UNIVERSITY OF COPENHAGEN
FACULTY OF SCIENCE

PhD thesis

Local, global, and relative problems in arithmetic intersection theory

Nuno Hultberg

Advisor: Fabien Pazuki and Lars Kühne

Submitted: September 30, 2024

This thesis has been submitted to the PhD School of The Faculty of Science, University of Copenhagen

Nuno Hultberg
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK–2100 København Ø
Denmark

nuno.hultberg@gmail.com

This thesis has been submitted to the PhD School of The Faculty of Science, University of Copenhagen.

Submission date:	September 30, 2024
Defence date:	October 25, 2024
Advisors:	Fabien Pazuki, University of Copenhagen
	Lars Kühne, University College Dublin
Assessment Committee:	
	Asger Törnquist, University of Copenhagen
	Klaus Künnemann, University of Regensburg
	José Ignacio Burgos Gil, ICMAT Madrid

ISBN: 978-87-7125-230-9

ACKNOWLEDGEMENTS

My sincerest apologies go to the three people I forgot to mention in my acknowledgements.[1]

I will first thank my advisors Fabien Pazuki and Lars Kühne. Fabien always encouraged me to organize events, talk to other mathematicians and most importantly to write down my results in a timely manner. Conversations with Lars often lasted a couple of hours longer than expected due to us thinking hard about some problem that turned out to be embarrassingly easy or deceivingly hard. You both spent a lot of time on me and on reading my manuscripts and complement each other perfectly as advisors.

Another important thank goes to my coauthors. Andreas Mihatsch was previously my master thesis supervisor and later my first collaborator. His clear vision of a project and style in writing are something I aspire to. Pablo and Michal, I thank for being patient with me and teaching me basic model theory. It was a pleasure to work with people of a very different background than myself. This is something I want to continue in the future.

I would like to thank Philipp Habegger for hosting me in Basel during my change of research environment. I further thank Harry Schmidt, Robert Wilms and Gerold Schefer for interesting discussions and specially Gerold for taking me to choir and making me feel more at home in Basel.

I thank the Arakelov geometry community at large for numerous discussions and a welcoming atmosphere. Amongst you a special thank goes to Westlake University and Huayi Chen. I further thank the referees of my thesis José Burgos Gil and Klaus Künnemann for their interest in my work. I thank José specifically for discussions on toric varieties and positivity in Arakelov geometry and Klaus Künnemann for his detailed comments on most of the thesis.

An unusual mathematical thanks goes out to my brother Bror. In conversations over the phone, he has explained some convex analysis to me including the minimax theorem. I have tried very hard to understand when you explain about metric entropy, beta-expansions or whatever math subject you are dealing with now. Unfortunately, I

[1]This is stolen from Roberto Gualdi's thesis who has written some of the nicest acknowledgements I've ever read.

ended up not using the minimax theorem in the final version of my toric bundle paper, but if we continue speaking this much maths it's a matter of time until you start proving my results.

My mathematical discussions in Copenhagen have enriched me as a mathematician. In this regard, I thank in particular Bai, Desirée, Mate, Maxime and Vignesh for numerous algebraic geometry and toric variety discussions. I thank all organizers of events and seminars for making Copenhagen a mathematically vibrant place. I lastly thank the administration for their quick help and in general for just being nice.

These three years have, of course, consisted of more than just maths. The personal thanks to my friends at the University of Copenhagen far surpass the mathematical ones. I have had lots of fun both in and out of uni! I thank my badminton friends Ida, Jay, Max and Villads for fun on and off the court. I also thank my friends from Bonn for keeping in touch even though I'm a disaster at picking up the phone. A thank also goes out to Zhao Binbin for showing me around Hangzhou and the surprise of making an actual friend during my three week stay. Some of you have been a psychological support beyond just having fun. I hope I have let you feel my gratefulness.

I am very grateful to my family. My parents and brother have always been a source of strength to me. Living in Denmark has meant being closer to much more family. I thank Lotte for invitations and company at the opera and Tine for hikes and dinners shared with my favourite bird. I thank you for your general interest in my well-being.

A special thank goes to my grandmother: for finding me a place to live, for taking care of practical things when I couldn't get myself to do it, for numerous dinners, trips to the museum and time on the farm. It was a pleasure to feel that it was a priority for you to spend time with me. I am grateful for your flexibility in our arrangements and your infectious joy. I thank Tommy for taking a part in this. You two have made my stay in Copenhagen even better than it would have already been otherwise.

ENGLISH ABSTRACT

This thesis consists of four independent articles. The overarching theme, arithmetic intersections, will present itself from very distinct angles.

Article [HM23] is an outlier in the sense that it is local in nature. It proves a conjectural identity between intersection numbers on Rapoport-Zink spaces and central derivatives of local orbital integrals, known as an arithmetic fundamental lemma(AFL), in the case of general linear groups over the quaternion division algebra.

Articles [Hul23], [Hul24] and [DHS24] belong to the field of Arakelov geometry. Article [Hul23] studies arithmetic properties of algebraic extensions of $\mathbb{Q}$ through its Northcott number. We extend finiteness results from fields satisfying the Northcott property to fields with big enough Northcott number and provide examples of infinite extensions of $\mathbb{Q}$ with finitely many CM points.

In Article [Hul24], we study the Arakelov geometry of toric bundles in a systematic way. The purpose is two-fold. On one hand it provides us with a new class of examples that can be studied explicitly. On the other, toric bundles contain many varieties of interest such as semiabelian varieties and their compactifications. We compute the Okounkov body and Boucksom-Chen transform of toric line bundles on toric bundles in terms of information on the base. We prove a formula for intersection numbers in terms of convex geometry data, an arithmetic relative BKK theorem.

Lastly, we study arithmetic intersection numbers in families in Article [DHS24]. We associate to finite type schemes over globally valued fields topological spaces and prove continuity of arithmetic intersection numbers on them. We apply this to prove a conjecture of Gualdi and Sombra on the height of complete intersections in toric varieties.

DANSK ABSTRACT

Denne afhandling består af fire uafhængige artikler. Deres fælles emne, aritmetisk snitteori, præsenterer sig fra vidt forskellige vinkler.

Artikel [HM23] afviger fra de andre artikler, idet det behandlede spørgsmål er lokalt. Den beviser en formodet identitet mellem snittal på Rapoport-Zink-rum og centrale afledninger af lokale orbitalintegraler, kendt som et aritmetisk fundamentalt lemma (AFL), i tilfælde af generelle linære grupper over en kvaternion divisionsalgebra.

Artiklerne [Hul23], [Hul24] og [DHS24] hører til feltet Arakelov geometri. Artikel [Hul23] behandler aritmetiske egenskaber af algebraiske udvidelser af $\mathbb{Q}$ ved hjælp af deres Northcott-tal. Vi udvider endelighedsresultater fra legemer med Northcott-egenskaben til legemer med tilstrækkeligt stort Northcott-tal og giver eksempler på uendelige algebraiske udvidelser af $\mathbb{Q}$ med endelig mange CM punkter.

I Artikel [Hul24], undersøger vi Arakelov geometrien af toriske bundter på systematisk vis. Det har to primære formål. For det første er toriske bundter en ny klasse af eksempler, som kan undersøges eksplicit. For det andet, findes der eksempler på toriske bundter af uafhængig interesse, såsom semiabelske varieteter og deres kompaktificeringer. Vi beregner Okounkov-legemerne og Boucksom-Chen-transformationerne af toriske linjebundter på toriske bundter i afhængighed af information om basen. Vi beviser en konveks-geometrisk formel for snittal, en relativ aritmetisk BKK-sætning.

Til sidst, undersøger vi aritmetiske snittal i familier i Artikel [DHS24]. Vi associerer topologiske rum til skemaer af endelig type over globalt valuerede legemer og beviser kontinuiteten af det aritmetiske snittal på dem. Vi andvender det til at bevise en formodning af Gualdi og Sombra om højden af transverse snit i toriske varieteter.

THESIS STATEMENT

The thesis is based on four articles with varying coauthors. The version of the article [Hul23] and [HM23] are improvements upon the cited versions by incorporating helpful comments of Klaus Künnemann and an anonymous reviewer respectively. Both of the articles [DHS24] and [Hul24] were not publicly available before the submission of the thesis.

References

[DHS24] Pablo Destic, Nuno Hultberg, and Michał Szachniewicz. *Continuity of heights in families and complete intersections in toric varieties*. 2024.

[HM23] Nuno Hultberg and Andreas Mihatsch. *A linear AFL for quaternion algebras*. 2023. arXiv: 2308.02458 [math.NT].

[Hul23] Nuno Hultberg. *Fields with few small points*. 2023. arXiv: 2307.00297 [math.NT].

[Hul24] Nuno Hultberg. *Arakelov geometry of toric bundles: Okounkov bodies and BKK*. 2024.

Contents

Introduction 13

Papers

Fields with few small points 25

A linear AFL for quaternion algebras 45

Continuity of heights in families and complete intersections in toric varieties 69

Arakelov geometry of toric bundles: Okounkov bodies and BKK . . 103

Introduction

INTRODUCTION

This thesis consists of four independent articles. The overarching theme, arithmetic intersections, will present itself from very distinct angles. The aim of this introduction is to shine a light on the connections that nonetheless exist. For a summary and statement of the results contained in each of the articles we refer to the respective introductions.

The starting point of much of modern number theory is the observation that the rational numbers $\mathbb{Q}$ share many properties of function fields of curves. Just as any meromorphic function on a complete curve has the same number of zeroes and poles, any $f \in \mathbb{Q}^\times$ satisfies the product formula

$$\prod_{p \in M_{\mathbb{Q}}} |f|_p = 1,$$

where the product goes over all normalized absolute values of $\mathbb{Q}$. This suggests considering the usual absolute value as corresponding to a further prime ∞. The natural next step is to study varieties over $\mathbb{Q}$ using relative algebraic geometry and intersection theory over $\operatorname{Spec} \mathbb{Z} \cup \{\infty\}$.[1]

The first instance of such an extension of algebraic geometry goes back to Arakelov who introduced intersections on arithmetic surfaces in [Ara76]. Faltings applied the newly developed Arakelov geometry to prove Mordell's conjecture, see [Fal83]. Arakelov geometry was later extended by Gillet and Soulé to varieties of arbitrary dimension.

1. **Heights.** An implicit application of arithmetic intersection theory predating Arakelov geometry can be found in the theory of Weil heights. Let us explain them from a geometric point of view.

Let X be a projective variety over $\mathbb{Q}$ and L a line bundle over X. Imagine that $\mathcal{X}$ is a proper model of X over $\operatorname{Spec} \mathbb{Z} \cup \{\infty\}$ and $\mathcal{L}$ is a model of L over $\mathcal{X}$. Then, to any closed point $x \in X$ corresponds a curve $\overline{\{x\}} \subset \mathcal{X}$. The height of $\xi \in X(\bar{\mathbb{Q}})$ is defined as $ht(x) = \frac{\widehat{\deg}_{\mathcal{L}}(\overline{\{x\}})}{\deg_L(x)}$, where x is the underlying closed point of ξ. However, before the advent of arithmetic intersection theory there was no way to make sense of what it means to choose a model of L. Still, the Weil height machine allowed to associate to the pair (X, L) an equivalence class of functions $h : X(\bar{\mathbb{Q}}) \to \mathbb{R}$, where two functions are deemed equivalent if they differ by a bounded function. Due to Northcott's theorem this is often sufficient in order to prove finiteness results.

[1] This is not a well-defined mathematical object. The notation is only used as an analogy to help build intuition.

Theorem 1.1 (Northcott)**.** Let X be a projective variety over a number field K. Let L be an ample line bundle on X and h_L a Weil height for L, i.e. a function $h_L : X(\bar{\mathbb{Q}}) \to \mathbb{R}$ in the equivalence class of functions associated to L by the Weil height machine. Then, for any $C \in \mathbb{R}$ the set

$$\{x \in X(K) | h_L(x) < C\}$$

is finite.

A particularly important example of a height is the logarithmic Weil height.

Definition 1.2. Let K be a number field and M_K the set of normalized absolute values. By normalized we understand that they extend the absolute values on $\mathbb{Q}_p$ and $\mathbb{R}$ respectively. The logarithmic Weil height of a point $x = [x_0 : \cdots : x_n] \in \mathbb{P}^n(K)$ is defined as

$$h(x) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \max\{\log |x_0|_v, \ldots, \log |x_n|\}.$$

Note that the above sum is finite and doesn't depend on the choice of representative. For $x \in \mathbb{P}^n(\bar{\mathbb{Q}})$ we define the Weil height by choosing a number field K over which x is defined. The height does not depend on the choice of K. We define the Weil height on $\bar{\mathbb{Q}}$ by viewing it as a subset of $\mathbb{P}^1(\bar{\mathbb{Q}})$.

This means that in order to show finiteness results for points over number fields it suffices to give height bounds. Going beyond the Weil height machine allows us to study the height properties of algebraic extensions of $\mathbb{Q}$ that do not satisfy the conclusion of Northcott's theorem, the Northcott property.

Definition 1.3 (Northcott number)**.** For a subset $S \subseteq \bar{\mathbb{Q}}$ of the algebraic numbers we define the Northcott number of S with respect to a function $f : \bar{\mathbb{Q}} \to [0, \infty)$ as

$$\mathcal{N}_f(S) = \inf\{t \in [0, \infty) | \#\{\alpha \in S; f(\alpha) < t\} = \infty\}.$$

We follow the convention that $\inf \emptyset = \infty$. We call $\mathcal{N}_f(S) \in [0, \infty]$ the Northcott number of S.

In the first article of this thesis we study the arithmetic implications of bounds on Northcott numbers, when we choose f to be the logarithmic Weil height h or the weighted Weil height $\deg^\gamma h$ for $\gamma \in \mathbb{R}$. The algebraic extensions with prescribed Northcott number constructed in [PTW21] and [OS22] provide us with examples of fields with few small points in various senses of the word. The following theorem illustrates such an application.

Theorem 1. *There are uncountably many algebraic field extensions of $\mathbb{Q}$ containing only finitely many CM j-invariants.*

2. **Arithmetic fundamental lemma and height pairings.** The topic of the arithmetic fundamental lemma stands out within the thesis as a completely local question. However, its global motivations are closely related to Arakelov geometry. For a survey on the context of the arithmetic fundamental lemma we refer to [Zha24] which has inspired the following short overview.

Let X be a smooth projective variety over a number field F. Denote by $CH^i(X)$ the codimension i Chow group. Let H denote a Weil cohomology theory. Then, there is a cycle class map $c : CH^i(X) \to H^{2i}(X)$ and we define the group of homologically trivial cycles $CH^i(X)_0$ as $\ker(c)$. For $i + j = \dim X + 1$, Beilinson and Bloch conjecture the existence of a height pairing

$$CH^i(X)_0 \times CH^j(X)_0 \to \mathbb{R}.$$

This is defined conditionally on the existence of a proper regular integral model and the special fibres satisfying the standard conjectures. Note that Grothendieck's standard conjectures are stated for smooth projective varieties and the special fibres might be singular. We hence need to be more specific about this condition. Specifically, we assume the model has strictly semistable reduction and its strata satisfy the standard conjectures, i.e. the intersection of any collection of irreducible components of the special fibre is smooth projective and satisfies the standard conjectures.

The Arakelov intersection theory of Gillet-Soulé provides one strategy to construct such a pairing. Let $\mathcal{X}$ be a regular integral model. Then, there is an intersection pairing

$$\widehat{\mathrm{CH}}^i(\mathcal{X}) \times \widehat{\mathrm{CH}}^j(\mathcal{X}) \to \mathbb{R}$$

for $i + j = \dim X + 1$. We would then like to extend homologically trivial cycles in a flat way. Note that this makes complete sense for divisors, i.e. we require the metrics at all places to be flat. On curves, this allows us to recover the Néron-Tate pairing.

The Beilinson-Bloch conjectures relate the height pairing to special values of motivic L-functions associated to X, thus generalizing the BSD conjecture. For special cycles on Shimura varieties one can apply automorphic methods such as (arithmetic) relative trace formulae to simplify the problem. While the existence of the Beilinson-Bloch pairing is still conjectural the relationship to the arithmetic intersection product on integral models suggests a decomposition into local pieces. After applying Rapoport-Zink uniformization one finally arrives at the fully local question, to relate intersections on Rapoport-Zink spaces(moduli spaces of deformations of p-divisible groups) and local orbital integrals. In [HM23], we prove an instance of such a relationship in the form of an arithmetic fundamental lemma by reducing it to a different arithmetic fundamental lemma.

3. **Globally valued fields.** Heights or arithmetic intersection numbers have traditionally been studied from a discrete viewpoint. An exception to this general rule is Silverman's study of the variation of heights in families in [Sil83] and work it inspired such as [Tat83] and [Gre89]. The key obstruction to the continuous study of heights, however, is not addressed: The classical topological spaces associated to a variety X over $\bar{\mathbb{Q}}$ are not suited to study maps to $\mathbb{R}$. The key idea is to study collections of valuations in a more conceptual way reminiscent of valued rings. The following definition is taken directly from [DHS24, Definition 2.1].

Definition 3.1. A *globally valued field* (abbreviated GVF) is a field F together with a height function $h : \mathbb{A}(F) \to \mathbb{R} \cup \{-\infty\}$, where $\mathbb{A}(F)$ denotes the disjoint union of $\mathbb{A}^n(F)$ for all $n \in \mathbb{N}$, satisfying the following axioms, for some *Archimedean error* $e \geq 0$.

Height of zero:	$\forall x \in F^n$,	$h(x) = -\infty \Leftrightarrow x = 0$
Height of one:		$h(1,1) = 0$
Invariance:	$\forall x \in F^n$, $\forall \sigma \in \mathrm{Sym}_n$,	$h(\sigma x) = h(x)$
Additivity:	$\forall x \in F^n$, $\forall y \in F^m$,	$h(x \otimes y) = h(x) + h(y)$
Monotonicity:	$\forall x \in F^n$, $\forall y \in F^m$,	$h(x) \leq h(x, y)$
Triangle inequality:	$\forall x, y \in F^n$,	$h(x + y) \leq h(x, y) + e$
Product formula:	$\forall x \in F^\times$,	$h(x) = 0$

Here $\otimes$ denotes the Segre product, i.e., $(x_1, \dots, x_n) \otimes (y_1, \dots, y_m) = (x_i \cdot y_j : 1 \leq i \leq n, 1 \leq j \leq m)$. Note that such height factors through $h : \mathbb{P}^n(F) \to \mathbb{R}_{\geq 0}$ for each n. We write $\mathrm{ht}(x) := h[x : 1]$ for $x \in F$.

This definition is motivated by the example $F = \bar{\mathbb{Q}}$ together with the function that associates to a tuple $(x_0, \dots, x_n)$ the Weil height of the corresponding point $[x_0 : \cdots : x_n] \in \mathbb{P}^n(\bar{\mathbb{Q}})$.

Arising first as a space of quantifier-free types we can associate to a finite type scheme X over a GVF K a locally compact Hausdorff space X_{GVF} which we refer to as the GVF analytification. Let us provide its definition that is highly reminiscent of the Berkovich analytification, cf. [DHS24].

Definition 3.2. Let X be a finite type scheme over a (non-trivial) GVF K. Then, the GVF analytification of X_{GVF} is defined as a set to be

$$\{(x, h) \mid x \in X, h_x \text{ GVF structure on } \kappa(x)\}.$$

We equip X_{GVF} with the weakest topology such that

(1) The map $\pi : X_{\mathrm{GVF}} \to X$ is continuous onto X with the Zariski topology.
(2) For every open $U \subset X$, and every tuple $(f_1, \dots, f_n) \in \mathcal{O}_X(U)^n$, the map $(x, h) \mapsto h(f_1, \dots, f_n)$ is continuous on $\pi^{-1}(D(f_1, \dots, f_n))$, where $D(f_1, \dots, f_n) \subset U$ is the open where at least one of the f_i does not vanish.

We extend the intersection theory of adelically metrized line bundles to projective varieties over a GVF. We prove the continuity of these global intersection numbers in families, i.e. the intersection numbers define a continuous function on the GVF analytification and apply it to prove the following conjecture of Roberto Gualdi.

Theorem 3.3 ([DHS24]). Let $f_1, \ldots, f_m$ be Laurent polynomials in n variables with coefficients in a number field K and let T be a proper toric variety with torus $\mathbb{T} = \mathbb{G}^n \subset T$. Denote by V_i the hypersurface defined by f_i and by ρ_i its Ronkin function. Let $(\zeta_{1,j}, \ldots, \zeta_{m,j})_j$ be a generic sequence of small points in $\mathbb{T}^m$ for the Weil height and let $\overline{D}_0, \ldots, \overline{D}_{n-m}$ be semipositive toric adelic divisors on T with associated local roof functions $\theta_{0,v}, \ldots, \theta_{n-m,v}$. Then,

$$\lim_{j \to \infty} h_{\overline{D}_0, \ldots, \overline{D}_{n-m}}(\zeta_{1,j} V_1 \cap \cdots \cap \zeta_{m,j} V_m) = \sum_{v \in \mathrm{M}_K} n_v MI_M(\theta_{0,v}, \ldots, \theta_{n-m,v}, \rho_1^\vee, \ldots, \rho_m^\vee).$$

Let us focus our attention on GVF analytifications over a number field K. Then, $X(\bar{\mathbb{Q}}) \subset X_{\mathrm{GVF}}$ is dense. This is equivalent to the existential closedness of $\bar{\mathbb{Q}}$ in the theory of globally valued fields, proven in [Sza23]. On the other hand, the Northcott property of K is equivalent to $X(K)$ being nowhere dense. For infinite algebraic extension F, one may ask for about more subtle questions in the spirit of [Hul23].

4. **Toric bundles.** Convex geometry has proven to be a useful tool in algebraic geometry as exemplified by the proof of differentiability of volume via Okounkov bodies in [LM09]. There are, however, so far only few examples of explicitly calculated Okounkov bodies and their arithmetic analogues. The goal of [Hul24] is to extend the combinatorial approach from toric varieties to toric bundles. To do this we construct arithmetic toric bundles in the spirit of [CT01] and study them combinatorially.

Once the Okounkov body and the Boucksom-Chen transform are calculated one may read off their height and their successive minima. A further contribution of the work on toric bundles is the establishment of an arithmetic bundle BKK theorem. We hope that continuation of work in this direction can establish further cases of the arithmetic standard conjectures. In the article [DHS24] an intersection problem occurs that is the intersection of line bundles of the form $\widehat{\rho}(\overline{L})$, but for differing metric structures on the torus bundle. It may be interesting to extend the arithmetic bundle BKK theorem to this setting.

The article in particular allows us to compute heights and successive minima on semiabelian varieties, thereby recovering work of Chambert-Loir in [Cha00]. The work suggests the existence of preferred choices of compactifications of semiabelian varieties. These will, however, no longer be given by a projective variety, but only by a limit along birational models in the spirit of [YZ24]. It is worth studying whether this is convenient in the study of compactifications of the moduli space of polarized abelian varieties.

5. **Possibilities for future research.** We note that contrary to the rest of the thesis this section is dedicated to wild speculation. It is intended to highlight problems that may benefit from the methods developed in the articles as well as new problems arising in their context.

- Analytic geometry over globally valued fields: The definition of the GVF analytification is highly reminiscent of the definition of a Berkovich space. This suggests the possibility for a global analytic geometry. Moreover one may define analogues of adic spaces over GVFs by allowing higher rank heights.
- Infinite algebraic extensions using the GVF analytification: One can recover the Northcott property of an extension F of $\mathbb{Q}$ as a topological property of F-points in the GVF analytification. It is interesting to study topological properties of F-points for other extensions. For instance, for which extensions is the closure in the GVF analytification restricted to the generic point discrete or convex? Which fields F satisfy that if F-points are Zariski dense they are also analytically dense?
- Compactifications of the GVF analytification: Contrary to our expectation from an analytification, the GVF analytification does not send proper varieties to compact topological spaces. On the other hand, it sends all varieties to Hausdorff spaces, even the ones that are not separated. Constructing some bigger space might allow to interpret more equidistribution results, such as the one for singular moduli, in terms of Arakelov geometry.
- Semiabelian varieties: A compactification of a semiabelian variety is essentially given as a polytope in the cocharacter space of its torus. A polarization on the abelian quotient endows the cocharacter space with a norm. This is preserved under homomorphisms of semiabelian varieties. Hence the obvious choice for a convex body in the cocharacter space is the unit ball. This is not a polytope, but it is still possible to study it using the theory of Yuan and Zhang. This may be helpful for uniformity results.
- Beyond toric bundles: Many constructions do not rely on the objects being toric bundles. Even in the case of toric varieties it is natural to consider $\mathbb{T}'$-linearized sheaves on a toric variety with torus $\mathbb{T}$ and a homomorphism $\mathbb{T}' \to \mathbb{T}$. This perspective clarifies the formula for pullback of $\mathbb{T}$-linearized line bundles to closed subvarieties. It may be interesting to extend methods to further groups, flag bundles and other setups. We note further that in the setting of toric bundles there is the possibility to consider what happens when varying the metric.

- Positivity in Arakelov geometry: It is arguably even more crucial to study positivity properties in Arakelov geometry than in classical algebraic geometry. This is because objects of interest like adelically metrized line bundles of Zhang, cf. [Zha95], are defined using a limiting procedure. In order to have a valid definition of intersection numbers one needs positivity assumptions. This is subtle even in apparently simple situations considered in [Hul24].

References

[Ara76] S. Ju. Arakelov. "Intersection theory of divisors on an arithmetic surface". English. In: *Math. USSR, Izv.* 8 (1976), pp. 1167–1180. ISSN: 0025-5726. DOI: 10.1070/IM1974v008n06ABEH002141.

[Cha00] Antoine Chambert-Loir. "Points de petite hauteur sur les variétés semi-abéliennes". In: *Annales Scientifiques de l'École Normale Supérieure* 33.6 (2000), pp. 789–821. ISSN: 0012-9593. DOI: https://doi.org/10.1016/S0012-9593(00)01053-3. URL: https://www.sciencedirect.com/science/article/pii/S0012959300010533.

[CT01] Antoine Chambert-Loir and Yuri Tschinkel. "Torseurs arithmétiques et espaces fibrés". In: *Rational points on algebraic varieties*. Vol. 199. Progr. Math. Birkhäuser, Basel, 2001, pp. 37–70. ISBN: 3-7643-6612-5.

[DHS24] Pablo Destic, Nuno Hultberg and Michał Szachniewicz. *Continuity of heights in families and complete intersections in toric varieties*. 2024.

[Fal83] G. Faltings. "Endlichkeitssätze für abelsche Varietäten über Zahlkörpern." ger. In: *Inventiones mathematicae* 73 (1983), pp. 349–366. URL: http://eudml.org/doc/143051.

[Gre89] William Green. "Heights in families of abelian varieties". In: *Duke Math. J.* 58.3 (1989), pp. 617–632. ISSN: 0012-7094,1547-7398. DOI: 10.1215/S0012-7094-89-05829-8. URL: https://doi.org/10.1215/S0012-7094-89-05829-8.

[HM23] Nuno Hultberg and Andreas Mihatsch. *A linear AFL for quaternion algebras*. 2023. arXiv: 2308.02458 [math.NT].

[Hul23] Nuno Hultberg. *Fields with few small points*. 2023. arXiv: 2307.00297 [math.NT].

[Hul24] Nuno Hultberg. *Arakelov geometry of toric bundles: Okounkov bodies and BKK*. 2024.

[LM09] Robert Lazarsfeld and Mircea Mustaţă. "Convex bodies associated to linear series". In: *Ann. Sci. Éc. Norm. Supér. (4)* 42.5 (2009), pp. 783–835. ISSN: 0012-9593,1873-2151. DOI: 10.24033/asens.2109. URL: https://doi.org/10.24033/asens.2109.

[OS22] Masao Okazaki and Kaoru Sano. *Northcott numbers for the weighted Weil heights*. 2022. DOI: 10.48550/ARXIV.2204.04446. URL: https://arxiv.org/abs/2204.04446.

[PTW21] Fabien Pazuki, Niclas Technau and Martin Widmer. *Northcott numbers for the house and the Weil height*. 2021. DOI: 10.48550/ARXIV.2107.09027. URL: https://arxiv.org/abs/2107.09027.

[Sil83] Joseph H. Silverman. In: *Journal für die reine und angewandte Mathematik* 1983.342 (1983), pp. 197–211. DOI: doi:10.1515/crll.1983.342.197. URL: https://doi.org/10.1515/crll.1983.342.197.

[Sza23] Michał Szachniewicz. *Existential closedness of $\overline{\mathbb{Q}}$ as a globally valued field via Arakelov geometry*. 2023. arXiv: 2306.06275 [math.LO].

[Tat83] J. Tate. "Variation of the Canonical Height of a Point Depending on a Parameter". In: *American Journal of Mathematics* 105.1 (1983), pp. 287–294. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2374389 (visited on 23/09/2024).

[YZ24] Xinyi Yuan and Shou-Wu Zhang. *Adelic line bundles on quasi-projective varieties*. 2024. arXiv: 2105.13587 [math.NT]. URL: https://arxiv.org/abs/2105.13587.

[Zha24] Wei Zhang. *High dimensional Gross–Zagier formula: a survey*. 2024. arXiv: 2402.17656 [math.NT]. URL: https://arxiv.org/abs/2402.17656.

[Zha95] Shouwu Zhang. "Small points and adelic metrics". In: *J. Algebraic Geom.* 4.2 (1995), pp. 281–300. ISSN: 1056-3911,1534-7486.

Papers

Fields with few small points

FIELDS WITH FEW SMALL POINTS

NUNO HULTBERG

Abstract. Let X be a projective variety over a number field K endowed with a height function associated to an ample line bundle on X. Given an algebraic extension F of K with a sufficiently big Northcott number, we can show that there are finitely many cycles in $X_{\bar{\mathbb{Q}}}$ of bounded degree defined over F. Fields F with the required properties were explicitly constructed in [PTW22] and [OS22], motivating our investigation. We point out explicit specializations to canonical heights associated to abelian varieties and selfmaps of $\mathbb{P}^n$. We apply similar methods to the study of CM-points. As a crucial tool, we introduce a refinement of Northcott's theorem.

There have recently been advances on the study of height properties of algebraic extensions of $\mathbb{Q}$ in [PTW22] and [OS22]. Let $\mathcal{N}$ denote the Northcott number with respect to the logarithmic Weil height. The Northcott number $\mathcal{N}(S)$ is defined in Definition 1.1. It is the smallest limit point of heights of a subset of $S \subseteq \bar{\mathbb{Q}}$ and ∞ if S satisfies the Northcott property. The key result of their work is the following theorem.

Theorem 0.1 (Theorem 1.3 [OS22])**.** *For every $t \in [0, \infty]$ there exist sequences of prime numbers $(p_i)_{i\in\mathbb{N}}$, $(q_i)_{i\in\mathbb{N}}$, and $(d_i)_{i\in\mathbb{N}}$ such that the field $F = \mathbb{Q}((\frac{p_i}{q_i})^{1/d_i} | i \in \mathbb{N})$ satisfies $\mathcal{N}(F) = t$.*

Remark 0.2. While not stated, everything in [OS22] can be done over an arbitrary number field K. For this, think of K as the first step in the tower.

The full strength of this result is not necessary for our purposes. Instead we opt for the simpler construction of [PTW22].

Theorem 0.3 (Theorem 1.3 [PTW22])**.** *For every $t \in [0, \infty)$ there exist sequences of prime numbers $(p_i)_{i\in\mathbb{N}}$ and $(d_i)_{i\in\mathbb{N}}$ such that p_i^{1/d_i} converges to $\exp(2t)$ and the p_i are strictly increasing.*

Given such a sequence, the field $F = \mathbb{Q}({p_i}^{1/d_i} | i \in \mathbb{N})$ satisfies $t \leq \mathcal{N}(F) \leq 2t$.

We can show the abundance of extensions of K with large Northcott number as a formal consequence of the above theorem, i.e. using it as a blackbox.

2020 *Mathematics Subject Classification.* 11G50, 14G40, 11R04, 11G15.

Key words and phrases. heights, small points, Bogomolov property, Northcott number, singular moduli.

Lemma 1. *Let $C > 0$ be a constant and K a number field. Then there exist uncountably many algebraic extensions F of K such that $\mathcal{N}(F) > C$.*

For fields satisfying the Northcott property the finiteness of cycles of bounded degree and height is known. It is natural to ask whether a similar result can be extended to fields with known Northcott number.

Let (X, L) be a pair consisting of a variety over a number field K and a line bundle on said variety. In order to state our theorems more elegantly, we write $D(V) = (\dim(V) + 1)\deg(V)$ for equi-dimensional cycles V on $X_{\bar{K}}$. The line bundle implicit in this notation will be clear from context. Going forward, all cycles will be assumed equi-dimensional and effective throughout the article. We call a cycle F-rational if it is fixed by the action of $\mathrm{Gal}(\bar{K}/F)$.

Theorem 1. *Let X be a projective scheme over a number field K endowed with an admissible adelically metrized line bundle $\bar{L}$ whose underlying line bundle L is ample. Let $d \in \mathbb{N}$ and $C > 0$ be constants. Then there exists a constant $R > 0$ such that, for all algebraic extensions F of K, such that its Northcott number satisfies $\mathcal{N}(F) > d(C + R)$, we obtain the following.*

There are only finitely many F-rational cycles V on X such that $D(V) \leq d$ and $h_{\bar{L}}(V) < CD(V)$.

Remark 0.4. Regardless of this theorem, we can't expect to have only finitely many subvarieties defined over even a number field K as the Northcott property holds only for subvarieties of bounded degree. An example of the failure of the Northcott property without bound on the degree are the subvarieties $\overline{\{(z, z^n)\}} \subseteq \mathbb{P}^2$. They are all distinct, defined over the base field and have canonical height 0.

We will now give some specializations of interest with explicit constants.

Theorem 2. *Consider $\mathbb{P}^n$ over a number field K endowed with the canonical toric height $\hat{h}$. Let $d \in \mathbb{N}$ and $C > 0$ be constants. Let F be an extension of K, such that its Northcott number satisfies*

$$\mathcal{N}(F) > d\left(C + \frac{7}{2}n\log 2 + \sum_{i=1}^{n}\frac{1}{2i} + \log 2\right).$$

Then there are only finitely many F-rational cycles V on $\mathbb{P}^n_K$ such that $D(V) \leq d$ and $\hat{h}(V) < CD(V)$.

Theorem 3. *Let A be an abelian variety of dimension g over a number field K endowed with an ample symmetric line bundle $\mathcal{M}$. Let L denote the extension of K generated by*

$$\ker\left(A \xrightarrow{[16]} A \xrightarrow{p_{\mathcal{M}}} A^{\vee}\right),$$

where $p_{\mathcal{M}}$ denotes the polarization morphism associated to $\mathcal{M}$. Then there is an embedding Θ of A into $\mathbb{P}^n$ defined over L with associated line bundle $\mathcal{M}^{\otimes 16}$. Denote by h_2 the l^2-logarithmic Weil height and by $\hat{h}_{\mathcal{M}}$ the Néron-Tate height associated to $\mathcal{M}$.

Let $d \in \mathbb{N}$ and $C > 0$ be constants. If F is an extension of L, such that its Northcott number satisfies

$$\mathcal{N}(F) > \frac{d}{16}\left(C + 4^{g+1}h_2(\Theta_{\mathcal{M}^{\otimes 16}}(0_A)) + 3g\log 2 + \sum_{i=1}^{n}\frac{1}{2i} + \log 2\right),$$

then there are only finitely many F-rational cycles V on A_L such that $D(V) \leq d$ and $\hat{h}_{\mathcal{M}}(V) < CD(V)$. In particular, there are only finitely many torsion points and abelian subvarieties with $D(V) \leq d$ defined over F.

