be a category which has weak sheafification with respect to J (this property ensures that the constant sheaf functor $\underline{(-)} : \mathcal{A} \to \mathsf{Sh}_J(\mathcal{C}, \mathcal{A})$ exists). We say that an $\mathcal{A}$-valued sheaf $\mathcal{F}$ on $\mathcal{C}$ is *constant* if it is in the essential image of the constant sheaf functor, i.e. if there exists an object X of A and an isomorphism $\underline{X} \cong \mathcal{F}$.

Proposition 3.4. *Let $\mathcal{C}$, J, and $\mathcal{A}$, be like in Definition 3.3. Suppose that $\mathcal{C}$ has a terminal object $*$ and that the constant sheaf functor $\mathcal{A} \to \mathsf{Sh}_J(\mathcal{C}, \mathcal{A}), X \mapsto \underline{X}$ is fully faithful. A sheaf $\mathcal{F}$ on $\mathcal{C}$ is constant if and only if the counit induces an isomorphism $\underline{\mathcal{F}(*)} \to \mathcal{F}$.*

Proof. This is a direct application of Proposition 2.1. □

The main result of this section is Proposition 3.6. To prove it, we first need a technical lemma:

Lemma 3.5. *Let $\mathcal{C}$ be a category equipped with a Grothendieck topology J, let $\mathcal{A}$ and $\mathcal{B}$ be categories with weak sheafification with respect to J, and let $U : \mathcal{A} \to \mathcal{B}$ be a functor which preserves sheafification and such that U has sheaf composition with respect to J. Suppose that the constant sheaf functors $\mathcal{A} \to \mathsf{Sh}_J(\mathcal{C}, \mathcal{A})$ and $\mathcal{B} \to \mathsf{Sh}_J(\mathcal{C}, \mathcal{B})$ are both fully faithful, and that $\mathcal{C}$ has a terminal object $*$. Let F be an $\mathcal{A}$-valued sheaf on $\mathcal{C}$. Denote by $\varepsilon^{\mathcal{A}}$ the counit of the constant sheaf adjunction between $\mathcal{A}$ and $\mathsf{Sh}_J(\mathcal{C}, \mathcal{A})$ and $\varepsilon^{\mathcal{B}}$ the counit of the constant sheaf between $\mathcal{B}$ and $\mathsf{Sh}_J(\mathcal{C}, \mathcal{B})$. Let F_U denote the sheaf composition functor associated to U. There is a commutative diagram*

$$\begin{array}{ccc} \underline{U(F(*))} & & \\ \downarrow & \searrow^{\varepsilon^{\mathcal{B}}_{U \circ F}} & \\ U \circ \underline{F(*)} & \xrightarrow[F_U(\varepsilon^{\mathcal{A}}_F)]{} & U \circ F \end{array},$$

where the vertical arrow is an isomorphism.

Proof. We start by extending the diagram in Construction 2.11 by the constant presheaf functors:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{U} & \mathcal{B} \\ {\scriptstyle \mathsf{cst}_{-}}\downarrow & & \downarrow{\scriptstyle \mathsf{cst}_{U(-)}} \\ \mathsf{PSh}(\mathcal{C}, \mathcal{A}) & \xrightarrow{U \circ} & \mathsf{PSh}(\mathcal{C}, \mathcal{B}) \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle G_2} \\ \mathsf{Sh}_J(\mathcal{C}, \mathcal{A}) & \xrightarrow[F_U]{} & \mathsf{Sh}_J(\mathcal{C}, \mathcal{B}) \end{array}$$

The upper square is strictly commutative by definition, and the lower square is commutative up to isomorphism by Proposition 2.12. For any $X \in \mathcal{A}$ we obtain an isomorphism $U \circ \underline{X} \cong \underline{U(X)}$ by applying this natural isomorphism at the object X. In particular, by taking X as $F(*)$, we obtain the vertical arrow in the triangle in the lemma statement.

To verify that the triangle commutes, it suffices to check the equality after precomposing with the unit of the sheafification adjunction (this amounts to applying the hom set equivalence of the adjunction). After precomposing with the unit, it becomes a question of unfolding the definitions of the maps to see that it commutes. □

Proposition 3.6. *Let $\mathcal{C}$ be a category equipped with a Grothendieck topology J, let $\mathcal{A}$ and $\mathcal{B}$ be categories with weak sheafification with respect to J, and let $U : \mathcal{A} \to \mathcal{B}$ be a functor which preserves sheafification and such that U has sheaf composition with respect to J. Suppose that the constant sheaf functors $\mathcal{A} \to \mathsf{Sh}_J(\mathcal{C}, \mathcal{A})$ and $\mathcal{B} \to \mathsf{Sh}_J(\mathcal{C}, \mathcal{B})$ are both fully faithful, and that $\mathcal{C}$ has a terminal object $*$. Let F be an $\mathcal{A}$-valued sheaf on $\mathcal{C}$. Then F is constant if and only if $U \circ F$ is constant.*

Proof. With notation as in Lemma 3.5, this amounts to showing that

$$\varepsilon^{\mathcal{A}}_F : \underline{F(*)} \to F$$

is an isomorphism if and only if

$$\varepsilon^{\mathcal{B}}_{U \circ F} : \underline{U(F(*))} \to U \circ F$$

is an isomorphism. We know that F_U reflects isomorphisms by Proposition 2.9, so we conclude using Lemma 3.5. □

4. The functors from Set to CondSet

The constant sheaf functor

$$\underline{(-)} : \mathsf{Set} \to \mathsf{CondSet}$$

is left adjoint to $U : \mathsf{CondSet} \to \mathsf{Set}$, the functor which maps a condensed set X to the underlying set $X(*)$, by Construction 3.2. Recall that a condensed set X is *discrete* if it is in the essential image of this functor, i.e. if there exists a set X' and an isomorphism $X \cong \underline{X'}$.

The goal of this section is to construct an isomorphism between $\underline{(-)}$ and the functor

$$L : \mathsf{Set} \to \mathsf{CondSet}$$
$$X \mapsto \mathsf{LocConst}(-, X).$$

This will yield a proof of Theorem A.

Theorem A. *A condensed set is discrete if and only if it is in the essential image of the functor*

$$L : \mathsf{Set} \to \mathsf{CondSet},$$

i.e. if there exists some set X' and an isomorphism of condensed sets

$$X \cong \mathsf{LocConst}(-, X')$$

The desired isomorphism is obtained by constructing an adjunction $L \dashv U$, and using the fact that adjoints are unique up to isomorphism.

The unit is easy to define (see Construction 4.1) and obviously an isomorphism, giving full faithfulness once we have established the adjunction. Defining the counit is the hard part (see Construction 4.2 and §4.1).

Construction 4.1. *Unit.* We construct a natural transformation

$$\mathsf{Id}_{\mathsf{Set}} \to U \circ L.$$

This amounts to giving a map $X \to \mathsf{LocConst}(*, X)$ for every set X and proving naturality in X. The desired map is just the one which takes an element $x \in X$ to the corresponding constant map. Naturality is easy, and we also easily see that this natural transformation is in fact an isomorphism.

Construction 4.2. *Components of the counit.* Let X be a condensed set and let S be a compact Hausdorff space. We construct a map

$$\mathsf{LocConst}(S, X(*)) \to X(S)$$

as follows: A locally constant map $f : S \to X(*)$ corresponds to a finite partition of S into closed (and hence compact Hausdorff) subsets $S_1, \ldots, S_n$ (these are the fibres of f). We have isomorphisms

$$X(S) \cong X(S_1 \amalg \cdots \amalg S_n) \cong X(S_1) \times \cdots \times X(S_n),$$

so giving an element of $X(S)$ is the same as giving an element of each of $X(S_1), \ldots, X(S_n)$. Let $x_i \in X(*)$ denote the element such that $S_i = f^{-1}\{x_i\}$, and let $g_i : S_i \to *$ be the unique map. Then our element of $X(S_i)$ will be $X(g_i)(x_i)$.

4.1. **Naturality of the counit.** We need to show that the map defined in Construction 4.2 as the components of the counit really defines a natural transformation of functors $L \circ U \to \mathsf{Id}_{\mathsf{CondSet}}$, where $U : \mathsf{CondSet} \to \mathsf{Set}$ is the underlying set functor. We start with naturality in the compact Hausdorff space S and then proceed to show that it is natural in the condensed set X. This is where it starts to really matter how one sets up the formalized proof. A key lemma which is easy to prove and is used repeatedly in these naturality proofs is Lemma 4.3.

Lemma 4.3. *Let X and Y be condensed sets, S a compact Hausdorff space and $f : S \to Y(*)$ a locally constant map. Let*

$$S \cong S_1 \amalg \cdots \amalg S_n$$

be the corresponding decomposition of S into the fibres of f. Denote by $\iota_i : S_i \to S$ the inclusion map. Let $x, y \in X(S)$ and suppose that for all i,

$$X(\iota_i)(x) = X(\iota_i)(y).$$

Then $x = y$.

Proof. The condition simply says that x and y are equal when considered as elements of the product

$$X(S_1) \times \cdots \times X(S_n).$$

□

To be able to successfully use this lemma in the formalization, it is important to be careful with setting everything up. In LEAN, the statement is as follows:

```
lemma presheaf_ext (X : (CompHausLike.{u} P)ᵒᵖ ⥤ Type max u w)
    [PreservesFiniteProducts X] (x y : X.obj ⟨S⟩)
    [HasExplicitFiniteCoproducts.{u} P]
    (h : ∀ (a : Fiber f), X.map (sigmaIncl f a).op x = X.map (sigmaIncl f a).op y) :
    x = y := ...
```

The assumption is literally phrased such that one needs to check that for every fibre $f^{-1}\{x\}$ (denoted by `a : Fiber f` in the Lean statement), X applied to the inclusion map $f^{-1}\{x\} \to S$ agrees on the two elements. The terms of type `Fiber f` are defined as actual subtypes of S and the maps `sigmaIncl f a` are the inclusion maps.

Lemma 4.4. *The proposed counit defined in Construction 4.2 is natural in the compact Hausdorff space S.*

Proof. Fix a condensed set X. Given a compact Hausdorff space S we denote by ε_S the map

$$\mathsf{LocConst}(S, X(*)) \to X(S)$$

defined in Construction 4.2. We need to show that for every continuous map $g : T \to S$ of compact Hausdorff spaces, the diagram

$$\begin{array}{ccc} \mathsf{LocConst}(S, X(*)) & \xrightarrow{\varepsilon_S} & X(S) \\ {\scriptstyle \circ g}\downarrow & & \downarrow{\scriptstyle X(g)} \\ \mathsf{LocConst}(T, X(*)) & \xrightarrow[\varepsilon_T]{} & X(T) \end{array}$$

commutes. Let $f : S \to X(*)$ be a locally constant map. We need to show that

$$X(g)(\varepsilon_S(f)) = \varepsilon_T(f \circ g).$$

We apply Lemma 4.3 with $Y = X$ to the locally constant map $f \circ g : T \to X(*)$. Let

$$T \cong T_1 \amalg \cdots \amalg T_n$$

be the decomposition of T into the fibres of $f \circ g$ and denote by $\iota_i^T : T_i \to T$ the inclusion maps. Now we need to show that

$$X(\iota_i^T)\left(X(g)(\varepsilon_S(f))\right) = X(\iota_i^T)\left(\varepsilon_T(f \circ g)\right).$$

Let $x_i \in X(*)$ be the element of which T_i is the fibre and let $t_i : T_i \to *$ denote the unique map. Then the above simplifies to

$$X(g \circ \iota_i^T)\left(\varepsilon_S(f)\right) = X(t_i)(x_i).$$

Now, let

$$S \cong S_1 \amalg \cdots \amalg S_m$$

be the decomposition of S into the fibres of f. Denote by $\iota_j^S : S_j \to S$ the inclusion maps. Let j be such that $S_j = f^{-1}(x_i)$. Then g restricts to a continuous map $g' : T_i \to S_j$, and we have $g \circ \iota_i^T = \iota_j^S \circ g'$. Letting $s_j : S_j \to *$ denote the unique map, we now have

$$\begin{aligned} X(g \circ \iota_i^T)\left(\varepsilon_S(f)\right) &= X(g')\left(X(\iota_j^S)(\varepsilon_S(f))\right) \\ &= X(g')\left(X(s_j)(x_i)\right) \\ &= X(s_j \circ g')(x_i) \\ &= X(t_i)(x_i) \end{aligned}$$

as desired. ◻

Lemma 4.5. *The proposed counit defined in Construction 4.2 is natural in the condensed set X.*

Proof. Let S be a compact Hausdorff space. Given any condensed set X, denote by ε_X the map

$$\mathsf{LocConst}(S, X(*)) \to X(S)$$

defined in Construction 4.2. Let $g : X \to Y$ be a morphism of condensed sets. We need to show that the diagram

$$\begin{array}{ccc} \mathsf{LocConst}(S, X(*)) & \xrightarrow{\varepsilon_X} & X(S) \\ {\scriptstyle g_* \circ}\downarrow & & \downarrow{\scriptstyle g_S} \\ \mathsf{LocConst}(S, Y(*)) & \xrightarrow[\varepsilon_Y]{} & Y(S) \end{array}$$

commutes (here, g_T denotes the component of the natural transformation g at a compact Hausdorff space T). Let $f : S \to X(*)$ be a locally constant map. We need to show that

$$g_S(\varepsilon_X(f)) = \varepsilon_Y(g_* \circ f).$$

We apply Lemma 4.3 with $X = Y$ to the locally constant map $g_* \circ f : S \to Y(*)$. Let

$$S \cong S_1 \amalg \cdots \amalg S_n$$

be the decomposition of S into the fibres of $g_* \circ f$ and denote by $\iota_i : S_i \to S$ the inclusion maps. Now we need to show that

$$Y(\iota_i)\left(g_S(\varepsilon_X(f))\right) = Y(\iota_i)\left(\varepsilon_Y(g_* \circ f)\right).$$

Let $y_i \in Y(*)$ be the element of which S_i is the fibre and let $s_i : S_i \to *$ denote the unique map. Then the above simplifies to

$$Y(\iota_i)\left(g_S(\varepsilon_X(f))\right) = Y(s_i)(y_i).$$

This is a question of proving equality of two elements of the set $Y(S_i)$. We apply Lemma 4.3 again, this time to the locally constant map $f \circ \iota_i : S_i \to X(*)$ (and the roles of X and Y swapped with respect to the names of the condensed sets in the lemma statement). Let

$$S_i \cong S_{i,1} \amalg \cdots \amalg S_{i,m}$$

be the decomposition of S_i into the fibres of $f \circ \iota_i$ and denote by $\iota_{i,j} : S_{i,j} \to S_i$ the inclusion maps and by $s_{i,j} : S_{i,j} \to *$ the unique map. Then we need to show that

$$Y(\iota_i \circ \iota_{i,j})\left(g_S(\varepsilon_X(f))\right) = Y(s_{i,j})(y_i).$$

By naturality of g the goal now becomes

$$g_{S_{i,j}}(X(\iota_i \circ \iota_{i,j})(\varepsilon_X(f))) = Y(s_{i,j})(y_i)$$

Now, let

$$S \cong S'_1 \amalg \cdots \amalg S'_r$$

be the decomposition of S into the fibres of f. Denote by $\iota'_k : S'_k \to S$ the inclusion maps. There exists a k such that $S_{i,j} \subseteq S'_k$, denote the inclusion map by ι'. Let $x'_k \in X(*)$ be the element of which S'_k is the fibre, and denote by $s'_k : S'_k \to *$ the unique map. Then we have

$$\begin{aligned} X(\iota_i \circ \iota_{i,j})(\varepsilon_X(f)) &= X(\iota'_k \circ \iota')(\varepsilon_X(f)) \\ &= X(\iota')(X(\iota'_k)(\varepsilon_X(f))) \\ &= X(\iota')(X(s'_k)(x'_k)) \\ &= X(s_{i,j})(x'_k) \end{aligned}$$

The goal now becomes

$$g_{S_{i,j}}(X(s_{i,j})(x'_k)) = Y(s_{i,j})(y_i)$$

Applying naturality of g again, it reduces to

$$Y(s_{i,j})(g_*(x'_k)) = Y(s_{i,j})(y_i),$$

so it suffices to show that $g_*(x'_k) = y_i$. Let $s \in S_{i,j}$ be some element. Then since $s \in S'_k$, we have $f(s) = x'_k$. Thus, since $s \in S_i$, we have $g_*(x'_k) = g_*(f(s)) = y_i$, as desired. □

4.2. **Triangle identities.** To prove that the natural transformations defined in Constructions 4.1, 4.2 form the unit and counit of the desired adjunction, we need to verify the triangle identities. Throughout this subsection, we let η_X denote the component of the unit at a set X and $\varepsilon_{X,S}$ the component of the counit at a condensed set X and compact Hausdorff space S. Recall the notation $L : \mathsf{Set} \to \mathsf{CondSet}$ for the functor mapping a set X to the sheaf of locally constant maps to X and $U : \mathsf{CondSet} \to \mathsf{Set}$ for the underlying set functor, mapping a condensed set X to $X(*)$.

Lemma 4.6. *Let X be a set. We have*

$$\varepsilon_{L(X)} \circ L(\eta_X) = \mathsf{id}_{L(X)}$$

Proof. We need to show that for every compact Hausdorff space S and every locally constant map $f : S \to X$,

$$\varepsilon_{L(X),S}(\eta_X \circ f) = f.$$

We apply Lemma 4.3 with X and Y both set to $L(X)$, to the locally constant map $\eta_X \circ f : S \to \mathsf{LocConst}(*, X)$. Let

$$S \cong S_1 \amalg \cdots \amalg S_n$$

be the decomposition of S into the fibres of $\eta_X \circ f$ and denote by $\iota_i : S_i \to S$ the inclusion maps. Now we need to show that

$$L(X)(\iota_i)\left(\varepsilon_{L(X),S}(\eta_X \circ f)\right) = L(X)(\iota_i)(f)$$

Let $x_i \in \mathsf{LocConst}(*, X)$ be the element of which S_i is the fibre and let $s_i : S_i \to *$ denote the unique map. Then the above simplifies to

$$x_i \circ s_i = f \circ \iota_i.$$

Let $s \in S_i$ be arbitrary. We have

$$x_i(s_i(s)) = f(s) = f(\iota_i(s))$$

as desired. □

Lemma 4.7. *Let X be a condensed set. We have*

$$U(\varepsilon_X) \circ \eta_{U(X)} = \mathsf{id}_{U(X)}$$

Proof. Let $x \in X(*)$. We need to show that

$$\varepsilon_{X,*}(\eta_{X(*)}(x)) = x.$$

This is trivial. □

Remark 4.8. The proof of Lemma 4.7 in Lean actually uses Lemma 4.3, trivially decomposing $*$ into a disjoint union corresponding to the locally constant map $\eta_{X(*)}(x)$. This shows how important Lemma 4.3 is in proving anything at all about the values of the counit.

4.3. **The adjunction.**

Construction 4.9. All the above assembles into an adjunction

$$L \dashv U,$$

with unit defined in Construction 4.1, counit given by Construction 4.2 together with Lemmas 4.4 and 4.5, and triangle identities by Lemmas 4.6 and 4.7.

Construction 4.10. By uniqueness of adjoints, we obtain a natural isomorphism

$$L \cong \underline{(-)},$$

where $\underline{X}$ denotes the constant sheaf at a set X.

