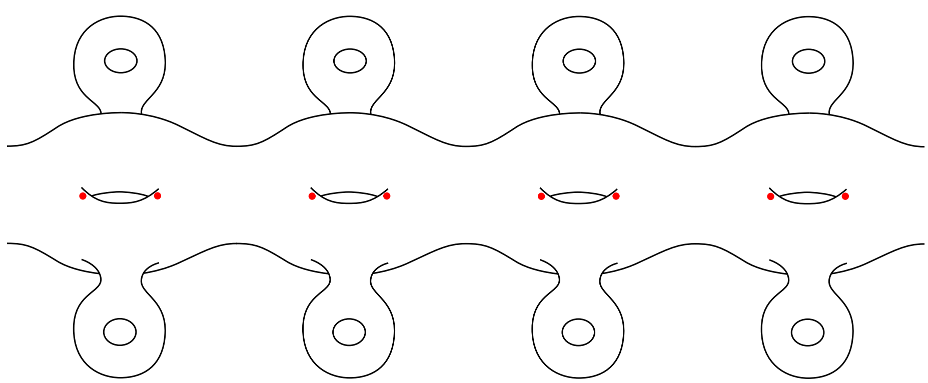
PhD Thesis

Equivariant cobordism categories and the homology of moduli spaces of equivariant manifolds

University of Copenhagen



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Abstract

The goal of this thesis is to study the moduli space $\mathcal{M}^G(M)$ associated to a smooth compact manifold M equipped with an action of a finite group G. This space is homotopy equivalent to the classifying space of $\mathrm{Diff}^G(M)$ the topological group of equivariant diffeomorphisms of M. We prove that under some connectivity conditions, its homology is often given by that of an infinite loop space in the stable range, answering a question raised by Galatius-Szucs in [GS21]. We strongly rely on the work of Galatius-Randal-Williams ([GR17a], [GR17b]) on the homology of moduli spaces of high dimensional manifolds, which gave such a stable computation in the non equivariant setting. Our proof relies on the existence of an isotropy separation sequence at the level of equivariant cobordism categories à la Steimle.

Resumé

Målet med denne afhandling er at studere modulirummet $\mathcal{M}^G(M)$ for en glat kompakt mangfoldighed M udstyret med en virkning af en endelig gruppe G. Dette rum er homotopiækvivalent med klassificerenderummet for $\mathrm{Diff}^G(M)$, den topologiske gruppe af ækvivariante diffeomorfier af M. Vi beviser, at under visse sammenhængsforhold er dens homologi ofte givet ved homologien af et uendeligt løkkerum i det stabile område, hvilket besvarer et spørgsmål, der blev rejst af Galatius og Szucs i [GS21]. Vores arbejde er stærkt afhængigt af Galatius og Randal-Williams' forskning ([GR17a], [GR17b]) om homologien af modulirum for højdimensionelle mangfoldigheder, som gav sådan en stabil udregning i det ikke-ækvivariante tilfælde. Vores bevis bygger på eksistensen af en isotropi-separationsfølge i rammerne af ækvivariante cobordismekategorier à la Steimle.

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I was fortunate to be supervised by not just one, but two wonderful academic supervisors, Søren Galatius and Nathalie Wahl. Although I initially began working primarily with Nathalie, I gradually became more engaged with Søren's mathematics during my first year of PhD studies and ultimately chose to continue my main project with him. Nevertheless, Nathalie remained an invaluable source of support throughout, always willing to discuss mathematics and assist whenever I needed it. Over the years, I have been consistently amazed by Søren's ability to understand my questions and provide the clearest possible answers, significantly aiding my progress each time we met. I would like to extend my warmest thanks to both of them for being so helpful and kind to me.

Many people contributed to making these three years as enjoyable as they were. I am grateful to Alexander Kupers for his hospitality and the engaging discussions about equivariant embedding calculus during my two-month visit to Toronto. I also thank Ismael Sierra, who helped me with some of the algebra needed for my computations and who has always shown interest in my projects since we first met early in my PhD journey. Thank you, Ismael, for being a great listener and an equally great, albeit occasionally annoying, friend. Greetings also go to Oscar Randal-Williams for suggesting a reference that was crucial in completing the final section of my article. My colleagues from the Copenhagen team were also a big part in making my PhD experience so enjoyable. I could always rely on Maxime Ramzi to untangle my confusion about higher structures, on Fadi Mezher for various manifold-related discussions, on Jan Steinebrunner, who assisted me on several occasions, and on Branko Juran who was always interested in my project, and with whom I had many enlightening conversations about G-manifolds and equivariant cobordism categories. For reading versions of my paper/thesis and coming with helpful remarks and suggestions, I thank my two supervisors Søren and Nathalie, as well as Fadi, Branko and Azélie Picot. However, the experience would not have been the same without the laughter. For this, I thank the entire Copenhagen topology team, including Alexis Aumonier, Kaif Hillman and Adriano Córdova who graduated before me, as well as those who kept me laughing from outside Copenhagen: Samuel Muñoz-Echániz, Étienne Batelier, and the math team from Toronto. I also enjoyed sharing an office with Bai Qingyuan, with whom I had great discussions about both mathematics and music. Thanks for introducing me to Clown Core.

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When I wasn't at the university or at home with my pet bunny Raclette, I was either obsessing over one of my many hobbies or hanging out with my awesome roommates and partying with friends outside the university. They were also a significant part of my time in Copenhagen. As for my hobbies, I began baking bread for the university during my first year and organized many pizza parties at the department. Later, I betrayed everyone and started spending most of my time at the bouldering gym. Fortunately, several people from the math department shared this interest—though not everyone. The apartment I lived in was far from a five-star hotel suite, but I always felt at home thanks to Matteo, Mariya, Pedro, Philipp and Stamatis, with whom I shared some of the greatest laughs. I would also like to thank Sara for being my best—and most reliable—bouldering friend. My thoughts also go to Christmas, undoubtedly one of my luckiest encounters.

My journey toward a PhD in mathematics began early in my studies, and I was not alone in this process. Nine years ago, when I arrived in Paris to start a "prépa maths," I met Adam, who would become my best friend. We followed the same mathematical path until late in the second year of our Master's program. In particular, we discovered the field of topology and the concept of manifolds together. We have now both completed our PhD theses, each focusing on different approaches to the study of these objects. I consider Adam to be the person who has most influenced the way I see and appreciate mathematics. Thank you, Adam, for growing with me mathematically and for achieving our goals together. I also want to extend my gratitude to our group of friends from ENS, especially Thomas and Jean, without whom these four years would not have been the same.

Finally, I would like to thank my close family—Maman, Papa, Gabrielle, and Simon—for being the best family one could hope for. Even as we, the children, grow and pursue our own paths, our close bond will remains a natural part of who we are.

Pierre Elis
Copenhagen, August 2024.

Contents

Chapter 1

Introduction

History and motivation

Over the past decade, significant advances have been made in understanding moduli spaces of manifolds and their stable homology. Given a smooth compact manifold W, its associated moduli space denoted by $\mathcal{M}(W)$ is a classifying space for smooth parametrized families of manifolds that are pointwise diffeomorphic to W. These moduli spaces are of great importance in the field of algebraic topology of manifolds, first intrinsically—since they arise naturally in the study manifolds and their invariants—but also because they help understand the group of diffeomorphisms $\mathrm{Diff}(W)$ of the manifold W. Notably $\mathrm{Diff}(W)$ and $\mathcal{M}(W) \simeq B\mathrm{Diff}(W)$ share the same homotopy groups up to a shift of degrees.

The most basic instance of such moduli spaces is the case of manifolds of dimension 0. For a set X of cardinality n, the space $\mathcal{M}(X) \simeq B\Sigma_n$ has been well understood since the 1960s, when Nakaoa showed that they satisfy homological stability as the cardinality increases by computing their homology ([Nak60],[Nak61]). Moreover, more can be said: the Barratt-Priddy theorem ([BP72]) identifies the homotopy type of the *stabilized moduli space* $\operatorname{hocolim}_{n\to\infty} B\Sigma_n$ with the infinite loop space $\Omega_0^\infty\mathbb{S}$ after plus construction, where $\mathbb{S}$ is the sphere spectrum.

In 1985, [Har85] showed that the homology of the moduli space of orientable surfaces with a boundary component stabilises as the genus increases, more precisely if $W_{g,1}$ denotes the orientable surface of genus g with one boundary component, the map of *oriented moduli spaces* $\mathcal{M}_{\partial,+}(W_{g,1}) \to \mathcal{M}_{\partial,+}(W_{g+1,1})$, induced by gluing $W_{1,1} := S^1 \times S^1 \setminus D^2$ along part of the boundary, is a homology isomorphism in a range of degrees growing w.r.t. g. A natural question was then to give a description of the stabilised moduli space $\operatorname{hocolim}_{g\to\infty} \mathcal{M}_\partial^+(W_{g,1})$. Mumford conjectured in 1983 in [Mum83] that its homology with rational coefficients is given by a polynomial algebra $\mathbb{Q}[\kappa_i]$, where the degree $2i > 0$ classes κ_i are the so-called *Miller-Morita-Mumford* classes and have a geometric description in terms of characteristic classes of associated vector bundles. This conjecture was proven in 2007 by Madsen and Weiss in [MW07], where they identify that space with some infinite loop space after plus construction, as it had been suggested in 1997 by Tillmann ([Til97]). Building on that, Madsen and Weiss along with Galatius and Tillmann extended these ideas in the celebrated article [GMT+09], where they offer a new proof of the Mumford conjecture, more conceptual, coming with a clear strategy in two steps:

- In general, given a dimension d, they consider the topological category $\mathcal{C}^d$ whose objects are given by smooth closed manifolds of dimension $d-1$ and whose morphisms are given by cobordisms between them. The classifying space of this category $B\mathcal{C}^d$ admits a map of a geometric nature – the parametrised Thom-Pontrjagin construction – into an infinite loop space, which is shown to be an equivalence in great generality. Namely, there are variants $\mathcal{C}_\theta^d$ that one can consider, given θ any tangential structure for instance an orientation. They show that the parametrized Thom-Pontrjagin construction is an equivalence $B\mathcal{C}_\theta^d \to \Omega^{\infty-1}MT\theta$, where $MT\theta$

is the so-called *Madsen-Tillmann spectrum* associated to θ. Given a surface S (without boundary to ease the notations), its oriented moduli space admits a natural map

$$\mathcal{M}_+(S) \to \Omega_0 B\mathcal{C}^2_+ \to \Omega^\infty_0 MTSO(2) \tag{1.1}$$

This map was later reinterpreted in a geometric way in [GR10] and has since then been referred to in the literature as the *scanning map*.

- Then, *parametrised surgeries* can be performed on the category $\mathcal{C}^2_+$ to show that the above map – in the presence of a boundary component – becomes an equivalence after taking colimit over $g \to \infty$ and plus construction, by an application of the group completion theorem.

The Mumford conjecture is then a consequence of Harer stability, together with the computation $H^*(\Omega^\infty_0 MTSO(2); \mathbb{Q}) \cong \mathbb{Q}[\kappa_i]$.

The philosophy of this strategy has been used to describe various stable moduli spaces, for instance certain moduli spaces of graphs leading to computations of the stable homology of $\mathrm{Aut}(F_n)$ in [Gal11], or higher dimensional analogs of the moduli space of surfaces. The latter forms the starting point of this thesis.

In a series of papers ([GR14], [GR17a],[GR17b]), Galatius and Randal-Williams generalized the parametrized surgery argument of [GMT+09] to manifolds of even dimensions $d = 2k \geq 6$, and proved relevant homological stability results for manifolds which are simply connected. The non simply connected case was later worked out by [Fri17] under some finiteness conditions. Collectively, these results provide a systematic method for calculating the stable homology of moduli spaces of even-dimensional manifolds in dimensions $d \geq 6$. The range of stability involves a notion of *θ-genus* $g^\theta(W, l_W)$ associated to a manifold with θ-structure (W, l_W). Ignoring the tangential structure, this notion of genus is a natural generalisation of that for surfaces. Namely, let $W_{g,1}$ be the connect sum of g copies of $S^k \times S^k$, to which one removes a disk D^{2k}. The genus of W is the largest g such that there exists an embedding $W_{g,1} \hookrightarrow W$. An informal statement summarizing a portion of these results can be presented as follows.

Theorem (Galatius-Randal-Williams,Friedrich)**.** *Let θ be a tangential structure, let (W, l_W) be a manifold of dimension $d = 2k \geq 6$ equipped with a θ-structure l_W which is k-connected, and suppose that $\pi_1(W)$ is virtually polycylic. Then the homology of the* moduli space with θ-structure *$\mathcal{M}_\theta(W, l_W)$ can be computed in a range, namely there exists an explicit linear map $r_W : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ depending only on $\pi_1(W)$ such that $r_W(x) \xrightarrow[x\to\infty]{} \infty$ and the so called* scanning map *(a generalisation of (1.1))*

$$\mathcal{M}_\theta(W, l_W) \to \Omega^\infty MT\theta$$

is an isomorphism in homology in degrees $* \leq r_W(g^\theta(W, l_W))$ onto the path component that it hits, for all local systems of coefficients. An analogous statement in the presence of a non empty boundary holds.

In this thesis, we explore an equivariant analog of the above statement.

Summary of the research article

This section is an attempt of an informal and ergonomic summary of the main part of this thesis, consisting of the actual paper. We apologize for the redundancy with its internal introduction.

Let G be a finite group. In [GS21] the authors defined a category which we shall denote by $\mathcal{C}^G_{\theta_d}$, where objects are smooth closed $(d-1)$-dimensional manifolds with an action of G, that also have some additional *tangential structure* encoded by the subscript θ_d. A morphism between two such G-manifolds is given by an equivariant cobordism of dimension d between them, with a compatible tangential structure. They prove that the homotopy type of this category is given by the infinite loop-space $\Omega^{\infty-1}(MT\theta_d)^G$ where $MT\theta_d$ is a genuine G-Thom spectrum and $(-)^G$ denotes genuine G-fixed points. Given (M, l_M) a closed G-manifold of dimension d with a θ_d-structure, there is an equivariant scanning map

$$\mathcal{M}^G_\theta(M, l_M) \to \Omega^\infty(MT\theta_d)^G$$

analog to the non-equivariant one. They raise the following natural question.

Question. [GS21, section 7] In general what can be said about the map $\mathcal{M}^G_\theta(M, l_M) \to \Omega^\infty_0(MT\theta)^G$? Typically, is it an isomorphism in homology in a certain range of degrees ?

In the paper below we give a systematic treatment of this question and show that this map does indeed induce an isomorphism in homology in a range of degrees with all local coefficient systems, under some hypotheses analogous to the non-equivariant case adressed in [GR17a] and [GR17b]. Our approach is based on isotropy separation sequences, which also allow to have a new view on the main result of [GS21] although we miss one step to conclude a full new proof.

We now describe the contents of the joint paper. We consider a slight variation of the categories defined in [GS21], namely we assemble them into one unique category $\mathcal{C}^G_\theta$. It is defined in the same way as above, with the difference that the manifolds are allowed to be of any finite dimension, possibly varying along their path components. This category is well-suited for the systematic study of G-manifolds, as the set of fixed points of a G-manifold is itself a manifold in that broader sense. Although this category has not been considered in the literature before to the author's knowledge, note that it does not encode any new homotopical information as we show that there is an equivalence of categories

$$(-)_{\mathbb{N}}\colon \mathcal{C}^G_\theta \to \prod_{d\geq 0}{}' \mathcal{C}^G_{\theta_d}$$

into the restricted product of the usual ones, where a G-manifold M is sent to the finitely supported family $(M_d)_{d\geq 0}$ of its d-dimensional parts.

We start by establishing some properties of these categories for varying G and θ, in particular we study how they interact with respect to taking H-fixed points for $H \leq G$. This will later allows us to say something about the moduli space associated to a G-manifold M with θ-structure, which can be seen as a path component of some morphism space of $\mathcal{C}_\theta^G$. Some constructions and new definitions arise naturally as follows.

(i) Given θ an equivariant tangential structure and $H \leq G$ we define an associated *H-fixed points structure* θ^H and construct a well defined H-fixed points functor

$$F^H : \mathcal{C}_\theta^G \to \mathcal{C}_{\theta^H}^{W_G H}$$

where $W_G H := N_G H/H$ stands for the Weyl group of H in G, which acts naturally on the H-fixed points.

(ii) Given $\mathcal{F}$ a family of subgroups of G, there is a corresponding (non full) subcategory $\mathcal{C}_{\theta,\mathcal{F}}^G$ of $\mathcal{C}_\theta^G$ consisting of those manifolds the points of which have isotropy contained in $\mathcal{F}$. If the family $\mathcal{F}$ contains only the trivial subgroup $* \leq G$, we denote the corresponding category by $\mathcal{C}_{\theta,\mathrm{free}}^G$. If H is a maximal subgroup inside $\mathcal{F}$, then the residual action of $W_G H$ on the H-fixed points is free, hence F^H refines to a functor

$$F^H : \mathcal{C}_{\theta,\mathcal{F}}^G \to \mathcal{C}_{\theta^H,\mathrm{free}}^{W_G H}$$

Let H be maximal in a family $\mathcal{F}$ of subgroups of G, we define a new family $\mathcal{F}-(H)$ by removing all conjugates of H inside $\mathcal{F}$. There is a canonical inclusion functor $i\colon \mathcal{C}_{\theta,\mathcal{F}-(H)}^G \hookrightarrow \mathcal{C}_{\theta,\mathcal{F}}^G$.

The main technical result of this paper is the following property about F^H.

Proposition. *The functor $F^H : \mathcal{C}_{\theta,\mathcal{F}}^G \to \mathcal{C}_{\theta^H,\mathrm{free}}^{W_G H}$ is a cocartesian fibration.*

By applying the additivity theorem of Steimle in [Ste21] we deduce

Theorem A. *With the notations above, there is a fiber sequence of spaces*

$$B\mathcal{C}_{\theta,\mathcal{F}-(H)}^G \xrightarrow{Bi} B\mathcal{C}_{\theta,\mathcal{F}}^G \xrightarrow{BF^H} B\mathcal{C}_{\theta^H,\mathrm{free}}^{W_G H}$$

which we call isotropy separation sequence for equivariant cobordism categories.

A version of this sequence exists at the higher level of equivariant cobordism categories themselves. In particular we show an isotropy separation sequence for spaces of equivariant nullbordisms, and conclude the proof of our main result.

Theorem B (see the corresponding Theorem B of the joint paper for a precise statement)**.** *Let (W, l_W) be a closed G-manifold with θ-structure. Under some conditions, the* equivariant scanning map $\mathcal{M}^G(W, l_W) \to \Omega_0^\infty(MT\theta)^G$ *is acyclic in a range of degrees.*

The range of degrees we get has an explicit formula, which we explain in particular in Corollary 2.3.8.

We apply our theorem in some examples to give concrete computations of the homology of certain equivariant moduli spaces. The first example is pictured on the front page of this thesis: it is a surface with an action of C_2. Our second example a class of hypersurfaces in $\mathbb{C}P^4$ called Fermat hypersurfaces. In the non-equivariant setting, they have been considered in [GR19] where the authors compute the stable homology of their oriented moduli space. Fermat hypersurfaces have a natural action of the symmetric group Σ_5, and restricting to a 3-cycle gives an interesting action of C_3 the fixed points of which are a surface of high genus. We compute the stable homology of their equivariant oriented moduli space.

Perspectives for future research

Diffeomorphisms of equivariant disks

By a work of Kupers ([Kup19]), the knowledge of the homology of $\mathcal{M}(W_{g,1})$ can be used as an input to show some finiteness properties of the higher homotopy groups of $\mathrm{Diff}_\partial(D^{2k})$ (if $2k \neq 4$). Namely, the author shows that the so-called *Weiss fiber sequence* admits a delooping

$$B\mathrm{Diff}_\partial(M) \to B\mathrm{Emb}^{\cong}_{1/2\partial}(M) \to B^2\mathrm{Diff}_\partial(D^{2k})$$

given M a compact manifold of dimension $2k$ and an embedding $D^{2k-1} \hookrightarrow \partial M$. In the middle term, $\mathrm{Emb}^{\cong}_{1/2\partial}(M)$ stands for those self-embeddings of M preserving a neighborhood $\partial M \setminus \mathrm{Int}(D^{2k})$ and which are isotopic through such embeddings to a diffeomorphism fixing a neighborhood of the boundary.

As $B^2\mathrm{Diff}_\partial(D^{2k})$ is simply connected, it it enough to show that its homology is finitely generated as this will imply the same for its homotopy groups. If M is taken to be $W_{g,1}$ for large g, then in a range of degrees (which can be made as big as desired), the homology of $B\mathrm{Diff}_\partial(W_{g,1})$ is finitely generated by the calculations of Galatius Randal-Williams. On the other end, embedding calculus can be used to describe the path component of the identity map in $\mathrm{Emb}^{\cong}_{1/2\partial}(M)$. The last ingredient needed is an input about finiteness properties of $\pi_0(W_{g,1})$.

The strategy above can be made equivariant, giving some hope to understand $\pi_k(\mathrm{Diff}^G(D(V \oplus \mathbb{R})))$ given V an odd-dimensional representation of G. The delooped Weiss fiber sequence also holds in the equivariant setting, as it turns out Kuper's construction can be reproduced almost verbatim. This gives a fiber sequence

$$B\mathrm{Diff}^G_\partial(M) \to B\mathrm{Emb}^{G,\cong}_{1/2\partial}(M) \to B^2\mathrm{Diff}_\partial(D(V \oplus \mathbb{R}))$$

given an equivariant embedding $D(V) \hookrightarrow \partial M$, V being a $(2k-1)$-dimensional representation of G. By the main result of this thesis, the homology of $B\mathrm{Diff}^G_\partial(M)$ is often finitely generated in a range of degrees. On the other hand, equivariant embedding calculus can be setup and shown to converge in a similar way to the classical setting, using geometric isotropy separation sequences at the level of embeddings as well as at the level of k-th derivatives. The last missing step would be to understand $\pi_0(\mathrm{Diff}^G_\partial(M))$ for some relevant G-manifolds M, which is an interesting topic of further research which was suggested to us by Sander Kupers.

Homological stability for equivariant moduli spaces

One of the unsatisfactory aspects of this thesis is that it does not tackle the question of homological stability for equivariant moduli spaces. The reason for that is that we could not find a proper way to phrase it in decent generality. Stabilisation maps can be defined in some easy cases, but in general it is not clear what it means to "increase genus". From our main theorem B, it seems clear that a relevant notion of genus could be that of (H,V)-genus $g_{(H,V)}(W)$ for varying (H,V), defined as the usual genus of the building block $W_{(H,V),\partial}/W_G H$.

Question. Given a G-manifold W such that $W^{(H,V)} \neq \emptyset$, is there always a well-defined glueing operation $W \mapsto W'$ such that $g_{(H,V)}(W') = g_{(H,V)}(W) + k_{(H,V)}$ for some fixed $k_{(H,V)} > 0$, inducing a continuous morphism $\mathrm{Diff}^G(W) \to \mathrm{Diff}^G(W')$? In the cases where such an operation exists, when does it exhibit homological stability phenomena ?

It can be seen by working with examples that we should not necessarily expect $k_{(H,V)}$ to be equal to 1 (for example starting with S^2 with a C_3-action consisting of a $2\pi/3$-rotation along an axis). An interesting question would be to minimize such an $r_{(H,V)}$.

On the other hand, there are positive signs that a general form of homological stability should be satisfied. In [GR17b], the authors show a form of stability which is prone to generalisation using isotropy separation sequences.

Proposition (Consequence of [GR17b, Corollary 1.7],[Fri17])**.** *Let (W, l_W) be a manifold of dimension $d = 2k \geq 6$ such that l_W is k-connected, and let M be a θ-bordism $M\colon P \rightsquigarrow \partial W$ such that $(M, \partial W)$ is $(k-1)$-connected. Suppose that the fundamental groups of W and $W' := W \cup_{\partial W} M$ are virtually polycylic. Then, there is an increasing map $r_{W,M,\theta}\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ such that $r_{W,M,\theta}(x) \xrightarrow[x\to\infty]{} \infty$ and the gluing map*

$$\mathcal{M}_{\theta,\partial}(W, l_W) \to \mathcal{M}_{\theta,\partial}(W \cup_{\partial W} M, l_W \cup_{\partial W} l_M)$$

induces an isomorphism in homology with abelian coefficient systems in degrees $* \leq r_{W,M,\theta}(g^\theta(W, l_W))$.

Suppose now that all manifolds in the statement above come with an action of a finite group G. By using techniques in this thesis, one can show that in the commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{\theta,\partial}(W, l_W) & \longrightarrow & \mathcal{M}_{\theta,\partial}(W \cup_{\partial W} M, l_W \cup_{\partial W} l_M) \\ \downarrow F^G & & \downarrow F^G \\ \mathcal{M}_{\theta^G,\partial}(W^G, l_W^G) & \longrightarrow & \mathcal{M}_{\theta^G,\partial}(W^G \cup_{\partial W^G} M^G, l_W^G \cup_{\partial W} l_M^G) \end{array}$$

the vertical homotopy fibers can be identified with a disjoint union of stabilization maps of the same manifolds after removing fixed points (in particular the size of the allowed isotropy decreases). By induction, and using a comparison of Serre spectral sequences with abelian coefficients, this gives an equivariant analog of the theorem above, after carefully chosing relevant hypotheses.

Note that homology stability for equivariant moduli spaces has been subject of recent research. For instance [BQV23] shows a version of homological stability for equivariant configuration spaces, which can be considered as a zero-dimensional case of the general problem.

Miscellaneous suggestions

An interesting generalization of the homotopy cartesian square of categories we get by taking fixed points would be to adapt it for extended equivariant cobordism categories. We expect this to work in the same way as the $(\infty, 1)$-case, although we do not know if F^H remains a cocartesian fibration in that setting. This would be a natural question in the context of the equivariant cobordism hypothesis.

Another natural question is about the homotopy type of embedded equivariant cobordism categories as we define them in this thesis. An answer in the classical setting is given in [Ran10], where the author identifies this homotopy type with the space of compactly supported sections of a certain bundle. We expect that the equivariant analog could be studied by using isotropy separation sequences and reducing to the theorem of Randal-Williams.

Chapter 2

Research article

THE STABLE HOMOLOGY OF MODULI SPACES OF EQUIVARIANT MANIFOLDS

PIERRE ELIS

ABSTRACT. We prove a formula for the cohomology of equivariant moduli spaces of manifolds equipped with an action of a finite group G, and provide concrete examples of computations. The proof consists in constructing an isotropy separation sequence for equivariant moduli spaces, which we also extend to one at the level of cobordism categories.

Introduction

Let G be a finite group, we define the cobordism category $\mathcal{C}^G$ of all *G-manifolds*. Objects in this category are closed smooth manifolds equipped with a smooth action of G, and are topologized as the union

$$\coprod_{[M]} B\mathrm{Diff}^G(M)$$

where $[M]$ ranges over isomorphism classes of G-manifolds and $\mathrm{Diff}^G(M)$ stands for the topological group of G-equivariant diffeomorphisms of M. A morphism between two G-manifolds is essentially a compact cobordism between them, also equipped with a smooth action of G in a compatible way. Morphisms are given a topology in a similar way, and a slight adjustment allows to define an associative composition resulting in a weakly unital topological category.

A variant $\mathcal{C}^G_\theta$ is also defined in the presence of an *equivariant tangential structure* θ. Given $H \leq G$, taking H-fixed points induces a continuous functor

$$F^H : \mathcal{C}^G_\theta \to \mathcal{C}^{W_G H}_{\theta^H}$$

where θ^H is the *H-fixed point structure* associated to θ, and $W_G H$ is the Weyl group of H in G. In this paper we prove that F^H is a cocartesian fibration, which has several consequences of interest.

A fibre sequence of classifying spaces

First, we prove an isotropy separation sequence for the classifying spaces of equivariant cobordism categories. A particular case of it is the following.

Theorem (A). *Consider the (non full) subcategory $\mathcal{C}^G_{\theta,\mathcal{P}} \hookrightarrow \mathcal{C}^G_\theta$ of those manifolds which have an action of G with proper isotropy. There is a homotopy fibre sequence*

$$B\mathcal{C}^G_{\theta,\mathcal{P}} \to B\mathcal{C}^G_\theta \to B\mathcal{C}_{\theta^G}.$$

As suggested by its title, this theorem is directly related to the isotropy separation sequence for genuine G-spectra. As a matter of fact, it is easily seen as a consequence of [GS21] which identifies the homotopy type of the d-dimensional equivariant cobordism category as $B\mathcal{C}^G_{\theta,d} \simeq \Omega^{\infty-1}(MT\theta)^G$, where $MT\theta$ is a genuine G-Thom spectrum. However we do not appeal to op. cit. in the proof of Theorem A, which allows to give a new understand of their result modulo a missing argument.

Corollary ([GS21]). *Let $MT\theta$ be the genuine G-Thom spectrum associated to θ. There is a weak equivalence*

$$B\mathcal{C}^G_\theta \xrightarrow{\simeq} \Omega^{\infty-1}(MT\theta)^G.$$

The homology of equivariant moduli spaces

A second consequence of F^H being a cocartesian fibration is a formula for the cohomology of $B\mathrm{Diff}^G_\theta(W, l_W)$ in a range of degrees, for (W, l_W) a closed smooth manifold with a smooth action of G together with a θ-structure.

In the series of papers [GR17a], [GR17b] (together with [Fri17] for the non simply-connected case) the authors show that under some conditions on a closed smooth manifold M equipped with a θ-structure l_M, the so-called *scanning map*

$$B\mathrm{Diff}_\theta(M, l_M) \to \Omega^\infty_0 MT\theta$$

becomes r-connected after passing to homology with any local coefficient system, for r a certain natural number depending on (M, l_M) and diverging according to its *θ-genus*. Say that (M, l_M) verifying this property is *r-stable*.

The r-stability (M, l_M) is typically ensured by requiring the map $l_M \colon M \to B$ to be k-connected, when M is of dimension $2k$. In the equivariant setting, we introduce the notion of $\frac{1}{2}$-connectedness which generalises the one described above. We show in Corollary 2.3.8 how to compute the homology of $\mathcal{M}^G_\theta(M, l_M)$ when l_M is $\frac{1}{2}$-connected. Namely we construct finite collections $(\theta_i)_{i\in I}$ of tangential structures, and $(M_i, l_i)_{i\in I}$ of compact smooth (non-equivariant) manifolds with structures which we call *building blocks associated to* (M, l_M), such that the following holds.

Proposition (simplified statement of Corollary 2.3.8)**.** *Under some conditions on θ, the scanning map*

$$\mathcal{M}_\theta^G(M, l_M) \to \Omega_0^\infty(MT\theta)^G$$

is acyclic in a range of degrees growing w.r.t. the genus of the building blocks associated to (M, l_M), under the assumption that l_M is $\frac{1}{2}$-connected.

In practice this range is quite explicit and can be computed as the minimum of the ranges of stability of those building blocks. It follows from a general result which we state as our Theorem B.

Theorem (B – simplified statement)**.** *Suppose that for all $i \in I$, (M_i, l_i) is r_i-stable, for some $r_i \in \mathbb{N}$. Then, (M, l_M) is r-stable for $r = \min_{i \in I} r_i$, in the sense that the equivariant scanning map*

$$B\mathrm{Diff}_\theta^G(M, l_M) \to \Omega_0^\infty(MT\theta)^G$$

is r-connected after passing to homology with all local coefficients.

Up to homotopy, the collection $(M_i)_{i \in I}$ can be described as $(M_{(H,V)}/W_G H)_{(H,V)}$ indexed by subgroups $H \leq G$ and a certain equivalence class of H-representations V. Here, $M_{(H,V)}$ denotes the subspace of M consisting of those points with pure H-isotropy, and tangential H-representation equivalent to V. The orbits are taken over the action of the Weyl group of H in G, acting freely on $M_{(H,V)}$.