A similar result may be obtained for dynamical systems on projective space.

Theorem 4. *Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a selfmap of degree $D \geq 2$, defined over a number field K. Denote by $\hat{h}$ the canonical height associated to f and the tautological line bundle. Let $d \in \mathbb{N}$ and $C > 0$ be constants. Let F be an extension of K, such that its Northcott number satisfies*

$$\mathcal{N}(F) > d\left(C + C_1(n,D)h(f) + C_2(n,D) + \sum_{i=1}^{n}\frac{1}{2i}\right),$$

where $h(f)$ is the height of the coefficients of f as a projective tuple and

$$C_1(n,D) = 5nD^{n+1}, \quad C_2(n,D) = 3^n n^{n+1}(2D)^{n2^{n+4}D^n}.$$

Then there are only finitely many F-rational effective divisors V on $\mathbb{P}^n_K$ such that $\deg(V) \leq d$ and $\hat{h}(V) < CD(V)$. In particular, there are only finitely many preperiodic hypersurfaces of degree $\leq d$ defined over F.

Remark 0.5. Based on the ideas in [Ing22], a result that is linear in $\deg(V)$ should be possible in any codimension. At the present moment we may use [Hut19, Theorem 4.12], which yields a bound exponential in $\deg(V)$.

Remark 0.6. If we restrict to geometrically irreducible closed subsets we can improve the bound on the Northcott number by $d\log 2$ in Theorems 1, 2 and by $d\log 2/16$ in Theorem 3. The statement of Theorem 4 cannot be improved.

We lastly consider an application to CM points on the modular curve. These are not small points in the usual sense. For this reason it is necessary to consider weighted Weil heights.

Theorem 5. *There are uncountably many algebraic field extensions of $\mathbb{Q}$ containing only finitely many CM j-invariants.*

The author is not aware of other examples of infinite algebraic extensions of $\mathbb{Q}$ known to contain only finitely many CM j-invariants.

In the first section we introduce Northcott numbers and their behaviour under field extension. Lastly we deduce Lemma 1.

The second section will deal with various notions of height and the bounds on their differences. At the end we will see how Theorems 1 and 2 follow from these bounds.

The third section contains the applications to abelian varieties and dynamical systems on projective space.

At last, we construct infinite algebraic extensions of $\mathbb{Q}$ over which only finitely many CM points are defined.

Acknowledgements

I thank Fabien Pazuki for his guidance and mathematical discussions. I am specially grateful for his suggestion to consider also positive dimensional subvarieties and pointing me to references.

I thank Desirée Gijón Gómez for helpful comments on drafts of this article.

I thank Ricardo Menares for nice conversations at the journées arithmetiques 2023 and his suggestion to consider CM points.

I lastly thank Martin Widmer for pointing out that [OS22] is more general than I originally credited it to be and the audience at the Atelier ANR J-invariant for feedback on a wrong attribution.

1. Northcott numbers

In this section, we introduce Northcott numbers of subsets of $\bar{\mathbb{Q}}$, which allows us to refine Northcott's theorem (see [DZ08, Theorem 2.1]) to a statement on Northcott numbers that we call the Northcott inequality. We conclude the section with a proof of Lemma 1.

Definition 1.1 (Northcott number)**.** For a subset $S \subseteq \bar{\mathbb{Q}}$ of the algebraic numbers we define the Northcott number of S with respect to a function $f : \bar{\mathbb{Q}} \to [0, \infty)$ as

$$\mathcal{N}_f(S) = \inf\{t \in [0, \infty) | \#\{\alpha \in S; f(\alpha) < t\} = \infty\}.$$

We follow the convention that $\inf \emptyset = \infty$. We call $\mathcal{N}(S) \in [0, \infty]$ the Northcott number of S.

Remark 1.2. Our main focus is on the case that $f = h$ is the logarithmic Weil height. In this case, we omit the h from the notation.

Example 1.3. Let K be a number field. Then by Northcott's theorem $\mathcal{N}(K) = \infty$. On the other hand, $\mathcal{N}(\bar{\mathbb{Q}}) = 0$.

We now state and prove the Northcott inequality.

Theorem 1.4 (Northcott inequality)**.** *Let F be a field with Northcott number $\mathcal{N}(F) = C$. Then the set of algebraic numbers X of degree $\leq d$ over F satisfies $\mathcal{N}(X) \geq \frac{C - d\log 2}{d2^d}$.*

Proof. Let $\epsilon > 0$. Let Y_ϵ be the set of algebraic numbers x of height $\leq \frac{C-d\log 2}{d2^d} - \epsilon = B_\epsilon$ satisfying $[F(x) : F] \leq d$. It is enough to show that the set Y_ϵ is finite for any $\epsilon > 0$. Let $x \in Y_\epsilon$. Then the at most d conjugates of x over F are also elements of Y_ϵ. The coefficients of the minimal polynomial of x over F are elementary symmetric functions in these conjugates. We can bound the height of the coefficients by

$$d2^d B_\epsilon + d\log 2 = C - \epsilon d 2^d$$

using the properties of the height (see [BG06, Prop. 1.5.15]). Let $x, x_1, \dots, x_r \in \bar{\mathbb{Q}}$ and $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, then

$$h(\sigma(x)) = h(x) \tag{1}$$

$$h(x_1 + \cdots + x_r) \leq h(x_1) + \cdots + h(x_r) + \log r \tag{2}$$

$$h(x_1 \dots x_r) \leq h(x_1) + \cdots + h(x_r). \tag{3}$$

However, by assumption on F, there are only finitely many such coefficients, thus showing the finiteness of Y_ϵ. □

Remark 1.5. The optimal bound we may obtain with these methods is $\min_{0\leq j\leq d} \frac{C - \log\binom{d}{j}}{\binom{d}{j} j}$.

In [PTW22, Lemma 5] they notice that the house shares the crucial properties necessary to perform the proof of Theorem 1.4. By combining the ideas of [PTW22, Lemma 5] and Theorem 1.4 we obtain.

Lemma 1.6. *Let $f : \bar{\mathbb{Q}} \to [0, \infty)$ be a function. Denote by $\mathcal{N}_f(S)$ the Northcott number of a subset $S \subseteq \bar{\mathbb{Q}}$ with respect to f. Suppose that f satisfies*

$$f(\sigma(x)) = f(x) \tag{4}$$

$$f(x_1 + x_2) \leq F(f(x_1), f(x_2)) \tag{5}$$

$$f(x_1 x_2) \leq F(f(x_1), f(x_2)) \tag{6}$$

for some continuous function $F : \mathbb{R}^2 \to [0, \infty)$ and all $x_1, x_2 \in \bar{\mathbb{Q}}$ and $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Then there exists a continuous function $G : [0, \infty] \to [0, \infty]$ with $G(\infty) = \infty$ depending only on F and an auxiliary natural number d such that the following holds. Let $U \subseteq \bar{\mathbb{Q}}$ and let $S \subseteq \bar{\mathbb{Q}}$ be the subset of numbers satisfying monic polynomials with coefficients in U of degree bounded by d. Then

$$\mathcal{N}_f(S) \geq G(\mathcal{N}_f(U)).$$

Let us be more explicit in the case of the house. The house is defined as follows.

$$\overline{|-|} : \bar{\mathbb{Q}} \to [0, \infty) \tag{7}$$

$$\alpha \mapsto \max_{\sigma : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}} |\sigma(\alpha)| \tag{8}$$

Lemma 1.7. *Let F be a field such that $\mathcal{N}_{\overline{|-|}}(\mathcal{O}_F) = C$. Then the set of algebraic integers X of degree $\leq d$ over F satisfies $\mathcal{N}_{\overline{|-|}}(X) \geq \frac{C^{1/d}}{2^d}$.*

Proof. The proof is analogous to that of Theorem 1.4 using the properties

$$\overline{|\sigma(x)|} = \overline{|x|} \tag{9}$$

$$\overline{|x_1 + x_2|} \leq \overline{|x_1|} + \overline{|x_2|} \tag{10}$$

$$\overline{|x_1 x_2|} \leq \overline{|x_1|}\,\overline{|x_2|}. \tag{11}$$

for $x_1, x_2 \in \bar{\mathbb{Q}}$ and $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ □

Remark 1.8. We may improve the constant to $\min_{0 \leq j \leq d} \frac{C^{1/j}}{\binom{d}{j}}$.

This approach, of course, can be used to upper bound Northcott numbers, as well.

Corollary 1.9. *Suppose a field K has a field extension F of degree d satisfying $\mathcal{N}(F) = C$. Then $\mathcal{N}(K) \leq Cd2^d + d \log 2$.*

Remark 1.10. Again we may improve the bound. Here the best possible bound is $\min_{0 \leq j \leq d} \binom{d}{j} jC + \log \binom{d}{j}$.

Example 1.11. We may apply this to the field extension $\mathbb{Q}^{\mathrm{tr}}(i)/\mathbb{Q}^{\mathrm{tr}}$ of the totally real numbers. In [ADZ14, Example 5.3] it is shown that

$$\alpha_k = \left(\frac{2-i}{2+i}\right)^{1/k}$$

is a sequence of points with height tending to zero in $\mathbb{Q}^{\mathrm{tr}}(i)$. In particular, $\mathcal{N}_h(\mathbb{Q}^{\mathrm{tr}}(i)) = 0$. Hence $\mathcal{N}(\mathbb{Q}^{\mathrm{tr}}) \leq \log 2 \approx 0.693$. The best known bound is the one in [Smy80] ($\mathcal{N}(\mathbb{Q}^{tr}) \leq 0.2732\ldots$).

Remark 1.12. The bound in the specific case of the totally real numbers is not sharp and may be improved. Using that the conjugates of α_k equidistribute around the unit circle we may see that $h(\alpha_k + \overline{\alpha_k}) \to \int_0^1 \max\{2\log|\cos(\pi x)|, 0\} \approx 0.323.$[1]

We can prove Lemma 1.

[1]This constant also appears as the Mahler measure of the polynomial $1 + x + y$, computed by Smyth in [Boy80] and as the Arakelov-Zhang pairing $\langle x^2, 1-(1-x)^2 \rangle$ in [PST12]. It equals $\frac{3\sqrt{3}}{4\pi} L(2, \chi)$, where χ is the nontrivial quadratic character modulo 3.

Lemma 1. *Let $C > 0$ be a constant and K a number field. Then there exist uncountably many algebraic extensions F of K such that $\mathcal{N}(F) > C$.*

Proof. When the ground field is $\mathbb{Q}$, this follows immediately by the work of [OS22] or [PTW22] quoted at the beginning of the introduction.

Consider now the case of an arbitrary number field K and write $d = [K : \mathbb{Q}]$. We may use Theorem 1.4 to obtain that for fields F satisfying $\mathcal{N}(F) > d2^dD + d\log 2$ the composite field KF satisfies $\mathcal{N}(KF) > D$. Over $\mathbb{Q}$, there are uncountably many fields satisfying $\mathcal{N}(F) > d2^dD + d\log 2$. Hence it suffices to show that KF are distinct for distinct F.

For this let us consider fields of the form $F = \mathbb{Q}({p_i}^{1/d_i} | i \in \mathbb{N})$, where all p_i and d_i are distinct primes. We can find an extension F of the above form that further satisfies that ${p_i}^{1/d_i}$ tends to $\exp 2t$ for some $t > d2^dD + d\log 2$. This satisfies the conditions of 0.3 and hence $\mathcal{N}(F) \geq t$. Let $t' \neq t$ and F' be an extension $\mathbb{Q}({p'_i}^{1/d'_i} | i \in \mathbb{N})$ with the same conditions of F, but with ${p'_i}^{1/d'_i}$ going to $\exp 2t'$. We need to show that KF cannot contain F'. Now F' contains infinitely many ${p'_i}^{1/d'_i}$ that are not contained in F. When $d'_i > [K : \mathbb{Q}]$, then also ${p'_i}^{1/d'_i} \notin KF$. □

Theorem 1.13. *Let $C > 0$ be a constant and K a number field. Then there exist uncountably many algebraic extensions F of K such that $\mathcal{N}_{\lceil - \rceil}(\mathcal{O}_F) > C$.*

Proof. Fields F with prescribed value for $\mathcal{N}_{\lceil - \rceil}(\mathcal{O}_F)$ are constructed in [PTW22, Theorem 1]. The same argument as above applies since the fields are of similar form. □

1.1. **Relative Northcott numbers.** In [Oka22], Northcott numbers are considered in a relative setting. The following simplified statement of their result suffices for our needs.

Theorem 1.14 ([Oka22] Thm. 1.7.)**.** *There exists a field L satisfying $\mathcal{N}(L) = 0$ such that, for every $t \in (0, \infty]$, there exist sequences of prime numbers $(p_i)_{i\in\mathbb{N}}$, $(q_i)_{i\in\mathbb{N}}$, and $(d_i)_{i\in\mathbb{N}}$ such that the field $F = L((\frac{p_i}{q_i})^{1/d_i} | i \in \mathbb{N})$ satisfies $\mathcal{N}(F \setminus L) = t$.*

Lemma 1.15. *Let $L \subseteq F \subseteq \bar{\mathbb{Q}}$ be fields satisfying $\mathcal{N}(L) = c$ and $\mathcal{N}(F \setminus L) = t$. Then there exists no $x \in F \setminus L$ satisfying $h(x) < t - c$.*

Proof. We notice that the set $F \setminus L$ is closed under multiplication by elements in $L^\times$. Suppose $x \in F\setminus L$ satisfies $h(x) < t-c$. Let $\epsilon > 0$ be such that $h(x)+2\epsilon < t-c$. Then for any of the infinitely many $y \in L^\times$ satisfying $h(y) \leq c+\epsilon$, yx lies in $F \setminus L$ and satisfies $h(yx) \leq h(y) + h(x) < t - c - 2\epsilon + c + \epsilon = t - \epsilon$. This contradicts the assumption $\mathcal{N}(F \setminus L) = t$. □

Using the lemma above we can state and prove our results in a relative setting. Theorem 2, for instance, would take the following form.

Theorem 1.16. *Consider $\mathbb{P}^n$ over an algebraic extension $L/\mathbb{Q}$ endowed with the canonical toric height $\hat{h}$. Let $d \in \mathbb{N}$ and $C > 0$ be constants. Suppose that $\mathcal{N}(L) = c$. Let F be an extension of K, such that its relative Northcott number satisfies*

$$\mathcal{N}(F \setminus L) > d\left(C + \frac{7}{2}n\log 2 + \sum_{i=1}^{n} \frac{1}{2i} + \log 2\right) + c.$$

Then all F-rational cycles V on $\mathbb{P}^n_K$ such that $D(V) \leq d$ and $\hat{h}(V) < CD(V)$ are already defined over K.

2. Heights

This section will contain an overview of some different notions of heights and the bounds on their differences. The two notions of heights we will consider are Arakelov heights, which are defined using arithmetic intersection theory, and Philippon heights, whose definition relies on Chow forms of subvarieties of projective space. While Arakelov heights have conceptual advantages, Philippon height will be crucial to obtain information on the height of a subvariety from the arithmetic of its field of definition.

As a link between these two notions we use canonical heights. Canonical heights may be considered as Arakelov heights, but can at the same time be obtained from Philippon heights by a limit procedure. We will lastly apply this study to prove Theorems 1 and 2.

2.1. **Arakelov heights and adelic metrics.** We now introduce the notions in Arakelov geometry needed in this text. For a more comprehensive survey, we refer to [Cha21].

Let X be a proper scheme over $\mathbb{Q}$. For all places $v \leq \infty$ we may associate an analytic space X_v^{an} over $\mathbb{Q}_v$. For $v = \infty$ we set $X_\infty^{\text{an}} = X(\mathbb{C})/F_\infty$, where F_∞ denotes complex conjugation. For $v < \infty$ the definition of the analytification is due to Berkovich in [Ber12]. For all v this is a compact metrizable, locally contractible topological space containing $X(\mathbb{C}_v)/\operatorname{Aut}(\mathbb{C}_v/\mathbb{Q}_v)$ as a dense subspace. Further, it's equipped with the structure of a locally ringed space with a valued structure sheaf $\mathcal{O}_{X_v^{\text{an}}}$, i.e. to each $f \in \mathcal{O}_{X_v^{\text{an}}}(U)$ we can associate an absolute value function $|f| : U \to \mathbb{R}_+$ that is continuous in a way that is compatible with restrictions. We define $X_{\text{ad}} = \coprod_{v \leq \infty} X_v^{\text{an}}$.

We now define the structure of an adelic metric on a line bundle L on X. An *adelic metric* is a collection of compatible v-adic metrics. A v-adic metric on a line bundle L_v^{an} on X_v^{an} is the association of a norm function $||s||_v : U \to \mathbb{R}_+$ to every section $s \in L_v^{\text{an}}(U)$ compatible with restriction. Being a norm function means compatibility with multiplication by holomorphic functions and that $||s||_v$ only vanishes when s does. Tensor products and inverses of line bundles with v-adic metrics are canonically endowed with v-adic metrics. The absolute value endows the trivial bundle with a v-adic metric at all places.

The compatibility conditions for adelic metrics reflect the global nature of X. A proper model $(\mathcal{X}, \mathcal{L})$ of (X, L) over $\operatorname{Spec}\mathbb{Z}$ induces continuous v-adic metrics at all finite places. For a collection of continuous v-adic metrics to form an adelic metric we demand it agrees with the metrics induced by $(\mathcal{X}, \mathcal{L})$ at all but finitely many places. A metric family on a line bundle L is called *algebraic* if it is induced by a model of some tensor power $L^{\otimes n}$ of L, cf. [Cha21, Example 4.4].

Not all adelically metrized line bundles can be studied equally well. It is often helpful to impose algebraicity and positivity conditions. A notion fulfilling these requirements is semipositivity. *Semipositive metrics* are limits of algebraic metrics with a positivity condition, cf. [Cha21, Definition 5.1]. Important examples of semipositive metrics are the canonical metrics obtained from polarized dynamical systems. An adelic line bundle is called *admissible* if it can be represented as the difference of semipositive adelic line bundles.

We can easily define the height of a point $P \in X(\bar{\mathbb{Q}})$ in terms of adelic metrics. Let $\bar{L}$ be an adelically metrized line bundle on X with underlying line bundle L and $P \in X(\bar{\mathbb{Q}})$. This point defines a point P_v in the Berkovich space X_v^{an} for all v. The height of a point $P \in X(\bar{\mathbb{Q}})$ with respect to an adelically metrized line bundle $\bar{L}$ on X is defined as $h_{\bar{L}}(P) = -\sum_{v \leq \infty} \log ||s(P_v)||_v$, where s is a meromorphic section of L with no poles or zeroes at P.

More generally, the height of irreducible closed subsets of $X_{\bar{\mathbb{Q}}}$ is defined using arithmetic intersection theory. Given an irreducible closed subset $Z \subseteq X_{\bar{\mathbb{Q}}}$ of dimension d, we recall the height to be the arithmetic intersection number

$$h_{\bar{L}}(Z) = \widehat{\deg}_{\bar{L}}(Z) = \widehat{\deg}(\hat{c}_1(\bar{L})^{d+1}|Z).$$

We do not follow the convention of [Cha21] since we would like a notion which is additive in cycles. Our convention differs from that of [Cha21] by the factor $D(V) = (\dim(V) + 1)\deg(V)$.

2.2. **Heights under the variations of metrics.**

We will now introduce a lemma comparing the heights with respect to two admissible metrics.

Lemma 2.1. *Let X be a proper scheme over $\mathbb{Q}$ endowed with a line bundle L. Let $\bar{L}$ and $\bar{L}'$ be admissible adelic metrics on L. Then there exists a constant $C \in \mathbb{R}$ such that for all closed integral subschemes $V \subseteq X_{\bar{\mathbb{Q}}}$ we have*

$$|h_{\bar{L}}(V) - h_{\bar{L}'}(V)| \leq CD(V).$$

If L is ample and the metrics are algebraic, the admissibility assumption can be omitted.

Proof. This follows from [Cha21, Prop. 5.3.], a limit argument and linearity. The second case is Prop. 3.7 loc.cit. . In order to follow our convention we multiply the bounds by $D(V)$. □

2.3. **Philippon height.** There is an alternative definition of heights of subvarieties of projective space introduced by Philippon in his papers [Phi91], [Phi94] and [Phi95]. The Philippon height is obtained from the coefficients of the Chow form of the variety. This viewpoint is important in order to obtain information on the height of a subvariety from the arithmetic of its field of definition. We do not consider the case of weighted projective spaces. For more details we refer to Philippon's original papers. The heights in his different papers differ in the contribution of the infinite places. We will follow [Phi95].

In order to define the Philippon height of a subvariety of projective space we need to first define its Chow form. This is done using projective duality. Let K be a field and V be a closed geometrically irreducible subvariety of $\mathbb{P}^n_K$ of dimension r. Denote the variety parametrizing linear hyperplanes in $\mathbb{P}^n$, i.e. the projective dual of $\mathbb{P}^n$, by $\mathbb{P}^{n,\vee}$. The subvariety X of $(\mathbb{P}^{n,\vee})^{r+1}$ consisting of the tuples of hyperplanes $(H_0, \ldots, H_r)$ such that $H_0 \cap \ldots H_r \cap V \neq \emptyset$ is a hypersurface. In fact, it is the vanishing locus of a multihomogeneous polynomial over K of degree $\deg V$ in the coordinates of each factor. This polynomial f, defined up to multiplication by a scalar, is called the *Chow form* of V. If K is a number field we may now proceed to define the Philippon height of V. Given the Chow form we define

$$h_{Ph}(V) := \frac{1}{[K:\mathbb{Q}]} \sum_v [K_v : \mathbb{Q}_v] \log \mathrm{M}_v(f).$$

Here $\mathrm{M}_v(f)$ is defined as the maximum v-adic absolute value of the coefficients of f when v is a finite place. For the archimedean places we define

$$\log \mathrm{M}_v(f) = \int_{(S^{n+1})^{r+1}} \log |\sigma_v(f)| \sigma_{n+1}^{\wedge(r+1)} + D(V) \sum_{i=1}^{n} \frac{1}{2i}.$$

Here σ_v denotes a choice of complex embedding for the place v. S^{n+1} denotes the unit sphere in $\mathbb{C}^{n+1}$, while σ_{n+1} denotes the invariant probability measure on S^{n+1}. We define a variant of the Philippon height $\tilde{h}_{Ph}$ by taking the contribution at an archimedean place to be the maximum modulus of the coefficients instead.

We need to compare the Philippon height with this variant in order to deduce from the Northcott number of a field something about the height of projective varieties defined over said field. Philippon attributes such a comparison to Lelong [Lel92, Théorème 4]. We state it now.

Lemma 2.2. *Let $V \subseteq \mathbb{P}^n_{\bar{\mathbb{Q}}}$ be an integral closed subvariety, then we have the inequalities*

$$0 \leq h_{Ph}(V) - \tilde{h}_{Ph}(V) \leq D(V) \sum_{i=1}^{n} \frac{1}{2i} = D(V)c(n).$$

Lastly we need to compare Philippon's heights with the toric canonical height on projective space. This allows us to relate Arakelov heights with Philippon heights. The following statement is taken from [DP99, Prop 2.1].

Proposition 2.3. *Let $V \subseteq \mathbb{P}^n_{\bar{\mathbb{Q}}}$ be a closed irreducible subset. Let $\hat{h}$ denote canonical toric height on $\mathbb{P}^n$. Then*

$$|\hat{h}(V) - h_{Ph}(V)| \leq D(V)\frac{7}{2}n \log 2.$$

2.4. **Cycles.** It may be useful to consider the height of general equi-dimensional F-rational cycles for a field $F \subseteq \bar{\mathbb{Q}}$. Since the components of an F-rational cycle C are not necessarily defined over F, a further lemma is required to relate its height to the arithmetic of F.

Let $C_{\bar{\mathbb{Q}}} = \sum n_i V_i$, for irreducible V_i, be a F-rational cycle on $\mathbb{P}^n$. Its Chow form is defined to be

$$f_C = \prod f_{V_i}^{n_i}.$$

Up to scalar, f_C has coefficients in F. Let us define the Philippon height of a cycle C by applying Philippon's construction to f_C. We can define $\tilde{h}_{Ph}$ in the analogous way.

The resulting height isn't linear with respect to addition of cycles. To address this issue we invoke an inequality on the height of products of polynomials.

Theorem 2.4 ([BG06] Thm 1.6.13)**.** *Let $f_1, \dots, f_m$ be polynomials in n variables, d the sum of partial degrees of $f = f_1 \dots f_m$ and let h denote the logarithmic Weil height of the coefficients of a polynomial considered as a projective tuple. Then*

$$|h(f) - \sum_{j=1}^{m} h(f_j)| \leq d \log 2.$$

Lemma 2.5. *Let $C = \sum n_i V_i$ be a homogeneous cycle of $\mathbb{P}^n_{\bar{\mathbb{Q}}}$. Then*

$$|\tilde{h}_{Ph}(C) - \sum n_i \tilde{h}_{Ph}(V_i)| \leq D(C) \log 2.$$

Proof. We apply the theorem to $f_C = \prod f_{V_i}^{n_i}$ and obtain that $d = (\dim(C) + 1)\deg(C)$. □

2.5. **Small subvarieties of projective space.** In this section we prove Theorems 2 and 1 on small subvarieties.

Theorem 2. *Consider $\mathbb{P}^n$ over a number field K endowed with the canonical toric height $\hat{h}$. Let $d \in \mathbb{N}$ and $C > 0$ be constants. Let F be an extension of K, such that its Northcott number satisfies*

$$\mathcal{N}(F) > d\left(C + \frac{7}{2}n\log 2 + \sum_{i=1}^{n}\frac{1}{2i} + \log 2\right).$$

Then there are only finitely many F-rational cycles V on $\mathbb{P}^n_K$ such that $D(V) \leq d$ and $\hat{h}(V) < CD(V)$.

Proof. Let $V = \sum n_i V_i$ be an F-rational homogeneous cycle. Then its Chow form f_V has coefficients in F. As such, we know that $h(f_V) \leq \mathcal{N}(F) - \epsilon$ for only finitely many cycles. By Lemma 2.5 there can only be finitely many cycles satisfying $\sum n_i \tilde{h}_{Ph}(V_i) \leq \mathcal{N}(F) - D(V)\log 2 - \epsilon$. Consequently there are only finitely many V with $\sum n_i h_{Ph}(V_i) + \epsilon \leq \mathcal{N}(F) - D(V)\,(c(n) + \log 2)$ by Lemma 2.2. Moreover, there are only finitely many V such that

$$\hat{h}(V) + \epsilon = \sum n_i \hat{h}(V_i) + \epsilon \leq \mathcal{N}(F) - D(V)\left(\frac{7}{2}n\log 2 + c(n) + \log 2\right)$$

by Proposition 2.3. Under the assumption that $C > \frac{\mathcal{N}(F)}{d} - \frac{7}{2}n\log 2 - c(n) - \log 2$ we obtain that there are only finitely many F-rational cycles V on $\mathbb{P}^n_K$ such that $D(V) \leq d$ and $\hat{h}(V) < CD(V)$. By rearranging the inequality, we conclude the theorem. □

We easily obtain Theorem 1 as a consequence.

Theorem 1. *Let X be a projective scheme over a number field K endowed with an admissible adelically metrized line bundle $\bar{L}$ whose underlying line bundle L is ample. Let $d \in \mathbb{N}$ and $C > 0$ be constants. Then there exists a constant $R > 0$ such that, for all algebraic extensions F of K, such that its Northcott number satisfies $\mathcal{N}(F) > d(C + R)$, we obtain the following.*

There are only finitely many F-rational cycles V on X such that $D(V) \leq d$ and $h_{\bar{L}}(V) < CD(V)$.

Proof. We need to compare the heights on X with heights on projective varieties. For this we replace $\bar{\mathcal{L}}$ by its n-th power such that the underlying line bundle is very ample. Let $X \hookrightarrow \mathbb{P}^k$ be an embedding associated to $\mathcal{L}$. Pulling back the canonical toric metric on $\mathcal{O}(1)$ induces an adelic metric on $\mathcal{L}$, which we denote $\tilde{\mathcal{L}}$.

Then by Lemma 2.1 the height associated to $\tilde{\mathcal{L}}$ only differs from the one associated to $\bar{\mathcal{L}}$ by an amount bounded by $R'D(V)$ for some constant R'. Now the result follows from Theorem 2. □

3. Applications to dynamical systems

Specializations of our main theorem can be obtained by applying more specific height bounds. The arguments required to obtain these specializations are adaptations of the proof of Theorem 2, which will only be sketched.

The dynamical systems to be considered in greater detail are the ones given by multiplication on abelian varieties and selfmaps of projective space. We start out with a more general situation considered in the foundational paper of Call and Silverman([CS93]).

In their setup, X is a smooth projective variety over a number field K endowed with a selfmap ϕ and a divisor class $\eta \in \mathrm{Pic}(X) \otimes \mathbb{R}$ satisfying $\phi^*\eta = \alpha\eta$ for some $\alpha > 1$. Suppose h is a Weil function associated with η. Then there is a constant R such that $|h \circ \phi - \alpha h| \leq R$. Let $\hat{h}$ denote the canonical height for η and ϕ. Then the following holds.

Proposition 3.1 ([CS93] Proposition 1.2)**.** *For every $P \in X(\bar{K})$, the following inequality holds:*

$$|\hat{h}(P) - h(P)| \leq \frac{R}{\alpha - 1}.$$

Note that we can't expect to have finitely many small points for arbitrary η, as an associated Weil function might not even be bounded below. We may, however, by adapting the proof of Theorem 2 obtain the following statement.

Proposition 3.2. *In the current setting, suppose that η is very ample and h is induced by the canonical toric height under some embedding into projective space. Let F be an algebraic extension of K satisfying $\mathcal{N}(F) > C + \frac{R}{\alpha-1}$. Then there are only finitely many points $P \in V(F)$ such that $\hat{h}(P) \leq C$.*

Proof. We adapt the proof of Theorem 2. We bound the height of a point in projective space from below by the height of one of its coordinates and use the bound in Proposition 3.1. □

3.1. Small subvarieties of abelian varieties.

In order to study small points on abelian varieties, we embed them into projective space using a variant of the theta embedding, first introduced in [Mum66]. For a more detailed overview of its properties, see [DP02]. We will then apply a bound on the difference of the canonical height to the Philippon height from loc.cit. to deduce a result on small points of abelian varieties.

Let A be a g-dimensional abelian variety defined over a number field K. Let $\mathcal{M}$ be an ample symmetric line bundle on A. Then $\mathcal{M}^{\otimes 16}$ is very ample. David and Philippon choose sections that yield the embedding $\Theta_{\mathcal{M}^{\otimes 16}}$, or simply Θ, into $\mathbb{P}^N$. It is inspired by the embedding of Mumford in [Mum66], but differs from it. As such, it is not defined over K itself, but over the field generated by

$$\ker\left(A \xrightarrow{[16]} A \xrightarrow{p_{\mathcal{M}}} A^\vee\right),$$

where $p_{\mathcal{M}}$ denotes the polarization morphism associated to $\mathcal{M}$.

In this setting, we have the following comparison of heights.

Proposition 3.3 ([DP02] Proposition 3.9.)**.** *Let V be an integral closed subvariety of $A_{\bar{K}}$ and let h_2 denote the l^2-logarithmic Weil height. Then*

$$|\hat{h}_{\mathcal{M}^{\otimes 16}}(V) - h_{Ph}(\Theta(V))| \leq c_0(\Theta)D(V).$$

Here, $c_0(\Theta) = 4^{g+1}h_2(\Theta(0_A)) + 3g\log 2$.

Theorem 3. *Let A be an abelian variety of dimension g over a number field K endowed with an ample symmetric line bundle $\mathcal{M}$. Let L denote the extension of K generated by*

$$\ker\left(A \xrightarrow{[16]} A \xrightarrow{p_{\mathcal{M}}} A^\vee\right),$$

where $p_{\mathcal{M}}$ denotes the polarization morphism associated to $\mathcal{M}$. Then there is an embedding Θ of A into $\mathbb{P}^n$ defined over L with associated line bundle $\mathcal{M}^{\otimes 16}$. Denote by h_2 the l^2-logarithmic Weil height and by $\hat{h}_{\mathcal{M}}$ the Néron-Tate height associated to $\mathcal{M}$.

Let $d \in \mathbb{N}$ and $C > 0$ be constants. If F is an extension of L, such that its Northcott number satisfies

$$\mathcal{N}(F) > \frac{d}{16}\left(C + 4^{g+1}h_2(\Theta_{\mathcal{M}^{\otimes 16}}(0_A)) + 3g\log 2 + \sum_{i=1}^{n}\frac{1}{2i} + \log 2\right),$$

then there are only finitely many F-rational cycles V on A_L such that $D(V) \leq d$ and $\hat{h}_{\mathcal{M}}(V) < CD(V)$. In particular, there are only finitely many torsion points and abelian subvarieties with $D(V) \leq d$ defined over F.

Proof. We adapt the proof of Theorem 2. The main differences are that Proposition 3.3 applies to $\hat{h}_{\mathcal{M}^{\otimes 16}} = 16\hat{h}_{\mathcal{M}}$ instead of directly to $\hat{h}_{\mathcal{M}}$ and that the Θ-embedding of A is not defined over its field of definition K, but only over $L = K\left(\ker\left(A \xrightarrow{[16]} A \xrightarrow{p_{\mathcal{M}}} A^\vee\right)\right)$. □

Remark 3.4. The l^2-logarithmic Weil height $h_2(\Theta_{\mathcal{M}^{\otimes 16}}(0_A))$ in the theorem is compared to the Faltings height of the abelian variety in [Paz12]. This allows for a phrasing of the theorem that does not reference the theta embedding. In [DP02] the quantity $h(\Theta_{\mathcal{M}^{\otimes 16}}(0_A))$ is denoted by $h(A)$ which may lead to confusion with the Philippon height of A, see [DP02, Notation 3.2.].

3.2. **Small subvarieties with respect to dynamical systems on $\mathbb{P}^n$.** Another case in which explicit bounds on difference of heights exist are divisors on $\mathbb{P}^n$ with a canonical height from a selfmap. In fact, [Ing22] proves the following statement.

Theorem 3.5. *Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a morphism of degree $d \geq 2$ defined over $\bar{\mathbb{Q}}$. Let V be an effective divisor on $\mathbb{P}^n$, then*

$$|\hat{h}_f(V) - h_{Ph}(V)| \leq (C_1(n,d)h(f) + C_2(n,d))D(V),$$

where $h(f)$ is the height of the coefficients of f as a projective tuple. Moreover, one may choose

$$C_1(n,d) = 5nd^{n+1}, \quad C_2(n,d) = 3^n n^{n+1}(2d)^{n2^{n+4}d^n}.$$

For simplicity he states the theorem only for hypersurfaces, but claims there to be no conceptual obstruction to its generalization.

This leads to Theorem 4.