Proposition 4.11. *The functor $L : \mathsf{Set} \to \mathsf{CondSet}$ (and hence also $\underline{(-)}$ because of Construction 4.10) is fully faithful.*

Proof. This follows from the fact that the unit of the adjunction in Construction 4.9 is an isomorphism. □

4.4. **Generality.** The formalization of the results described in this section was in fact mostly done for sheaves on a site of the form `CompHausLike P` as described in §2.2, and only at the very end specialized to condensed sets and light condensed sets.

Some conditions on the predicate `P` were required. A few were required simply for the coherent topology to exist, but the two most important ones were

```
[∀ (S : CompHausLike.{u} P) (p : S → Prop), HasProp P (Subtype p)],
```

and

```
[HasExplicitFiniteCoproducts P].
```

The first one says that every subtype of an object of the category `CompHausLike P` satisfies the predicate `P`. This could in fact be weakened to only requiring clopen subsets to satisfy it. This more restrictive assumption turned out to be more convenient to work with, because the weaker one required carrying around too many proofs that certain subsets were clopen. It was still good enough, because in the application, `P` is either `True`, or the predicate saying that the space is totally disconnected and second countable. These are both stable under taking subspaces. This is an example of the importance of finding the "correct generality" for formalized statements. One wants the results to hold in the greatest generality possible, while not making an unnecessary effort for more generality when the payoff is small.

The second one says that finite disjoint unions of spaces in the category `CompHausLike P` exist, and form a coproduct.

The purpose of both these assumptions is explained by the discussion following Lemma 4.3. We need to be able to write an object of this category as a coproduct of the fibres of a locally constant map out of it, and we want the components of this coproduct to be literal subtypes of it.

5. The colimit characterization

Informally, one can state the main result of this section as follows: a condensed set X is discrete if and only if for every profinite set $S = \varprojlim_i S_i$, the canonical map $\varinjlim_i X(S_i) \to X(S)$ is an isomorphism. One needs to clarify, however, what the phrase "for every profinite set $S = \varprojlim_i S_i$" means. This is done in Construction 5.5 and Proposition 5.6 below.

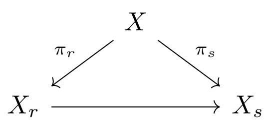
The condition that the canonical map $\varinjlim_i X(S_i) \to X(S)$ is an isomorphism is phrased in Lean as saying that a cocone is colimiting (this is equivalent to the statement about the canonical map being an isomorphism, but more convenient in formalization as explained in Remark 5.10 below). We say that a presheaf X on $\mathsf{Profinite}$ satisfies *the colimit condition* if this holds (Definition 5.9).

5.1. **Profinite sets as limits.** The material in this section was mostly formalized by Adam Topaz and Calle Sönne, and was already in mathlib before the start of this project. We do not write out informal proofs and refer instead to the Lean code for details.

Definition 5.1. Let X be a topological space. A *discrete quotient* of X is an equivalence relation on X with open fibres. Let r be a discrete quotient on X and denote by X_r its set of equivalence classes. We obtain a *projection map* $\pi_r : X \to X_r$, $x \mapsto [x]$. Give X_r the discrete topology. Then the projection map $\pi_r : X \to X_r$ is a quotient map.

Construction 5.2. Let r, s be two discrete quotients of a topological space X. We say that $r \leq s$ if for all $x, y \in X$, $r(x, y) \implies s(x, y)$. This gives the set of discrete quotients of X the structure of an inf-semilattice, i.e. a partial order with greatest lower bound. This makes the category structure inherited from this order cofiltered.

Remark 5.3. One can interpret the order on the discrete quotients as follows. Using the same notation as in Definition 5.1 and Construction 5.2, if $r \leq s$, then there exists a map $X_r \to X_s$ making the triangle



commute.

Construction 5.4. Let S be a profinite set. We define a functor from the category of discrete quotients of S to the category of finite sets by mapping a discrete quotient i of S to the set S_i of equivalence classes (this lands in finite sets because S is compact), and a morphism $i \leq j$ to the map $S_i \to S_j$ described in Remark 5.3.

Construction 5.5. We define a cone on the functor $i \mapsto S_i$ described in Construction 5.4 in the category $\mathsf{Profinite}$. The cone point is the profinite set S and the projection maps are the projections $\pi_i : S \to S_i$ defined in Definition 5.1.

Proposition 5.6. *The cone described in Construction 5.5 is limiting.*

Lemma 5.7. *Let $f : \varprojlim_i T_i \to S$ be a locally constant map from a profinite set S to a cofiltered limit of profinite sets T_i. Then f factors through one of the projections $\pi_i : \varprojlim_i T_i \to T_i$.*

5.2. **The colimit condition.** When we talk about "a profinite set $S = \varprojlim_i S_i$", we mean that S is written as the limit of its discrete quotients as described in §5.1.

Construction 5.8. Let X be a condensed set and let S be a profinite set. We define a cocone on the functor $i \mapsto X(S_i)$ from the opposite category of discrete quotients of S to Set. The cocone point is $X(S)$ and the coprojections are given by the maps $X(\pi_i) : X(S_i) \to X(S)$.

Definition 5.9. A presheaf of sets X on $\mathsf{Profinite}$ *satisfies the colimit condition* if for every profinite set S, the cocone described in Construction 5.8 is colimiting.

Construction 5.8 and Definition 5.9 do not exist as explicit declarations in MATHLIB. The colimit condition is expressed as the assumption

```
∀ S : Profinite, IsColimit <| F.mapCocone S.asLimitCone.op
```

where `S.asLimitCone.op` is the cocone obtained by taking the opposite of the cone in Construction 5.5, and `F` is any presheaf on $\mathsf{Profinite}$.

Remark 5.10. It is clear that a condensed set X satisfies the colimit condition if and only if for every profinite set $S = \varprojlim_i S_i$ the canonical map $\varinjlim_i X(S_i) \to X(S)$ is an isomorphism (indeed, defining said "canonical map" requires the data of a cocone with cocone point $X(S)$ on the functor). This is closer to the way one would state the condition in informal mathematics. In formalized mathematics, it is often more convenient to work directly with cones and cocones.

We can now state the main theorem of this section more concisely and more precisely.

Theorem B. *A condensed set X is discrete if and only if it satisfies the colimit condition.*

5.3. **Proof.** The forward direction follows easily from material that was already in mathlib prior to this work. The argument is the following. Suppose X is discrete. Then there is a set Y and an isomorphism $X \cong \mathsf{LocConst}(-, Y)$. What one needs to show is that given a profinite set $S = \varprojlim_i S_i$, each locally constant map $f : S \to Y$, factors through a projection map $\pi_i : S \to S_i$, and if it factors through both π_i and π_j, then there exists a k with π_k larger than both π_i and π_j, such that f factors through π_k. This follows from Lemma 5.7 and surjectivity of the projection maps.

The difficult part is the other direction. Suppose that X is a finite-product preserving presheaf on $\mathsf{Profinite}$ satisfying the colimit condition. It suffices to show that X is isomorphic to the presheaf $\mathsf{LocConst}(-, X(*))$. Applying this to condensed sets, we get the characterization that a condensed set X is discrete if and only if its underlying presheaf on $\mathsf{Profinite}$ satisfies the colimit condition, as desired.

It is not enough to give pointwise isomorphisms $X(S) \cong \mathsf{LocConst}(S, X(*))$ for each fixed profinite set S; we need to give such isomorphisms that are natural in S. As an intermediate step, we try to make the assignment $S \mapsto \varinjlim_i X(S_i)$ functorial in the profinite set S. The closest we can get to this is by using Kan extensions.

The strategy now is to construct a sequence of isomorphisms as indicated

$$X \cong \mathrm{Lan}_{\iota^{\mathrm{op}}}(X \circ \iota^{\mathrm{op}}) \cong \mathrm{Lan}_{\iota^{\mathrm{op}}}\left(\mathsf{LocConst}(-, X(*)) \circ \iota^{\mathrm{op}}\right) \cong \mathsf{LocConst}(-, X(*))$$

where $\mathrm{Lan}_G H$ denotes the left Kan extension of H along G and $\iota : \mathsf{FinSet} \to \mathsf{Profinite}$ is the inclusion functor. Once this is established, we have proved Theorem B.

The first and third isomorphisms

$$X \cong \mathrm{Lan}_{\iota^{\mathrm{op}}}(X \circ \iota^{\mathrm{op}})$$

and

$$\mathrm{Lan}_{\iota^{\mathrm{op}}}\left(\mathsf{LocConst}(-, X(*)) \circ \iota^{\mathrm{op}}\right) \cong \mathsf{LocConst}(-, X(*))$$

come from Construction 5.12 below (the condensed set $\mathsf{LocConst}(-, X(*))$ satisfies the colimit condition as explained at the beginning of this proof, and X does so by assumption).

The middle isomorphism

$$\mathrm{Lan}_{\iota^{\mathrm{op}}}(X \circ \iota^{\mathrm{op}}) \cong \mathrm{Lan}_{\iota^{\mathrm{op}}}\left(\mathsf{LocConst}(-, X(*)) \circ \iota^{\mathrm{op}}\right)$$

comes from Construction 5.11 below, together with uniqueness of left Kan extensions up to isomorphism.

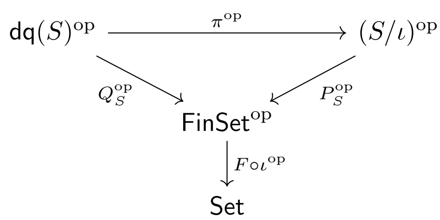
Construction 5.11. We construct an isomorphism $X \circ \iota^{\mathrm{op}} \cong \mathsf{LocConst}(-, X(*)) \circ \iota^{\mathrm{op}}$. Consider the functor $F : \mathsf{FinSet} \to \mathsf{Set}$ which takes a finite set Y to the set $X(*)^Y$ of maps $Y \to X(*)$. We proceed by showing that both functors are isomorphic to this functor. The isomorphism $F \cong \mathsf{LocConst}(-, X(*)) \circ \iota^{\mathrm{op}}$ is trivial. The component of the isomorphism $X \circ \iota^{\mathrm{op}} \cong F$ at a finite set Y is the following composition of isomorphisms

$$X(Y) \cong X\left(\coprod_{y \in Y} *\right) \cong \prod_{y \in Y} X(*) \cong X(*)^Y.$$

For the proof of naturality, we refer to the Lean code.

Construction 5.12. Let $F : \mathsf{Profinite} \to \mathsf{Set}$ be a presheaf satisfying the colimit condition (see Definition 5.9). We construct an isomorphism $F \cong \mathrm{Lan}_{\iota^{\mathrm{op}}}(F \circ \iota^{\mathrm{op}})$.

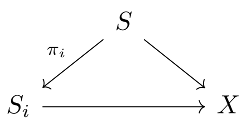
To explain the construction of this isomorphism more clearly, we introduce some notation. For a profinite set S, denote by $\mathsf{dq}(S)$ the poset of discrete quotients of S, and by $Q_S : \mathsf{dq}(S) \to \mathsf{FinSet}$ the functor $i \mapsto S_i$ defined in Construction 5.4. Let S/ι denote the over category consisting of morphisms $S \to \iota(X)$ in $\mathsf{Profinite}$ where X runs over all finite sets. Denote by $P_S : S/\iota \to \mathsf{FinSet}$ the projection functor. Note that there is an obvious functor $\pi : \mathsf{dq}(S) \to S/\iota$ defined by mapping i to $\pi_i : S \to S_i$. The triangle in the diagram



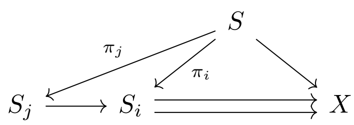
is strictly commutative, and constructing the desired isomorphism amounts to constructing an isomorphism

$$\mathrm{colim}(F \circ \iota^{\mathrm{op}} \circ Q_S^{\mathrm{op}}) \cong \mathrm{colim}(F \circ \iota^{\mathrm{op}} \circ P_S^{\mathrm{op}})$$

which is natural in the profinite set S. Omitting the proof of naturality, it suffices, by Proposition 2.6, to show that the functor π is initial. This amounts to showing that for every $S \to X \in S/\iota$, i.e. for every continuous map from S to a finite discrete set X, there exists a discrete quotient i of S with a map $S_i \to X$ such that the triangle



commutes, and that for every discrete quotient i of S with two commutative triangles as above, there is a $j \le i$ equalizing the two parallel maps in the diagram



The former condition follows from Lemma 5.7 and the latter is trivial; one can always take $j = i$ because the projection map π_i is surjective.

5.4. **Formalization.** To prove the colimit characterization of discrete condensed sets, we actually formalized more general constructions than those described informally in §5.1-§5.3. In what follows, we describe the formalization of this general setup. Throughout this section, three dots "..." in a block of LEAN code signify an omitted proof.

The functor π described in Construction 5.12 is defined for any cone on a functor to the category of finite sets, and proved initial if the indexing category is cofiltered, the cone is limiting, and each projection map is epimorphic:

```
variable {I : Type u} [SmallCategory I] {F : I ⥤ FintypeCat}
    (c : Cone <| F ⋙ toProfinite)

def functor : I ⥤ StructuredArrow c.pt toProfinite where
  obj i := StructuredArrow.mk (c.π.app i)
  map f := StructuredArrow.homMk (F.map f) (c.w f)

lemma functor_initial [IsCofiltered I] (hc : IsLimit c) [∀ i, Epi (c.π.app i)] :
    Initial (functor c) := by
  ...
```

Then, given a presheaf G on the category of profinite sets, valued in any category, a cocone is defined on the functor $F \circ \iota^{\mathrm{op}} \circ P_S^{\mathrm{op}}$ (here the notation is again borrowed from Construction 5.12):

```
variable {C : Type*} [Category C] (G : Profiniteᵒᵖ ⥤ C)

def cocone (S : Profinite) :
    Cocone (CostructuredArrow.proj toProfinite.op ⟨S⟩ ⋙ toProfinite.op ⋙ G) where
  pt := G.obj ⟨S⟩
  ι := {
    app := fun i ↦ G.map i.hom
    naturality := ... }

example : G.mapCocone c.op = (cocone G c.pt).whisker (functorOp c) := rfl
```

The `example` below the definition is important: it shows that the cocone obtained by applying the presheaf `G` to the cone `c` is *definitionally equal* to the cocone obtained by whiskering `cocone G c.pt` with the functor π^{op} (in general, equality of cocones is not a desirable property to work with, but definitional equality is good). This means that under the same conditions that make the functor π initial, together with an assumption that `c` becomes a colimiting after applying `G`, the cocone defined above is also colimiting:

```
def isColimitCocone (hc : IsLimit c) [∀ i, Epi (c.π.app i)]
    (hc' : IsColimit <| G.mapCocone c.op) : IsColimit (cocone G c.pt) :=
  (functorOp_final c hc).isColimitWhiskerEquiv _ _ hc'
```

We also provide the dual results for a covariant functor G out of Profinite. As an application of the dual (applied to the identity functor from Profinite to itself), we obtain a new way to write a profinite set as a cofiltered limit of finite discrete sets ↗.

6. Discrete condensed R-modules

The goal of this section is to show that given any ring R, a condensed R-module is discrete if and only if its underlying condensed set is discrete.

Consider the functor

$$L : \mathsf{Mod}_R \to \mathsf{CondMod}_R$$
$$M \mapsto \mathsf{LocConst}(-, M)$$

from R-modules to condensed R-modules. We would like to make the analogous constructions to the ones in §4. A bad approach to this would be to try to repeat the whole story. With that approach, showing that L is left adjoint to the forgetful functor $\mathsf{CondMod}_R \to \mathsf{Mod}_R$ would be significantly harder. One would have to prove that the counit is R-linear, and this would mean keeping track of the fibres of two locally constant maps and their addition. This is ill-suited for formalization, and the approach of reducing the problem for condensed modules to the case of underlying condensed sets is both easier, and provides the opportunity to formalize much more general results (e.g. those in §3).

We start by proving that the constant sheaf functor $\mathsf{Mod}_R \to \mathsf{CondMod}_R$ is fully faithful. Once that is established the result follows from Proposition 3.6 together with Proposition 2.15, because the underlying condensed set of a condensed R module is given by postcomposing with the forgetful functor from R-modules to sets.

Construction 6.1. We construct a natural isomorphism

$$\underline{(-)} \cong L.$$

For each R-module M, we need to give an isomorphism of condensed R-modules

$$\underline{M} \cong \mathsf{LocConst}(-, M).$$

This is given by the composition

$$\underline{M} \cong \underline{\mathsf{LocConst}(*, M)} \cong \mathsf{LocConst}(-, M)$$

where the first isomorphism is given by applying the constant sheaf functor to the obvious isomorphism of R-modules

$$M \cong \mathsf{LocConst}(*, M)$$

and the second isomorphism is given by the counit of the constant sheaf adjunction. It remains to prove two things; that this counit is an isomorphism, and that the isomorphism constructed this way is natural in M.

Lemma 6.2. *Let M be an R-module. The counit of the constant sheaf adjunction applied at the condensed R-module $\mathsf{LocConst}(-, M)$ is an isomorphism.*

Proof. Denote this map by

$$\varepsilon_{L(M)} : \underline{\mathsf{LocConst}(*, M)} \to \mathsf{LocConst}(-, M)$$

Since the forgetful functor $U : \mathsf{Mod}_R \to \mathsf{Set}$ reflects isomorphisms, the "forgetful functor" $F_U : \mathsf{CondMod}_R \to \mathsf{CondSet}$ (given as the sheaf-composition functor associated to U) also reflects isomorphisms by Proposition 2.9(3). It is therefore enough to check that $F_U(\varepsilon_{L(M)})$ is an isomorphism. Using Lemma 3.5, it suffices to show that $\varepsilon'_{U \circ L(M)}$ is an isomorphism, where ε' denotes the counit of the constant sheaf adjunction for condensed sets. But this is the same as saying that the condensed set $\mathsf{LocConst}(-, M)$ is discrete, which we already know. □

Lemma 6.3. *The isomorphism in Construction 6.1 is natural in the R-module M.*

Proof. This follows by unfolding the definitions and using some straightforward naturality arguments. We refer to the Lean code for more details. □

Lemma 6.4. *The functor $L : \mathsf{Mod}_R \to \mathsf{CondMod}_R$ is fully faithful.*

Proof. Using construction 6.1, we obtain an adjunction $L \dashv U$ where U is the forgetful functor $\mathsf{CondMod}_R \to \mathsf{Mod}_R$. The obvious isomorphism of R-modules

$$M \cong \mathsf{LocConst}(*, M)$$

is clearly natural in M, i.e. it induces a natural isomorphism $\mathrm{Id} \cong U \circ L$. Now the result follows from Corollary 2.3. □

Lemma 6.4 together with the general Proposition 3.6 concludes the proof of the main result (it has been established in §2.3 and §4 that Proposition 3.6 applies in this situation):

Theorem C. *Let R be a ring. A condensed R-module is discrete if and only if its underlying condensed set is discrete.* □

7. Conclusion

We summarize the characterization of discrete condensed sets in Theorem 7.1, and that of discrete condensed modules over a ring in Theorem 7.2.