General tangential structures

When working out concrete examples of computation for a G-manifold (M, l_M) with θ-structure, it is rare that Theorem B can be applied directly as l_M will in general not be $\frac{1}{2}$-connected. In the non-equivariant setting, a systematic solution to this issue is described in [GR17b] and is as follows: given a non-equivariant closed $2k$-manifold (W, l_W) with θ-structure, consider a factorisation $W \xrightarrow{l'_W} B' \xrightarrow{u} B$ where l'_W is k-connected and u is k-truncated. Then, the moduli spaces $\mathcal{M}_\theta(W, l_W)$ and $\mathcal{M}_{\theta \circ u}(W, l'_W)$ lie in a homotopy fibre sequence which allows calculations for $\mathcal{M}_\theta(W, l_W)$.

We show an equivariant version of this fibre sequence in Proposition 2.3.16. A corollary is the following.

Proposition. *Suppose that there exists a factorisation*

$$l_M \colon M \xrightarrow{l'_M} B' \xrightarrow{u} B$$

such that l'_M is a $\frac{1}{2}$-connected cofibration and u is a $\frac{1}{2}$-truncated fibration. Define $\mathrm{Aut}(u)$ to be the topological monoid of those homotopy equivalences B'

over u. If $u \circ \theta$ satisfies the conditions of Corollary 2.3.8 and M satisfies the gap hypothesis, there exists a map

$$\mathcal{M}_\theta^G(M, l_M) \to \Omega^\infty(MT\theta')^G // \mathrm{Aut}(u)$$

which onto the path component that it hits is acyclic in a range of degrees growing w.r.t. to the genera of the building blocks associated to (M, l'_M).

The gap hypothesis on an G-manifold is defined for instance in [Sch06]. It is quite restrictive and intuitively means that the dimension of the fixed points grows exponentially (as a power of 2) along inclusions $H \leq K \leq G$.

2.1 Preliminaries and statement of the results

In this section we give the definitions as well as the basic properties of the moduli spaces of G-manifolds for a finite group G, and the equivariant cobordism category. We first define the equivariant notion of tangential structure θ and discuss how they interact with taking fixed points and orbits. We then define the moduli space associated to a G-manifold with θ-structure, and use it to define the cobordism category of all compact G-manifolds with θ-structure $\mathcal{C}_\theta^G$. Finally we recall some major results of [GMT+09],[GS21],[GR17a],[GR17b] about these moduli spaces and the category $\mathcal{C}_\theta^G$.

2.1.1 Homotopy theory of G-spaces and G-vector bundles

We first recall some natural model structures one may consider on equivariant spaces. The need for model structures in this paper comes from the fact that we use pointset models for our categories and in particular for moduli spaces of manifolds. Once our main definitions have been set, it will most often be possible to abstract away from model structures and work within the underlying ∞-categories directly.

Model theoretic considerations By convention our topological spaces are supposed to be compactly generated and weakly Hausdorff. We denote by Top the category of such spaces endowed with the Quillen model structure. Given a general topological group Π, a *family of subgroups* $\mathcal{F}$ of Π is by definition a collection of *closed* subgroups of Π stable under conjugation and taking subgroup. The category Top_Π of spaces equipped with a Π-action admit a cofibrantly generated model structure depending on $\mathcal{F}$ called the $\mathcal{F}$-model structure, such that

- weak equivalences are G-maps inducing weak-equivalences on H-fixed points for all $H \in \mathcal{F}$,
- fibrations are G-maps which are fibrations on H-fixed points for all $H \in \mathcal{F}$,
- generating cofibrations – resp. acyclic cofibrations are the maps $\Pi/\Lambda \times S^{k-1} \hookrightarrow \Pi/\Lambda \times D^k$, $k \geq 0$ – resp. $\Pi/\Lambda \times D^k \hookrightarrow \Pi/\Lambda \times D^k \times [0,1]$, $k \geq 0$, where $\Lambda \in \mathcal{F}$.

This result is folklore as stated, and was generalised in [Ste16]. For this model structure, cofibrant objects are in particular $\mathcal{F}$-spaces, that is, they have isotropy contained in $\mathcal{F}$.

For G a finite group, and $\mathcal{F}$ the family of all subgroups, this model structure is called *genuine equivariant*. We endow Top_G with the latter by default. When $\mathcal{F}$ is the trivial family, it is called *Borel equivariant*, and we denote by $\mathrm{Top}_{G,\mathrm{Borel}}$ the associated model category.

Given $d \geq 0$, we consider the category $\mathrm{Top}_{G\times O(d)}$ endowed with a certain $\mathcal{F}$-model structure, namely for the choice of family $\mathcal{F} := \mathcal{G}_d$ the family of *graph subgroups*. A graph subgroup $\Lambda \leq G \times O(d)$ is a subgroup of the form

$$\Lambda_\rho = \{(g, \rho(g))|g \in H\}$$

for some subgroup $H \leq G$ and some group morphism $\rho\colon H \to O(d)$. Graph subgroups are exactly those subgroups $\Lambda \leq G \times O(d)$ such that $\Lambda \cap O(d) = *$, also called $O(d)$*-free* $(G\times O(d))$*-spaces*. Let $\mathcal{G}_d$ be the family of graph subgroups of $G \times O(d)$. We will call this structure the *graph model structure*, and denote the associated model category by $\mathrm{Top}_{G\times O(d),\mathcal{G}_d}$.

Let Π be a general topological group and $\mathcal{F}$ be a family of (closed) subgroups of Π. Recall that a *universal* $\mathcal{F}$*-space* is a cofibrant space $E\mathcal{F} \in \mathrm{Top}_\Pi$ which is an $\mathcal{F}$-space, and such that $(E\mathcal{F})^H$ is weakly contractible for all $H \in \mathcal{F}$. This property characterizes it uniquely inside the homotopy category $h\mathrm{Top}_\Pi$. A Π-space admits an equivariant map to $E\mathcal{F}$ if and only if it is an $\mathcal{F}$-space and if so, this map is unique up to homotopy.

For $\Pi = G\times O(d)$, we are interested in the universal $\mathcal{G}_d$-space $E\mathcal{G}_d$ which we also denote by $E_GO(d)$. Its $O(d)$-orbits form a G-space $B_GO(d)$ which is well defined inside $h\mathrm{Top}_G$. Those spaces have a geometric description as follows. Let $\mathcal{U}_G$ be the direct sum of all finite dimensional orthogonal representations of G, one in each isomorphism class $\mathcal{U}_G := \bigoplus_{[V]} V$. Any orthogonal G-representation isomorphic to $\mathcal{U}_G$ will be called a *complete* G*-universe*.

Lemma 2.1.1. *The space* $\mathrm{St}_d(\mathcal{U}_G)$ *of orthogonal* d*-frames inside* $\mathcal{U}_G$ *is a universal* $\mathcal{G}_d$*-space. Hence the Grassmannian* $\mathrm{Gr}_d(\mathcal{U}_G) := \mathrm{St}_d(\mathcal{U}_G)/O(d)$ *is a model for* $B_GO(d)$ *in* Top_G.

Proof. This is a consequence of [Sch18, Proposition 1.1.26] □

We also recall another model for G-spaces which will be convenient to apply our main theorem in section 2.4. Given $X, Y \in \mathrm{Top}_G$, let $\mathrm{Map}^G(X,Y) \in \mathrm{Top}$ be the fixed points of the internal mapping space $\mathrm{Map}_G(X,Y) \in \mathrm{Top}_G$.

Theorem (Elmendorf (e.g. [MC96, Theorem 3.2])**.** *Let* $\mathcal{O}_G$ *be the category with objects the coset projections* G/H*,* $H \leq G$ *and maps given by* G*-equivariant maps. The functor*

$$\Phi\colon \mathrm{Top}_G \to \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top}), \quad X \mapsto \mathrm{Map}^G(-, X)$$

is a Quillen equivalence, for the projective model structure on the right.

Note that $\mathrm{Map}_G(G/H, X) \cong X^H$, so that the theorem above means that a G-space is determined by its fixed points for all subgroups of H and the maps between them. This will allow more flexibility in defining G-spaces by specifying their underlying Borel G-space and their fixed points separately.

G-vector bundles In this short paragraph we follow the exposition of [MC96, Chapter VII].

Definition 2.1.2. Given Π a topological group, a G-equivariant principal Π-bundle is the datum of a principal Π-bundle $p\colon E \to B$ which is G-equivariant as map of spaces, such that G acts through maps of principle Π-bundles.

In particular in this definition, p is the projection on Π-orbits from a Π-free $(G\times\Pi)$-space E. This point of view gives an equivalent definition after requiring the local triviality of the projection map p.

Remark 2.1.3. Usually in the theory of equivariant bundles, stronger local triviality conditions are imposed on bundles, taking the action G into account (see e.g. [GS21, Section 2.1.]). Because we took G finite, equivariant local triviality in the sense of op. cit. follows from the non-equivariant one.

The following is a consequence of [Las82, Theorem 2.14].

Proposition 2.1.4. *Let B be a G-space which is paracompact. Then, there is a bijection between the set of equivariant homotopy classes of maps $[B, B_GO(d)]_G$, and the set of equivariant isomorphism classes of G-equivariant $O(d)$-principal bundles over B.*

When B is a finite complex, this property can be seen as a particular case of general Quillen equivalence

$$\mathrm{Top}_{G\times O(d),\mathcal{G}_d} \to \mathrm{Top}_G/B_GO(d)$$

given by taking $O(d)$-orbits of the projection $-\times E\mathcal{G}_d \to E\mathcal{G}_d$, where on the right is the slice model structure.

Definition 2.1.5. A *G-equivariant d-vector bundle*, or *G-vector bundle of dimension d*, is a d-vector bundle $\xi\colon E \to B$ such that E and B are G-spaces, ξ is G-equivariant and G acts through bundle maps.

Given $p\colon E \to B$ a G-equivariant $O(d)$-principal bundle, we get a G-equivariant d-vector bundle $E \times_{O(d)} \mathbb{R}^d \to B$ over B in the usual way.

2.1.2 Tangential structures and cobordism categories

The representation $\mathcal{U}_G \cong \mathbb{R} \times \mathcal{U}_G$ is a complete G-universe. Hence $\mathrm{Gr}_d(\mathbb{R}\times\mathcal{U}_G)$ is a model for $B_GO(d)$.

Definition 2.1.6. The model category of *equivariant tangential structures* is defined as $\mathrm{Top}_G/\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)$, where $\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G) := \sqcup_{d\geq 0}\mathrm{Gr}_d(\mathbb{R}\times\mathcal{U}_G)$. A typical object will be denoted as $\theta\colon B \to \mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)$.
When V is a representation of G, we more generally define an *equivariant tangential structure for manifolds in* V as the datum of a G-fibration $\theta\colon B \to \mathrm{Gr}(V)$.

Note that $\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)$ is a classifying space for disjoint unions of G-vector bundles of varying dimensions. Hence the the model category of equivariant tangential structures is also that of *G-vector bundles* in that sense.

Taking pullback along the inclusions $\mathrm{Gr}_d(\mathbb{R}\times\mathcal{U}_G) \hookrightarrow \mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)$ gives a map $\theta \mapsto \theta_{d\in\mathbb{N}}$ inducing an equivalence of categories

$$\mathrm{Top}_G/\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G) \cong \prod_{d\geq 0}\mathrm{Top}_G/\mathrm{Gr}_d(\mathbb{R}\times\mathcal{U}_G)$$

Definition 2.1.7. Identifying θ and $(\theta_d)_{d\in\mathbb{N}}$ through this isomorphism, we say θ is *of dimension* d if $\theta = \theta_d$.

Remark 2.1.8. By the discussion in the previous section, the model category of equivariant tangential structures is Quillen equivalent to $\prod_{d\geq 0}\mathrm{Top}_{G\times O(d),\mathcal{G}_d}$. In [GS21], the authors define an equivariant tangential structure of dimension d as the datum of a $(G\times O(d))$-space, but define equivalences between such structures as equivariant maps which are equivalences on H-fixed points for all $H \leq G$. With this convention, the underlying homotopy theory is not that of G-spaces over $B_G O(d)$, and the observations they make in their Remark 6.7 are not true as stated. The issue is solved however by considering the graph model structure instead.

The cobordism category of G-manifolds The manifolds we consider in this paper are smooth, possibly with boundary and occasionally (if specified) with corners. By definition, a point in the interior a manifold $x \in M$ has a neighbourhood which is homeomorphic to $\mathbb{R}^d$ for some $d \geq 0$. We do not require d to be constant along the points of M. A G-manifold is a manifold which is equipped with a smooth action of G. With the convention above, when M is a G-manifold, its fixed points M^G also form a G-manifold.

Definition 2.1.9. Given M a G-manifold, and $d \geq 0$, define M_d to be the union of those path components which have dimension d.

Although $\mathbb{R}\times\mathcal{U}_G$ is not finite dimensional, it is the colimit of the G-manifolds $\mathbb{R}\times V$, for V ranging over isomorphism classes of finite dimensional orthogonal representations of G. We will say that $M \subset \mathbb{R}\times\mathcal{U}_G$ is a G-submanifold if it is a G-submanifold of $\mathbb{R}\times V$ for some V as above.

Let $W \subseteq \mathbb{R}\times\mathcal{U}_G$ be a G-submanifold. Then, the association $x \mapsto T_xW$ induces a G-map $\tau_W\colon W \to \mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)$ which we call the *Gauss map* of W. This map classifies the equivariant tangent bundle of W. If instead we are given P a G-submanifold of $\mathcal{U}_G$, the same process applied to $\mathbb{R}\times P \subseteq \mathbb{R}\times\mathcal{U}_G$

and composed with $P \xrightarrow{(0,p)} \mathbb{R} \times P$ gives a G-map $\varepsilon \oplus \tau_P \colon P \to \mathrm{Gr}_d(\mathbb{R} \times \mathcal{U}_G)$, which classifies the equivariant vector bundle $\varepsilon \oplus TP$. The Gauss map depends continuously on the submanifold, in the sense that the maps

$$\varepsilon \oplus \tau \colon \mathrm{Emb}^G(P, \mathcal{U}_G) \to \mathrm{Map}^G(P, \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G))$$

and

$$\tau \colon \mathrm{Emb}^G(W, \mathbb{R} \times \mathcal{U}_G) \to \mathrm{Map}^G(W, \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G))$$

are continous where on the left is the strong Whitney topology, and on the right is the compact open topology.

Definition 2.1.10. An equivariant θ-bundle over a G-space X is the datum of a map $X \to B$. The underlying equivariant vector bundle is given by composing with $\theta \colon B \to \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G)$.

Using the notations from the discussion above, a θ-structure on $W \subseteq \mathbb{R} \times \mathcal{U}_G$ is the datum of a θ-bundle $l_W \colon W \to B$ such that the underlying G-vector bundle is given by τ_W.

A θ-structure on $P \subseteq \mathcal{U}_G$ is the datum of a θ-bundle $l_P \colon P \to B$ such that the underlying G-vector bundle is given by $\varepsilon \oplus \tau_P$.

Note that in either case, a θ-structure on M is equivalent to the data of θ_d-structures on M_d for all $d \geq 0$. Also, given θ an equivariant tangential structure, taking pullback by $\iota \colon \mathcal{U}_G \hookrightarrow \mathbb{R} \times \mathcal{U}_G$ the inclusion of $\{0\} \times \mathcal{U}_G$ induces $\iota^*\theta$ an equivariant tangential structure for manifolds in $\mathcal{U}_G$. The last definition can be rephrased in saying that $\hat{l}_P$ is a $\iota^*\theta$-structure on P.

Although this paper mainly focuses on studying equivariant cobordism categories in the usual sense, it will be convenient for some proofs to define the more general equivariant analog of the notion of *embedded cobordism categories* as studied in [Ran10]. Consider M a G-submanifold of $\mathcal{U}_G$, and define the cobordism category of G-manifolds with θ-structure embedded inside M as follows.

Definition 2.1.11. Let $\mathcal{C}^G_{\theta,\varepsilon}(M)$ be the non-unital category with objects given by

(i) a closed G-submanifold P of M, and
(ii) a G-map $\hat{l}_P \colon P \to B$ such that $\varepsilon \oplus \tau_P = \hat{l}_P \circ \theta$

A morphism from $(P, \hat{l}_P)$ to $(Q, \hat{l}_Q)$ is given by

(i) an $s \in (2\varepsilon, \infty)$,
(ii) a G-submanifold $W \subset [0, s] \times M$, such that W intersects $([0, \varepsilon) \times M) \cup ((s - \varepsilon, s] \times M)$ in $([0, \varepsilon) \times P) \cup ((s - \varepsilon, s] \times Q)$,
(iii) a θ-structure $l_W \colon W \to B$ on W restricting to $\hat{l}_P \circ \mathrm{pr}$ on $[0, \varepsilon) \times P$ and to $\hat{l}_Q \circ \mathrm{pr}$ on $(s - \varepsilon, s] \times Q$.

We put a topology on the set of objects by identifying it as a subspace of

$$\coprod_{[P]} \Big(\mathrm{Emb}^G(P, M) \times_{\mathrm{Map}^G(P, \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G))} \mathrm{Map}^G(P, B) \Big) / \mathrm{Diff}^G(P)$$

The set of morphisms $\mathrm{mor}(\mathcal{C}^G_{\theta,\varepsilon}(M))$ is toplogized as a subspace of

$$(0,+\infty) \times \bigsqcup_{[W]} \mathrm{Emb}^G(W,[0,1]\times M) \times_{\mathrm{Map}^G(W,\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G))} \mathrm{Map}^G(W,B)/\mathrm{Diff}^G(W)$$

Morphisms in $\mathcal{C}^G_{\theta,\varepsilon}(M)$ are called *ε-collared.* We define $\mathcal{C}^G_{\theta}(M)$ as the following (filtered) colimit of topological categories

$$\mathcal{C}^G_{\theta}(M) := \operatorname*{colim}_{\varepsilon\to 0} \mathcal{C}^G_{\theta,\varepsilon}(M)$$

The set of morphisms in $\mathcal{C}^G_{\theta}(M)$ coincides with that in $\mathcal{C}^G_{\theta,\varepsilon}(M)$ where condition (ii) in Definition 2.1.11 is verified *for some* ε, and is endowed with the colimit topology. This description holds in good generality as we recall for completeness.

Lemma 2.1.12. *Let $\mathcal{D}\colon I \to$ nuTopCat be a filtered diagram non-unital topological categories. Then the natural maps* $\mathrm{colim}(\mathrm{ob}\circ\mathcal{D}) \to \mathrm{ob}(\mathrm{colim}(\mathcal{D}))$ *and* $\mathrm{colim}(\mathrm{mor}\circ\mathcal{D}) \to \mathrm{mor}(\mathrm{colim}(\mathcal{D}))$ *are isomorphisms.*

Proof. The nerve functor nuTopCat $\to \mathrm{Fun}(\Delta^{op}_{inj},\mathrm{Top})$ valued in semisimplicial topological spaces induces an equivalence onto the subcategory of those objects verifying the Segal condition. As the Segal condition involves a limit diagram and filtered colimits commute with limits, the result follows. □

Remark 2.1.13. An alternative way of chosing the topology on $\mathrm{mor}(\mathcal{C}^G_{\theta,\varepsilon}(M))$ is to first consider the subspace of

$$(0,+\infty) \times \bigsqcup_{[W]} \mathrm{Emb}^G(W,[0,1]\times M) \times_{\mathrm{Map}^G(W,\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G))} \mathrm{Map}^G(W,B)$$

of those embeddings such that the image of W is ε-collared, and then take the orbits for the action of $\mathrm{Diff}^G(W)$. Because this subspace is closed and saturated, the two topologies on the quotient coincide.

Strictly speaking, the definitions don't allow us yet to chose $M = \mathcal{U}_G$. In order to consider more complicated infinite dimensional ambiant spaces in general, we introduce the following definition.

Definition 2.1.14. A *increasing union of G-submanifolds of $\mathcal{U}_G$* is the datum of a subspace $M \subseteq \mathcal{U}_G$ such that there exists an increasing sequence of G-subrepresentations $(V_n)_{n\geq 0}$ of $\mathcal{U}_G$ such that $M_n := M \cap V_n$ is a G-submanifold of V_n and $\cup_{n\geq 0} V_n = \mathcal{U}_G$. We say that such a sequence $(V_n)_{n\geq 0}$ *realises M as an increasing union of G-submanifolds of $\mathcal{U}_G$.*

Fix M such a subspace of $\mathcal{U}_G$, an embedding of a G-manifold N inside of M is the datum of a map $f\colon N \to M$ such that there exists a sequence $(V_n)_{n\geq 0}$ realising M as a union of G-submanifolds of $\mathcal{U}_G$ and $n \geq 0$ such that f factors through $M_n = M \cap V_n$ and the resulting map is a G-embedding. Note that if this is true for one such sequence, it is true for all.

As an example, $\mathcal{U}_G$ is an increasing union of G-submanifolds of $\mathcal{U}_G$ in the obvious way. One other example which we will use later on is given by $S(\mathcal{U}_G)$ the unit sphere of $\mathcal{U}_G$. Any increasing sequence of finite dimensional subrepresentations $(V_n)_{n\geq 0}$ such that $\cup_{n\geq 0} V_n = \mathcal{U}_G$ realises $S(\mathcal{U}_G)$ as an union of G-submanifolds of $\mathcal{U}_G$, as $S(\mathcal{U}_G) \cap V_n = S(V_n)$ is the unit sphere in V_n which is a G-manifold.

Definition 2.1.15. Let $(V_n)_{n\geq 0}$ realise M as an increasing union of G-submanifolds of $\mathcal{U}_G$, we define $\mathcal{C}_\theta^G(M) := \operatorname{colim}_{n\geq 0} \mathcal{C}_\theta^G(M \cap V_n)$ where the colimit is taken inside the category of non-unital topological categories.

Note that for any other choice of sequence $(V'_n)_{n\geq 0}$ yields a category which homeomorphic the the one above hence we omit it from the notation.

When $M = \mathcal{U}_G$ (resp. $M = \mathbb{R}^\infty$), we simply write $\mathcal{C}_\theta^G$ (resp. $\mathcal{C}_\theta$) for $\mathcal{C}_\theta^G(M)$ (resp. $\mathcal{C}_\theta(M)$).

Notation 2.1.16. Given $\mathcal{C}$ a category and $f \in \operatorname{mor}(\mathcal{C})$, we write $\mathcal{C}(f)$ for the path component of $\operatorname{mor}(\mathcal{C})$ spanned by f. In the case of a bordism (W, l_W, s) in the morphism space of one of the cobordism categories defined above, we drop the letter s from the notations. We shall write $\mathcal{C}_\theta^G(W, l_W; V)$ instead of $\mathcal{C}_\theta^G(V)(W, l_W)$ when W is embedded inside $\mathbb{R}_{\geq 0} \times V$.

When $W : \partial W \rightsquigarrow \emptyset$ we write $\mathcal{M}_{\partial,\theta}^G(W, l_W)$ instead, and we drop ∂ from the notation when $\partial W = \emptyset$.

If P a G-manifold with $i^*\theta$-structure, $\mathcal{N}_{\theta,\partial}^G(P, \hat{l}_P)$ denotes the space of all morphisms $\mathcal{C}_\theta^G((P, \hat{l}_P), \emptyset)$, which is a union of $\mathcal{M}_{\partial,\theta}^G(W, l_W)$ where W ranges over diffeomorphism classes of equivariant θ-nulbordisms of $(P, \hat{l}_P)$.

If (W, l_W) is a compact manifold with an action of G and a θ-structure, then $\mathcal{M}_{\partial,\theta}^G(W, l_W)$ is a model for $B\mathrm{Diff}_{\partial,\theta}^G(W, l_W)$, the classifying space for concordance classes of smooth G-manifold bundles $E \to X$ with θ-structure and fibre (W, l_W) over compact base manifolds X, together with an equivariant trivialisation $X \times (\partial W \times [0, \varepsilon)) \subseteq E$ over X as θ-manifolds for some $\varepsilon > 0$, as defined in [GR19, section 12.2.2].

Remark 2.1.17. The topologies chosen above make $\mathcal{C}_\theta^G$ into a non-unital topological category. It is weakly-unital in the sense of [Ste21]. Moreover, it is locally fibrant in the sense of op. cit., which one proves using equivariant isotopy extension. Given such a category, there is an associated marked semiSegal space in the sense of [Har15], which gives rise to an ∞-category after completion. Also, given F a continuous functor between such categories which preserves weak units, then F gives rise to a functor between the associated ∞-categories. If moreover F is a weak equivalence on objects and on morphisms, then the induced functor between ∞-categories is an equivalence. This translation can be applied at several places in this paper to deduce statements about equivariant cobordism ∞-categories.

Proposition 2.1.18. *Taking pullback along* $\mathrm{Gr}_d(\mathbb{R} \times \mathcal{U}_G) \subseteq \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G)$ *gives a continuous functor*

$$(-)_d : \mathcal{C}_\theta^G \to \mathcal{C}_{\theta_d}^G$$

There is an induced functor into the restricted product

$$(-)_{\mathbb{N}}\colon \mathcal{C}_\theta^G \to \operatorname*{colim}_{D\subseteq\mathbb{N}} \prod_{d\in D} \mathcal{C}_{\theta_d}^G = \prod\nolimits'_{d\in\mathbb{N}} \mathcal{C}_{\theta_d}^G$$

where the filtered colimit is indexed over finite subsets $D \subset \mathbb{N}$ and taken inside the category of non-unital topological categories. This functor is a weak-equivalence on objects and on morphisms.

Corollary 2.1.19. *Define $MT\theta$ as $\oplus_{d\in\mathbb{N}} MT\theta_d$. Then, there is an equivalence $B\mathcal{C}_\theta^G \simeq \Omega^{\infty-1}(MT\theta)^G$.*

Proof. (of 2.1.19) Apply 2.1.18 and use the equivalence

$$\Omega^{\infty-1} MT\theta \simeq \prod\nolimits'_{d\in\mathbb{N}} \Omega^{\infty-1} MT\theta_d$$

together with the main theorem of [GS21]. □

Proof of the proposition. First note that if M is an object or a morphism in $\mathcal{C}_\theta^G$, then $M_d = \emptyset$ for all except a finite number of $d \in \mathcal{N}$, by compactness. Hence the functor $(-)_{\mathbb{N}}$ does take values in the restricted product. We prove the equivalence for morphisms, the proof for objects is similar. Consider a diagram

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{\alpha} & \operatorname{mor}\left(\mathcal{C}_\theta^G\right) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle (-)_{\mathbb{N}}} \\ D^k & \xrightarrow{\beta} & \operatorname{mor}\left(\prod'_{d\in\mathbb{N}} \mathcal{C}_\theta^G\right) \end{array}$$

The colimit defining restricted product is filtered, hence

$$\operatorname{mor}\left(\prod\nolimits'_{d\in\mathbb{N}} \mathcal{C}_\theta^G\right) \cong \operatorname*{colim}_{D\subseteq\mathbb{N}} \prod_{d\in D} \operatorname{mor}\left(\mathcal{C}_\theta^G\right) \cong \operatorname*{colim}_{D\subseteq\mathbb{N}} \operatorname*{colim}_{\varepsilon\to 0} \prod_{d\in D} \operatorname{mor}(\mathcal{C}_{\theta,\varepsilon}^G)$$

By connectedness and compactness, β lands in one path component of $\prod_{d\in D} \operatorname{mor}(\mathcal{C}_{\theta,\varepsilon}^G)$ for some $D \subseteq \mathbb{N}$ finite and for some $\varepsilon > 0$. Such a path component is of the form $\prod_{d\in D} \mathcal{C}_{\theta,\varepsilon}^G(W_d, l_{W_d})$ for some compact G-bordisms (W_d, l_{W_d}) with θ-structure embedded inside $\mathbb{R}_{\geq 0} \times \mathcal{U}_G$ Again by compactness and by finiteness of D, β actually lands in $\prod_{d\in D} \mathcal{C}_{\theta,\varepsilon}^G(W_d, l_{W_d}; V)$ for some finite dimensional representation $V \subseteq \mathcal{U}_G$. Let ρ be a smooth function $D^k \to [0,1]$ with $\rho(z) = 0$ iff $z \in S^{k-1}$, and let $\mathbb{R}x$ be a trivial summand of $\mathcal{U}_G - V$. Then,

$$H(z,t) = (\beta_d(z) + td\rho(z)x)_{d\in D}$$

is a homotopy from β to a map $\beta' := H(-,1)$ such that the manifolds $\beta'_d(z)$, $d \in D$ are pairwise disjoint. This homotopy induces a homotopy of squares, from the one we started with to a new square such that taking disjoint union gives a lift. □

Remark 2.1.20. Let $(W, l_W) = (W_d, l_{W_d})_{d\in\mathbb{N}}$ be a G-manifold with θ-structure. From the proof of Proposition 2.1.18, one can see that the natural map

$$\mathcal{M}^G_{\partial,\theta}(W, l_W) \to \prod_{d\in\mathbb{N}} \mathcal{M}^G_{\partial,\theta}(W_d, l_{W_d})$$

is an equivalence, where the product on the right really is a finite product.

Remark 2.1.21. We have only defined the notion of a θ-structure on a manifold when it is given as a submanifold of $\mathcal{U}_G$ or $\mathbb{R} \times \mathcal{U}_G$. If M is now any closed G-manifold, fix a G-map $\tau\colon M \to B_G O(d)$ – in particular make a choice of a model for $B_G O(d)$ – which classifies its equivariant tangent bundle. If $\theta\colon B \to B_G O(d)$ is a G-fibration, a θ-structure on M as the datum of a map $l_M\colon M \to B$ is such that $\tau = \theta \circ l_M$. Consider $\mathcal{M}^G_\theta(M, \tau, l_M)$ to be the path component of $\mathrm{Map}^G_{\mathrm{Top}_G/B_G O(d)}(M, B)//\mathrm{Diff}^G(M)$ containing l_M. Then $\mathcal{M}^G_\theta(M, \tau, l_M)$ coincides with the moduli space previously defined, in the sense that if M is embedded inside $\mathcal{U}_G$, $\tau_M\colon M \to \mathrm{Gr}_d(\mathbb{R} \times \mathcal{U}_G)$ is the equivariant Gauss map associated to the embedding $M \hookrightarrow \mathcal{U}_G \hookrightarrow \mathbb{R} \times \mathcal{U}_G$ and $\theta\colon B \to \mathrm{Gr}_d(\mathbb{R} \times \mathcal{U}_G)$ is a d-dimensional equivariant tangential structure,then there is an equivalence

$$\mathcal{M}^G_\theta(M, \tau_M, l_M) \simeq \mathcal{M}^G_\theta(M, l_M)$$

Remark 2.1.22. Another flexibility we have is the choice of model for the category of G-spaces. The definitions above also make sense if Top_G is replaced by $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top})$, and we will allows ourselves to switch between models when it is convenient to.

The fixed point structure Given $H \leq G$, the space $(\mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G))^H$ is equal to the space of orthogonal H-representations embedded inside $\mathbb{R} \times \mathcal{U}_G$. Applying H-fixed points further gives a map

$$(-)^H\colon (\mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G))^H \to \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G^H)$$

Note that this map is also equivariant, for the residual action of the Weyl group $W_G H$ of H in G, i.e. the quotient of its normaliser $W_G H = N_G H/H$. Remarking that the $W_G H$-representation $\mathcal{U}_G^H$ is a complete $W_G H$-universe, we might as well take the definition $\mathcal{U}_{W_G H} := \mathcal{U}_G^H$ as a convention.

Definition 2.1.23. Let $\theta\colon B \to \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G)$ be an equivariant tangential structure. For $H \leq G$, define the H-fixed points structure $F^H(\theta)$ as

$$F^H(\theta)\colon B^H \xrightarrow{\theta^H} (\mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G))^H \xrightarrow{(-)^H} \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_{W_G H})$$

which is as such a $W_G H$-equivariant structure.

Abusing notations, we shall write θ^H instead of $F^H(\theta)$ when this does not lead to any confusion.

With this definition, we see that the H-fixed points of a manifold with θ-structure inherit a θ^H-structure, compatible with the residual $W_G H$-action.

Definition 2.1.24. Taking H-fixed points and remembering the residual action of $W_G H$ defines a functor

$$F^H : \mathcal{C}^G_\theta \to \mathcal{C}^{W_G H}_{\theta^H}$$

Recall that the space of H-fixed points of $B_G O(d)$ is equivalent to a disjoint union (e.g. [MC96])

$$B_G O(d)^H \simeq \coprod_{[V]\in \mathrm{Rep}_H(d)} B\mathrm{Aut}^H(V) \tag{2.1}$$

where $\mathrm{Rep}_H(d)$ is the set of isomorphism classes of d-dimensional real representations of H.