Theorem 4. *Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a selfmap of degree $D \geq 2$, defined over a number field K. Denote by $\hat{h}$ the canonical height associated to f and the tautological line bundle. Let $d \in \mathbb{N}$ and $C > 0$ be constants. Let F be an extension of K, such that its Northcott number satisfies*

$$\mathcal{N}(F) > d\left(C + C_1(n,D)h(f) + C_2(n,D) + \sum_{i=1}^{n} \frac{1}{2i}\right),$$

where $h(f)$ is the height of the coefficients of f as a projective tuple and

$$C_1(n,D) = 5nD^{n+1}, \quad C_2(n,D) = 3^n n^{n+1}(2D)^{n2^{n+4}D^n}.$$

Then there are only finitely many F-rational effective divisors V on $\mathbb{P}^n_K$ such that $\deg(V) \leq d$ and $\hat{h}(V) < CD(V)$. In particular, there are only finitely many preperiodic hypersurfaces of degree $\leq d$ defined over F.

Proof. We adapt the proof of Theorem 2. Note that Theorem 3.5 applies directly to cycles, so the results in section 2.5 are not needed. □

4. Application to special points

While special points on Shimura varieties are not small in the usual sense, our approach can still deduce a finiteness result for CM points on the modular curve defined over certain infinite extensions. To this end, we will use weighted Weil heights.

We have some information on the height of special points on the modular curve from [Bre01]. The result on the degree is a restating of the Brauer-Siegel theorem.

Proposition 4.1 (Proposition 2.1 [Bre01])**.** *Let $x \in \bar{\mathbb{Q}}$. If the elliptic curve E_x of j-invariant x has complex multiplication we denote $\Delta(x) = |\operatorname{Disc}(\operatorname{End}(E_x))|$.*

(1) If x is CM, then $[\mathbb{Q}(x) : \mathbb{Q}] = \Delta(x)^{1/2+o(1)}$.
(2) There exists an effectively computable constant C such that if x is CM, $h(x) \leq \pi\Delta(x)^{1/2} + C$.

Remark 4.2. In fact, the proof of part 2 computes the asymptotic of the house as the discriminant grows: $\overline{|x|} \approx \exp(\pi\Delta(x)^{1/2})$.

Let $\gamma \in \mathbb{R}$, $x \in \bar{\mathbb{Q}}$. Then, the weighted Weil height h_γ is defined by $h_\gamma(x) = \deg(x)^\gamma h(x)$. We may consider Northcott numbers of subsets $S \subseteq \bar{\mathbb{Q}}$ for varying γ. For a set $S \subseteq \bar{\mathbb{Q}}$, define the sets

$$I_0(S) = \{\gamma \mid \mathcal{N}_{h_\gamma}(S) = 0\}, \quad I_\infty(S) = \{\gamma \mid \mathcal{N}_{h_\gamma}(S) = \infty\}.$$

We can summarize the work of [OS22] as follows.

Theorem 4.3. *The sets $I_0(S)$ and $I_\infty(S)$ are (in the case of $I_0(S)$ possibly empty) rays. They satisfy $(1, \infty) \subseteq I_\infty$ and $I(S) = \sup I_0 = \inf I_\infty$. For $\gamma \in (-\infty, 1)$ and $c \in [0, \infty]$ one can construct a field F such that $I(F) = \gamma$ and $\mathcal{N}_{h_\gamma}(F) = c$.*

We phrase a Corollary of Theorem 4.1 in terms of weighted Weil heights.

Corollary 4.4. *The set of CM points S satisfies $I(S) \leq -1$.*

Theorem 5. *There are uncountably many algebraic field extensions of $\mathbb{Q}$ containing only finitely many CM j-invariants.*

Proof. Any field F satisfying $I(F) < 1$ constructed in Theorem 4.3 fulfills the conditions. □

Remark 4.5. Using Remark 4.2 we see that the corresponding properties for the weighted house suffice for the conclusion. However, the counterpart to Theorem 4.3 has not yet been proven in this setting.

References

[ADZ14] Francesco Amoroso, Sinnou David and Umberto Zannier. "On fields with the Property (B)". In: *Proceedings of the American Mathematical Society* 142 (June 2014). DOI: 10.1090/S0002-9939-2014-11925-3.

[Ber12] Vladimir Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields.* 33. American Mathematical Soc., 2012.

[BG06] Enrico Bombieri and Walter Gubler. *Heights in Diophantine Geometry.* New Mathematical Monographs. Cambridge University Press, 2006. ISBN: 9780521712293.

[Boy80] David Boyd. “Speculations Concerning the Range of Mahler’s Measure”. In: *Canadian Mathematical Bulletin* 24.4 (1980), pp. 453–469. DOI: 10.4153/CMB-1981-069-5.

[Bre01] Florian Breuer. “Heights of CM points on complex affine curves”. In: *Ramanujan J.* 5.3 (2001), pp. 311–317. ISSN: 1382-4090,1572-9303. DOI: 10.1023/A:1012982812988.

[Cha21] Antoine Chambert-Loir. “Chapter VII: Arakelov Geometry, Heights, Equidistribution, and the Bogomolov Conjecture”. In: *Arakelov Geometry and Diophantine Applications*. Ed. by Emmanuel Peyre and Gaël Rémond. Cham: Springer International Publishing, 2021, pp. 299–328. ISBN: 978-3-030-57559-5. DOI: 10.1007/978-3-030-57559-5_8.

[CS93] Gregory Call and Joseph Silverman. “Canonical heights on varieties with morphisms”. In: *Compositio Mathematica* 89.2 (1993), pp. 163–205.

[DP02] Sinnou David and Patrice Philippon. “Minorations des hauteurs normalisées des sous-variétés de variétés abeliennes II”. In: *Commentarii Mathematici Helvetici* 77 (2002), pp. 639–700.

[DP99] Sinnou David and Patrice Philippon. “Minorations des hauteurs normalisées des sous-variétés des tores”. fre. In: *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze* 28.3 (1999), pp. 489–543. URL: http://eudml.org/doc/84386.

[DZ08] Roberto Dvornicich and Umberto Zannier. “On the properties of Northcott and of Narkiewicz for fields of algebraic numbers”. In: *Functiones et Approximatio Commentarii Mathematici* 39 (Nov. 2008). DOI: 10.7169/facm/1229696562.

[Hut19] Benjamin Hutz. “Good reduction and canonical heights of subvarieties”. In: *Mathematical Research Letters* 25.6 (2019), pp. 1837–1863. ISSN: 1073-2780. DOI: 10.4310/MRL.2018.v25.n6.a7.

[Ing22] Patrick Ingram. *Explicit canonical heights for divisors relative to endomorphisms of* $\mathbb{P}^N$. 2022. arXiv: 2207.07206 [math.NT].

[Lel92] Pierre Lelong. “Mesure de Mahler des polynômes et majoration par convexité”. In: *Comptes rendus de l’Académie des sciences. Série 1, Mathématique* 315.2 (1992), pp. 139–142.

[Mum66] David Mumford. “On the Equations Defining Abelian Varieties. I.” In: *Inventiones mathematicae* 1 (1966), pp. 287–354. URL: http://eudml.org/doc/141838.

[Oka22] Masao Okazaki. *Relative Northcott numbers for the weighted Weil heights*. 2022. arXiv: 2206.05440 [math.NT].

[OS22] Masao Okazaki and Kaoru Sano. *Northcott numbers for the weighted Weil heights*. 2022. arXiv: 2204.04446 [math.NT].

[Paz12] Fabien Pazuki. “Theta height and Faltings height”. en. In: *Bulletin de la Société Mathématique de France* 140.1 (2012), pp. 19–49. DOI: 10.24033/bsmf.2623. URL: http://numdam.org/articles/10.24033/bsmf.2623/.

[Phi91] Patrice Philippon. “Sur des hauteurs alternatives. I.” In: *Mathematische Annalen* 289.2 (1991), pp. 255–284. URL: http://eudml.org/doc/164780.

[Phi94] Patrice Philippon. “Sur des hauteurs alternatives. II”. fr. In: *Annales de l'Institut Fourier* 44.4 (1994), pp. 1043–1065. DOI: 10.5802/aif.1426.

[Phi95] Patrice Philippon. “Sur des hauteurs alternatives III”. In: *Journal de mathématiques pures et appliquées* 74.4 (1995), pp. 345–365.

[PST12] Clayton Petsche, Lucien Szpiro and Thomas Tucker. “A dynamical pairing between two rational maps”. In: *Transactions of the American Mathematical Society* 364.4 (2012), pp. 1687–1710. ISSN: 00029947. URL: http://jstor.org/stable/41524901.

[PTW22] Fabien Pazuki, Niclas Technau and Martin Widmer. “Northcott numbers for the house and the Weil height”. In: *Bulletin of the London Mathematical Society* 54.5 (2022), pp. 1873–1897. DOI: 10.1112/blms.12662.

[Smy80] Christopher Smyth. “On the measure of totally real algebraic integers”. In: *Journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics* 30.2 (1980), pp. 137–149. DOI: 10.1017/S1446788700016426.

Nuno Hultberg. University of Copenhagen, Institute of Mathematics, Universitetsparken 5, 2100 Copenhagen, Denmark; ORCiD: orcid.org/0000-0003-0097-0499

Email address: nh@math.ku.dk

A linear AFL for quaternion algebras

A LINEAR AFL FOR QUATERNION ALGEBRAS

NUNO HULTBERG AND ANDREAS MIHATSCH

Abstract. We prove new fundamental lemma and arithmetic fundamental lemma identities for general linear groups over quaternion division algebras. In particular, we verify the transfer conjecture and the arithmetic transfer conjecture from [11] in cases of Hasse invariant 1/2.

Contents

1. Introduction 1
2. Statement of Results 2
3. Fundamental Lemma 9
4. Arithmetic Fundamental Lemma 16
References 21

1. Introduction

Fix a non-archimedean local field F and some $n \geq 1$. The linear arithmetic fundamental lemma (AFL) conjecture of Q. Li [7] states a family of identities between derivatives of orbital integrals on $GL_{2n}(F)$ and intersection numbers on moduli spaces of strict formal O_F-modules of height $2n$. It has a global motivation which is parallel to that of W. Zhang's unitary AFL [14] and which is related to the trace formula comparison of Leslie–Xiao–Zhang [6]. We refer to [15] and the introduction of [11] for this global aspect and henceforth focus on the local setting.

While the unitary AFL has been proved [16, 12, 13, 17], the linear AFL conjecture is still open. It is, however, known to hold when $n \leq 2$, see [8]. Moreover, both the conjecture and its validity for $n \leq 2$ have been extended to a biquadratic setting by Howard–Li [3] and Li [9]. A non-basic version of both the linear and the biquadratic AFL is formulated and reduced to the basic setting in [10].

The linear AFL concerns orbital integrals for hyperspecial test functions and moduli spaces for GL_{2n} for hyperspecial level (good reduction). In a recent article, Li and the second author formulated a variant that relates parahoric test functions on the analytic side with moduli spaces for central simple algebras on the intersection-theoretic side.

In the present article, we consider this variant in the case of Hasse invariant 1/2. More precisely, we prove a fundamental lemma (FL) type and an AFL type statement for the group $GL_n(B)$ where B/F is a quaternion division algebra. We now state these results in a vague form, together with references to their precise formulations.

Date: September 30, 2024.

Theorem A (Fundamental Lemma, see Theorem 2.7)**.** *Let* $\gamma \in GL_{2n}(F)$ *and* $g \in GL_n(B)$ *be regular semi-simple matching elements. Then*

$$\mathrm{Orb}(\gamma, 1_{\mathrm{Par}}, 0) = \mathrm{Orb}(g, 1_{GL_n(O_B)}).$$

Here, the orbital integral on the left is with respect to the subgroup $GL_n(F \times F)$. The one on the right is with respect to the subgroup $GL_n(E)$ for a fixed embedding $E \to B$ of an unramified quadratic extension E/F. The test function 1_{Par} on the left hand side is the indicator function of the standard $(n \times n)$-block parahoric subgroup of GL_{2n}. Precise definitions will be given in §2.

In order to state our arithmetic result, we also need the datum of a 1-dimensional strict O_E-module $\mathbb{Y}$ of O_E-height n. The Serre tensor construction $O_B \otimes_{O_E} \mathbb{Y}$ then has dimension 2 and O_F-height $4n$.

Theorem B (Arithmetic Fundamental Lemma, see Theorem 2.9)**.** *Let* $\gamma \in GL_{2n}(F)$ *and* $g \in \mathrm{Aut}^0_B(O_B \otimes_{O_E} \mathbb{Y})$ *be regular semi-simple matching elements. Assume that the linear AFL holds for* $\mathbb{Y}$*. Then*

$$\frac{d}{ds}\Big|_{s=0} \mathrm{Orb}(\gamma, 1_{\mathrm{Par}}, s) = 2\,\mathrm{Int}(g)\log(q). \tag{1.1}$$

We refer to §2.4 for the definition of the intersection number $\mathrm{Int}(g)$. Using that the linear AFL is known for all $\mathbb{Y}$ whose connected factor has O_E-height ≤ 2 we obtain the following corollary.

Corollary (see Corollary 2.10)**.** *Assume that the connected factor of* $\mathbb{Y}$ *has* O_E*-height* ≤ 2*. Then, for every pair of regular semi-simple matching elements* $\gamma \in GL_{2n}(F)$ *and* $g \in \mathrm{Aut}^0_B(O_B \otimes_{O_E} \mathbb{Y})$*,*

$$\frac{d}{ds}\Big|_{s=0} \mathrm{Orb}(\gamma, 1_{\mathrm{Par}}, s) = 2\,\mathrm{Int}(g)\log(q).$$

In particular, this verifies new cases of the arithmetic transfer (AT) conjecture [11, Conjecture 1.5]. We remark that [11, Conjecture 1.5] involves an unspecified correction term. Identity (1.1) shows that we expect this term to vanish for Hasse invariant 1/2.

Our proofs of Theorems A and B are by reduction to the Guo–Jacquet FL [2], the quadratic base change FL, and the linear AFL. On the orbital integral side, this reduction relies on a combinatorial interpretation in terms of lattice counts. On the intersection-theoretic side, it relies on a moduli-theoretic translation between the intersection problems for $GL_{2n}(F)$ and $GL_n(B)$.

2. Statement of Results

The definitions and conventions that follow are taken from [11, §2 – §4], but specialized to general linear groups over quaternion algebras. We begin this section by recalling the invariant polynomial (§2.1) and by fixing our setting (§2.2). Next, we give the definitions of the relevant orbital integrals (§2.3) and intersection numbers (§2.4). Then we state our main results (§2.5).

2.1. **Regular semi-simple orbits.** We will consider several instances of the following kind. Let F be a field and let E/F be an étale quadratic extension with Galois conjugation σ. Let D be a central simple algebra (CSA) over F of degree $2n$ together with an embedding $E \to D$. Let $C = \mathrm{Cent}_E(D)$ be the centralizer which is a CSA over E of degree n.

In this situation, $C^\times \times C^\times$ acts on $D^\times$ by $(h_1, h_2) \cdot g = h_1^{-1} g h_2$. An element $g \in D^\times$ is called regular semi-simple if its orbit for this action is Zariski closed and its stabilizer of minimal dimension. (This minimal dimension is equal to n.) According to Jacquet–Rallis [4] and Guo [2, §1], the regular semi-simple orbits can be characterized as follows:

Let $D = D_+ \oplus D_-$ be the decomposition into E-linear and E-conjugate linear components. That is, $D_+ = C$ and $D_- = \{x \in D \mid xa = \sigma(a)x \text{ for } a \in E\}$. Let $g = g_+ + g_-$ denote the corresponding decomposition of an element $g \in D^\times$. For $g \in D^\times$ with $g_+ \in D^\times$, set $z_g = g_+^{-1} g_-$.

Definition 2.1. Assume that $g \in D^\times$ with $g_+ \in D^\times$. The reduced characteristic polynomial $\mathrm{charred}_{D/F}(z_g^2; T) \in F[T]$ is always a square. Define the invariant of g as its unique monic square root,

$$\mathrm{Inv}(g; T) := \mathrm{charred}_{D/F}(z_g^2; T)^{1/2} \in F[T]. \tag{2.1}$$

The polynomial $\mathrm{Inv}(g; T)$ is monic, of degree n, and satisfies $\mathrm{Inv}(g; 1) \neq 0$. An element $g \in D^\times$ is regular semi-simple if and only if both g_+, g_- lie in $D^\times$ and $\mathrm{Inv}(g; T)$ is a separable polynomial. The invariant polynomial classifies orbits in the sense that for two regular semi-simple elements $g_1, g_2 \in D^\times$,

$$C^\times g_1 C^\times = C^\times g_2 C^\times \iff \mathrm{Inv}(g_1; T) = \mathrm{Inv}(g_2; T).$$

2.2. **Setting and notation.** For the rest of this article, we fix an integer $n \geq 1$ and a non-archimedean local field F with uniformizer π. Let q denote its residue field cardinality, v its normalized valuation and $|x| = q^{-v(x)}$ its normalized absolute value.

We denote by $K = F \times F$ the split quadratic extension of F. The diagonal embedding $K \to M_{2n}(F)$, $(a, b) \mapsto \mathrm{diag}(a \cdot 1_n, b \cdot 1_n)$ is of the kind considered in §2.1 and has centralizer $M_n(K)$. Let

$$(G', \ H') = (GL_{2n}(F), \ GL_n(K))$$

be the corresponding pair of linear groups. For $\gamma \in G'$, the notion of being regular semi-simple and the invariant $\mathrm{Inv}(\gamma; T)$ are meant with respect to the $(H' \times H')$-action. Let $G'_{\mathrm{rs}} \subset G'$ denote the set of regular semi-simple elements.

We denote by E/F an unramified quadratic extension, by $\sigma \in \mathrm{Gal}(E/F)$ the non-trivial element, and by $\eta : F^\times \to \{\pm 1\}$, $\eta(x) = (-1)^{v(x)}$ the character associated to E. Let O_F and O_E be the rings of integers in F and E.

Let B_λ/F be a quaternion algebra over F of Hasse invariant $\lambda \in \{0, 1/2\}$. For both possibilities $B \in \{B_0, B_{1/2}\}$, we fix a maximal order $O_B \subset B$ and an embedding $O_E \to O_B$. The resulting diagonal embedding $E \to M_n(B)$ is of the type considered in §2.1 and has centralizer $M_n(E)$. Let

$$(G_\lambda, \ H) = (GL_n(B_\lambda), \ GL_n(E))$$

be the corresponding pair of linear groups. For $g \in G_\lambda$, the notion of being regular semi-simple and the invariant $\mathrm{Inv}(g;T)$ are meant with respect to the $(H \times H)$-action. Let $G_{\lambda,\mathrm{rs}} \subset G_\lambda$ be the subset of regular semi-simple elements.

2.3. **Orbital integrals.**

2.3.1. *The case of G'.* We define two characters η and $|\cdot|$ on H' by

$$\eta,\ |\cdot| : H' \longrightarrow \mathbb{C}^\times, \quad \eta((a,b)) := \eta(\det(ab^{-1})), \quad |(a,b)| := |\det(ab^{-1})|.$$

For a regular semi-simple element $\gamma \in G'_{\mathrm{rs}}$, we denote by

$$(H' \times H')_\gamma = \{(h_1,h_2) \in H' \times H' \mid h_1\gamma = \gamma h_2\} \tag{2.2}$$

the stabilizer of γ. Set $L_\gamma = F[z_\gamma^2]$ which is an étale extension of degree n of F because γ is regular semi-simple. There is the identity $L_\gamma^\times = H' \cap \gamma^{-1}H'\gamma$, so $(H' \times H')_\gamma$ can be identified with the torus $L_\gamma^\times$. We normalize the Haar measures on $H' \times H'$ and $(H' \times H')_\gamma$ by

$$\mathrm{Vol}(GL_n(O_K)) = \mathrm{Vol}(O_{L_\gamma}^\times) = 1. \tag{2.3}$$

For a regular semi-simple element $\gamma \in G'_{\mathrm{rs}}$, a test function $f' \in C_c^\infty(G')$ and a complex parameter $s \in \mathbb{C}$, we can now define the orbital integral

$$\mathrm{Orb}(\gamma, f', s) := \Omega(\gamma, s) \cdot \int_{\frac{H' \times H'}{(H' \times H')_\gamma}} f'(h_1^{-1}\gamma h_2)|h_1h_2|^s \eta(h_2) d(h_1,h_2). \tag{2.4}$$

Here, the so-called transfer factor $\Omega(\gamma, s) \in \pm q^{\mathbb{Z}\cdot s}$ ensures that the definition only depends on the orbit $H'\gamma H'$. It is defined by

$$\Omega\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, s\right) = \eta(\det(cd^{-1})) \cdot |\det(b^{-1}c)|^s$$

and satisfies $\Omega(h_1^{-1}\gamma h_2, s) = |h_1h_2|^s\eta(h_2)\Omega(\gamma, s)$. The definition in (2.4) moreover relies on [3, Lemma 3.2.3.] which states that η and $|\cdot|$ have trivial restriction to $(H' \times H')_\gamma$. We will be interested in the central value and the central derivative only, so we define

$$\mathrm{Orb}(\gamma, f') := \mathrm{Orb}(\gamma, f', 0) \quad \text{and} \quad \partial\mathrm{Orb}(\gamma, f') := \frac{d}{ds}\bigg|_{s=0} \mathrm{Orb}(\gamma, f', s). \tag{2.5}$$

2.3.2. *The case of G_λ.* Let $G \in \{G_0, G_{1/2}\}$ be one of the two possible groups. For a regular semi-simple element $g \in G_{\mathrm{rs}}$, we denote by $(H \times H)_g$ its stabilizer. It can be identified with the torus $L_g^\times$, where $L_g = F[z_g^2]$ as before. We normalize the Haar measures on $H \times H$ and $(H \times H)_g$ by

$$\mathrm{Vol}(GL_n(O_E)) = \mathrm{Vol}(O_{L_g}^\times) = 1. \tag{2.6}$$

For a regular semi-simple element $g \in G_{\mathrm{rs}}$ and a test function $f \in C_c^\infty(G)$, we put

$$\mathrm{Orb}(g, f) := \int_{\frac{H \times H}{(H \times H)_g}} f(h_1^{-1}gh_2) d(h_1,h_2). \tag{2.7}$$

2.4. **Intersection numbers.** We next define two families of intersection numbers, one for each of the possible Hasse invariants $\lambda \in \{0, 1/2\}$. For $\lambda = 0$, these are the intersection numbers that occur in the linear AFL from [10]. For $\lambda = 1/2$, these are a special case of the intersection numbers from [11].

Let $B = B_\lambda$ in the following. Let $\breve{F}$ be the completion of a maximal unramified extension of F with ring of integers $O_{\breve{F}}$ and residue field $\mathbb{F}$; fix an embedding $E \to \breve{F}$. As a further datum, let $\mathbb{Y}$ be a strict π-divisible O_E-module of dimension 1 and O_E-height n over $\mathbb{F}$, see [10, Definition 3.1]. We denote its O_E-action by $\iota : O_E \to \mathrm{End}(\mathbb{Y})$. Recall that the connected-étale sequence of any such $\mathbb{Y}$ has a unique splitting, meaning there is a canonical isomorphism

$$\mathbb{Y} = \mathbb{Y}^\circ \times \mathbb{Y}^{\mathrm{et}}$$

where $\mathbb{Y}^\circ$ is the connected component of the identity of $\mathbb{Y}$ and $\mathbb{Y}^{\mathrm{et}}$ its maximal étale quotient. Let n° and n^{et} be the O_E-heights of $\mathbb{Y}^\circ$ and $\mathbb{Y}^{\mathrm{et}}$. Then $n = n^\circ + n^{\mathrm{et}}$, every $1 \le n^\circ \le n$ may occur, and $(n^\circ, n^{\mathrm{et}})$ characterize $\mathbb{Y}$ uniquely up to isomorphism.

Define $(\mathbb{X}, \kappa)$ as the Serre tensor product $\mathbb{X} = O_B \otimes_{O_E} \mathbb{Y}$ with O_B-action $\kappa(x) = x \otimes \mathrm{id}_{\mathbb{Y}}$. Note that $\mathbb{X}$ also carries the O_B-linear O_E-action $\mathrm{id}_{O_B} \otimes \iota$ which we again denote by ι.

Remark 2.2. Assume that $\lambda = 0$. Then any choice of isomorphism $O_B \cong M_2(O_F)$ provides a decomposition of $\mathbb{X}$ as $\mathbb{X} = \mathbb{Y}^{\oplus 2}$ (Morita equivalence). In this way, the ensuing definitions for $\lambda = 0$ reduce to the ones in [10].

We consider the rings of quasi-endomorphisms $C_{\mathbb{Y}} = \mathrm{End}^0_E(\mathbb{Y}, \iota)$ and $D_{\lambda,\mathbb{Y}} = \mathrm{End}^0_B(\mathbb{X}, \kappa)$, as well as the groups $H_{\mathbb{Y}} = C_{\mathbb{Y}}^\times$ and $G_{\lambda,\mathbb{Y}} = D_{\lambda,\mathbb{Y}}^\times$. By functoriality of the Serre tensor construction, there is a group homomorphism

$$H_{\mathbb{Y}} \longrightarrow G_{\lambda,\mathbb{Y}}, \quad g \longmapsto \mathrm{id}_{O_B} \otimes g.$$

The structure of these groups is as follows. Let $\mathbb{X} = \mathbb{X}^\circ \times \mathbb{X}^{\mathrm{et}}$ be the connected-étale decomposition of $\mathbb{X}$. There are no homomorphisms between connected and étale π-divisible groups over $\mathbb{F}$, so

$$C_{\mathbb{Y}} = C_{\mathbb{Y}}^\circ \times C_{\mathbb{Y}}^{\mathrm{et}}, \qquad D_{\lambda,\mathbb{Y}} = D_{\lambda,\mathbb{Y}}^\circ \times D_{\lambda,\mathbb{Y}}^{\mathrm{et}}, \tag{2.8}$$

where the factors denote the E-linear (resp. B-linear) quasi-endomorphisms of the factors of $\mathbb{Y}$ and $\mathbb{X}$. Then

$$\begin{array}{rclcrcl} C_{\mathbb{Y}}^\circ & \cong & C_{1/n^\circ} & \qquad & D_{\lambda,\mathbb{Y}}^\circ & \cong & D_{1/2n^\circ+\lambda} \\ C_{\mathbb{Y}}^{\mathrm{et}} & \cong & M_{n^{\mathrm{et}}}(E) & & D_{\lambda,\mathbb{Y}}^{\mathrm{et}} & \cong & M_{n^{\mathrm{et}}}(B). \end{array}$$

Here, C_{1/n° denotes a central division algebra (CDA) of Hasse invariant $1/n^\circ$ over E, and $D_{1/2n^\circ+\lambda}$ denotes a CSA of degree $2n^\circ$ and Hasse invariant $1/2n^\circ + \lambda$ over F. Note that the two pairs $(C_{\mathbb{Y}}^\circ, D_{\lambda,\mathbb{Y}}^\circ)$ and $(C_{\mathbb{Y}}^{\mathrm{et}}, D_{\lambda,\mathbb{Y}}^{\mathrm{et}})$ are of the type considered in §2.1.

Definition 2.3. An element $g = (g^\circ, g^{\mathrm{et}}) \in G_{\lambda,\mathbb{Y}}$ is called regular semi-simple if its components g° and g^{et} are regular semi-simple with respect to $H_{\mathbb{Y}}^\circ$ and $H_{\mathbb{Y}}^{\mathrm{et}}$, and if its invariant, defined as

$$\mathrm{Inv}(g; T) = \mathrm{Inv}(g^\circ; T)\,\mathrm{Inv}(g^{\mathrm{et}}; T) \in F[T],$$

is a separable polynomial. Let $G_{\lambda,\mathbb{Y},\mathrm{rs}} \subset G_{\lambda,\mathbb{Y}}$ be the subset of regular semi-simple elements.

We now associate an intersection number to each $g \in G_{\lambda,\mathbb{Y},\mathrm{rs}}$. Let $\mathcal{N}$ be the RZ space for $\mathbb{Y}$: By definition, this means that it is the formal scheme over $O_{\breve{F}}$ that represents the functor

$$\mathcal{N}(S) = \left\{ (Y, \iota, \rho) \;\middle|\; \begin{array}{c} (Y,\iota)/S \text{ a strict } O_E\text{-module} \\ \rho : \overline{S} \times_{\operatorname{Spec}\mathbb{F}} (\mathbb{Y}, \iota) \longrightarrow \overline{S} \times_S (Y,\iota) \text{ a quasi-isogeny} \end{array} \right\}. \tag{2.9}$$

Here and in the following, $\overline{S} = \mathbb{F} \otimes_{O_{\breve{F}}} S$ denotes the special fiber of S.

Similarly, let $\mathcal{M}_\lambda$ be the RZ space for $(\mathbb{X}, \kappa)$. Recall for its definition that an O_B-action on a 2-dimensional strict O_F-module X over a $\operatorname{Spf} O_{\breve{F}}$-scheme S is called special if the two $\kappa(O_E)$-eigenspaces of $\operatorname{Lie}(X)$ are both locally free of rank 1 as $\mathcal{O}_S$-modules.[1] Then $\mathcal{M}_\lambda$ is the formal scheme over $\operatorname{Spf} O_{\breve{F}}$ that represents the functor

$$\mathcal{M}_\lambda(S) = \left\{ (X, \kappa, \rho) \;\middle|\; \begin{array}{c} (X,\kappa)/S \text{ a special } O_B\text{-module} \\ \rho : \overline{S} \times_{\operatorname{Spec}\mathbb{F}} (\mathbb{X}, \kappa) \longrightarrow \overline{S} \times_S (X,\kappa) \text{ a quasi-isogeny} \end{array} \right\}. \tag{2.10}$$

It is clear that if (Y, ι) is a 1-dimensional strict O_E-module, then $O_B \otimes_{O_E} (Y, \iota)$ is a special O_B-module. Thus we obtain a morphism

$$\begin{aligned} \mathcal{N} &\longrightarrow \mathcal{M} \\ (Y,\,\iota,\,\rho) &\longmapsto (O_B \otimes_{O_E} Y,\ \kappa(x) := x \otimes \mathrm{id}_Y,\ \mathrm{id}_{O_B} \otimes \rho). \end{aligned} \tag{2.11}$$

This is a closed immersion by [11, Proposition 4.15]. There are right actions of $H_{\mathbb{Y}}$ on $\mathcal{N}$ and of $G_{\lambda,\mathbb{Y}}$ on $\mathcal{M}_\lambda$ by

$$h \cdot (Y, \iota, \rho) = (Y, \iota, \rho h) \quad \text{and} \quad g \cdot (X, \kappa, \rho) = (X, \kappa, \rho g).$$

The morphism (2.11) is equivariant with respect to $H_{\mathbb{Y}} \to G_{\lambda,\mathbb{Y}}$. Moreover, the structure of $\mathcal{N}$ as formal scheme is easy to describe:

$$\mathcal{N} \cong \coprod_{\mathbb{Z} \times GL_{n^{\mathrm{et}}}(E)/GL_{n^{\mathrm{et}}}(O_E)} \operatorname{Spf} O_{\breve{F}}[\![t_1, \dots, t_{n-1}]\!], \tag{2.12}$$

where the indexing can be chosen compatibly with the $H_{\mathbb{Y}}^{\mathrm{et}}$-action for a fixed identification $H_{\mathbb{Y}}^{\mathrm{et}} \cong GL_{n^{\mathrm{et}}}(E)$. In particular, $\mathcal{N}$ is formally smooth of relative dimension $n-1$ over $\operatorname{Spf} O_{\breve{F}}$.

Concerning $\mathcal{M}_\lambda$, it is known to be locally formally of finite type over $\operatorname{Spf} O_{\breve{F}}$ and regular of dimension $2n$, see [11, Proposition 4.13]. Using (2.11), we can thus view $\mathcal{N}$ as cycle in middle dimension on $\mathcal{M}_\lambda$.

For a regular semi-simple element $g \in G_{\lambda,\mathbb{Y},\mathrm{rs}}$, we define the intersection locus

$$\mathcal{I}(g) := \mathcal{N} \cap g \cdot \mathcal{N}. \tag{2.13}$$

For such g, we also denote by $g = g_+ + g_-$ the decomposition of g into E-linear and E-conjugate linear components, where $\iota : E \to D_{\lambda,\mathbb{Y}}$ comes from the definition of $(\mathbb{X}, \kappa)$. Set $z_g = g_+^{-1} g_-$ and $L_g = F[z_g^2]$. Since $L_g^\times = H_{\mathbb{Y}} \cap g^{-1} H_{\mathbb{Y}} g$ as subgroups of $G_{\lambda,\mathbb{Y}}$, the $L_g^\times$-action on $\mathcal{M}$ preserves the intersection locus $\mathcal{I}(g)$. Let $\Gamma \subset L_g^\times$ be a free discrete subgroup such that $L_g^\times = \Gamma \times O_{L_g}^\times$. It acts without fixed points on $\mathcal{M}_\lambda$.

Proposition 2.4. *Let $g \in G_{\lambda,\mathbb{Y}}$ and $\Gamma \subset L_g^\times$ be as before. The quotient $\Gamma \backslash (\mathcal{N} \cap g \cdot \mathcal{N})$ is an artinian scheme.*

[1] This definition goes back to Drinfeld [1].

Proof. By [11, Proposition 4.18],[2] the quotient $\Gamma\backslash\mathcal{I}(g)$ is a proper scheme over $\mathrm{Spec}\, O_{\breve F}$ with empty generic fiber. Since the maximal reduced subscheme $\mathcal{N}_{\mathrm{red}}$ is 0-dimensional, cf. (2.12), this scheme has to be artinian. □

The general definition of intersection numbers in [11, Definition 4.21] now specializes to taking the length:

Definition 2.5. For a regular semi-simple element $g \in G_{\lambda,\mathbb{Y},\mathrm{rs}}$, choose Γ as before and define

$$\mathrm{Int}(g) := \mathrm{len}_{O_{\breve F}}\left(\mathcal{O}_{\Gamma\backslash\mathcal{I}(g)}\right).$$

2.5. **FL and AFL.** We consider the following test functions. For the case $\lambda = 0$, i.e. for $B \cong M_2(F)$, we define

$$f_0' = 1_{GL_{2n}(O_F)} \in C_c^\infty(G') \quad \text{and} \quad f_0 = 1_{GL_{2n}(O_F)} \in C_c^\infty(G).$$

For the case $\lambda = 1/2$, we first define the parahoric subgroup

$$\mathrm{Par} = \left\{ \begin{pmatrix} GL_n(O_F) & \pi\, M_n(O_F) \\ M_n(O_F) & GL_n(O_F) \end{pmatrix} \right\} \subset GL_{2n}(O_F). \tag{2.14}$$

Let $1_{\mathrm{Par}} \in C_c^\infty(G')$ be its characteristic function and let $h \in H'$ be the element $\mathrm{diag}(\pi 1_n, 1_n)$. Then put

$$f_{1/2}'(\,\cdot\,) = 1_{\mathrm{Par}}(h\,\cdot\,) \in C_c^\infty(G') \quad \text{and} \quad f_{1/2} = 1_{GL_n(O_B)} \in C_c^\infty(G). \tag{2.15}$$

The orbital integrals of $f_{1/2}'$ and 1_{Par} are related by

$$\mathrm{Orb}(\gamma, f_{1/2}', s) = q^{-ns}\,\mathrm{Orb}(\gamma, 1_{\mathrm{Par}}, s). \tag{2.16}$$

The advantage of $f_{1/2}'$ over 1_{Par} is that its orbital integral satisfies the completely symmetric functional equation (see [11, Proposition 3.19])

$$\mathrm{Orb}(\gamma, f_{1/2}', -s) = \varepsilon_{1/2}(\gamma)\,\mathrm{Orb}(\gamma, f_{1/2}', s) \tag{2.17}$$

where the sign is defined by

$$\varepsilon_{1/2}(\gamma) = (-1)^n\,(-1)^r, \quad r = v\left(\mathrm{det}_{M_n(B)/F}(z_\gamma)\right). \tag{2.18}$$

Definition 2.6. Two regular semi-simple elements $\gamma \in G'_{\mathrm{rs}}$ and $g \in G_{\lambda,\mathrm{rs}}$ (resp. γ and $g \in G_{\lambda,\mathbb{Y},\mathrm{rs}}$) are said to match if

$$\mathrm{Inv}(\gamma; T) = \mathrm{Inv}(g; T).$$

Theorem 2.7 (Fundamental Lemma)**.** *For every regular semi-simple $\gamma \in G'_{\mathrm{rs}}$,*

$$\mathrm{Orb}(\gamma, f_\lambda') = \begin{cases} \mathrm{Orb}(g, f_\lambda) & \textit{if there exists a matching } g \in G_{\lambda,\mathrm{rs}} \\ 0 & \textit{otherwise.} \end{cases}$$

The case $\lambda = 0$ is well-known (Guo–Jacquet FL) and due to Guo [2]. Our addition is the case $\lambda = 1/2$ whose proof will be given in §3. We now turn to the central derivatives:

[2]Strictly speaking, [11, Proposition 4.18] is formulated only for p-adic F. However, Proposition 2.4 is known when $\lambda = 0$ by [10, Lemma 3.7] and the comparison between the cases $\lambda = 0$ and $\lambda = 1/2$ in §4 gives an alternative proof of Proposition 2.4 that applies to all F.