Theorem 7.1. *Let $L : \mathsf{Set} \to \mathsf{CondSet}$ denote the functor that takes a set X to the sheaf of locally constant maps $\mathsf{LocConst}(-, X)$, and for a set X, denote by $\underline{X}$ the constant sheaf at X. Recall that these functors are both left adjoint to the underlying set functor $U : \mathsf{CondSet} \to \mathsf{Set}$, which takes a condensed set X to its underlying set $X(*)$. The following conditions on a condensed set X are equivalent.*

(1) X is discrete, i.e. there exists a set Y and an isomorphism $X \cong \underline{Y}$.
(2) X is in the essential image of the functor L
(3) The component at X of the counit of the adjunction $\underline{(-)} \dashv U$ is an isomorphism.
(4) The component at X of the counit of the adjunction $L \dashv U$ is an isomorphism.
(5) For every profinite set $S = \varprojlim_i S_i$, the canonical map $\varinjlim_i X(S_i) \to X(S)$ is an isomorphism.

Proof. The equivalence of (1)-(4) is established in §4 where the isomorphism between the functors L and $\underline{(-)}$ is constructed, and full faithfulness proved. For the equivalence with (5), see §5. □

Theorem 7.2. *Let R be a ring. Denote by $L : \mathsf{Mod}_R \to \mathsf{CondMod}_R$ the functor that takes an R-module X to the sheaf of R-modules of locally constant maps $\mathsf{LocConst}(-, X)$, and for an R-module X, denote by $\underline{X}$ the constant sheaf at X. These functors are both left adjoint to the underlying module functor $U : \mathsf{CondMod}_R \to \mathsf{Mod}_R$, which takes a condensed R-module X to its underlying module $X(*)$. The following conditions on a condensed R-module X are equivalent.*

(1) X is discrete, i.e. there exists an R-module N and an isomorphism $X \cong \underline{N}$.
(2) X is in the essential image of the functor L
(3) The component at X of the counit of the adjunction $\underline{(-)} \dashv U$ is an isomorphism.
(4) The component at X of the counit of the adjunction $L \dashv U$ is an isomorphism.
(5) For every profinite set $S = \varprojlim_i S_i$, the canonical map $\varinjlim_i X(S_i) \to X(S)$ is an isomorphism.

Proof. The proof of the equivalence of (1)-(4) is identical to the condensed set case. For the equivalence with (5), we use the fact that a condensed module is discrete if and only if its underlying set is (Theorem C), and that the forgetful functor from R-modules to sets preserves and reflects filtered colimits. □

Remark 7.3. The variants of Theorems 7.1 and 7.2 for light condensed sets and modules also hold. The statements are identical except that the last condition becomes

(5) For every light profinite set $S = \varprojlim_{n \in \mathbb{N}} S_n$, the canonical map $\varinjlim_n X(S_n) \to X(S)$ is an isomorphism.

Acknowledgements

This work began as my first formalization project, depending on the definition of condensed objects in the Liquid Tensor Experiment (LTE). The work presented here is a much improved version of it, depending only on MATHLIB. This improvement would not have been possible without the help of many people, to which I am very grateful and will try to list here.

First, I would like to thank everyone involved in LTE for pioneering the formalization of condensed mathematics.

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Project report — Solid K-theory of p-adic Banach algebras

This chapter contains an account of partial progress made in late 2021 and early 2022 on studying the condensed K-theory of p-complete rings. The material appearing here was written at that time and revised in 2024. Parts of it appeared as the notes [Asg22a, Asg22b] on the author's website.

SOLID K-THEORY OF p-ADIC BANACH ALGEBRAS

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Abstract. This is a report on partial progress in understanding the solidification of the condensed K-theory spectra of p-complete rings arising as the rings of integers of p-adic Banach algebras, mostly for the two examples $\mathcal{O}_{\mathbb{C}_p}$ and $\mathbb{Z}[T]_p^\wedge \simeq \mathcal{O}_{\mathbb{Q}_p\langle T\rangle}$. Along the way, we develop the foundations of solid spectra, and prove some general results about condensed rings coming from p-adic Banach algebras.

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1. Introduction

Algebraic K-theory is an important spectrum-valued invariant of rings which has been widely studied. It admits a promotion to an invariant of condensed rings, by taking values in condensed spectra. This is a way to keep track of the topological information, which is of course lost when taking the classical algebraic K-theory, which we may call *discrete K-theory* to distinguish it from its condensed counterpart. In general, the condensed K-theory spectrum of a condensed ring is badly behaved, in the same way as the tensor product of condensed abelian groups (this tensor product is related to K_2 in many cases). A way to tame it, which is useful especially in non-archimedean contexts, is to *solidify* it. This is an analogue for condensed spectra of the solidification of condensed abelian groups, which has been used in the literature (see e.g. [19, Proposition 10.6], [10, §1.2.2], and [1]). We develop the foundations of solid spectra in some detail in this report. Foundational material on this theory has been somewhat lacking in the literature, although [1] gives a proper definition. Our definition of solid spectra is different, but closely related. We make some remarks on how to compare the two approaches towards the end of §2.2.2.

Solidification in some sense erases archimedean information. For example, the solidification of the condensed abelian group $\mathbb{R}$ vanishes ([20, Corollary 6.1 (iii)]). More recently, it has been shown by Ko Aoki [1] that solid K-theory of real and complex Banach algebras is discrete.

Less is known about condensed K-theory of non-archimedean rings. For a p-complete ring R, solidified condensed K-theory admits a canonical comparison map

$$K^{\blacksquare}(R) \to \varprojlim_{n \in \mathbb{N}} K^{\delta}(R/p^n).$$

A natural question to ask is whether this map is an equivalence. It is known to be true in the simplest case $R = \mathbb{Z}_p$ and in general with finite coefficients. We consider two examples, $R = \mathcal{O}_{\mathbb{C}_p}$ and $R = \mathbb{Z}[T]_p^{\wedge}$, which are more complicated. Even studying the lower K-groups in these cases turns out to be non-trivial.

In §2, the aforementioned foundational theory of solid spectra is established, and in §3, we prove some general results about the condensed rings $\mathcal{O}_{\mathbb{C}_p}$ and $\mathbb{Z}[T]_p^{\wedge}$, and more generally those coming from p-adic Banach algebras. In §4, we go on to apply the results in §3 to take the first steps towards understanding $K^{\blacksquare}(R)$ for our two examples, by studying the lower K-groups.

Notation and terminology. We will use the theory of ∞-categories as developed for quasicategories in [15, 16, 17]. For objects X, Y of an ∞-category $\mathcal{C}$, we denote by $\mathrm{Map}(X, Y)$ the *mapping anima* of morphisms (1-cells) $X \to Y$. This is an object of the ∞-category An of *anima*, called *spaces* in [15]. If $\mathcal{C}$ is a stable ∞-category, $\mathrm{Map}(X, Y)$ admits an enhancement to spectra. We will denote this *mapping spectrum* by $\mathrm{map}(X, Y)$. This is an object of the stable ∞-category Sp of *spectra*, which is defined as the stabilization of An in the sense of [16, §1.4.2]. The mapping spectrum $\mathrm{map}(X, Y)$, roughly speaking, contains the same information as all the mapping anima $\mathrm{Map}(X[n], Y)$ for $n \in \mathbb{Z}$, where $X[n]$ is the shift functor, defined as the n-th power of the suspension or loop functor[1], depending on if n is positive or negative (see [16, §1.1.2] for details). More precisely, the homotopy groups of $\mathrm{map}(X, Y)$ in degrees ≥ 0 agree with those of $\mathrm{Map}(X, Y)$, and since

$$\pi_k \,\mathrm{map}(X, Y) \simeq \pi_0 \,\mathrm{map}(X[-k], Y),$$

we obtain all the homotopy groups of $\mathrm{map}(X, Y)$ from those of the mapping anima of the form $\mathrm{Map}(X[n], Y)$ for $n \in \mathbb{Z}$.

All rings are assumed to be commutative unless otherwise specified. We denote by Sp the stable symmetric monoidal ∞-category of spectra. Its tensor product is classically called *smash product*. The unit for this symmetric monoidal structure is the *sphere spectrum*, denoted by $\mathbb{S}$.

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2. Solid spectra

Let $\mathcal{C}$ be an ∞-category. The ∞-category of condensed objects of $\mathcal{C}$, $\mathsf{Cond}(\mathcal{C})$, is the category of hypercomplete sheaves (see [15, §6.5]) on the site of profinite sets with finite families of jointly surjective maps as covers.[2] As in the 1-categorical situation, the condensed objects have a more explicit description as follows: A condensed object in $\mathcal{C}$ is a presheaf $X : \mathsf{Profinite}^{\mathrm{op}} \to \mathcal{C}$ such that

(i) X preserves finite products: in other words, for every finite family (T_i) of profinite sets, the natural map

$$X\Big(\coprod_i T_i\Big) \longrightarrow \prod_i X(T_i)$$

is an equivalence.

[1] In particular, a shift is a finite limit or a finite colimit

[2] There are the usual set-theoretic issues described in [20, Appendix to Lecture 2]. We resolve these by using a universe bump as described in [4, 5]. This is similar to fixing a cut-off cardinal κ, and considering sheaves on the site of κ-small profinite sets.

(ii) For every hypercover $S_\bullet \to T$ in $\mathsf{Profinite}$ with respect to the regular topology, i.e. for every simplicial object $S_\bullet$ in $\mathsf{Profinite}$ with a surjection $S_0 \to T$ and such that for every n, the map $S_{n+1} \to \operatorname{cosk}_n(\operatorname{sk}_n S_\bullet)_{n+1}$ is surjective, the natural map

$$X(T) \to \varprojlim_{n\in\Delta} X(S_n)$$

is an equivalence.

This characterization follows from [17, Proposition A.5.7.2]. The characterization of condensed objects as presheaves on the site of extremally disconnected compact Hausdorff spaces, satisfying condition (i) above, also holds in this setting.

The higher analogue of condensed sets is *condensed anima*, i.e. condensed objects in the ∞-category An of *anima*, often called *spaces*.

In the case where $\mathcal{C}$ is Sp, the ∞-category of spectra, we can equivalently regard the category CondSp of *condensed spectra* as the category of hypercomplete sheaves of spectra on the site of profinite sets, or as the stabilization of CondAn in the sense of [16, §1.4.2] (there called the category of *spectrum objects*). There is a t-structure on CondSp, whose heart is equivalent to the category of condensed abelian groups, and has the property that $\pi_n(X)(T) \simeq \pi_n(X(T))$ for every condensed spectrum X and extremally disconnected set T. It follows from this and classical results in algebraic topology that homotopy groups commute with arbitrary products of connective condensed spectra, and in particular, an arbitrary product of connective condensed spectra is connective (as is the case for ordinary spectra). The proofs of the statements in this paragraph can all be extracted from the general theory developed in [15, 16, 17].

2.1. **Discrete condensed anima and spectra.** In this section we give a useful characterization of discrete condensed anima (Proposition 2.3). As a corollary, we deduce the same characterization for discrete condensed spectra.

If $\mathcal{C}$ is the ∞-category of anima or that of spectra, then the global sections functor $\mathsf{Cond}(\mathcal{C}) \to \mathcal{C}$, $X \mapsto X(*)$ admits a fully faithful left adjoint, which we denote by $X \mapsto \underline{X}$. This is given by the constant hypercomplete sheaf. See [5, Construction 2.2.12].

Definition 2.1. An object $X \in \mathsf{Cond}(\mathcal{C})$ is *discrete* if it belongs to the essential image of the functor $\underline{(-)}$.

Proposition 2.2. *The presheaf of ∞-categories on the site of profinite sets, $T \mapsto \mathsf{Sh}(T)$, is a hypercomplete sheaf.*

Proof. This follows from [13, Corollary 2.8 and Example 1.28]. □

Proposition 2.3. *Let Y be a condensed anima. The following are equivalent*

(1) Y is discrete.

(2) For every profinite set $T = \varprojlim_i T_i$, the canonical right hand map in

$$\varinjlim_i \prod_{t\in T_i} Y(\{t\}) \leftarrow \varinjlim_i Y(T_i) \to Y(T)$$

is an equivalence. The left hand map is an equivalence by assumption; the fact that Y is a sheaf implies that it preserves finite products, and T_i is a finite coproduct of its singleton subsets.

Proof. We let X be an anima and define a presheaf $\tilde{X}$ of anima on the site of profinite sets, by informally letting

$$\tilde{X}(T) = \varinjlim_i \prod_{t\in T_i} X$$

for every profinite set $T = \varprojlim_i T_i$ (formally, this is defined as the left Kan extension of the yoneda presheaf on finite sets along the inclusion of finite sets in profinite sets). We claim that the canonical map $\tilde{X} \to \underline{X}$ is an equivalence. Granting this, the result follows. Indeed, if Y satisfies (1), then there is an anima X and an equivalence $\underline{X} \simeq Y$. In particular $Y(*) \simeq X$ (the unit of the adjunction $\underline{(-)} \dashv (Y \mapsto Y(*))$ gives the equivalence, see e.g. [5, Construction 2.2.12]), so

$$\varinjlim_i \prod_{t\in T_i} Y(*) \simeq \tilde{X}(T) \simeq \underline{X}(T) \simeq Y(T).$$

Conversely, if Y satisfies (2), then $\underline{Y(*)} \simeq Y$, because by the result we are going to prove, $\underline{Y(*)}$ satisfies (2).

We now turn to proving the equivalence $\tilde{X} \simeq \underline{X}$, and start by showing that $\tilde{X}$ is a hypercomplete sheaf.

Proposition 2.2 gives, for any two sheaves $\mathcal{F}, \mathcal{G}$ on the topological space T and any hypercover $S_\bullet \to T$, an equivalence

$$\mathrm{Map}_{\mathsf{Sh}(T)}(\mathcal{F}, \mathcal{G}) \simeq \varprojlim_{n \in \Delta} \mathrm{Map}_{\mathsf{Sh}(S_n)}(f_n^* \mathcal{F}, f_n \mathcal{G})$$

where $f_n : S_n \to T$ is the obvious map (this is a general fact about how mapping anima are computed in a limit of ∞-categories). For an anima Y and topological space Z, denote by $\underline{Y}_Z$ the constant sheaf at Y on Z. Applying the above equivalence to $\mathcal{F} = \underline{*}_T$ and $\mathcal{G} = \underline{X}_T$ we get an equivalence

$$\underline{X}_T(T) \simeq \varprojlim_{n \in \Delta} \underline{X}_{S_n}(S_n)$$

so in order to show that $\tilde{X}$ is a hypercomplete sheaf, it suffices to show that for every profinite set T, $\underline{X}_T(T) \simeq \tilde{X}(T)$.

Now we fix the topological space T, which is assumed to be profinite, and throughout the rest of this proof, the word (pre)sheaf will mean (pre)sheaf of anima on the topological space T.

Let X_T denote the constant presheaf $U \mapsto X$ on T. The plus construction ([15, Construction 6.2.2.9]) gives a presheaf $X_T^\dagger$ on T. We will finish this proof by providing two equivalences

$$\tilde{X}(T) \simeq X_T^\dagger(T) \simeq \underline{X}_T(T).$$

For the second one, we will prove that $X_T^\dagger$ is a sheaf on the topological space T; this is enough because it already satisfies the universal property of the sheafification by [15, Lemma 6.2.2.14]. We start by the first one.

Let us analyze the definition of $X_T^\dagger$ in our situation. Let P denote the poset of open subsets of T, let $\mathsf{Cov}(P)$ denote the poset of pairs $(U, \mathcal{U})$ where U is an open subset of T and $\mathcal{U}$ is an open cover of U, ordered such that $(U, \mathcal{U}) \leq (V, \mathcal{V})$ whenever $U \subseteq V$ and $\mathcal{U}$ is a refinement of $\mathcal{V}$. Finally, let P^+ be the poset of triples $(U, \mathcal{U}, U')$ where $(U, \mathcal{U}) \in \mathsf{Cov}(P)$ and $U' \in \mathcal{U}$, ordered such that $(U, \mathcal{U}, U') \leq (V, \mathcal{V}, V')$ whenever $(U, \mathcal{U}) \leq (V, \mathcal{V})$ in $\mathsf{Cov}(P)$ and $U' \subseteq V'$.

There are three projection functors,

- $e : P^+ \to P$, $(U, \mathcal{U}, U') \mapsto U'$,
- $\pi : P^+ \to \mathsf{Cov}(P)$, $(U, \mathcal{U}, U') \mapsto (U, \mathcal{U})$,
- $\rho : \mathsf{Cov}(P) \to P$, $(U, \mathcal{U}) \mapsto U$.

For a presheaf $\mathcal{F}$ on T, we then have $\mathcal{F}^\dagger = \rho_! \pi_* e^* \mathcal{F}$, where e^* is pullback along e, π_* is the right adjoint of the pullback along π (given by right Kan extension) and $\rho_!$ is the left adjoint of the pullback along ρ (given by left Kan extension).

In the case of $\mathcal{F} = X_T$ a constant presheaf, e^* simply takes it to the corresponding constant presheaf on P^+. Suppose now that U is an open subset of T and that $\mathcal{U} = \{U_j\}_{j \in J}$ is a disjoint open covering of U. The limit formula for right Kan extensions then gives that for a presheaf $\mathcal{F} : P^{+\,\mathrm{op}} \to \mathsf{Set}$,

$$\pi_* \mathcal{F}(U, \mathcal{U}) \simeq \prod_{j \in J} \mathcal{F}(U, \mathcal{U}, U_j)$$

(this is because $\mathcal{U}$ is disjoint and thus the limit in the formula is taken over a discrete category).

The colimit which calculates the value of $\rho_! \mathcal{F}$ for a presheaf $\mathcal{F} : \mathsf{Cov}(P)^{\mathrm{op}} \to \mathsf{Set}$ at an open $U \subset T$ is indexed by a category equivalent to the poset of open covers of U ordered by refinement. Now, given a clopen subset V of T, the poset of open covers of V ordered by refinement contains a cofinal subset given by the finite disjoint clopen covers of V. Indeed, it suffices to show that every open cover $\mathcal{U}$ of V admits a refinement which consists of pairwise disjoint clopen subsets. First, because the clopen subsets form a basis for the topology, we may write each element of $\mathcal{U}$ as a union of clopen sets. Then by compactness, we take a finite subcover of that clopen cover of V. Such a finite clopen covering of V admits a refinement of pairwise disjoint clopen subsets, by taking intersections and complements. Thus, given a clopen subset V of T and a presheaf $\mathcal{F} : \mathsf{Cov}(P)^{\mathrm{op}} \to \mathsf{Set}$, we have

$$\rho_! \mathcal{F}(V) \simeq \varinjlim_{\mathcal{V}} \mathcal{F}(V, \mathcal{V}),$$

where the colimit is taken over the poset of finite, disjoint clopen covers of V, ordered by refinement. Applying this sequence of functors to our constant presheaf X_T gives that

$$X_T^\dagger(T) \simeq \varinjlim_{\mathcal{V}} \prod_{V \in \mathcal{V}} X \simeq \tilde{X}(T),$$

where the latter isomorphism follows from the cofinality arguments in [2, §5] (because the poset of finite disjoint clopen covers is identified with that of discrete quotients of T).