In terms of our model $\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)$ for $B_G O(\bullet)$, this equivalence is reflected by an equivariant homeomorphism $\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)^H \cong \coprod_{[V]\in\mathrm{Rep}_H} \mathrm{Gr}_{(H,V)}(\mathbb{R}\times\mathcal{U}_G)$, where the summands are defined as the space of sub-H-representations of $\mathbb{R}\times\mathcal{U}_G$ which are abtractly isomorphic to V. Note that the action of $W_G H$ isn't clear from this formula, as isomorphism classes of H-representations might have non trivial orbits along the action of $W_G H$ (acting by conjugation). Hence, given such an isomorphism class *up to $W_G H$-conjugation* $[V] \in \mathrm{Rep}_H/W_G H$, we define $\mathrm{Gr}_{(H,[V])}(\mathbb{R}\times\mathcal{U}_G)$ as the union of the summands above, along the $W_G H$-orbits of V. Then, there is a $W_G H$-equivariant splitting $\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)^H \cong \coprod_{[V]\in\mathrm{Rep}_H/W_G H} \mathrm{Gr}_{(H,[V])}(\mathbb{R}\times\mathcal{U}_G)$.

Definition 2.1.25. Let θ be a G-equivariant tangential structure, $H \le G$ and $[V] \in \mathrm{Rep}_H/W_G H$ an isomorphism class of H-representations up to $W_G H$-conjugation. Define $B^{(H,V)}$ as the pullback

$$B^{(H,[V])} := \mathrm{Gr}_{(H,[V])}(\mathbb{R}\times\mathcal{U}_G) \times_{\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)^H} B^H$$

and $\theta^{(H,[V])}$ as

$$\theta^{(H,[V])} \colon B^{(H,[V])} \to B^H \to \mathrm{Gr}(\mathbb{R}\times\mathcal{U}_{W_G H}).$$

More generally, taking pullback along this inclusion induces a continous functor

$$F^{(H,V)} \colon \mathcal{C}^G_\theta \to \mathcal{C}^{W_G H}_{\theta^{(H,[V])}}$$

On morphisms, this functor maps (W, l_W) to $(W^{(H,[V])}, l_W^{(H,[V])})$ defined as the submanifold of of those points $x \in W^H$ such that $[T_x W] \in \mathrm{Rep}_H/W_G H$ is equal to $[V]$, with the structure described above.

2.1.3 Isotropy separation for cobordism categories

Let $\mathcal{F}$ be a family of subgroups of G.

Definition 2.1.26. Let $\mathcal{C}^G_{\theta,\mathcal{F}}$ be the subcategory of $\mathcal{C}^G_\theta$ on those objects and morphisms consisting of manifolds the points of which have isotropy contained in $\mathcal{F}$.

This is not a full subcategory, but is an inclusion of path components on objects as well as on morphisms.

Remark 2.1.27. Given an equivariant tangential structure $\theta\colon B \to \mathrm{Gr}(\mathcal{U}_G)$, consider the product structure

$$\theta \times E\mathcal{F}\colon B \times E\mathcal{F} \xrightarrow{\mathrm{pr}} B \xrightarrow{\theta} \mathrm{Gr}(\mathcal{U}_G)$$

Recall that the space of G-maps from a G-space X to $E\mathcal{F}$ is either contractible if all points of X have isotropy contained in $\mathcal{F}$, or empty. This implies the existence of a canonical dotted arrow

$$\begin{array}{ccc} \mathcal{C}^G_{\theta\times E\mathcal{F}} & \xrightarrow{\mathrm{pr}} & \mathcal{C}^G_{\theta} \\ \downarrow & \nearrow & \\ \mathcal{C}^G_{\theta,\mathcal{F}} & & \end{array}$$

which is a weak equivalence on objects and on morphisms. Hence $\mathcal{C}^G_{\theta,\mathcal{F}}$ is also (equivalent to) a cobordism category of G-manifold with some equivariant tangential structure, namely $\theta \times E\mathcal{F}$.

Let $H \in \mathcal{F}$ which is maximal for the inclusion, and denote $\mathcal{F} - (H)$ for the family F where the conjugacy class (H) of H has been taken out.

By maximality of H, the functor F^H factors as

$$\begin{array}{ccc} \mathcal{C}^G_{\theta,\mathcal{F}} & \xrightarrow{F^H} & \mathcal{C}^{W_G H}_{\theta^H,\mathrm{free}} \\ \downarrow & & \downarrow \\ \mathcal{C}^G_{\theta} & \xrightarrow{F^H} & \mathcal{C}^{W_G H}_{\theta^H} \end{array}$$

where $\mathcal{C}^{W_G H}_{\theta^H,\mathrm{free}}$ is a notation for $\mathcal{C}^{W_G H}_{\theta^H,\{\{e\}\}}$.

Theorem A. *The functors*

$$\mathcal{C}^G_{\theta,\mathcal{F}-(H)} \xrightarrow{i} \mathcal{C}^G_{\theta,\mathcal{F}} \xrightarrow{F^H} \mathcal{C}^{W_G H}_{\theta^H,\mathrm{free}}$$

induce a homotopy fibre sequence based at $\emptyset$

$$B\mathcal{C}^G_{\theta,\mathcal{F}-(H)} \xrightarrow{i} B\mathcal{C}^G_{\theta,\mathcal{F}} \xrightarrow{F^H} B\mathcal{C}^{W_G H}_{\theta^H,\mathrm{free}}$$

We call this result *isotropy seperation sequence for equivariant cobordism categories*, by analogy to the corresponding sequence of spectra which we describe in section 2.2.1. Our ideas and proofs are strongly inspired from [Ste21]. The latter describes conditions under which a certain strict pullback of categories induces a homotopy pullback of classifying spaces, and applies it to give a new proof of the classical Genauer sequence of [Gen11]. We shall adapt his proof strategy in our equivariant setting, and then argue how to see the Genauer sequence as a particular case of Theorem A.

A corollary Given θ a d-dimensional G-equivariant tangential structure, we define in section 2.2.3 a non-equivariant structure θ_{hG} called *homotopy orbit structure*, which is informally described as

$$\theta_{hG}\colon B_{hG} \to (B_G O(d))_{hG} \simeq BG \times BO(d) \to BO(d)$$

where the first map is induced on homotopy orbits from θ, and where the equivalence holds because $B_G O(d)$ is Borel equivalent to $BO(d)$ with trivial action. The precise definition of θ_{hG} allows us to make the association $(M, l_M) \mapsto (M/G, (l_M)_{hG})$ functorial on the free G-manifold M with θ-structure.

Proposition 2.1.28. *Taking quotient for the action of G induces a functor*

$$\mathcal{C}^G_{\theta,\mathrm{free}} \xrightarrow{\simeq} \mathcal{C}_{\theta_{hG}}$$

which is an equivalence of objects and on morphisms.

Corollary 2.1.29. *The fibre sequence of Theorem A can be rewritten as*

$$B\mathcal{C}^G_{\theta,\mathcal{F}-(H)} \xrightarrow{i} B\mathcal{C}^G_{\theta,\mathcal{F}} \xrightarrow{F^H} B\mathcal{C}_{\theta^H_{hW_G H}}$$

This corollary makes it possible to decompose $B\mathcal{C}^G_{\theta,\mathcal{F}}$ into two simpler pieces: the base is a cobordism category of non-equivariant manifolds, and the fibre is a cobordism category of manifolds having smaller isotropy. This allows proofs by induction, in particular, we give a new approach to understanding the following main result [GS21].

Theorem ([GS21])**.** *There is an equivalence*

$$B\mathcal{C}^G_{\theta} \xrightarrow{\simeq} \Omega^{\infty-1}(MT\theta)^G$$

Our proof uses the description of the homotopy type of the non-equivariant cobordism category in [GMT+09], which serves as base case for the induction as well as a key ingredient in the induction step. In turn, this proof is very different from the original one in [GS21], where the authors develop some equivariant delooping machinery to adapt the non-equivariant proof of [GR10]. In that sense, our proof may be considered as more elementary, although we do not get results for the homotopy type of the cobordism category of manifolds embedded in a finite dimensional representation like the authors in [GS21] do.

Remark 2.1.30. Theorem A can be slightly generalised, considering the subfunctor $F^{(H,[V])}$ of F^H, for $[V] \in \mathrm{Rep}_H/W_G H$. We will not need this fact.

2.1.4 Moduli spaces of equivariant manifolds

Let W be a G-manifold and $H \leq G$ be a maximal in the family $\mathcal{F}_W := \{H \leq G \mid W^H \neq \emptyset\}$. The residual action of $W_G H$ on $W^{(H,[V])}$

is free, for all $[V] \in \mathrm{Rep}_H/W_GH$. If W is equipped with a θ-structure l_W, then the quotient $W^{(H,[V])}/W_GH$ comes with a $\theta^{(H,V)}_{hW_GH}$-structure $\left(l_W^{(H,V)}\right)_{hW_GH}$ as in Proposition 2.1.28.

For $K > H$, let $o(W^{(K)})$ be the image of a G-equivariant tubular neighbourhood of $\cup_{g\in G} W^{gKg^{-1}}$ inside W^H, and define

$$W_{(H),\partial} = W - \bigcup_{K>H} o(W^{(K)})$$

After rounding corners, we get a G-manifold with boundary $W_{(H),\partial} \subseteq W_{(H)}$ such that the inclusion is an isotopy equivalence where $W_{(H)}$ is the subspace of W of those points having pure isotropy H. As before, there is a W_GH-splitting

$$W_{(H),\partial} = \bigsqcup_{[V]\in \mathrm{Rep}_H/W_GH} W_{(H,[V]),\partial}$$

where $W_{(H,[V]),\partial}$ is a subspace of $W^{(H,[V])}$.

Definition 2.1.31. For $(H,[V])$ as above, we define

$$W_{(H,[V]),hW_GH} := W_{(H,[V]),\partial}/W_GH$$

This is a compact manifold with boundary, and comes with a $\theta^{(H,[V])}_{hW_GH}$-structure $(l_W)^{(H,[V])}_{hW_GH}$. We refer to this collection of compact manifolds with tangential structure as the *building blocks* associated to (W, l_W).

The homology of equivariant moduli spaces Let θ be an non-equivariant tangential structure, and let (W, l_W) be a nullbordism in $\mathcal{C}_\theta$, i.e. $(W, l_W) \in \mathcal{N}_\theta(P, \hat{l}_P)$ for $(P, \hat{l}_P)$ its boundary. The so-called *scanning map* is the composition

$$s\colon \mathcal{M}_{\partial,\theta}(W, l_W) \subseteq \mathcal{C}_\theta((P, \hat{l}_P), \emptyset) \to \Omega_{[(P,\hat{l}_P),\emptyset]} B\mathcal{C}_{d,\theta} \xrightarrow{\simeq} \Omega_{[(P,\hat{l}_P),\emptyset]} \Omega^{\infty-1} MT\theta$$

where on the right is the space of paths from $(P, \hat{l}_P)$ to $\emptyset$ inside the classifying space of $B\mathcal{C}_\theta$, equivalent to $\Omega^{\infty-1}MT\theta$ by Corollary 2.1.19 applied to $G = *$. In many situations, this scanning map is r-acyclic for some interesting r, in the following sense.

Definition 2.1.32. Let $f\colon X \to Y$ be a continuous map and $r \in \mathbb{N}$, we say that f is r-acyclic if for all local coefficient system $\mathcal{L}$ on Y, the map induced on homology $H_*(X; f^*\mathcal{L}) \to H_*(Y; \mathcal{L})$ is an isomorphism for $* < r$ and an epimorphism for $* = r$.

In cases of interest, r is a diverging function of the *θ-genus* of (W, l_W), which implies increasingly better understanding of the homology as this genus increases.

Note that although the literature deals with the case of a d-dimensional tangential structure θ, results for general θ follow because the map s splits as a product

$$s \simeq (s_d)_{d\geq 0} \colon {\prod}'_{d\geq 0} \mathcal{M}_{\partial,\theta}(W_d, l_{W_d}) \to \Omega_{[(P_d,\hat{l}_{P_d}),\emptyset]} {\prod}'_{d\geq 0} \Omega^{\infty-1} MT\theta_d$$

with θ of dimension d, hence everything takes places dimension-wise.

Supposing W is a non-equivariant manifold of a fixed dimension, r-acyclicity type theorems include

(i) The case where W is of dimension $2n \geq 6$, is simply connected, and $l_W \colon W \to B$ is n-connected, by [GR17b, Corollary 1.8]. In this case the range is given by
$$3r(W, l_W) = g^\theta(W, l_W) - 1$$
where the θ-genus of $g^\theta(W, l_W)$ is defined as the maximal number of ways of embedding disjoint copies of $W_{1,1} = D^{2n}\#S^n \times S^n$ with an *admissible* θ-structure.

(ii) The case where W is a connected manifold of dimension $2n \geq 6$ and, more generally, $\pi_1(W)$ is *virtually polycylic*, by [Fri17, Theorem 4.12], together with the proof of [GR17a, Corollary 1.8]. In this case the range is given by
$$3r(W, l_W) = g^\theta(W, l_W) - h(\pi_1(W)) - 3$$
where $g^\theta(W, l_W)$ is defined as above and $h(\pi_1(W))$ is the *Hirsch length* of the fundamental group of W.

(iii) The case where W is of dimension 0 and B the base space of θ is path-connected, by [Kra19, Theorem D]. In this case, the range is given by
$$3r(W, l_W) = n$$
where n is the cardinality of W.

(iv) The case where W is of dimension 2, under some hypothesis on θ, and in the case where $\pi_1(MT\theta) = *$ (so that all coefficient systems are constant). The homological stability statement (with constant coefficients only) is proved in [Ran16, Theorems 7.1 and 7.2], and the range is a complicated function depending on θ, but not on l_W.

Let θ be an equivariant tangential structure. We make a last definition to ease the statement of our main result.

Definition 2.1.33. Let (W, l_W) be a compact G-manifold with θ-structure, and let $r \in \mathbb{N}$. We say that (W, l_W) is r-stable if the scanning map

$$s \colon \mathcal{M}^G_{\partial,\theta}(W, l_W) \to \Omega_{[(P,\hat{l}_P),\emptyset]} \Omega^{\infty-1}(MT\theta)^G$$

is r-acyclic onto the path component that it hits.

Theorem B. *Let (W, l_W) be a compact G-manifold with θ-structure. Suppose that in the collection of building blocks associated to (W, l_W) indexed by $(H, [V])$ where $H \leq G$ and $[V] \in \mathrm{Rep}_H/W_G H$, all elements are respectively $r_{(H,[V])}$-stable.*

Then, (W, l_W) is $r_{(W,l_W)}$-stable where

$$r_{(W,l_W)} := \min_{(H,V)} r_{(H,[V])}$$

where the minimum is taken over the $(H, [V])$ such that $W_{(H,[V]),\partial} \neq \emptyset$.

2.2 Theorem A and a corollary

As explained in the previous section, our Theorem A is very similar to [Ste21, Theorem 4.1], which proves a homotopy fibre sequence for cobordism categories with corners, the Genauer sequence. The original proof of this fibre sequence in [Gen11] was done by realising the homotopy types of the categories in play as loop spaces of some spectra, and by remaking that the spectra themselves lie in a fibre sequence. However, the extra knowledge of their homotopy type is not needed to derive the fibre sequence, rather, the general additivity result [Ste21, Theorem 2.3] can be applied to give a direct proof, which is what Steimle did in op. cit.

We first give a sketch of proof of Theorem A using the stable homotopy theoretic argument, and then focus on our proof of interest following the work of Steimle.

2.2.1 Isotropy separation for spectra

In this section we explain the link between isotropy separation for cobordism categories as in Theorem A, and that of genuine G-spectra. This leads to a first (sketch of) proof of Theorem A. Some extra care is needed to verify the commutativity of certain diagrams, we have made the choice of not writing the full details as we will give another proof in the next section. This section is independent from the next ones and can be skipped.

Let $\mathcal{F}$ be a family of subgroups of a finite group G, and let $H \leq G$. For X a genuine G-spectrum, X^H canonically maps to the geometric fixed points $\Phi^H(X)$, and Φ^H is symmetric monoidal. Moreover, H is normal in $N_G H$ so that there is a natural equivalence $X^{N_G H} \simeq (X^H)^{W_G H}$. Suppose that H is maximal in $\mathcal{F}$ and consider the composition $\Psi_H^{W_G H}$ given by

$$\begin{aligned}
(X \otimes E\mathcal{F}_+)^G &\to (X \otimes E\mathcal{F}_+)^{N_G H} \\
&\xrightarrow{\simeq} ((X \otimes E\mathcal{F}_+)^H)^{W_G H} \\
&\to \Phi^H(X \otimes E\mathcal{F}_+)^{W_G H} \\
&\xrightarrow{\simeq} (\Phi^H(X) \otimes (E\mathcal{F}_+)^H)^{W_G H} \\
&\xrightarrow{\simeq} (\Phi^H(X) \otimes EW_G H_+)^{W_G H}
\end{aligned}$$

Lemma 2.2.1. *For $H \leq G$ maximal in $\mathcal{F}$, there is a fibre sequence*

$$(X \otimes E(\mathcal{F} - (H))_+)^G \to (X \otimes E\mathcal{F}_+)^G \to (\Phi^H(X) \otimes EW_GH_+)^{W_GH}$$

*for every genuine G-spectrum X, where the first map is induced by the canonical G-map $E(\mathcal{F} - (H)) \to E\mathcal{F}$, and the second map is the one described above. We refer to it as the $(\mathcal{F}, (H))$-*isotropy separation sequence*.*

Remark that in the case where $\mathcal{F}$ is the family of all (closed) subgroups of G and $H = G$, the sequence above takes the form

$$(X \otimes E\mathcal{P})^G \to X^G \to \Phi^G(X)$$

which is the classical isotropy separation sequence for G-spectra.

Proof. The maps in play commute with colimits in the variable X, so we are left with the case where $X = \Sigma^\infty_+ G/K$, where $(K) \leq (H)$, which follows from a comparison of the tom Dieck splittings. □

By [GS21], the vertical arrows in the following commutative square (in Ho(Top)) are equivalences

$$\begin{array}{ccc} B\mathcal{C}^G_{\theta \times E(\mathcal{F}-(H))} & \xrightarrow{Bi} & B\mathcal{C}^G_{\theta \times E\mathcal{F}} \\ \downarrow \simeq & & \downarrow \simeq \\ \Omega^{\infty-1}(MT(\theta \times E(\mathcal{F} - (H))))^G & \longrightarrow & \Omega^{\infty-1}(MT(\theta \times E\mathcal{F}))^G \end{array}$$

Also, there is an equivalence

$$B\mathcal{C}^{W_GH}_{\theta^H \times EW_GH} \xrightarrow{\simeq} \Omega^{\infty-1}(MT(\theta^H \times EW_GH))^{W_GH}$$

The equivalence of W_GH-spectra $\Phi^H(MT\theta) \xrightarrow{\simeq} MT(\theta^H)$ composed with $\Psi^{W_GH}_H$ gives a map

$$MT(\theta \times E\mathcal{F})^G \to MT(\theta^H \times EW_GH)^{W_GH}$$

and it can be verified that the following diagram commutes up to homotopy

$$\begin{array}{ccc} B\mathcal{C}^G_{\theta \times E\mathcal{F}} & \xrightarrow{BF^H} & B\mathcal{C}^{W_GH}_{\theta^H \times EW_GH} \\ \downarrow \simeq & & \downarrow \simeq \\ \Omega^{\infty-1}MT(\theta \times E\mathcal{F})^G & \longrightarrow & \Omega^{\infty-1}MT(\theta^H \times EW_GH)^{W_GH} \end{array}$$

Sketch of a first proof of Theorem A. Consider the (homotopy) commutative diagram

$$\begin{array}{ccccc} B\mathcal{C}^G_{\theta,\mathcal{F}-(H)} & \longrightarrow & B\mathcal{C}^G_{\theta,\mathcal{F}} & \longrightarrow & B\mathcal{C}^{W_GH}_{\theta^H,\text{free}} \\ \uparrow & & \uparrow & & \uparrow \\ B\mathcal{C}^G_{\theta \times E(\mathcal{F}-(H))} & \longrightarrow & B\mathcal{C}^G_{\theta \times E\mathcal{F}} & \longrightarrow & B\mathcal{C}^{W_GH}_{\theta^H \times EW_GH} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^{\infty-1}(MT(\theta \times E(\mathcal{F} - (H))))^G & \longrightarrow & \Omega^{\infty-1}(MT(\theta \times E\mathcal{F}))^G & \longrightarrow & \Omega^{\infty-1}(MT(\theta^H \times EW_GH)^{W_GH} \end{array}$$

The bottom row is obtained by applying the functor $\Omega^{\infty-1}$ to the maps of spectra

$$MT(\theta \times E(\mathcal{F} - (H)))^G \to MT(\theta \times E\mathcal{F})^G \to MT(\theta^H \times EW_G H)^{W_G H}$$

Moreover there is an equivalence of genuine G-spectra $MT(\vartheta \times A) \simeq MT\vartheta \otimes A_+$ for all equivariant tangential structure ϑ and G-space A, hence this sequence is equivalent to the $(\mathcal{F}, (H))$-isotropy separation sequence applied to $MT\theta$. We conclude that bottom row of the diagram above is a homotopy fibre sequence. The map from the middle row to the top row is an equivalence as in Lemma 2.1.27, which concludes the proof. □

2.2.2 The main proof of Theorem A

This section is the heart of this paper, where we prove in particular that the functor $F^H : \mathcal{C}^G_\theta \to \mathcal{C}^{W_G H}_{\theta^H}$ is a cocartesian fibration. This will be the main input in the proof of both theorems A and B. We first recall some definitions as well as the main result of [Ste21]. We then prove our Theorem A and use it to give a new view on the main result of [GS21].

2.2.2.1 A plan of proof

We first recall the setting of [Ste21] and state their Theorem 2.3.

Definition 2.2.2. Let $\mathcal{C}$ and $\mathcal{D}$ be non-unital topological categories. $\mathcal{C}$ is *weakly unital* if all $x \in \mathrm{ob}(\mathcal{C})$ admits a *weak unit* which is a morphism $u \in \mathcal{C}(x, x)$ such that u and $u \circ u$ lie in the same path-component. A continuous functor $F: \mathcal{C} \to \mathcal{D}$ is

- a level fibration if the map induced on nerves is a level-wise Serre fibration,
- a local fibration if both maps

$$\begin{array}{rcl} (F, s, t) & : & \mathrm{mor}(\mathcal{C}) \to \mathrm{mor}(\mathcal{D}) \times_{\mathrm{ob}(\mathcal{D})^2} \mathrm{ob}(\mathcal{C})^2 \\ F & : & \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D}) \end{array}$$

  are Serre fibrations,
- a cocartesian fibration if for all $g: d \to d' \in \mathrm{mor}(\mathcal{D})$ and $c \in \mathrm{ob}(\mathcal{C})$ such that $F(c) = d$, there exists $f: c \to c' \in \mathrm{mor}(\mathcal{C})$ such that
  - $F(f) = g$,
  - f is F-cocartesian in the sense that for all $t \in \mathrm{ob}(\mathcal{C})$ the square

$$\begin{array}{ccc} \mathcal{C}(c, t) & \xrightarrow{-\circ f} & \mathcal{C}(c', t) \\ \downarrow F & & \downarrow F \\ \mathcal{D}(d, F(t)) & \xrightarrow{-\circ g} & \mathcal{D}(d', F(t)) \end{array}$$

  is homotopy cartesian.

A non-unital topological category $\mathcal{C}$ is called locally fibrant if the unique functor $\mathcal{C} \to *$ is a local fibration, i.e. if $\mathrm{mor}(\mathcal{C}) \xrightarrow{(s,t)} \mathrm{ob}(\mathcal{C})^2$ is a Serre fibration.

- a cartesian fibration if F^{op} is a cocartesian fibration,
- a bicartesian fibration if it is both a cartesian and a cocartesian fibration.

If both categories are in addition weakly unital, F is *weakly unital* if it sends weak units to weak units.

Remark 2.2.3. Several definitions of weakly unital topological categories exist, as well as several characterisations of weak units. They are proven to coincide in [Har15], a summary is given in [Ste22, Fact 4.8].

We will use the following result.

Theorem. *([Ste21, Theorem 2.3]) Let $P\colon \mathcal{C} \to \mathcal{D}$ and $I\colon \mathcal{D}' \to \mathcal{D}$ be two continuous weakly unital functors between locally fibrant weakly unital topological categories. Suppose that P is a level fibration and bicartesian fibration. Then taking classifying space transforms the pullback square*

$$\begin{array}{ccc} \mathcal{D}' \times_{\mathcal{D}} \mathcal{C} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow P \\ \mathcal{D}' & \xrightarrow{I} & \mathcal{D} \end{array}$$

into a homotopy pullback square.

Let G be a finite group, $\mathcal{F}$ be a family of subgroups of G and let $H \in \mathcal{F}$ be maximal. Given θ be an equivariant tangential structure, Theorem A states that there is a homotopy fibre sequence

$$B\mathcal{C}^G_{\theta,\mathcal{F}-(H)} \to B\mathcal{C}^G_{\theta,\mathcal{F}} \to B\mathcal{C}^{W_G H}_{\theta^H,\mathrm{free}}$$

based at $\emptyset$. By abuse of notation let us also note $\emptyset$ for the subcategory of $\mathcal{C}^{W_G H}_{\theta^H,\mathrm{free}}$ spanned by the empty manifold, with morphisms given by $(s, \emptyset)$, $s > 0$. Note that morphisms in $\emptyset$ are exactly the weak units of $\emptyset \in \mathrm{ob}(\mathcal{C}^{W_G H}_{\theta^H,\mathrm{free}})$. There is a strict pullback diagram

$$\begin{array}{ccc} \mathcal{C}^G_{\theta,\mathcal{F}-(H)} & \xhookrightarrow{I} & \mathcal{C}^G_{\theta,\mathcal{F}} \\ \downarrow & & \downarrow F^H \\ \emptyset & \hookrightarrow & \mathcal{C}^{W_G H}_{\theta^H,\mathrm{free}} \end{array}$$

All the non-unital topological categories in play are weakly unital, and locally fibrant as a consequence of equivariant isotopy extension. The functors I and F^H are also continous and weakly unital.

Supposing we can show that F^H is a level and bicartesian fibration, Steimle's theorem applies and immediatly gives a proof of Theorem A, as $B(\emptyset \hookrightarrow \mathcal{C}^{W_G H}_{\theta^H,\mathrm{free}})$ is homotopic to the inclusion of the point $\{\emptyset\} \to B\mathcal{C}^{W_G H}_{\theta^H,\mathrm{free}}$.

Although F^H turns out not to be a level fibration, we shall define other categories (for all $(\theta, \mathcal{F}, H)$ where H is maximal in $\mathcal{F}$)

$$\begin{array}{ccc}
\mathcal{C}^{G,\square_H}_{\theta,\mathcal{F}} & \xrightarrow[\hookrightarrow]{\simeq} & \mathcal{C}^{G}_{\theta,\mathcal{F}} \\
\downarrow{\scriptstyle F^H_\square} & & \downarrow{\scriptstyle F^H} \\
\mathcal{C}^{W_G H}_{\theta^G,sm} & \xrightarrow[\hookrightarrow]{\simeq} & \mathcal{C}^{W_G H}_{\theta^H}
\end{array}$$

such that

- the horizontal inclusions are equivalences on objects and morphisms,
- the restriction $F^H_\square$ is a local fibration – in particular a level fibration,
- the map induced after taking pullback along the inclusion of $\emptyset$ is equal to the inclusion of categories $\mathcal{C}^{G,\square_H}_{\theta,\mathcal{F}-(H)} \hookrightarrow \mathcal{C}^{G}_{\theta,\mathcal{F}-(H)}$

Secondly, we show that the functor $F^H_\square$ is a bicartesian fibration (hence so is F^H). Applying Theorem 2.2.2.1 to $F^H_\square$ then yields a proof of Theorem A.

2.2.2.2 Technical preparations

In this paragraph we define the categories $\mathcal{C}^{G,\square_H}_{\theta,\mathcal{F}}$ and $\mathcal{C}^{W_G H}_{\theta^G,sm}$ and show that the induced functor $F^H_\square$ is a local fibration.

Lemma 2.2.4. *Let W be a G-manifold embedded inside $\mathbb{R} \times \mathcal{U}_G$ and $H \leq G$, then W intersects $\mathbb{R} \times \mathcal{U}_G^H$ in an orthogonal way in the following sense: for $x \in W^H$, there is a splitting*

$$T_x W = T_x(W^H) \oplus [T_x W \cap (\mathcal{U}_G^{H,\perp})]$$

where $\mathcal{U}_G^{H,\perp}$ is the orthogonal complement of $\mathcal{U}_G^H$ inside $\mathcal{U}_G$. This identifies $TW_{|W^H} \cap (W^H \times \mathcal{U}_G^{H,\perp})$ with $N_W W^H$, the normal bundle of W^H in W.

Proof. We wish to show that the orthogonal complement N of $T_x(W^H)$ in $T_x W$ is contained in $\mathcal{U}_G^{H,\perp}$. The orthogonal projection $p\colon N \to \mathcal{U}_G^H$ is equivariant, and N has no trivial summand, so p is null, e.g. by the Schur lemma. □

Corollary 2.2.5. *Let W is a G-manifold embedded inside $\mathbb{R} \times \mathcal{U}_G$ and $H \leq G$, then the normal bundle $N_W W^H$ is naturally equivariantly embedded inside $W^H \times (\mathcal{U}_G^{H,\perp}) \subseteq \mathbb{R} \times \mathcal{U}_G$.*

When W is given as a G-submanifold of $\mathbb{R} \times \mathcal{U}_G$ we shall then see $N_W W^H$ as an embedded (non compact, with non compact boundary) G-submanifold $\mathbb{R} \times \mathcal{U}_G$.

Definition 2.2.6. Let M be a G-submanifold of $\mathbb{R}\times\mathcal{U}_G$ with θ-bundle $f\colon M\to B$, and $\delta>0$. Say that (M,f) has *δ-neat H-fixed points* if it satisfies the condition

$$M\cap[(\mathbb{R}\times\mathcal{U}_G^H)\times D_\delta(\mathcal{U}_G^{H,\perp})]=D_\delta(N_M M^H)$$

as G-manifolds with θ-bundle, where the θ-bundle structure on $D_\delta(N_M M^H)$ is given by the composition

$$D_\delta(N_M M^H)\xrightarrow{\text{pr}}M^H\xrightarrow{f^H}B^H\to B$$

Let $\mathcal{C}_{\theta,\varepsilon}^{G,\square_H^\delta}$ be the non-full subcategory of $\mathcal{C}_{\theta,\varepsilon}^G$ of those objects and morphisms which have δ-neat H-fixed points, and define

$$\mathcal{C}_{\theta,\varepsilon}^{G,\square_H}:=\operatorname*{colim}_{\delta>0}\mathcal{C}_{\theta,\varepsilon}^{G,\square_H^\delta};\quad \mathcal{C}_{\theta}^{G,\square_H}:=\operatorname*{colim}_{\varepsilon>0}\mathcal{C}_{\theta,\varepsilon}^{G,\square_H}.$$

Remark 2.2.7. Despite the notion of H-neatness being specific to the subgroup H, note that it is equivalent to require it for all conjugates of H, so that the condition is really a property of a neighbourhood of $G\times_{N_GH}W^H$ inside $\mathbb{R}\times\mathcal{U}_G$.

The category $\mathcal{C}_{\theta,\varepsilon}^{G,\square_H^\delta}$ is locally fibrant, which is proven in an analogous way as for $\mathcal{C}_\theta^G$, hence so are its colimits over $\delta>0$ resp. $\varepsilon>0$.