Conjecture 2.8 (Arithmetic Fundamental Lemma)**.** For every regular semi-simple element $\gamma \in G'_{\mathrm{rs}}$,

$$\partial\mathrm{Orb}(\gamma, f'_\lambda) = \begin{cases} 2\,\mathrm{Int}(g)\log(q) & \text{if there is a strict } O_E\text{-module } (\mathbb{Y},\iota) \text{ and } g \in G_{\lambda,\mathbb{Y},\mathrm{rs}} \text{ that matches } \gamma \\ 0 & \text{otherwise.} \end{cases} \tag{2.19}$$

The AT conjecture for general CSAs [11, Conjecture 1.5] also includes an unspecified correction term that cannot be omitted in general. Conjecture 2.8 here, which is for Hasse invariant 1/2, is hence stronger in the sense that this correction term is conjectured to vanish.

For $\lambda = 0$, Conjecture 2.8 is the linear AFL conjecture from [10]. The vanishing part of (2.19) is known to hold in general, see [10, Corollary 2.18]. Moreover, Identity (2.19) is known for all $g \in G_{0,\mathbb{Y},\mathrm{rs}}$ with $\mathbb{Y}$ such that $n^\circ \le 2$. This statement is [10, Corollary 1.3] and goes back to Li [8] who verified the case $n \le 2$.

Our main result here is a reduction of the case $\lambda = 1/2$ in Conjecture 2.8 to the case $\lambda = 0$:

Theorem 2.9. *(1) The vanishing part of Conjecture 2.8 holds.*
(2) If Conjecture 2.8 holds for $\lambda = 0$, then it also holds for $\lambda = 1/2$. More precisely, assume that Identity (2.19) *holds for $\lambda = 0$ and all $g \in G_{0,\mathbb{Y},\mathrm{rs}}$ for some $\mathbb{Y}$. Then Identity* (2.19) *also holds for $\lambda = 1/2$ and all $g \in G_{1/2,\mathbb{Y},\mathrm{rs}}$.*

Corollary 2.10 (to Theorem 2.9 and [10, Corollary 1.3])**.** *Identity* (2.19) *holds in all cases with $\lambda = 1/2$ and with $\mathbb{Y}$ such that $n^\circ \le 2$.* □

Our proofs of Theorems 2.7 and 2.9 are by expressing the occurring orbital integrals and intersection numbers for $\lambda = 1/2$ in terms of orbital integrals and intersection numbers for $\lambda = 0$. This is made precise by the following result.

Theorem 2.11. *(1) Assume that $\gamma \in G'_{\mathrm{rs}}$ is regular semi-simple and that $\pi^{-n}\mathrm{Inv}(\gamma;\pi T)$ lies in $T^n + \pi O_F[T]$. Then there exists a regular semi-simple element $\widetilde{\gamma} \in G'_{\mathrm{rs}}$ with $\mathrm{Inv}(\widetilde{\gamma};T) = \pi^{-n}\mathrm{Inv}(\gamma;\pi T)$ and*

$$\mathrm{Orb}(\gamma, f'_{1/2}, s) = \mathrm{Orb}(\widetilde{\gamma}, f'_0, s).$$

(2) Assume that $g \in G_{1/2,\mathrm{rs}}$ and that $\pi^{-n}\mathrm{Inv}(g;\pi T) \in T^n + \pi O_F[T]$. Then there exists a regular semi-simple $\widetilde{g} \in G_{0,\mathrm{rs}}$ with $\mathrm{Inv}(\widetilde{g};T) = \pi^{-n}\mathrm{Inv}(g;\pi T)$ and

$$\mathrm{Orb}(g, f_{1/2}) = \mathrm{Orb}(\widetilde{g}, f_0)).$$

(3) Assume that $g \in G_{1/2,\mathbb{Y},\mathrm{rs}}$ is regular semi-simple with $\pi^{-n}\mathrm{Inv}(g;\pi T) \in T^n + \pi O_F[T]$. Then there exists a regular semi-simple $\widetilde{g} \in G_{0,\mathbb{Y},\mathrm{rs}}$ with $\mathrm{Inv}(\widetilde{g};T) = \pi^{-n}\mathrm{Inv}(g;\pi T)$ and

$$\mathrm{Int}(g) = \mathrm{Int}(\widetilde{g}).$$

The proof of Theorem 2.11 will be constructive in the sense that we work with explicit orbit representatives γ resp. g and then define concrete elements $\widetilde{\gamma}$ and $\widetilde{g}$.

The condition on the invariants of $\gamma \in G'$ resp. $g \in G_{1/2}$ or $g \in G_{1/2,\mathbb{Y}}$ in Theorem 2.11 can also be formulated by saying that $z_\gamma^2/\pi \in L_\gamma$ resp. $z_g^2/\pi \in L_g$ are topologically nilpotent. It is clear that Theorem 2.11 immediately implies Theorems 2.7 and 2.9 for such elements. Moreover, the cases where z_γ^2/π or z_g^2/π are not integral over O_F are trivial: Here, all orbital integrals or intersection numbers in question vanish, a fact proven in Corollary 3.4, Lemma 3.10 and Lemma 4.5. This leaves the edge cases where z_γ^2/π or z_g^2/π are integral but not topologically nilpotent. We will prove some auxiliary results (Corollary 3.6, Lemma 3.11, Proposition 4.8) that allow to split off the non-topologically nilpotent part and to treat it separately.

3. Fundamental Lemma

In this section we prove parts (1) and (2) of Theorem 2.11. As a corollary we will obtain the fundamental lemma for $GL_n(B)$ in Theorem 2.9. Our main tool is the combinatorial interpretation of orbital integrals in terms of lattice counts.

3.1. **Orbital integrals on G'.** Our aim in this section is to relate the orbital integrals $\mathrm{Orb}(\gamma, f'_{1/2}, s)$ and $\mathrm{Orb}(\gamma, f'_0, s)$ for regular semi-simple $\gamma \in G'_{\mathrm{rs}}$. They both only depend on the double coset $H'\gamma H'$. Hence it suffices to consider elements of the form $\gamma(x) = \left(\begin{smallmatrix} 1 & x \\ 1 & 1 \end{smallmatrix}\right)$, where $x \in GL_n(F)$ and where $1 \in GL_n(F)$ denotes the unit element. Recall that we define $z_\gamma = \gamma_+^{-1}\gamma_-$ and $L_\gamma = F[z_\gamma^2]$. Note that $z_{\gamma(x)} = \left(\begin{smallmatrix} 0 & x \\ 1 & 0 \end{smallmatrix}\right)$ and $z_{\gamma(x)}^2 = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$. So it is by definition that the invariant polynomial $\mathrm{Inv}(\gamma(x), T)$ of $\gamma(x)$ equals the characteristic polynomial of x. We next recall some definitions and results from [11], but specialized to our quaternion algebra setting. Let $K = F \times F$ be as in §2.2.

Definition 3.1 ([11, Definition 3.20])**.** Let $\mathcal{L}$ denote the set of O_K-lattices in F^{2n}. For $\gamma \in G'_{\mathrm{rs}}$ regular semi-simple and $\lambda \in \{0, 1/2\}$, we define $\mathcal{L}_\lambda(\gamma)$ in the following way. For $\lambda = 0$, we put

$$\mathcal{L}_0(\gamma) := \{\Lambda \in \mathcal{L} \mid \gamma\Lambda \in \mathcal{L}\}. \tag{3.1}$$

For $\lambda = 1/2$, and where $(\pi, 1) \in O_K$ is the element $\mathrm{diag}(\pi, 1)$, we put

$$\mathcal{L}_{1/2}(\gamma) := \{\Lambda \in \mathcal{L} \mid z_\gamma^2\Lambda \subseteq z_\gamma(\pi, 1)\Lambda \subseteq \pi\Lambda\}. \tag{3.2}$$

These two sets equal the set $\mathcal{L}(\gamma)$ from [11, Definition 3.20] in the two cases of Hasse invariant 0 and 1/2. In the case of Hasse invariant 1/2, we have furthermore used the equivalent description in [11, (3.26)].

Note that the O_K-lattices $\Lambda \in \mathcal{L}$ are precisely the direct sums $\Lambda_+ \oplus \Lambda_-$ of O_F-lattices $\Lambda_+, \Lambda_- \subset F^n$. We use this to give a more concrete description of $\mathcal{L}_{1/2}(\gamma(x))$:

Definition 3.2. For $x \in GL_n(F)$, define

$$\mathcal{L}(x) := \{(\Lambda_+, \Lambda_-) \text{ pairs of } O_F\text{-lattices in } F^n \mid x\Lambda_- \subseteq \Lambda_+ \subseteq \Lambda_-\}. \tag{3.3}$$

Since $z_{\gamma(x)} = \left(\begin{smallmatrix} 0 & x \\ 1 & 0 \end{smallmatrix}\right)$, it follows directly from definitions that

$$\begin{aligned} \mathcal{L}(x/\pi) &\xrightarrow{\sim} \mathcal{L}_{1/2}(\gamma(x)) \\ (\Lambda_+, \Lambda_-) &\longmapsto \Lambda_+ \oplus \Lambda_-. \end{aligned}$$

Definition 3.3 ([11, Definition 3.20])**.** For a regular semi-simple element $\gamma \in G'_{\mathrm{rs}}$ and a lattice $\Lambda \in \mathcal{L}_0(\gamma)$, we put

$$\Omega(\gamma, \Lambda, s) := \Omega(h_1^{-1}\gamma h_2, s), \tag{3.4}$$

where $h_1, h_2 \in H'$ are chosen such that $\Lambda = h_1 O_F^{2n}$ and $\gamma\Lambda = h_2 O_F^{2n}$.

For $x \in GL_n(F)$, we let $L_x = F[x] \subseteq M_n(F)$ be the generated F-algebra. For a pair $\Lambda = (\Lambda_+, \Lambda_-) \in \mathcal{L}(x)$, we denote by $R_\Lambda \subset L_x$ the stabilizer of Λ under the diagonal action of L_x on lattices. We can now state the combinatorial interpretation of orbital integrals from [11, Equation (3.25)] of the test function $f'_{1/2}$:

$$\mathrm{Orb}(\gamma(x), f'_{1/2}, s) = q^{-ns} \sum_{\Lambda=(\Lambda_+,\Lambda_-)\in\mathcal{L}(x/\pi)/L_x^\times} [O_{L_x}^\times : R_\Lambda^\times]\ \Omega(\gamma(x),\, \Lambda_+ \oplus \Lambda_-,\, s). \quad (3.5)$$

Corollary 3.4. *If x/π is not integral over O_F, then the orbital integral $\mathrm{Orb}(\gamma(x), f'_{1/2}, s)$ vanishes identically.*

Proof. The set $\mathcal{L}(x/\pi)$ is empty if x/π is not integral. □

Assume now that x is topologically nilpotent. Then [11, Lemma 3.21 (1)] states that there is a bijection

$$\begin{aligned} \mathcal{L}(x) &\xrightarrow{\sim} \mathcal{L}_0(\gamma(x)) \\ (\Lambda_+, \Lambda_-) &\longmapsto \Lambda_+ \oplus \Lambda_-. \end{aligned}$$

Identity [11, Equation (3.25)] then specializes to

$$\mathrm{Orb}(\gamma(x), 1_{GL_{2n}(O_F)}, s) = \sum_{\Lambda=(\Lambda_+,\Lambda_-)\in\mathcal{L}(x)/L_x^\times} [O_{L_x}^\times : R_\Lambda^\times]\ \Omega(\gamma(x),\, \Lambda_+ \oplus \Lambda_-,\, s). \quad (3.6)$$

Combining (3.5) and (3.6) yields a proof of Theorem 2.11 (1):

Proof of Theorem 2.11 (1). First recall the statement: Let $\gamma \in G'_{\mathrm{rs}}$ be regular semi-simple. The statement we would like to prove only depends on the orbit of γ, so we may assume without loss of generality that $\gamma = \gamma(x)$ for some $x \in GL_n(F)$. Set $\widetilde{x} = x/\pi$. The assumption is

$$\mathrm{char}(\widetilde{x}; T) = \pi^{-n}\mathrm{Inv}(\gamma(x), \pi T) \in T^n + \pi O_F[T]$$

which implies that $\widetilde{x}$ is topologically nilpotent. In particular, $\widetilde{\gamma} := \gamma(\widetilde{x})$ lies in $GL_{2n}(F)$ and is regular semi-simple. It satisfies $\mathrm{Inv}(\widetilde{\gamma}; T) = \pi^{-n}\mathrm{Inv}(\gamma; \pi T)$ and our task is to show that

$$\mathrm{Orb}(\gamma, f'_{1/2}, s) = \mathrm{Orb}(\widetilde{\gamma}, 1_{GL_{2n}(O_F)}, s). \quad (3.7)$$

To this end, we compare the two quantities $\Omega(\gamma, \Lambda_+ \oplus \Lambda_-, s)$ and $\Omega(\widetilde{\gamma}, \Lambda_+ \oplus \Lambda_-, s)$ for $(\Lambda_+, \Lambda_-) \in \mathcal{L}(\widetilde{x})$. Choose h_1 and h_2 in H' such that $h_1 O_F^{2n} = \Lambda_+ \oplus \Lambda_-$ and $h_2 O_F^{2n} = \gamma(\Lambda_+ \oplus \Lambda_-)$. Write $h_1^{-1} = \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right)$ and $h_2 = \left(\begin{smallmatrix} c & \\ & d \end{smallmatrix}\right)$. We obtain from Definition 3.3 that

$$\begin{aligned} \Omega(\widetilde{\gamma},\, \Lambda_+ \oplus \Lambda_-,\, s) &= \Omega\left(\begin{pmatrix} ac & axd/\pi \\ bc & bd \end{pmatrix}, s\right) \\ &= \Omega\left(\begin{pmatrix} ac & axd \\ bc & bd \end{pmatrix}, s\right) q^{-ns} = \Omega(\gamma,\, \Lambda_+ \oplus \Lambda_-,\, s) q^{-ns}. \end{aligned} \quad (3.8)$$

Substituting this into (3.6) for $\widetilde{\gamma}$ yields (3.5) for γ which proves (3.7) as desired. □

Note that if $(\Lambda_+, \Lambda_-) \in \mathcal{L}(x)$, then both Λ_+ and Λ_- are $O_F[x]$-modules. For this reason, we next focus on the ring $O_F[x]$, in particular on its idempotents.

Lemma 3.5. *Assume that $x \in GL_n(F)$ is integral over O_F. Then there is a unique way to write $O_F[x]$ as a product $R_0 \times R_1$ such that the image (x_0, x_1) of x has the property that x_0 is topologically nilpotent and $x_1 \in R_1^\times$ a unit.*

Proof. As x was assumed to be integral over O_F, the ring $O_F[x]$ is of the form $O_F[T]/(P(T))$ for some monic polynomial $P(T) \in O_F[T]$. We consider the reduction $\overline{P}$ of P modulo π. It will factor as $\overline{P} = T^m f(T)$ with $f(0) \neq 0$. By Hensel's lemma, this factorization lifts to a factorization of P which defines the desired factorization $R_0 \times R_1$. □

Corollary 3.6. *Let $x \in GL_n(F)$ have the property that $\widetilde{x} = x/\pi$ is integral over O_F. Let $O_F[\widetilde{x}] = R_0 \times R_1$ be the factorization from Lemma 3.5 with respect to $\widetilde{x}$ and let (x_0, x_1) denote the components of x. Let $F^n = V_0 \times V_1$ be the corresponding factorization of F^n and fix isomorphisms $V_i \cong F^{n_i}$. Then $\gamma = \gamma(x)$ lies in G'_{rs} while $\gamma_0 = \gamma(x_0)$ and $\gamma_1 = \gamma(x_1)$ lie in $GL_{2n_i}(F)_{\mathrm{rs}}$. There is an identity of orbital integrals*

$$\mathrm{Orb}(\gamma, f'_{1/2}, s) = \mathrm{Orb}(\gamma_0, f'_{1/2,n_0}, s)\,\mathrm{Orb}(\gamma_1, f'_{1/2,n_1}, s) \tag{3.9}$$

where the test functions on the right hand side are meant in the sense of (2.15) *but on $GL_{2n_i}(F)$.*

Proof. Every $O_F[\widetilde{x}]$-lattice is a direct sum of an R_0-lattice and an R_1-lattice. It is easily seen that this defines a bijection $\mathcal{L}(\widetilde{x}) \xrightarrow{\sim} \mathcal{L}(x_0/\pi) \times \mathcal{L}(x_1/\pi)$. Furthermore, the definition of $\Omega(\gamma, \Lambda_+ \oplus \Lambda_-, s)$ is multiplicative in such direct sums. The desired identity then follows from (3.5). □

Identity (3.7) already covers the factor $\mathrm{Orb}(\gamma_0, f'_{1/2,n_0}, s)$, so we now turn to the factor $\mathrm{Orb}(\gamma_1, f'_{1/2,n_1}, s)$. For elements $x \in GL_n(F)$ that are regular semi-simple in the usual sense, we consider the conjugation orbital integral

$$\mathrm{Orb}(x, 1_{GL_n(O_F)}) = \int_{GL_n(F)/L_x^\times} 1_{GL_n(O_F)}(y^{-1}xy)dy, \tag{3.10}$$

where $L_x = F[x]$ as before and where dy is the Haar measure on $GL_n(F)/L_x^\times$ that is normalized by

$$\mathrm{Vol}(GL_n(O_F)) = \mathrm{Vol}(O_{L_x}^\times) = 1. \tag{3.11}$$

Lemma 3.7. *Assume that $x \in GL_n(F)$ has the property that $\widetilde{x} = x/\pi$ is integral over O_F with $\det(\widetilde{x}) \in O_F^\times$. Then there is an identity of orbital integrals*

$$\mathrm{Orb}(\gamma(x), f'_{1/2}, s) = \mathrm{Orb}(\widetilde{x}, 1_{GL_n(O_F)}). \tag{3.12}$$

Proof. We again use (3.5) to express the left hand side. A general identity (see [11, Lemma 3.23]) states that the transfer factor of a lattice $\Lambda = \Lambda_+ \oplus \Lambda_- \in \mathcal{L}_0(\gamma)$ is given by

$$\Omega(\gamma, \Lambda_+ \oplus \Lambda_-, s) = (-1)^{[(\gamma\Lambda)_- : z\Lambda_+] + [(\gamma\Lambda)_- : \Lambda_-]} q^{([(\gamma\Lambda)_+ : z\Lambda_-] + [(\gamma\Lambda)_- : z\Lambda_+])s}, \tag{3.13}$$

where $z = z_\gamma$ and where $[\Lambda_1 : \Lambda_2]$ denotes the length of Λ_1/Λ_2. In the situation of (3.5), we apply this formula to the element $\gamma = \gamma(x)$ and a lattice $\Lambda = \Lambda_+ \oplus \Lambda_-$ with $(\Lambda_+, \Lambda_-) \in \mathcal{L}(\widetilde{x})$. We have that $z = \left(\begin{smallmatrix} 0 & x \\ 1 & 0 \end{smallmatrix}\right)$ and that $\gamma\Lambda = \Lambda$ because x is topologically nilpotent under the assumption $\det(\widetilde{x}) \in O_F^\times$. The assumption moreover implies that any $\mathcal{L}(\widetilde{x})$ is the set of lattice pairs (Λ_+, Λ_-) that satisfy

$$\widetilde{x}\Lambda_+ = \Lambda_- = \Lambda_+.$$

We see that $z\Lambda_+ = \Lambda_-$ and $z\Lambda_- = \pi\Lambda_+$ for all such (Λ_+, Λ_-). Substituting this in (3.13) gives $\Omega(\gamma(x), \Lambda_+ \oplus \Lambda_-, s) = q^{ns}$ for all $(\Lambda_+, \Lambda_-) \in \mathcal{L}(\widetilde{x})$ and hence

$$\mathrm{Orb}(\gamma, f'_{1/2}, s) = q^{ns}q^{-ns} \sum_{\Lambda=(\Lambda_+,\Lambda_-)\in\mathcal{L}(\widetilde{x})} [O_{L_x}^\times : R_\Lambda^\times]. \tag{3.14}$$

This is precisely the combinatorial description of $\mathrm{Orb}(\widetilde{x}, 1_{GL_n(O_F)})$ and the proof is complete. □

3.2. **Orbital integrals on G_λ.** Recall that E/F is an unramified quadratic extension and that B_λ denotes a quaternion algebra over F of Hasse invariant $\lambda \in \{0, 1/2\}$ with an embedding $E \to B_\lambda$. Recall that $\sigma \in \mathrm{Gal}(E/F)$ denotes the non-trivial element. Our aim in this section is to relate the orbital integrals $\mathrm{Orb}(-, f_0)$ and $\mathrm{Orb}(-, f_{1/2})$, where $f_\lambda \in C_c^\infty(G_\lambda)$ is the characteristic function of $GL_n(O_{B_\lambda})$. Put $\varepsilon = 2\lambda \in \{0, 1\}$ and fix an element $\varpi \in O_{B_\lambda}$ that satisfies

$$\varpi^2 = \pi^\varepsilon \quad \text{and} \quad \varpi a = \sigma(a)\varpi \;\text{ for } a \in E. \tag{3.15}$$

For $x \in GL_n(E)$, we define $g(x) = 1 + x\varpi \in M_n(B_\lambda)$. This element lies in $G_{\lambda,\mathrm{rs}}$ if and only if the characteristic polynomial of $z_{g(x)}^2 = x\sigma(x)\pi^\varepsilon \in GL_n(E)$ is separable and does not vanish at 0 or 1. In this case, $\mathrm{Inv}(g(x); T) = \mathrm{char}(x\sigma(x)\pi^\varepsilon; T)$. Since $\mathrm{Orb}(g, f_\lambda)$ only depends on the double coset HgH, we may restrict attention to group elements of the form $g(x)$.

Definition 3.8. Denote by $\sigma : E^n \to E^n$ the coordinate-wise Galois conjugation. For $x \in GL_n(E)$, define $\mathcal{L}^\sigma(x)$ as the set of O_E-lattices $\Lambda \subset E^n$ that satisfy

$$x\sigma(\Lambda) \subseteq \Lambda. \tag{3.16}$$

Moreover, define $L_x \subset M_n(E)$ as the subalgebra $F[x\sigma(x)\pi^\varepsilon]$. Then $L_x^\times$ acts by multiplication on $\mathcal{L}^\sigma(x)$.

Lemma 3.9. *Let $\lambda \in \{0, 1/2\}$ be any and let $B = B_\lambda$. Let $g = 1 + x\varpi \in G_{\lambda,\mathrm{rs}}$ be regular semi-simple. Suppose further that $x\sigma(x)\pi^\varepsilon$ is topologically nilpotent. Then there is the identity*

$$\mathrm{Orb}(g(x), 1_{GL_n(O_B)}) = \sum_{\Lambda\in\mathcal{L}^\sigma(x)/L_x^\times} [O_{L_x}^\times : R_\Lambda^\times]. \tag{3.17}$$

Here $R_\Lambda \subseteq L_x$ denotes the order that stabilizes Λ.

Proof. The assumption that $x\sigma(x)\pi^\varepsilon = (x\varpi)^2$ is topologically nilpotent implies that the determinant $\det(g(x))$ lies in $O_F^\times$. (Here and in the following, the determinant is meant in the sense of the reduced norm $GL_n(B) \to F^\times$.) Hence the condition $h_1^{-1}gh_2 \in GL_n(O_B)$ holds if and only if $h_1^{-1}gh_2 \in M_n(O_B)$. Since $h_1^{-1}gh_2 = h_1^{-1}h_2(1 + h_2^{-1}x\varpi h_2)$ and since $h_2^{-1}x\varpi h_2$ is topologically nilpotent by assumption, this is equivalent to $h_1^{-1}h_2 \in GL_n(O_E)$ and $h_2^{-1}x\sigma(h_2) \in M_n(O_E)$. Given a pair (h_1, h_2) with $h_1^{-1}h_2 \in GL_n(O_E)$, consider the lattice $\Lambda = h_1O_E^n = h_2O_E^n$. Then $h_2^{-1}x\sigma(h_2)$ lies in $M_n(O_E)$ if and only if $x\sigma(\Lambda) \subseteq \Lambda$. Rewriting the definition of $\mathrm{Orb}(g, 1_{GL_n(O_B)})$ in this way gives (3.17). □

Lemma 3.10. *Let $g \in G_{1/2,\mathrm{rs}}$ and $B = B_{1/2}$. The orbital integral $\mathrm{Orb}(g, 1_{GL_n(O_B)})$ vanishes if z_g^2/π is not integral over O_F.*

Proof. Since $\mathrm{Orb}(g, 1_{GL_n(O_B)})$ only depends on the orbit of g and since every regular semi-simple orbit contains a representative of the form $g(x)$ we may assume without loss of generality that $g = g(x)$ for some $x \in GL_n(E)$. In this case $z_g^2/\pi = x\sigma(x)$. Moreover, $GL_n(O_B) = GL_n(O_E) + M_n(O_E)\varpi$ because B is the division algebra. So $h_1^{-1}h_2 + h_1^{-1}x\sigma(h_2)\varpi \in GL_n(O_B)$ with $h_i \in GL_n(E)$ can only hold if $h_1^{-1}h_2 \in GL_n(O_E)$ and $h_2^{-1}x\sigma(h_2) \in M_n(O_E)$. The second condition implies that $x\sigma(x)$ is integral over O_F because it is conjugate to $h_2^{-1}x\sigma(h_2)\sigma(h_2^{-1}x\sigma(h_2))$. This was to be shown. □

Proof of Theorem 2.11 (2). Let $g \in G_{1/2,\mathrm{rs}}$ be such that $\pi^{-n}\mathrm{Inv}(g;\pi T) \in T^n + \pi O_F[T]$. Denote by $\varpi_\lambda \in B_\lambda$ the element fixed in (3.15). Without loss of generality we may assume that $g = 1 + x\varpi_{1/2}$ for some $x \in GL_n(E)$. The assumption on g is then equivalent to $x\sigma(x)$ being topologically nilpotent. The element $\widetilde{g} = 1 + x\varpi_0$ hence lies in $G_{0,\mathrm{rs}}$ and satisfies $\mathrm{Inv}(\widetilde{g}; T) = \pi^{-n}\mathrm{Inv}(g;\pi T)$. Lemma 3.9 applies to both g and $\widetilde{g}$, and yields

$$\mathrm{Orb}(g, 1_{GL_n(O_B)}) = \mathrm{Orb}(\widetilde{g}, 1_{GL_{2n}(O_F)}) \tag{3.18}$$

as desired. □

We next prove a factorization of the orbital integral that is analogous to that in Lemma 3.6.

Lemma 3.11. *Let $B = B_{1/2}$ and assume that $g(x) \in G_{1/2,\mathrm{rs}}$ is such that $x\sigma(x)$ is integral over O_F. Let $O_F[x\sigma(x)] = R_0 \times R_1$ be as in Lemma 3.5. Let $E^n = V_0 \times V_1$ be the induced decomposition of E^n which is preserved by x. Write (x_0, x_1) for its components and choose isomorphisms $V_i \cong E^{n_i}$. Put $g_i = 1 + x_i\varpi \in GL_{n_i}(B)$. Then there is an identity of orbital integrals*

$$\mathrm{Orb}(g, 1_{GL_n(O_B)}) = \mathrm{Orb}(g_0, 1_{GL_{n_0}(O_B)})\,\mathrm{Orb}(g_1, 1_{GL_{n_1}(O_B)}). \tag{3.19}$$

Proof. Every $O_F[x\sigma(x)]$-lattice is a direct sum of an R_0-lattice and an R_1-lattice. This defines a bijection $\mathcal{L}^\sigma(x) \xrightarrow{\sim} \mathcal{L}^\sigma(x_0) \times \mathcal{L}^\sigma(x_1)$. The desired identity now follows from Lemma 3.9 which can be applied to all three orbital integrals because $x\sigma(x)$, $x_0\sigma(x_0)$ and $x_1\sigma(x_1)$ are all integral by assumption. □

Identity (3.18) already covers the factor $\mathrm{Orb}(g_0, 1_{GL_{n_0}(O_B)})$, so we now turn to the factor $\mathrm{Orb}(g_1, 1_{GL_{n_1}(O_B)})$. Let $x \in GL_n(O_E)$ be an element that is regular semi-simple with respect to the σ-twisted conjugation action $(y, x) \mapsto y^{-1}x\sigma(y)$. Its stabilizer then equals $L_x^\times$ where $L_x = F[x\sigma(x)]$. We normalize the Haar measure dy on $GL_n(E)/L_x^\times$ by

$$\mathrm{Vol}(GL_n(O_E)) = \mathrm{Vol}(O_{L_x}^\times) = 1 \tag{3.20}$$

and define a twisted orbital integral by

$$\mathrm{Orb}^\sigma(x, 1_{GL_n(O_E)}) := \int_{GL_n(E)/L_x^\times} 1_{GL_n(O_E)}(y^{-1}x\sigma(y))dy. \tag{3.21}$$

Lemma 3.12. *Let $B = B_{1/2}$ and let $x \in GL_n(E)$ be such that $x\sigma(x)$ is integral over O_F with $\det(x\sigma(x)) \in O_F^\times$. Then there is the identity*

$$\mathrm{Orb}(g(x), 1_{GL_n(O_B)}) = \mathrm{Orb}^\sigma(x, 1_{GL_n(O_E)}). \tag{3.22}$$

Proof. The assumption that $x\sigma(x)$ is integrally invertible implies that $\mathcal{L}^\sigma(x)$ is the set of O_E-lattices $\Lambda \subseteq E^n$ such that $x\sigma(\Lambda) = \Lambda$. In this case, the lattice counting expression in Lemma 3.9 equals $\mathrm{Orb}^\sigma(x, 1_{GL_n(O_E)})$. □

3.3. **Vanishing orders.** The FL and the AFL both include vanishing statements. In order to prove these, we now recall some results from [10] for the case $\lambda = 0$.

By regular semi-simple invariant (of degree n), we mean a degree n polynomial $\delta \in F[T]$ that is monic, separable and satisfies $\delta(1)\delta(0) \neq 0$. These are precisely the polynomials that arise as invariant polynomials of elements $\gamma \in G'_{\mathrm{rs}}$. Given a regular semi-simple $\delta \in F[T]$, we define

$$L_\delta := F[z^2]/(\delta(z^2)) \quad \text{and} \quad B_\delta := (E \otimes_F L_\delta)[z]/(z(a \otimes b) = (\sigma(a) \otimes b)z)_{a\in E,\ b\in L_\delta}. \tag{3.23}$$

Then L_δ is an étale F-algebra of degree $n = \deg(\delta)$ by separability, and B_δ/L_δ is a quaternion algebra. Moreover, B_δ contains E by construction. Let $L_\delta = \prod_{i\in I} L_i$ and $B_\delta = \prod_{i\in I} B_i$ be the factorizations of L_δ and B_δ according to the idempotents in L_δ. The algebraic vanishing order of δ from [10, Definition 2.15] is defined as the integer

$$\mathrm{ord}_0(\delta) := \#\{i \in I \mid B_i \text{ is a division algebra}\}.$$

One of the main results of [10] is a factorization formula for $\mathrm{Orb}(\gamma, 1_{GL_{2n}(O_F)}, s)$. It implies, see [10, Corollary 2.17], that for every regular semi-simple $\gamma \in G'_{\mathrm{rs}}$,

$$\mathrm{ord}_{s=0} \mathrm{Orb}(\gamma, 1_{GL_{2n}}(O_F), s) \geq \mathrm{ord}_0(\mathrm{Inv}(\gamma)). \tag{3.24}$$

Our aim is to formulate an analogous result in the case of invariant $\lambda = 1/2$.

Definition 3.13. Let $\delta \in F[T]$ be a regular semi-simple invariant of degree n and let L_δ, B_δ be defined as in (3.23). Let $B = B_{1/2}$ be the quaternion division algebra over F. Define

$$\mathrm{ord}_{1/2}(\delta) := \#\{i \in I \mid B_i \not\cong B \otimes_F L_i\}.$$

In other words, $\mathrm{ord}_{1/2}(\delta)$ is the number of indices such that B_i is split and $[L_i : F]$ odd, or such that B_i is division and $[L_i : F]$ even. It is checked with a simple case distinction that

$$\mathrm{ord}_{1/2}(\delta(T)) = \mathrm{ord}_0(\pi^{-n}\delta(\pi T)). \tag{3.25}$$

Corollary 3.14. *Let $\gamma \in G'_{\mathrm{rs}}$ be a regular semi-simple element. Then*

$$\mathrm{ord}_{s=0} \mathrm{Orb}(\gamma, f'_{1/2}, s) \geq \mathrm{ord}_{1/2}(\mathrm{Inv}(\gamma)). \tag{3.26}$$

Proof. If z_γ^2/π is not integral over O_F, then $\mathrm{Orb}(\gamma, f'_{1/2}, s) = 0$ by Corollary 3.4 and there is nothing to prove. So assume that z_γ^2/π is integral and consider the orbital integral factorization from Corollary 3.6,

$$\mathrm{Orb}(\gamma, f'_{1/2}, s) = \mathrm{Orb}(\gamma_0, f'_{1/2}, s)\,\mathrm{Orb}(\gamma_1, f'_{1/2}, s).$$

Since $\mathrm{Inv}(\gamma) = \mathrm{Inv}(\gamma_0)\mathrm{Inv}(\gamma_1)$, we have

$$\mathrm{ord}_{1/2}(\mathrm{Inv}(\gamma)) = \mathrm{ord}_{1/2}(\mathrm{Inv}(\gamma_0)) + \mathrm{ord}_{1/2}(\mathrm{Inv}(\gamma_1)).$$

It hence suffices to show (3.26) for γ_0 and γ_1 separately, meaning we may either assume z_γ^2/π to be topologically nilpotent or integrally invertible.