We turn to proving that $X_T^\dagger$ is a sheaf. Since the topological space T is coherent (see [15, above Proposition 6.5.4.4]; the compact open subsets are the same as the clopen subsets in this case, since T is a compact Hausdorff space; the clopens are stable under finite intersections and form a basis for the topology on T), it suffices by [15, Theorem 7.3.5.2] to show that $X_T^\dagger(\varnothing) \simeq *$ and that for every pair of clopen subsets $U, V \subset T$, the diagram

$$\begin{array}{ccc} X_T^\dagger(U \cap V) & \longrightarrow & X_T^\dagger(U) \\ \downarrow & & \downarrow \\ X_T^\dagger(V) & \longrightarrow & X_T^\dagger(U \cup V) \end{array}$$

is a pullback square. The condition on the empty set is clear since it is mapped to an empty product, which is equivalent to a point, and the condition on the square above follows from the fact that filtered colimits commute with finite limits.

We have now established that $\tilde{X}$ is a hypercomplete sheaf. It is now simple to prove the desired equivalence: We show that for any extremally disconnected set T,

$$\underline{X}(T) \simeq \underline{X}_T(T).$$

We denote by $\underline{X}^T$ the sheaf on the extremally disconnected set T defined by its values on a basis of clopen subsets by $\underline{X}^T(U) = \underline{X}(U)$ (this is a sheaf because it comes from the pullback of the sheaf $\underline{X}$ on the site of extremally disconnected sets over T along the obvious functor from the poset of clopen subsets of T). Further, it receives a map from the presheaf X_T (since X_T is given at the clopen subsets of T by restriction of the constant presheaf X on the site of extremally disconnected sets, and the sheafification of a presheaf receives a map from that presheaf). It follows that there is a map

$$\underline{X}_T \to \underline{X}^T.$$

But it is clear that this map is an equivalence (this can be checked on stalks, where it becomes the identity map $X \to X$), and so we are done. □

Proposition 2.4. *Let Y be a condensed anima. The characterization of Y as discrete in Proposition 2.3 also holds with "profinite" replaced by "extremally disconnected".*

Proof. With the notation of the proof of Proposition 2.3, showing that $\tilde{X}$ is a hypercomplete sheaf becomes a question of showing that it preserves finite products, which is clear because filtered colimits commute with finite products (and in particular does not rely on Proposition 2.2 and the deep results it depends on). The rest of the proof is the same. □

Corollary 2.5. *Let Y be a condensed spectrum. The following are equivalent*

(1) Y is discrete

(2) For every profinite set $T = \varprojlim_i T_i$, the canonical map

$$\varinjlim_i \bigoplus_{t \in T_i} Y(\{t\}) \simeq \varinjlim_i Y(T_i) \to Y(T)$$

is an equivalence.

(3) For every extremally disconnected set $T = \varprojlim_i T_i$, the canonical map

$$\varinjlim_i \bigoplus_{t \in T_i} Y(\{t\}) \simeq \varinjlim_i Y(T_i) \to Y(T)$$

is an equivalence.

Proof. Let X be a spectrum and define $\tilde{X}$ like in the proof of Proposition 2.3. The only difficult thing to do is to prove that $\tilde{X}$ is a hypercomplete sheaf, but for that it is enough to prove that for every spectrum Y, the presheaf of anima

$$T \mapsto \mathrm{Map}(Y, \tilde{X}(T))$$

is a hypercomplete sheaf. It suffices to check this on the (compact) generators of Sp; namely shifts of the sphere spectrum. But by compactness of Y we have that the presheaf above is actually equivalent to

$$\widetilde{\mathrm{Map}(Y, X)}$$

which is a hypercomplete sheaf by the proof of Proposition 2.3. □

2.2. **Solid structures.**

2.2.1. *Solid abelian groups.* We recall a few of the main results of the theory of solid abelian groups, see [20, Lectures V and VI] for a detailed account. The results we state without proof are proved there. See also [19, Lecture II] for another discussion including more motivation behind the theory.

Construction 2.6. For a profinite set $T = \varprojlim_i T_i$, we let

$$\mathbb{Z}[T]^{\blacksquare} := \varprojlim_i \mathbb{Z}[T_i].$$ [3]

It comes equipped with a canonical natural map $\mathbb{Z}[T] \to \mathbb{Z}[T]^{\blacksquare}$.

Definition 2.7. A condensed abelian group M is *solid* if the map

$$\mathrm{Hom}\left(\mathbb{Z}[T]^{\blacksquare}, M\right) \to \mathrm{Hom}\left(\mathbb{Z}[T], M\right), \tag{1}$$

induced from the natural map in 2.6, is an isomorphism. An object C of $\mathcal{D}(\mathsf{CondAb})$ is *solid* if the corresponding map

$$R\,\mathrm{Hom}\left(\mathbb{Z}[T]^{\blacksquare}, C\right) \to R\,\mathrm{Hom}\left(\mathbb{Z}[T], C\right) \tag{2}$$

is an equivalence.

Remark 2.8. It follows from the general theory that

(1) A condensed abelian group M is solid if and only if the object $M[0] \in \mathcal{D}(\mathsf{CondAb})$ is solid.
(2) An object C of $\mathcal{D}(\mathsf{CondAb})$ is solid if and only if its condensed cohomology groups $H^i(C)$ are solid abelian groups for all i.
(3) An object is solid if and only if the internal versions of (1) and (2) hold (i.e. with Hom replaced by $\underline{\mathrm{Hom}}$ and $R\,\mathrm{Hom}$ replaced by $R\underline{\mathrm{Hom}}$).

For $T = \varprojlim_i T_i$ a profinite set, we consider the condensed abelian group $\underline{\mathrm{Hom}}\left(\mathbb{Z}[T], \mathbb{Z}\right)$. This is in fact a discrete condensed abelian group:

Lemma 2.9. *For every profinite set T, there is a set I and an isomorphism of condensed abelian groups*

$$\underline{\mathrm{Hom}}\left(\mathbb{Z}[T], \mathbb{Z}\right) \simeq \bigoplus_{i \in I} \mathbb{Z}.$$

Proof. See [3] or [20, Theorem 5.4] for a proof of the equivalence

$$\underline{\mathrm{Hom}}\left(\mathbb{Z}[T], \mathbb{Z}\right)(*) \simeq C(T, \mathbb{Z}) \simeq \left(\bigoplus_{i \in I} \mathbb{Z}\right)(*).$$

To see that it extends to an isomorphism of the condensed abelian groups, we use the characterization of discrete condensed objects, described for condensed anima and spectra in §2.1 and proved formally for

[3] This is made functorial by defining it as a right Kan extension in the obvious way, similar to $\tilde{X}$ in the proof of Proposition 2.3.

condensed sets and modules in [2], to show that $\underline{\mathrm{Hom}}(\mathbb{Z}[T],\mathbb{Z})$ is in fact discrete. Indeed, its S-valued points for a profinite set $S = \varprojlim_j S_j$ are

$$\begin{aligned}\underline{\mathrm{Hom}}(\mathbb{Z}[T],\mathbb{Z})(S) &\simeq \mathrm{Hom}(\mathbb{Z}[T\times S],\mathbb{Z})\\ &\simeq \varinjlim_{i,j} \mathrm{Hom}(\mathbb{Z}[T_i\times S_j],\mathbb{Z})\\ &\simeq \varinjlim_{j}\varinjlim_{i} \mathrm{Hom}(\mathbb{Z}[T_i\times S_j],\mathbb{Z})\\ &\simeq \varinjlim_{j} \mathrm{Hom}\left(\mathbb{Z}[\varprojlim_i T_i\times S_j],\mathbb{Z}\right)\\ &\simeq \varinjlim_{j} \underline{\mathrm{Hom}}(\mathbb{Z}[T],\mathbb{Z})(S_j),\end{aligned}$$

since $\mathbb{Z}$ is discrete. □

Remark 2.10. One can similarly show, by using the fact that filtered colimits of condensed sets computed objectwise on extremally disconnected sets, that for every discrete condensed abelian group A and profinite set T, the canonical map

$$\varinjlim_i \underline{\mathrm{Hom}}(\mathbb{Z}[T_i],A)\to \underline{\mathrm{Hom}}(\mathbb{Z}[T],A)$$

is an equivalence. This extends to discrete condensed spectra using the characterization in §2.1, and will be used in that setting in the section 2.2.2.

Corollary 2.11. *For every profinite set T, there is a set I and an isomorphism of condensed abelian groups*

$$\mathbb{Z}[T]^{\blacksquare}\simeq \prod_{i\in I}\mathbb{Z}.$$

Proof. We have

$$\mathbb{Z}[T]^{\blacksquare}\simeq \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(\mathbb{Z}[T],\mathbb{Z}),\mathbb{Z})\simeq \underline{\mathrm{Hom}}\left(\bigoplus_{i\in I}\mathbb{Z},\mathbb{Z}\right)\simeq \prod_{i\in I}\mathbb{Z}.$$

The first isomorphism follows from 2.10, in this case $\varinjlim_i \underline{\mathrm{Hom}}(\mathbb{Z}[T_i],\mathbb{Z})\simeq \underline{\mathrm{Hom}}(\mathbb{Z}[T],\mathbb{Z})$. The second one follows from the isomorphism $\bigoplus_{i\in I}\mathbb{Z}\simeq \underline{\mathrm{Hom}}(\mathbb{Z}[T],\mathbb{Z})$. The last isomorphism is a priori only true on underlying abelian groups, but it follows formally for the condensed abelian groups by showing that they corepresent the same functor on CondAb. □

Proposition 2.12. *For every profinite set T, $\mathbb{Z}[T]^{\blacksquare}$ is solid.* □

We denote by SolidAb the full subcategory of CondAb spanned by the solid abelian groups.

Theorem 2.13. *([20, Theorem 5.8])*

(1) The category of solid abelian groups is an abelian category generated by compact projectives of the form $\prod_{i\in I}\mathbb{Z}$. Further, the fully faithful inclusion i : SolidAb $\hookrightarrow$ CondAb preserves all limits, colimits and extensions and has a left adjoint denoted $M\mapsto M^{\blacksquare}$ which is a colimit-preserving extension of $\mathbb{Z}[T]\mapsto \mathbb{Z}[T]^{\blacksquare}$, and as such, unique up to unique isomorphism.

(2) The functor $i:\mathcal{D}(\mathsf{SolidAb})\to\mathcal{D}(\mathsf{CondAb})$ induced from i above is fully faithful, and its essential image is spanned by the solid objects of $\mathcal{D}(\mathsf{CondAb})$. It admits a left adjoint $C\mapsto C^{L\blacksquare}$ which is the left derived functor of the solidification functor on condensed abelian groups. Also, it is a colimit-preserving extension of $\mathbb{Z}[T]\mapsto\mathbb{Z}[T]^{L\blacksquare}$ and as such, unique up to contractible choice. An object $C\in\mathcal{D}(\mathsf{CondAb})$ is solid if and only if $H^i(C)$ is solid for all i. □

Theorem 2.14. *There is a unique way to endow the category* SolidAb *with a symmetric monoidal tensor product $\otimes^{\blacksquare}$, making the functor $M\mapsto M^{\blacksquare}$ symmetric monoidal.*

Remark 2.15. A similar statement to 2.14 holds in the derived setting (see 2.34).

2.2.2. *Solid spectra.* The theory of solid spectra can be developed analogously to that of solid abelian groups. Our most important results (Lemmas 2.19 and 2.22) will be tools to reduce statements to analogues for solid abelian groups. We obtain the main two results 2.32 and 2.34, analogues to 2.13 and 2.14.

Construction 2.16. For a profinite set $T = \varprojlim_i T_i$, we let

$$\mathbb{S}[T]^{\blacksquare} := \varprojlim_i \mathbb{S}[T_i]^{4}.$$

It comes equipped with a canonical natural map $\mathbb{S}[T] \to \mathbb{S}[T]^{\blacksquare}$.

Definition 2.17. A condensed spectrum X is *solid* if the map

$$\text{map}\left(\mathbb{S}[T]^{\blacksquare}, X\right) \to \text{map}\left(\mathbb{S}[T], X\right) \tag{3}$$

induced from the natural map in 2.16, is an equivalence.

Construction 2.18. For a condensed spectrum X and profinite set T, we consider the internal condensed mapping spectrum $\underline{\text{map}}\left(\mathbb{S}[T], X\right)$. For the discrete condensed spectra $\mathbb{S}$ and $\mathbb{Z}$, we have a unique map of commutative algebras $\mathbb{S} \to \mathbb{Z}$ in $\mathsf{CondSp}^{\otimes}$, yielding a map

$$\underline{\text{map}}\left(\mathbb{S}[T], \mathbb{S}\right) \to \underline{\text{map}}\left(\mathbb{S}[T], \mathbb{Z}\right).$$

Further, since $\underline{\text{map}}\left(\mathbb{S}[T], \mathbb{Z}\right)$ is a $\mathbb{Z}$-module in condensed spectra, we have a map

$$\underline{\text{map}}\left(\mathbb{S}[T], \mathbb{S}\right) \otimes_{\mathbb{S}} \mathbb{Z} \to \underline{\text{map}}\left(\mathbb{S}[T], \mathbb{Z}\right).$$

Lemma 2.19. *The map*

$$\underline{\text{map}}\left(\mathbb{S}[T], \mathbb{S}\right) \otimes_{\mathbb{S}} \mathbb{Z} \to \underline{\text{map}}\left(\mathbb{S}[T], \mathbb{Z}\right)$$

from 2.18 is an equivalence.

Proof. First a note on how to interpret the right hand side in the equivalence. For the sake of this argument, we will temporarily denote the Eilenberg-Maclane spectrum associated to an abelian group A by HA and not just A. We have:

$$\begin{aligned}\underline{\text{map}}\left(\mathbb{S}[T], H\mathbb{Z}\right) &\simeq \underline{\text{map}}_{H\mathbb{Z}}\left(\mathbb{S}[T] \otimes_{\mathbb{S}} H\mathbb{Z}, H\mathbb{Z}\right) \\ &\simeq R\underline{\text{Hom}}\left(\mathbb{Z}[T], \mathbb{Z}\right) \\ &\simeq H\underline{\text{Hom}}\left(\mathbb{Z}[T], \mathbb{Z}\right)\end{aligned}$$

where in the final step, we used the fact that the condensed cohomology of a profinite set is concentrated in degree 0. In other words, the internal mapping spectrum in question lies in the heart of the t-structure on derived condensed abelian groups.

Now for the actual proof of the lemma, we use the fact that $\mathbb{S}$ and $\mathbb{Z}$ are discrete, and by the analogue of 2.10 for spectra, we can write

$$\underline{\text{map}}\left(\mathbb{S}[T], \mathbb{S}\right) \simeq \varinjlim_i \underline{\text{map}}\left(\mathbb{S}[T_i], \mathbb{S}\right) \simeq \varinjlim_i \bigoplus_{T_i} \mathbb{S},$$

$$\underline{\text{map}}\left(\mathbb{S}[T], \mathbb{Z}\right) \simeq \varinjlim_i \underline{\text{map}}\left(\mathbb{S}[T_i], \mathbb{Z}\right) \simeq \varinjlim_i \bigoplus_{T_i} \mathbb{Z},$$

and use the fact that the tensor product commutes with colimits in each variable. □

Corollary 2.20. *For every profinite set T, there is a set I and an equivalence of condensed spectra*

$$\underline{\text{map}}\left(\mathbb{S}[T], \mathbb{S}\right) \simeq \bigoplus_{i \in I} \mathbb{S}.$$

[4] This is made functorial by defining it as a right Kan extension in the obvious way, in the same way as Construction 2.6

Proof. We need to produce a map of spectra

$$\bigoplus_{i\in I}\mathbb{S}\to \operatorname{map}(\mathbb{S}[T],\mathbb{S})$$

which after extension of scalars along $\mathbb{S}\to\mathbb{Z}$ becomes the equivalence

$$\bigoplus_{i\in I}\mathbb{Z}\simeq \operatorname{map}(\mathbb{S}[T],\mathbb{Z})\simeq \operatorname{map}(\mathbb{S}[T],\mathbb{S})\otimes_{\mathbb{S}}\mathbb{Z}$$

that we already have. This is enough because the condensed spectra we are working with (also the internal mapping spectra, as the proof of Lemma 2.9 shows for the internal hom in condensed abelian groups — the proof translates readily to the setting of spectra) are in fact discrete, and a map $X\to Y$ of connective spectra is an equivalence if and only if the induced $X\otimes_{\mathbb{S}}\mathbb{Z}\to Y\otimes_{\mathbb{S}}\mathbb{Z}$ is an equivalence. For each $i\in I$ we have a map of $\mathbb{Z}$-module spectra

$$p_i:\mathbb{Z}\to\bigoplus_{i\in I}\mathbb{Z}\simeq \operatorname{map}(\mathbb{S}[T],\mathbb{Z})$$

which gives a

$$\tilde{p}_i:\mathbb{S}\to \operatorname{map}(\mathbb{S}[T],\mathbb{Z})$$

by restriction of scalars along $\mathbb{S}\to\mathbb{Z}$. This is an element of $\pi_0\operatorname{map}(\mathbb{S}[T],\mathbb{Z})$. Since π_0 commutes with filtered colimits and finite direct sums, and $\pi_0\mathbb{S}\simeq\mathbb{Z}$, we can use the characterization given by Corollary 2.5 to obtain the corresponding element of $\pi_0\operatorname{map}(\mathbb{S}[T],\mathbb{S})$. These assemble into the desired map

$$\bigoplus_{i\in I}\mathbb{S}\to \operatorname{map}(\mathbb{S}[T],\mathbb{S})\,.$$

□

Corollary 2.21. *For every profinite set T, there is a set I and an equivalence of condensed spectra*

$$\mathbb{S}[T]^{\blacksquare}\simeq\prod_{i\in I}\mathbb{S}.$$

Proof. The proof is identical to the one for abelian groups 2.11. □

We are now ready to prove the lemma which will be our main tool in reducing the proofs of results in the theory of solid spectra to known results about solid abelian groups.

Lemma 2.22. *For every profinite set T,*

$$\mathbb{S}[T]^{\blacksquare}\otimes_{\mathbb{S}}\mathbb{Z}\simeq\mathbb{Z}[T]^{\blacksquare}$$

Remark 2.23. Before proving the lemma, we note that it identifies solid objects of the derived category of condensed abelian groups and solid spectra with the structure of a $\mathbb{Z}$-module in condensed spectra: Indeed, if C is a $\mathbb{Z}$-module in condensed spectra, or equivalently an object of $\mathcal{D}(\mathsf{CondAb})$, then by the lemma and restriction of scalars,

$$\begin{aligned}\operatorname{map}(\mathbb{S}[T]^{\blacksquare},C)&\simeq \operatorname{map}_{\mathbb{Z}}(\mathbb{S}[T]^{\blacksquare}\otimes_{\mathbb{S}}\mathbb{Z},C)\\&\simeq R\operatorname{Hom}(\mathbb{Z}[T]^{\blacksquare},C)\,.\end{aligned}$$

Further, the analogous equivalence is true when $\mathbb{S}[T]^{\blacksquare},\mathbb{Z}[T]^{\blacksquare}$ are replaced by $\mathbb{S}[T],\mathbb{Z}[T]$. We conclude that C is solid as a spectrum if and only if it is solid as an object of $\mathcal{D}(\mathsf{CondAb})$.

In particular, a condensed abelian group A is solid if and only if the Eilenberg-Maclane spectrum A is solid.

Remark 2.24. Lemma 2.22 is [1, Lemma 2.20]. The proof there relies on the fact that $\mathbb{Z}$ is *pseudocoherent* (see e.g. [9, §4.1]) as a spectrum, without spelling out an explicit proof of that fact. What we present here is essentially the same proof, giving more details and using a more low-tech approach. Pseudocoherence of spectra translates to the property in Lemma 2.25 below.