Lemma 2.2.8. *The natural functor $I\colon\mathcal{C}_{\theta,\varepsilon}^{G,\square_H^\delta}\subseteq\mathcal{C}_\theta^G$ is an equivalence on objects and on morphisms. Hence the functors induced after taking colimits over $\delta>0$ or $\varepsilon>0$ are as well.*

Proof. We prove it for morphisms, the argument is analogous for objects. Let (W,l_W,s) be a morphism in $\mathcal{C}_{\theta,\varepsilon}^{G,\square_H^\delta}$, we show that this functor is an equivalence after restriction to $\mathcal{C}_{\theta,\varepsilon}^{G,\square_H^\delta}(W,l_W)$ (note that it is injective on π_0). Consider first the case where there is no tangential structure, and define $\mathrm{Emb}_{\partial,\varepsilon}^{G,\square_H^\delta}(W,[0,-]\times\mathcal{U}_G)$ as the subspace of $\mathrm{Emb}_\partial^G(W,[0,1]\times\mathcal{U}_G)\times(0,+\infty)$ consisting of those (f,s) such that the image of f has δ-neat H-fixed points, and ε/s-collared boundary. The functor I factors as $\mathcal{C}_{\theta,\varepsilon}^{G,\square_H^\delta}\to\mathcal{C}_{\theta,\varepsilon}^G\to\mathcal{C}_\theta^G$ and it is clear that the rightmost map is an equivalence, as the natural functor $\mathcal{C}_{\theta,\varepsilon}^G\to\mathcal{C}_{\theta,\varepsilon'}^G$ is an equivalence for $\varepsilon>\varepsilon'$. On the other hand the leftmost map, restricted to $\mathcal{C}_{\theta,\varepsilon}^{G,\square_H^\delta}(W,l_W)$, is given up to homeomorphism by the $\mathrm{Diff}_\partial^G(W)$-orbits of the inclusion

$$\mathrm{Emb}_{\partial,\varepsilon}^{G,\square_H^\delta}(W,[0,-]\times\mathcal{U}_G)\hookrightarrow\mathrm{Emb}_{\partial,\varepsilon}^G(W,[0,-]\times\mathcal{U}_G)\tag{2.2}$$

where on the right the case $\delta=0$ is omitted from the notation. Consider a square

$$\begin{array}{ccc}S^{r-1}&\xrightarrow{\beta}&\mathrm{Emb}_{\partial,\varepsilon}^{G,\square_H^\delta}(W,[0,-]\times\mathcal{U}_G)\\ \downarrow&&\downarrow\\ D^r&\xrightarrow{\alpha}&\mathrm{Emb}_{\partial,\varepsilon}^G(W,[0,-]\times\mathcal{U}_G)\end{array}$$

We shall deform this diagram by a continuous path of squares, to one admitting a lift. We construct this path in two steps. First, consider $\chi\colon [0,+\infty) \to [0,1]$ be a smooth function with $\chi \equiv 1$ near 0, supported on $[0,1]$, and let $\eta\colon N_W W^H \hookrightarrow W$ be a tubular neighbourhood of W^H inside W. By compactness of $[0,1]\times D^r$ and the openness of embeddings for the strong Whitney topology, there exists $a>0$ such that for all $z \in D^r$,

$$t \mapsto (1 - t\chi(\| - \|/a))\alpha_z \circ \eta + t\chi(\| - \|/a)T_0(\alpha_z \circ \eta)$$

is a path of embeddings $N_W W^H \to [0,1]\times \mathcal{U}_G$, where $T_0(\alpha_z\circ\eta)\colon N_W W^H \to [0,1]\times\mathcal{U}_G$ is the map $(w,v)\mapsto T_{w,0}\alpha_z(0,v)$. Close to the zero-section $W^H \hookrightarrow N_W W^H$, these embeddings are linear embeddings of $N_W W^H$ hence their images have δ'-neat H-fixed points for some $\delta' > 0$. Far from the zero-section, they coincide with α_z, hence they extend to a path of embeddings $W \to [0,1]\times \mathcal{U}_G$. This gives a first homotopy of squares, to one where α admits a lift to $\mathrm{Emb}_{\partial,\varepsilon}^{G,\square_H^{\delta'}}(W,[0,-]\times\mathcal{U}_G)$. Continue this homotopy with a radial rescaling of the subspace $\mathcal{U}_G^{H,\perp}$ until all images of W have δ-neat H-fixed points, and we have proved that the inclusion (2.2) is a weak equivalence in the case without tangential structures.

Because θ is a G-fibration, this inclusion is also an equivalence after taking pullack with

$$\mathrm{Map}_\partial^G(W,\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)) \xrightarrow{\theta_*} \mathrm{Map}_\partial^G(W,B)$$

The space $\mathcal{C}_{\theta,\varepsilon}^{G,\square_H^\delta}(W,l_W)$ is a quotient of $\mathrm{Emb}_{\partial,\theta,\varepsilon}^{G,\square_H^\delta}((W,l_W),[0,-]\times\mathcal{U}_G)$ defined as the subspace of the pullback

$$\mathrm{Emb}_{\partial,\varepsilon}^{G,\square_H^\delta}(W,[0,-]\times\mathcal{U}_G)\times_{\mathrm{Map}_\partial^G(W,\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G))}\mathrm{Map}_\partial^G(W,B)$$

consisting of those (ν,l_ν) such that $l_\nu\circ\nu^{-1}$ is fibrewise constant on $\nu(W)\cap[0,1]\times\mathcal{U}_G^H\times D_\delta(\mathcal{U}_G^{H,\perp})$ in W, and agreeing with $l_{\partial W}\circ\nu^{-1}$ on $\nu(W)\cap([0,\varepsilon/s]\cup[1-\varepsilon/s,1])\times\mathcal{U}_G$ in W. The inclusion of this subspace is an equivalence: the condition on the structure can be achieved by a continuous deformation with the same strategy as above. □

Suppose that $(M_H \subseteq \mathbb{R}\times\mathcal{U}_G^H, f_H)$ a W_GH-submanifold with equipped with a θ^H-bundle, which is an object (resp. a morphism) of $\mathcal{C}_{\theta^H}^{W_GH}$. Because it classifies the smooth W_GH-vector bundle $\varepsilon\oplus TM$ (resp. TM), the composition

$$M \xrightarrow{f_H} B^H \xrightarrow{\theta^H} (\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G))^H \to \mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G^H)$$

is *smooth*, in the sense that if factors as a smooth map to $\sqcup_{d\in D}\mathrm{Gr}_d(V)$, for some finite $D\subseteq\mathbb{N}$ and some finite dimensional W_GH-subrepresentation V of $\mathbb{R}\times\mathcal{U}_G^H$, followed by the inclusion in $\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G^H)$. If we further suppose that (M_H,f_H) is equal to $F^H(M,f)$ for (M,f) some object (resp. morphism) of $\mathcal{C}_\theta^G$, then the shorter composition

$$M_H \xrightarrow{f_H} B^H \xrightarrow{\theta^H} (\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G))^H$$

is smooth because it classifies $\varepsilon \oplus TM_{|M_H}$ (resp. $TM_{|M_H}$).

Definition 2.2.9. Let $\mathcal{C}^{W_G H}_{\theta^H,sm}$ be the subcategory spanned those objects and morphisms (M, f) such that f is *smooth for* θ^H, meaning that the composition

$$M \xrightarrow{f} B^H \xrightarrow{\theta^H} (\mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G))^H$$

is smooth in the sense described above.

We will need the following smooth approximation tool.

Lemma 2.2.10. *Let K be a finite complex, M be a G-manifold, $p\colon E \to X$ be a smooth G-manifold bundle over the compact G-manifold X, and let $f\colon K \times M \to E$ be a G-map such that $p \circ f_k$ is smooth for all $k \in K$. Then, there exists an equivariant homotopy $H\colon [0,1] \times K \times M \to E$ over p such that $H_k(0,-) = f_k$ and $H_k(1,-)\colon M \to E$ is smooth for all k.*

Remark 2.2.11. The lemma above also has a relative version if we are given $A \subset M$ a closed subset on which the maps f_k are already smooth.

Proof. The manifold bundle p is isomorphic to a G-submanifold bundle of $X \times \mathcal{U}_G$ by the classification theorem for G-manifold bundles. By conjugating with such an isomorphism, we can suppose that p is already of this form. For all $x \in X$ there exists a small neighbourhood U_x of x over which the bundle p is trivial, which implies that p over U_x is given by a G-embedding $U_x \times E_x \to E \hookrightarrow X \times \mathcal{U}_G$ over X. Then there exists $\varepsilon_x > 0$ such that all points $(y, v) \in U_x \to \mathcal{U}_G$ such that $d(v, E_y) \leq \varepsilon$ have a unique projection on E_y. As X is compact, it is the union of a finite number of opens U_{x_i}, and we define $\varepsilon := \min_i \varepsilon_{x_i}$. Let U be the subspace of those $(x, v) \in X \times \mathcal{U}_G$ such that $d(v, E_x) < \varepsilon$. Then, the projection map $p\colon E \to X$ can be extended to U, and $p\colon U \to X$ is a sub fibre bundle of $X \times \mathcal{U}_G$ the fibres of which are open in $\mathcal{U}_G$. There exists an equivariant deformation of $f\colon K \times M \to E$ inside $X \times \mathcal{U}_G$ which at time 1 is smooth. By compactness of K we can take it small enough that it lands inside U. If H is a homotopy realising this deformation, then the composition $p \circ H$ satisfies the properties we were looking for. ☐

Lemma 2.2.12. *The inclusion $\mathcal{C}^{W_G H}_{\theta^H,sm} \subseteq \mathcal{C}^{W_G H}_{\theta^H}$ is an equivalence on objects and on morphisms.*

Proof. We show it for morphisms, the argument is analogous for objects. It is enough to show that given a compact $W_G H$-manifold W_H, the inclusion

$$\mathrm{Map}^{W_G H}_{\partial,sm}(W_H, B^H) \subseteq \mathrm{Map}^{W_G H}_{\partial}(W_H, B^H)$$

is an equivalence over $\mathrm{Map}^{W_G H}(W_H, \mathrm{Gr}(\mathbb{R} \times \mathcal{U}^H_G))$, where the first space is the space of smooth maps for θ^H. Let $f\colon K \to \mathrm{Map}^{W_G H}_{\partial}(W_H, B^H)$ where K is a connected finite complex.

As K is connected, $\theta \circ \bar{f}\colon K \times W_H \to \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G)^H$ lands in one path component of $\mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G)^H$, which is given by $\mathrm{Gr}_V(\mathbb{R} \times \mathcal{U}_G)$, the subspace of those sub H-representations of $\mathbb{R} \times \mathcal{U}_G$ that are isomorphic to V for some $V \in \mathrm{Rep}_H$. By compactness of $K \times W_H$, we can factor this map further through an inclusion $\mathrm{Gr}_V(V')$, for V' a large enough finite-dimensional sub-H-representation of $\mathbb{R} \times \mathcal{U}_G$ containing a copy of V. The map

$$(-)^H : \mathrm{Gr}_V(V') \to \mathrm{Gr}_{V^H}((V')^H)$$

is a $W_G H$-equivariant manifold bundle with typical fibre isomorphic to $\mathrm{Gr}_{V-V^H}(V' - (V')^H)$. Hence, the (relative version of the) lemma above applies and gives a $W_G H$-equivariant homotopy which makes the maps f_k smooth after composition with θ. The last step is to lift this homotopy to B^H, which is possible because θ^H is a $W_G H$-fibration. □

Let (M_H, f_H) be an object resp. a morphism of $\mathcal{C}^{W_G H}_{\theta^H, sm}$. To it we associate a vector bundle embedded in $\mathbb{R} \times \mathcal{U}_G$ as

$$N(M_H, f_H) := \{(x, v) \in M_H \times \mathcal{U}_G^{H,\perp} \mid v \in \mathrm{pr}_{H,\perp} \circ \theta^H \circ l_{M_H}(x)\}$$

where $\mathrm{pr}_{H,\perp}$ is the orthogonal projection on $\mathcal{U}_G^{H,\perp}$. Defined as such, it is an $N_G H$-submanifold of $\mathbb{R} \times \mathcal{U}_G$, as well as an $N_G H$-vector bundle over M^H.

Consider its translates over the cosets of $N_G H$ in G,

$$N_G(M_H, f_H) := G \times_{N_G H} N(M_H, f_H)$$

which is a G-vector bundle over $G \times_{N_G H} M_H$, G-equivariantly embedded inside $\mathbb{R} \times \mathcal{U}_G$. It comes with a θ-bundle structure $N(f_H)$ defined by

$$N(f_H) \colon N_G(M_H, f_H) \xrightarrow{\mathrm{pr}} G \times_{N_G H} M_H \xrightarrow{f_H} G \times_{N_G H} B^H \to B$$

We shall write $N_G^{<r}(M_H, f_H)$ resp. $N_G^{\leq r}(M_H, f_H)$ resp. $N_G^{=r}(M_H, f_H)$ for the subspace of those (x, v) such that $\|v\| < r$, $\|v\| \leq r$ and $\|v\| = r$ respectively.

Lemma 2.2.13. *The fixed point functor*

$$F^H : \mathcal{C}_\theta^{G, \square_H} \to \mathcal{C}^{W_G H}_{\theta^H, sm}$$

is a local fibration.

Proof. Let $(W, l_W, s) \colon (P, \hat{l}_P) \to (Q, \hat{l}_Q)$ be a morphism in $\mathcal{C}_\theta^{G, \square_H}$, we want to show that the maps

$$\begin{array}{ccccc}
 & \overset{F^H}{\curvearrowright} & X & \longrightarrow & \mathcal{C}_\theta^{G,\square_H}(P, \hat{l}_P) \times \mathcal{C}_\theta^{G,\square_H}(Q, \hat{l}_Q) \\
\mathcal{C}_\theta^{G,\square_H}(W, l_W) & & \downarrow & \lrcorner & \downarrow \\
 & & \mathcal{C}^{W_G H}_{\theta^H, sm}(W^H, l_W^H) & \longrightarrow & \mathcal{C}^{W_G H}_{\theta^H, sm}(P^H, \hat{l}_P^H) \times \mathcal{C}^{W_G H}_{\theta^H, sm}(Q^H, \hat{l}_Q^H)
\end{array}$$

and

$$F^H : \mathcal{C}_\theta^{G,\square_H}(P,\hat{l}_P) \to \mathcal{C}_{\theta^H,sm}^{W_G H}(P^H,\hat{l}_P^H)$$

are Serre fibrations, where X is a short name for the pullback. We show it for the first map, the second will follow by a subargument. Let $(f_t, g_t, h_t)\colon [0,1]^k \to \mathcal{C}_{\theta,sm}^{W_G H}(W^H, l_W^H) \times \mathcal{C}_\theta^{G,\square_H}(P,\hat{l}_P) \times \mathcal{C}_\theta^{G,\square_H}(Q,\hat{l}_Q)$ $(t \in [0,1])$ be a map valued in the pullback, and suppose that we are given a lift through F^H, $w_0\colon [0,1]^k \to \mathcal{C}_\theta^{G,\square_H}(W, l_W)$ at time 0. We wish to extend w_0 to a path w_t extending (f_t, g_t, h_t).

By compactness of $[0,1]\times[0,1]^k$, we can assume that the maps f_t, g_t and h_t are valued in the morphism spaces of the corresponding decorated cobordism categories, where boundaries are ε-collard and H-fixed points are ε-neat for small enough $\varepsilon > 0$. The first step will be to lift all of those maps to certain spaces of embeddings.

There is a principal fibre bundle

$$\mathrm{Diff}_\partial(W) \to \mathrm{Emb}_{\partial,\theta,\varepsilon}^{G,\square_H^\varepsilon}((W,l_W),[0,-]\times\mathcal{U}_G) \to \mathcal{C}_{\theta,\varepsilon}^{G,\square_H^\varepsilon}(W,l_W)$$

Because $[0,1]^k$ is contractible, we may consider a section

$$\bar{w}_0\colon [0,1]^k \to \mathrm{Emb}_{\partial,\theta,\varepsilon}^{G,\square_H^\varepsilon}((W,l_W),[0,-]\times\mathcal{U}_G)$$

over w_0. Analogous principal bundles exist for the spaces $\mathcal{C}_{\theta,sm,\varepsilon}^{W_G H}(W^H, l_W^H)$, $\mathcal{C}_{\theta,\varepsilon}^{G,\square_H^\varepsilon}(P,\hat{l}_P) \times \mathcal{C}_{\theta,\varepsilon}^{G,\square_H^\varepsilon}(Q,\hat{l}_Q)$ and $\mathcal{C}_{\theta^H,sm,\varepsilon}^{W_G H}(P^H,\hat{l}_P^H) \times \mathcal{C}_{\theta^H,sm,\varepsilon}^{W_G H}(Q^H,\hat{l}_Q^H)$, which in turns gives a fibration with base given by the pullback space X, and total space given by the appropriate pullback of spaces of embeddings. Using that, with the section $\tilde{w}_0$ as starting point, consider lifts

$$(\overline{f}_t,\overline{g}_t,\overline{h}_t)\colon [0,1]^k \to \begin{cases} \mathrm{Emb}_{\partial,\theta^H,sm,\varepsilon}^{W_G H}((W^H,l_W^H),[0,-]\times\mathcal{U}_G^H) \\ \mathrm{Emb}_{\theta,\varepsilon}^{G,\square_H^\varepsilon}((P,\hat{l}_P),\mathcal{U}_G) \\ \mathrm{Emb}_{\theta,\varepsilon}^{G,\square_H^\varepsilon}((Q,\hat{l}_Q),\mathcal{U}_G) \end{cases}$$

of (f_t, g_t, h_t) with values in the pullback. We now show, at the level of embeddings, that $(\overline{f}_t,\overline{g}_t,\overline{h}_t)$ admits a lift through F^H starting at $\bar{w}_0$. After a straight-line homotopy of the ambiant space and making ε smaller, we may assume that length of the bordisms given by the images of $\bar{f}_t$ are always equal to 1. By the inverse function theorem, we can find $\varepsilon' < \varepsilon$ and $c_P^{\varepsilon'}\colon [0,\varepsilon']\times P \hookrightarrow W$, $c_Q^{\varepsilon'}\colon [1-\varepsilon',1]\times Q \hookrightarrow W$ two collars, such that $w_0 \circ c_P$ and $w_0 \circ c_Q$ are given by $\overline{[0,\varepsilon']} \times g_0$ resp. $\overline{[1-\varepsilon',1]} \times h_0$. Let $\eta\colon N_W W^H \hookrightarrow W$ be an equivariant tubular neighbourhood of $G \times_{N_G H} W^H$ inside W. The embeddings $\bar{f}_t$ linearly extends to G-equivariant embeddings of $G \times_H D_{\varepsilon'}(N_W W^H)$ inside $[0,1]\times\mathcal{U}_G$. In turn, letting $A_{\varepsilon'}$ be the union of the images of $c_P^{\varepsilon'}$, $c_Q^{\varepsilon'}$ and η restricted to vectors of length $< \varepsilon'$, there is a preferred way of defining the wanted lift $\tilde{w}_t$ restricted to $A_{\varepsilon'}$. Note that the closure $\overline{A}_{\varepsilon'} \subseteq W$ is not a manifold itself, however it is an equivariant submanifold with corners. By perturbating it slightly we can find an equivariant submanifold with boundary $A \subseteq W$ to which it is isotopy equivalent.

The restriction map

$$\mathrm{Emb}^G_\partial(W, [0,1] \times \mathcal{U}_G) \to \mathrm{Emb}^G_\partial(A, [0,1] \times \mathcal{U}_G)$$

is a fibre bundle, where embeddings in the right side are relative $\partial W \subseteq \partial A$. The path lifting property for this bundle gives a lift of $(\overline{f}_t, \overline{g}_t, \overline{h}_t)$ as wanted, if we first let θ-structures aside. In order to also lift the tangential structures, it is enough to argue that a lift exists in the following diagram

$$\begin{array}{ccc} \{0\} \times [0,1]^k & \longrightarrow & \mathrm{Map}^G_\partial(W, B) \\ \downarrow & & \downarrow {\scriptstyle i^* \times \theta_*} \\ [0,1] \times [0,1]^k & \longrightarrow & \mathrm{Map}^G_\partial(A, B) \times_{\mathrm{Map}^G_\partial(A, \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G))} \mathrm{Map}^G_\partial(W, \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G)) \end{array}$$

The inclusion $i \colon A \to W$ is a G-cofibration, and $\theta \colon B \to \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G)$ is a G-fibration, so we can conclude by the pullback-power axiom. □

Remark 2.2.14. Let $\mathcal{C}^{G,\square_H}_{\theta,\mathcal{F}}$ be the subcategory of $\mathcal{C}^{G,\square_H}_{\theta}$ on those manifolds having isotropy in the family $\mathcal{F}$, and suppose $H \in \mathcal{F}$ is maximal. Then, F^H restricts to a local fibration $F^H \colon \mathcal{C}^{G,\square_H}_{\theta,\mathcal{F}} \to \mathcal{C}^{W_G H}_{\theta^H, sm, \mathrm{free}}$.

2.2.2.3 The equivariant Genauer sequence

In the proof of Theorem A, we shall use an equivariant version of [Ste21, Lemma 4.4]. Together with their general additivity result, this lemma is one of the main technical inputs in their geometric proof of the Genauer sequence. Although we don't need an equivariant Genauer sequence in our proof of Theorem A per se, we do state it anyway as it is arguably interesting on its own.

Cobordisms with corners We define equivariant cobordism categories where morphisms are manifolds that have corners of local form $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^k$ (after forgetting the action of G), and discuss why the Genauer sequence of [Ste21, Theorem 4.8]. holds in the equivariant setting.

Definition 2.2.15. Let $\theta \colon B \to \mathrm{Gr}(\mathbb{R}^2 \times \mathcal{U}_G)$ be an equivariant tangential structure for manifolds inside $\mathbb{R}^2 \times \mathcal{U}_G$. Letting $\iota_0, \iota_1 \colon \mathbb{R} \times \mathcal{U}_G \hookrightarrow \mathbb{R}^2 \times \mathcal{U}_G$ be the two canonical inclusions, we define the equivariant tangential structures $\iota_0^*\theta$ and $\iota_1^*\theta$ by taking pullback along ι_0 and ι_1.

For this paragraph we fix a choice of θ as above.

Definition 2.2.16. Let M an increasing union of G-submanifolds of $\mathcal{U}_G$, we define the *equivariant cobordism category of manifolds with corners* $\mathcal{C}^G_{\theta,\langle 2\rangle}(M)$ in a similar way to Definition 2.1.11. Let $\varepsilon, \delta > 0$, we first define a category $\mathcal{C}^{G,\delta}_{\theta,\varepsilon,\langle 2\rangle}(M)$ with objects given by

(i) a compact G-submanifold P of $[0,+\infty)\times M$ with boundary which is δ-neat in the sense that $\partial P = P\cap\{0\}\times M$ and

$$\partial P\times[0,\delta] = P\cap([0,\delta]\times M)$$

(ii) a G-map $\hat{l}_P\colon P\to B$ such that $\varepsilon\oplus\tau_P = \theta\circ\hat{l}_P$ (in other words a $\iota_0^*\theta$-structure) which is δ-neat in the sense that

$$(\hat{l}_P)_{|P\cap([0,\delta]\times M)} = \partial P\times[0,\delta]\xrightarrow{pr}\partial P\xrightarrow{\hat{l}_P} B$$

A morphism from $(P,\hat{l}_P)$ and $(Q,\hat{l}_Q)$ is given by

(i) an $s\in(2\varepsilon,+\infty)$,

(ii) a compact G-submanifold $W\subseteq[0,s]\times[0,+\infty)\times M$ with corners such that W is (ϵ,δ)-neat in the following sense. First we define

$$\partial_0 W := W\cap(\{0,s\}\times[0,+\infty)\times M),\quad \partial_1 W := W\cap([0,s]\times\{0\}\times M)$$

We require that

(a) W intersects $([0,\varepsilon]\times[0,+\infty)\times M)\cup((s-\varepsilon,s]\times[0,+\infty)\times M)$ in $([0,\varepsilon)\times P)\cup((s-\varepsilon,s]\times Q)$ (ε-neatness for ∂_0),

(b) $\partial_1 W\times[0,\delta] = W\cap([0,\varepsilon]\times[0,\delta]\times M)$,

(iii) an equivariant θ-structure $l_W\colon W\to B$ on W restricting to $\hat{l}_P$ on $\{0\}\times P$ and to $\hat{l}_Q$ on $\{s\}\times Q$, which is (ϵ,δ)-neat in the sense that

(a) the restriction of l_W to $W\cap(([0,\varepsilon]\cup[s-\varepsilon,s])\times[0,+\infty)\times M$ is given by

$$([0,\varepsilon)\times P)\cup((s-\varepsilon,s]\times Q)\xrightarrow{pr} P\cup Q\xrightarrow{\hat{l}_P\sqcup\hat{l}_Q} B$$

(b) the restriction of l_W to $W\cap(([0,s])\times[0,\delta]\times M$ is given by

$$\partial_1 W\times[0,\delta]\xrightarrow{pr}\partial_1 W\xrightarrow{l_W} B$$

We make $\mathcal{C}^{G,\delta}_{\theta,\varepsilon,\langle 2\rangle}(M)$ into a topological category in the same way as we did in Definition 2.1.11, and define

$$\mathcal{C}^{G}_{\theta,\langle 2\rangle}(M) := \operatorname{colim}_{\delta,\epsilon>0}\mathcal{C}^{G,\delta}_{\theta,\varepsilon,\langle 2\rangle}(M)$$

When (W,l_W) is a morphism in $\mathcal{C}^G_{\theta,\langle 2\rangle}$, $\partial_1 W$ inherits a $\iota_1^*\theta$-structure. Moreover, this association assembles into a continuous functor

$$\partial_1\colon\mathcal{C}^G_{\theta,\langle 2\rangle}(M)\to\mathcal{C}^G_{\iota_1^*\theta}(M)$$

We state the following lemma in full generality – namely for all tangential structures θ – although it will be convenient to first prove it in the case where θ is trivial, and postpone the rest of proof to the later Remark 2.2.20.

Lemma 2.2.17. $\partial_1 \colon \mathcal{C}^G_{\theta,\langle 2\rangle}(M) \to \mathcal{C}^G_{\iota_1^*\theta}(M)$ *is a cocartesian fibration.*

Proof (in the absence of tangential structures). In this simpler case, the proof is almost verbatim the same as [Ste21, Lemma 4.4]. The homeomorphism $\mathbf{B}\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0}$ constructed in op. cit. induces an equivariant homeomorphism denoted by the same letter

$$\mathbf{B}\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0} \times M \to \mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0} \times M$$

by applying the identify map on M. With this modification, the proof of [Ste21, Lemma 4.4]. still is still valid and the present lemma follows. □

Note that the embedded cobordism category $\mathcal{C}^G_\theta(\mathbb{R}_{>0} \times M)$ can be identified with the strict fibre of ∂_1 based at $\emptyset$.

Corollary 2.2.18. *The functors*

$$\mathcal{C}^G_\theta(\mathbb{R}_{>0} \times M) \hookrightarrow \mathcal{C}^G_{\theta,\langle 2\rangle}(M) \xrightarrow{\partial_1} \mathcal{C}^G_{\iota_1^*\theta}(M)$$

induce a homotopy fibre sequence based at $\emptyset$

$$B\mathcal{C}^G_\theta(\mathbb{R}_{>0} \times M) \to B\mathcal{C}^G_{\theta,\langle 2\rangle}(M) \xrightarrow{B\partial_1} B\mathcal{C}^G_{\iota_1^*\theta}(M)$$

Proof. First note that the categories in play are weakly unital and locally fibrant. The functor ∂_1 is a level fibration, which is proven in an way analogous to F^H in Lemma 2.2.13. Because the categories in play are self dual, Lemma 2.2.17 imples that ∂_1 is a bicartesian fibration and the result follows from the additivity theorem [Ste21], Theorem 2.3. □

2.2.2.4 The proof of Theorem A

We now state and show our main result about the functor F^H.

Proposition 2.2.19. *The fixed point functor*

$$F^H_\square : \mathcal{C}^{G,\square_H}_\theta \to \mathcal{C}^{W_G H}_{\theta^H, sm}$$

is a cocartesian fibration. Hence F^H *also is.*

As we explained earlier, this proposition is analogous to [Ste21, Lemma 4.4], which as we argued in Lemma 2.2.17 can be generalised equivariantly. We shall reduce the proof of Proposition 2.2.19 to the latter.

A morphism (W, l_W) out of an object (P, l_P) of $\mathcal{C}^{G,\square_H}_\theta$ contains a small cylinder $[0,\varepsilon] \times (P, l_P)$, and if $F^H(W, l_W)$ factors through (W_H, l_{W_H}), W also contains a thickening $N^{\leq\varepsilon}_G(W_H, l_{W_H})$ for small enough ε, by H-neatness. Hence in some sense, (W, l_W) factors through $[0,\varepsilon] \times (P, l_P) \cup N^{\leq\varepsilon}_G(W_H, l_{W_H})$ which is a bordism with corners between (P, l_P) and another manifold with corners X. An adaptation of the smoothing procedure used in [Ste21] will modify this bordism into an actual morphism in $\mathcal{C}^{G,\square_H}_\theta$, between (P, l_P) and a smoothened version of X, which will be shown to be F^H-cocartesian.

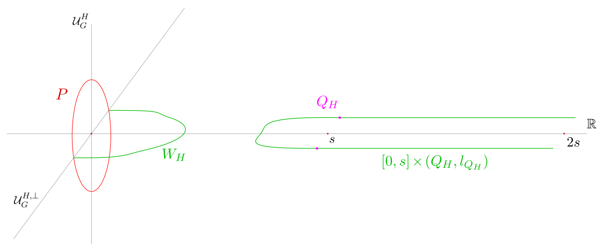


Figure 2.1: The datum to be lifted.

Proof. We first study the case where $\theta\colon \mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)\xrightarrow{=}\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)$ is the trivial structure. Note that the fixed point structure θ^H is not the trivial one however. Let (P,l_P) be an object of $\mathcal{C}_\theta^{G,\square_H}$, (Q_H,l_{Q_H}) be an object of $\mathcal{C}_{\theta^H,sm}^{W_GH}$ and let (W_H,l_{W_H},s) be a morphism in $\mathcal{C}_{\theta^H,sm}^{W_GH}$ between $F^H(P,l_P)$ and (Q_H,l_{Q_H}), as in figure 2.2.2.4. We shall construct an F^H-cocartesian lift of the morphism $([0,s]\times(Q_H,l_{Q_H}))\circ(W_H,l_{W_H})$, as in [Ste21], where we forget the length of the bordisms in the notation. By local fibrancy of F^H, it will follow that (W_H,l_{W_H}) itselfs admits a cocartesian lift.

Fix $\varepsilon < s$ small enough so that P has ε-neat H-fixed points, and the morphism (W_H,l_{W_H},s) has ε-collared incoming boundary. Define $T_{\varepsilon,H}$ to be the subspace $[0,\varepsilon]\times\mathcal{U}_G\cup\mathbb{R}_{\geq 0}\times\mathcal{U}_G^H\times D_{\leq\epsilon}(\mathcal{U}_G^{H,\perp})$ of $\mathcal{U}_G$, and $S_{\varepsilon,H}$ to be the subspace $[\varepsilon,+\infty)\times\mathcal{U}_G^H\times(\mathcal{U}_G^{H,\perp}-D_{<\epsilon}(\mathcal{U}_G^{H,\perp}))$, which we identify with $[\varepsilon,+\infty)^2\times\mathcal{U}_G^H\times S(\mathcal{U}_G^{H,\perp})$. Denote the intersection of $T_{\varepsilon,H}$ and $S_{\varepsilon,H}$ by $\partial_{\varepsilon,H}$, so that $\mathbb{R}_{\geq 0}\times\mathcal{U}_G = T_{\varepsilon,H}\cup_{\partial_{\varepsilon,H}} S_{\varepsilon,H}$. We define

$$W_{\leq\varepsilon} := [0,\varepsilon]\times P\cup N_G^{\leq\varepsilon}(W_H,l_{W_H})$$

and the longer version

$$W'_{\leq\varepsilon} := [0,\varepsilon]\times P\cup N_G^{\leq\varepsilon}(([0,s]\times(Q_H,\hat{l}_{Q_H}))\circ(W_H,l_{W_H}))$$

as well as their corner part $W_{=\varepsilon} := W_{\leq\varepsilon}\cap\partial_{\varepsilon,H}$, and $W'_{=\varepsilon} := W'_{\leq\varepsilon}\cap\partial_{\varepsilon,H}$. We now follow the construction of [Ste21, Lemma 4.4] applied to the manifold with corners $W_{=\varepsilon}$. Namely, we use the bending homeomorphism $\mathbf{B}$ of op. cit., which we see as a map $\mathbf{B}\colon[\varepsilon,+\infty)^2\to[\varepsilon,+\infty)^2$, with $\mathbf{B}(\varepsilon,s)=(\varepsilon,0)$ and $\mathbf{B}(\varepsilon,0)=(s,0)$. Applying this map on the first two factors and the identity elsewhere gives an equivariant homeomorphism

$$\mathbf{B}\colon S_{\varepsilon,H}\xrightarrow{\cong}S_{\varepsilon,H}$$

The space $Y:=\mathbf{B}^{-1}(W_{=\varepsilon})$ is a smooth G-submanifold of $\{\varepsilon\}\times[\varepsilon,+\infty)\times\mathcal{U}_G^H\times S(\mathcal{U}_G^{H,\perp})$, and we define $W'_{\geq\varepsilon}:=\mathbf{B}([\varepsilon,s]\times Y)\subseteq[\varepsilon,2s]\times[\varepsilon,+\infty)\times\mathcal{U}_G^H\times S(\mathcal{U}_G^{H,\perp})$. Finally we define $W':=W'_{\leq\varepsilon}\cup W'_{\geq\varepsilon}$. Then, W' is indeed a solution to the lifting problem we started with, if we omit the tangential structures.