If z_γ^2/π is topologically nilpotent, then we can apply Theorem 2.11 (1): For any element $\widetilde{\gamma} \in G'_{\mathrm{rs}}$ with $\mathrm{Inv}(\widetilde{\gamma}; T) = \pi^{-n}\mathrm{Inv}(\gamma; \pi T)$, we have $\mathrm{Orb}(\gamma, f'_{1/2}, s) =$

$\mathrm{Orb}(\widetilde{\gamma}, 1_{GL_{2n}(O_F)}, s)$. Using (3.25), the desired vanishing statement (3.26) follows directly from (3.24).

If z_γ^2/π is integral over O_F and $\det(z_\gamma^2/\pi) \in O_F^\times$, then

$$\mathrm{ord}_{1/2}(\mathrm{Inv}(\gamma; T)) = \mathrm{ord}_0(\pi^{-n}\mathrm{Inv}(\gamma; \pi T)) = 0$$

because z_γ^2/π then lies in $O_{L_\gamma}^\times$ and is hence a norm from $E \otimes_F L_\gamma$. (This uses that E/F is unramified.) The inequality $\mathrm{ord}_{s=0} \mathrm{Orb}(\gamma, f'_{1/2}, s) \geq 0$ holds trivially and there is nothing to prove in this case. This completes the argument. (We remark that if $z_\gamma^2/\pi \in O_{L_\gamma}^\times$, then Corollary 3.7 states that $\mathrm{Orb}(\gamma, f'_{1/2}, s)$ is a constant independent of s.) □

3.4. **Proof of Theorem 2.7.** We can finally deduce the fundamental lemma for $\lambda = 1/2$.

Proof of Theorem 2.7. Let $\gamma \in G'_{\mathrm{rs}}$. We need to show the identity

$$\mathrm{Orb}(\gamma, f'_{1/2}) = \begin{cases} \mathrm{Orb}(g, f_{1/2}) & \text{if there exists a matching } g \in G_{1/2} \\ 0 & \text{otherwise.} \end{cases} \tag{3.27}$$

We first note that if $\pi^{-n}\mathrm{Inv}(\gamma; \pi T) \notin O_F[T]$, then both sides of the equation vanish. This follows from Corollary 3.4 and from Lemma 3.10, respectively. Hence we assume from now on that $\pi^{-n}\mathrm{Inv}(\gamma; \pi T) \in O_F[T]$ or, equivalently, that z_γ^2/π is integral. Consider L_δ and B_δ as in Equation (3.23) for $\delta = \mathrm{Inv}(\gamma; T)$. Let $L_\delta = \prod_i L_i$ and $B_\delta = \prod_i L_i$ be their factorizations according to idempotents. By [11, Corollary 2.8] there exists an element $g \in G_{1/2,\mathrm{rs}}$ that matches γ if and only if $\mathrm{ord}_{1/2}(\delta) = 0$, see Definition 3.13. In particular, if there is no matching element $g \in G_{1/2,\mathrm{rs}}$, then $\mathrm{Orb}(\gamma, f'_{1/2}) = 0$ by Corollary 3.14. This proves the vanishing part of (3.27).

We henceforth consider the case that there is an element $g \in G_{1/2,\mathrm{rs}}$ that matches γ. Assuming that $\gamma = 1 + z_\gamma$ and that $g = 1 + z_g$, let $\gamma = (\gamma_0, \gamma_1)$ and $g = (g_0, g_1)$ be the components of γ and g such that $z_{\gamma_0}^2/\pi$ and $z_{g_0}^2/\pi$ are topologically nilpotent, and $z_{\gamma_1}^2/\pi$ and $z_{g_1}^2/\pi$ integrally invertible (Corollaries 3.7 and 3.12). Then γ_0 matches g_0 and γ_1 matches g_1 as can be seen by using the isomorphism $O_F[z_\gamma^2] \cong O_F[z_g^2]$ that sends z_γ^2 to z_g^2. By Lemmas 3.6 and 3.11, the two sides of (3.27) factor and the desired equality becomes

$$\mathrm{Orb}(\gamma_0, f'_{1/2,n_0}) \, \mathrm{Orb}(\gamma_1, f'_{1/2,n_1}) \overset{?}{=} \mathrm{Orb}(g_0, 1_{GL_{n_0}(O_B)}) \, \mathrm{Orb}(g_1, 1_{GL_{n_1}(O_B)}).$$

Here, $B = B_{1/2}$. We can prove this identity factor-by-factor, meaning we may assume that z_γ^2/π is topologically nilpotent or that $z_\gamma^2/\pi \in O_{L_\gamma}^\times$.

Assume first that z_γ^2/π is topologically nilpotent. Then we may apply Theorem 2.11 (1) and (2): Let $\widetilde{\gamma} \in G'_{\mathrm{rs}}$ and $\widetilde{g} \in G_{0,\mathrm{rs}}$ be such that $\mathrm{Inv}(\widetilde{\gamma}; T) = \pi^{-n}\delta(\pi T) = \mathrm{Inv}(\widetilde{g}; T)$. We obtain

$$\mathrm{Orb}(\gamma, f'_{1/2}) = \mathrm{Orb}(\widetilde{\gamma}, 1_{GL_{2n}(O_F)}) = \mathrm{Orb}(\widetilde{g}, 1_{GL_{2n}(O_F)}) = \mathrm{Orb}(g, 1_{GL_n(O_B)})$$

where the middle equality is the Guo–Jacquet FL, i.e. the case $\lambda = 0$ of Theorem 2.7.

Assume now that $z_\gamma^2/\pi \in O_{L_\gamma}^\times$. Without loss of generality, we may assume that $\gamma = \gamma(x)$ for some $x \in GL_n(F)$ and $g = 1 + x'\varpi$ for some $x' \in GL_n(E)$. The fact that γ and g match translates to the identity

$$\mathrm{char}_F(x/\pi; T) = \mathrm{char}_E(x'\sigma(x'); T).$$

In other words, x/π and x' match in the sense of the quadratic base change FL, see [5]. We obtain from Lemmas 3.7 and 3.12, as well as the base change FL that

$$\mathrm{Orb}(\gamma, f'_{1/2}) = \mathrm{Orb}(x/\pi, 1_{GL_n(O_F)}) = \mathrm{Orb}^\sigma(x', 1_{GL_n(O_E)}) = \mathrm{Orb}(g, 1_{GL_n(O_B)}).$$

The proof of Theorem 2.7 is now complete. □

4. Arithmetic Fundamental Lemma

It is left to prove Theorem 2.11 (3) and Theorem 2.9 which is the aim of this final section. Its structure is analogous to that of §3: We first relate the intersection problems for $\lambda = 0$ and $\lambda = 1/2$ by analyzing their moduli descriptions (§4.1). Then we prove a factorization result for $\mathrm{Int}(g)$ in the case of $\lambda = 1/2$ (§4.2). Combining both techniques we obtain the proof of Theorem 2.9 (§4.3). Our notation in this section is the same as in §2.4.

4.1. **Description of $\mathcal{I}(g)$.** Let $\lambda \in \{0, 1/2\}$ and let $g \in G_{\lambda,\mathbb{Y},\mathrm{rs}}$ be a regular semi-simple element. Our first aim is to give a more explicit description of $\mathcal{I}(g)$. Let $B = B_\lambda$ be the quaternion algebra of invariant λ. Let $\varpi \in B^\times$ be the element from (3.15). Recall that this means that ϖ is chosen such that $\varpi a = \sigma(a)\varpi$ for all $a \in E$ and such that $\varpi^2 = \pi^\varepsilon$ where $\varepsilon = 2\lambda$. Then $O_B = O_E \oplus \varpi O_E$. So for every strict O_E-module $\mathbb{Y}$, we obtain the coordinates $\mathbb{X} = \mathbb{Y} \oplus \varpi\mathbb{Y}$. With respect to this decomposition, the O_B-action $\kappa : O_B \to \mathrm{End}(\mathbb{X})$ is given by

$$a + b\varpi \longmapsto \begin{pmatrix} a & b\pi^\varepsilon \\ \sigma(b) & \sigma(a) \end{pmatrix}, \quad a, b \in E.$$

The O_B-linear endomorphisms of $\mathbb{X}$ then have the presentation

$$D_{\lambda,\mathbb{Y}} = \left\{ \begin{pmatrix} x & \pi^\varepsilon y \\ y & x \end{pmatrix} \;\middle|\; x \in C_\mathbb{Y},\ y \in \mathrm{End}^0_F(\mathbb{Y}) \text{ s.th. } ya = \sigma(a)y \text{ for } a \in E \right\}. \tag{4.1}$$

Let $g = \left(\begin{smallmatrix} x & \pi^\varepsilon y \\ y & x \end{smallmatrix}\right) \in G_{\lambda,\mathbb{Y}} = D^\times_{\lambda,\mathbb{Y}}$ be an element such that both x and y are invertible. Then z_g takes the form

$$z_g = \begin{pmatrix} & \pi^\varepsilon x^{-1}y \\ x^{-1}y & \end{pmatrix}. \tag{4.2}$$

It follows from this that the invariant polynomial of g is

$$\mathrm{Inv}(g; T) = \mathrm{charred}_{C_\mathbb{Y}/E}\left(\pi^\varepsilon (x^{-1}y)^2; T\right).$$

Definition 4.1. For an element $z \in \mathrm{End}^0_F(\mathbb{X})$, we denote by $\mathcal{Z}(z) \subseteq \mathcal{M}_\lambda$ the closed formal subscheme with functor of points description

$$\mathcal{Z}(z)(S) = \{(X, \kappa, \rho) \in \mathcal{M}_\lambda(S) \mid \rho z \rho^{-1} \in \mathrm{End}(X)\}.$$

We analogously define $\mathcal{Z}(w) \subseteq \mathcal{N}$ for an endomorphism $w \in \mathrm{End}^0_F(\mathbb{Y})$.

Lemma 4.2. *Let $g \in G_{\lambda,\mathbb{Y},\mathrm{rs}}$ be a regular semi-simple element such that z_g is topologically nilpotent. Then, as closed formal subschemes of $\mathcal{M}_\lambda$,*

$$\mathcal{I}(g) = \mathcal{N} \cap \mathcal{Z}(z_g).$$

Furthermore, writing $z_g = \left(\begin{smallmatrix} & \pi^\varepsilon w \\ w & \end{smallmatrix}\right)$ as in (4.2), there is the following identity of closed formal subschemes of $\mathcal{N}$:

$$\mathcal{N} \cap \mathcal{Z}(z_g) = \mathcal{Z}(w).$$

Proof. The first identity is a special case of [11, Proposition 4.23 (2)] which we may apply because z_g is topologically nilpotent by assumption. The second identity follows directly from the definitions of $\mathcal{Z}(z_g)$ and $\mathcal{Z}(w)$. □

Corollary 4.3. *Part (3) of Theorem 2.11 holds.*

Proof. Let $g \in G_{1/2,\mathbb{Y},\mathrm{rs}}$ be an element whose invariant has the property that $\pi^{-n}\mathrm{Inv}(g;\pi T)$ lies in $T^n + \pi O_F[T]$. We need to construct an element $\widetilde{g} \in G_{0,\mathbb{Y},\mathrm{rs}}$ such that both $\mathrm{Inv}(\widetilde{g},T) = \pi^{-n}\mathrm{Inv}(g,\pi T)$ and $\mathrm{Int}(g) = \mathrm{Int}(\widetilde{g})$.

Given g, we define $\widetilde{g} \in D_{0,\mathbb{Y}}$ by the following relation:

$$g = \begin{pmatrix} x & \pi y \\ y & x \end{pmatrix} \quad \text{and} \quad \widetilde{g} := \begin{pmatrix} x & y \\ y & x \end{pmatrix}.$$

Since g is regular semi-simple, $g_+ = \left(\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right)$ and $g_- = \left(\begin{smallmatrix} & \pi y \\ y & \end{smallmatrix}\right)$ are both invertible. Then $\widetilde{g}_+ = g_+$ and $\widetilde{g}_- = \left(\begin{smallmatrix} & y \\ y & \end{smallmatrix}\right)$ are invertible as well. The assumption on $\mathrm{Inv}(g;T)$ implies that $\mathrm{Inv}(g;\pi) \neq 0$, so $\widetilde{g}$ lies in $G_{0,\mathbb{Y}}$ and has invariant polynomial

$$\mathrm{Inv}(\widetilde{g};T) = \pi^{-n}\mathrm{Inv}(g;\pi T).$$

This polynomial is separable by assumption on g, so $\widetilde{g}$ is regular semi-simple. Moreover, $z_{\widetilde{g}}^2$ is topologically nilpotent since $\mathrm{Inv}(\widetilde{g};T) \equiv T^n$ modulo $\pi O_F[T]$. We may now apply Lemma 4.2 twice to see that

$$\begin{aligned} \mathcal{I}(\widetilde{g}) &= \mathcal{N} \cap_{\mathcal{M}_0} \mathcal{Z}\left(\left(\begin{smallmatrix} & y \\ y & \end{smallmatrix}\right)\right) \\ &= \mathcal{Z}(y) \\ &= \mathcal{N} \cap_{\mathcal{M}_{1/2}} \mathcal{Z}\left(\left(\begin{smallmatrix} & \pi y \\ y & \end{smallmatrix}\right)\right) \\ &= \mathcal{I}(g). \end{aligned} \tag{4.3}$$

Note that $z_g^2 = \pi z_{\widetilde{g}}^2$, so the two F-algebras $L_g = F[z_g^2]$ and $L_{\widetilde{g}} = F[z_{\widetilde{g}}^2]$ agree as subalgebras of $C_{\mathbb{Y}}$. The isomorphism in (4.3) is then equivariant with respect to the action of $L_g^\times = L_{\widetilde{g}}^\times$. Choosing the same subgroup $\Gamma \subset L_g^\times = L_{\widetilde{g}}^\times$ in the definition of the intersection number (Definition 2.5), we obtain $\Gamma\backslash\mathcal{I}(g) = \Gamma\backslash\mathcal{I}(\widetilde{g})$ and hence the identity

$$\mathrm{Int}(g) = \mathrm{Int}(\widetilde{g})$$

as was to be shown. □

4.2. **Factorization of** $\mathrm{Int}(g)$**.** Recall that $\mathbb{Y} = \mathbb{Y}^\circ \times \mathbb{Y}^{\mathrm{et}}$ and $\mathbb{X} = \mathbb{X}^\circ \times \mathbb{X}^{\mathrm{et}}$ denote the connected-étale decompositions of $\mathbb{Y}$ and $\mathbb{X}$. We define RZ spaces $\mathcal{N}^\circ$, $\mathcal{N}^{\mathrm{et}}$, $\mathcal{M}_\lambda^\circ$ and $\mathcal{M}_\lambda^{\mathrm{et}}$ in complete analogy to our definitions of $\mathcal{N}$ and $\mathcal{M}$ in (2.9), (2.10), but using the objects $\mathbb{Y}^\circ$, $\mathbb{Y}^{\mathrm{et}}$, $\mathbb{X}^\circ$ and $\mathbb{X}^{\mathrm{et}}$ instead of $\mathbb{Y}$ and $\mathbb{X}$. For $g = (g^\circ, g^{\mathrm{et}}) \in G_{\lambda,\mathbb{Y},\mathrm{rs}}$, we can then also define

$$\mathcal{I}(g^\circ) := \mathcal{N}^\circ \cap g^\circ(\mathcal{N}^\circ) \quad \text{and} \quad \mathcal{I}(g^{\mathrm{et}}) := \mathcal{N}^{\mathrm{et}} \cap g^{\mathrm{et}}(\mathcal{N}^{\mathrm{et}})$$

where the intersections happen on $\mathcal{M}_\lambda^\circ$ and $\mathcal{M}_\lambda^{\mathrm{et}}$.

Lemma 4.4. *The set* $\mathcal{I}(g)(\mathbb{F})$ *has the product structure*

$$\mathcal{I}(g)(\mathbb{F}) = \mathcal{I}(g^\circ)(\mathbb{F}) \times \mathcal{I}(g^{\mathrm{et}})(\mathbb{F}). \tag{4.4}$$

Proof. Every strict O_F-module over $\mathbb{F}$ is, in a unique way, the product of its identity connected component and its maximal étale quotient. This decomposition is functorial in all respects, giving (4.4) from definitions. □

Lemma 4.5. *Let $g \in G_{1/2,\mathbb{Y},\mathrm{rs}}$ be regular semi-simple and such that $\mathcal{I}(g) \neq \emptyset$. Then z_g^2/π is integral over O_F.*

Proof. By the product structure on $\mathcal{I}(g)(\mathbb{F})$ from Lemma 4.4, $\mathcal{I}(g)$ being non-empty implies that both $\mathcal{I}(g^\circ)$ and $\mathcal{I}(g^{\mathrm{et}})$ are non-empty. Since $\mathbb{Y}^\circ$ has no étale factor by definition, [11, Proposition 4.23 (1)] applies and states that z_g° is topologically nilpotent. Then $\mathcal{I}(g^\circ) = \mathcal{N}^\circ \cap \mathcal{Z}(z_g^\circ)$ by Lemma 4.2. Writing $g = \left(\begin{smallmatrix} x & \pi y \\ y & x \end{smallmatrix}\right)$ as in (4.1) (here we used that we consider the case $\lambda = 1/2$), we obtain from

$$z_g^\circ = \begin{pmatrix} & \pi w^\circ \\ w^\circ & \end{pmatrix}, \qquad w = x^{-1}y, \tag{4.5}$$

that $\mathcal{I}(g^\circ) = \mathcal{Z}(w^\circ)$ where the right hand side is a closed formal subscheme of $\mathcal{N}$. Thus $\mathcal{I}(g^\circ) \neq \emptyset$ implies that w° is integral. Since $(z_g^\circ)^2/\pi = (w^\circ)^2$, this shows that $(z_g^\circ)^2/\pi$ is integral as claimed.

We are left to prove that $(z_g^{\mathrm{et}})^2/\pi$ is integral. Passing from $\mathbb{Y}^{\mathrm{et}}$ and $\mathbb{X}^{\mathrm{et}}$ to Tate modules, we may identify $\mathcal{I}(g^{\mathrm{et}})(\mathbb{F})$ with the set $\mathcal{L}(\varpi^{-1}z_g^{\mathrm{et}})$ from Definition 3.8. By Lemma 3.10, this set being non-empty implies $(z_g^{\mathrm{et}})^2/\pi$ integral. The proof is now complete. □

Lemma 4.6. *Let $g^\circ \in G^\circ_{1/2,\mathbb{Y},\mathrm{rs}}$ be a regular semi-simple element such that $(z_g^\circ)^2/\pi$ is integral. Then $(z_g^\circ)^2/\pi$ is even topologically nilpotent.*

Proof. We consider the two cases n° even or odd separately. Assume first that n° is even. Then $D^\circ_{1/2,\mathbb{Y}}$ is a CDA over F of degree $2n^\circ$ and Hasse invariant $(n^\circ+1)/2n^\circ$. Thus L_g° is a field extension of degree n° of F. As $E \otimes_F L_g^\circ$ embeds into $D^\circ_{1/2,\mathbb{Y}}$, this tensor product has to be a field, so the inertia degree of L_g over F is odd. This implies that its ramification index is even. Moreover, B_g° embeds into $D^\circ_{1/2,\mathbb{Y}}$ and is hence a division algebra. It follows that $(z_g^\circ)^2 \in (L_g^\circ)^\times$ is not a norm from $E \otimes_F L_g^\circ$ and hence has odd valuation. Since the ramification index is even, $(z_g^\circ)^2/\pi$ has odd valuation as well. So $(z_g^\circ)^2/\pi \in O^\times_{L_g^\circ}$ is impossible as was to be shown.

Now we consider the case that n° is odd. Then $D_{1/2,\mathbb{Y}} \cong M_2(Q)$ where Q is a CDA over F of degree n° and Hasse invariant $(n^\circ+1)/2n^\circ$. The étale F-algebra L_g°, which has degree n°, again has to be a field because there cannot exist an embedding $L_g \to M_2(Q)$ otherwise. (This argument used that n° is odd.) Then one obtains that $B_g^\circ \cong M_2(L_g^\circ)$ because it equals the centralizer of L_g° in $M_2(Q)$. This means that $(z_g^\circ)^2 \in L_g^\circ$ is a norm from $E \otimes_F L_g^\circ$ which is equivalent to $(z_g^\circ)^2$ having even valuation. The degree $[L_g^\circ : F]$ is odd, so it follows that $(z_g^\circ)^2/\pi$ has odd valuation and thus cannot lie in $O^\times_{L_g^\circ}$ as was to be shown. □

Construction 4.7. Assume that $g = (g^\circ, g^{\mathrm{et}}) \in G_{1/2,\mathbb{Y},\mathrm{rs}}$ satisfies that z_g^2/π is integral. By Lemma 4.6, $(z_g^\circ)^2/\pi$ is even topologically nilpotent. Let $R = O_F[z_g^2/\pi] \subset C_{\mathbb{Y}}$ be the O_F-algebra generated by z_g^2/π. Using Hensel's Lemma as in Lemma 3.5, there is a unique factorization $R = R_0 \times R_1$ such that, when writing $z_g^2/\pi = (\zeta_0, \zeta_1)$, the component ζ_0 is topologically nilpotent and $\zeta_1 \in R_1^\times$. Since $(z_g^\circ)^2/\pi$ is topologically nilpotent, the projection $R \to R_1$ factors through the projection map

$$F[z_g^2/\pi] = F[(z_g^\circ)^2/\pi] \times F[(z_g^{\mathrm{et}})^2/\pi] \longrightarrow F[(z_g^{\mathrm{et}})^2/\pi].$$

Let $\mathbb{Y} = \mathbb{Y}_0 \times \mathbb{Y}_1$ be the decomposition of $\mathbb{Y}$ up to isogeny with respect to the idempotents defining $R = R_0 \times R_1$. By what was just said, $\mathbb{Y}_1$ is an étale π-divisible O_E-module. The centralizer in $G_{1/2,\mathbb{Y},\mathrm{rs}}$ of the idempotent $(1,0) \in R$ is thus of the form $J_0 \times J_1$ with

$$J_0 = G_{1/2,\mathbb{Y}_0} \quad \text{and} \quad J_1 = G_{1/2,\mathbb{Y}_1} \cong GL_{n_1}(B).$$

Here, n_1 is the O_E-height of $\mathbb{Y}_1$. Let $(g_0, g_1) \in J_{0,\mathrm{rs}} \times J_{1,\mathrm{rs}}$ be a pair of regular semi-simple elements such that $H_\mathbb{Y}(g_0, g_1)H_\mathbb{Y} = H_\mathbb{Y} g H_\mathbb{Y}$. Such a pair exists: For example, after an $H_\mathbb{Y}$-translation, we may assume that $g = 1 + z_g$ in which case g commutes with z_g. Then g itself lies in $J_0 \times J_1$.

Proposition 4.8. *Assume $g \in G_{1/2,\mathbb{Y},\mathrm{rs}}$ is such that z_g^2/π is integral over O_F. Let $(g_0, g_1) \in J_{0,\mathrm{rs}} \times J_{1,\mathrm{rs}}$ be as in Construction 4.7. Then the intersection number of g factors as*

$$\mathrm{Int}(g) = \mathrm{Int}(g_0)\,\mathrm{Orb}(g_1, 1_{GL_{n_1}(O_B)}). \tag{4.6}$$

Proof. All three quantities in (4.6) only depend on the invariants of the three elements g, g_0 and g_1. So we may assume that, g has the form $\left(\begin{smallmatrix} 1 & \pi y \\ y & 1 \end{smallmatrix}\right)$. By Lemma 4.2,

$$\mathcal{I}(g) = \mathcal{N} \cap \mathcal{Z}\left(\left(\begin{smallmatrix} & \pi y \\ y & \end{smallmatrix}\right)\right) = \mathcal{Z}(y)$$

where $\mathcal{Z}(y) \subset \mathcal{N}$ is the subspace of all (Y, ι, ρ) such that $\rho y \rho^{-1} \in \mathrm{End}(Y)$. Moreover, since $y^2 = z_g^2/\pi$, the ring R in Construction 4.7 agrees with $O_F[y^2]$. Using its idempotents, every $(Y, \iota, \rho) \in \mathcal{Z}(y)$ factors as $(Y_0, \iota_0, \rho_0) \times (Y_1, \iota_1, \rho_1)$ where the two triples lie in $\mathcal{I}(g_0)$ and $\mathcal{I}(g_1)$. Furthermore, we may choose $\Gamma \subset L_g^\times = F[\pi y^2]^\times$ as $\Gamma = \Gamma_0 \times \Gamma_1$ with $\Gamma_i \subset F[z_{g_i}^2]^\times$. In this way, we obtain the factorization

$$\Gamma \backslash \mathcal{I}(g) \cong (\Gamma_0 \backslash \mathcal{I}(g_0)) \times_{\mathrm{Spf}\, O_{\breve{F}}} (\Gamma_1 \backslash \mathcal{I}(g_1)). \tag{4.7}$$

Since $\mathbb{Y}_1$ is an étale π-divisible O_E-module, $\mathcal{I}(g_1)$ is étale over $\mathrm{Spf}\, O_{\breve{F}}$. Moreover, passing to Tate modules, $\mathcal{I}(g_1)(\mathbb{F})$ may be identified with the set $\mathcal{L}(\varpi^{-1} z_{g_1})$ from Definition 3.8. The formal scheme $\Gamma_1 \backslash \mathcal{I}(g_1)$ is then a disjoint union of $\mathrm{Orb}(g_1, 1_{GL_{n_1}(O_B)})$ many copies of $\mathrm{Spf}\, O_{\breve{F}}$. Thus (4.6) follows from (4.7). □

4.3. **Proof of Theorem 2.9.**

Proof of Theorem 2.9 (Vanishing Part). Let $\gamma \in G'_{\mathrm{rs}}$ be regular semi-simple with invariant $\delta = \mathrm{Inv}(\gamma)$. We claim that there exists a strict O_E-module $\mathbb{Y}$ over $\mathbb{F}$ and a matching element $g \in G_{1/2,\mathbb{Y},\mathrm{rs}}$ if and only if $\mathrm{ord}_{1/2}(\delta) = 1$. Indeed, this condition is by definition equivalent to

$$L_\delta \cong L_0 \times L_1, \quad B_\delta \cong (B_{1/2} \otimes_F L_0) \times B_1 \tag{4.8}$$

where L_0 is an étale F-algebra, L_1 a field extension of F, and B_1/L_1 the quaternion algebra that is not isomorphic to $B_{1/2} \otimes_F L_1$. Assuming this condition is met, let $\mathbb{Y}$ be the unique strict O_E-module such that $n^\circ = [L_1 : F]$ and $n^{\mathrm{et}} = [L_0 : F]$. Then

$$D_{1/2,\mathbb{Y}} \cong M_{[L_0:F]}(B) \times D^\circ_{1/2,\mathbb{Y}}. \tag{4.9}$$

Here, $D^\circ := D^\circ_{1/2,\mathbb{Y}}$ is a CSA of degree $2n^\circ$ with Hasse invariant $(n^\circ + 1)/2n^\circ$. There are two cases: If n° is even, then B_1 and D° are both division algebras. If n° is odd, then $B_1 = M_2(L_1)$ and D° is the ring of (2×2)-matrices over a CDA. In both cases, there exists an F-algebra embedding $B_1 \to D^\circ$ and hence an F-algebra embedding $B_\delta \to D_{1/2,\mathbb{Y}}$. By [11, Corollary 2.8] (2), resp. its extension to semi-simple F-algebras, this is equivalent to the existence of an element $g \in G_{1/2,\mathbb{Y},\mathrm{rs}}$ with $\mathrm{Inv}(g;T) = \delta(T)$.

Conversely, assume that there exists an embedding $\beta : B_\delta \to D_{1/2,\mathbb{Y}}$ where $D_{1/2,\mathbb{Y}}$ is as in (4.9). Then $\beta(B_\delta)$ agrees with the centralizer of $\beta(L_\delta)$ for dimension reasons [11, Proposition 2.6 (3)] which implies that B_δ takes the form in (4.8), and hence that $\mathrm{ord}_{1/2}(\delta) = 1$. This finishes the prove of our claim.

The vanishing statement is now obtained as follows. Assume there is no $\mathbb{Y}$ with matching $g \in G_{1/2,\mathbb{Y},\mathrm{rs}}$. By the claim, this means $\mathrm{ord}_{1/2}(\delta) = 0$ or $\mathrm{ord}_{1/2}(\delta) \geq 2$. In the first case, we apply the functional equation

$$\mathrm{Orb}(\gamma, f'_{1/2}, -s) = \varepsilon_{1/2}(\gamma)\,\mathrm{Orb}(\gamma, f'_{1/2}, s)$$

from (2.17). The sign $\varepsilon_{1/2}(\gamma)$ can be seen to equal $(-1)^{\mathrm{ord}_{1/2}(\delta)}$. Thus $\partial\mathrm{Orb}(\gamma, f'_{1/2}) = 0$ if $\mathrm{ord}_{1/2}(\delta) = 0$. In the second case, the vanishing of $\partial\mathrm{Orb}(\gamma, f'_{1/2})$ is implied by Corollary 3.14. □

Proof of Theorem 2.9 (Reduction to the linear AFL). Fix $\mathbb{Y}$ and a regular semi-simple element $g \in G_{1/2,\mathbb{Y},\mathrm{rs}}$. Let $\gamma \in G'_{\mathrm{rs}}$ be a matching element. Assuming the linear AFL for all strict O_E-modules with the same connected height n°, we need to see that

$$\partial\mathrm{Orb}(\gamma, f'_{1/2}) = 2\,\mathrm{Int}(g)\log(q). \tag{4.10}$$

The condition that γ and g match is by definition equivalent to assuming that z_γ^2 and z_g^2 have the same characteristic polynomial. Thus z_γ^2/π is integral if and only if z_g^2/π is integral. By Corollary 3.4 and by Lemma 4.5, both sides of (4.10) vanish if these elements are not integral. So from now on we assume that z_g^2/π and z_γ^2/π are integral.

We may assume that γ and g take the form $\gamma = 1 + z_\gamma$ and $g = 1 + z_g$. Let $\gamma = (\gamma_0, \gamma_1)$ and $g = (g_0, g_1)$ be the components of γ and g such that $z_{\gamma_0}^2/\pi$ and $z_{g_0}^2/\pi$ are topologically nilpotent, and $z_{\gamma_1}^2/\pi$ and $z_{g_1}^2/\pi$ integrally invertible. Then γ_0 matches g_0 and γ_1 matches g_1.

We write $f'_{1/2,n_0}$ and $f'_{1/2,n_1}$ for the Parahoric test functions on $GL_{2n_0}(F)$ and $GL_{2n_1}(F)$. By Lemma 3.7, $\partial\mathrm{Orb}(\gamma_1, f'_{1/2,n_1}) = 0$. By Corollary 3.6, we hence obtain that

$$\partial\mathrm{Orb}(\gamma, f'_{1/2}) = \partial\mathrm{Orb}(\gamma_0, f'_{1/2,n_0})\,\mathrm{Orb}(\gamma_1, f'_{1/2,n_1}). \tag{4.11}$$

By Proposition 4.8, we also have the factorization

$$\mathrm{Int}(g) = \mathrm{Int}(g_0)\,\mathrm{Orb}(g_1, 1_{GL_{n_1}(O_B)}). \tag{4.12}$$

By the fundamental lemma (Theorem 2.7),

$$\mathrm{Orb}(\gamma_1, f'_{1/2,n_1}) = \mathrm{Orb}(g_1, 1_{GL_{n_1}(O_B)}).$$

Thus, Identity (4.10) follows if we can prove

$$\partial\mathrm{Orb}(\gamma_0, f'_{1/2,n_0}) = 2\,\mathrm{Int}(g_0)\log(q).$$

Since $z_{\gamma_0}^2/\pi$ and $z_{g_0}^2/\pi$ are topologically nilpotent, by Theorem 2.11, there are two elements $\widetilde{\gamma}_0 \in GL_{2n_0}(F)_{\mathrm{rs}}$ and $\widetilde{g}_0 \in G_{0,\mathbb{Y}_0,\mathrm{rs}}$ such that

$$\partial\mathrm{Orb}(\widetilde{\gamma}_0, 1_{GL_{2n_0}(O_F)}) = \partial\mathrm{Orb}(\gamma_0, f'_{1/2,n_0}) \quad \text{and} \quad \mathrm{Int}(\widetilde{g}_0) = \mathrm{Int}(g_0).$$

The linear AFL for $\mathbb{Y}_0$, whose connected part has height n°, precisely states that

$$\partial\mathrm{Orb}(\widetilde{\gamma}_0, 1_{GL_{2n_0}(O_F)}) = 2\,\mathrm{Int}(\widetilde{g}_0)\log(q)$$

and the proof is complete. □

References

[1] V. G. Drinfeld, *Coverings of p-adic symmetric domains*, Funkcional. Anal. i Priložen. **10** (1976), no. 2, 29–40.

[2] J. Guo, *On a generalization of a result of Waldspurger*, Canad. J. Math. **48** (1996), no. 1, 105–142.

[3] B. Howard, Q. Li, *Intersections in Lubin–Tate space and biquadratic Fundamental Lemmas*, Preprint (2020), arXiv:2010.07365.

[4] H. Jacquet, S. Rallis, *Uniqueness of linear periods*, Compositio Math. **102** (1996), no. 1, 65–123.

[5] R. Kottwitz, *Base change for unit elements of Hecke algebras*, Compositio Math. **60** (1986), no.2, 237–250.

[6] S. Leslie, J. Xiao, W. Zhang, *Periods and heights for unitary symmetric pairs*, in preparation.

[7] Q. Li, *An intersection number formula for CM cycles in Lubin–Tate towers*, Preprint (2018), arXiv:1803.07553, to appear in Duke Math. J.

[8] Q. Li, *A computational proof of the linear Arithmetic Fundamental Lemma for GL_4*, Preprint (2019), arXiv:1907.00090, to appear in Canad. J. Math.

[9] Q. Li, *A computational proof for the biquadratic Linear AFL for GL_4*, in preparation.

[10] Q. Li, A. Mihatsch, *On the linear AFL: The non-basic case*, Preprint (2022), arXiv:2208.10144.

[11] Q. Li, A. Mihatsch, *Arithmetic Transfer for inner forms of GL_{2n}*, Preprint (2023), arXiv:2307.11716.

[12] A. Mihatsch, *Local Constancy of Intersection Numbers*, Algebra Number Theory, **16** (2022), no. 2, 505–519.

[13] A. Mihatsch, W. Zhang, *On the Arithmetic Fundamental Lemma Conjecture over a general p-adic field*, Preprint (2021).

[14] W. Zhang, *On arithmetic fundamental lemmas*, Invent. Math. **188** (2012), no. 1, 197–252.

[15] W. Zhang, *Periods, cycles, and L-functions: a relative trace formula approach*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, 487–521, World Sci. Publ., Hackensack, NJ, 2018.

[16] W. Zhang, *Weil representation and arithmetic fundamental lemma*, Ann. of Math. (2) **193** (2021), no. 3, 863–978.

[17] Z. Zhang, *Maximal parahoric arithmetic transfers, resolutions and modularity* Preprint (2021), arXiv:2112.11994.