Recall that a spectrum is *finite* if it can be written as a finite colimit of shifts of $\mathbb{S}$. The finite spectra are compact objects of Sp.

Lemma 2.25. *For every connective spectrum X with finitely generated homotopy groups, there is a sequence of spectra $(X_n)_{n\in\mathbb{N}}$ with maps $X_n \to X$ such that the induced map $\tau_{\leq n}X_n \to \tau_{\leq n}X$ is an equivalence for all n.*

Proof. We let $X_{-1} \to X$ be a finite spectrum mapping to X which is surjective on π_0 (it can be taken as a finite direct sum of copies of $\mathbb{S}$). Obviously, since X_{-1} and X are connective, the induced map $\tau_{\leq -1}X_{-1} \to \tau_{\leq -1}X$ is an equivalence.

We let F_{-1} be the fibre of $X_{-1} \to X$. It is connective because $F_{-1}[1]$ is 1-connective as the cofibre of $X_{-1} \to X$ which is surjective on π_0. Also, F_{-1} has finitely generated π_0, because $\pi_1 X$ and $\pi_0 C_0$ are finitely generated and $\pi_0 F_0$ sits between them in the long exact sequence.

For the inductive step, suppose we have defined $X_{n-1} \to X$ with X_{n-1} a finite spectrum, such that $\tau_{\leq n-1}X_{n-1} \to \tau_{\leq n-1}$ is an equivalence, and such that X_{n-1} is surjective on π_n.

We immediately see that $F_{n-1} = \mathrm{fib}(X_{n-1} \to X)$ is n-connective with finitely generated π_n.

We let $Y_n \to F_{n-1}$ be a map from a finite direct sum of copies of $\mathbb{S}[n]$, surjective on π_n, and X_n the cofibre of $Y_n \to X_{n-1}$. The map $X_n \to X$ comes from the universal property of X_n as a cofibre. The following diagram consists of parts of the two long exact sequences corresponding to the fibre sequences $F_{n-1} \to X_{n-1} \to X$ and $Y_n \to X_{n-1} \to X_n$:

$$\begin{array}{ccccccc}
\pi_{n+1}X_{n-1} & \longrightarrow & \pi_{n+1}X_n & \longrightarrow & \pi_n Y_n & \longrightarrow & \pi_n X_{n-1} \\
\| & & \downarrow & & \downarrow & & \| \\
\pi_{n+1}X_{n-1} & \longrightarrow & \pi_{n+1}X & \longrightarrow & \pi_n F_{n-1} & \longrightarrow & \pi_n X_{n-1}
\end{array}$$

By a diagram chase, the map $\pi_{n+1}X_n \to \pi_{n+1}X$ is an epimorphism.

Inductively we get a sequence $X_0 \to X_1 \to X_2 \to \cdots$, such that $X_n \to X$ is surjective on π_{n+1} and X_n is a finite spectrum for all n. We will show that

$$\pi_k(X_n) \to \pi_k(X)$$

is an isomorphism for all $k \leq n$. Now if we consider the diagram

$$\begin{array}{ccccccccc}
\pi_k Y_n & \longrightarrow & \pi_k X_{n-1} & \longrightarrow & \pi_k X_n & \longrightarrow & \pi_{k-1} Y_n & \longrightarrow & \pi_{k-1} X_{n-1} \\
\downarrow & & \| & & \downarrow & & \downarrow & & \| \\
\pi_k F_{n-1} & \longrightarrow & \pi_k X_{n-1} & \longrightarrow & \pi_k X & \longrightarrow & \pi_{k-1} F_{n-1} & \longrightarrow & \pi_{k-1} X_{n-1}
\end{array}$$

obtained from the long exact sequences corresponding to the fibre sequences $Y_n \to X_{n-1} \to X_n$ and $F_{n-1} \to X_{n-1} \to X$, we see that the vertical morphism furthest to the left is an epimorphism (it is actually an isomorphism when $k < n$, but not necessarily when $k = n$) while the other three vertical morphisms excluding the middle one are isomorphisms (in $\pi_{k-1}Y_n \to \pi_{k-1}F_{n-1}$, we have now established that both source and target are 0). Thus, the vertical morphism in the middle is an isomorphism, as desired. □

Remark 2.26. The proof of Lemma 2.25, together with the ∞-categorical Dold-Kan correspondence ([16, §1.2.4]) shows that every connective spectrum X with finitely generated homotopy groups can be written as a geometric realization of a simplicial spectrum consisting of finite spectra. We will not use this result here.

Remark 2.27. The proof of Lemma 2.25 shows the stronger result that there is a sequence of finite spectra

$$X_0 \to X_1 \to X_2 \to \cdots$$

with colimit X, such that $\tau_{\leq n}X_n \to \tau_{\leq n}X$ is an equivalence for all n.

Lemma 2.28. *Let $\mathcal{C}$ and $\mathcal{D}$ be stable ∞-categories, each equipped with a t-structure. Let F be an exact functor $\mathcal{C} \to \mathcal{D}$, which takes connective objects to connective objects Let*

$$X_0 \to X_1 \to X_2 \to \cdots$$

be a sequence of connective spectra with colimit X such that for all n, the map $\tau_{\leq n}X_n \to \tau_{\leq n}X$ is an equivalence. Then

$$\varinjlim_n F(X_n) \to F(X)$$

is an equivalence.

Proof. Note that by exactness, F preserves shifts (as these are given by a pushout or pullback). Therefore, F takes $\mathcal{C}_{\geq n}$ to $\mathcal{D}_{\geq n}$ for every n. By exactness again, and since F preserves connectivity, we have that

$$\operatorname{cofib}\left(F(X_m) \to F\left(X\right)\right) \in \mathcal{D}_{\geq m+1}$$

and thus also,

$$\operatorname{fib}\left(F(X_m) \to F\left(X\right)\right) \in \mathcal{D}_{\geq m}.$$

Therefore, the map $F(X_m) \to F\left(\varinjlim_n D(n)\right)$ is an isomorphism on π_k for $k < m$. By varying m, the statement about the colimit follows. □

Remark 2.29. Using the ∞-categorical Dold-Kan correspondence, we can prove that a functor satisfying the hypotheses of Lemma 2.28 preserves geometric realizations of simplicial spectra consisting of connective objects. We will note use this fact here.

Proof of Lemma 2.22. We use the equivalence $\mathbb{S}[T]^{\blacksquare} \simeq \prod_{i\in I} \mathbb{S}$ and the corresponding one for $\mathbb{Z}$. Apply Lemma 2.25 to the connective spectrum $\mathbb{Z}$ and write $\mathbb{Z}$ as a colimit of finite spectra

$$X_0 \to X_1 \to X_2 \to \cdots$$

such that $\tau_{\leq n} X_n \to \tau_{\leq n} X$ is an equivalence for all n. Since $\mathbb{Z}$ is discrete as a condensed spectrum, this holds in condensed spectra as well. We need to show that

$$\prod_{i\in I} \mathbb{S} \otimes \mathbb{Z} \simeq \prod_{i\in I} \mathbb{Z}.$$

In other words

$$\varinjlim_n \left(\prod_{i\in I} \mathbb{S} \otimes X_n\right) \to \prod_{i\in I} (\varinjlim_n X_n).$$

Since each X_n is a finite colimit of shifts of $\mathbb{S}$ (because products — in fact all limits — are exact in a stable ∞-category), this becomes a question of commuting the colimit with the colimit $\varinjlim_n X_n$. But this follows from Lemma 2.28 applied to the product functor $\mathsf{CondSp}^I \to \mathsf{CondSp}$. □

Corollary 2.30. *If a condensed spectrum X has solid homotopy groups, then X is a solid spectrum.*

Proof. First suppose X is bounded below. It is easy to see that solid spectra are closed under limits, extensions ($\operatorname{map}(X, -)$ is an exact functor for every X) and shifts ($\operatorname{map}(-,-)$ commutes with shifts in the appropriate way). Writing $X \simeq \varprojlim_n \tau_{\leq n} X$ we then see that it suffices to show that $\tau_{\leq n} X$ is solid for all n. We can prove this by induction using the fibre sequences

$$(\pi_n X)[n] \to \tau_{\leq n} X \to \tau_{\leq n-1} X$$

where the inductive step follows from the assumption that all homotopy groups are solid (as spectra by Remark 2.23) and the fact that solid spectra are closed under extensions and shifts. The fact that X is bounded below allows us to get the induction started.

To extend to the general case, we note that for every $d \in \mathbb{Z}$,

$$\pi_d \operatorname{map}\left(\mathbb{S}[T]^{\blacksquare}, X\right) \to \pi_d \operatorname{map}\left(\mathbb{S}[T], X\right)$$

only depends on $\tau_{\geq -d} X$ because $\mathbb{S}[T]^{\blacksquare}$ and $\mathbb{S}[T]$ are connective. □

Proposition 2.31. *For every profinite set T, $\mathbb{S}[T]^{\blacksquare}$ is solid.*

Proof. Because of the equivalence with a product of copies of $\mathbb{S}$, it suffices to check that the spectrum $\mathbb{S}$ is solid. By 2.30, it suffices to show that $\pi_n \mathbb{S}$ is a solid abelian group for all n. This is clear because these are all built from $\mathbb{Z}$ (which is solid, see [20, Proposition 5.7]) by colimits, limits and extensions, and solid abelian groups are closed under these by 2.13. □

Theorem 2.32. *The category of solid spectra is a stable ∞-category generated under shifts and colimits by compact projectives of the form $\prod_{i\in I}\mathbb{S}$. Further, the fully faithful inclusion $i : \mathsf{SolidSp} \hookrightarrow \mathsf{CondSp}$ preserves all limits, colimits and extensions and has a left adjoint denoted $X \mapsto X^{\blacksquare}$ which is a colimit-preserving extension of $\mathbb{S}[T] \mapsto \mathbb{S}[T]^{\blacksquare}$, and as such, unique up to contractible choice. Furthermore, a condensed spectrum X is solid if and only if all of its homotopy groups $\pi_n X$ are solid abelian groups.*

Proof. We have already noted that solid spectra are closed under limits and extensions. Closure under colimits follows from the fact that $\mathsf{SolidSp}$ is generated under colimits by the objects of the form $\mathbb{S}[T]^{\blacksquare}[n]$ for T profinite and $n \in \mathbb{Z}$, which is the main body of the proof that now follows.

We apply [15, Proposition 5.5.4.15] to the collection S of all morphisms $\mathbb{S}[T]^{\blacksquare}[n] \to 0$, where T is profinite and $n \in \mathbb{Z}$. It implies that we have a functor $L : \mathsf{CondSp} \to \mathsf{CondSp}$ and for each condensed spectrum X a map $X \to L(X)$ belonging to the strongly saturated class of maps generated by S, in particular, belongs to the full subcategory of $\mathrm{Fun}(\Delta^1, \mathsf{CondSp})$ generated under colimits by S (see [15, Definition 5.5.4.5]). This implies that the fibre $F(X)$ of $X \to L(X)$ belongs to the full subcategory of CondSp generated under colimits by $\mathbb{S}[T]^{\blacksquare}[n]$, T profinite, $n \in \mathbb{Z}$.

Furthermore, $L(X)$ is S-local, meaning (see [15, Definition 5.5.4.1]) that for every profinite set T and every integer n, the map

$$0 \simeq \mathrm{Map}\left(0, L(X)\right) \to \mathrm{Map}\left(\mathbb{S}[T]^{\blacksquare}[n], L(X)\right)$$

is an equivalence. This implies that

$$\pi_k \, \mathrm{map}\left(\mathbb{S}[T]^{\blacksquare}, L(X)\right) \simeq 0$$

for all k. In particular, the spectrum of maps $\mathbb{S}[T]^{\blacksquare} \to L(X)$ is zero.

We conclude that the map

$$\mathrm{map}\left(\mathbb{S}[T]^{\blacksquare}, F(X)\right) \to \mathrm{map}\left(\mathbb{S}[T]^{\blacksquare}, X\right)$$

is an equivalence.

We want to show that $F(X)$ is solid. We show that its homotopy groups are solid, by showing that the full subcategory $\mathsf{SolidSp}'$ of CondSp, spanned by the condensed spectra whose homotopy groups are solid, is closed under colimits. This indeed proves that $F(X)$ has solid homotopy groups, since it is generated under colimits by the objects $\mathbb{S}[T]^{\blacksquare}[n]$, which have solid homotopy groups.

Now for the proof that $\mathsf{SolidSp}'$ is stable under colimits, it suffices to show that it is stable under cofibres and direct sums. Since homotopy groups commute with direct sums, the latter is clear. If $Y \to Z \to W$ is a fibre sequence of condensed spectra such that Y and Z have solid homotopy groups, then by a diagram chase on the diagram obtained from applying the functors $\mathrm{Hom}(\mathbb{Z}[T]^{\blacksquare}, -)$ and $\mathrm{Hom}(\mathbb{Z}[T], -)$ to the long exact sequence in homotopy, the homotopy groups of W are solid.

If X is solid, the map $F(X) \to X$ must be an equivalence, because for every profinite set T, we have

$$\begin{aligned} F(X)(T) &\simeq \mathrm{map}\left(\mathbb{S}[T], F(X)\right) \\ &\simeq \mathrm{map}\left(\mathbb{S}[T]^{\blacksquare}, F(X)\right) \\ &\simeq \mathrm{map}\left(\mathbb{S}[T]^{\blacksquare}, X\right) \\ &\simeq \mathrm{map}\left(\mathbb{S}[T], X\right) \\ &\simeq X(T). \end{aligned}$$

Therefore, X is a colimit of shifts of the objects $\mathbb{S}[T]^{\blacksquare}$. Conversely, since shifts of $\mathbb{S}[T]^{\blacksquare}$ have solid homotopy groups, any colimit of such objects has solid homotopy groups as we established above. We have now established that the category of solid spectra is generated under colimits and shifts by the objects $\mathbb{S}[T]^{\blacksquare}$, and that the solid spectra are precisely those with solid homotopy groups.

By the mapping property, the objects $\mathbb{S}[T]^{\blacksquare}$, for T extremally disconnected, are compact projectives. Since a retract of a product of copies of $\mathbb{S}$ is again such a product (see argument in [20, Proof of Corollary 6.1]), we have the desired statement about the objects of the form $\prod_{i\in I}\mathbb{S}$ being compact projective generators.

As for the solidification functor $X \mapsto X^{\blacksquare}$ left adjoint to the inclusion, its existence is clear by closure under limits. Let's denote it by G temporarily and show that $G(\mathbb{S}[T]) \simeq \mathbb{S}[T]^{\blacksquare}$ for all profinite sets T. For every solid spectrum X, we have

$$\mathrm{map}\left(G\left(\mathbb{S}[T]\right), X\right) \simeq \mathrm{map}\left(\mathbb{S}[T], X\right) \simeq \mathrm{map}\left(\mathbb{S}[T]^{\blacksquare}, X\right)$$

so $\mathbb{S}[T]^{\blacksquare}$ and $G(\mathbb{S}[T])$ represent the same functor on $\mathsf{SolidSp}$ and are hence equivalent.

Finally, the fact that $\mathsf{SolidSp}$ is a stable ∞-category follows from closure under fibres, cofibres and shifts (see [16, §1.1.3]), which we have established. □

Now we can make a comparison with the approach to solid spectra in [1, §2.4]. That paper defines a category $\mathsf{SolSp}_{\geq 0}$ as the full subcategory of connective condensed spectra generated by products of $\mathbb{S}$ under colimits and extensions. It follows from the general theory, as explained in [1, Proof of Theorem 2.19] and [19, Lecture 12], that this definition is equivalent to the connective part of our $\mathsf{SolidSp}$. The full category of solid spectra in [1] is then defined as the stabilization of $\mathsf{SolSp}_{\geq 0}$ (this is also what is done in general for $\mathcal{D}(\mathcal{A}, \mathcal{M})$ for an analytic animated ring $\mathcal{A}$ in [19]). Thus, to prove the two approaches equivalent, it remains to prove that $\mathsf{SolidSp}$ is the stabilization of its connective part. We will not do that here, but in any case, the definitions clearly agree for connective spectra.

Lemma 2.33. *A condensed spectrum X is solid if and only if for every profinite set T, the map*

$$\underline{\mathrm{map}}\,(\mathbb{S}[T]^{\blacksquare}, X) \to \underline{\mathrm{map}}\,(\mathbb{S}[T], X)$$

is an equivalence.

Proof. Evaluating at the point, one direction is clear. For the other one, if X is solid, then X has solid homotopy groups. Now the proof of Corollary 2.30 carries over to prove the condition on the internal mapping spectra, also using the analogous result for condensed abelian groups ([20, Corollary 6.1(iv)]). □

Construction 2.34. We finish this section by promoting the adjunction

$$\mathsf{CondSp} \underset{i}{\overset{(-)^{\blacksquare}}{\rightleftarrows}} \mathsf{SolidSp}$$

to a symmetric monoidal adjunction

$$\mathsf{CondSp}^{\otimes} \underset{i^{\otimes}}{\overset{(-)^{\blacksquare,\otimes}}{\rightleftarrows}} \mathsf{SolidSp}^{\otimes},$$

in a unique (up to contractible choice) way. We will denote the tensor product obtained on $\mathsf{SolidSp}$ by $\otimes^{\blacksquare}$ and in fact we will have

$$X \otimes^{\blacksquare} Y \simeq (X \otimes Y)^{\blacksquare}.$$

Let W be the collection of morphisms $X \to Y$ in CondSp such that the induced $X^{\blacksquare} \to Y^{\blacksquare}$ is an equivalence. Then we see that

$$\mathsf{SolidSp} \hookrightarrow \mathsf{CondSp} \to \mathsf{CondSp}[W^{-1}]$$

is an equivalence by [16, Example 1.3.4.3]. Thus by [16, Proposition 4.1.7.4] it suffices to show that if $X \to Y$ becomes an equivalence after solidification, and Z is a condensed spectrum, then the induced maps

$$(X \otimes Z)^{\blacksquare} \to (Y \otimes Z)^{\blacksquare}, \qquad (Z \otimes X)^{\blacksquare} \to (Z \otimes Y)^{\blacksquare}$$

are equivalences. We only consider the former, as the latter is completely analogous. Since we know that

$$(X^{\blacksquare} \otimes Z)^{\blacksquare} \to (Y^{\blacksquare} \otimes Z)^{\blacksquare}$$

is an equivalence, it suffices to show that for every condensed spectrum X, the map

$$(X \otimes Z)^{\blacksquare} \to (X^{\blacksquare} \otimes Z)^{\blacksquare},$$

induced by the unit of the adjunction, is an equivalence. Thanks to Lemma 2.33, to prove that the solidification functor is symmetric monoidal, one can copy [20, Proof of Theorem 6.2] replacing $\mathbb{Z}$ by $\mathbb{S}$ and Hom by map. The uniqueness statement about $\otimes^{\blacksquare}$ follows from [16, Remark 4.1.7.5].

2.3. **Solidification and p-completion.** Let M be a condensed abelian group. Since condensed abelian groups form an abelian category, one can, for an integer n, define the *multiplication by n map* as the identity times n. In this case it is given objectwise on extremally disconnected sets by multiplication by n. We denote by M/n the cokernel of this map. Then $(M/n)(T) \simeq M(T)/n$ for every extremally disconnected set T since all limits and colimits are computed objectwise on extremally disconnected sets.