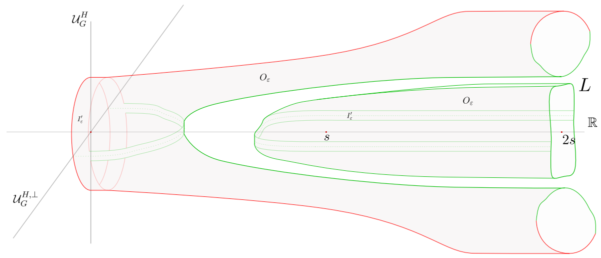


Figure 2.2: The lift.

To define a θ-structure l'_W on W' which lifts the one on W_H and Q_H and coincides with the one prescribed on P, we start defining its restriction on $W'_{\leq\varepsilon}$. On $[0,\varepsilon]\times P \subseteq W'_{\leq\varepsilon}$, we define it as $[0,\varepsilon]\times P \xrightarrow{\text{pr}} P \xrightarrow{\hat{l}_P} B$. On $N_G^{\leq\varepsilon}(([0,s]\times(Q_H,\hat{l}_{Q_H}))\circ(W_H,l_{W_H}))$, we also define it as a projection followed by

$$([0,s]\times(Q_H,\hat{l}_{Q_H}))\circ(W_H,l_{W_H}) \xrightarrow{([0,s]\times\hat{l}_{Q_H})\cup l_{W_H}} B$$

The cylindricity hypotheses of the structures in play imply that these two maps glue to form a map $l'_{W_{\leq\varepsilon}}\colon W'_{\leq\varepsilon}\to B$. Because $W'_{\leq\varepsilon}\hookrightarrow W$ is an acyclic G-cofibration and θ is a G-fibration, we can make an arbitrary (but unique up to homotopy) choice of structure l_W on W by extending $l_{W'_{\leq\varepsilon}}$. Defined as such, $(W,l_W,2s)$ is an equivariant smooth bordism, such that $F^H(W,l_W)=([0,s]\times(Q_H,\hat{l}_{Q_H}))\circ(W_H,l_{W_H})$, between $(P,\hat{l}_P)$ and another equivariant θ-manifold that we denote by $(L=L(P,l_P,W_H,l_{W_H}),\hat{l}_L)$, satisfying $F^H((L,\hat{l}_L))=(Q_H,\hat{l}_{Q_H})$. This construction is pictured in 2.2.2.4.

It now remains to show that (W',l'_W) is F^H-cocartesian. That is, given $(R,\hat{l}_R)$ an object of $\mathcal{C}_\theta^{G,\square_H}$, we need to prove that the following square is homotopy cartesian

$$\begin{array}{ccc}
\mathcal{C}_\theta^{G,\square_H}((L,\hat{l}_L),(R,\hat{l}_R)) & \xrightarrow{-\circ W'} & \mathcal{C}_\theta^{G,\square_H}((P,\hat{l}_P),(R,\hat{l}_R)) \\
\downarrow{\scriptstyle F^H} & & \downarrow{\scriptstyle F^H} \\
\mathcal{C}_{\theta^H,sm}^{W_GH}((Q_H,\hat{l}_{Q_H}),(R^H,\hat{l}_R^H)) & \xrightarrow{(W')^H} & \mathcal{C}_{\theta^H,sm}^{W_GH}((P^H,\hat{l}_P^H),(R^H,\hat{l}_R^H)).
\end{array}$$

Again, we first consider the case where θ is the trivial structure $\theta\colon \mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)\xrightarrow{=}\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)$. We still keep θ in the notations however, because the fixed points structures are not trivial structures.

We show it on vertical fibres: given $(V,l_V,t)\in\mathcal{C}_{\theta^H,sm}^{W_GH}((Q_H,\hat{l}_{Q_H}),(R^H,\hat{l}_R^H))$ we show that the induced map on homotopy fibres

$$-\circ(W',l'_W)\colon \frac{\mathcal{C}_\theta^{G,\square_H}((L,\hat{l}_L),(R,\hat{l}_R))}{(V,l_V)} \to \frac{\mathcal{C}_\theta^{G,\square_H}((P,\hat{l}_P),(R,\hat{l}_R))}{(V,l_V)\circ((W')^H,l_{W'}^H)} \tag{2.3}$$

is an equivalence.

Using Lemma 2.2.8, we can make V ε-collared, and the same lemma reduces the problem to showing that the map above is an equivalence on the category $\mathcal{C}_{\theta,\varepsilon}^{G,\square^\varepsilon_H}$ where all bordism are ε-collared and all manifolds have ε-neat H-fixed points.

We now introduce corners in order to apply results of [Ste21]. Consider the ambiant space $U = \mathcal{U}_G^H \times S(\mathcal{U}_G^{H,\perp})$. Note that any increasing sequence $(V_n)_{n\geq 0}$ of subrepresentations of $\mathcal{U}_G$ with $\cup_{n\geq 0} V_n = \mathcal{U}_G$ realises U as an increasing sequence of G-submanifolds of $\mathcal{U}_G$. There is a functor cut: $\mathcal{C}_{\theta,\varepsilon}^{G,\square^\varepsilon_H} \to \mathcal{C}_{\theta,\langle 2\rangle}^G(U)$ with the following behaviour on morphisms (M,s) (there is no tangential structure to consider): first intersect (M,s) with

$$[\varepsilon, s-\varepsilon] \times \mathcal{U}_G^H \times (\mathcal{U}_G^{H,\perp} - D_{<\epsilon}(\mathcal{U}_G^{H,\perp})) \subseteq \mathbb{R} \times \mathcal{U}_G$$

The result can be identified with a subspace of $[\varepsilon,+\infty)^2 \times U$ and lastly apply a translation by $(-\varepsilon,-\varepsilon)$ on the two first coordinates, which gives a bordism of length $s-2\varepsilon$. The functor is defined similarly on objects.

In the same way, we define a *normal boundary* functor $\partial_n \colon \mathcal{C}_{\theta^H,sm}^{W_G H} \to \mathcal{C}_\theta^G(U)$ which on morphisms takes a G-bordism with structure (M_H, l_{M_H}, s) and outputs the intersection of $N_G^{=\varepsilon}(M_H, l_{M_H}) \subseteq [0,+\infty) \times \{\varepsilon\} \times U$ with $[\varepsilon, s-\varepsilon] \times \{\varepsilon\} \times U$, composed with a translation of $(-\varepsilon,-\varepsilon)$ on the first two variables, with the induced structure.

We can now relate our situation to that of Lemma 2.2.17 as the following diagram commutes

$$\begin{array}{ccc}
\mathcal{C}_{\theta,\varepsilon}^{G,\square^\varepsilon_H} & \xrightarrow{\text{cut}} & \mathcal{C}_{\theta,\langle 2\rangle}^G(U) \\
\downarrow{\scriptstyle F^H} & & \downarrow{\scriptstyle \partial_1} \\
\mathcal{C}_{\theta^H,sm}^{W_G H} & \xrightarrow{\partial_n} & \mathcal{C}_\theta^G(U)
\end{array}$$

By Lemma 2.2.17, ∂_1 is a cocartesian fibration (note that we are using the version of that lemma which we have already proven). Moreover, an inspection of the proof of [Ste21, Lemma 4.4] shows that $\mathrm{cut}(W')$ is ∂_1-cocartesian. Consider the induced square

$$\begin{array}{ccc}
\dfrac{\mathcal{C}_{\theta,\varepsilon}^{G,\square^\varepsilon_H}((L,\hat{l}_L),(R,\hat{l}_R))}{(V,l_V)} & \xrightarrow{-\circ(W',l_{W'})} & \dfrac{\mathcal{C}_{\theta,\varepsilon}^{G,\square^\varepsilon_H}((P,\hat{l}_P),(R,\hat{l}_R))}{(V,l_V)\circ((W')^H, l^H_{W'})} \\
\downarrow{\scriptstyle \text{cut}} & & \downarrow{\scriptstyle \text{cut}} \\
\dfrac{\mathcal{C}_{\theta,\langle 2\rangle}^G(\mathrm{cut}(L,\hat{l}_L),\mathrm{cut}(R,\hat{l}_R))}{\partial_n(V,l_V)} & \xrightarrow{-\circ\mathrm{cut}(W',l_{W'})} & \dfrac{\mathcal{C}_{\theta,\langle 2\rangle}^G(\mathrm{cut}(P,\hat{l}_P),\mathrm{cut}(R,\hat{l}_R))}{\partial_n((V,l_V)\circ((W')^H, l^H_{W'}))}
\end{array}$$

where implicitely the upper line are morphisms over F^H and the lower are morphisms over ∂_1. By design, the vertical maps are homeomorphisms. Because

∂_1 is a cocartesian fibration and $\mathrm{cut}(W')$ is ∂_1-cocartesian, the lower horizontal map is an equivalence. Hence, the upper map is as well.

Lastly we show that the result still holds in the presence of θ-structures. Note that all the steps in the proof still work except the last one, namely we have still not proven that ∂_1 is cocartesian in the presence of tangential structures.

We now show that the map (2.3) is still an equivalence for general θ. Comparing this map with the one without structure gives a commutative square

$$\begin{array}{ccc} \dfrac{\mathcal{C}_{\theta,\varepsilon}^{G,\square^{\varepsilon}_H}((L,\hat{l}_L),(R,\hat{l}_R))}{(V,l_V)} & \xrightarrow{-\circ(W',l_{W'})} & \dfrac{\mathcal{C}_{\theta,\varepsilon}^{G,\square^{\varepsilon}_H}((P,\hat{l}_P),(R,\hat{l}_R))}{(V,l_V)\circ((W')^H,l^H_{W'})} \\ \downarrow & & \downarrow \\ \dfrac{\mathcal{C}_{\varepsilon}^{G,\square^{\varepsilon}_H}(L,R)}{(V,(l_V)_{triv})} & \xrightarrow{-\circ W'} & \dfrac{\mathcal{C}_{\varepsilon}^{G,\square^{\varepsilon}_H}(P,R)}{(V,(l_V)_{triv})\circ((W')^H,(l^H_{W'})_{triv})} \end{array} \tag{2.4}$$

where for $l_M\colon M\to B^H$, $(l_M)_{triv}$ is the associated $\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)^H$-structure, and where the vertial maps forget the tangential structures resp. take the associated $\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)^H$-structure. Note that given (C,l_C,s) an element of the bottom left, the homotopy fibre of the left hand vertical map based at the latter is given by the space of dotted lifts in the square

$$\begin{array}{ccc} C_{\leq\varepsilon} & \longrightarrow & B \\ \big\downarrow & \nearrow & \big\downarrow \theta \\ C & \longrightarrow & \mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G) \end{array}$$

where $C_{\leq\varepsilon}$ is defined as $[0,\varepsilon]\times P\cup\left(N_G^{\leq\varepsilon}((l_V)_{triv})\right)\cup[s-\varepsilon,s]\times Q$. A similar description holds for the homotopy fibre of the right hand map of (2.4) based at $(C,l_C)\circ(W',l'_W)$, where $C_{\leq\varepsilon}$ is replaced by $W'\cup_L C_{\leq\varepsilon}$. Because W' is topologically a cylinder between P and L, the map induced on homotopy fibres is a weak equivalence. As $-\circ(W',l'_W)$ is surjective on path components, we deduce that it is an equivalence. Hence the result follows. □

Remark 2.2.20. In the proof above we have first shown that F^H is a cocartesian fibration in the absence of tangential structures, and have then used a homotopy-theoretic argument to conclude that it is still so in general. The same argument can be used to prove that ∂_1 is a cocartesian fibration for all tangential structures, which proves Lemma 2.2.17 in full generality.

Proof. (of Theorem A) The functor F^H is clearly continuous and weakly unital. By Lemma 2.2.13, it is a local fibration, and so is also a level fibration. By Lemma 2.2.19, it is a cocartesian fibration, and because the categories in play are self-dual, it is in turn a cartesian fibration as well. We have gathered the required hypotheses to apply [Ste21, Theorem 2.3] which concludes the proof. □

Remark 2.2.21. Using Remark 2.1.17, a model independant consequence of Theorem A can be stated. By Lemma 2.2.13, the square

$$\begin{array}{ccc} \mathcal{C}^G_{\theta,\mathcal{F}-(H)} & \longrightarrow & \mathcal{C}^G_{\theta,\mathcal{F}} \\ \downarrow & & \downarrow F^H \\ * & \xrightarrow{\emptyset} & \mathcal{C}^{W_G H}_{\theta^H,\text{free}} \end{array}$$

becomes homotopy cartesian after passing to marked Segal spaces. It turns out that it is also homotopy cartesian after completion by the argument in [CDH+23, Proof of Theorem 2.5.1], namely because F^H is an isofibration which is implied by Lemma 2.2.19. Hence it induces a pullback of associated ∞-categories.

Moreover, it remains a pullback when the arrow $* \xrightarrow{\emptyset} \mathcal{C}^{W_G H}_{\theta^H,\text{free}}$ is replaced with the inclusion of any object (P_H, l_{P_H}), by an adaptation of the proof of [GR14, Proposition 2.16].

2.2.3 A new approach to the result of [GS21]

In this section we explain how Theorem A can give a new understanding of the main theorem of [GS21]. We miss an argument to provide a new full proof of the latter, but believe there is hope in this direction. We first prove a lemma identifying cobordism categories of free manifolds with non-equivariant ones.

When W is a free G-manifold with θ-structure, the quotient manifold W/G ought to inherit some structure as well. We define below the *homotopy orbit structure* θ_{hG} associated to θ.

The inclusion of fixed points $\mathbb{R}^\infty \hookrightarrow \mathcal{U}_G$ induces a map $\mathrm{Gr}(\mathbb{R}\times\mathbb{R}^\infty) \to \mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)$ which is a Borel G-equivalence. As a consequence, there is an equivalence over BG

$$\mathrm{Gr}(\mathbb{R}\times\mathbb{R}^\infty)\times BG \xrightarrow{\simeq} (\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G))_{hG} \tag{2.5}$$

In order to have a well-defined association $\theta \mapsto \theta_{hG}$, and in turn a functor $\mathcal{C}^G_{\theta,\text{free}} \to \mathcal{C}_{\theta_{hG}}$, we wish to define a map in the other direction of this equivalence.

Remark that a free G-manifold $W \subseteq \mathbb{R}\times\mathcal{U}_G$ is in particular embedded inside $\mathbb{R}\times EG(\mathcal{U}_G)$, where $EG(\mathcal{U}_G)$ is the subspace of $\mathcal{U}_G$ consisting of those points which have free isotropy. As the latter is an open subspace, it is clearly an increasing union of G-submanifolds of $\mathcal{U}_G$ in the sense of Definition 2.1.14. It turns out that $EG(\mathcal{U}_G)$ has the G-equivariant homotopy type of EG. Hence the projection $p\colon EG(\mathcal{U}_G) \to BG(\mathcal{U}_G) := EG(\mathcal{U}_G)/G$ is a model for the universal principal G-bundle $EG \to BG$. More generally, for $\mathcal{F}$ a family of subgroups of G, the following holds.

Lemma 2.2.22. *Let $E\mathcal{F}(\mathcal{U}_G)$ be the open subspace of $\mathcal{U}_G$ of those vectors having isotropy contained in $\mathcal{F}$. Then, $E\mathcal{F}(\mathcal{U}_G)$ has the G-equivariant homotopy type of $E\mathcal{F}$.*

Remark 2.2.23. As a consequence, θ-manifolds M embedded inside $\mathbb{R} \times \mathcal{U}_G$ which have isotropy contained in $\mathcal{F}$ canonically come with an explicit choice of $\theta \times E\mathcal{F}$-structure, given by the inclusion $M \hookrightarrow \mathbb{R} \times E\mathcal{F}$. This defines a functor $\mathcal{C}^G_{\theta,\mathcal{F}} \to \mathcal{C}^G_{\theta\times E\mathcal{F}}$ which is a section of the equivalence of Remark 2.1.27.

Proof. We show it when $\mathcal{F} = \{*\}$ i.e. for $EG(\mathcal{U}_G)$, the general statement follows by induction. Clearly $EG(\mathcal{U}_G)^H = \emptyset$ if $H \neq *$. Now let $f\colon K \to EG(\mathcal{U}_G)$ be a map from a compact space, then f lands in some finite dimensional representation $V \subseteq \mathcal{U}_G$. Take $x \in EG(\mathcal{U}_G) - V$, and first deform f by $f_t = f + tx$. Then, contracting V to $\{0\}$ gives a homotopy from f_1 to the constant map equal to x. □

From now on, we will use the strict model for the Borel construction X_{hG} of a topological space X with G-action $X_{hG} := (X \times \mathbb{R} \times EG(\mathcal{U}_G))/G$. In particular, the projection $p\colon EG \to BG$ will now designate the smooth principal G-bundle $\mathbb{R} \times EG(\mathcal{U}_G) \to (\mathbb{R} \times EG(\mathcal{U}_G))/G$.

We now construct a map $EG(\mathcal{U}_G)/G \to \mathbb{R}^\infty$, which can be thought of as a smooth embedding in some sense. Letting ρ be the regular representation of G, there is an isomorphism of orthogonal G-representations $\mathcal{U}_G \cong \mathrm{colim}_{n\geq 0}\rho^{\oplus n}$. By induction $n \geq 0$ and using the relative Whitney embedding theorem, we define smooth embeddings $EG(\rho^{\oplus n})/G \hookrightarrow \mathbb{R}^{k_n}$ in such a way that the diagram

$$\begin{array}{ccc} EG(\rho^{\oplus n})/G & \hookrightarrow & \mathbb{R}^{k_n} \\ \downarrow & & \downarrow \\ EG(\rho^{\oplus n+1})/G & \hookrightarrow & \mathbb{R}^{k_{n+1}} \end{array}$$

where the map $\mathbb{R}^{k_n} \to \mathbb{R}^{k_{n+1}}$ is the inclusion of the k_n first coordinates. These maps assemble to a well defined map $e\colon EG(\mathcal{U}_G)/G \to \mathbb{R}^\infty$. Define a G-map h as

$$h\colon \begin{cases} (\mathbb{R} \times EG(\mathcal{U}_G)) \times \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G) & \to & (\mathbb{R} \times EG(\mathcal{U}_G)) \times \mathrm{Gr}(\mathbb{R} \times \mathbb{R}^\infty) \\ (x, V) & \mapsto & (x, T_x(e \circ p)(V)) \end{cases} \tag{2.6}$$

For $\eta\colon W \to \mathbb{R} \times \mathcal{U}_G$ an embedding of a free G-manifold, the composition

$$h \circ (\eta, T\eta)\colon W \to (\mathbb{R} \times EG(\mathcal{U}_G)) \times \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G)$$

induces after taking G-orbits a map

$$W/G \xrightarrow{[h\circ(\eta,T\eta)]/G} (\mathbb{R}\times EG(\mathcal{U}_G)/G)\times \mathrm{Gr}(\mathbb{R}\times\mathbb{R}^\infty) \xrightarrow{((\mathrm{id},e),\mathrm{id})} (\mathbb{R}\times\mathbb{R}^\infty)\times\mathrm{Gr}(\mathbb{R}\times\mathbb{R}^\infty) \tag{2.7}$$

which has the property of being equal to the product of an embedding $\eta_G\colon W/G \to \mathbb{R}\times\mathbb{R}^\infty$ with its associated Gauss map $\tau_\eta\colon W/G \to \mathrm{Gr}(\mathbb{R}\times\mathbb{R}^\infty)$.

By Lemma 2.2.22, the map h induces on strict G-orbits a map

$$\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)_{hG} \xrightarrow{h_*} \mathrm{Gr}(\mathbb{R}\times\mathbb{R}^\infty)\times BG$$

which goes in the reverse direction of 2.5 as wanted.

Definition 2.2.24. Let $\theta : B \to \mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)$ be an equivariant tangential structure. The *homotopy orbit structure* assiciated to θ is the following (non-equivariant) tangential structure

$$\theta_{hG}\colon B_{hG} \to (\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G))_{hG} \xrightarrow{h_*} \mathrm{Gr}(\mathbb{R}\times\mathbb{R}^\infty)\times BG \xrightarrow{\mathrm{pr}} \mathrm{Gr}(\mathbb{R}\times\mathbb{R}^\infty)$$

When (W, l_W) is a free G-submanifold of $\mathbb{R}\times\mathcal{U}_G$ with θ-structure, we identify its quotient W/G with its image in $\mathbb{R}\times\mathbb{R}^\infty$ through the embedding e, so that W/G becomes a submanifold of $\mathbb{R}\times\mathbb{R}^\infty$ equipped with a θ_{hG}-structure which we denote by $(l_W)_{hG}$.

Lemma 2.2.25. *Taking G-orbits induces a functor*

$$\mathcal{C}^G_{\theta,\mathrm{free}} \xrightarrow{(-)_{hG}} \mathcal{C}_{\theta_{hG}}$$

which is a weak equivalence on objects and on morphisms.

Remark 2.2.26. In particular there is an equivalence $B\mathcal{C}^G_{\theta\times EG} \xrightarrow{\simeq} B\mathcal{C}_{\theta_{hG}}$. Because $MT(\theta_{hG})$ is equivalent to the homotopy orbits spectrum $(MT\theta)_{hG}$, this translates to what we believe to be a geometric intepretation of the Adams isomorphism $(MT\theta\otimes EG_+)^G \xrightarrow{\simeq} (MT\theta)_{hG}$ after applying $\Omega^{\infty-1}$.

Proof. It is clear that the constructions of 2.7 and in Definition 2.2.24 produces a well-defined continuous functor as claimed. As usual we show that the induced map of morphism spaces is a weak equivalence, as the proof for objects is similar. We also first treat the case where θ is the trivial structure, in which case θ_{hG} is the product structure $\mathrm{Gr}(\mathbb{R}\times\mathbb{R}^\infty)\times BG \to \mathrm{Gr}(\mathbb{R}\times\mathbb{R}^\infty)$ which we denote by $/BG$.

A path component of the space $\mathrm{mor}(\mathcal{C}^G_{free})$ has the homotopy type of

$$\mathrm{Emb}^G_\partial(W, [0,1]\times EG(\mathcal{U}_G))/\mathrm{Diff}^G_\partial(W)$$

where W is a G-manifold with a given decomposition $\partial W = \partial_0 W \sqcup \partial_1 W$.

Taking strict G-orbits induces a map φ

$$\mathrm{Emb}^G_\partial(W, [0,1]\times EG(\mathcal{U}_G))/\mathrm{Diff}^G_\partial(W) \to \mathrm{Emb}_\partial(W/G, [0,1]\times EG(\mathcal{U}_G)/G)/\mathrm{Diff}_\partial(W/G)$$

which we show is an equivalence. First note that it is a bijection, as more generally free G-subspaces of $\mathbb{R}\times\mathcal{U}_G$ i.e. G-subspaces of $EG = \mathbb{R}\times EG(\mathcal{U}_G)$ are in one-to-one correspondance with subspaces of $BG = (\mathbb{R}\times EG(\mathcal{U}_G))/G$, with inverse given by taking the pullback along the projection $EG \to BG$.

We claim that the map $\tilde{\varphi}$, defined as φ before passing to orbits for diffeomorphism groups, is a Serre fibration. Indeed consider a diagram

$$\begin{array}{ccc} \{0\} \times [0,1]^k & \xrightarrow{\alpha} & \mathrm{Emb}^G_\partial(W, [0,1] \times EG(\mathcal{U}_G)) \\ \big\downarrow & & \big\downarrow \tilde{\varphi} \\ [0,1] \times [0,1]^k & \xrightarrow{\beta_t} & \mathrm{Emb}_\partial(W/G, [0,1] \times EG(\mathcal{U}_G)/G) \end{array}$$

which we can rewrite as a diagram of G-maps

$$\begin{array}{ccc} \{0\} \times [0,1]^k \times W & \xrightarrow{\overline{\alpha}} & [0,1] \times EG(\mathcal{U}_G) \\ \big\downarrow & & \big\downarrow pr \\ [0,1] \times [0,1]^k \times W/G & \xrightarrow{\overline{\beta}_t} & [0,1] \times EG(\mathcal{U}_G)/G \end{array} \tag{2.8}$$

Note that the right vertical map is a covering map, so there exists a unique solution $\overline{\alpha}_t$ to the lifting problem above, and it is equivariant by uniqueness. It induces a map $\alpha'_t \colon [0,1] \times [0,1]^k \to \mathrm{Map}^G_\partial(W/G, [0,1] \times EG(\mathcal{U}_G)/G)$ and we wish to show that α'_t factors through the forgetful map

$$\mathrm{Emb}^G_\partial(W/G, [0,1] \times EG(\mathcal{U}_G)/G) \to \mathrm{Map}^G_\partial(W/G, [0,1] \times EG(\mathcal{U}_G)/G)$$

The fact that α'_t lands in $\mathrm{Emb}^G_\partial(W/G, [0,1] \times EG(\mathcal{U}_G)/G)$ as a subset is due to the fact that the lift of a topological embedding through a covering map is still a topological embedding, moreover in our case the covering map is a local diffeomorphism which implies that the lift of an immersion is an immersion. The continuity of the induced map

$$\alpha_t \colon [0,1] \times [0,1]^k \to \mathrm{Emb}^G_\partial(W/G, [0,1] \times EG(\mathcal{U}_G)/G)$$

can be seen using the characterisation of the weak Whitney topology in terms of sections of jet spaces. Hence $\tilde{\varphi}$ is a Serre fibration. Because the projections from $\mathrm{Emb}^G_\partial(W, [0,1] \times EG(\mathcal{U}_G))$ (resp. $\mathrm{Emb}_\partial(W/G, [0,1] \times EG(\mathcal{U}_G)/G)$) onto the orbits mod $\mathrm{Diff}^G_\partial(W)$ (resp. $\mathrm{Diff}_\partial(W/G)$) are fibre bundles hence Serre fibrations, the map φ induced on orbits from $\tilde{\varphi}$ also is. Because it is bijective, it is a weak equivalence.

We now consider the $\mathrm{Diff}(W/G)$-equivariant map of locally trivial $\mathrm{Diff}(W/G)$-spaces

$$\psi \colon \begin{array}{c} \mathrm{Emb}_\partial(W/G, [0,1] \times EG(\mathcal{U}_G)/G) \\ \big\downarrow (e, \psi_2) \\ \mathrm{Emb}_\partial(W/G, [0,1] \times \mathbb{R}^\infty) \times \mathrm{Map}(W/G, BG) \end{array}$$

where ψ_2 is the forgetful map from embeddings to continuous maps, composed with postcomposition by the inclusion $[0,1] \times EG(\mathcal{U}_G)/G \subseteq BG$. The space $\mathrm{Emb}_\partial(W/G, [0,1] \times \mathbb{R}^\infty)$ is contractible, on the other hand the weak Whitney embedding theorem implies that the map $\mathrm{Emb}_\partial(W/G, [0,1] \times EG(\mathcal{U}_G)/G) \to \mathrm{Map}_\partial(W/G, [0,1]\times EG(\mathcal{U}_G)/G)$ is an equivalence. Therefore ψ is an equivalence, and so is the map induced on $\mathrm{Diff}(W/G)$-orbits.

As a consequence, the map $\mathrm{mor}(\mathcal{C}^G_{free}) \to \mathrm{mor}(\mathcal{C}_{/BG})$ is a weak equivalence onto the path components that it hits. It is also surjective on π_0, by approximating the map $W/G \to BG$ to an embedding, and chosing a lift of the composition $W \to W/G \to BG$ along the smooth covering map $EG \to BG$. Hence it is an equivalence.

We now finish the proof by considering the case of general tangential structures. Let θ be an equivariant tangential structure, we consider the commutative diagram

$$\begin{array}{ccc} \mathrm{mor}(\mathcal{C}^G_{\theta,\mathrm{free}}) & \xrightarrow{(-)_{hG}} & \mathrm{mor}(\mathcal{C}_{\theta_{hG}}) \\ \downarrow f & & \downarrow \bar{f} \\ \mathrm{mor}(\mathcal{C}^G_{free}) & \xrightarrow{(-)_{hG}} & \mathrm{mor}(\mathcal{C}_{/BG}) \end{array}$$

where the vertical left map forgets the structure, and the right one composes the θ_{hG}-structures with the canonical map $p\colon B_{hG} \to BG$. We can check as before that the upper horizontal map is surjective on π_0. Now let (W, s) be element in $\mathrm{mor}(\mathcal{C}^G_{free})$, the homotopy fibre of f based at that point is equivalent to the (maybe empty space) $\mathrm{Map}^G_{\partial,/\mathrm{Gr}(\mathbb{R}\times\mathcal{U}_G)}(W, B)$. On the other hand, the homotopy fibre of $\bar{f}$ based at $(W, s)_{hG}$ is equivalent to

$$\mathrm{hofib}(\mathrm{Map}_{\partial,/\mathrm{Gr}(\mathbb{R}\times\mathbb{R}^\infty)}(W/G, B_{hG})) \to \mathrm{Map}_{\partial,/\mathrm{Gr}(\mathbb{R}\times\mathbb{R}^\infty)}(W/G, BG))$$

which is a formula for the mapping space between W/G and B_{hG} in the category of spaces over $BG \times \mathrm{Gr}(\mathbb{R} \times \mathbb{R}^\infty) \simeq \mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G)_{hG}$ with the appropriate boundary condition. The result follows by a slight modification of the equivalence of homotopy theories

$$\mathrm{Top}_{G,\mathrm{Borel}}/\mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G) \xrightarrow{\simeq} \mathrm{Top}/\mathrm{Gr}(\mathbb{R} \times \mathcal{U}_G)_{hG}$$

to take the boundary condition into account, because W is Borel G-cofibrant. □

Note that we have proved Theorem A without any reference to [GS21]. We can now give a new insight on the main theorem of [GS21]. Let $MT\theta$ be the genuine Thom G-spectrum associated to θ. Recall that their main shows that there is a weak equivalence

$$B\mathcal{C}^G_\theta \xrightarrow{\simeq} \Omega^{\infty-1}(MT\theta)^G$$

We have seen that taking $E\mathcal{F}(\mathcal{U}_G)$ as a model for $E\mathcal{F}$, there is a natural functor $\mathcal{C}^G_{\theta,\mathcal{F}} \to \mathcal{C}^G_{\theta\times E\mathcal{F}}$ which is an equivalence on objects and morphism, hence an equivalence on classifying spaces. Composing with the scanning maps, we get a commutative diagram

$$\begin{array}{ccc} B\mathcal{C}^G_{\theta,\mathcal{F}-(H)} & \longrightarrow & \Omega^{\infty-1}(MT(\theta\times E(\mathcal{F}-(H))))^G \\ \big\downarrow & & \big\downarrow \\ B\mathcal{C}^G_{\theta,E\mathcal{F}} & \longrightarrow & \Omega^{\infty-1}(MT(\theta\times E\mathcal{F}))^G \\ \big\downarrow & & \big\downarrow \\ B\mathcal{C}^{W_GH}_{\theta^H,\mathrm{free}} & \longrightarrow & \Omega^{\infty-1}(MT(\theta^H\times EW_GH))^{W_GH} \end{array}$$

By Lemma 2.2.25, $\mathcal{C}^{W_GH}_{\theta^H,\mathrm{free}}$ is equivalent to $\mathcal{C}_{\theta^H_{hW_GH}}$ on objects and morphisms. There are isomorphisms in the homotopy category of spectra

$$(MT(\theta^H\times EW_GH))^{W_GH}\simeq (MT(\theta^H)\otimes EW_GH_+)^{W_GH}$$

and

$$MT(\theta^H_{hW_GH})\simeq MT(\theta^H)_{hW_GH}$$

hence there is a square in the homotopy category

$$\begin{array}{ccc} B\mathcal{C}^{W_GH}_{\theta^H,\mathrm{free}} & \xrightarrow{\quad\simeq\quad} & B\mathcal{C}_{\theta^H_{hW_GH}} \\ \big\downarrow & & \big\downarrow{\simeq} \\ \Omega^{\infty-1}(MT(\theta^H)\otimes EW_GH_+)^{W_GH} & \overset{\alpha}{\dashleftarrow} & \Omega^{\infty-1}MT(\theta^H)_{hW_GH} \end{array} \tag{2.9}$$

where α is defined so that it makes the diagram commute, inverting the top horizontal equivalence and the right vertical map which is an equivalence by [GMT+09].