Nuno Hultberg, University of Copenhagen, Department of Mathematical Sciences, Universitetsparken 5, 2100 København, Denmark

Email address: nh@math.ku.dk

Andreas Mihatsch, Universität Bonn, Mathematisches Institut, Endenicher Allee 60, 53115 Bonn, Germany

Email address: mihatsch@math.uni-bonn.de

Continuity of heights in families and complete intersections in toric varieties

CONTINUITY OF HEIGHTS IN FAMILIES AND COMPLETE INTERSECTIONS IN TORIC VARIETIES

PABLO DESTIC, NUNO HULTBERG AND MICHAŁ SZACHNIEWICZ

Abstract. We study the variation of heights of cycles in flat families over number fields or, more generally, globally valued fields. To a finite type scheme S over a GVF K we associate a locally compact Hausdorff space S_{GVF} which we refer to as the GVF analytification of S. For a flat projective family $\mathcal{X} \subset \mathbb{P}^n_S \to S$, we prove that $(s \in S_{\mathrm{GVF}}) \mapsto \mathrm{ht}(\mathcal{X}_s)$ is continuous.

As an application, we prove Roberto Gualdi's conjecture on limit heights of complete intersections in toric varieties.

Contents

1. Introduction 1
1.1. Acknowledgments 4
2. Preliminaries 4
2.1. Globally valued fields 4
2.2. GVF analytification 6
2.3. Polarisations 7
2.4. Convex geometry and toric varieties 8
3. Intersection product 12
3.1. Lattice divisors 12
3.2. Heights of resultants 15
3.3. Definability of adelic intersection product 18
3.4. Integrable divisors over globally valued fields 19
4. Average intersections 22
Appendix A. Mahler measures on complex polynomials 25
References 31

1. Introduction

In classical algebraic geometry Bezout's theorem states that generically the intersection of n hypersurfaces of degrees $d_1, \dots, d_r$ in $\mathbb{P}^n$ is of degree $d_1 \dots d_r$. If the hypersurfaces are defined over a number field, one may ask whether it is possible to compute the Weil height of the intersection in terms of their arithmetic complexity. This is important, because estimates on Weil heights can lead to finiteness theorems about rational points. A striking example of this philosophy appears in (Proposition 2.17 of) [Fal91]. Another example is the arithmetic Bezout's theorem obtained in [BGS94], however it only gives an upper bound for the height of an intersection. Since Weil heights are also connected to special values of L-functions and periods (see e.g. Section 4 of [PP24], or [Mai00; CM00]) it is desirable to have formulas for their exact values. The starting point of our considerations is [Gua18b], where such a formula was given, for the height of a single hypersurface ($r = 1$) in a toric variety, with respect to a semipositively metrized toric divisor.

However, Roberto Gualdi observed in his thesis [Gua18a] that it is not possible to extend a result of the same nature to $r \geq 2$. In fact, he gives examples of polynomials with the same associated arithmetic data, defining cycles of differing heights. As a remedy to this issue, he suggested to consider average heights of cycles with prescribed arithmetic data and formulated the following conjecture, which we prove in this article.

Theorem 1.1 (Theorem 4.5)**.** Let $f_1, \ldots, f_m$ be Laurent polynomials in n variables with coefficients in a number field K and let T be a proper toric variety with torus $\mathbb{T} = \mathbb{G}^n \subset T$. Denote by V_i the hypersurface defined by f_i and by ρ_i its Ronkin function. Let $(\zeta_{1,j}, \ldots, \zeta_{m,j})_j$ be a generic sequence of small points in $\mathbb{T}^m$ for the Weil height and let $\overline{D}_0, \ldots, \overline{D}_{n-m}$ be semipositive toric adelic divisors on T with associated local roof functions $\theta_{0,v}, \ldots, \theta_{n-m,v}$. Then,

$$\lim_{j\to\infty} \widehat{\deg}(\overline{D}_0, \ldots, \overline{D}_{n-m} \mid \zeta_{1,j}V_1 \cap \cdots \cap \zeta_{m,j}V_m)$$

$$= \sum_{v \in \mathrm{M}_K} n_v \, \mathrm{MI}_M(\theta_{0,v}, \ldots, \theta_{n-m,v}, \rho_1^\vee, \ldots, \rho_m^\vee).$$

The original conjecture can be found in [Gua18a, Conjecture 6.4.4]. Let us briefly describe its contents. The arithmetic degree of a suitably generic complete intersection subvariety in a toric variety can be computed as an arithmetic intersection number of adelic toric divisors. The formulation of the conjecture is in terms of a convex geometry identity for this intersection number involving mixed integrals. Mixed integrals and further convex geometry tools are discussed in Section 2.4.

Roberto Gualdi and Martin Sombra have together proved partial results in this direction. In [GS23] they prove the above result in the case $T = \mathbb{P}^2$, $m = 2$ and $f_1(x_1, x_2) = f_2(x_1, x_2) = x_1 + x_2 + 1$. Moreover, for this example, they compute that both sides of the identity in question are equal to the intriguing value $\frac{2\zeta(3)}{3\zeta(2)}$. They have also solved the $m = 2$ case of the conjecture in [GS24]. Their methods fundamentally differ from ours. In particular, their approach relies on local logarithmic equidistribution as in [DH22] while we modify the problem in such a way that we can apply Yuan's equidistribution theorem from [Yua08].

Said modification can be conveniently phrased in the framework of globally valued fields (abbreviated GVF). Globally valued fields were introduced by Ben Yaacov and Hrushovski to serve as a theory of fields with multiple valuations satisfying the product formula. As such, globally valued fields are closely related to proper adelic curves as defined and developed in [CM20] by Chen and Moriwaki, see [Ben+24, Corollary 1.3] for the precise relationship. Globally valued fields, however, form a theory in unbounded continuous logic from [Ben08], and are therefore amenable to methods from model theory.

The original motivation of this article was to prove the definability of intersection products of divisors over globally valued fields, i.e., that arithmetic intersection numbers parametrized over a base form a quantifier-free definable formula. For the benefit of Arakelov geometers reading this article, we will phrase this as a continuity result.[1] First, let us state a version of the 'continuity of heights' over $\overline{\mathbb{Q}}$ that does not require any additional definitions.

Theorem 1.2. *Let $\pi : \mathcal{X} \to S$ be a surjective morphism of projective varieties over $\overline{\mathbb{Q}}$, generically of relative dimension d. Let $(s_i)_i$ be a generic sequence of points in $S(\overline{\mathbb{Q}})$ such that for every adelic line bundle $\overline{M}$ on S the value $h_{\overline{M}}(s_i)$ converges. Let $\overline{L}_0, \ldots, \overline{L}_d$ be integrable line bundles on $\mathcal{X}$. Then, the arithmetic intersection number on fibers*

$$\widehat{\deg}(\overline{L}_0 \ldots \overline{L}_d | \mathcal{X}_{s_i})$$

converges. Its limit can be described as a certain arithmetic intersection number over a GVF.

If $h_{\overline{M}}(s_i)$ converges to $\overline{L}^{\dim S}\overline{M}$ for some arithmetically nef line bundle $\overline{L}$ and every adelically metrized line bundle $\overline{M}$, then $\widehat{\deg}(\overline{L}_0 \ldots \overline{L}_d | \mathcal{X}_{s_i})$ converges to the intersection number $(\pi^*\overline{L})^{\dim S}\overline{L}_0 \ldots \overline{L}_d$.

The first part corresponds to Theorem 3.1 (which we state in detail below) and the latter part to Proposition 3.26, essentially due to Chen and Moriwaki [CM21, Proposition 4.5.1]. Equidistribution theorems in Arakelov geometry provide us with various examples of sequences to which the above theorem can be applied. We note in particular Yuan's equidistribution in the form of [Cha21, Lemma 8.2] gives such examples.

Before stating the 'continuity of heights' Theorem 3.1 let us recall its algebro-geometric counterpart which is the following constructibility/constantness result.

Theorem 1.3 (Corollary III 9.10 [Har77])**.** *Let $\mathcal{X} \to S$ be a flat projective family of varieties over a field K of relative dimension d. Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Then, the degree of the fibres $\deg_{\mathcal{L}}(\mathcal{X}_s)$ is locally constant.*

Note that this is equivalent to the map $s \mapsto \deg_{\mathcal{L}}(\mathcal{X}_s)$ being a continuous function on S with the Zariski topology. The same result is implied for the intersection of d line bundles. In our result the Zariski topology is replaced by a GVF analytic topology, and the degree is replaced by height. Let us be more precise.

We associate to a finite type scheme S over a GVF K, a locally compact Hausdorff space S_{GVF}, called its GVF analytification. This is done similarly to the Berkovich analytification, however with valuations replaced by global heights (see Definition 2.1 and Section 2.2). For $K = \overline{\mathbb{Q}}$, a generic sequence s_i in $S(\overline{\mathbb{Q}}) \subset S_{\mathrm{GVF}}$ converges to a point in S_{GVF} if and only if the value $h_{\overline{M}}(s_i)$ converges for every adelically metrized line bundle $\overline{M}$ on S. In Definition 3.21, for a flat family $\mathcal{X} \to S$ over a GVF K, we introduce the group of global line bundles on $\mathcal{X}$ over S. Following [Zha95a], we define semipositivity and integrability in this context. For example, the group of globally integrable line bundles on $\mathcal{X}$ over S is denoted by $\widehat{\mathrm{Pic}}_{\mathbb{Q}}^{\mathrm{int}}(\mathcal{X}/S)$. In Proposition 3.22 we define an intersection paring on tuples of globally integrable line bundles and we prove that is satisfies the following.

Theorem 1.4 (Theorem 3.1)**.** *Let $\mathcal{X} \to S$ be a flat projective morphism of finite type schemes over a GVF K of relative dimension d. Let $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_d \in \widehat{\mathrm{Pic}}_{\mathbb{Q}}^{\mathrm{int}}(\mathcal{X}/S)$.*

[1] We thank Rémi Reboulet for pointing out the similarity to the existence of a Deligne pairing. In fact, in the number field case our result is likely to be implied by the Deligne pairing in [YZ24, Theorem 4.1.3]. Our approach has the advantage of its simplicity.

Then, the map

$$\begin{aligned} S_{\mathrm{GVF}} &\to \mathbb{R} \\ s &\mapsto \widehat{\deg}(\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_d | \mathcal{X}_s) \end{aligned}$$

is continuous.

Equivalently, this means that the intersection product $\widehat{\deg}(\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_d | \mathcal{X}_s)$ (for $s \in S(F)$ with $K \subset F$ being a GVF extension) can be defined by a quantifier-free formula in the GVF language, with parameters from the base-field K. We believe that this fact may be important in axiomatizing the model companion of globally valued fields (if it exists).

The structure of this text is the following. In Section 2 we introduce globally valued fields, GVF analytifications, and give some examples. We also present necessary notions from Arakelov geometry of toric varieties. In Section 3 we use the theory of adelic curves to define the intersection product over arbitrary GVF and prove Theorem 3.1. We also prove Theorem 3.24 which relates global integrable line bundles (in our new sense) to integrable adelic line bundles over a number field (in the sense of Zhang). In Section 4 we use the previous results and perform calculations which allow us to conclude with Theorem 4.5. In Appendix A we prove an estimate on a variant of the Mahler measure of a polynomial, needed in the proof of Theorem 3.1. We believe this is known to experts, however we could not find a suitable reference. The appendix is self-contained and elementary.

1.1. **Acknowledgments.** We would like to thank our advisors Itaï Ben Yaacov, Ehud Hrushovski and Fabien Pazuki for various discussions and comments regarding this work. Special thanks go to Roberto Gualdi and Martin Sombra, for sharing with us their draft [GS24] and for their helpful remarks. Moreover, we would like to thank the organizers and the participants of the 2024 Students' Conference on Non-Archimedean, Tropical and Arakelov Geometry, for their interest in this project. We are grateful to Rémi Reboulet for pointing out the connection to Deligne pairings. The third author would like to thank Douglas Molin for discussions about limit heights.

2. Preliminaries

This section will serve both to recall definitions and theorems, as well as to state and prove technical results related to the definitions just recalled.

2.1. **Globally valued fields.** Globally valued fields are a theory in (unbounded) continuous logic designed to model fields with multiple valuations and a product formula. They are closely related to adelic curves.

There are multiple ways to describe globally valued fields, see [Ben+24] for an introduction. The simplest definition is the following.

Definition 2.1. A *globally valued field* (abbreviated GVF) is a field F together with a height function $h : \mathbb{A}(F) \to \mathbb{R} \cup \{-\infty\}$, where $\mathbb{A}(F)$ denotes the disjoint union of $\mathbb{A}^n(F)$ for all $n \in \mathbb{N}$, satisfying the following axioms, for some *Archimedean* *error* $e \geq 0$.

Height of zero:	$\forall x \in F^n$,	$h(x) = -\infty \Leftrightarrow x = 0$
Height of one:		$h(1,1) = 0$
Invariance:	$\forall x \in F^n$, $\forall \sigma \in \mathrm{Sym}_n$,	$h(\sigma x) = h(x)$
Additivity:	$\forall x \in F^n$, $\forall y \in F^m$,	$h(x \otimes y) = h(x) + h(y)$
Monotonicity:	$\forall x \in F^n$, $\forall y \in F^m$,	$h(x) \leq h(x, y)$
Triangle inequality:	$\forall x, y \in F^n$,	$h(x + y) \leq h(x, y) + e$
Product formula:	$\forall x \in F^\times$,	$h(x) = 0$

Here $\otimes$ denotes the Segre product, i.e., $(x_1, \ldots, x_n) \otimes (y_1, \ldots, y_m) = (x_i \cdot y_j : 1 \leq i \leq n, 1 \leq j \leq m)$. Note that such height factors through $h : \mathbb{P}^n(F) \to \mathbb{R}_{\geq 0}$ for each n. We write $\mathrm{ht}(x) := h[x : 1]$ for $x \in F$.

If F is countable, these can also be seen as equivalence classes of proper adelic curve structures on F (originally defined in [CM20]).

Definition 2.2. A *proper adelic curve* is a field F together with a measure space $(\Omega, \mathcal{A}, \nu)$ and with a map $(\omega \mapsto |\cdot|_\omega) : \Omega \to M_F$ to the space of absolute values on F, such that for all $a \in F^\times$ the function

$$\omega \mapsto \omega(a) := -\log |a|_\omega$$

is in $L^1(\nu)$ with integral zero.

If F is equipped with a proper adelic curve structure, then one can define a GVF structure on it, by putting

$$h(x_1, \ldots, x_n) := \int_\Omega -\min_i(\omega(x_i)) d\nu(\omega).$$

On the other hand, if F is countable, any GVF structure on F is represented by some proper adelic curve structure in this way ([Ben+24, Corollary 1.3]).

There is yet another equivalent way to describe a GVF structure on a field which turns out convenient for our purposes. We give here this definition in the case where F is countable. The general definition can be found in [Ben+24].

Definition 2.3. A *lattice group* is a partially ordered abelian group $(G, +, \leq)$ such that every pair of elements $x, y \in G$ has a greatest lower bound, denoted $x \wedge y$. Since the order may be recovered from the binary operator $\wedge$, we may also call the triple $(G, +, \wedge)$ a lattice group.

We equip the space $(M_F)^\mathbb{R}$ with the structure of a lattice group such that $f : M_F \to \mathbb{R}$ is positive if and only if $f(|\cdot|) \geq 0$ for each absolute value $|\cdot|$. We call the space of *lattice divisors* over F, denoted $\mathrm{LDiv}_\mathbb{Q}(F)$, the divisible lattice subgroup of $(M_F)^\mathbb{R}$ generated by elements of the form $\widehat{\mathrm{div}}(x) : |\cdot| \mapsto -\log |x|$ for $x \in F^\times$. These generators are called *principal* lattice divisors.

Then, GVF structures on F correspond to so called *GVF functionals*, which are linear functionals

$$l : \mathrm{LDiv}_\mathbb{Q}(F) \to \mathbb{R}$$

that are non-negative on the positive cone, and are zero on principal lattice divisors.

Assume that F is a finitely generated extension of $\mathbb{Q}$. In [Sza23] and [Ben+24] an Arakelov theoretic interpretation of the lattice $\mathrm{LDiv}_\mathbb{Q}(F)$ was given. More precisely, $\mathrm{LDiv}_\mathbb{Q}(F)$ embeds into

$$\mathrm{ADiv}_\mathbb{Q}(F) = \varinjlim \mathrm{ADiv}_\mathbb{Q}(\mathcal{X}),$$

where $\mathrm{ADiv}_{\mathbb{Q}}(\mathcal{X})$ is the group of arithmetic $\mathbb{Q}$-divisors of C^0-type on $\mathcal{X}$, and the union is taken over the system of all arithmetic varieties $\mathcal{X}$ (i.e., normal, integral, flat and projective over $\mathrm{Spec}(\mathbb{Z})$) with an isomorphism $\kappa(\mathcal{X}) \simeq F$. In [Sza23] a group lattice structure on $\mathrm{ADiv}_{\mathbb{Q}}(F)$ was defined, so that the embedding is a morphism of group lattices.

Let X be projective scheme over $\mathbb{Q}$ whose function field is isomorphic to F. To avoid confusion, we use the name *Zhang divisors/Zhang line bundles* for Cartier divisors on X equipped with adelic Green functions/line bundles with adelic metrics, in the sense of [Cha21, Remark (4.8)] (originally defined by Zhang [Zha95b]). We denote the group of Zhang divisors on X by $\mathrm{ZDiv}(X)$ and by $\mathrm{ZDiv}_{\mathbb{Q}}(X)$ its tensor with $\mathbb{Q}$. Using the same methods as in [Sza23], one also defines a lattice structure on

$$\mathrm{ZDiv}_{\mathbb{Q}}(F) = \varinjlim \mathrm{ZDiv}_{\mathbb{Q}}(X),$$

where the union is taken over the system of all X as above (with maps respecting the isomorphisms $\kappa(X) \simeq F$). Then the group lattice $\mathrm{LDiv}_{\mathbb{Q}}(F)$ has a natural embedding into $\mathrm{ZDiv}_{\mathbb{Q}}(F)$. Moreover, every GVF functional extends uniquely to a positive functional on $\mathrm{ZDiv}_{\mathbb{Q}}(F)$ so we get the following.

Corollary 2.4. On a finitely generated field F over $\mathbb{Q}$ there is a natural bijection

$$\big\{\text{GVF functionals } \mathrm{ZDiv}_{\mathbb{Q}}(F) \to \mathbb{R}\big\} \longleftrightarrow \big\{\text{GVF structures on } F\big\},$$

where by GVF functionals we mean the linear ones that are non-negative on the effective cone, and are zero on principal Zhang divisors.

For $\overline{D} \in \mathrm{ZDiv}_{\mathbb{Q}}(X)$ and $x \in X(\overline{\mathbb{Q}})$ we write $h_{\overline{D}}(x)$ for the height of x with respect to $\overline{D}$. A GVF functional l determines a GVF structure on a field by the formula

$$h(x_1, \dots, x_n) := l\big(-\bigwedge_{i=1}^{n} \widehat{\mathrm{div}}(x_i)\big).$$

Globally valued fields form a category where maps are embeddings of fields respecting height functions. Equivalently, an extension of fields $K \subset F$ induces an embedding $\mathrm{LDiv}_{\mathbb{Q}}(K) \subset \mathrm{LDiv}_{\mathbb{Q}}(F)$ and GVF structures on F extending a given GVF structure on K are precisely extensions of GVF functionals. Let us point out the following fact.

Lemma 2.5. [Ben+24, Lemma 10.2] For any $e \geq 0$ there is a unique GVF structure on $\mathbb{Q}$ (and on any subfield of $\overline{\mathbb{Q}}$) satisfying $\mathrm{ht}(2) = e \cdot \log 2$.

We refer to the GVF structure with $e = 1$ as the standard one.

2.2. **GVF analytification.**

Now, we describe a construction which recovers the space of quantifier-free types in the theory of globally valued fields, as in [Ben+24].

Definition 2.6. Let K be a non-trivial GVF (i.e. there exists $a \in K$ such that $h[1 : a] \neq 0$), and let X be a finite type scheme over K. We define the *GVF analytification* of X over K, denoted $X_{\mathrm{GVF},K}$ (or X_{GVF} if the base GVF is implied), to be the set of couples $x = (\pi(x), h_x)$ where $\pi(x)$ is a point of X, and h_x is a height on $\kappa(\pi(x))$ extending the height on K. If $x \in X_{\mathrm{GVF},K}$ and $U \subset X$ is an open containing $\pi(x)$, we will also denote by h_x the map

$$h_x : \begin{array}{ccc} \mathbb{A}(\mathcal{O}_X(U)) & \to & \mathbb{R} \cup \{-\infty\} \\ (f_1, \dots, f_n) & \mapsto & h_x(f_1(\pi(x)), \dots, f_n(\pi(x))) \end{array}.$$

We equip $X_{\mathrm{GVF},K}$ with the weakest topology such that

(1) The map $\pi : X_{\mathrm{GVF},K} \to X$ is continuous onto X with the Zariski topology.
(2) For every open $U \subset X$, and every tuple $(f_1, \dots, f_n) \in \mathcal{O}_X(U)^n$, the map $x \mapsto h_x(f_1, \dots, f_n)$ is continuous on $\pi^{-1}(D(f_1, \dots, f_n))$, where $D(f_1, \dots, f_n) \subset U$ is the open where at least one of the f_i does not vanish.

Remark 2.7. One can in fact prove that for every tuple of local sections $(f_1, \dots, f_n) \in \mathcal{O}_X(U)^n$, the map $x \mapsto h_x(f_1, \dots, f_n)$ is continuous on U as a whole. Moreover, in (1) one could demand that π is continuous with respect to the constructible topology on X and it would yield the same space. In particular, it follows that $\mathbb{A}_{\mathrm{GVF}}$ can be naturally identified with the space of quantifier-free n-types $S_n^{\mathrm{qf}}(K)$, see [Ben+24, Construction 11.15].

Remark 2.8. Let $K \subset F$ be a GVF extension. Then, there is a canonical *analytification map* $X(F) \to X_{\mathrm{GVF},K}$, defined by taking $x \in X(F)$ to the point (x, h_x), where h_x is the restriction of the height on F to $\kappa(x)$. The image of x by this map will be denoted x^{an}.

2.3. **Polarisations.** Here we present how arithmetic intersection theory can induce GVF structures and how to interpret Yuan's equidistribution result as certain kind of uniqueness of a GVF structure.

Definition 2.9. Let F be a finitely generated characteristic zero field equipped with a GVF structure. We say that this GVF structure comes from a *polarisation* $(X, \overline{H}_1, \dots, \overline{H}_d)$, if X is a normal projective variety of dimension d over $\mathbb{Q}$ with function field F and $\overline{H}_i \in \mathrm{ZDiv}_{\mathbb{Q}}(X)$ are arithmetically nef Zhang divisors, such that the GVF functional

$$l : \mathrm{ZDiv}_{\mathbb{Q}}(F) \to \mathbb{R}$$

is given by

$$l(\overline{D}) = \overline{H}_1 \cdot \ldots \cdot \overline{H}_d \cdot \overline{D}.$$

The name "polarisation" comes from [Mor00, Section 3.1], however here we also allow not necessarily model Zhang divisors. We also use the term polarisation if instead of $\overline{H}_i$'s we are given the corresponding Zhang line bundles $\mathcal{O}(\overline{H}_i)$. If the $\overline{H}_i$ occurs with multiplicity k_i we denote by $(X, \overline{H}_1^{k_1}, \dots, \overline{H}_r^{k_r})$ for the corresponding polarisation.

Remark 2.10. A polarisation $(X, \overline{H}_1, \dots, \overline{H}_d)$ induces a GVF structure structure on F that extends the standard GVF structure on $\mathbb{Q}$ (i.e. satisfies $\mathrm{ht}(2) = \log 2$) if and only if the geometric intersection number satisfies $H_1 \cdot \ldots \cdot H_d = 1$.

Yuan's equidistribution [Yua08] gives examples of polarised GVF structures. Let us recall a version of it from [Cha21, Lemma (8.2)].

Theorem 2.11. Let X be a projective variety of dimension d over $\mathbb{Q}$. Fix a semipositive Zhang divisor $\overline{D}$ with D ample and $\overline{D}^{d+1} = 0$. For any generic sequence $x_n \in X(\overline{\mathbb{Q}})$ with $h_{\overline{D}}(x_n) \to 0$, and any $\overline{M} \in \mathrm{ZDiv}_{\mathbb{Q}}(X)$ we have

$$\lim_n h_{\overline{M}}(x_n) = \frac{\overline{D}^d \cdot \overline{M}}{\deg(D)}.$$

For the next two corollaries, fix the context of the above theorem. Also, denote by F the function field of X.

Corollary 2.12. There is a unique GVF functional l on F extending the standard one on $\mathbb{Q}$ and satisfying $l(\overline{D}) = 0$.

Proof. Fix a GVF functional l on F with the above properties and a Zhang divisor $\overline{M} \in \mathrm{ZDiv}_{\mathbb{Q}}(F)$. Since the quantities from the assumptions of Theorem 2.11 are birational invariant, without loss of generality $\overline{M} \in \mathrm{ZDiv}_{\mathbb{Q}}(X)$. By existential closedness of $\overline{\mathbb{Q}}$ from [Sza23, Theorem A] there is a generic sequence of elements $x_n \in X(\overline{\mathbb{Q}})$ such that $h_{\overline{D}}(x_n) \to l(\overline{D}) = 0$ and $h_{\overline{M}}(x_n) \to l(\overline{M})$. By Theorem 2.11 we get that

$$l(\overline{M}) = \frac{\overline{D}^d \cdot \overline{M}}{\deg(D)}.$$

On the other hand, l given by this formula on X and its blowups, is a GVF functional, which finishes the proof. □

Corollary 2.13. Fix a generic sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the assumptions from Theorem 2.11. The sequence $x_n^{\mathrm{an}} \in X_{\mathrm{GVF}}$ converges to the point $x_\infty^{\mathrm{an}} \in X_{\mathrm{GVF}}$ defined by the generic point $x_\infty \in X(F)$ together with the GVF structure l on F. Moreover, l is induced by the polarisation $(X, \overline{D}, \dots, \overline{D})$.

Yuan's equidistribution theorem can be applied to the case when $X = \mathbb{P}^d_K$ over a number field K and $\mathcal{O}(1)$ endowed with the Weil metrics(denoted by $\overline{\mathcal{O}(1)}$).

Corollary 2.14. There is a unique GVF structure on $\overline{\mathbb{Q}}(x_1, \dots, x_d)$ extending the standard one on $\overline{\mathbb{Q}}$ and satisfying $\mathrm{ht}(x_1) = \cdots = \mathrm{ht}(x_d) = 0$.

Corollary 2.15. Let $\pi_i : \prod_{i=1}^e \mathbb{P}^d_K \to \mathbb{P}^d_K$ be the i-th projection and let $\overline{L} = \sum_i \pi_i^* \overline{\mathcal{O}(1)}$. Let $x_n \in \prod_{i=1}^e \mathbb{P}^d_K(\overline{\mathbb{Q}})$ be a generic sequence of small points, i.e., satisfying $h_{\overline{L}}(x_n) \to 0$. Let F be the function field of $\prod_{i=1}^e \mathbb{P}^d_K$ with the GVF structure coming from the polarisation $(\prod_{i=1}^e \mathbb{P}^d_K, \pi_1^*\overline{\mathcal{O}(1)}^d, \dots, \pi_e^*\overline{\mathcal{O}(1)}^d)$. Then

$$\lim_n x_n^{\mathrm{an}} = x_\infty^{\mathrm{an}},$$

where $x_\infty^{\mathrm{an}} \in \mathbb{P}^d_{\mathrm{GVF}}$ is defined by the generic point $x_\infty \in \mathbb{P}^d_K(F)$. Moreover, naturally identifying F with $K(x_{ij} : i \leq d, j \leq e)$, the GVF structure on F is the unique one satisfying $\mathrm{ht}(x_{ij}) = 0$ and $\mathrm{ht}(2) = \log 2$.

2.4. **Convex geometry and toric varieties.** This section contains basic notions on Ronkin metrics on toric varieties and some calculations in convex geometry. For more details on Ronkin metrics we suggest [Gua18b]. For an in depth treatment of the Arakelov geometry of toric varieties, see [BPS14].

Let $\mathbb{T} \cong \mathbb{G}_m^n$ be a split torus over a number field K. Denote by N its co-character lattice and by M its character lattice. Let X_Σ denote the proper toric variety associated to a complete fan Σ in $N_\mathbb{R}$. Let $\mathbb{T} \cong X_0 \subset X_\Sigma$ denote the open $\mathbb{T}$-orbit in X_Σ.

Definition 2.16. A *toric Cartier divisor* on a toric variety X is defined to be a Cartier divisor which is invariant under the action of the torus $\mu : \mathbb{T} \times X \to X$. This means that a Cartier divisor D is toric if $\mu^* D = \pi^* D$, where π denotes the projection map.

These are in bijection with virtual polytopes. This bijection is explained and proven in [BPS14, Section 3.3].

Definition 2.17. A *virtual support function* or *virtual polytope* with respect to a fan Σ on $N_\mathbb{R}$ is a function $N_\mathbb{R} \to \mathbb{R}$ that is linear and integral on each cone in Σ.

Let v be a place of K. A v-adic Green's function g_v for a toric Cartier divisor D is called toric if its restriction to $(X_0)^{\mathrm{an}}$ factors through the tropicalization map $(X_0)^{\mathrm{an}} \to N_\mathbb{R}$, defined in [BPS14, Section 4.1].

Theorem 2.18. [BPS14, Theorem 4.8.1.(1)] Let D be the toric divisor associated to the virtual support function Ψ. Then, the space of v-adic Green's functions for D is in bijection with continuous functions $\psi : N_\mathbb{R} \to \mathbb{R}$ such that $\psi - \Psi$ is bounded. Concave functions correspond to semipositive metrics under this bijection.

If $\overline{D}_v$ is a toric divisor with a semipositive v-adic Green's function, Legendre-Fenchel duality associates to ψ a concave function on the polytope $D_\Psi \subset M_\mathbb{R}$ associated to Ψ. It is called the *roof function* of $\overline{D}_v$ and denoted by $\theta_{\overline{D}_v}$. For details, we refer to [BPS14, Theorem 4.8.1].

A collection of functions $(\psi_v)_{v\in \mathrm{M}_K}$ defines a Zhang metric on the toric divisor corresponding to Ψ if for all v the difference $|\psi_v - \Psi|$ is bounded and for almost all v we have $\psi_v = \Psi$. We call $\overline{D} \in \mathrm{ZDiv}(X_\Sigma)$ a *toric Zhang divisor*, if D is a toric Cartier divisor, and the metrics on $\overline{D}$ come from a collection of functions $(\psi_v)_{v\in \mathrm{M}_K}$ satisfying the above condition.

The vector space $M_\mathbb{R}$ carries a Haar measure normalized in such a way that M has covolume 1. We associate to a compact convex set Δ its volume $\mathrm{vol}(\Delta)$ with respect to the Haar measure. Recall that the Minkowski sum of subsets S_1, S_2 of a vector space V is defined by

$$S_1 + S_2 = \{s_1 + s_2 \mid s_1 \in S_1, s_2 \in S_2\}.$$

The volume is a homogeneous polynomial on the space of compact convex sets with Minkowski addition. It can therefore be polarized, cf. [BPS14, Definition 2.7.14].

Definition 2.19. The mixed volume is a multilinear form on compact convex sets defined by

$$\mathrm{MV}(\Delta_1, \dots, \Delta_n) = \sum_{j=1}^{n} (-1)^{n-j} \sum_{1\le i_1<\cdots<i_j\le n} \mathrm{vol}(\Delta_{i_1} + \cdots + \Delta_{i_j}).$$

It satisfies $\mathrm{MV}(\Delta, \dots, \Delta) = n! \,\mathrm{vol}(\Delta)$.

Given a concave function θ on a compact convex set Δ, we associate to it its integral $\int_\Delta \theta(m)dm$. Given concave functions θ_1 on Δ_1 and θ_2 on Δ_2, we define their sup-convolution by

$$\theta_1 \boxplus \theta_2(m) = \sup_{m_1+m_2=m} \theta(m_1) + \theta(m_2).$$

It defines a concave function on $\Delta_1 + \Delta_2$. It is defined precisely such that the hypograph of $\theta_1 \boxplus \theta_2$ is the Minkowski sum of the hypographs of θ_1 and θ_2. One has a similar polarization as in the case of mixed volumes, cf. [BPS14, Definition 2.7.16].

Definition 2.20. The mixed integral is a multilinear form on concave functions $\theta_0, \dots, \theta_n$ on compact convex sets $\Delta_0, \dots, \Delta_n$ defined by

$$\mathrm{MI}(\theta_0, \dots, \theta_n) = \sum_{j=0}^{n} (-1)^{n-j} \sum_{0\le i_0<\cdots<i_j\le n} \int_{\Delta_{i_1}+\cdots+\Delta_{i_j}} \theta_{i_1} \boxplus \cdots \boxplus \theta_{i_j}(m)dm.$$

It satisfies $\mathrm{MI}(\theta, \ldots, \theta) = (n+1)! \int_\Delta \theta(m) dm$.

These notions are helpful to express arithmetic intersection numbers combinatorially.

Theorem 2.21. [BPS14, Theorem 5.2.5] Let $\overline{D}_0, \ldots, \overline{D}_n$ be semipositive toric Zhang divisors on X_Σ. Then,

$$\widehat{\deg}(\overline{D}_0, \ldots, \overline{D}_n | X_\Sigma) = \sum_{v \in \mathrm{M}_K} n_v \, \mathrm{MI}_M(\theta_{0,v}, \ldots, \theta_{n,v}),$$

where $\theta_{i,v}$ is the roof function of $\overline{D}_{i,v}$, for every $i = 0, \ldots, n$ and $v \in \mathrm{M}_K$.

For a nonzero Laurent polynomial $f \in K[M]$, we denote by $V(f)$ its vanishing locus on X_Σ. Let $NP(f)$ be the Newton polytope of f, i.e. the convex hull of the position of its non-zero coefficients. Then, $NP(f)$ defines a Cartier divisors on a suitable toric modification of X_Σ by [BPS14, Section 3.4] such that f gives rise to a regular section by [BPS14, Section 3.4]. It vanishes precisely on $V(f)$. This amounts to an easy check that the section restricts to a nontrivial section on codimension 1 toric subvarieties, see [Gua18b, Theorem 4.3]. We now define the Ronkin metric on the divisor $D_{NP(f)}$ corresponding to $NP(f)$. For this we use that the fibers $\mathrm{trop}^{-1}(u)$ of the tropicalization map $\mathrm{trop} : \mathbb{T}_v^{\mathrm{an}} \to N_\mathbb{R}$ carry a natural choice of probability measure σ_u. In the non-Archimedean case, it is concentrated on the Gauss point over u. In the archimedean case, it is induced by the Haar measure on $(S^1)^n$.

Definition 2.22. [Gua18b, Definition 2.7] Let f be a nonzero Laurent polynomial over K. The Ronkin function of f over a place v is the map $\rho_f : N_\mathbb{R} \to \mathbb{R}$ defined by

$$\rho_f : u \mapsto \int_{\mathrm{trop}^{-1}(u)} -\log |f(x)| d\sigma_u(x).$$

By [Gua18b, Proposition 2.10], ρ_f is a concave continuous function with bounded difference from $\Psi_{NP(f)}$. By Theorem 2.18, it defines a semipositive Green's function for $D(f)$. The collection of v-adic Ronkin functions gives rise to a Zhang divisor R_f by [Gua18b, Lemma 5.11].