Definition 2.35. Let M be a condensed abelian group. The *p-completion* of M is the condensed abelian group

$$M_p^\wedge = \varprojlim_k M/p^k$$

where the maps

$$M/p^{k+1} \to M/p^k$$

are simply reduction modulo p^k (objectwise on extremally disconnected sets). There is a canonical map $M \to M_p^\wedge$ and if this map is an isomorphism, M is *p-complete.*

Definition 2.36. A condensed abelian group M is *torsion-free* if for all $n \in \mathbb{N}$, the multiplication by n map $M \to M$ is a monomorphism.

Remark 2.37. It is clear that M is torsion-free if and only if $M(T)$ is torsion-free for every extremally disconnected set T.

Proposition 2.38. *A condensed abelian group M is torsion-free if and only if it is flat.*

Proof. Suppose M is torsion-free. Then for every profinite set T, the abelian group $M(T)$ is torsion-free and thus flat. Therefore, the functor of objectwise tensoring with M on the presheaf category is left exact, and since sheafification left exact, we conclude that $- \otimes M$ is a left exact functor on condensed abelian groups, i.e. M is flat as a condensed abelian group.

Suppose M is flat. Then tensoring the injection $n\mathbb{Z} \hookrightarrow \mathbb{Z}$ with M gives the result. □

Theorem 2.39. *If M is a torsion-free, p-complete condensed abelian group such that M/p is a discrete condensed abelian group, then there is an isomorphism*

$$M \simeq \left(\bigoplus_I \mathbb{Z}\right)_p^\wedge$$

Proof. By picking a basis for the (discrete) $\mathbb{F}_p$-vector space M/p we are producing a map of abelian groups

$$\bigoplus_I \mathbb{Z} \to M(*)$$

which is an equivalence modulo p. Since $\bigoplus_I \mathbb{Z}$ is discrete, this actually produces a map of condensed abelian groups

$$\bigoplus_I \mathbb{Z} \to M$$

(which factors through the canonical map of condensed abelian groups $M(*) \to M$, which is still an equivalence modulo p). Using the exact sequence

$$0 \to \mathbb{Z}/p^k \to \mathbb{Z}/p^{k+1} \to \mathbb{Z}/p^k \to 0$$

and flatness of M, we see that the map

$$\bigoplus_I \mathbb{Z} \to M$$

is an equivalence modulo p^k for all $k \in \mathbb{N}$. Therefore, we have an equivalence

$$\left(\bigoplus_I \mathbb{Z}\right)_p^\wedge \to M_p^\wedge \simeq M$$

as desired. □

Throughout this section, $\otimes^{\blacksquare}$ denotes the tensor product of solid spectra defined in Construction 2.34. When applied to derived solid abelian groups, this agrees with the derived solid tensor product, denoted $\otimes^{L\blacksquare}$ in [20].

Remark 2.40. By [20, Example 6.4], the p-adic integers $\mathbb{Z}_p$ form an idempotent in the category of solid abelian groups: $\mathbb{Z}_p \otimes^{\blacksquare}_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathbb{Z}_p$ (also in the derived sense). This means that for $\mathbb{Z}_p$-modules M, N in solid spectra, we have

$$M \otimes^{\blacksquare}_{\mathbb{Z}_p} N \simeq M \otimes^{\blacksquare}_{\mathbb{Z}} N$$

Proposition 2.41. *For every profinite set $T = \varprojlim_i T_i$ with T_i finite,*

$$\mathbb{Z}_p[T]^{\blacksquare} \simeq \varprojlim_i \mathbb{Z}_p[T_i]$$

Proof. Call a ring A *good* if it has the desired property

$$A[T]^{\blacksquare} \simeq \varprojlim_i A[T_i].$$

Since

$$\mathbb{Z}[T]^{\blacksquare} = \prod_{i \in I} \mathbb{Z},$$

a ring A is good if and only if

$$A \otimes^{\blacksquare} \prod_{i \in I} \mathbb{Z} \simeq \prod_{i \in I} A.$$

Since $\mathbb{Z}[[X]] \simeq \prod_{n \in \mathbb{N}} \mathbb{Z}$, the ring $\mathbb{Z}[[X]]$ is good by [20, Proposition 6.3]. We have a short exact sequence

$$0 \to \mathbb{Z}[[X]] \to \mathbb{Z}[[X]] \to \mathbb{Z}_p \to 0$$

and we conclude that $\mathbb{Z}_p$ is good too. □

Lemma 2.42. *Let $X = \varprojlim_i X_i$ be a* prodiscrete *condensed set, i.e. a cofiltered limit of discrete condensed sets X_i, taken in condensed sets. Then one can write X as the filtered colimit of profinite sets $T = \varprojlim_i T_i$ where $T_i \subset X_i$ is a finite subset for all i.*

Proof. We may write X (indeed, any condensed set) as a colimit of representables, i.e. as a colimit of profinite sets regarded as condensed sets. This colimit is taken over the category of profinite sets over X. Given a map $S \to X$ of condensed sets where S is a profinite set. This map is given by continuous (i.e. locally constant) maps $S \to X_i$ for every i, compatible with the colimit. Each of these maps has finite image by compactness of S. Thus, the image of $S \to X$ is profinite and of the form $\varprojlim_i T_i$ where $T_i \subset X_i$ is a finite subset for all i. So each map from a profinite set to X factors through a profinite set of the desired form, and thus the colimit can be taken over these. □

Theorem 2.43. *Let I and J be sets. There is an equivalence*

$$\left(\bigoplus_I \mathbb{Z}\right)^{\wedge}_p \otimes^{\blacksquare} \left(\bigoplus_J \mathbb{Z}\right)^{\wedge}_p \simeq \left(\bigoplus_{I \times J} \mathbb{Z}\right)^{\wedge}_p$$

Proof. We write

$$\left(\bigoplus_I \mathbb{Z}\right)^{\wedge}_p = \varprojlim_k \bigoplus_I \mathbb{Z}/p^k$$

and note that each $\bigoplus_I \mathbb{Z}/p^k$ is discrete. Therefore, we may apply Lemma 2.42 and write $(\bigoplus_I \mathbb{Z})^{\wedge}_p$ as a colimit of profinite subsets. In fact, we can write it as a colimit of profinite subgroups of the form $A = \varprojlim_k A_k$ where

$$A_k = \bigoplus_{I_k} \mathbb{Z}/p^k$$

and $I_k \subset I$ is finite. We can even arrange that $I_k \subset I_{k+1}$ for all k. By cofinality of the diagonal in $\mathbb{N} \times \mathbb{N}$, we have further

$$\mathbb{Z}_p \left[\varprojlim_k I_k\right]^{\blacksquare} \simeq \varprojlim_k \mathbb{Z}_p[I_k] \simeq \varprojlim_k \left((\mathbb{Z}/p^k)[I_k]\right) \simeq \varprojlim_k A_k \simeq A$$

We note that if $(I_k)_{k\in\mathbb{N}}$ and $(J_k)_{k\in\mathbb{N}}$ are families of finite subsets of I resp. J with $I_k \subset I_{k+1}$ resp. $J_k \subset J_{k+1}$ for all k, then

$$\mathbb{Z}_p\left[\varprojlim_k I_k\right]^{\blacksquare} \otimes^{\blacksquare} \mathbb{Z}_p\left[\varprojlim_k J_k\right]^{\blacksquare} \simeq \mathbb{Z}_p\left[\varprojlim_k I_k \times \varprojlim_k J_k\right]^{\blacksquare} \simeq \mathbb{Z}_p\left[\varprojlim_k (I_k \times J_k)\right]^{\blacksquare} \qquad (*)$$

Now, for any set K, we have observed that

$$\left(\bigoplus_K \mathbb{Z}\right)_p^{\wedge} = \varinjlim_{(K_k)} \mathbb{Z}_p\left[\varprojlim_k K_k\right]^{\blacksquare}$$

where the colimit is taken over all families $(K_k)_{k\in\mathbb{N}}$ such that $K_k \subset K_{k+1}$ are finite subsets of K for all k. In fact, in the case $K = I \times J$, the colimit can, by cofinality, be taken over those families with $K_k = I_k \times J_k$ with I_k, J_k finite subsets of I, J respectively, each forming increasing families. By $(*)$ and the fact that $\otimes^{\blacksquare}$ commutes with colimits in each variable, we are done. $\square$

Remark 2.44. The proof of 2.43 shows that if we have torsion-free condensed abelian groups A and B such that A/p, B/p are discrete and

$$A \simeq \varprojlim_n A/p^n, \qquad B \simeq \varprojlim_n B/p_n,$$

then

$$A \otimes^{\blacksquare} B \simeq \varprojlim_n \left(A/p^n \otimes B/p^n\right).$$

This is because, in the notation of the proof, $A/p \simeq \bigoplus_I \mathbb{Z}/p$; it follows that $A/p^n \simeq \bigoplus_I \mathbb{Z}/p^n$, and the analogous identity for B.

3. p-adic Banach algebras

Let p be a prime number which we fix throughout the text. Consider the following conditions on a commutative algebra A over $\mathbb{Q}_p$:

(i) There is a norm $\|-\|$ on A satisfying the ultrametric inequality for all $f, g \in A$:

$$\|f+g\| \le \max\{\|f\|, \|g\|\}$$

and the norm restricted to $\mathbb{Q}_p$ gives the usual p-adic absolute value.

(ii) The norm is fully multiplicative, i.e. for all $f, g \in A$, $\|fg\| = \|f\|\,\|g\|$.

(iii) A is complete with respect to this norm.

(iv) The unit ball centred at 1 is contained in the group of units of the ring of integers of A:

$$B(1,1) \subset \mathcal{O}_A^{\times}$$

(here $\mathcal{O}_A$ simply denotes the elements of norm ≤ 1).

Conditions (i)-(iii) say that A is a commutative Banach algebra over $\mathbb{Q}_p$ with fully multiplicative norm. Condition (iv) enables the definition of a p-adic logarithm, which is a homomorphism from a multiplicative group contained in the group of units of the ring of integers, and exponential, partially defined on A.

The trivial example $A = \mathbb{Q}_p$ obviously satisfies all these conditions. Two other examples to keep in mind are $A = \mathbb{C}_p$, the p-adic complex numbers, and $A = \mathbb{Q}_p\langle T\rangle$, the ring of formal power series whose coefficients form null sequences in $\mathbb{Q}_p$. Note that we don't require all elements of $\mathcal{O}_A$ of norm 1 to be units. We prove below that these two examples satisfy all the conditions as well.

Proposition 3.1. *Suppose A is a commutative algebra over $\mathbb{Q}_p$ satisfying conditions (i)-(iv) above. The power series*

$$\log(1+x) = \sum_{n\ge 0} (-1)^{n+1}\frac{x^n}{n}$$

converges if and only if $\|x\| < 1$, and the map $\log : B(1,1) \to B(0,1)$ is a group homomorphism from the multiplicative group on the left to the additive group on the right.

The power series

$$\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}$$

converges if and only if $\|x\| < r_p := p^{-\frac{1}{p-1}}$. *Let* $x \in A$, $\|x\| < r_p$. *Then*

$$\log(\exp(x)) = x \qquad \textit{and} \qquad \exp(\log(1+x)) = 1+x.$$

In other words, the maps log *and* exp *define inverse group isomorphisms* $B(1, r_p) \leftrightarrow B(0, r_p)$ *where the group on the left is multiplicative and the group on the right is additive.*

Proof. This is proved in the case $A = \mathbb{Q}_p$ in [12, Proposition 5.7.8]. One can verify that the proofs work for any algebra A satisfying (i)-(iv). □

The existence of the p-adic logarithm and exponential allow us to analyse the abelian group structure of the units of $\mathcal{O}_A$, which we do in detail in §3.2 and §3.3 for $A = \mathbb{C}_p$ and $A = \mathbb{Q}_p\langle T\rangle$. In §3.1 we prove some general results about this type of Banach algebra and study the condensed structure in more detail. In §3.4, we recall the notion of k-fold stability of rings and its relation to lower K-groups. We study the relationship between stability of $\mathcal{O}_A$ and its sections as a condensed ring.

3.1. Generalities.

Lemma 3.2. $\mathcal{O}_A$ *is* p*-complete, i.e.* $\mathcal{O}_A \simeq \varprojlim_n \mathcal{O}_A/p^n$ *as discrete rings.*

Proof. The map $\mathcal{O}_A \to \varprojlim_n \mathcal{O}_A/p^n$ is injective because if $x, y \in \mathcal{O}_A$ are congruent mod p^n for all n, then $\|x-y\| \leq p^{-n}$ for all n and hence $x = y$. It is also surjective: Let $x \in \varprojlim_n \mathcal{O}_A/p^n$, and for each n let x_n be a representative of the image of x in $\mathcal{O}_A/p^n$. Then x_{n+1} is congruent to x_n mod p^n and hence $\|x_{n+1} - x_n\| \leq p^{-n} \to 0$ as $n \to \infty$. The limit of the sequence x_n is thus mapped to x by the canonical map $\mathcal{O}_A \to \varprojlim_n \mathcal{O}_A/p^n$, and we are done. □

Lemma 3.3. $\mathcal{O}_A/p$ *is discrete as a condensed abelian group (in other words, the canonical map* $\mathcal{O}_A \to \mathcal{O}_A/p$ *is locally constant).*

Proof. It suffices to show that every convergent sequence in $\mathcal{O}_A$ maps to an eventually constant sequence in $\mathcal{O}_A/p$. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\mathcal{O}_A$ and let $N \in \mathbb{N}$ be such that for all $n \geq N$, $\|x_{n+1} - x_n\| < \|p\| = 1/p$. Since p is invertible in A there is a $y \in A$ such that $x_{n+1} - x_n = py$. Because of that equality, $\|y\| \leq 1$ so in fact $y \in \mathcal{O}_A$ and hence x_{n+1} and x_n map to the same coset in $\mathcal{O}_A/p$. □

Lemma 3.4. *The ring isomorphism in Lemma 3.2 is a homeomorphism when the* $\mathcal{O}_A/p^k$ *are given the discrete topology. In other words,* $\mathcal{O}_A$ *is* p*-complete as a condensed ring.*

Proof. The canonical map $\mathcal{O}_A \to \varprojlim_n \mathcal{O}_A/p^n$ is continuous because it is after composition with each projection (by Lemma 3.3). To show that it is a homeomorphism it suffices to show that a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{O}_A$, which maps to a convergent sequence in the limit, is convergent. Each reduction modulo p^k is convergent in the discrete topology and hence eventually constant. Thus, for each k, there is an N such that for all $n \geq N$, $\|x_{n+1} - x_n\| < \|p^k\| = 1/p^k$, and we conclude that $(x_n)_{n\in\mathbb{N}}$ is convergent. □

Corollary 3.5. *For every profinite set* T,

$$C(T, \mathcal{O}_A) \simeq \varprojlim_n C(T, \mathcal{O}_A)/p^n$$

Lemma 3.6. *An element* $x \in \mathcal{O}_A$ *is a unit if and only if its reduction modulo* p *is a unit in* $\mathcal{O}_A/p$.

Proof. Let $r \in \mathcal{O}_A$. If r is a unit then it is clear that $\bar{r}$ is a unit in $\mathcal{O}_A/p$. Conversely, suppose $\bar{r}$ is a unit in $\mathcal{O}_A/p$. Then there is an $s \in \mathcal{O}_A$ such that

$$rs - 1 \in p\mathcal{O}_A,$$

in particular $\|rs-1\| < 1$. Thus, $rs \in B(1,1) \subseteq \mathcal{O}_A^\times$ by condition (iv). But this implies that r is a unit as well. □

Lemma 3.7. *For every profinite set $T = \varprojlim_i T_i$,*

$$C(T, \mathcal{O}_A)/p^k \simeq C(T, \mathcal{O}_A/p^k).$$

Proof. Since $\mathcal{O}_A/p^k$ is discrete for all k (Lemma 3.3), we have

$$C(T, \mathcal{O}_A/p^k) \simeq \varinjlim_i \prod_{t \in T_i} \mathcal{O}_A/p^k.$$

Thus,

$$C(T, \mathcal{O}_A^{\mathrm{disc}})/p^k \simeq C(T, \mathcal{O}_A/p^k)$$

where $\mathcal{O}_A^{\mathrm{disc}}$ denotes the ring $\mathcal{O}_A$ equipped with the discrete topology. Now it suffices to show that for every continuous map $f : T \to \mathcal{O}_A$, there is a continuous map $\tilde{f} : T \to \mathcal{O}_A^{\mathrm{disc}}$ with the same reduction modulo p^k. Denote by $\overline{f}$ the reduction of f mod p^k. We can write T as a clopen partition of the fibres of $\overline{f}$ and define $\tilde{f}$ as a constant map on each part on the partition. More precisely, given a nonempty fibre of $\overline{f}$, we pick an element t_0 of the fibre, and let $\tilde{f}$ be the constant map $f(t_0)$ on the whole fibre. This map $\tilde{f}$ is locally constant and has the same reduciton modulo p^k as f, as desired. □

Lemma 3.8. *If R is a discrete ring, then $C(T, R)^\times$ consists of precisely those continuous functions f whose image is contained in $R^\times$.*

Proof. We need to prove that if $f(T) \subset R^\times$, the map

$$\frac{1}{f} : T \to R$$

is continuous. For every $r \in R$,

$$\left(\frac{1}{f}\right)^{-1} \{r\} = f^{-1}\left\{\frac{1}{r}\right\}$$

is open, so we are done. □

Theorem 3.9. *An element $f \in C(T, \mathcal{O}_A)$ is a unit if and only if its reduction mod p is a unit.*

Proof. If f is a unit, its reduction $\overline{f}$ is clearly a unit. Conversely, if $\overline{f} : T \to \mathcal{O}_A/p$ is a unit, then $\overline{f}(t)$ is a unit in $\mathcal{O}_A/p$ for all $t \in T$ so by Lemma 3.6, $f(t) \in \mathcal{O}_A^\times$ for all $t \in T$. Therefore, $f(t)$ is a unit mod p^k for all k. By Lemma 3.8, the reductions of f mod p^k are all units in the rings $C(T, \mathcal{O}_A/p^k)$, which by Lemma 3.7 are isomorphic to $C(T, \mathcal{O}_A)/p^k$. By uniqueness of inverses, these assemble into an inverse in the limit, i.e. an inverse to f in $C(T, \mathcal{O}_A)$ (cf. Corollary 3.5). □

Corollary 3.10.

$$C(T, \mathcal{O}_A^\times) \simeq C(T, \mathcal{O}_A)^\times.$$

□

3.2. **The units of $\mathcal{O}_{\mathbb{C}_p}$.** We let $\mathbb{C}_p$ denote the p-adic complex numbers. These are constructed as follows. Extend the p-adic absolute value of $\mathbb{Q}_p$ to its algebraic closure $\overline{\mathbb{Q}_p}$ e.g. as described in [12, §6.3]. This $\mathbb{Q}_p$-algebra is not a complete metric space, so we complete it with respect to this p-adic absolute value. This gives us an algebraically closed field $\mathbb{C}_p$ which is complete with respect to the p-adic absolute value. See [12, §6.8].

Define its ring of integers $\mathcal{O}_{\mathbb{C}_p}$ as

$$\mathcal{O}_{\mathbb{C}_p} = \{z \in \mathbb{C}_p : |z| \leq 1\}.$$

Here, $|\cdot|$ denotes the p-adic absolute value. It is related to the p-adic valuation $v_p(-)$ in that

$$|x| = p^{-v_p(x)}$$

and in particular, for an integer n, $v_p(n)$ is the highest exponent such that $p^{v_p(n)}$ divides n.