On the other hand, the Adams map is an isomorphism

$$MT(\theta^H)_{hW_GH}\xrightarrow{\simeq}(MT(\theta^H)\otimes EW_GH_+)^{W_GH} \tag{2.10}$$

If if α is homotopic to the Adams map after applying loops, then it is possible to derive a new proof of the result of [GS21] by induction on the family of subgroups $\mathcal{F}$. Hence we raise this as an open question.

Question 1. Is the map α homotopic the Adams isomorphism 2.10 after applying $\Omega^{\infty-1}$?

2.3 The homology of equivariant moduli spaces

2.3.1 Proof of Theorem B

In this section we give the proof of our main Theorem B. Let (W, l_W) be an equivariant compact θ-manifold with boundary $(P, \hat{l}_P)$, we wish to compute the homology of the moduli space $\mathcal{M}^G_{\partial,\theta}(W, l_W)$.

Let $\mathcal{F}_W$ be the family of subgroups of G consisting of those H such that $W^H \neq \emptyset$, and let H be maximal inside $\mathcal{F}$. Note that $\mathcal{M}^G_{\partial,\theta}(W, l_W)$ is a path component of the bigger space of nullbordisms $\mathcal{N}^G_{\theta,\mathcal{F}}(P, \hat{l}_P)$ of $(P, \hat{l}_P)$. Consider the H-fixed points map

$$F^H : \mathcal{N}^G_{\theta,\mathcal{F}}(P, \hat{l}_P) \to \mathcal{N}^{W_G H}_{\theta^H,\mathrm{free}}(P^H, \hat{l}^H_P)$$

The following lemma can be considered as an isotropy separation sequence for nullbordism spaces.

Lemma 2.3.1. *Let $(C, l_C) \colon (P, \hat{l}_P) \rightsquigarrow (S, \hat{l}_S)$ be the F^H-cocartesian lift of (W^H, l^H_W) starting at $(P, \hat{l}_P)$ which we constructed in the proof of Lemma 2.2.19. Then, there is a homotopy fibre sequence based at (W^H, l^H_W)*

$$\mathcal{N}^G_{\theta,\mathcal{F}-(H)}(S, \hat{l}_S) \xrightarrow{(C,l_C)\circ -} \mathcal{N}^G_{\theta,\mathcal{F}}(P, \hat{l}_P) \xrightarrow{F^H} \mathcal{N}^{W_G H}_{\theta^H,\mathrm{free}}(P^H, \hat{l}^H_P)$$

Proof. This is a direct consequence of the fact that $(C, \hat{l}_C)$ is F^H-cocartesian, proved in Lemma 2.2.19. □

Corollary 2.3.2. *With the same notations, there is a commutative diagram of homotopy fibre sequences based at (W^H, l^H_W)*

$$\begin{array}{ccc}
\mathcal{N}^G_{\theta,\mathcal{F}-(H)}(S, \hat{l}_S) & \longrightarrow & \Omega_{[(S,\hat{l}_S),\emptyset]} B\mathcal{C}^G_{\theta,\mathcal{F}-(H)} \\
\downarrow {\scriptstyle (C,l_C)\circ -} & & \downarrow {\scriptstyle (C,l_C)\cdot -} \\
\mathcal{N}^G_{\theta,\mathcal{F}}(P, \hat{l}_P) & \longrightarrow & \Omega_{[(P,\hat{l}_P),\emptyset]} B\mathcal{C}^G_{\theta,\mathcal{F}} \\
\downarrow {\scriptstyle F^H} & & \downarrow {\scriptstyle F^H} \\
\mathcal{N}^{W_G H}_{\theta^H,\mathrm{free}}(P^H, \hat{l}^H_P) & \longrightarrow & \Omega_{[(P^H,\hat{l}^H_P),\emptyset]} B\mathcal{C}^{W_G H}_{\theta^H,\mathrm{free}}
\end{array}$$

Proof. By Theorem A and applying path spaces, the bottom row is a homotopy fibre sequence based at (W^H, l^H_W). Both squares commute by definition. □

Proof of Theorem B. We prove the statement by induction on $\mathcal{F}_W$. If $\mathcal{F}_W = \{*\}$, then the action on W is free and the theorem follows from Lemma 2.2.25. Assume that the theorem has been proven for all manifolds (W', l'_W) such that $\mathcal{F}_{W'} < \mathcal{F}_W$.

Let H be maximal in $\mathcal{F}_W$. Restricting the diagram given by Corollary 2.3.2 (and keeping the notations therein) to the path component $\mathcal{M}^G_{\partial,\theta}(W, l_W)$ gives a new map of fibre sequences

$$\begin{array}{ccc} F & \longrightarrow & \Omega_{[(S,\hat{l}_S),\emptyset]} B\mathcal{C}^G_{\theta,\mathcal{F}-(H)} \\ \big\downarrow{\scriptstyle (C,l_C)\circ -} & & \big\downarrow{\scriptstyle (C,l_C)\cdot -} \\ \mathcal{M}^G_{\partial,\theta}(W,l_W) & \longrightarrow & \Omega_{[(P,\hat{l}_P),\emptyset]} B\mathcal{C}^G_{\theta,\mathcal{F}} \\ \big\downarrow{\scriptstyle F^H} & & \big\downarrow{\scriptstyle F^H} \\ \mathcal{M}^{W_GH}_{\partial,\theta^H}(W^H,l^H_W) & \longrightarrow & \Omega_{[(P^H,\hat{l}^H_P),\emptyset]} B\mathcal{C}^{W_GH}_{\theta^H,\mathrm{free}} \end{array} \tag{2.11}$$

The fibre F is a union of path components of $\mathcal{N}^G_{\theta,\mathcal{F}-(H)}(S,\hat{l}_S)$, hence a union of moduli spaces $\mathcal{M}^G_{\theta,\partial}(W_i,l_i)$ for some G-manifolds with θ-structure (W_i,l_i). By construction, namely by the proof of Theorem 2.2.19, those manifolds are all equivariantly isotopy-equivalent to W where an equivariant tubular neighbourhood of W^H has been removed. [1]. In particular, the collection of their building blocks is a subcollection of that of W, hence they satisfy the hypotheses of Theorem B with the same ranges as for W. As $\mathcal{F}_{W_i} < \mathcal{F}_W$, the induction hypothesis applies to them and we deduce that the top map in (2.11) is $r_{H,\perp}$-acyclic, where $r_{H,\perp}$ is the minimum of $r_{K,[V]}$ over those $K \in \mathcal{F}_W - (H)$ and $[V] \in \mathrm{Rep}_K/W_GK$.

On the other hand, for all $[V] \in \mathrm{Rep}_H/W_GH$ such that $W_{(H,[V])} \neq \emptyset$, the manifold $(W_{(H,V),\partial}, l_W^{(H,[V])})$ is $r_{(H,V)}$-stable w.r.t. the tangential structure $\theta^{(H,V)}_{hW_GH}$ for some $r_{(H,V)} \in \mathbb{Z}_{\geq 0}$, by the hypotheses of the theorem and the base case of the inducion. Hence their union $(W_{(H),hW_GH}, (l_W)^H_{W_GH})$ is $r_{(H)}$-stable where $r_{(H)}$ is defined as the minimum of $r_{(H,[V])}$ over the $V \in \mathrm{Rep}_H/W_GH$ for which $W^{(H,[V])} \neq \emptyset$. As H is maximal in $\mathcal{F}_W$, $W_{(H),\partial}$ is actually equal to W^H, and we deduce that the bottom map in (2.11)

$$\mathcal{M}^{W_GH}_{\theta^H,\partial}(W^H,l^H_W) \to \Omega_{[(P^H,\hat{l}^H_P),\emptyset]} B\mathcal{C}^{W_GH}_{\theta^H,\mathrm{free}}$$

is $r_{(H)}$-acyclic. The proof follows by the following spectral sequence argument. □

Lemma 2.3.3. *Consider a commutative diagram of spaces*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle h} \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

such that the two rows are homotopy fibre sequences based at all possible points. If f is r-acyclic and h is r'-acyclic, then h is $\min(r,r')$-acyclic.

[1] Note that moreover, the number of path components is given by the size of $\mathrm{coim}\, \pi_0(\mathrm{Diff}^G_\theta(W,l_W) \to \mathrm{Diff}^{W_GH}_{\theta^H}(W^H,l^H_W))$

Proof. The map of fibre sequences induces a map between associated Serre spectral sequences with local coeffients. On the E^2-pages, this map is an isomorhism in degrees $0 \leq p, q \leq \min(r, r')$ hence it remains an isomorphism on the E^∞-pages in the same degrees. The proof follows. □

Remark 2.3.4. Note that in the proof of Theorem B, acyclicity of the scanning maps is needed at each step of the induction so that the spectral sequence argument can be carried out. If one wishes to simply show that the equivariant scanning map for $\mathcal{M}^G_{\partial,\theta}(W, l_W)$ is an integral homology isomorphism in a range of degrees, there is a priori no simplification of our strategy. If $G = C_p$ though, something more can be said.

If in the situation Lemma 2.3.3, if the map h is r'-acyclic but f induces an r-connected map after applying integral chains only, then the same is true for g with range $\min(r, r')$ by the same spectral sequence argument. Hence, to compute the homology of $\mathcal{M}^{C_p}_\theta(W, l_W)$ with integral coefficients in a range of degrees, it is enough to show r-stability for the fixed points $(W^{C_p}, l_W^{C_p})$ and to show a range of stability with respect to integral coefficients for $(W_{*,hC_p}, (l_W)_{hC_p})$. We will use this remark for the first example of computation in section 2.4.

2.3.2 Applying Theorem B in practice

As pointed out in section 2.1.4, there are several conditions under which an ordinary manifold with structure (M, l_M) can be shown to be r-stable. The case of dimension 0 i.e. of a finite set X is well known, c.f. [Kra19]. The case of an even dimension $d = 2k \geq 2$ is usually studied in two complementary ways, first by showing that the scanning map becomes an equivalence onto the path component that it hits after stabilizing the moduli space ([GR17b, Theorem 1.5]), and then by using a homological stability input. The first step is covered in great generality by op. cit., and can be carried out under the condition that the tangential structure $l_M : M \to B$ is k-connected. On the other hand, the precise range given by the homological stability input depends on the dimension (2 or $d \geq 6$) and on the fundamental group ([Ran16], [GR17a], [Fri17]), and no case of stability is known in dimension 4. The case of dimension 2 is treated in a way which is somehow less uniform that the others, hence we state the following corollary of Theorem B in a way which avoids complications in that dimension.

In practice, we apply Theorem B under a slightly more explicit form. Recall that for $H \leq G$ there is an equivalence

$$B_G O(\bullet)^H \simeq \coprod_{V \in \mathrm{Rep}_H} B\mathrm{Aut}(V)$$

Definition 2.3.5. Let f be a map between two G-spaces over $B_G O(\bullet)$, we define $f^{(H,V)}$ as the pullback of f^H along the inclusion $B\mathrm{Aut}(V) \hookrightarrow B_G O(\bullet)^H$. Say that f is $\frac{1}{2}$-*connected* if for all $H \leq G$, $f^{(H,V)}$ is $\frac{1}{2}\dim V^H$-connected.

Definition 2.3.6. Let M be a G-manifold. Say that M satisfies the *gap hypothesis* if for all (H,V), $H \leq G$, $V \in \mathrm{Rep}_H$, the inclusion of (non-equivariant) submanifolds $M^{(H',V')} \hookrightarrow M^{(H,V)}$ is of codimension at least $\frac{1}{2}\dim V^H + 1$ whenever $(H',V') > (H,V)$, i.e. whenever there exists a proper (not surjective) embedding of G-representations $G \times_H V \hookrightarrow G \times_{H'} V'$.

Remark 2.3.7. As in applications we always require that fixed points are even-dimensional, the way we state the gap hypothesis is equivalent to that in [Sch06] referred there as the *standard gap hypothesis*.

Let (M, l_M) be a d-dimensional compact G-manifold with θ-structure. Let $g_i := g^{\theta_i}(M_i, l_{M_i})$ for $i \in I$, where $(M_i, l_{M_i})_{i\in I}$ is the collection of building blocks associated to (M, l_M).

Corollary 2.3.8 (of Theorem B)**.** *Let $d = 2k \geq 0$ and $\theta\colon B \to B_G O(d)$ be an equivariant tangential structure such that*

(i) $B^{(H,V)}/W_G H$ *is connected if* $\dim V^H = 0$,
(ii) $\theta^{(H,V)})$ *satisfies the hypotheses of [Ran16, Theorem 7.1] and* $MT(\theta^{(H,V})$ *is simply-connected if* $\dim V^H = 2$
(iii) $B^{(H,V)} = \emptyset$ *if* $\dim V^H$ *is odd or equal to* 4,
(iv) $\pi_1(B^{(H,V)})$ *is virtually polycyclic for all basepoints if* $\dim V^H \geq 6$

There exists a function $r_\theta\colon \mathbb{Z}^I_{\geq 0} \to \mathbb{Z}_{\geq 0}$ *depending only on* θ *such that* $r_\theta(x) \xrightarrow[|x|\to\infty]{} \infty$ *and the scanning map*

$$\mathcal{M}^G_{\theta,\partial}(M, l_M) \to \Omega^\infty_0(MT\theta)^G$$

is $r_\theta((g_i)_{i\in I})$*-acyclic under the assumption that* l_M *is* $\frac{1}{2}$*-connected and* M *satisfies the gap hypothesis.*

Remark 2.3.9. The genus of a $2k$-manifold M is defined as largest g such that it admits an embedding of $W_{g,1} = (S^k \times S^k)^{\# g} \setminus D^{2k}$. In dimension 0, $W_{g,1}$ is a set of size $2g+1$, hence $g(M) = \frac{1}{2}(\# M - 1)$. Also note that in that dimension, in the presence of a path-connected tangential structure, the θ-genus is remains the same.

Remark 2.3.10. The function r_θ can be explicitly described in practice. Namely we can take $r_\theta((g_i)_{i\in I}) = \min_{i\in I} r_{\theta_i}(g_i)$, where r_{θ_i} is one of the following functions, according to what (H,V)-building block (W_i, l_i) corresponds to.

- if $\dim V^H = 0$, we can take $r_{\theta,i}(g) = \frac{1}{3}(2g+1)$ by [Kra19],
- if $\dim V^H = 2$, there is no systematic answer in the literature so $r_{\theta,i}$ has to be worked out case by case according to θ,
- if $\dim V^H \geq 6$ then we can take $r_{\theta,i}(g) = \frac{1}{3}(g - h(\pi_1(B^{(H,V)})) - 3)$ where the middle term is the Hirsch length of $\pi_1(B^{(H,V)})$ (or the minimum of

those along the path components of $B^{(H,V)}$ if it is not connected), by [Fri17]. Note that in the last case, in practice one would have to consider the Hirsch length of the fundamental group of $B^{(H,V)}_{hW_GH}$, but these happen to coincide because G is finite.

Proof. By the gap hypothesis, for all (H,V) the inclusion $M_{(H,V)} \hookrightarrow M^{(H,V)}$ is $\frac{1}{2}\dim V^H$-connected, hence the hypotheses ensures that all building blocks (M_i, l_i) fall under the range of application of one of the papers cited above. More precisely, for M_i of dimension 0, condition (1) implies that (M_i, l_i) is r-stable with range $r = n/3$ where n is the cardinality of M_i. In the case when M_i is 2-dimensional, the ad hoc condition (2) gives the existence of a certain range of stability only for integral coefficients by [Ran16, Theorem 7.1], on the other hand the simple connectedness of $MT\theta$ implies that there are no more general coefficients possible. Condition (3) together with the $\frac{1}{2}$-connectedness of l_M implies that M has no fixed points of dimension 4 or odd. Lastly, if $\pi_1(B^{(H,V)})$ is virtually polycyclic then so is $\pi_1(B^{(H,V)}_{hW_GH})$, so that building blocks M_i of dimension $2k \geq 6$ can be handled by [Fri17, Theorem 4.12]. Hence the conclusion follows from Theorem B. □

2.3.3 Rational coefficients

The cohomology of the spectrum $(MT\theta)^G$ can be described in a convenient way over $\mathbb{Q}$. In general, given X a genuine G-spectrum, there is a rational equivalence

$$X \xrightarrow{\simeq,\mathbb{Q}} \bigoplus_{(H)\leq G} (\Phi^H X)_{hW_GH}$$

where (H) ranges over the conjugacy classes of subgroups of G, and Φ^H denotes geometric H-fixed points, for example by [Wim19, Theorem 3.1]. There is an equivalence $\Phi^H(MT\theta) \simeq MT(\theta^H)$ which follows from the fact that Φ^H commutes with Σ^∞_+ and with colimits. Hence using the notations we introduced in section 2.1.2, this implies

Lemma 2.3.11. *Given θ an equivariant tangential structure, there is a rational equivalence*

$$(MT\theta)^G \xrightarrow{\simeq,\mathbb{Q}} \bigoplus_{(H,[V])} MT(\theta^{H,[V]}_{hW_GH})$$

where $(H,[V])$ ranges over conjugacy classes of subgroups $H \leq G$ and isomorphism classes of finite dimensional H-representations up to W_GH-conjugacy $[V] \in \mathrm{Rep}_H/W_GH$. □

This lemma is essentially a reformulation of Lemma 7.5 in [GS21]. With our notations, Corollary 7.6 of op. cit. can be stated as follows

Corollary 2.3.12 (see [GS21, Corollary 7.6])**.** *Let (H) be a conjugacy class of subgroups of G and let $[V] \in \mathrm{Rep}_H/W_GH$. The tangential structure $\theta^{(H,[V])}_{hW_GH}$ is* *a map*

$$\theta^{(H,[V])}\colon B^{(H,[V])}_{hW_GH} \to BO(\dim V^H)$$

Let $w_1^{(H,[V])}$ *be the orientation character of* $\theta^{(H,[V])}_{hW_GH}$ *and* $\mathbb{Q}^{(H,[V])}$ *be the associated local system on* $B^{(H,[V])}_{hW_GH}$.

For all $n > \dim V^H$ *and* $c \in H^n(B^{(H,[V])}_{hW_GH}; \mathbb{Q}^{(H,[V])})$, *there is an associated* MMM class $\kappa_{H,[V],c} \in H^{n-\dim V^H}(\Omega_0^\infty(MT\theta)^G; \mathbb{Q})$.

If B^H *is of finite type for all* H, *the induced map*

$$\mathbb{Q}[\kappa_{H,[V],c}] \to H^*(\Omega_0^\infty(MT\theta)^G; \mathbb{Q})$$

is an isomorphism, where on the left is the polynomial algebra generated by the MMM classes assciated to each element of a basis of $\bigoplus_{(H,[V])} H^{>\dim V^H}(B^{(H,[V])}_{hW_GH}; \mathbb{Q}^{(H,[V])})$.

Remark 2.3.13. As G is finite, the natural map $B^{(H,[V])} \to B^{(H,[V])}_{hW_GH}$ exhibits $H_*(B^{(H,[V])}_{hW_GH}; \mathbb{Q})$ as the quotient $H_*(B^{(H,[V])}; \mathbb{Q})/W_GH$.

Hence in the case of the structure $\mathfrak{o}_2$ given by $BSO(2) \to BO(2)$, Lemma 2.3.11 gives a rational equivalence of spectra

$$(MT\mathfrak{o}_2)^G \xrightarrow{\simeq,\mathbb{Q}} \bigoplus_{(H)\leq G} \left(MTSO(2) \oplus \bigoplus_{\substack{[H\to SO(2)] \bmod W_GH \\ \text{non zero}}} \Sigma^\infty_+ BSO(2) \right)$$

which is a correction to the formula given in [GS21, end of section 7].

Note that as in the non-equivariant setting, the class $\kappa_{H,[V],c}$ has a geometric interpretation. We saw in Lemma 2.2.8 that the inclusion of categories $\mathcal{C}_\theta^{G,\square^\delta_H} \hookrightarrow \mathcal{C}_\theta^G$ was an equivalence on objects and morphisms given any $\delta > 0$, more generally we can see by induction that the inclusion of $\bigcap_{(H)\leq G} \mathcal{C}_\theta^{G,\square^\delta_H}$ is an equivalence, defined as the intersection on objects and on morphisms. An object resp. a morphism in this category is in particular a G-manifold embedded in $\mathcal{U}_G$ resp. $\mathbb{R}\times\mathcal{U}_G$ in a *straight* way, in the sense of [Was69, page 143]. Such a manifold W can be cut into pieces that are equivariantly isotopy-equivalent to $W_{(H,[V]),\partial}$ for varying $(H,[V])$. This yields in particular a well-defined map between moduli spaces

$$\mathcal{M}^G_{\theta,\partial}(W, l_W) \xleftarrow{\simeq} \bigcap_{(H)\leq G} \mathcal{C}_\theta^{G,\square^\delta_H}(W, l_W) \to \prod_i \mathcal{M}_{\theta_i}(W_i, l_i)$$

where $(W_i, l_i) = (W_{(H,[V]),hW_GH}, (l_W^{(H,[V])})_{hW_GH})$ for varying $(H,[V])$. Hence, a G-manifold bundle (with tangential structure) with fibre (W, l_W) gives rise to a collection of non-equivariant manifold bundles with fibres given by the building blocks of (W, l_W). The classical non-equivariant MMM classes of these manifold bundles correspond to the ones in Corollary 2.3.12.

2.3.4 Replacements of structures

In many cases, the manifold (W, l_W) does not verify the conditions of Corollary 2.3.8. However, in some situations, there exists another tangential structure $\theta' \colon B' \to \mathrm{Gr}(\mathcal{U}_G)$ mapping to θ via a map u, and a factorisation

$$l_W \colon W \xrightarrow{l'_W} B' \xrightarrow{u} B$$

such that the θ'-manifold (W, l'_W) satisfies the conditions of Theorem B. We will see that under some condition on the factorisation, it can be possible to recover information about $\mathcal{M}^G_{\partial,\theta}(W, l_W)$ from the calculation of $\mathcal{M}^G_{\partial,\theta'}(W, l'_W)$. This is an adaptation of the non-equivariant situation, dealt in [GR17a, Theorem 9.4]. In this section we will work in the framework of ∞-categories to avoid technicalities of model structures. In particular all mapping spaces in this section are derived. Let S be $(\infty$-)category of spaces, and $\mathsf{S}_G := \mathrm{Fun}(\mathcal{O}_G^{op}, \mathsf{S})$ be that of G-spaces.

Given $A \xrightarrow{l} X \xrightarrow{u} Y \in A/\mathsf{S}_G/Y$, we write $\mathrm{Aut}^G(u, l)$ for the subspace of its endomorphisms spanned by equivalences. For $l = l'_{\partial W} \colon \partial W \to B'$ and $u \colon B' \to B$, $\mathrm{Aut}^G(u, l'_{\partial W})$ acts on $\mathrm{Map}^G_{\partial W/\mathsf{S}_G/B_G O(\bullet)}(W, B')$ by postcomposition, and composition with u refines to an $\mathrm{Aut}^G(u, l'_{\partial W})$-invariant map

$$\mathrm{Map}^G_{\partial W/\mathsf{S}_G/B_G O(\bullet)}(W, B') \xrightarrow{-\circ u} \mathrm{Map}^G_{\partial W/\mathsf{S}_G/B_G O(\bullet)}(W, B)$$

Furthermore, this map also refines to a $\mathrm{Diff}^G_\partial(W)$-equivariant map so that there is an induced map of moduli spaces

$$\mathcal{M}^G_{\partial,\theta'}(W, l'_W) \xrightarrow{-\circ u} \mathcal{M}^G_{\partial,\theta}(W, l_W)$$

In the non-equivariant setting, [GR17a, Theorem 9.4] shows that there is a fibre sequence involving $\mathcal{M}_{\partial,\theta'}(W, l'_W)$ and $\mathcal{M}_{\partial,\theta}(W, l_W)$ which allows to translate computations for one of the two moduli spaces to computations for the other. This requires l'_W to be n-connected and u to be n-truncated, where W is of dimension $2n$. A factorisation $l_W = u \circ l_W \colon W \to B$ as such is given by the n-th stage of the Moore Postnikov tower of l_W and is unique up to homotopy. It can be functorial in the arrow $W \to B$, and we denote by τ_n the induced functor

$$\tau_n \colon \mathrm{Fun}(\Delta^1, \mathcal{S}) \to \mathrm{Fun}(\Delta^1, \mathcal{S}) \times_{\mathcal{S}} \mathrm{Fun}(\Delta^1, \mathcal{S})$$

The equivariant setting is similar, and a natural idea is to apply Moore Postnikov decomposition levelwise on the map $l_W \colon W \to B$ of functors $\mathcal{O}_G^{op} \to \mathsf{S}$ with stages depending on the dimension of each fixed point stratum, using the functoriality of τ_n. Unfortunately, such a factorisation may not always exist, as the functors τ_n do not come with a given natural transformation $\tau_n \implies \tau_m$ when $m > n$. We introduce the following definition to take this remark into account.

Definition 2.3.14. An orthogonal partial factorisation system on an ∞-category $\mathcal{C}$ is the datum of two classes of maps L, R of $\mathcal{C}$ stable under composition and retract, such that for any arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $f \in L$ and $g \in R$, f is left orthogonal to g in the sense of factorisation systems.

Let (L, R) be an orthogonal partial factorisation system on S_G and suppose that $u \in R$. Let $\mathrm{Map}^G_{\partial W/\mathsf{S}_G/B_G O(\bullet)}(W, B')_L$ be the subspace of those maps belonging to L, and let $\mathcal{M}^G_{\partial,\theta'}(W, l'_{\partial W})_L$ denote its homotopy $\mathrm{Diff}_\partial(W)$-orbits.

Proposition 2.3.15. *With the notations above, there is a* $\mathrm{Diff}_\partial(W)$*-equivariant equivalence*

$$\mathrm{Map}^G_{\partial W/\mathsf{S}_G/\mathrm{Gr}(\mathcal{U}_G)}(W, B')_L /\!\!/ \mathrm{Aut}^G(u, l'_{\partial W}) \to \mathrm{Map}^G_{\partial W/\mathsf{S}_G/\mathrm{Gr}(\mathcal{U}_G)}(W, B)$$

onto the path component that it hits. Hence, so is the map induced homotopy orbits

$$\mathcal{M}^G_{\partial,\theta'}(W, l'_{\partial W})_L /\!\!/ \mathrm{Aut}^G(u, l'_{\partial W}) \xrightarrow{-\circ u} \mathcal{M}^G_{\partial,\theta}(W, l_{\partial W})$$

Proof. This proposition is analogous to [GR17a, Lemma 9.2], and so is its proof. The homotopy fibre over l_W of the map

$$\mathrm{Map}^G_{\partial W/\mathsf{S}_G/\mathrm{Gr}(\mathcal{U}_G)}(W, B')_L \to \mathrm{Map}^G_{\partial W/\mathsf{S}_G/\mathrm{Gr}(\mathcal{U}_G)}(W, B) \tag{2.12}$$

is identified with the (derived) space of equivariant fillings

$$\begin{array}{ccc} \partial W & \xrightarrow{l_{\partial W}} & B' \\ \downarrow & \nearrow & \downarrow u \\ W & \xrightarrow{l_W} & B \end{array}$$

belonging to L, i.e. the mapping space $\mathrm{Map}^G_{\partial W/\mathsf{S}_G/B}(W, B')_L$. We show that this space is either empty or an $\mathrm{Aut}^G(u, l'_{\partial W})$-torsor. If it is not empty, let $l_W\colon W \to B'$ be an element of it, and consider the map

$$\mathrm{Map}^G_{\partial W/\mathsf{S}_G/B}(B', B') \xrightarrow{g \mapsto g \circ l_W} \mathrm{Map}^G_{\partial W/\mathsf{S}_G/B}(W, B')$$

We claim that it is an equivalence. Indeed, the homotopy fibre of this map taken at l is given by $\mathrm{Map}^G_{W/\mathsf{S}_G/B}(B', B')$, the space fillings of the diagram

$$\begin{array}{ccc} W & \xrightarrow{l} & B' \\ \downarrow l_W{}' & \nearrow & \downarrow u \\ B' & \xrightarrow{u} & B \end{array} \tag{2.13}$$

The latter is contractible because l'_W and u are orthogonal. Moreover, the map composing with l_W send the subspace $\mathrm{Aut}^G(u, l'_{\partial W}) \subseteq \mathrm{Map}^G_{\partial W/\mathsf{S}_G/B}(B', B')$ onto $\mathrm{Map}^G_{\partial W/\mathsf{S}_G/B}(W, B')_L$, indeed L is stable under composition and contains equivalences, and if $l\colon W \to B'$ is in $\mathrm{Map}^{G}_{\partial W/\mathsf{S}_G/B}(W, B')_L$, then the diagram (2.13) admits a lift f and the uniqueness of lifts up to contractible choice shows that it is an equivalence.

So, the fibres of (2.12) are either empty, or an $\mathrm{Aut}^G(u, l'_{\partial W})$-torsor, hence the result follows. Note that all maps are also $\mathrm{Diff}_\partial(W)$-equivariant, and that the actions of $\mathrm{Diff}_\partial(W)$ and $\mathrm{Aut}^G(u, l'_{\partial W})$ commute, so that we also get $\mathrm{Diff}_\partial(W)$-equivariance after taking homotopy orbits for the other action. □

Applying the orbit-stabiliser theorem, we get

Corollary 2.3.16. *Let $\Gamma \leq \mathrm{Aut}^G(u, l'_{\partial W})$ be the the submonoid on those maps $f\colon B' \to B'$ which preserve the θ'-structure l'_W of W up to an equivariant θ-diffeomorphism of (W, l_W). Then there is an equivalence*

$$\mathcal{M}^G_{\partial,\theta'}(W, l'_W)/\!/\Gamma \xrightarrow{-\circ u} \mathcal{M}^G_{\partial,\theta}(W, l_W)$$

and in particular, a fibre sequence

$$\mathcal{M}^G_{\partial,\theta'}(W, l'_W) \xrightarrow{-\circ u} \mathcal{M}^G_{\partial,\theta}(W, l_W) \to B\Gamma$$

Remark 2.3.17. The fibre sequence of moduli space above also has an analog at the level of infinite loop spaces: there is also an action of $\mathrm{Aut}^G(u, l'_{\partial W})$ on $MT\theta'$ compatible with the scanning map, and if $\mathrm{Aut}^G(u, l'_{\partial W})_{[W, l'_W]}$ denotes the union of path components that preserve $[W, l'_W] \in \pi_0(MT\theta')$, there is an associated fibre sequence

$$\Omega^\infty MT\theta' \to \Omega^\infty MT\theta \to B\mathrm{Aut}^G(u, l'_{\partial W})_{[W, l'_W]}$$

receieving a map from the first one.