Theorem 2.23. (variation of [Gua18b, Theorem 5.12]) Let X_Σ be a proper toric variety. Let f be a Laurent polynomial with vanishing locus Z such that $NP(f)$ defines a divisor on X_Σ. Let $\overline{D}_0, \ldots, \overline{D}_{n-1}$ be semipositive toric Zhang divisors on X_Σ. Then,

$$\widehat{\deg}(\overline{D}_0, \ldots, \overline{D}_{n-1} | Z) = \widehat{\deg}(\overline{D}_0, \ldots, \overline{D}_{n-1}, R_f | X_\Sigma).$$

For future use, we will relate the Ronkin functions and the Ronkin roof function of Laurent polynomials related by maps of tori.

Definition 2.24. Let $\gamma : V \to W$ be a homomorphism of finite dimensional real vector spaces and $f : V \to \mathbb{R} \cup \{-\infty\}$ be a closed concave function with compact support. We define the direct image of f along γ by

$$\gamma_* f(w) = \max_{v \in \gamma^{-1}(w)} f(v).$$

It is a closed concave function with compact domain in W.

Lemma 2.25. Let $\gamma : M \to M'$ be a map of lattices. Then, this induces a map on the rings of Laurent polynomials $K[\gamma] : K[M] \to K[M']$. Let $f \in K[M] \setminus \{0\}$. Then,

$$\rho_{K[\gamma](f)} = \rho_f \circ \gamma^\vee$$

and

$$\rho^\vee_{K[\gamma](f)} = \gamma_* \rho^\vee_f(m).$$

Proof. The first statement follows readily from the definitions. The second statement follows from the first using [BPS14, Proposition 2.3.8(1)]. ☐

We now prove generalizations of Lemma 1.11 and Proposition 1.12 in [Gua18b]. We apply this to write the formulas in the preceding section in the form used in [Gua18a, Conjecture 1].

Lemma 2.26. Let $\Delta_1, \ldots, \Delta_k$ be polytopes contained in a k-dimensional rational subspace L and denote by π the projection away from this subspace to the quotient P. Let $Q_1, \ldots, Q_{n-k}$ be polytopes in $M_\mathbb{R}$. Then,

$$\mathrm{MV}_M(\Delta_1, \ldots, \Delta_k, Q_1, \ldots, Q_{n-k}) = \mathrm{MV}_L(\Delta_1, \ldots, \Delta_k) \cdot \mathrm{MV}_P(\pi(Q_1), \ldots, \pi(Q_{n-k})).$$

Proof. By the definition of mixed volume one obtains

$$\begin{aligned} &\mathrm{MV}_M(\Delta_1, \ldots, \Delta_k, Q_1, \ldots, Q_{n-k}) \\ &= \sum_{j=1}^{n-k} (-1)^{n-j} \sum_{1 \le i_1 < \cdots < i_j \le n-k} \sum_{I \subset \{1,\ldots,k\}} (-1)^{|I|} \operatorname{vol}(\Delta_I + Q_{i_1} + \cdots + Q_{i_j}). \end{aligned}$$

Here Δ_I is taken to denote $\sum_{i \in I} \Delta_i$. It now suffices to show that

$$\sum_{I \subset \{1,\ldots,k\}} (-1)^{|I|} \operatorname{vol}(\Delta_I + Q) = (-1)^k \mathrm{MV}_L(\Delta_1, \ldots, \Delta_k) \operatorname{vol}(\pi(Q)).$$

For this take any $p \in P$ and denote by Q_p the preimage of p in Q. We may view $\sum_{I \subset \{1,\ldots,k\}} (-1)^{|I|} \operatorname{vol}(\Delta_I + Q)$ as an integral over $\pi(Q)$, namely as

$$\int_{\pi(Q)} \sum_{I \subset \{1,\ldots,k\}} (-1)^{|I|} \operatorname{vol}(\Delta_I + Q_p).$$

In order to conclude, we need to show that $\sum_{I \subset \{1,\ldots,k\}} (-1)^{|I|} \operatorname{vol}(\Delta_I + R) = \mathrm{MV}_L(\Delta_1, \ldots, \Delta_k)$ for any polytope R in the k-dimensional subspace spanned by the Δ_i. For this we decompose the expression by writing out the volumes as mixed volumes and order by the number of R occurring in the expansion.

$$\begin{aligned} &\sum_{I \subset \{1,\ldots,k\}} (-1)^{|I|} \operatorname{vol}(\Delta_I + R) \\ &= \sum_{s=0}^{k} \binom{k}{s} \sum_{I \subset \{1,\ldots,k\}} (-1)^{|I|} \sum_{1 \le j_1 \le \cdots \le j_{k-s} \le k, j_i \in I} \binom{k-s}{J} \mathrm{MV}(\Delta_{j_1}, \ldots, \Delta_{j_{k-s}}, R, \ldots, R). \end{aligned}$$

Here $\binom{k-s}{J}$ is taken to denote the number of partitions of $k-s$ elements into partitions of form J, i.e. the multinomial coefficient for $k-s$ and $\#\{i | j_i = m\}$. We reorder the sum to sum over size $k-s$ multisets of $\{1, \ldots, k\}$. Fix a multiset

$1 \le j_1 \le \cdots \le j_{k-s} \le k$ of order $k-s$ in elements of $\{1,\dots k\}$. Then, its contribution to the sum is

$$\binom{k}{s}\binom{k-s}{J}\operatorname{MV}(\Delta_{j_1},\dots,\Delta_{j_{k-s}},R,\dots,R)\sum_{I\supseteq J}(-1)^{|I|},$$

where containment is understood on the level of underlying sets. By comparing to the expansion of $\prod_{I\setminus J}(1-1)$ we see that this vanishes for $J \neq I$ and is $(-1)^k$ for $J = I$. □

Lemma 2.27. Let the g_i be concave functions on polytopes Q_i and Δ_i as before. Then,

$$\operatorname{MI}_M(\iota_{\Delta_1},\dots,\iota_{\Delta_k},g_1,\dots,g_{n-k+1}) = \operatorname{MV}_L(\Delta_1,\dots,\Delta_k)\cdot \operatorname{MI}_P(\pi_* g_1,\dots,\pi_* g_{n-k+1}).$$

Proof. The reduction to the previous Lemma is precisely as in [Gua18b, Proposition 1.12]. □

3. Intersection product

In this section we study the intersection product defined in [CM21] and prove that it varies continuously in flat families. More precisely, we prove the following theorem.

Theorem 3.1. Let $\mathcal{X} \to S$ be a flat projective morphism of finite type schemes over a GVF K of relative dimension d. Let $\overline{\mathcal{L}}_0,\dots,\overline{\mathcal{L}}_d \in \widehat{\operatorname{Pic}}^{\operatorname{int}}_{\mathbb{Q}}(\mathcal{X}/S)$ be globally integrable line bundles on $\mathcal{X}$ over S. Then, the map

$$\begin{aligned} S_{\mathrm{GVF}} &\to \mathbb{R} \\ s &\mapsto \widehat{\deg}(\overline{\mathcal{L}}_0,\dots,\overline{\mathcal{L}}_d|\mathcal{X}_s) \end{aligned}$$

is continuous.

This can be applied in various settings of interest to number theorists. This can be seen for instance by the following corollary.

Corollary 3.2. Suppose that S is projective over $\mathbb{Q}$ and $\mathcal{X} \to S$ a projective morphism. Then, any integrable Zhang line bundle on $\mathcal{X}$ is a globally integrable line bundle in $\widehat{\operatorname{Pic}}^{\operatorname{int}}_{\mathbb{Q}}(\mathcal{X}/S)$.

Proof. It is a weaker property to be an element of $\widehat{\operatorname{Pic}}^{\operatorname{int}}_{\mathbb{Q}}(\mathcal{X}/S)$ than of $\widehat{\operatorname{Pic}}^{\operatorname{int}}_{\mathbb{Q}}(\mathcal{X})$. Hence, the statement follows from Theorem 3.24. □

Remark 3.3. The property of being globally integrable is preserved under base change. In particular, one may apply Theorem 3.1 to integrable Zhang divisors on a projective $\mathcal{X}$ after restricting to the flat locus of $\mathcal{X} \to S$.

3.1. Lattice divisors.

Let K be a countable GVF. We can assume that it is represented by a proper adelic curve $(K,(\Omega,\mathcal{A},\nu),\phi)$ with $\Omega = M_K$, the trivial absolute value having zero mass, and the restriction of the measure to the archimedean places $\nu|_{\Omega_\infty}$ being supported at normalized valuations (i.e., satisfying $v(2) = -\log 2$). Moreover, whenever we consider a GVF extension $K \subset F$ in this subsection, we assume that F is also countable and the GVF structure on F is induced by an adelic curve structure with the same properties (see [Ben+24, Section 9]).

We recall ideas from the theory of adelic curves only briefly. We refer to [CM20; CM21] for details. Let us start by recalling a definition.

Definition 3.4. Let X be a finite type K-scheme with a line bundle L. A *metric family* on L is a family $\varphi = (\varphi_\omega)_{\omega\in\Omega}$, where each φ_ω is a continuous metric on L_ω on X_ω^{an}. Here X_ω^{an} is the Berkovich analytification of $X_\omega = X \otimes_K K_\omega$, where K_ω is the completion of K with respect to the absolute value $\omega \in \Omega = M_K$. We call a pair $\overline{L} = (L, \varphi)$ a *metrized line bundle* on X over K.

This is as in [CM21, Definition 4.1.4], but we also allow non-projective X. Also, we naturally extend this definition to $\mathbb{Q}$-line bundles. Similarly, we extend the definition of a *Green function family* of a ($\mathbb{Q}$-)Cartier divisor on X from [CM21, Definition 4.2.1], to a non-projective X, word for word. A ($\mathbb{Q}$-)Cartier divisor with a Green function family is called a *metrized divisor*.

Chen and Moriwaki introduce *adelic line bundles* as a subset of metrized line bundles respecting the global nature of the adelic curve, see [CM21, Definition 4.1.9]. There are two conditions. Firsty, the variation of metrics along ω has to be measurable. This condition does not occur for number fields since their set of places is discrete. Finally, one needs a condition requiring that the family of metrics has finite distance to arising from a global model. This condition is referred to as being dominated. In order to obtain an intersection theory, one needs to demand the metrics at each place to be *integrable*, i.e. the difference of *semipositive* metrics. We follow the definition of semipositivity from [CM20, Section 2.3]. Note that this only allows for semipositive metrics on semiample line bundles.

Definition 3.5. Consider $\mathbb{P}^n = \mathbb{P}^n_K$ with coordinates $x_0, \dots, x_n$. We equip the anti-tautological line bundle $\mathcal{O}(1)$ with two families of metrics $\varphi = (|\cdot|_{\phi_\omega})_{\omega\in\Omega}, \psi = (|\cdot|_{\psi_\omega})_{\omega\in\Omega}$ defined in the following way. For a section $s \in H^0(\mathbb{P}^n, \mathcal{O}(1))$ identified with a linear form A we have

$$|s(z)|_{\varphi_\omega} := \frac{|A(z)|_\omega}{\max(|z_0|_\omega, \dots, |z_n|_\omega)},$$

for $z \in \mathbb{P}^{n,\mathrm{an}}_\omega$ and any $\omega \in \Omega$. The metric ψ is defined in the same way for non-Archimedean ω, but for archimedean ω we set

$$|s(z)|_{\psi_\omega} := \frac{|A(z)|_\omega}{\sqrt{\sum_{i=0}^n |z_i|_\omega^2}},$$

for all $z \in \mathbb{P}^{n,\mathrm{an}}_\omega$. We call φ, ψ the Weil and the Fubini-Study metric respectively and use the notation $\overline{\mathcal{O}(1)} = (\mathcal{O}(1), \varphi), \overline{\mathcal{O}(1)}^{\mathrm{FS}} = (\mathcal{O}(1), \psi)$. We use the same notation for pullbacks of these adelic line bundles on $\mathbb{P}^n_S$ for any finite type K-scheme S. Both the Weil and the Fubini-Study metric define semipositive adelic line bundles.

Recall that for a place $\omega \in \Omega$ and Green's functions ϕ, ψ for a divisor D we denote by $d_\omega(\phi, \psi)$ the sup-norm of $\phi - \psi$ and call it the local distance of ϕ and ψ. For different metrics on a divisor over an adelic curve we denote by $d(\phi, \psi)$ the global distance of ϕ and ψ. It is defined as the upper integral $\int^+ d_\omega(\phi_\omega, \psi_\omega)\nu(d\omega)$ over the local distances.

From the point of view of globally valued fields the Weil metric is the fundamental object. We need to relate it to the Fubini-Study height to invoke calculations from that setting.

Lemma 3.6. Let $\alpha_n : \mathbb{P}^r \to \mathbb{P}^s$ be the n-th Veronese map. Consider two metrics on the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^r$, namely the Weil metric φ and the metric $\sqrt[n]{\alpha_n^*\psi}$ which is the n-th root of the pullback of the Fubini-Study metric from $\mathcal{O}(1)$ on $\mathbb{P}^s$. Then for non-Archimedean places the two metrics are the same and for archimedean places $\sigma \in \Omega$ we have

$$d_\sigma(\varphi, \sqrt[n]{\alpha_n^*\psi}) \leq \frac{r \log n}{n},$$

for $n \geq r+1$.

Proof. We only calculate the archimedean case. Consider the section x_0 of $\mathcal{O}(1)$ on $\mathbb{P}^r$. By definition we have

$$|x_0(z)|_\varphi = \frac{|z_0|_\sigma}{\max(|z_0|_\sigma, \ldots, |z_n|_\sigma)}$$

and

$$|x_0(z)|_{\sqrt[n]{\alpha_n^*\psi}} = \sqrt[n]{\frac{|z_0|_\sigma^n}{\sqrt{\sum_{|I|=n} |z^I|_\sigma^2}}} = \frac{|z_0|_\sigma}{\sqrt[2n]{\sum_{|I|=n} |z^I|_\sigma^2}}.$$

Hence we calculate

$$\left| -\log \frac{|x_0(z)|_\varphi}{|x_0(z)|_{\sqrt[n]{\alpha_n^*\psi}}} \right| = \left| \log \frac{\max(|z_0|_\sigma, \ldots, |z_n|_\sigma)}{\sqrt[2n]{\sum_{|I|=n} |z^I|_\sigma^2}} \right|.$$

This is bounded by

$$|\log \sqrt[2n]{s+1}| = \left| \log \sqrt[2n]{\binom{r+n}{n}} \right| \leq \frac{1}{2n} |\log(r+1)n^r| \leq \frac{r \log n}{n},$$

where in the first inequality we use the fact that $\binom{r+n}{n} \leq (r+1)n^r$ and the second inequality holds for $n \geq r+1$. □

Lemma 3.7. Let $f : Q \to P$ be a morphism of projective schemes over K and let $\overline{L}$ be an adelic line bundle on P. Then $f^*\overline{L}$ is an adelic line bundle on Q.

If $\overline{L}$ is semipositive or integrable, so is $f^*\overline{L}$.

Proof. The fact that $f^*\overline{L}$ is an adelic line bundle is found in [CM24, Section 2.8.3 and 2.9.5]. The semipositivity assertion follows from [CM24, Lemma 6.1.2]. □

Let us fix a projective morphism $\pi : \mathcal{X} \to S$ of finite type K-schemes, where S is not necessarily projective.

Definition 3.8. We say that a metrized family $\overline{\mathcal{L}}$ on $\mathcal{X}$ is *simple over* S, if there is a closed embedding $j : \mathcal{X} \to \mathbb{P}^n_S$ over S (for some n), such that $\overline{\mathcal{L}} = j^*\overline{\mathcal{O}(1)}$. The elements of the $\mathbb{Q}$-vector space of metrized $\mathbb{Q}$-line bundles on $\mathcal{X}$ generated by simple ones are called *lattice line bundles on* $\mathcal{X}$ *over* S and are denoted by $\mathrm{LPic}_{\mathbb{Q}}(\mathcal{X}/S)$. The space of metrized $\mathbb{Q}$-divisors coming from rational sections of such metrized $\mathbb{Q}$-line bundles is denoted by $\mathrm{LDiv}_{\mathbb{Q}}(\mathcal{X}/S)$ and its elements are called *lattice divisors on* $\mathcal{X}$ *over* S. If S is equal to $\mathrm{Spec}(K)$, we omit it in the notation. Moreover, if we want to emphasize the dependence on the GVF K, we use the notation $\mathrm{LPic}_{\mathbb{Q}}(\mathcal{X}/S)_K$.

Remark 3.9. It follows from Lemma 3.7 that simple metrized line bundles on $\mathcal{X}$ over S are semipositive on fibers. More precisely, if $K \subset F$ is a GVF extension, and $s \in S(F)$, then a simple metrized line bundle $\overline{\mathcal{L}}$ over S coming from an embedding

$j : \mathcal{X} \to \mathbb{P}^n_S$ induces an adelic line bundle $\overline{\mathcal{L}}_s = j_s^* \overline{\mathcal{O}(1)}$ which is semipositive on $\mathcal{X}_s$ with respect to the GVF F.

We remark that for any GVF extension F/K, a finite type F-scheme T, and a morphism of K-schemes $T \to S$, there is a base-change map

$$\mathrm{LPic}_{\mathbb{Q}}(\mathcal{X}/S)_K \to \mathrm{LPic}_{\mathbb{Q}}(\mathcal{X}_T/T)_F.$$

In particular for $s \in S(F)$, there is a specialisation map

$$\mathrm{LPic}_{\mathbb{Q}}(\mathcal{X}/S)_K \to \mathrm{LPic}_{\mathbb{Q}}(\mathcal{X}_s)_F.$$

Let us describe these maps more precisely. Let $\overline{\mathcal{L}} = (\mathcal{L}, \varphi) \in \mathrm{LPic}_{\mathbb{Q}}(\mathcal{X}/S)_K$, for $\varphi = (\varphi_\omega)_{\omega \in M_K}$. This means that each φ_ω is a metric on $\mathcal{L}_\omega$ over $\mathcal{X}_\omega^{\mathrm{an}}$. To get a family of metrics $\psi = (\psi_v)_{v \in M_F}$ on the base-change $\mathcal{L}_T$ of $\mathcal{L}$ via the morphism $\mathcal{X}_T \to \mathcal{X}$ one proceeds as in [CM21, Example 4.1.8] (which works the same when $K' = F/K$ is not algebraic). Equivalently, one could take an embedding $j : \mathcal{X} \to \mathbb{P}^n_S$ that realises $\overline{\mathcal{L}} = j^* \overline{\mathcal{O}(1)}$ (or two embeddings such that it comes from the difference of pullbacks) and define $(\mathcal{L}_T, \psi)$ via the pullback of $\overline{\mathcal{O}(1)}$ through the map $j_T : \mathcal{X}_T \to \mathbb{P}^n_T$.

3.2. **Heights of resultants.** Chen and Moriwaki have constructed an intersection product of integrable adelic Cartier divisors in [CM21]. In this subsection we look closely at its definition which uses heights of certain resultants.

Theorem 3.10. [CM21, Theorem B] Let X be a projective scheme of pure dimension d over a GVF K. Then, there is a multilinear *adelic intersection product*

$$\mathrm{LPic}_{\mathbb{Q}}(X)^{d+1} \to \mathbb{R}.$$

Proof. Given finitely many elements of $\mathrm{LPic}_{\mathbb{Q}}(X)$ we can replace K by a countable subfield over which the corresponding embeddings to projective spaces (and X) are defined. Then we can represent the GVF K by an adelic curve. Since lattice line bundles are integrable, an arithmetic intersection number is defined by [CM21, Theorem B]. We show that the product on $\mathrm{LPic}_{\mathbb{Q}}(F)$ only depends on the induced GVF structure on K in Corollary 3.15. □

We write the intersection number of lattice line bundles $\overline{L}_0, \dots, \overline{L}_d$ as $\overline{L}_0 \cdots \overline{L}_d$. We observe that the adelic intersection product is determined by its values on tuples of simple line bundles. Let us fix a tuple of such simple adelic Cartier divisors on a projective scheme X and analyse how to calculate their intersection.

Assume we are given closed embeddings $\xi_i : X \to \mathbb{P}^{r_i} = \mathbb{P}(V_i)$ for $i = 0, \dots, d$, where V_i is a $(r_i + 1)$-dimensional vector space over K with a distinguished basis. For a natural number n, denote by $\xi_i^{\otimes n} : X \to \mathbb{P}^{r_i(n)}$ the composition of ξ_i with the n-th Veronese map $\mathbb{P}^{r_i} = \mathbb{P}(V_i) \to \mathbb{P}(S^n V_i) = \mathbb{P}^{r_i(n)}$ and write $V_i(n) := S^n V_i$. Note that in this case we have

$$\dim V_i(n) = r_i(n) + 1 = \binom{r_i + n}{n} = O(n^{r_i}).$$

For each n we define the line bundle $L_i(n)$ to be the pullback $\xi_i^{\otimes n,*} \mathcal{O}(1)$. We pull back the Weil metric and the Fubini-Study metric to obtain adelic line bundles $\overline{L_i(n)}$ and $\overline{L_i(n)}^{\mathrm{FS}}$ respectively. For $n = 1$, we omit the n in the notation. We note that there is a canonical isomorphism $L_i(n) \cong L_i^{\otimes n}$.

Let $\delta_i(n)$ be the intersection number $L_0(n)\cdots L_{i-1}(n)\cdot L_{i+1}(n)\cdots L_d(n)$ and set $\delta_i=\delta_i(1)$. Since $L_i(n)\cong L_i^{\otimes n}$, we have $\delta_i(n)=n^d\cdot\delta_i$. Let

$$W(n):=S^{\delta_0(n)}(V_0(n)^\vee)\otimes_K\ldots\otimes_K S^{\delta_d(n)}(V_d(n)^\vee)$$

and note that using distinguished bases of V_i for $i=0,\ldots,d$ we can naturally interpret elements of $W(n)$ as polynomials of multi-degree $(\delta_0(n),\ldots,\delta_d(n))$ on $V_0(n)\times\cdots\times V_d(n)$. There is a unique (up to scaling) element $R_n\in W(n)$ such that it vanishes on $(v_0,\ldots,v_d)$ if and only if the intersection $X\cap Z(v_0)\cap\cdots\cap Z(v_d)$ is non-empty (as a scheme), where $Z(v_i)$ is the pullback to X of the hyperplane in $\mathbb{P}(V_i(n))$ defined by the zero-set of the linear form v_i. We call it the resultant of X with respect to embeddings $\xi_i^{\otimes n}$. It determines a unique element $R_n\in\mathbb{P}(W(n))$ whose height calculates the adelic intersection product in the followng way.

Remark 3.11. [CM21, Remark 4.2.13]

$$\overline{L_0(n)}^{\mathrm{FS}}\cdot\ldots\cdot\overline{L_d(n)}^{\mathrm{FS}}=\int_{\Omega\setminus\Omega_\infty}\log\|R_n\|_\omega\nu(d\omega)$$

$$+\int_{\Omega_\infty}\nu(d\sigma)\int_{\mathbb{S}_0(n)_\sigma\times\cdots\times\mathbb{S}_d(n)_\sigma}\log|(R_n)_\sigma(z_0,\ldots z_d)|\eta_{\mathbb{S}_0(n)_\sigma}(dz_0)\otimes\cdots\otimes\eta_{\mathbb{S}_d(n)_\sigma}(dz_d)$$

$$+\nu(\Omega_\infty)\frac{1}{2}\sum_{i=0}^{d}\delta_i(n)\sum_{l=1}^{r_i(n)}\frac{1}{l},$$

where we use the notation from the cited remark, but with an additional variable n. This means that $\mathbb{S}_i(n)_\sigma$ is the unit sphere in $V_i(n)_\sigma$ with the sphere measure $\eta_{\mathbb{S}_i(n)_\sigma}$. Moreover, for a non-Archimedean $\omega\in\Omega$, the norm $\|\cdot\|_\omega$ is the maximum of coefficients norm, with respect to the distinguished basis of $W(n)$, for example by [CM24, Proposition A.2.2]. Later we write $\eta(dz)$ for $\eta_{\mathbb{S}_0(n)_\sigma}(dz_0)\otimes\cdots\otimes\eta_{\mathbb{S}_d(n)_\sigma}(dz_d)$ and z for the tuple $z_0,\ldots,z_d$ (here n,d and σ are implicit).

Lemma 3.12. The adelic intersection product satisfies

$$\lim_n\frac{1}{n^{d+1}}\overline{L_0(n)}^{\mathrm{FS}}\cdot\ldots\cdot\overline{L_d(n)}^{\mathrm{FS}}=\overline{L_0(n)}\cdot\ldots\cdot\overline{L_d(n)}.$$

Proof. It suffices to show that the metrics on $\frac{1}{n}\overline{L_i(n)}^{\mathrm{FS}}$ converge with respect to the global distance to the metrics on $\overline{L_i}$. By Lemma 3.6 the global distance satisfies $d(\frac{1}{n}\overline{L_i(n)}^{\mathrm{FS}},\overline{L_i})\leq\nu(\Omega_\infty)\cdot\frac{r_i\log n}{n}$. ☐

We use the above formula and the lemma to calculate $\overline{D}_0\cdot\ldots\cdot\overline{D}_d$ through resultants. More precisely we show the following.

Proposition 3.13. In the above context we have

$$\overline{L}_0\cdot\ldots\cdot\overline{L}_d=\lim_n\frac{1}{n^{d+1}}\operatorname{ht}(R_n),$$

where we treat R_n's as tuples, using the distinguished basis of $W(n)$.

Proof. By Lemma 3.12 we only need to show that

$$|\overline{L_0(n)}^{\mathrm{FS}}\cdot\ldots\cdot\overline{L_d(n)}^{\mathrm{FS}}-\operatorname{ht}(R_n)|=o(n^{d+1}).$$

First we express $\mathrm{ht}(R_n)$ in a form of an integral

$$\mathrm{ht}(R_n) = \int_{\Omega} \log \|R_n\|_{\omega} \nu(d\omega)$$

$$= \int_{\Omega\setminus\Omega_\infty} \log \|R_n\|_{\omega} \nu(d\omega) + \int_{\Omega_\infty} \log \|R_n\|_{\sigma} \nu(d\sigma),$$

where $\|\cdot\|_\omega, \|\cdot\|_\sigma$ denote the maximum of coefficients norms (for a non-Archimedean $\omega \in \Omega$ or archimedean $\sigma \in \Omega_\infty$), with respect to the distinguished basis of $W(n)$.

Claim 3.14. We have

$$\nu(\Omega_\infty)\frac{1}{2}\sum_{i=0}^{d} \delta_i(n) \sum_{l=1}^{r_i(n)} \frac{1}{l} = O(n^d \log n).$$

In particular, when divided by n^{d+1} converges to zero, for $n \to \infty$.

Proof. This follows from the fact that $\delta_i(n) = n^d \delta_i$ and

$$\sum_{l=1}^{r_i(n)} \frac{1}{l} = O(\log r_i(n)) = O(\log n^{r_i}) = O(\log n).$$

□

By Remark 3.11 we will be done if we show that

$$\Big| \int_{\mathbb{S}_0(n)_\sigma \times \cdots \times \mathbb{S}_d(n)_\sigma} \log |(R_n)_\sigma(z)| \eta(dz) - \log \|R_n\|_\sigma \Big| = o(n^{d+1}),$$

where the constant is independent of $\sigma \in \Omega_\infty$. But by Proposition A.1 applied to the polynomial $(R_n)_\sigma$, we have

$$\Big| \int_{\mathbb{S}_0(n)_\sigma \times \cdots \times \mathbb{S}_d(n)_\sigma} \log |(R_n)_\sigma(z)| \eta(dz) - \log \|R_n\|_\sigma \Big| \leqslant \sum_{i=0}^{d} \delta_i(n) \left(\log(r_i(n)+1) + \sum_{k=1}^{r_i(n)-1} \frac{1}{k} \right),$$

where $\delta_i(n) = n^d \delta_i$ and $\log(r_i(n)+1) + \sum_{k=1}^{r_i(n)-1} \frac{1}{k} = O(\log(r_i(n))) = O(\log n)$ for all $i \leqslant d$, so

$$\sum_{i=0}^{d} \delta_i(n) \left(\log(r_i(n)+1) + \sum_{k=1}^{r_i(n)-1} \frac{1}{k} \right) = O(n^d \log n) = o(n^{d+1}),$$

where the given bound does not depend on σ. □

Corollary 3.15. The intersection product on $\mathrm{LPic}_{\mathbb{Q}}(X)$ only depends on the induced GVF structure on K.

Remark 3.16. By analysing precisely the proof of Proposition 3.13, one can see that in fact it shows existence of an absolute constant C such that

$$\Big| \overline{L}_0 \cdot \ldots \cdot \overline{L}_d - \frac{1}{n^{d+1}} \mathrm{ht}(R_n) \Big| \leq C \cdot (1 + \nu(\Omega_\infty)) \cdot \max_i r_i \delta_i \cdot \frac{\log n}{n}.$$

Note that the number $\nu(\Omega_\infty)$ can be expressed as $\frac{\mathrm{ht}(2)}{\log 2}$ with respect to the induced GVF structure on K.

3.3. **Definability of adelic intersection product.** In this section we prove Theorem 3.1 for lattice line bundles. Let $\pi : \mathcal{X} \to S$ be a flat projective morphism with d-dimensional fibers. Suppose $S = \operatorname{Spec} A$ is an affine variety (so that A is an integral domain).

Let $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_d \in \operatorname{LPic}(\mathcal{X}/S)$ be simple over S with embeddings $\alpha_i : \mathcal{X} \to \mathbb{P}^{k_i}_S$. For a field-valued point $s \in S(F)$ and any object Q over S, the notation $Q(s)$ denotes its base change to s. Denote by δ_i the intersection number

$$\deg(\mathcal{L}_0(s) \cdot \ldots \cdot \mathcal{L}_{i-1}(s) \cdot \mathcal{L}_{i+1}(s) \cdot \ldots \cdot \mathcal{L}_d(s) | \mathcal{X}_s)$$

for any $s \in S(F)$ in any field extension $K \subset F$. It is independent of the choice of s since the family $\pi : \mathcal{X} \to S$ is flat.

Lemma 3.17. There is a family of polynomials R_n with coefficients in A defined up to scalar in $A^\times$, such that for all $s \in S(F)$ we have

$$R_n(s) = R_{n,s},$$

where $R_{n,s}$ is the resultant R_n from Subsection 3.2, defined for the scheme $\mathcal{X}_s$ equipped with the family of embeddings $(\alpha_i)_s : \mathcal{X}_s \to \mathbb{P}^{k_i}_{\kappa(s)}$ for $i = 0, \ldots, d$.

Proof. This is probably standard, but we could not find a reference so we sketch a proof here, based on the construction of resultants from [CM21, Section 1.6]. We use notation from loc.cit but with the base-field k replaced by A, X over k replaced by $\mathcal{X}$ over $S = \operatorname{Spec}(A)$, and L_i's replaced by $\mathcal{L}_i$'s.

First note that the whole Section 1.5 and Section 1.6 up to Proposition 1.6.2 of [CM21] go through word-for-word over A. It remains to prove the analogue of [CM21, Proposition 1.6.2] over A. This boils down to calculating the cycles

$$q_*(c_1(p^*\mathcal{L}_0) \cdots c_1(p^*\mathcal{L}_{i-1}) c_1(q^* q_i^*(\mathcal{O}_{E_i^\vee}(1))) c_1(p^*\mathcal{L}_{i+1}) \cdots c_1(p^*\mathcal{L}_d) \cap [\mathcal{X} \times_S \check{\mathbb{P}}])$$
$$= c_1(q_i^*(\mathcal{O}_{E_i^\vee}(1))) \cdot q_*(c_1(p^*\mathcal{L}_0) \cdots c_1(p^*\mathcal{L}_{i-1}) c_1(p^*\mathcal{L}_{i+1}) \cdots c_1(p^*\mathcal{L}_d) \cap [\mathcal{X} \times_S \check{\mathbb{P}}])$$

We look at the diagram

$$\begin{array}{ccc} \mathcal{X} \times_S \check{\mathbb{P}} & \xrightarrow{p} & \mathcal{X} \\ \downarrow q & & \downarrow r \\ \check{\mathbb{P}} & \xrightarrow{s} & S \end{array}$$

and use flat base-change to get the equality

$$q_*(c_1(p^*\mathcal{L}_0) \cdots c_1(p^*\mathcal{L}_{i-1}) c_1(p^*\mathcal{L}_{i+1}) \cdots c_1(p^*\mathcal{L}_d) \cap [\mathcal{X} \times_S \check{\mathbb{P}}])$$
$$= s^* r_*(c_1(\mathcal{L}_0) \cdots c_1(\mathcal{L}_{i-1}) c_1(\mathcal{L}_{i+1}) \cdots c_1(\mathcal{L}_d) \cap [\mathcal{X}]).$$

Let η be the generic point of S. By the flat base-change for the localisation map $\eta \to S$, we get

$$r_*(c_1(\mathcal{L}_0) \cdots c_1(\mathcal{L}_{i-1}) c_1(\mathcal{L}_{i+1}) \cdots c_1(\mathcal{L}_d) \cap [\mathcal{X}])$$
$$= \deg(\mathcal{L}_0(\eta) \cdots \mathcal{L}_{i-1}(\eta) \mathcal{L}_{i+1}(\eta) \cdots \mathcal{L}_d(\eta))[S]$$

which is equal to $\delta_i \cdot [S]$. Hence the cycle in question is equal to

$$c_1(q_i^*(\mathcal{O}_{E_i^\vee}(1))) s^*(\delta_i \cdot [S]) = c_1(q_i^*(\mathcal{O}_{E_i^\vee}(1))) \cap \delta_i[\check{\mathbb{P}}] = c_1(q_i^*(\mathcal{O}_{E_i^\vee}(\delta_i))) \cap [\check{\mathbb{P}}],$$

which finishes the proof as in [CM21, Proposition 1.6.2].

□

Proposition 3.18. Let $\mathcal{X} \to S$ be a flat projective morphism of finite type schemes over a GVF K of relative dimension d. Let $\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d \in \mathrm{LPic}(\mathcal{X}/S)$ be lattice line bundles on X over S. Then, the map

$$\begin{aligned} S_{\mathrm{GVF}} &\to \mathbb{R} \\ s &\mapsto \widehat{\deg}(\overline{\mathcal{L}}_0(s), \dots, \overline{\mathcal{L}}_d(s)|\mathcal{X}_s) \end{aligned}$$

is continuous.

Proof. By Remark 2.7 (continuity with respect to the constructible topology) we may without loss of generality assume that S is an affine variety and the line bundles are simple over the base S.

Fix a net $(s_i)_i \in S_{\mathrm{GVF}}$ and $s \in S_{\mathrm{GVF}}$ such that $s_i \to s$. Using the notation from Lemma 3.17, put

$$I_{n,i} = \frac{1}{n^{d+1}} \mathrm{ht}(R_n(s_i)), \quad I_n = \frac{1}{n^{d+1}} \mathrm{ht}(R_n(s)),$$

$$I_i = \lim_n \frac{1}{n^{d+1}} \mathrm{ht}(R_n(s_i)), \quad I = \lim_n \frac{1}{n^{d+1}} \mathrm{ht}(R_n(s)).$$

We need to show that $\lim_i I_i = I$. Pick a positive number ε. First, note that there is a natural number n such that

$$|I - I_n| < \varepsilon$$

and

$$|I_i - I_{n,i}| < \varepsilon$$

for all i. Indeed, this is possible because by Remark 3.16 the above differences are bounded by $\frac{\log n}{n}$ times an absolute multiple of $(1 + \nu(\Omega_\infty)) \max_i k_i \delta_i$. Next, note that for i big enough we have

$$|I_{n,i} - I_n| < \varepsilon$$

because for a fixed n, $\mathrm{ht}(R_n(y))$ is a continuous function on S_{GVF}. Hence together we get

$$|I - I_i| < 3\varepsilon,$$

which finishes the proof as $\varepsilon > 0$ was arbitrary. □

3.4. **Integrable divisors over globally valued fields.** For applications, it is useful to consider not only lattice line bundles, but also to allow certain limit metrics. We will define global line bundles and globally semipositive line bundles over a GVF in the spirit of Zhang line bundles. Let $\mathcal{X} \to S$ be a projective morphism of finite type schemes over a GVF K.