Denote by $\mathfrak{m} \subset \mathcal{O}_{\mathbb{C}_p}$ the maximal ideal, i.e.

$$\mathfrak{m} = \{z \in \mathbb{C}_p : |z| < 1\},$$

and by $\mathcal{O}_{\mathbb{C}_p}^\times \subset \mathcal{O}_{\mathbb{C}_p}$ the group of units. We have

$$\mathcal{O}_{\mathbb{C}_p}^\times \simeq \{z \in \mathbb{C}_p : |z| = 1\}.$$

We note that $1+\mathfrak{m} \subset \mathcal{O}_{\mathbb{C}_p}^{\times}$, is in fact a subgroup. The ring $\mathbb{C}_p$ is by construction complete with respect to the absolute value and forms a Banach algebra over $\mathbb{Q}_p$ satisfying conditions (i)-(iv).

The goal of this section is to prove a result 3.20 about the structure of $\mathcal{O}_{\mathbb{C}_p}^{\times}$ as a condensed abelian group. More precisely, we prove that the p-adic logarithm map from Proposition 3.1 gives rise to a split short exact sequence

$$0 \to \mathbb{Q}/\mathbb{Z} \to \mathcal{O}_{\mathbb{C}_p}^{\times} \to \mathbb{C}_p \to 0$$

of condensed abelian groups.

Lemma 3.11. *The group $1+\mathfrak{m} \subset \mathcal{O}_{\mathbb{C}_p}^{\times}$ is divisible, and thus injective as a $\mathbb{Z}$-module.*

Proof. See [18, Proposition III.4.5.1] □

Proposition 3.12. *There is a continuous homomorphism $\exp : \mathbb{C}_p \to 1+\mathfrak{m}$ extending the original p-adic exponential defined on $B(0, r_p)$.*

Proof. This is [18, Proposition V.4.4], but we repeat the argument here:

First we note that for $x \in B(0, r_p)$, we have $|\exp(x)| = 1$ and $|\exp(x) - 1| = |x| < r_p \leq 1$, so exp lands in $B(1,1) = 1+\mathfrak{m}$. Thus the homomorphism exp defined on the subgroup $B(0, r_p)$ of $\mathbb{C}_p$, with target the divisible abelian group $1+\mathfrak{m}$, extends to all of $\mathbb{C}_p$ by injectivity of the target. Continuity of the extension follows from continuity at 0. □

Lemma 3.13. *Let $x \in \mathbb{C}_p$. Then $x^{p^n} \to 1$ as $n \to \infty$ if and only if $x \in 1+\mathfrak{m}$.*

Proof. See [18, Proposition III.4.5.2] □

Remark 3.14. The injectivity of divisible abelian groups relies on Zorn's lemma. In particular, the extension of the exponential is not unique. However, we have the following corollary.

Corollary 3.15. *For any exponential function*

$$\exp : \mathbb{C}_p \to 1+\mathfrak{m}$$

we have $\log \circ \exp = \mathrm{id}$. In particular, any exponential function is injective, and the logarithm is a surjective group homomorphism.

Proof. This is [18, Corollary V.4.4], but we repeat the argument here:

Let $x \in \mathbb{C}_p$ and pick a large enough integer n such that $p^n x < r_p$. Then

$$p^n \log(\exp(x)) = \log(\exp(x)^{p^n}) = \log(\exp(p^n x)) = p^n x$$

and thus $\log(\exp(x)) = x$. □

The rest of this section (§3.2) can essentially be extracted from [18, V.4.5], but we give a more structured, self-contained account here.

Proposition 3.16. *Denote by $\mu_{p^\infty} \subset \mathbb{C}_p$ the roots of unity whose order is a power of p. We have*

(1) $\mu_{p^\infty} \subset 1+\mathfrak{m}$,
(2) μ_{p^∞} is a discrete subspace of $\mathbb{C}_p$.

Proof. Let $\zeta \in \mu_{p^\infty}$ be a primitive p^t-th root of unity, $t \geq 1$. We want to show that

$$|\zeta - 1| = |p|^{1/\phi(p^t)}$$

where ϕ denotes the Euler ϕ-function (and in particular $\phi(p^t) = p^{t-1}(p-1)$). This will prove (1) because $|p|^{1/\phi(p^t)} < 1$ and (2) because $|p|^{1/\phi(p^t)} \geq |p|^{1/(p-1)}$ so it shows that 1 is an isolated point of μ_{p^∞}, and since it's a subgroup of $\mathbb{C}_p^{\times}$ this implies that every point is isolated.

Denote by Φ_m the m-th cyclotomic polynomial. Let P_t denote the set of primitive p^t-th roots of unity in $\mathbb{C}_p$. We start by showing that

$$\Phi_{p^{t+1}}(X) = \Phi_{p^t}(X^p).$$

Indeed, the degrees agree, and the roots agree since x is a primitive p^{t+1}-st root of unity if and only if x^p is a primitive p^t-th root of unity. We conclude that

$$\Phi_{p^{t+1}}(1) = \Phi_{p^t}(1) = \cdots = \Phi_p(1) = p.$$

Now we turn to proving the statement:

$$|p| = |\Phi_{p^t}(1)| = \left|\prod_{\zeta \in P_t} (\zeta - 1)\right| = \prod_{\zeta \in P_t} |\zeta - 1|.$$

If we can show that all of the factors $\zeta - 1$ have the same absolute value, then for any one particular of them, we the desired equality

$$|\zeta - 1| = |p|^{1/\phi(p^t)}.$$

To see that $|1 - \zeta_1| = |1 - \zeta_2|$ for $\zeta_1, \zeta_2 \in P_t$, we use the fact that there exist integers i, j prime to p^t such that $\zeta_2 = \zeta_1^j$ and $\zeta_1 = \zeta_2^i$. Now,

$$\left|\frac{1 - \zeta_1^j}{1 - \zeta_1}\right| = |1 + \zeta_1 + \cdots + \zeta_1^{j-1}| \leq \max\{|\zeta_1^i| : i = 0, \dots, j-1\} = 1.$$

In the same way, we get

$$\left|\frac{1 - \zeta_2^i}{1 - \zeta_2}\right| \leq 1$$

and we conclude that $|\zeta_1 - 1| = |\zeta_2 - 1|$ □

Proposition 3.17. *The kernel of* $\log : 1 + \mathfrak{m} \to \mathbb{C}_p$ *is* μ_{p^∞}.

Proof. It is clear that μ_{p^∞} is contained in the kernel. Conversely, let $x = 1 + t \in 1 + \mathfrak{m}$ be in this kernel. We know that $x^{p^n} \to 1$ as $n \to \infty$ by 3.13. We take n large enough so that $|x^{p^n} - 1| < r_p$, and note that x^{p^n} is still in the kernel. Recall that $|\log(1+y)| = |y|$ for every y with $|y| < r_p$ and thus $0 = |\log(x^{p^n})| = |x^{p^n} - 1|$, so $x^{p^n} = 1$, i.e. $x \in \mu_{p^\infty}$. □

Proposition 3.18. *Denote by* $\mu_{(p)}$ *the roots of unity whose order is prime to* p. *We have*

(1) $\mathcal{O}_{\mathbb{C}_p}^\times \simeq \mu_{(p)} \times (1 + \mathfrak{m})$ *(direct product of abelian groups)*

(2) $\mu_{(p)}$ *is a discrete subspace of* $\mathbb{C}_p$.

Proof. Consider the ring homomorphism of reduction mod $\mathfrak{m}$,

$$\varepsilon : \mathcal{O}_{\mathbb{C}_p} \to \mathcal{O}_{\mathbb{C}_p}/\mathfrak{m} \simeq \overline{\mathbb{F}}_p.$$

It is continuous when the target is considered as a discrete space, since the fibers are of the form $x + \mathfrak{m} = B(x, 1)$, and in particular open. When restricted to the units $\mathcal{O}_{\mathbb{C}_p}^\times$, we want to show that ε gives a surective group homomorphism $\mathcal{O}_{\mathbb{C}_p}^\times \to \overline{\mathbb{F}}_p^\times$. Let $\alpha \in \overline{\mathbb{F}}_p^\times$. Then there is an n such that $\alpha \in \mathbb{F}_{p^n}^\times$ and thus an m dividing $p^n - 1$ (and hence prime to p) such that α is a primitive m-th root of unity. Let $a \in \mathcal{O}_{\mathbb{C}_p}$ be a representative of the coset corresponding to α, i.e. $\varepsilon(a) = \alpha$. Then $a^m - 1 \in \mathfrak{m}$ and thus a is a unit (since $a^m \in 1 + \mathfrak{m} \subset \mathcal{O}_{\mathbb{C}_p}^\times$). Furthermore, if we let $f(X) = X^m - 1$, since m is a unit, we have

$$|f'(a)| = |m||a^{m-1}| = 1$$

and since $a^m - 1 \in \mathfrak{m}$,

$$|f(a)| = |a^m - 1| < 1 = |f'(a)|^2.$$

Thus, Hensel's lemma applies, and we see that there is a unique $x \in B(a, 1) = a + \mathfrak{m}$ such that $x^m - 1 = 0$. The fact that $x \in a + \mathfrak{m}$ means that $\varepsilon(x) = \varepsilon(a) = \alpha$. We have in fact shown that ε restricts to a continuous group isomorphism $\mu_{(p)} \to \overline{\mathbb{F}}_p^\times$, proving (2). Moreover, the kernel of the group homomorphism $\varepsilon_{|\mathcal{O}_{\mathbb{C}_p}^\times}$ is by definition $1 + \mathfrak{m}$, and thus we have

$$\mu_{(p)} \simeq \mathcal{O}_{\mathbb{C}_p}^\times / (1 + \mathfrak{m})$$

yielding (1). □

Remark 3.19. We note that μ, the group of all roots of unity, is a direct product of $\mu_{(p)}$ and μ_{p^∞}. The above then shows that $\mu \subset \mathbb{C}_p$ is a discrete subspace. Also, we know that μ is isomorphic to the discrete abelian group $\mathbb{Q}/\mathbb{Z}$. All of the above thus gives Theorem 3.20

Theorem 3.20. *We have an isomorphism of condensed abelian groups*

$$\mathcal{O}_{\mathbb{C}_p}^{\times} \simeq \mathbb{C}_p \oplus \mathbb{Q}/\mathbb{Z}$$

3.3. **The units of $\mathbb{Z}[T]_p^\wedge$.** We turn our attention to a condensed ring that is somewhat similar to $\mathcal{O}_{\mathbb{C}_p}$, namely $\mathbb{Z}[T]_p^\wedge$, the p-completion of the ring of polynomials in one variable T with coefficients in $\mathbb{Z}$. The starting point, however, is the Banach algebra $\mathbb{Q}_p\langle T\rangle$ with a certain norm defined below, and $\mathbb{Z}[T]_p^\wedge$ will be shown to be the "ring of integers" (or rather, closed unit disk) of this algebra.

Definition 3.21. Define $\mathbb{Q}_p\langle T\rangle$ as the ring of formal power series $\sum_{n\geq 0} a_n T^n$, with coefficients in $\mathbb{Q}_p$ such that, $a_n \to 0$ as $n \to \infty$.

Remark 3.22. The ring $\mathbb{Q}_p\langle T\rangle$ is one of the simplest examples of a *Tate algebra* or *affinoid algebra* (see [11] and [7]). As such, it has a norm which gives the condensed structure on $\mathbb{Z}[T]_p^\wedge$ which we want to study.

Proposition 3.23. *There is a norm on $\mathbb{Q}_p\langle T\rangle$ given by*

$$\left\| \sum_{n\geq 0} a_n T^n \right\| = \max\{|a_n| : n \in \mathbb{N}\}.$$

Moreover, $\mathbb{Q}_p\langle T\rangle$ is complete with respect to this norm, and the norm is multiplicative ($\|fg\| = \|f\|\,\|g\|$).

Proof. Note that since (a_n) is a null-sequence, the maximum is well defined. It is clear that this defines a norm. To show completeness, take a Cauchy sequence

$$(a_k)_{k\in\mathbb{N}} = \left(\sum_{n\in\mathbb{N}} a_{n,k} T^n\right)_{k\in\mathbb{N}}.$$

Then by definition of the norm, for every n the sequence $(a_{n,k})_k$ is Cauchy in $\mathbb{Q}_p$ and hence convergent; denote by a_n its limit. We just need to prove that $a_n \to 0$ as $n \to \infty$. But this is standard: the space of null-sequences is complete with respect to the sup norm. □

Proposition 3.24. *The ring $\mathbb{Z}[T]_p^\wedge$ is isomorphic to the closed unit ball in $\mathbb{Q}_p\langle T\rangle$, and consists of precisely those formal power series in $\mathbb{Q}_p\langle T\rangle$ with coefficients in $\mathbb{Z}_p$. Further, the topology is the same.*

Proof. Let $\mathbb{Z}_p\langle T\rangle \subset \mathbb{Q}_p\langle T\rangle$ denote the set of those power series whose coefficients are in $\mathbb{Z}_p$. For every n, reduction mod p^n gives a map

$$\mathbb{Z}_p\langle T\rangle \to \mathbb{Z}/p^n[T]$$

which then lifts to an injection

$$\mathbb{Z}_p\langle T\rangle \to \mathbb{Z}[T]_p^\wedge$$

(if a series is 0 mod p^n for all n, then it is 0 in $\mathbb{Z}_p\langle T\rangle$). For surjectivity, note that for any sequence (f_n) of polynomials where f_n has coefficients in $\mathbb{Z}/p^n$ and $f_n = f_{n+1} \bmod p^{n+1}$, we get coefficient-wise a sequence of elements of $\mathbb{Z}_p$ tending to 0. □

Proposition 3.25. *The set of units in $\mathbb{Z}[T]_p^\wedge$ consists of those power series $\sum_{n\geq 0} a_n T^n$ with $|a_0| = 1$ and $|a_n| < 1$ for $n > 1$.*

Proof. See [7, 5.1.3, Proposition 1]. □

We have now established that $\mathbb{Q}_p\langle T\rangle$ satisfies conditions (i)-(iv) and can therefore use the logarithm and exponential to analyse the abelian group structure on the set of units.

Proposition 3.26. *The additive abelian group $\mathbb{Z}[T]_p^\wedge$ is isomorphic to its subgroup $B(0, r_p)$.*

Proof. Let $q = 4$ if $p = 2$, $q = p$ otherwise. Since the value group of $\mathbb{Q}_p\langle T\rangle$ is the same as that of $\mathbb{Q}_p$, we have that $B(0, r_p) = q\mathbb{Z}[T]_p^\wedge \subset \mathbb{Q}_p\langle T\rangle$ and hence isomorphic as additive groups. □

Theorem 3.27. *There is a split exact sequence of condensed abelian groups*

$$0 \to \mathbb{Z}[T]_p^\wedge \to (\mathbb{Z}[T]_p^\wedge)^\times \to (\mathbb{Z}/q)^\times \to 0$$

where $q = 4$ if $p = 2$ and $q = p$ otherwise.

Proof. The injection $\mathbb{Z}[T]_p^\wedge \to (\mathbb{Z}[T]_p^\wedge)^\times$ is multiplication by q followed by the exponential (this is a continuous embedding by the work we've done above). The surjection $(\mathbb{Z}[T]_p^\wedge)^\times \to (\mathbb{Z}/q)^\times$ comes from the ring map $\mathbb{Z}[T]_p^\wedge \to \mathbb{Z}[T]_p^\wedge/q\mathbb{Z}[T]_p^\wedge$ (this ring map is continuous as it has open fibres). Now it suffices to show that $\mathbb{Z}[T]_p^\wedge/q\mathbb{Z}[T]_p^\wedge \simeq (\mathbb{Z}/q)[T]$ since the units in the polynomial ring are the units in the base ring, but this isomorphism is clear. This surjection has a section given by the inclusion of the roots of unity in $\mathbb{Z}_p$, which are well known to form a cyclic group with $\phi(q)$ elements, i.e. isomorphic to $(\mathbb{Z}/q)^\times$.

The fact that the exact sequence holds for the condensed (in this case topological) abelian groups follows from all the remarks on continuity in the proof. □

3.4. **Stability.**

Definition 3.28. Let R be a ring. A pair of elements $(a, b) \in R \times R$ is *unimodular* if $aR + bR = R$.

Definition 3.29. Let k be a positive integer. A commutative ring R is *k-fold stable* if it satisfies the following condition. Given k unimodular pairs $(a_i, b_i) \in R \times R$, $i = 1, \dots, k$, there exists an $r \in R$ such that for all i, $a_i + rb_i$ is a unit in R.

Proposition 3.30. *Let $k \geq 1$. If R is a k-fold stable ring and I an ideal of R, then R/I is k-fold stable.*

Proof. We start by proving the following: if $a, b, x, y \in R$, then there is an $r \in R$ such that the pair $(a + r(ax + by - 1), b)$ is unimodular in R. This is actually the special case $m = 1$, $n = 3$, $b_1 = a$, $b_2 = b$ and $b_3 = ax + by - 1$ of [23, Theorem 1]. The following argument is a specialization of the proof found there. The pair $(a, ax + 1)$ is unimodular, hence there exists an $r \in R$ such that $a + r(ax - 1)$ is a unit. Thus the pair $(a + r(ax + 1), b)$ is unimodular. We have

$$\begin{pmatrix} a + r(ax + by - 1) \\ b \end{pmatrix} = \begin{pmatrix} 1 & ry \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a + r(ax - 1) \\ b \end{pmatrix}$$

and since the matrix on the right-hand side is invertible, the pair $(a + r(ax + by - 1), b)$ is unimodular.

Now let the pairs $(a_i, b_i) \in R \times R$, $i = 1, \dots, k$ be such that their images in R/I are unimodular. Then there exist elements $x_i, y_i \in R$, $i = 1, \dots, k$, such that $a_i x_i + b_i y_i - 1 \in I$ for all i, and by the first paragraph of the proof we can find $r_i \in R$ such that $(a_i + r_i(a_i x_i + b_i y_i - 1), b_i)$ is unimodular in R for all i. By k-fold stability of R there exists an $s \in R$ such that for all i,

$$a_i + r_i(a_i x_i + b_i y_i - 1) + sb_i$$

is a unit in R. Hence, its image $\overline{a_i + sb_i}$ in R/I is a unit and we are done. □

Proposition 3.31. *A local ring is k-fold stable if and only if its residue field has at least $k + 1$ elements.*

Proof. This proposition is stated more generally for semi-local rings in [22, 5.4(3)]. We give an elementary proof for this special case here.

Let R be a local ring with maximal ideal $\mathfrak{m}$. We start by proving that R is k-fold stable if and only if $R/\mathfrak{m}$ is k-fold stable.

If R is k-fold stable, then $R/\mathfrak{m}$ is k-fold stable by Proposition 3.30.

Conversely, suppose $R/\mathfrak{m}$ is k-fold stable. Let (a_i, b_i) be unimodular pairs, $i = 1, \dots, k$. Then the images $(\overline{a_i}, \overline{b_i})$ are unimodular pairs in $R/\mathfrak{m}$ so there is an r such that $\overline{a_i + rb_i}$ is a unit (i.e. is non zero) in $R/\mathfrak{m}$ for all i. In other words, $a_i + rb_i \notin \mathfrak{m}$ for all i. But $R^\times = R \setminus \mathfrak{m}$ so we conclude that R is k-fold stable.