Note that in the above proposition, all the spaces are spaces over $B_G O(\bullet)$, and the statement remains true if the orthogonal partial factorisation system is taken in the over-category $\mathsf{S}_G/B_G O(\bullet)$. There is a natural choice of such a system, which allows us to mimic [GR17a, Theorem 9.4] levelwise. Recall that for $H \leq G$ there is an equivalence

$$B_G O(\bullet)(G/H) \simeq \coprod_{V \in \mathrm{Rep}_H} B\mathrm{Aut}(V)$$

Define a map $\frac{1}{2}\colon B_G O(\bullet) \to \mathbb{Z}$ where $\mathbb{Z}$ is the constant presheaf equal to $\mathbb{Z}$, by $\frac{1}{2}_{G/H}(V) = \frac{1}{2}\dim(V^H)$ if $\dim(V^H)$ is even, and an arbitrary choice otherwise. Say that a map f of G-spaces over $B_G O(\bullet)$ is $\frac{1}{2}$-connected if for all $H \leq G$, $f_{G/H}$ is $\frac{1}{2}_{G/H}(V)$-connected after pulling back along the inclusion $B\mathrm{Aut}(V) \hookrightarrow B_G O(\bullet)(G/H)$, for all $V \in \mathrm{Rep}_H$. We define $\frac{1}{2}$-truncatedness in a similar way. Then, the classes of $\frac{1}{2}$-connected and $\frac{1}{2}$-truncated maps form an orthogonal partial factorisation system in $\mathsf{S}_G/B_G O(\bullet)$.

Proposition 2.3.18. *Let $d = 2k \geq 0$ and $\theta\colon B \to B_GO(d)$ be an equivariant tangential structure. Let (M, l_M) be a compact G-manifold with θ-structure and suppose that there exists a factorisation*

$$l_M\colon M \xrightarrow{l'_M} B' \xrightarrow{u} B$$

such that l'_M is a $\frac{1}{2}$-connected cofibration and u is a $\frac{1}{2}$-truncated fibration (for the genuine model structure). Suppose furthermore that the equivariant tangential structure $u \circ \theta$ satisfies the conditions of Corollary 2.3.8 and that M satisfies the gap hypothesis. Then, there exists a map

$$\mathcal{M}^G_\theta(M, l_M) \to \Omega^\infty_0(MT\theta')^G // \mathrm{Aut}(u, l'_{\partial M})_{[M, l'_M]}$$

which is $r_{\theta'}((g_i)_{i\in I})$-acyclic, where g_i denotes the θ'-genus of the i-th building block of (M, l'_M) and $r_{\theta'}$ is the output of Corollary 2.3.8 applied to (M, l'_M).

Proof. The (co)fibrancy conditions ensure that the the topological monoid $\mathrm{Aut}(u, l'_{\partial M})$ has the correct homotopy type. Hence this is a direct consequence of the discussion above and the proof of the non-equivariant version of this statement in [GR19, Corollary 4.6]. □

2.3.5 A note on Bredon homology

Our Theorem B concerns the space $\mathcal{M}^G_{\theta,\partial}(W, l_W)$ associated to (W, l_W), which can be seen as the G-fixed points of a G-space $\mathcal{M}_{\theta,\partial}(W, l_W)$. The latter is a path component of the morphism G-space of the cobordism G-*category* $\mathcal{C}_{\theta_d}$, defined by Galatius-Szűcs in [GS21]. What they show is an equivalence of genuine G-spaces $B\mathcal{C}_{\theta_d} \to \Omega^{\mathcal{U}_G - 1} MT\theta_d$. A natural question is the behaviour of the equivariant scanning map

$$\mathcal{M}_{\theta,\partial}(W, l_W) \to \Omega_{[(P,\hat{l}_P),\emptyset]}\Omega^{\mathcal{U}_G - 1} MT\theta$$

on homology with coeffients in a Mackey functor. Bredon homology of equivariant moduli spaces has already been considered in the literature, recently in [BQV23] for intance where the authors show a version of homological stability for equivariant configuration spaces.

It turns out that Corollary 2.3.8 can be adapted to take Mackey functor coefficients into account.

Corollary 2.3.19. *Suppose that θ is an equivariant tangential structure satisfying the hypotheses of Corollary 2.3.8, that l_W is $\frac{1}{2}$-connected and that W satisfies the gap hypothesis. Then, the scanning map*

$$\mathcal{M}_{\theta,\partial}(W, l_W) \to \Omega_{[(P,\hat{l}_P),\emptyset]}\Omega^{\mathcal{U}_G - 1} MT\theta$$

induces an isomorphism in $RO(G)$-graded homology with coefficients in all Mackey functors as long as the index (H, V) is such that $\dim V$ is in the same range as that given by Remark 2.3.10.

Proof. One shows by induction on the size of the family $\mathcal{F}$ that the map

$$\left(\Sigma^\infty_+(\mathcal{M}_{\theta,\partial}(W,l_W))\otimes A\otimes E\mathcal{F}_+\right)^G \to \left(\Sigma^\infty_+\Omega_{[(P,\hat{l}_P),\emptyset]}\Omega^{\mathcal{U}_G-1}MT\theta\otimes A\otimes E\mathcal{F}_+\right)^G \tag{2.14}$$

is r-connected where A is a Mackey functor, identified with its associated Eilenberg-MacLane spectrum, and r is the range given by Corollary 2.3.8 applied to (W,l_W). The base case corresponds, under the Adams isomorphism, to the map of spectra

$$\Sigma^\infty_+(\mathcal{M}_{\theta,\partial}(W,l_W)_{hG})\otimes A_{hG}\to\Sigma^\infty_+(\Omega_{[(P,\hat{l}_P),\emptyset]}\Omega^{\infty-1}MT\theta)_{hG}\otimes A_{hG}$$

The functor $D_{\geq 0}(\mathbb{Z})\to\mathsf{Sp}$ preserves colimits (where on the right is the category of spectra) so that A_{hG} can be computed in $D_{\geq 0}(\mathbb{Z})$-valued Mackey functors. Hence A_{hG} is a sum of (positive) suspensions of Eilenberg-MacLane spectra. Hence the map above is indeed $r_{(W,l_W)}$-connected by Corollary 2.3.8.

If the statement is true for a family $\mathcal{F}$ then given any $H\notin\mathcal{F}$ we consider the map induced between the two $(\mathcal{F}+(H),H)$-isotropy separation sequences, as described in section 2.2.1. The map induced on fibres is $r_{(W,l_W)}$-connected by the induction step, and on the base is given by the W_GH-orbits of the geometric H-fixed points. Before passing to W_GH-orbits, the latter is equivalent to the map

$$\Sigma^\infty_+B\mathrm{Diff}^H_{\theta,\partial}(W,l_W)\otimes\Phi^H(A)\to\Sigma^\infty_+\Omega_{[(P,\hat{l}_P),\emptyset]}\Omega^{\infty-1}(MT\theta)^H\otimes\Phi^H(A)$$

where we see (W,l_W) as an H-manifold by restricting the action of G. As Φ^H preserves colimits, the remark above about $(-)_{hG}$ also applies for it, hence $\Phi^H(A)$ is a sum of (positive) suspensions of Eilenberg-MacLane spectra. We now wish to apply Corollary 2.3.8 to the H-manifold (W,l_W), in a way that the range of stability is at least as good as that for (W,l_W) seen as a G-manifold. We denote this H-manifold with structure by $({}_HW,l_{{}_HW})$. The fact that the G-manifold W verifies the gap hypothesis implies that ${}_HW$ does as well, hence Corollary 2.3.8 applies to $({}_HW,l_{{}_HW})$. As for the range of stability, it can be verified that those for the building blocks of $({}_HW,l_{{}_HW})$ are greater or equal than those for the corresponding building blocks for (W,l_W): this follows from a case by case analysis using the explicit description or the range in Remark 2.3.10, and the fact that the θ-genus decreases along submanifold inclusions. Hence the map (2.14) is r-connected. If instead of taking genuine G-fixed points we take H-fixed points, it remains r-connected by the same argument. Hence, it is V-connected for $\dim V\leq r$. □

2.4 Examples of computation

In order to describe examples of computations, it will be convenient to use $\mathrm{Fun}(\mathcal{O}_G^{op},\mathrm{Top})$ as model for G-spaces by the Elmendorf theorem. As noted in Remark 2.2.22, our definitions and results also hold for that choice of model.

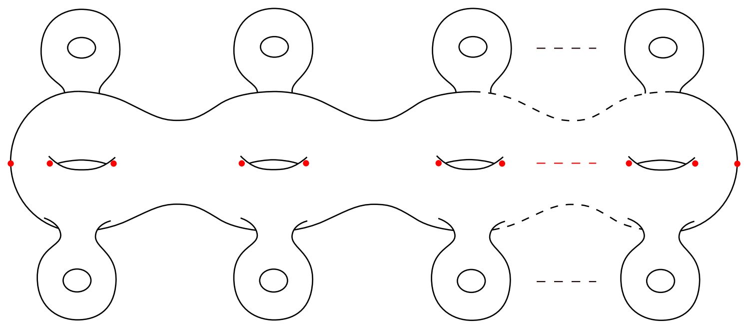


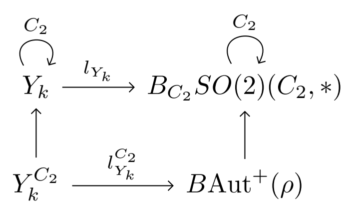
Figure 2.3: The C_2-surface Y_k, with fixed points marked in red.

2.4.1 An *ad hoc* example

We first show how to apply our main result in a sort of canonical example, taylored to satisfy all its hypotheses. Consider the C_2-surface Y_k defined as follows: first start with W_k the genus k surface endowed with the hyperelliptic involution. Then $W_k = W_1 \#_{C_2/C_2} W_1 \#_{C_2/C_2} \cdots \#_{C_2/C_2} W_1$, where the subscript means that the connected sum is done along a disc which is an equivariant tubular neigborhood of a fixed point. We now replace W_1 by $Y_1 := W_1 \#_{C_2/e} \{S^1 \times S^1 \times C_2\}$, i.e. the connect sum of W_1 with two copies of the torus, glued at a point of free orbit, where the induced action swaps the two tori. We then define Y_k to be equal to $Y_1 \#_{C_2/C_2} Y_1 \#_{C_2/C_2} \cdots \#_{C_2/C_2} Y_1$, see Figure 2.4.1.

The cardinality of the fixed points $Y_k^{C_2}$ is equal to $2k+2$ (note that it is a 0-dimensional manifold of genus k), and the genus of $(Y_k)_{*,\partial}/C_2 \simeq (Y_k - Y_k^{C_2})/C_2$ is equal to k.

We shall compute the rational cohomology of $B\mathrm{Diff}^{C_2}_{\mathfrak{o}}(Y_k, l_{Y_k})$ in the stable range, where $\mathfrak{o}$ denotes the equivariant tangential structure hit by Y_k inside $B_{C_2}SO(2)$. The map l_{Y_k} is the following morphism in $\mathrm{Fun}(\mathcal{O}_{C_2}^{op}, \mathrm{Top})$



where ρ is the tangential representation of the fixed points, namely the unique irreducible 2-dimensional representation of C_2. Note that $\mathrm{Aut}^+(\rho) \simeq SO(2)$. Because $BSO(2)$ is simply connected, we have that

- the C_2-manifold $(Y_k^{C_2}, l_{Y_k}^{C_2})$ is r-stable for $r = \#Y_k^{C_2}/3 = \frac{1}{3}(2k+2)$, by [Kra19],

- the map $(Y_k)_{*,\partial} \hookrightarrow Y_k \xrightarrow{l_{Y_k}} B_{C_2}SO(2)(C_2/*)$ is 1-connected, and as a consequence its C_2-homotopy orbits i.e. $(l_{Y_k})_{hC_2}\colon (Y_k)_{*,\partial}/C_2 \to B_{C_2}SO(2)_{hC_2} \simeq BSO(2) \times BC_2$ are as well.

The 2-dimensional tangential structure $\theta_2\colon BSO(2) \times BC_2 \to BO(2)$ is not one of those studied in [Ran16], so we still have to argue about homological stability for this structure. Although $\pi_1(MTSO(2) \otimes (BC_2)_+) \neq *$, so that local systems of coefficients might appear, it is enough by Remark 2.3.4 to study homological stability for integral coefficients in this case. The later turns out to be satisfied, and is a consequence of [Put23, Theorem A].

Proposition 2.4.1. *Let (S, l_S) be a oriented surface of genus g with at least one boundary component, where l_S is a θ_2-structure on S. Picking a path component of ∂S defines a stabilisation map*

$$\eta\colon B\mathrm{Diff}_{\partial,\theta_2}(S, l_S) \to B\mathrm{Diff}_{\partial,\theta_2}(S \natural W_{1,1}, l'_S)$$

where the map $l'_S\colon S\natural W_{1,1} \to BC_2$ is an extension of l_S which is nulhomotopic on $W_{1,1}$.

Then, $\eta_*\colon H_*(B\mathrm{Diff}_{\partial,\theta_2}(S, l_S); \mathbb{Z}) \to H_*(B\mathrm{Diff}_{\partial,\theta_2}(S\natural W_{1,1}, l'_S); \mathbb{Z})$ is an isomorphism for $* \leq \frac{g-4}{3}$, and is surjective for $* = \lfloor \frac{g-3}{3} \rfloor$.

Proof. By a theorem of Earle-Eells ([EE69]), for $S' \in \{S, S\natural W_{1,1}\}$, $\mathrm{Diff}_\partial(S')$ is discrete. Moreover, $\mathrm{Map}_\partial(S', BC_2)$ is also discrete because $\partial S' \neq \emptyset$, hence so is $\mathrm{Diff}_{\partial,\theta_2}(S', l_{S'})$. The map η_* can then be seen as a stabilisation map between mapping class groups with homology markings in the sense of [Put23].
Theorem A in op. cit. is a stability theorem for surfaces with only one boundary component. Although S can have several boundary components, the stabilisation map only uses one, and in that case the proof of [Put23, Theorem A] can be reproduced verbatim, ignoring the remaining components. Hence the proposition follows. □

Thus the hypotheses of Theorem B are satisfied modulo Remark 2.3.4, and we can deduce the following calculation.

Corollary 2.4.2. *The natural map $\mathcal{M}^{C_2}_{\mathfrak{o}}(Y_k, l_{Y_k}) \to \Omega^\infty_0(MT\mathfrak{o}^{C_2})$ inducing a homology isomorphism for all constant coefficients, in degrees $* \leq \frac{k-4}{3}$.*
In particular with rational coefficients, this homology coincides with the graded algebra $\mathbb{Q}[\kappa_i, \kappa'_j \mid i, j \geq 0]$ in these degrees, where

- *κ_i is of degree $2i$ and is the MMM class associated to the class c_{i+1} in $H^{2i+2}(\Omega^\infty_0 MTSO(2); \mathbb{Q})$ coming from an associated surface bundle,*
- *κ'_j is of degree $2j$ and is the MMM class associated to the class c_j in $H^{2j}(\Omega^\infty_0 \Sigma^\infty_+ BSO(2); \mathbb{Q})$ coming from the associated bundle of configurations labelled in $BSO(2)$*

by Corollary 2.3.12.

2.4.2 Hypersurfaces in $\mathbb{C}P^4$

In [GR19], the authors consider a complex manifold V_d, and compute the rational homology of its moduli space in a range. In this section, we show that V_d admits an action of the group C_3, and we apply Theorem B to compute the rational homology of its oriented equivariant moduli space in a range. We shall often refer to [GR19] in this section, as our argument is simply an extension of the latter.

The complex manifold $V_d \subseteq \mathbb{C}P^4$ is defined as the zero locus of a non degenerate polynomial of degree d on $\mathbb{C}P^4$. Its diffeomorphism type does not depend on the chosen polynomial, hence we can make the arbitrary choice of $P := X_0^d + X_1^d + \cdots + X_4^d$.

With this choice of P, the zero locus V_d clearly inherits an action of the symmetric group Σ_5, induced from the global action of Σ_5 on $\mathbb{C}P^4$ that permutes the coordinates.

The Σ_5-manifold V_d does not verify the assumptions of B, therefore we will consider a simpler subaction. Namely, let $C_3 \to \Sigma_5$ be the inclusion of a 3-cycle in Σ_5, and consider V_d as a C_3-manifold by restriction the action. We are interested in its *equivariantly oriented* moduli space, which is the classifying space for the topological group of those equivariant diffeomorphisms of V_d which preserving the orientation of V_d as well as that of its C_3-fixed points. This condition corresponds to a certain C_3-equivariant tangential structure.

For convenience, in this section we replace the orthogonal group $O(6)$ by its linear analog $\mathrm{GL}(6)$. Note that the inclusion $O(6) \hookrightarrow \mathrm{GL}(6)$ induces a weak equivalence $BO(6) \simeq B\mathrm{GL}(6)$. As V_d is a complex C_3-manifold of complex dimension 3, we make can make a choice of a C_3-map $\tau^{\mathbb{C}}_{V_d}\colon V_d \to B_{C_3}\mathrm{GL}(3,\mathbb{C})$ in $\mathrm{Fun}(\mathcal{O}^{op}_{C_3}, \mathrm{Top})$ such that $\tau^{\mathbb{C}}_{V_d}$ composed with the map $B_{C_3}\mathrm{GL}(3,\mathbb{C}) \xrightarrow{B\iota} B_{C_3}\mathrm{GL}(6)$ classifies its equivariant tangent bundle, where $\iota\colon \mathrm{GL}(3,\mathbb{C}) \to \mathrm{GL}(6)$ is the standard inclusion.

The map $B_{C_3}\mathrm{GL}_{>0}(6) \to B_{C_3}\mathrm{GL}(6)$ defines an equivariant tangential structure which we denote by $\mathfrak{o}$ where the subscript > 0 denotes positive determinant matrices; $l^{\mathbb{C}}_{V_d}$ induces a canonical choice of an $\mathfrak{o}$-structure $l^+_{V_d}$ on V_d. We wish to compute the cohomology of $\mathcal{M}^{C_3}_{\mathfrak{o}}(V_d, l^+_{V_d})$.

Following [GR19] and the hypotheses of B, we shall proceed as follows.

(i) Identify the diffeomorphism type of the fixed points $V_d^{C^3}$

(ii) Find an equivariant factorisation of $l^+_{V_d}\colon V_d \xrightarrow{l_{V_d}} B_d \xrightarrow{\theta_d} B_{C_3}\mathrm{GL}_{>0}(6)$ by an $\frac{1}{2}$-connected map followed by an $\frac{1}{2}$-truncated map, such that (V_d, l_{V_d}) satisfies the hypotheses of Theorem B.

(iii) Compute the $\theta_d^{C_3}$-genus of $V_d^{C_3}$, and the θ_{hC_3}-genus of $(V_d)_{*,\partial}/C^3 \simeq (V_d - V_d^{C^3})/C^3$ and deduce a lower bound for the stable range in Theorem B,

(iv) Compute $H^*(\mathcal{M}^{C_3}_{\theta_d}(V_d, l_{V_d}); \mathbb{Q})$ in the stable range,

(v) Deduce a computation of $H^*(\mathcal{M}^{C_3}_{\mathfrak{o}}(V_d, l_{V_d}); \mathbb{Q})$ in some range using Corollary 2.3.16.

We will first have to restrict to those d which are multiples of 3, and we eventually focus on those that are divisible by 6.

2.4.2.1 Preliminaries

From [GR19] we know that the 6-dimensional manifold V_d has genus equal to $d^4 - 5d^3 + 10d^2 - 10d + 4$, which is proved by first computing its Euler characteristic, equal to $d(10 - 10d + 5d^2 - d^3)$.

We reproduce the same argument to derive the genus of $V_d^{C^3}$, and then that of $(V_d)_{*,\partial}$.

We make the explicit choice of 3-cycle $(2,3,4)$ in $S_5 = \mathrm{Aut}\{0,1,2,3,4\}$. Then,

$$V_d^{C_3} = V_d \cap \{[x_0 : x_1 : x_2 : x_3 : x_4] = [x_0 : x_1 : x_3 : x_4 : x_2]\} \subseteq \mathbb{C}P^4$$

where the right-end term can be rewritten as the union

$$\{[x_0 : x_1 : x_2 : x_2 : x_2]\} \cup \{[0 : 0 : x_2 : jx_2 : j^2x_2]\} \cup \{[0 : 0 : x_2 : j^2x_2 : jx_2]\}$$

which we denote by $\mathbb{C}P^{2\prime} \cup \mathbb{C}P^{0\prime}_0 \cup \mathbb{C}P^{0\prime}_1$.

- $V_d \cap \mathbb{C}P^{2\prime}$ can be seen as a the zero locus of the non-degenerate polynomial

  $$P^{C_3} := X_0^d + X_1^d + 3X_2^d$$

  in $\mathbb{C}P^2$, so it has real dimension 2, and has the diffeomorphism type of the complex dimension 1 analogue of V_d.
- The two other spaces of the form $V_d \cap \mathbb{C}P^{0\prime}_i$ are either empty or a point, according to if the equation

  $$1 + \omega^d + \omega^{2d} = 0$$

  is satisfied or not, where $\omega = \sqrt[3]{-1}$. Thus, they are empty when $3|d$, and are not otherwise.

It follows that if d is not a multiple of 3, $V_d^{C^3}$ has components of dimension 0 with constant genus. Hence the conclusion of Theorem B is of little interest in this case. Therefore we shall from now on assume that d is divisible by 3.

Writing $H^2(V_d^{C^3};\mathbb{Z}) = \mathbb{Z}\langle u\rangle$, then $i^*x = du$ for x a generator of $H^2(\mathbb{C}P^2;\mathbb{Z})$, because P has degree d. Also, $V_d^{C^3}$ is a transverse pullback of the 0-section $\mathbb{C}P^2 \to \mathcal{O}(d)$, thus has normal bundle $i^*\mathcal{O}(d)$. Therefore there is an isomorphism of complex vector bundles

$$TV_d^{C^3} \oplus i^*\mathcal{O}(d) \oplus \underline{\mathbb{C}} \cong i^*(T\mathbb{C}P^2) \oplus \underline{\mathbb{C}} \cong i^*(\mathcal{O}(1)^{\oplus 3})$$

We deduce the following formula for the total Chern class

$$c(V_d^{C^3}) = i^*\left(\frac{(1+x)^3}{1+dx}\right)$$

so that $c_1(V_d^{C^3}) = (3-d)i^*x = (3d-d^2)u$.

Computing the Euler characteristic as $\langle [V_d^{C^3}], c_1(V_d^{C^3})\rangle$ gives

$$\chi(V_d^{C^3}) = 3d - d^2$$

Therefore, the real surface $V_d^{C^3}$ has genus

$$g(V_{3d}^{C^3}) = 2 - \chi(V_d^{C^3}) = d^2 - 3d + 2$$

2.4.2.2 Factoring $l^+_{V_d}$

Letting $[V]$ be the isomorphism class of the C_3-representation on the fixed points of V_d, the $\mathfrak{o}$-structure of V_d amounts to a diagram

$$\begin{array}{ccc} V_d & \xrightarrow{f} & B_{C_3}\mathrm{GL}_{>0}(6)(C_3/*) \\ \uparrow & & \uparrow \\ V_d^{C_3} & \xrightarrow{g} & BS\mathrm{Aut}(V) \end{array}$$

where the top map is C_3-equivariant.

Recall the following result from [GR19].

Theorem A ([GR19, section 12.5.3.3])**.** *Consider V_d as an non-equivariant manifold, together with its canonical orientation $\tau\colon V_d \to B\mathrm{GL}_{>0}(6)$. Then, the 3-stage of the Moore–Postnikov decomposition of τ is given by*

$$V_d \xrightarrow{l_{V_d}} B_d \xrightarrow{\theta} B\mathrm{GL}_{>0}(6)$$

where

$$\theta = \begin{cases} B\mathrm{Spin}^c(6) \to B\mathrm{GL}_{>0}(6) & \text{if } d \text{ is even} \\ B\mathrm{Spin}(6) \times K(\mathbb{Z},2) \to B\mathrm{GL}_{>0}(6) & \text{if } d \text{ is odd} \end{cases}$$

We will focus on the case when d is even for simplicity. That is, we will from now consider degrees which are divisible by 6. In order to make Theorem A equivariant for the action of C_3, we give a geometric description of the map l_{V_d}. In order to avoid dealing with hermitian metrics, we define $\Pi(n) := (\mathrm{GL}_{>0}(n) \times \mathbb{C}^*)/C_2$ which is a non orthogonal analog of $\mathrm{Spin}^c(n)$. By definition there is an inclusion of a compact subgroup $\mathrm{Spin}^c(n) \hookrightarrow \Pi(n)$ which is a homotopy equivalence.

There is a fibre sequence of Lie groups

$$\mathbb{C}^* \to \Pi(n) \to \mathrm{GL}_{>0}(n)$$

and a 2-sheeted covering map $p\colon \Pi(n) \to \mathrm{GL}_{>0}(n) \times \mathbb{C}^*$.

Let $\iota\colon \mathrm{GL}(k,\mathbb{C}) \to \mathrm{GL}_{>0}(2k)$ be the standard embedding, then the map $\iota \times \det\colon \mathrm{GL}(k,\mathbb{C}) \to \mathrm{GL}_{>0}(2k) \times \mathbb{C}^*$ induces on π_1 the map $\mathbb{Z} \to \mathbb{Z}/2 \oplus \mathbb{Z}$ which coincides with $\pi_1(p)$, so there exists a lift

$$\iota \times \det\colon \mathrm{GL}(k,\mathbb{C}) \xrightarrow{\rho} \Pi(2k) \xrightarrow{p} \mathrm{GL}_{>0}(2k) \times \mathbb{C}^*$$

Hence if M is a complex G-manifold, its complex equivariant Gauss map $\tau_M^{\mathbb{C}}\colon M \to B_G\mathrm{GL}(k,\mathbb{C})$ can be composed with $B_G\rho\colon B_G\mathrm{GL}(k,\mathbb{C}) \to B_G\Pi(2k)$ endowing M with an equivariant $\Pi(2k)$-structure. We shall denote the C_3-equivariant tangential structures $B_{C_3}(\Pi(n)) \to B_{C_3}\mathrm{GL}(n)$ by $\mathfrak{o}_\Pi$ for all n, the value of n will be clear from the context.

An $\mathfrak{o}_\Pi$-structure on a manifold can be twisted by a complex line bundle as follows. The kernel of the projection $\Pi(n) \to \mathrm{GL}_{>0}(n)$, isomorphic to $\mathbb{C}^*$, is contained in the centre of $\Pi(n)$. This induces a morphism given by multiplication $\mu\colon \Pi(n) \times \mathbb{C}^* \to \Pi(n)$. It follows after passing to classifying spaces that if l_M is a $\mathfrak{o}_\Pi$-structure on a manifold M, and $\mathbb{L}$ is a complex line bundle on M, there is a new $\mathfrak{o}_\Pi$-structure on M which we denote by $l_W \otimes \mathbb{L}$, the twist of l_W by $\mathbb{L}$. Letting $\det(l_M)$ denote the determinant bundle associated to the $\mathfrak{o}_\Pi$-structure l_M i.e. the line bundle classified by $B\det \circ l_M\colon M \to B\mathbb{C}^*$, the following equation about first Chern classes is satisfied ([GGK02, Proposition D.43]).

$$c_1(l_M \otimes \mathbb{L}) = c_1(\det l_M) + 2c_1(\mathbb{L}) \tag{2.15}$$

Note that the vector bundles $\mathcal{O}(k)$ on $\mathbb{C}P^4$ are equivariant for the action of Σ_5, hence for that of C_3.

Lemma 2.4.3. *Suppose that $6|d$ and let $o_d\colon V_d \to B_{C_3}\mathbb{C}^*$ be a C_3-map classifying the C_3-equivariant complex line bundle $i^*\mathcal{O}(d/2-2)$ on V_d. Define l_{V_d} to be the C_3-equivariant twist of the canonical $\mathfrak{o}_\Pi$-structure on V_d by $i^*\mathcal{O}(d/2-2)$, i.e. the composition*

$$l'_{V_d}\colon V_d \xrightarrow{(B\rho\circ\tau^{\mathbb{C}}_{V_d})\times o_d} B_{C_3}\Pi(6) \times B_{C_3}\mathbb{C}^* \to B_{C_3}\Pi(6)$$

Then, l'_{V_d} is 3-connected, and hence is a (Borel) C_3-equivariant model for the factorisation in Theorem A.

Note that for the statement above to make sense, we must make a choice of a classifying space functor $B_{C_3}\colon \mathrm{TopGrp} \to \mathrm{Fun}(\mathcal{O}^{op}_{C_3}, \mathrm{Top})$ which commutes with products, which we fix for the rest of this section.

Proof. Both spaces are simply connected. On $H^2(-;\mathbb{Z})$, the map is given by the first Chern class of the determinant bundle of l'_{V_d}. By the equation 2.15, it is given by $c_1(V_d)+2c_1(\mathcal{O}(d/2-2)) = 5-d+2(d/2-2) = 1$, so that the map is an isomorphism on H^2 and hence on π_2 by Hurewicz theorem. The space $B_{C_3}\Pi(6)$ has vanishing π_3 because $\Pi(6)$ is a Lie group, hence the proof follows. □

Let $[V]$ be the isomorphism class of the tangential representation of C_3 on the fixed points of V_d. The representation V comes from a complex representation, which can be checked to be isomorphic to $\lambda\colon C_3 \to U(3)$ given by

$$\sigma \mapsto A := \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}; \quad M = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

for σ a generator of C_3. The 6×6 matrix $\iota(A)$ is conjugate to the real block matrix

$$B := \begin{pmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{pmatrix}$$

and the centralizer of B in $\mathrm{GL}_{>0}(6)$ is isomorphic to $\mathrm{GL}_{>0}(2) \times \mathrm{GL}(2, \mathbb{C})$, embedded inside $\mathrm{GL}_{>0}(2) \times \mathrm{GL}_{>0}(4) \subset \mathrm{GL}_{>0}(6)$ by the identity on the first factor, and on the second factor by the map $\mathrm{GL}(2, \mathbb{C}) \to \mathrm{GL}_{>0}(4)$ induced by the map

$$\mathbb{C} \to M(2, \mathbb{R}), \quad a + \omega b \mapsto \begin{pmatrix} a & b \\ -b & a-b \end{pmatrix}$$

Note that this non-standard inclusion is conjugate to the standard one by the block matrix

$$P = \begin{pmatrix} I & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{pmatrix}; \quad C = \begin{pmatrix} \sqrt{3} & 0 \\ 1 & 2 \end{pmatrix}$$

In turn, the centralizer $C_\lambda(\mathrm{GL}_{>0}(6))$ is conjugate to the subgroup $\mathrm{GL}_{>0}(2) \times \mathrm{GL}(2, \mathbb{C}) \leq \mathrm{GL}_{>0}(6)$.

Remark 2.4.4. This conjugation happens inside $\mathrm{GL}_{>0}(6)$ and of course does not preserve the inclusion $O(6) \hookrightarrow \mathrm{GL}(6)$ which why we chose to use linear groups instead of orthogonal groups for this example.

For $\theta\colon B \to B_{C_3}\mathrm{GL}(6)$ an equivariant tangential structure and $l\colon V_d \to B$ a θ-structure on V_d, level-wise restricting to the path component which is hit by l gives a subfunctor of θ which we refer as *the tangential structure θ after restriction to V_d*.

Lemma 2.4.5. *There is a factorisation of equivariant tangential structures after restriction to V_d*

$$\mathfrak{o}_\Pi \to \theta \to \mathfrak{o}$$

such that

- *the map of Borel C_3-spaces underlying $\mathfrak{o}_\Pi \to \theta$ is an equivalence,*
- *the map on fixed points underlying $\theta \to \mathfrak{o}$ is given by the path-component inclusion of $BC_\lambda(\mathrm{GL}_{>0}(6))$ after restriction to V_d.*

In particular θ satisfies the hypotheses of Corollary 2.3.8. Moreover, there is an equivalence between the fixed point structure θ^{C_3} and the product structure $B\mathrm{GL}_{>0}(2) \times B\mathrm{GL}(2, \mathbb{C}) \to B\mathrm{GL}(2)$.

Proof. Let p be the 2-sheeted covering map $p\colon \Pi(6) \xrightarrow{\pi \times \det} \mathrm{GL}_{>0}(6) \times \mathbb{C}^*$, clearly $\pi^{-1}(C_\lambda(\mathrm{GL}_{>0}(6))) = p^{-1}(C_\lambda(\mathrm{GL}_{>0}(6) \times \mathbb{C}^*))$. There is an inclusion $\mathcal{C}_\lambda(\Pi(6)) \subseteq \pi^{-1}(C_\lambda(\mathrm{GL}_{>0}(6)))$ and on the other hand, $x \in \pi^{-1}(C_\lambda(\mathrm{GL}_{>0}(6)))$

iff $AxA^{-1}x^{-1} \in \ker p = \{I, I'\}$. We now show that this preimage is connected so that only I can be hit.

The inclusion $i\colon C_\lambda(\mathrm{GL}_{>0}(6)) \hookrightarrow \mathrm{GL}_{>0}(6)$ is conjugate to the standard one $\mathrm{GL}_{>0}(2) \times \mathrm{GL}(2,\mathbb{C}) \leq \mathrm{GL}_{>0}(6)$ so that the maps on π_1 coincide. Hence, as in the standard case, the map

$$C_\lambda(\mathrm{GL}_{>0}(6)) \xrightarrow{\pi\times\det} \mathrm{GL}_{>0}(6) \times \mathbb{C}^*$$

admits a lift through p. This gives a section of π along $C_\lambda(\mathrm{GL}_{>0}(6))$, so that $\pi^{-1}(C_\lambda(\mathrm{GL}_{>0}(6))) \simeq C_\lambda(\mathrm{GL}_{>0}(6)) \times U(1)$ in particular it is connected.

In turn we deduce that $\mathcal{C}_\lambda(\Pi(6)) = \pi^{-1}(C_\lambda(\mathrm{GL}_{>0}(6)))$, so that $\pi\colon \mathcal{C}_\lambda(\Pi(6)) \to C_\lambda(\mathrm{GL}_{>0}(6))$ is isomorphic to the projection $C_\lambda(\mathrm{GL}_{>0}(6)) \times \mathbb{C}^* \to C_\lambda(\mathrm{GL}_{>0}(6))$.

Define the group morphism

$$s\colon C_\lambda(\mathrm{GL}_{>0}(6)) \xrightarrow{\mathrm{id}\times *} C_\lambda(\mathrm{GL}_{>0}(6)) \times U(1) \cong \mathcal{C}_\lambda(\Pi(6))$$

We define $B \in \mathrm{Fun}(\mathcal{O}_{C_3}^{op}, \mathrm{Top})$, the base space of a C_3-equivariant tangential structure θ, as the middle column in the following diagram, which describes the factorisation $\mathfrak{o}_\Pi \to \theta \to \mathfrak{o}$ after restriction to V_d

$$\begin{array}{ccccc}
B_{C_3}\Pi(6)(C_3/*) & \longrightarrow & B_{C_3}\Pi(6)(C_3/*) & \longrightarrow & B_{C_3}\mathrm{GL}_{>0}(6)(C_3/*) \\
{\scriptstyle u}\uparrow & & {\scriptstyle Bs\circ u}\uparrow & & \uparrow \\
BC_\lambda(\Pi(6)) & \longrightarrow & BC_\lambda(\mathrm{GL}_{>0}(6)) & \longrightarrow & BC_\lambda(\mathrm{GL}_{>0}(6))
\end{array}$$

The tangential structure θ clearly verifies the first two properties claimed. The third follows from the fact that $C_\lambda(\mathrm{GL}_{>0}(6)) \hookrightarrow \mathrm{GL}_{>0}(6)$ is conjugate in $\mathrm{GL}_{>0}(6)$ to the subgroup $\mathrm{GL}_{>0}(2) \times \mathrm{GL}(2,\mathbb{C})$ and the conjugation preserves the projection to $\mathrm{GL}_{>0}(2)$. □

Lemma 2.4.6. *Suppose $6|d$. The $\mathfrak{o}_\Pi$-structure l'_{V_d} gives rise to a θ-structure l_{V_d} on V_d. This structure is $\frac{1}{2}$-connected and (V_d, l_{V_d}) satisfies the hypotheses of Corollary 2.3.8.*

Proof. By Lemma 2.4.3, the map induced $l_{V_d}(C_3/*)$ is 3-connected, also the map $l_{V_d}(C_3/C_3)$ is clearly 1-connected because the $BC_\lambda(\mathrm{GL}(6))$ is simply connected. Moreover V_d clearly satisfies the gap hypothesis. □

2.4.2.3 Estimating the genus of $V_d^{C_3}$ and $(V_d)_{*,\partial}$

Suppose $6|d$. The stable range in Corollary 2.3.8 depends on the θ^{C_3}-genus of $(V_d^{C_3}, l_{V_d})$, and the θ_{hC_3}-genus of $((V_d)_{*,hC_3}, (l_{V_d})_{hC_3})$.

We have seen that there is a 1-stage Moore Postnikov decomposition

$$V_d^{C_3} \xrightarrow{l'_{V_d}} B_d^{C_3} \to B\mathrm{GL}(6)^{C_3}$$

hence from [GR17b, Lemma 9.4], the θ^{C_3}-genus of $((V_d^{C_3}, l_{V_d}))$ coincides with its genus in the usual sense. From the earlier computation, we deduce

$$g^{\theta^{C_3}}((V_d^{C_3}, l_{V_d})) = d^2 - 3d + 2$$

We now wish to compute the θ_{hC_3}-genus of $((V_d)_{*,hC_3}, (l_{V_d})_{hC_3})$. We do not know how to compute it explicitely, rather, we shall prove a certainly unoptimal lower bound.

Technical preparations We explain how to generalize the argument of [GR17a, Remark 7.16] where the authors show how to approximate the θ-genus of a simply connected manifold using only computable algebraic invariants.

Suppose that we are given an oriented compact manifold M of dimension $2n$ with fundamental group G, and a θ-structure $l_M\colon M \to B$ which is n-connected. Smale-Hirsch theory identifies the $\mathbb{Z}[G]$-module $\pi_n(\mathrm{Fr}(M))$ with the set of regular homotopy classes of immersions of framed spheres inside M together with a path in $\mathrm{Fr}(M)$ to a prescribed point, denoted by $I_n^{\mathrm{fr}}(M)$. As explained in [GR17a, Definition 5.2], the latter can be equipped with an ε-symmetric $\mathbb{Z}[G]$-bilinear form λ for $\varepsilon = (-1)^n$ which computes intersections of spheres, and which can be refined into an ε-quadratic form $\mu\colon I_n^{\mathrm{fr}}(M) \to \mathbb{Z}[G]/(g - \varepsilon g^{-1})$ which computes self-intersections. Taking the maps induced in homology gives map $h_*\colon I_n^{\mathrm{fr}}(M) \to H_n(M;\mathbb{Z})$ which is compatible with the Poincaré pairing – note that there is no a priori no given quadratic refinement on the right.

Given R a ring with involution and (M, λ) an ε-symmetric R-module, recall that the (symmetric) Witt index $g_s(M, \lambda)$ is defined as the largest g such that there exists a decomposition $(M, \lambda) \cong (H, \lambda_h)^{\oplus g} \oplus (M', \lambda')$, for (H, λ_h) hyperbolic. When (M, λ, μ) is an ε-quadratic R-module, $g_q(M, \lambda, \mu)$ is defined in the same way with instead (H, λ_h, μ_h) hyperbolic for the ε-quadratic structure. We shall omit λ, μ from the notations when there is no confusion.

The proof of [Fri17, Theorem 4.7] shows that the genus $g(M)$ is equal to the quadratic Witt index of $I_n^{\mathrm{fr}}(M) = \pi_n(\mathrm{Fr}(M))$. In the presence of a θ-structure $l_M\colon M \to B$, recall that the θ-genus of (M, l_M) only counts those copies of $S^n \times S^n \setminus D^{2n}$ on which the pulled back structure is standard. Define $I_n^{\mathrm{fr}}(M, l_M) \subseteq I_n^{\mathrm{fr}}(M)$ as in [GR17a, Remark 7.16] to be the subset of those classes $[f\colon S^n \times D^n \hookrightarrow M]$ such that $f^* l_M$ is standard. Then the Smale-Hirsch theorem implies that the map taking homotopy classes

$$I_n^{\mathrm{fr}}(M, l_M) \to \ker\left(\pi_n(\mathrm{Fr}(M)) \xrightarrow{(l_M)_*} \mathrm{Fr}(\theta^*\gamma_{2n})\right)$$

is an isomorphism, where γ_{2n} is the universal $2n$-dimensional vector bundle. In the same way as before, there is an equality $g^\theta(M, l_M) = g_q(I^{\mathrm{fr}}(M, l_M))$ where the ε-quadratic structure is obtained by composing with the inclusion into $I^{\mathrm{fr}}(M)$.

The argument is then as follows: $\ker(\pi_n(\mathrm{Fr}(M)) \xrightarrow{(l_M)_*} \mathrm{Fr}(\theta^*\gamma_{2n})$ is equal to $\mathrm{Im}(\pi_{n+1}(\mathrm{Fr}(\theta^*\gamma_{2n}), \mathrm{Fr}(M)) \xrightarrow{\partial} \pi_n(\mathrm{Fr}(M)))$ so there is an inequality

$$g_s(\pi_{n+1}(\mathrm{Fr}(\theta^*\gamma_{2n}), \mathrm{Fr}(M))) \geq g_s(I^{\mathrm{fr}}(M, l_M)).$$

On the other hand, because l_M is n-connected the Hurewicz theorem implies

$$\pi_{n+1}(\mathrm{Fr}(\theta^*\gamma_{2n}), \mathrm{Fr}(M))) \cong \pi_{n+1}(\tilde{B}, \tilde{M}) \cong H_{n+1}(\tilde{B}, \tilde{M})$$

There is an exact sequence

$$H_{n+1}(\tilde{B}, \tilde{M}) \xrightarrow{\partial} H_n(\tilde{M}) \xrightarrow{(l_M)_*} H_n(\tilde{B})$$

and all maps are compatible with the ε-symmetric structures. To continue the estimation we shall use the following lemma the proof of which we learned from Ismael Sierra.

Lemma 2.4.7. *Let R be a ring with involution such that the unique map $\mathbb{Z} \to R$ is split as a $\mathbb{Z}$-module map. Let N be an abelian group generated by at most e elements, let (M, λ) be a ε-symmetric R-module and $p\colon \mathsf{M} \to \mathsf{N}$ be an $\mathbb{Z}$-module map. Then $g_s(\ker p) \geq g_s(\mathsf{M}, \lambda) - e$.*

Proof. We induct on the number of generators of N as a $\mathbb{Z}$-module. The case when $\mathsf{N} = 0$ is clear. Suppose that the statement is true when N is generated by $\leq e-1$ elements. Chose an isomorphism of abelian groups $\mathsf{N} \cong \mathsf{N}_1 \oplus \mathsf{N}_2$ where N_1 is generated by one element. Applying the induction hypothesis on the composition $p_2\colon \mathsf{M} \xrightarrow{p} \mathsf{N} \to \mathsf{N}_2$ shows the existence of $g-e+1$ hyperbolic summands contained in ker p_2. Let $(u_i, f_i)_{1\leq i\leq g-e+1}$ be a hyperbolic R-basis for these, and let $\mathsf{H}_{\mathbb{Z}}$ be the $\mathbb{Z}$-linear span of this basis. Identifying $\mathbb{Z}$ as a sub-$\mathbb{Z}$-module of R, note that the bilinear form λ factors through $\mathbb{Z}$ when restricted on $\mathsf{H}_{\mathbb{Z}}$, so that $(\mathsf{H}_{\mathbb{Z}}, \lambda)$ is a skew-symmetric $\mathbb{Z}$-module, which is clearly a sum of $g-e+1$ hyperbolic summands over $\mathbb{Z}$. Consider the map $p_1\colon : \mathsf{H}_{\mathbb{Z}} \xrightarrow{p} \mathsf{N} \to \mathsf{N}_1$. Because $\mathsf{H}_{\mathbb{Z}}$ is free over $\mathbb{Z}$, we can lift p_1 to $\mathbb{Z}$ given a choice of surjection $\mathbb{Z} \to \mathsf{N}_1$, forming a new map $\tilde{p}_1$. Its image is a submodule of $\mathbb{Z}$ which is of the form $d\mathbb{Z}$, $d \geq 0$. If $d = 0$ then we are done, otherwise we consider $\tilde{p}_1' := \tilde{p}_1/d$, which is surjective and verifies $\ker \tilde{p}_1' = \ker \tilde{p}_1 \subseteq \ker p_1$. Because $(\mathsf{H}_{\mathbb{Z}}, \lambda)$ is non-singular, there exists $x \in H_{\mathbb{Z}}$ such that $\tilde{p}_1' = \lambda(-, x)$. As $\tilde{p}_1'$ is surjective, x is unimodular and we deduce the existence of an automorphism h of $(\mathsf{H}_{\mathbb{Z}}, \lambda)$ such that $h(x) = u_{g-e+1}$. Consider now the family $(u_i', f_i')_{1\leq i\leq g-e} = (h^{-1}(u_i), h^{-1}(f_i))_{1\leq i\leq g-e}$. By design, all these vectors are orthogonal to x so that they lie in $\ker \tilde{p}_1'$. In turn, they lie in $\ker p$, and form a basis for $g-e$ hyperbolic summands inside $\ker p$ over $\mathbb{Z}$. It follows that they also form an R-basis of their R-linear span which is a basis for $g-e$ hyperbolic summands over R. The induction follows. ☐

Corollary 2.4.8. *Suppose that the abelian group $H_n(\tilde{B})$ is generated by at most e elements. Then, $g_s(I^{\mathrm{fr}}(M, l_M) \geq g_s(H_n(\tilde{M})) - e$.*

The next lemma will enable us to make an estimation of the ε-quadratic Witt index from the symmetric one.

Lemma 2.4.9. *Let R be as in Lemma 2.4.7 and suppose that $R/(r - \varepsilon\bar{r})$ is generated by at most e elements over $\mathbb{Z}$. Let $(\mathsf{M}, \lambda, \mu)$ be an ε-quadratic R-module. Then, $g_q(\mathsf{M}, \lambda, \mu) \geq g_s(\mathsf{M}, \lambda) - e$.*

Proof. Consider an inclusion $\mathsf{H}^{\oplus g} \hookrightarrow \mathsf{M}$. The quadratic form μ takes values in $\mathbb{Z}/2$. Let $\mu_h \colon \mathsf{H}^{\oplus g} \to R/(r - \varepsilon\bar{r})$ be the quadratic refinement of λ which makes $\mathsf{H}^{\oplus g}$ hyperbolic as an ε-quadratic module. Consider the difference $\mu - \mu_h \colon \mathsf{H}^{\oplus g} \to R/(r - \varepsilon\bar{r})$, which is a $\mathbb{Z}$-linear map. By Lemma 2.4.7, we can find a sum of $g - e$ hyperbolics for the ε-symmetric structure contained inside $\ker(\mu - \mu_h)$. On these, there is an equality $\mu = \mu_h$. It follows that they are also hyperbolic for the ε-quadratic structure. □

Putting everything together shows the following lemma.

Lemma 2.4.10 (Estimation of the θ-genus)**.** *Let M be a compact oriented $2n$-manifold which is connected with $\pi_1(M) = G$, and let $l_M \colon M \to B$ be an n-connected θ-structure. If $\mathbb{Z}[G]/(g - (-1)^n g^{-1})$ resp. $H_n(B;\mathbb{Z})$ are generated by at most e_0 resp. e_1 elements as abelian groups, then there is an inequality*

$$g^\theta(M, l_M) \geq g_s(H_n(\tilde{M};\mathbb{Z})) - e_0 - e_1$$

where $H_n(\tilde{M};\mathbb{Z}))$ is seen as a $(-1)^n$-symmetric $\mathbb{Z}[G]$-module. □

The θ-genus of $(V_d)_{*,hC_3}$ Note that for $G = C_3$, there is an isomorphism $\mathbb{Z}[G]/(g + g^{-1}) \cong \mathbb{Z}/2$. On the other hand, it turns out that the homology group $H_3(B\mathrm{Spin}^c(6);\mathbb{Z})$ vanishes. Indeed there is a fibre sequence

$$BU(1) \to B\mathrm{Spin}^c(6) \to BSO(6)$$

and the Serre spectral sequence identifies $H_3(B\mathrm{Spin}^c(6);\mathbb{Z})$ as a subgroup of $H_3(BSO(6);\mathbb{Z})$. Then of [Bro82, Theorem 1.5] implies that $H^3(BSO(6);\mathbb{Z}) \cong \mathbb{Z}/2$ generated by δw_2, so that $H_3(BSO(6);\mathbb{Z})$ is torsion. The same theorem implies $H^4(BSO(6);\mathbb{Z}) \cong \mathbb{Z}$ generated by p_1, so that $H_3(BSO(6);\mathbb{Z})$ is torsion free. In turn, $H_3(B\mathrm{Spin}^c(6)) = H_3(BSO(6)) = 0$. As $\Pi(6)$ deform retracts on $\mathrm{Spin}^c(6)$, lemma 2.4.10 implies

$$g^\theta((V_d)_{*,hC_3}, (l_{V_d})_{hC_3}) \geq g_s(H_3((V_d)_{*,\partial};\mathbb{Z})) - 1.$$

Note that $V_d^{C_3}$ is of codimension 4 in V_d, so the inclusion $(V_d)_{*,\partial} \hookrightarrow V_d$ is 3-connected. Hence the $g_s(H_3((V_d)_{*,\partial};\mathbb{Z})) \geq g_s(H_3(V_d))$ where $H_3(V_d)$ is seen as a skew-symmetric $\mathbb{Z}[C_3]$-module. We will use an explicit description of this module together with its Poincaré skew-symmetric form, proved in [Loo]. We thank Oscar Randal-Williams for pointing out this article to us.

Theorem ([Loo, Corollary 2.2]). *Let μ_d be the cyclic group of order d with generator u, there is an isomorphism $\mathbb{Z}[\zeta_d] \cong \mathbb{Z}[\mu_d]/1 + u + \cdots + d^{d-1}$ for $\zeta_d := e^{2i\pi/d}$.*

Letting $u_i = 1 \otimes \cdots \otimes u \otimes \cdots \otimes 1$ where u is in the ith position and $v := u_1 \cdot u_2 \cdot u_3 \cdot u_4$, there is an isomorphism of abelian groups

$$H_3(V_d) \cong \mathbb{Z}[\zeta_d]^{\otimes 4}/(1 + v + \cdots + v^{d-1})$$

where $v \in \mathbb{Z}[\mu_d]^{\otimes 4}$ is identified with its image in $\mathbb{Z}[\zeta_d]^{\otimes 4}$.

The C_3-action is given by permuting the generators u_2, u_3 and u_4, and the $\mathbb{Z}$-valued Poincaré pairing is given by

$$\lambda(x, y) = (x\bar{y} \cdot (1 - \bar{v})(1 - u_1)(1 - u_2)(1 - u_3)(1 - u_4))_0$$

where the involution $a \mapsto \bar{a}$ is induced by the inverse map $g \mapsto g^{-1}$ of μ_d, and for $r \in \mathbb{Z}[\mu_d]^{\otimes 4}$, r_0 denotes the projection of r on the split summand $\mathbb{Z} \hookrightarrow \mathbb{Z}[\mu_d]^{\otimes 4}$.

More explicitely, the pairing is given by $\lambda(x, y) = (x\bar{y}r)_0$ where

$$r = (\bar{u}_1 - u_1) + (\bar{u}_2 - u_2) + (\bar{u}_3 - u_3) + (\bar{u}_4 - u_4) - (\overline{u_1u_2} - u_1u_2) - (\overline{u_2u_3} - u_2u_3) - (\overline{u_3u_4u_2} - u_3u_4) - (\overline{u_4u_1} - u_4u_1) + (\overline{u_1u_2u_3} - u_1u_2u_3) + (\overline{u_2u_3u_4} - u_2u_3u_4) + (\overline{u_3u_4u_1} - u_3u_4u_1) + (\overline{u_4u_1u_2} - u_4u_1u_2) - (\overline{u_1u_2u_3u_4} - u_1u_2u_3u_4)$$

Let σ be a generator of C_3 and consider the $\mathbb{Z}[C_3]$-valued pairing

$$\lambda'(x, y) = \lambda(x, y) + \lambda(x, \sigma y)\sigma + \lambda(x, \sigma^2 y)\sigma^2$$

We wish to find hyperbolic summands inside the skew-symmetric $\mathbb{Z}[C_3]$-module $(H_3(V_d), \lambda')$. Suppose that d is even and $d \geq 6$. Identifying u_i with its image in $H_3(V_d)$, a direct computation shows that for all $a \geq 0$, the vectors

$$v_a := u_2^a, u_3^{a+2}, u_4^{a+3}, \quad v'_a := u_2^{a+1}, u_3^{a+2}, u_4^{a+3}$$

span a hyperbolic, which we denote by $H_a \subseteq H_3(V_d)$. Another analogous computation shows that $\lambda(x, y) = 0$ for all $x \in H_a$ and $y \in H_b$ if and only if $a - b \notin \{-1, 0, 1\} \pmod{d}$. As a consequence, there is a split inclusion of the sum of hyperbolic summands

$$H_0 \oplus H_2 \oplus \cdots H_{d-2} \hookrightarrow H_3(V_d)$$

From the previous discussion, we can now conclude the following estimation.

Proposition 2.4.11. $g^{\theta_{hC_3}}((V_d)_{*,hC_3}, (l_{V_d})_{hC_3}) \geq \frac{d}{2} - 1.$ □

2.4.2.4 The cohomology of $\mathcal{M}_\theta^{C_3}(V_d, l_{V_d})$

We can now state a first result about the cohomology of an equivariant moduli space associated to V_d.

Proposition 2.4.12. *Let $d > 0$ be divisible by d, and equip V_d with the θ-structure l_{V_d} defined in Lemma 2.4.6. Then the equivariant scanning map*

$$\mathcal{M}_\theta^{C_3}(V_d, l_{V_d}) \to \Omega^\infty(MT\theta)^{C_3}$$

is acyclic in degrees $* \leq \frac{d}{2} - 1$ onto the path component that it hits.

Proof. By Lemma 2.4.6, (V_d, l_{V_d}) satisfies the assumptions of Theorem B. Hence in this case the equivariant scanning map is acyclic in degrees $* \leq \min(r_1, r_2)$, where r_1 is the θ^{C_3}-genus of $V_d^{C_3}$ equal to $d^2 - 3d + 2$, and r_2 is the θ_{hC_3}-genus of $(V_d)_{*,hC_3}$ which we have bounded below by $\frac{d}{2} - 1$ in Proposition 2.4.11. This concludes the proof. □

Remark 2.4.13. As pointed out earlier, the estimation of the genus for $(V_d)_{*,hC_3}$ is in no mean sharp. Judging by the amount of $\mathbb{Q}[C_3]$-free summands inside $H_3((V_d)_{*,\partial}, \mathbb{Q})$ – a degree 3 polynomial in d – we expect that for d large enough

$$g^{\theta_{hC_3}}((V_d)_{*,\partial}/C_3, (l_{V_d})_{hC_3}) \geq g^{\theta^{C_3}}((V_d^{C_3}, l_{V_d}))$$

If this is the case, the range of degrees given by Proposition 2.4.12 can be improved to $* \leq d^2 - 3d + 2$.

The tangential structure $\theta \colon B \to B_{C_3}\mathrm{GL}(6)$ is such that $B \in \mathrm{Fun}(\mathcal{O}_{C_3}^{op}; \mathrm{Top})$ is levelwise simply connected. Also, there is a rational equivalence $B(C_3/*) \to B(C_3/*)_{hC_3}$ because C_3 is finite. Hence

$$H^*(B(C_3/*)_{hC_3}; \mathbb{Q}) \cong H^*(BSO(6) \times BS^1; \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, e, t]$$

by [GR19, section 5.3.3], where $t \in H^2(BS^1; \mathbb{Q})$ and $e \in H^6(BSO(6); \mathbb{Q})$ is the Euler class. The degrees are given by $|p_1| = 4$, $|p_2| = 8$, $|e| = 6$, $|t| = 2$. On the other hand,

$$H^*(B(C_3/C_3); \mathbb{Q}) \cong H^*(BSO(2) \times BU(2); \mathbb{Q}) \cong \mathbb{Q}[e', c_1, c_2]$$

with degrees $|e'| = 2$, $|c_1| = 2$, $|c_2| = 4$.

Let $\mathcal{B}$ be a basis for monomials in the variables p_1, p_2, e, t, and $\mathcal{B}'$ be a basis for those in the variables e', c_1, c_2. Then, Propostion 2.4.12 together with Proposition 2.3.12 imply that the natural map

$$\mathbb{Q}[\kappa_c, \kappa_{c'} \mid c \in \mathcal{B},\ c' \in \mathcal{B}',\ |c| > 6,\ |c'| > 2] \to H^*(\mathcal{M}'_\theta(V_d, l_{V_d}); \mathbb{Q})$$

is an isomorphism, in degrees $* \leq \frac{d}{2} - 2$ and an epimorphism for $* = \frac{d}{2} - 1$.

2.4.2.5 Change of tangential structure

In order to compute the cohomology of $\mathcal{M}_{\mathfrak{o}}^{C_3}(V_d, l_{V_d}^+)$ the last step is to compare it with the moduli space for the tangential structure θ. Forgetting the action of C_3 on V_d, this corresponds to [GR19, section 5.3.4]. By chance, it is possible to quickly reduce this change of structures to the non-equivariant computation worked out in details in op. cit.

The morphism of equivariant tangential structures $\theta \to \mathfrak{o}$ factors as

$$\mathfrak{o}_\Pi \xrightarrow{\quad f \quad} \mathfrak{o} \times BS^1 \longrightarrow \mathfrak{o}$$

$$\begin{array}{ccccc} B_{C_3}\Pi(6)(C_3/*) & \xrightarrow{f} & B_{C_3}\mathrm{GL}_{>0}(6)(C_3/*) \times BS^1 & \longrightarrow & B_{C_3}\mathrm{GL}_{>0}(6)(C_3/*) \\ \uparrow & & \uparrow & & \uparrow \\ BC_\lambda(\mathrm{GL}_{>0}(6)) & \xrightarrow{=} & BC_\lambda(\mathrm{GL}_{>0}(6)) & \xrightarrow{=} & BC_\lambda(\mathrm{GL}_{>0}(6)) \end{array}$$

where the columns are the descriptions of the structure as functors $\mathrm{Fun}(\mathcal{O}_{C_3}^{op}, \mathrm{Top})$, the maps to $B_{C_3}\mathrm{GL}(6)$ being the canonical ones. The Borel C_3-action on BS^1 is taken to be the trivial one. Following [GR19], we decompose the change of structures into two changes, one from θ to $\mathfrak{o} \times BS^1$, and the second from $\mathfrak{i} \times BS^1$ to $\mathfrak{o}$. In both case we reduce to the computations in [GR19].

Recall that $\mathrm{Aut}^{C_3}(f)$ is defined as the ∞-group formed by equivariant self-equivalences of $\mathfrak{o}_\Pi$ over $\mathfrak{o} \times BS^1$. The proof of Proposition 2.3.15 shows it coincides with the space of $\frac{1}{2}$-connected θ-structures l on V_d such that $f \circ l = f \circ l_{V_d}$. The latter is equivalent to the space of lifts

$$\begin{array}{ccc} & & \mathfrak{o}_\Pi \\ & \nearrow & \downarrow f \\ V_d & \xrightarrow{f \circ l_{V_d}} & \mathfrak{o} \times BS^1 \end{array}$$

where V_d is seen as an object of $\mathrm{Fun}(\mathcal{O}_{C_3}^{op}, \mathrm{Top})/B_{C_3}\mathrm{GL}(6)$. Evaluating this diagram at $C_3/C_3 \in \mathcal{O}_{C_3}^{op}$ gives a constant diagram, hence the space of lifts coincides with the space of lifts of the same diagram evaluated at $C_3/*$. Writing $\mathrm{Map}_{\mathrm{Borel}}^{C_3}$ resp. Map_{C_3} for the mapping space in Borel C_3-spaces resp. the internal mapping space in Top_{C_3}, there is a natural equivalence $\mathrm{Map}_{\mathrm{Borel}}^{C_3} \simeq (\mathrm{Map}_{C_3})^{hC_3}$ so that as a space, the following formula for $\mathrm{Aut}^{C_3}(f)$ holds

$$\mathrm{Aut}^{C_3}(f) \simeq \mathrm{hofib}_{f \circ l_{V_d}} \left(\mathrm{Map}_{C_3}(V_d, \mathfrak{o}_\Pi(C_3/*) \to \mathrm{Map}_{C_3}(V_d, \mathfrak{o}(C_3/*) \times BS^1\right)^{hC_3}$$

The discussion in [GR19, section 5.3.4] shows that the space of lifts in the non-equivariant case is rationally trivial. The fact that the map $X^{hG} \to X$ is rationally a monomorphism when G is finite implies that $\mathrm{Aut}^{C_3}(f)$ itself is rationally trivial. Hence Corollary 2.3.16 implies that the map

$$\mathcal{M}_\theta^{C_3}(V_d, l_{V_d}) \xrightarrow{f \circ -} \mathcal{M}_{\mathfrak{o} \times BS^1}^{C_3}(V_d, f \circ l_{V_d})$$

is a rational homotopy equivalence, as the subgroup $\Gamma \leq \mathrm{Aut}^{C_3}(f)$ has to be rationally trivial itself.

For the second change of structures, we use Corollary 2.3.16 which gives a fibre sequence

$$\mathcal{M}^{C_3}_{\mathfrak{o}\times BS^1}(V_d, f\circ l_{V_d}) \to \mathcal{M}^{C_3}_{\mathfrak{o}}(V_d, l^+_{V_d}) \to B\Gamma'$$

where Γ' is the subgroup of $\mathrm{Aut}^{C_3}(\mathrm{pr}\colon \mathfrak{o}\times BS^1 \to \mathfrak{o})$ consisting of those equivariant automorphisms which preserve $f\circ l_{V_d}$ up to equivariant oriented diffeomorphism of V_d. As in the first change of structures, the map pr is non-trivial only at the level of Borel C_3-spaces, so that the following formula holds

$$\begin{array}{rcl}\mathrm{Aut}^{C_3}(\mathrm{pr}) & \simeq & (\mathrm{Map}_{C_3}(B_{C_3}SO(6), \mathrm{Aut}(BS^1)))^{hC_3} \\ & \simeq & \mathrm{Map}(BSO(6)\times BC_3, \mathrm{Aut}(BS^1)) \\ & \simeq & \mathbb{Z}^\times \ltimes \mathrm{Map}(BSO(6)\times BC_3, K(\mathbb{Z},2)) \\ & \simeq & \mathbb{Z}^\times \ltimes K(\mathbb{Z},2)\times K(\mathbb{Z}/2,1)\end{array}$$

because the action of C_3 on $B_{C_3}SO(6)$ is Borel-trivial. By the same argument as [GR19], $\Gamma' \leq \mathrm{Aut}^{C_3}(\mathrm{pr})$ coincides with $K(\mathbb{Z},2)\times K(\mathbb{Z}/2,1)$. Hence the fibre sequence is of the form

$$\mathcal{M}^{C_3}_{\mathfrak{o}\times BS^1}(V_d, f\circ l_{V_d}) \to \mathcal{M}^{C_3}_{\mathfrak{o}}(V_d, l^+_{V_d}) \to K(\mathbb{Z},3)\times K(\mathbb{Z}/2,2) \tag{2.16}$$

The space $K(\mathbb{Z}/2,2)$ is rationally trivial, so that rationally we are lead to the same computation as in [GR19]. This proves

Proposition 2.4.14. *There is an isomorphism*

$$H_*(\mathcal{M}^{C_3}_{\mathfrak{o}}(V_d, l^+_{V_d});\mathbb{Q}) \cong \ker\left(d_3 \circlearrowleft \mathbb{Q}[\kappa_c \,|\, c\in\mathcal{B},\ |c|>6]\right) \otimes \mathbb{Q}[\kappa_{c'} \,|\, c'\in\mathcal{B}',\ |c'|>2]$$

in degrees $* \leq \frac{d}{2}-1$, where d_3 is the third differential in the Serre spectral sequence associated to (2.16).

More properties of d_3 are explained in [GR19].

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