Now to finish the proof of the proposition, we prove that a field K is k-fold stable if and only if it has at least $k + 1$ elements. Suppose K has k or fewer elements. Consider k unimodular pairs $(x, 1)$ where every $x \in K$ appears at least once. Then for every $r \in K$, we consider a pair where $x = -r$ and see that $x + r \cdot 1 = 0$ is not a unit, so K is not k-fold stable. Conversely, suppose K has at least $k + 1$ elements. Let (a_i, b_i), $i = 1, \dots, k$ be k unimodular pairs in K. In particular $(a_i, b_i) \neq (0, 0)$ for all i. Thus $a_i + rb_i$ is a degree 0 or 1 polynomial in r. Each of the k polynomials has at most one root in K and thus there is at least one $r \in K$ for which $a_i + rb_i \neq 0$, i.e. is a unit, for all i. □

Proposition 3.32. *Let $(R_j)_{j\in J}$ be a filtered system of k-fold stable rings. Then the colimit $\varinjlim_j R_j$ is k-fold stable.*

Proof. Let the pairs (a_i, b_i), $i = 1, \ldots, k$, be unimodular in $\varinjlim_j R_j$. Since J is filtered, we can find an R_j such that all the pairs (a_i, b_i) are images of unimodular pairs in R_j. By k-fold stability of R_j we are done. □

Proposition 3.33. *Let $(R_i)_{i\in I}$ be a family of k-fold stable rings indexed by any set I. Then the product $\prod_{i\in I} R_i$ is k-fold stable.*

Proof. This is obvious; everything is done coordinate-wise in the product. □

If A is a Banach algebra satisfying conditions (i)-(iv), we can deduce stability conditions for $C(T, \mathcal{O}_A)$ from stability conditions for $\mathcal{O}_A$, where T is a profinite set:

Theorem 3.34. *For every profinite set $T = \varprojlim_i T_i$, if $\mathcal{O}_A$ is k-fold stable then the ring $C(T, \mathcal{O}_A)$ is k-fold stable.*

Proof. By Proposition 3.30, the ring $\mathcal{O}_A/p$ is k-fold stable. Since $\mathcal{O}_A/p$ is discrete, we have that

$$C(T, \mathcal{O}_A/p) \simeq \varinjlim_i \prod_{t\in T_i} \mathcal{O}_A/p$$

is k-fold stable by Propositions 3.32 and 3.33. Now consider k unimodular pairs $(f_i, g_i) \in C(T, \mathcal{O}_A) \times C(T, \mathcal{O}_A)$. Then their reductions mod p are k unimodular pairs in $C(T, \mathcal{O}_A/p)$ and hence there exists an $r \in C(T, \mathcal{O}_A)$ such that $\overline{f_i + rg_i}$ is a unit in $C(T, \mathcal{O}_A/p)$ for all i. Thus, by 3.9, $f_i + rg_i$ is a unit in $C(T, \mathcal{O}_A)$ for all i and we are done. □

4. Solid K-theory

Let R be an ordinary ring. We will denote by $K^\delta(R)$ the *discrete* algebraic K-theory spectrum of R: this is just the usual (connective) algebraic K-theory spectrum. Now suppose R is a condensed ring. Then we denote by $K(R)$ the *condensed K-theory spectrum* of R. For each extremally disconnected compact Hausdorff space T, its T-valued points are simply $K(R)(T) := K^\delta(R(T))$ (to define it on all profinite sets, one needs to sheafify). Finally, we denote by $K^\blacksquare(R)$ the solidification of the condensed K-theory spectrum of R, i.e. $K^\blacksquare(R) := (K(R))^\blacksquare$. For details on solid spectra, see §2.

A rather optimistic conjecture one could state for the solid K-theory of a p-complete ring R is that it is continuous in the sense that the canonical map

$$K^\blacksquare(R) \to \varprojlim_{n\in\mathbb{N}} K^\delta(R/p^n) \tag{4}$$

is an equivalence. Two pieces of evidence for this conjecture are briefly mentioned in [8, Sessions 10-11]. These are the fact that it is true in the most simple case of $R = \mathbb{Z}_p$ and that it is true (at least in some cases) "with finite coefficients", i.e. modulo n for any integer $n \geq 1$. The proof for $\mathbb{Z}_p$ relies on compactness, so natural next examples to consider are $\mathcal{O}_{\mathbb{C}_p}$ and $\mathbb{Z}[T]^\wedge_p$, which are not compact.

We do not resolve the conjecture here for the two examples mentioned above. However, we describe some partial progress by computing the solidifications of the zeroth and first condensed K-groups of both, and prove a structural result about $K_2(\mathcal{O}_{\mathbb{C}_p})^\blacksquare$.

4.1. What evidence do the lower K-groups provide?

Of course, knowing the induced map of the canonical map (4) on homotopy groups obviously provides evidence of whether it is an equivalence or not. What we mean by the question in the title is what evidence do the solidifications of the condensed abelian groups K_0, K_1, K_2, etc. provide? A priori, $\pi_2(K^\blacksquare(R))$ is not identified with the condensed abelian group $K_2(R)^\blacksquare$. However, we have the following general result

Proposition 4.1. *Let X be a connective condensed spectrum and let n be a natural number. Suppose that the homotopy groups $\pi_k(X)$ are solid for $k < n$. Then the canonical map $(\pi_n X)^\blacksquare \to \pi_n X^\blacksquare$ is an equivalence.*

Proof. By Theorem 2.32, the condensed spectrum $\tau_{\leq n-1}X$ is solid. Thus, solidifying the fibre sequence in [16, Remark 1.2.1.8], we obtain a fibre sequence

$$(\tau_{\geq n}X)^{\blacksquare} \to X^{\blacksquare} \to \tau_{\leq n-1}X.$$

Using a long exact sequence argument, it now suffices to show that $\pi_n(\tau_{\geq n}X)^{\blacksquare} \simeq (\pi_n X)^{\blacksquare}$. More generally, we show that for an n-connective condensed spectrum, $\pi_n X^{\blacksquare} \simeq (\pi_n X)^{\blacksquare}$. It suffices to show that for every solid abelian group M,

$$\mathrm{Hom}(\pi_n X^{\blacksquare}, M) \simeq \mathrm{Hom}(\pi_n X, M).$$

This follows from the fact that $M[n]$ is solid as a condensed spectrum, and concentrated in degree n. □

Since the condensed K-groups $K_0(R)$ and $K_1(R)$ are both solid for $R = \mathcal{O}_{\mathbb{C}_p}$ and $R = \mathbb{Z}[T]^{\wedge}_p$ (see Corollaries 4.3 and 4.8), we conclude by Proposition 4.1 that the solidification of K_2 is identified with π_2 of the solidified K-theory spectrum in this case. This explains that computing the lower condensed K-groups or their solidification gives useful information about the left-hand side in 4. The right-hand side is supposedly better understood, and a complete description could be extracted from material in [6, 9, 14], at least for $R = \mathcal{O}_{\mathbb{C}_p}$. Extracting this description is a non-trivial task, which we will not attempt in the present work. It is reasonable to expect that this description would give

$$\pi_0\left(\varprojlim_n K^{\delta}(\mathcal{O}_{\mathbb{C}_p}/p^n)\right) \simeq \mathbb{Z},$$

$$\pi_1\left(\varprojlim_n K^{\delta}(\mathcal{O}_{\mathbb{C}_p}/p^n)\right) \simeq \mathcal{O}_{\mathbb{C}_p}^{\times},$$

and

$$\pi_2\left(\varprojlim_n K^{\delta}(\mathcal{O}_{\mathbb{C}_p}/p^n)\right) \simeq 0,$$

as condensed abelian groups. We emphasize the fact that we have not verified this in detail, and were this work to be completed, such a verification would be necessary.

4.2. $K_0^{\blacksquare}$ and $K_1^{\blacksquare}$.

Theorem 4.2. *Let A be a commutative algebra over $\mathbb{Q}_p$ satisfying conditions (i)-(iv) from §3. Then $K_0(\mathcal{O}_A)$ is a discrete condensed abelian group that agrees with $K_0^{\delta}(\mathcal{O}_A(*))$.*

Proof. We first note that for a discrete condensed ring R, the condensed abelian group $K_0(R)$ is discrete. This follows from the characterization of discrete condensed objects in §2.1, because K-theory commutes with filtered colimits and finite products, and so does π_0.

Next, recall [24, Lemma II.2.2]; it implies that if a ring R satisfies $R \simeq \varprojlim R/I^n$ where $I \subset R$ is an ideal, then $K_0(R) \simeq K_0(R/I)$.

Let $T = \varprojlim_i T_i$ be an extremally disconnected set. By the above and 3.5, we have

$$K_0(\mathcal{O}_A)(T) \simeq K_0(C(T, \mathcal{O}_A)) \simeq K_0(C(T, \mathcal{O}_A)/p) \simeq K_0(\mathcal{O}_A/p)^{\mathrm{disc}}(T) \simeq K_0(\mathcal{O}_A)^{\mathrm{disc}}(T)$$

□

Corollary 4.3.

$$K_0(\mathcal{O}_{\mathbb{C}_p}) \simeq \mathbb{Z} \simeq K_0(\mathbb{Z}[T]^{\wedge}_p)$$

as condensed abelian groups. In particular, these K_0-groups are discrete, and hence solid.

Proof. The result follows from 4.2 and the introduction of [24, Chapter II.2]; for $\mathcal{O}_{\mathbb{C}_p}$ because it is a local ring; for $\mathbb{Z}[T]^{\wedge}_p$ because

$$K_0(\mathbb{Z}[T]^{\wedge}_p) \simeq K_0(\mathbb{Z}[T]^{\wedge}_p/p)$$

and $\mathbb{Z}[T]^{\wedge}_p/p \simeq \mathbb{F}_p[T]$ is a PID. □

Lemma 4.4. *Let R be a 1-fold stable ring. Then for every $n > 1$ and every* unimodular row $(a_1, \ldots, a_n) \in R^n$ *(this just means that the ideal generated by the elements $a_1, \ldots, a_n$ is all of R), there exists an $(n-1)$-tuple $(b_1, \ldots, b_{n-1})$ such that $(a_1 + a_n b_1, \ldots, a_{n-1} + a_n b_{n-1}) \in R^{n-1}$ is a unimodular row.*

Proof. The ring $R/(a_2, \dots, a_{r-1})$ is 1-fold stable by 3.30. Let $b_1 \in R$ be such that $\overline{a_1 + a_n b_1}$ is a unit in the quotient $R/(a_2, \dots, a_{r-1})$ and let $b_2 = \cdots b_{r-1} = 0$. Then $(a_1 + a_n b_1, \dots, a_{n-1} + a_n b_{n-1}) \in R^{n-1}$ is clearly unimodular. ☐

Theorem 4.5. *Let R be a 1-fold stable ring. Then*

$$K_1^\delta(R) \simeq R^\times.$$

Proof. By 4.4, 1 *defines a stable range* for R in the language of [21], and so [21, Theorem 13.5(1)] gives that the natural map $R^\times \to K_1(R)$ is surjective. The determinant map $K_1(R) \to R^\times$ is a splitting of this map and hence an isomorphism. ☐

Proposition 4.6. *Let T be a profinite set. The ring $C(T, \mathcal{O}_{\mathbb{C}_p})$ is k-fold stable for all k.*

Proof. Since $\mathcal{O}_{\mathbb{C}_p}$ is local with infinite residue field $\overline{\mathbb{F}}_p$, it is k-fold stable for all k by Proposition 3.31. Therefore by 3.34, $C(T, \mathcal{O}_{\mathbb{C}_p})$ is k-fold stable for every k. ☐

Proposition 4.7. *Let T be a profinite set. The ring $C(T, \mathbb{Z}[T]_p^\wedge)$ is 1-fold stable.*

Proof. First we prove that $\mathbb{Z}[T]_p^\wedge$ is 1-fold stable. Let (a, b) be a unimodular pair in $\mathbb{Z}[T]_p^\wedge$. Then their reductions mod p form a unimodular pair in the euclidean ring $\mathbb{Z}[T]_p^\wedge/p \simeq \mathbb{F}_p[T]$, and thus by [21, Example, p. 203] there is an $r \in \mathbb{Z}[T]_p^\wedge$ such that $a + rb$ is a unit mod p, i.e. a unit in $\mathbb{Z}[T]_p^\wedge$.

For the ring $C(T, \mathbb{Z}[T]_p^\wedge)$, we now conclude by 3.34. ☐

Note however that the ring $\mathbb{Z}[T]_p^\wedge$ is not k-fold stable if $k > 1$. Indeed, it suffices to show that $\mathbb{Z}[T]_p^\wedge$ is not 2-fold stable and for that it suffices to show that $\mathbb{F}_p[T] \simeq \mathbb{Z}[T]_p^\wedge/p$ is not 2-fold stable. Consider the two unimodular pairs $(0, 1)$ and $(1, T)$. If $r \in \mathbb{F}_p[T]$ be such that $0 + r \cdot 1 = r$ is a unit, then $1 + rT$ is not a unit.

We have established:

Corollary 4.8.

$$K_1(\mathcal{O}_{\mathbb{C}_p}) \simeq \mathcal{O}_{\mathbb{C}_p}^\times \simeq \mathbb{C}_p \oplus \mathbb{Q}/\mathbb{Z}$$

and

$$K_1(\mathbb{Z}[T]_p^\wedge) \simeq (\mathbb{Z}[T]_p^\wedge)^\times \simeq \mathbb{Z}[T]_p^\wedge \oplus (\mathbb{Z}/q)^\times$$

as condensed abelian groups. In particular, these K_1-groups are solid.

4.3. $K_2(\mathcal{O}_{\mathbb{C}_p})^\blacksquare$.

Theorem 4.9. *Let R be a 5-fold stable ring. Then*

$$K_2^\delta(R) \simeq \frac{R^\times \otimes R^\times}{G},$$

where $G \subset R^\times \otimes R^\times$ is the subgroup generated by the elements of the form $x \otimes (1-x)$ where x and $1-x$ are both invertible elements of R.

Proof. See [22, Corollary 8.5]. ☐

We conclude that, as a condensed abelian group, $K_2(\mathcal{O}_{\mathbb{C}_p})$ is the sheafification of the presheaf which sends each extremally disconnected compact Hausdorff space T to

$$\frac{(C(T, \mathcal{O}_{\mathbb{C}_p}^\times) \otimes C(T, \mathcal{O}_{\mathbb{C}_p}^\times))}{G_T},$$

where G_T is the subgroup of $(C(T, \mathcal{O}_{\mathbb{C}_p}^\times) \otimes C(T, \mathcal{O}_{\mathbb{C}_p}^\times))$ generated by the elements $f \otimes (1-f)$ where for all $t \in T$, $f(t)$ and $1 - f(t)$ are both units. Denote by G the sheafification of the presheaf $T \mapsto G_T$. Then

$$K_2(\mathcal{O}_{\mathbb{C}_p})^\blacksquare \simeq \frac{\mathcal{O}_{\mathbb{C}_p}^\times \otimes^\blacksquare \mathcal{O}_{\mathbb{C}_p}^\times}{G^\blacksquare}.$$

This is the starting point in attempting to prove that $K_2(\mathcal{O}_{\mathbb{C}_p})^\blacksquare$ vanishes. As mentioned earlier, we do not prove this here, but we end this report by calculating the solid tensor product in the numerator (the derived version, which gives the underived one as the degree 0 part).

Construction 4.10. Since $\mathcal{O}_{\mathbb{C}_p}/p$ is discrete and $\mathcal{O}_{\mathbb{C}_p}$ is torsion-free and p-complete, we have an isomorphism

$$\mathcal{O}_{\mathbb{C}_p} \simeq \left(\bigoplus_I \mathbb{Z}\right)_p^\wedge$$

by Theorem 2.39. Thus by Theorem 2.43,

$$\mathbb{C}_p \otimes^{\blacksquare} \mathbb{C}_p \simeq \left(\bigoplus_{I\times I} \mathbb{Z}\right)_p^\wedge \left[\frac{1}{p}\right].$$

While this looks non-canonical, Remark 2.44 gives the following equivalence:

$$\mathcal{O}_{\mathbb{C}_p} \otimes^{\blacksquare} \mathcal{O}_{\mathbb{C}_p} \simeq \varprojlim_n \left(\mathcal{O}_{\mathbb{C}_p}/p^n \otimes \mathcal{O}_{\mathbb{C}_p}/p^n\right)$$

and thus

$$\mathbb{C}_p \otimes^{\blacksquare} \mathbb{C}_p \simeq \varprojlim_n \left(\mathcal{O}_{\mathbb{C}_p}/p^n \otimes \mathcal{O}_{\mathbb{C}_p}/p^n\right) \left[\frac{1}{p}\right].$$

Construction 4.11. For any profinite set T, the ring $\mathbb{C}_p(T)$ of continuous maps $T \to \mathbb{C}_p$ is divisible as an abelian group, i.e. has a (necessarily unique) vector space structure over $\mathbb{Q}$. Thus we have an equivalence of condensed spectra

$$\mathbb{C}_p \otimes \mathbb{Z} \simeq \mathbb{C}_p \otimes \mathbb{Q}$$

and we conclude that, as the solidification of the cofibre of the equivalence above,

$$\mathbb{C}_p \otimes^{\blacksquare} (\mathbb{Q}/\mathbb{Z}) \simeq 0.$$

Construction 4.12. The condensed abelian groups $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are all discrete. Thus the solid (derived) tensor product of any two of them agrees with the condensed (derived) tensor product which in turn agrees with the derived tensor product of underlying abelian groups. We have that $\mathbb{Q}$ is divisible and $\mathbb{Q}/\mathbb{Z}$ is torsion so $\mathbb{Q} \otimes^u \mathbb{Q}/\mathbb{Z} \simeq 0$ (here $\otimes^u$ denotes the underived tensor product). Since $\mathbb{Q}$ is a torsion-free abelian group, it is flat as a $\mathbb{Z}$-module and thus $\mathrm{Tor}_i(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) = 0$ for all $i > 0$. We conclude that $\mathbb{Q} \otimes \mathbb{Q}/\mathbb{Z} \simeq 0$ (here we mean the derived tensor product). Thus we have a cofibre sequence

$$\mathbb{Q}/\mathbb{Z} \to 0 \to (\mathbb{Q}/\mathbb{Z}) \otimes (\mathbb{Q}/\mathbb{Z})$$

which gives that the tensor product of the roots of units with itself is just a shift:

$$(\mathbb{Q}/\mathbb{Z}) \otimes^{\blacksquare} (\mathbb{Q}/\mathbb{Z}) \simeq (\mathbb{Q}/\mathbb{Z})[1].$$

We conclude:

Theorem 4.13.

$$\mathcal{O}_{\mathbb{C}_p}^\times \otimes^{L\blacksquare} \mathcal{O}_{\mathbb{C}_p}^\times \simeq (\mathbb{Q}/\mathbb{Z})[1] \oplus \left(\bigoplus_{I\times I} \mathbb{Z}\right)_p^\wedge \left[\frac{1}{p}\right] \simeq (\mathbb{Q}/\mathbb{Z})[1] \oplus \varprojlim_n \left(\mathcal{O}_{\mathbb{C}_p}/p^n \otimes \mathcal{O}_{\mathbb{C}_p}/p^n\right) \left[\frac{1}{p}\right].$$

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