Definition 3.19. A lattice line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ over S is called *semipositive* if for every $s \in S_{\mathrm{GVF}}$ the family of metrics $\varphi|_{\mathcal{X}_s}$ consists of semipositive metrics over almost all places $\omega \in M_{\kappa(s)}$.

Definition 3.20. Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$ over S. For every compact set $C \subset S_{\mathrm{GVF}}$, we define a pseudometric d_C on the space of metrics on $\mathcal{L}$. Let s be a section of $\mathcal{L}$ and let ϕ and ψ be two families of metrics on $\mathcal{L}$. Then, we define

$$d_C(\phi, \psi) = \sup_{z \in C} \int^+_{\Omega_{\kappa(z)}} \sup_{x \in \mathcal{X}^{\mathrm{an}}_{s,\omega}} |\log|s|_\phi - \log|s|_\psi| \nu(d\omega).$$

Note that the set of (semipositive) adelic divisors is closed under this norm. The norm allows us to define notions of global and integrable divisors on $\mathcal{X}$ over S.

Definition 3.21. A global line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ over S is defined to be a line bundle $\mathcal{L}$ with a metric family ϕ such that there is a sequence of lattice line bundles $\overline{\mathcal{L}}_k = (\mathcal{L}, \phi_k)$ such that

$$\lim_{k\to\infty} d_C(\phi_k, \phi) = 0$$

for every compact $C \subset S_{\mathrm{GVF}}$.

It is called globally semipositive if the metric families ϕ_k can be chosen semipositive. A global line bundle $\overline{\mathcal{L}}$ is called globally integrable if there are globally semipositive line bundles $\overline{\mathcal{L}}_+$ and $\overline{\mathcal{L}}_-$ and an isometry $\overline{\mathcal{L}} \cong \overline{\mathcal{L}}_+ \otimes \overline{\mathcal{L}}_-$.

We denote the isometry classes of global $\mathbb{Q}$-line bundles by $\widehat{\mathrm{Pic}}_{\mathbb{Q}}(\mathcal{X}/S)$. The subgroup of integrable line bundles is denoted by $\widehat{\mathrm{Pic}}_{\mathbb{Q}}^{\mathrm{int}}(\mathcal{X}/S)$. If $S = \operatorname{Spec} K$, we often omit it in the notation. We furthermore denote the distance between two metric families by d.

Proposition 3.22. Let X be a projective scheme of pure dimension d over a countable GVF K. The intersection product on lattice divisors extends to a pairing

$$\widehat{\mathrm{Pic}}_{\mathbb{Q}}^{\mathrm{int}}(X)^{d+1} \to \mathbb{R}.$$

Proof. By linearity, it suffices to construct the pairing for globally semipositive divisors. It suffices to show that on the set of semipositive lattice divisors the intersection product is continuous with respect to d.

Let $\overline{L} = (\mathcal{O}, \phi)$ be a lattice line bundle with trivial underlying line bundle and $d(\phi, 0) = C$ and let $\overline{L}_1, \dots, \overline{L}_d \in \mathrm{LPic}_{\mathbb{Q}}{}^+(X)$ be semipositive lattice line bundles. Then,

$$|\overline{L} \cdot \overline{L}_1 \cdots \overline{L}_d| \leq C \deg(\overline{L}_1 \cdots \overline{L}_d).$$

This can be read off from the interpretation of the intersection number as the integral over local heights

$$\begin{aligned}
|\overline{L} \cdot \overline{L}_1 \cdots \overline{L}_d| &= |\int_\Omega \int_{X_\omega^{\mathrm{an}}} \log|1|_{\phi,\omega}\; c_1(\overline{L}_{1,\omega}) \cdots c_1(\overline{L}_{d,\omega}) \nu(d\omega)| \\
&\leq \int_\Omega^+ \sup_{x \in X_\omega^{\mathrm{an}}} |\log|1|_{\phi,\omega}(x)| \deg(\overline{L}_1 \cdots \overline{L}_d) \nu(d\omega) \\
&\leq C \deg(\overline{L}_1 \cdots \overline{L}_d).
\end{aligned}$$

□

We are finally in the position to prove Theorem 3.1. We restate it for convenience.

Theorem 3.23. Let $\mathcal{X} \to S$ be a flat projective morphism of finite type schemes over a GVF K of relative dimension d. Let $\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d \in \widehat{\mathrm{Pic}}_{\mathbb{Q}}^{\mathrm{int}}(\mathcal{X}/S)$ be globally integrable divisors on $\mathcal{X}$ over S. Then, the map

$$\begin{aligned}
S_{\mathrm{GVF}} &\to \mathbb{R} \\
s &\mapsto \widehat{\deg}(\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d | \mathcal{X}_s)
\end{aligned}$$

is continuous.

Proof. We assume by linearity that $\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d$ are all semipositive. Let $C \subset S_{\mathrm{GVF}}$ be a compact subset. For $k \in \mathbb{N}$, let $\overline{\mathcal{L}}_0^k, \dots, \overline{\mathcal{L}}_d^k$ be semipositive lattice line bundles such that $d_C(\overline{\mathcal{L}}_i^k, \overline{\mathcal{L}}_i)$ converges to zero. Then, the functions $\widehat{\deg}(\overline{\mathcal{L}}_0^k, \dots, \overline{\mathcal{L}}_d^k | \mathcal{X}_s)$ converge uniformly to $\widehat{\deg}(\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d | \mathcal{X}_s)$ on C. Since the former are continuous by Proposition 3.18, the latter is, too. We are done since S_{GVF} is locally compact. □

For applications the following theorem is crucial. It follows from [Zha95a], but we have not found a suitable reference for the precise statement we need. Our reasoning uses some techniques from the proof of the arithmetic Demailly theorem in [QY23] or from [Cha17]. Note that we have the notion of semipositivity and integrability for both Zhang line bundles and global line bundles and that they are a priori different.

Theorem 3.24. Every integrable Zhang line bundle $\overline{L} = (L, \phi)$ on a projective variety over a number field is induced by an integrable global divisor.

Proof. We prove that arithmetically ample divisors are induced by integrable global divisors. Arithmetically ample divisors in turn are dense in semipositive Zhang divisors allowing us to finish the proof.

Definition 3.25. A hermitian line bundle $\overline{\mathcal{L}}$ over an arithmetic variety $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ is called arithmetically ample if

(1) $\mathcal{L}_{\mathbb{Q}}$ is ample,
(2) the metrics on $\overline{\mathcal{L}}$ are semipositive at each place,
(3) the height $\widehat{c}_1(\overline{\mathcal{L}}|_{\mathcal{Y}})^{\dim \mathcal{Y}} > 0$ for every irreducible horizontal subvariety $\mathcal{Y} \subseteq \mathcal{X}$.

A Zhang divisor is called arithmetically ample if it is induced by an arithmetically ample Hermitian line bundle.

We want to approximate an arithmetically ample arithmetic divisor $\overline{L}$ defined on the model $\mathcal{X}$ from below. For this we apply the maps $\iota_n : \mathcal{X} \to \mathbb{P}(H^0(nL))$. We endow the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(H^0(nL))$ with the metric h_n induced by the supremum norm on $H^0(nL)$. Semipositivity implies precisely that the induced metrics on L converge uniformly to $\overline{L}$ by [Zha95a, Theorem 3.5]. This is always semipositive at all places and the underlying line bundle is ample. It is arithmetically ample since the subspace of integral sections in $H^0(nL)$ has a basis $s_1, \dots, s_N$ consisting of strictly small integral sections, at least for n large enough. We are reduced to proving the claim for $(\mathcal{O}(1), h_n)$ on $\mathbb{P}(H^0(nL))$. At finite places, the metric on $\mathcal{O}(1)$ agrees with the Weil metric for the basis $s_1 \dots, s_N$ (as in [Sza23, Claim 3.1.15]). It remains to show the approximation at the infinite place. We can approximate the metric h_n by a smooth metric with everywhere positive curvature.

From now on we assume that L induces $\mathcal{O}(1)$ on some projective space $\mathbb{P}^n$ with Weil metrics at finite places and a smooth metric with everywhere positive curvature at ∞. By Dini's theorem it suffices to prove pointwise approximation, i.e. for every point $x \in X(\mathbb{C})$ and $\epsilon > 0$ there exists an integer $N > 0$ and a small integral section $s \in H^0(\mathcal{O}(N))$ such that $-\frac{1}{N} \log |s(x)| < \epsilon$.

We prove first that for arbitrarily big N we can find $l \in H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{O}(N))$ satisfying $-\log |l|_{\sup} \geq \epsilon$ and $-\frac{1}{N} \log |l(x)| < 2\epsilon$. Let $\overline{x}$ be the complex conjugate of x. We apply [Zha95a, Theorem 2.2] to $\mathcal{O}(1)$ and $Y = \{x, \overline{x}\}$ to obtain a holomorphic section s of $\mathcal{O}(N)$ with $-\log |s|_{\sup} = 0$ and $-\frac{1}{N} \log |s_N(x)|, -\frac{1}{N} \log |s_N(\overline{x})| < \epsilon/2$. The section $s_N \otimes \overline{s_N} \in H^0(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(2N))$ is then a section $l \in H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{O}(2N))$ satisfying $-\log |l|_{\sup} \geq 0$ and $-\frac{1}{2N} \log |l(x)| < \epsilon$. Rescaling proves the claim.

The vector space $H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{O}(N))$ has a norm given by the supremum norm on $X(\mathbb{C})$. The global sections over $\mathbb{Z}$ form a lattice $\Lambda_N = H^0(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{O}(N))$. By [Zha95a, Theorem 4.2], there exists $0 < r < 1$ such that for big enough N, there is a basis of Λ_N consisting of vectors of norm $< r^N$. For $r < r' < 1$, and big enough N it follows that for every $l \in H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{O}(N))$ there exists $l' \in \Lambda_N$ with $|l - l'|_{\sup} < (r')^N$. We apply this to the section l constructed in the previous paragraph. Then, l' eventually satisfies $|l'|_{\sup} \leq 1$. Furthermore, for small enough ϵ in the construction of l we can ensure $-\log |r'| > 2\epsilon$. Then, for big enough N we have $-\frac{1}{N} \log |l'(x)| < 3\epsilon$ proving the theorem. □

Let us present a result that allows to calculate the adelic intersection product over a GVF structure that comes from a polarisation, due to Chen and Moriwaki.

Let $(S, \overline{H}_1, \ldots, \overline{H}_n)$ be a polarisation inducing a GVF structure on $F = \mathbb{Q}(S)$. Let X be a d-dimensional projective variety over F which is the generic fiber of a projective morphism $\pi : \mathcal{X} \to S$. Fix globally integrable line bundles $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_d \in \widehat{\mathrm{Pic}}^{\mathrm{int}}_{\mathbb{Q}}(\mathcal{X}/S)_{\mathbb{Q}}$ and denote by $\overline{L}_0, \ldots, \overline{L}_d$ their restriction to $\widehat{\mathrm{Pic}}^{\mathrm{int}}_{\mathbb{Q}}(X)_F$.

Proposition 3.26. [CM21, Proposition 4.5.1] The following equality holds:

$$\widehat{\deg}(\overline{L}_0, \ldots, \overline{L}_d | X) = \overline{\mathcal{L}}_0 \cdot \ldots \cdot \overline{\mathcal{L}}_d \cdot \pi^* \overline{H}_1 \cdot \ldots \cdot \pi^* \overline{H}_n,$$

where the left hand side is the adelic intersection product over the globally valued field F and the right hand side is the arithmetic/Arakelov intersection product of integrable Zhang divisors on $\mathcal{X}$.

Proof. The case of the polarization and the line bundles $\overline{L}_i$ being defined on a model over $\mathbb{Z}$ is [CM21, Proposition 4.5.1]. Our version follows from continuity of the intersection product on semipositive line bundles and continuity of the intersection number in families, cf. Theorem 3.1. □

4. Average intersections

Let $f_1, \ldots, f_m$ be Laurent polynomials in n variables with coefficients in a number field K. Each f_i defines a hypersurface V_i inside a proper toric variety T with torus $\mathbb{T} = \mathbb{G}_m^n \subset T$. Let $(u_{1,j}, \ldots, u_{m,j})_j$ be a generic sequence of small points in $\mathbb{T}^m$ with respect to the Weil height on $\mathbb{T}^m \subset \mathbb{P}^{nm}$. For integrable Zhang divisors $\overline{D}_0, \ldots, \overline{D}_{n-m}$ on T we want to compute

$$\lim_{j \to \infty} \widehat{\deg}(\overline{D}_0, \ldots, \overline{D}_{n-m} | u_{1,j} V_1 \cap \cdots \cap u_{m,j} V_m).$$

Denote the coordinates of the i-th factor of $\mathbb{G}_m^n$ by $w_{1,i}, \ldots, w_{n,i}$. We let $V \subset T \times \mathbb{T}^m$ be the intersection of the vanishing loci of $f_i(z_1 w_{1,i}^{-1}, \ldots, z_n w_{n,i}^{-1})$. We note that under $V \to \mathbb{T}^m$ the generic fibre has dimension $n - m$. The map $V \to \mathbb{T}^m$ is flat of relative dimension $n - m$ over a dense Zariski open $U \subseteq \mathbb{T}^m$. We define global line bundles $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_{n-m} \in \widehat{\mathrm{Pic}}^{\mathrm{int}}_{\mathbb{Q}}(V/U)$ by pulling back $\mathcal{O}(\overline{D}_0), \ldots, \mathcal{O}(\overline{D}_{n-m})$ to $T \times \mathbb{T}^m$ and restricting to V. This makes sense by Theorem 3.24. By Theorem 3.1, the map

$$\begin{aligned} U_{\mathrm{GVF}} &\to \mathbb{R} \\ u &\mapsto \widehat{\deg}(\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_d | \mathcal{X}_u) \end{aligned}$$

is continuous. We note that on $(u_1,\ldots,u_m)\in U(\overline{K})$ the above map is given by

$$(u_1,\ldots,u_m)\mapsto \widehat{\deg}(\overline{D}_0,\ldots,\overline{D}_{n-m}|u_1V_1\cap\cdots\cap u_mV_m).$$

By Corollary 2.15, a generic small sequence $(u_{1,j},\ldots,u_{m,j})_j$ in $\mathbb{T}^m$ has the corresponding points on U_{GVF} converging to $K(w_{1,1},\ldots,w_{n,1},\ldots,w_{1,m},\ldots,w_{n,m})$ with the polarized GVF structure associated to $(\prod_{i=1}^m \mathbb{P}^n, \pi_1^*\overline{\mathcal{O}(1)}^n,\ldots,\pi_m^*\overline{\mathcal{O}(1)}^n)$. Denote this limit point by $\eta^{\mathrm{can}}\in U_{\mathrm{GVF}}$.

Since the right hand side is the intersection over a polarized GVF it can be computed up to birational modification as the height of $V\subset T\times\prod_{i=1}^m\mathbb{P}^n$ with respect to $\pi_h^*\overline{D}_0\ldots\pi_h^*\overline{D}_{n-m}\cdot(\sum\pi_i^*\overline{\mathcal{O}(1)})^{nm}$ by Proposition 3.26. In other words, we get the following.

Lemma 4.1. Suppose that $\overline{D}_0,\ldots,\overline{D}_{n-m}$ are integrable Zhang divisors on T over K. Then,

$$\lim_{j\to\infty}\widehat{\deg}(\overline{D}_0,\ldots,\overline{D}_{n-m}|u_{1,j}V_1\cap\cdots\cap u_{m,j}V_m)$$
$$=\widehat{\deg}(\pi_h^*\overline{D}_0\ldots\pi_h^*\overline{D}_{n-m}\cdot\pi_1^*\overline{\mathcal{O}(1)}^n\cdots\pi_m^*\overline{\mathcal{O}(1)}^n|V).$$

On the right hand side, the $\overline{D}_i$ should be viewed as adelic divisors pulled back to F.

We need to be slightly careful when applying Fubini's theorem in the non-Archimedean setting. This is because of the failure of $(X\times Y)^{\mathrm{an}}=X^{\mathrm{an}}\times Y^{\mathrm{an}}$. In our setting, there is a preferred very affine chart given by the torus in the toric variety on which we may apply Fubini.

We sketch an argument that one can always apply Fubini in such situations. This is based on [Sto21, Proposition 3.4.21]. If $\alpha\in A^{(\dim X,\dim X)}(X^{\mathrm{an}})$ and $\beta\in A^{(\dim Y,\dim Y)}(Y^{\mathrm{an}})$ are smooth forms that are defined on very affine charts of integration $U\subseteq X$ and $V\subseteq Y$. Then, $U\times V\subseteq X\times Y$ is a very affine chart of integration for $\pi_X^*\alpha\wedge\pi_Y^*\beta$. Furthermore, $\mathrm{trop}(U\times V)=\mathrm{trop}(U)\times\ \mathrm{trop}(V)$ by Rosenlicht's theorem. Then, one needs to prove that one has the product measure on each polyhedron in the tropicalization. This is done by adapting [Sto21, Lemma 3.4.16]. The general case follows by approximation. We refer to [Gub16] for an introduction to the theory of forms in the non-Archimedean setting.

Theorem 4.2. Let $\mathcal{R}_i$ denote the Ronkin divisor associated to $g_i=f_i(z_1w_{1,i}^{-1},\ldots,z_nw_{n,i}^{-1})$ on a suitable toric blowup X of $T\times\prod_{i=1}^m\mathbb{P}^n$. Let $\tilde{V}$ denote the common vanishing locus of the g_i. Then, we have an identity

$$\widehat{\deg}(\mathcal{R}_1\ldots\mathcal{R}_m\pi_h^*\overline{D}_0\ldots\pi_h^*\overline{D}_{n-m}\cdot\pi_1^*\overline{\mathcal{O}(1)}^n\cdots\pi_m^*\overline{\mathcal{O}(1)}^n|X)$$
$$=\widehat{\deg}(\pi_h^*\overline{D}_0\ldots\pi_h^*\overline{D}_{n-m}\cdot\pi_1^*\overline{\mathcal{O}(1)}^n\cdots\pi_m^*\overline{\mathcal{O}(1)}^n|\tilde{V}).$$

Proof. In order not to overburden notation we omit the superscript denoting the analytification.

Let s_i denote the distinguished section of $\mathcal{O}(\mathcal{R}_i)$. The sections g_is_i of $\mathcal{O}(\mathcal{R}_i)$ have common vanishing locus $\tilde{V}$. By the iterative definition of the height, the equality of intersection numbers is equivalent to the vanishing of the integrals occurring in the height computations. These are of the form

$$\int_{\mathrm{div}(g_1)\cap\cdots\cap\mathrm{div}(g_{r-1})}\log|g_rs_r|c_1(\mathcal{R}_{r+1})\ldots c_1(\mathcal{R}_m)\pi_h^*c_1(\overline{D}_0)\ldots\pi_h^*c_1(\overline{D}_{n-m})\prod_{i=1}^m\pi_i^*c_1(\overline{\mathcal{O}(1)}).$$

Let us write ω for $c_1(\mathcal{R}_{r+1})\dots c_1(\mathcal{R}_m)\pi_h^*c_1(\overline{D}_0)\dots\pi_h^*c_1(\overline{D}_{n-m})\prod_{i\neq r}\pi_i^*c_1(\overline{\mathcal{O}(1)})$. To each of them we can apply Fubini to obtain

$$\begin{aligned}
&\int_{\mathrm{div}(g_1)\cap\cdots\cap\mathrm{div}(g_{r-1})\cap X}\log|g_rs_r|\pi_r^*c_1(\overline{\mathcal{O}(1)})^n\omega\\
&=\int_{\left(\mathrm{div}(g_1)\cap\cdots\cap\mathrm{div}(g_{r-1})\cap(\mathbb{T}\times\prod_{i=1}^{r-1}\mathbb{T})\right)\times\mathbb{T}\times\prod_{i=r+1}^m\mathbb{T}}\log|g_rs_r|\pi_r^*c_1(\overline{\mathcal{O}(1)})^n\omega\\
&=\int_{\left(\mathrm{div}(g_1)\cap\cdots\cap\mathrm{div}(g_{r-1})\cap(\mathbb{T}\times\prod_{i=1}^{r-1}\mathbb{T})\right)\times\prod_{i=r+1}^m\mathbb{T}}\left(\int_{\mathbb{T}}\log|g_rs_r|c_1(\overline{\mathcal{O}(1)})^n\right)\omega\\
&=\int_{\left(\mathrm{div}(g_1)\cap\cdots\cap\mathrm{div}(g_{r-1})\cap(\mathbb{T}\times\prod_{i=1}^{r-1}\mathbb{T})\right)\times\prod_{i=r+1}^m\mathbb{T}}\left(\int_{\mathbb{T}}\log|g_rs_r|d\sigma_0(x)\right)\omega.
\end{aligned}$$

The first equality follows since a Zariski closed subset with empty interior is a nullset with respect to a measure associated to differential forms.

We claim that the inner integral $\int_{\mathbb{T}}\log|g_rs_r|\sigma_0(x)$ vanishes at each fibre. Recall that $g_r(t,t_1,\dots,t_m)=f_r(tt_r^{-1})$. Let π_r denote the projection from $\mathbb{T}\times\prod_{i=1}^m\mathbb{T}$ to the r-th component $\mathbb{T}_r$ in the second factor.

We compute

$$\begin{aligned}
\log|s_r(t,t_1,\dots,t_m)|&=\int_{\mathrm{trop}^{-1}(\mathrm{trop}(t,t_1,\dots,t_m))}-\log|g_r(x)|d\sigma_{(t,t_1,\dots,t_m)}(x)\\
&=\int_{\mathrm{trop}^{-1}(\mathrm{trop}(t,t_r))\subset(\mathbb{T}\times\mathbb{T}_r)^{\mathrm{trop}}}-\log|f_r(xx_r^{-1})|d\sigma_{(t,t_r)}(x,x_r)\\
&=\int_{\mathrm{trop}^{-1}(\mathrm{trop}(tt_r^{-1}))\subset\mathbb{T}_r^{\mathrm{trop}}}-\log|f_r(x)|d\sigma_{tt_r^{-1}}(x)=\rho_r(tt_r^{-1}).
\end{aligned}$$

Here, ρ_r denotes the Ronkin function for f_r and $\mathbb{T}_r$ denotes the r-th factor of the torus.

Consider the fibre over an element

$$(t,t_1,\dots,t_{r-1},t_{r+1},\dots,t_m)\in\left(\mathrm{div}(g_1)\cap\cdots\cap\mathrm{div}(g_{r-1})\cap(\mathbb{T}\times\prod_{i=1}^{r-1}\mathbb{T})\right)\times\prod_{i=r+1}^m\mathbb{T}.$$

Over this fibre we evaluate the integral

$$\begin{aligned}
\int_{\mathbb{T}}\log|g_r(t,t_1,\dots,t_m)s_r(t,t_1,\dots,t_m)|\sigma_0(t_r)&=\int_{\mathbb{T}}\log|f_r(tt_r^{-1})|+\rho_r(tt_r^{-1})\sigma_0(t_r)\\
&=-\rho_r(t)+\rho_r(t)=0.
\end{aligned}$$

□

Lemma 4.3. Let R_i denote the Ronkin line bundle associated to f_i and suppose it is already defined on T and assume $\overline{D}_0,\dots,\overline{D}_{n-m}$ are toric. Let X denote a suitable toric blow-up of $T\times\prod_{i=1}^m\mathbb{P}^n$ over which $\mathcal{R}_i$ are defined. We have an equality of intersection numbers

$$\widehat{\deg}(\mathcal{R}_1\dots\mathcal{R}_m\pi_h^*\overline{D}_0\dots\pi_h^*\overline{D}_{n-m}\cdot\pi_1^*\overline{\mathcal{O}(1)}^n\cdots\pi_m^*\overline{\mathcal{O}(1)}^n|X)$$

$$=\widehat{\deg}(R_1\dots R_m\overline{D}_0\dots\overline{D}_{n-m}|T)$$

Proof. By linearity, we assume that $\overline{D}_0, \dots, \overline{D}_{n-m}$ are all semipositive. Then, we interpret the left hand side in a combinatorial manner as in [BPS14, Theorem 5.2.5]. We then apply Lemma 2.27. The occurring polytopes are m times n copies of the unit simplex, one set of copies for each factor in $\prod_{i=1}^m \mathbb{P}^n$. It is immediate to see that the pushforward of the roof functions of the $\pi_h^*\overline{D}_i$ yield precisely the roof functions of the $\overline{D}_i$. Similarly, the pushforward of the roof function of $\mathcal{R}_i$ is the roof function of R_i by Lemma 2.25. □

Theorem 4.4. Let $f_1, \dots, f_m$ be Laurent polynomials in n variables with coefficients in a number field K and let T be a proper toric variety with torus $\mathbb{T} = \mathbb{G}_m^n \subset T$. Suppose that $NP(f_i)$ define divisors on T. Denote by V_i the hypersurface defined by f_i. Let $(\zeta_{1,j}, \dots, \zeta_{m,j})_j$ be a generic sequence of small points in $\mathbb{T}^m$ with respect to the Weil height and $\overline{D}_0, \dots, \overline{D}_{n-m}$ be integrable toric Zhang divisors on T. Then,

$$\lim_{j\to\infty} \widehat{\deg}(\overline{D}_0, \dots, \overline{D}_{n-m} \mid \zeta_{1,j}V_1 \cap \dots \cap \zeta_{m,j}V_m) = \widehat{\deg}(R_1 \dots R_m \overline{D}_0 \dots \overline{D}_{n-m}|T).$$

Proof. This is a combination of Lemma 4.1, Theorem 4.2 and Lemma 4.3. □

We finally prove Conjecture 6.4.4. in [Gua18a].

Theorem 4.5. Let $f_1, \dots, f_m$ be Laurent polynomials in n variables with coefficients in a number field K and let T be a proper toric variety with torus $\mathbb{T} = \mathbb{G}^n \subset T$. Denote by V_i the hypersurface defined by f_i and by ρ_i its Ronkin function. Let $(\zeta_{1,j}, \dots, \zeta_{m,j})_j$ be a generic sequence of small points in $\mathbb{T}^m$ for the Weil height and let $\overline{D}_0, \dots, \overline{D}_{n-m}$ be semipositive toric Zhang divisors on T with associated local roof functions $\theta_{0,v}, \dots, \theta_{n-m,v}$. Then,

$$\begin{aligned}&\lim_{j\to\infty} \widehat{\deg}(\overline{D}_0, \dots, \overline{D}_{n-m} \mid \zeta_{1,j}V_1 \cap \dots \cap \zeta_{m,j}V_m) \\ &= \sum_{v\in \mathrm{M}_K} n_v \,\mathrm{MI}_M(\theta_{0,v}, \dots, \theta_{n-m,v}, \rho_1^\vee, \dots, \rho_m^\vee).\end{aligned}$$

Proof. We apply the projection formula to restrict to the case, where the $NP(f_i)$ define divisors on T. Then, the conjecture follows from Theorem 4.4 and Theorem 2.21. □

Appendix A. Mahler measures on complex polynomials

In this appendix, we study some measures of complexity of complex polynomials. For a nonzero $P \in \mathbb{C}[X_1, \dots, X_n]$, we define the logarithmic Mahler measure

$$m(P) = \int_{[0,1]^n} \log |P(e^{2i\pi t_1}, \dots, e^{2i\pi t_n})| dt_1 \dots dt_n,$$

and the logarithmic Fubini-Study Mahler measure

$$m_{\mathbb{S}_n}(P) = \int_{\mathbb{S}_n} \log |P(z_1, \dots, z_n)| d\eta_n(z_1, \dots, z_n),$$

where $\mathbb{S}_n$ is the unit sphere in $\mathbb{C}^n$ for the usual Euclidean norm, and η_n is the spherical measure on $\mathbb{S}_n$, normalized so that $\eta_n(\mathbb{S}_n) = 1$.

In [Lel94], Pierre Lelong studied these two measures and gave a bound for the distance between them in terms of n and the degree of P.

In this appendix, we prove an analogue of Lelong's result in the space of polynomials $\mathbb{C}[\overline{X}_1, \ldots, \overline{X}_n]$, where each $\overline{X}_i$ is a tuple of abstract variables of length m_i. More precisely, we define the mixed Fubini-Study Mahler measure by

$$m_{\mathbb{S}_{m_1}\times\ldots\times\mathbb{S}_{m_n}}(P) = \int_{\mathbb{S}_{m_1}\times\ldots\mathbb{S}_{m_n}} \log|P(\overline{z}_1, \ldots, \overline{z}_n)| d\eta_{m_1}(\overline{z}_1) \wedge \ldots \wedge d\eta_{m_n}(\overline{z}_n).$$

Our goal is to prove the following proposition, where the norm $\|\cdot\|$ on $\mathbb{C}[X_1, \ldots, X_n]$ assigns to a polynomial the maximum absolute value of its coefficients.

Proposition A.1. Let $P \in \mathbb{C}[\overline{X}_1, \ldots, \overline{X}_n]$ be a nonzero polynomial, where each $\overline{X}_i$ is a tuple of abstract variables of length m_i. For all $i \leq n$, let d_i be the degree of P in $\overline{X}_i$. Then,

$$\left|m_{\mathbb{S}_{m_1}\times\ldots\times\mathbb{S}_{m_n}} - \log\|P\|\right| \leqslant \sum_{i=1}^{n} d_i \left(\log(m_i+1) + \frac{1}{2}\sum_{k=1}^{m_i-1}\frac{1}{k}\right).$$

Let us start with the simpler case where each m_i is equal to 1. Let $P \in \mathbb{C}[X_1, \ldots, X_n]$ be a nonzero polynomial of degree d.

Lemma A.2. Let

$$S(P) := \sup\{|P(z_1, \ldots, z_n)| \,:\, (z_1, \ldots, z_n) \in \mathbb{C}^n \text{ and } |z_i| \leqslant 1 \text{ for all } i\}.$$

Then,

$$\|P\| \leqslant S(P) \leqslant \binom{n+d}{n}\|P\|.$$

Proof. The rightmost inequality follows from the fact that a polynomial of degree d has at most $\binom{n+d}{n}$ nonzero coefficients.

For the other inequality, consider the integral

$$\begin{aligned}
I &= \int_{[0,1]^n} |P(e^{2i\pi t_1}, \ldots, e^{2i\pi t_n})|^2 \, dt_1 \ldots dt_n \\
&= \int_{[0,1]^n} P(e^{2i\pi t_1}, \ldots, e^{2i\pi t_n})\overline{P(e^{2i\pi t_1}, \ldots, e^{2i\pi t_n})} \, dt_1 \ldots dt_n \\
&= \int_{[0,1]^n} \left(\sum_{k,l\in\mathbb{N}^n} a_k\overline{a_l}\exp\left(2i\pi(k_1-l_1)t_1 + \ldots + 2i\pi(k_n-l_n)t_n\right)\right) dt_1 \ldots dt_n \\
I &= \sum_{k,l\in\mathbb{N}^n} a_k\overline{a_l}\int_{[0,1]^n} \exp\left(2i\pi(k_1-l_1)t_1 + \ldots + 2i\pi(k_n-l_n)t_n\right) \, dt_1 \ldots dt_n.
\end{aligned}$$

Now, for $k, l \in \mathbb{N}^n$ with $k \neq l$, there exists $1 \leqslant r \leqslant n$ with $k_r - l_r \neq 0$. So,

$$\begin{aligned}
&\int_{[0,1]^n} \exp\left(2i\pi(k_1-l_1)t_1 + \ldots + 2i\pi(k_n-l_n)t_n\right) \, dt_1 \ldots dt_n \\
&\qquad = \int_{[0,1]^{n-1}} \exp\left(\sum_{s\neq r} 2i\pi(k_s-l_s)t_s\right)\left(\int_0^1 e^{2i\pi(k_r-l_r)t_r}dt_r\right)\prod_{s\neq r} dt_s = 0,
\end{aligned}$$

and for $k = l$,

$$\int_{[0,1]^n} \exp\left(2i\pi(k_1-l_1)t_1 + \ldots + 2i\pi(k_n-l_n)t_n\right) \, dt_1 \ldots dt_n = 1.$$

So, finally

$$I = \sum_{k\in\mathbb{N}} |a_k|^2 \geqslant \|P\|^2.$$

But we also have

$$I = \int_{[0,1]^n} |P(e^{2i\pi t_1},\dots,e^{2i\pi t_n})|^2\, dt_1 \dots dt_n \leqslant S(P)^2.$$

Hence, $\|P\| \leqslant S(P)$. □

Remark A.3. In particular, Lemma A.2 implies that

$$S(P) = \lim_{m\to+\infty} \|P^m\|^{1/m}.$$

Lemma A.4. Assume that $n = 1$, i.e. $P = \sum_{k=0}^{d} a_k X^k \in \mathbb{C}[X]$. Then, for all $0 \leqslant k \leqslant d$:

$$|a_k| \leqslant \binom{d}{k} \exp(m(P))$$

Proof. Since $\mathbb{C}$ is algebraically closed, we can write $P = \lambda \prod_{i=1}^{\deg P} (X - \alpha_i)$, where $\lambda \in \mathbb{C}^\times$, $\alpha_1,\dots,\alpha_{\deg P} \in \mathbb{C}$. Now, by Jensen's formula, we have for each $1 \leqslant i \leqslant \deg P$,

$$\int_0^1 \log\left|e^{2i\pi t} - \alpha_i\right| dt = \max(0, \log|\alpha_i|).$$

So, by summing,

$$m(P) = \log|\lambda| + \sum_{i=1}^{\deg P} \max(0, \log|\alpha_i|).$$

Now, let $k \leqslant \deg P$. Then, the coefficient of X^k in P is equal to

$$a_k = (-1)^{\deg P - k}\lambda \sum_{I\subseteq\{1,\dots,\deg P\}} \prod_{i\in I} \alpha_i.$$

So, by the triangle inequality

$$|a_k| \leqslant |\lambda| \sum_{I\subseteq\{1,\dots,\deg P\}} \prod_{i\in I} |\alpha_i| \leqslant \binom{\deg P}{k} |\lambda| \prod_{i=1}^{\deg P} \max(1,\alpha_i) \leqslant \binom{d}{k} \exp(m(P).$$

□

Lemma A.5. Write $P = \sum_{m\in\mathbb{N}^n} a_m X_1^{m_1} \dots X_n^{m_n}$. Then, for every $m = (m_1,\dots,m_n) \in \mathbb{N}^n$, we have

$$|a_m| \leqslant \binom{d}{m_1,\dots,m_n} \exp(m(P)).$$

Proof. We prove this inequality by induction on n. The $n = 1$ case is the result of Lemma A.4, so we may assume $n \geqslant 2$. The result is also immediate if a_m is zero, so assume it is not. Write $P = \sum_{k=0}^{d} P_k X_n^k$, where the P_k are in $\mathbb{C}[X_1,\dots,X_{n-1}]$. Then, we may write

$$m(P) = \int_{[0,1]^{n-1}} m(P(e^{2i\pi t_1},\dots,e^{2i\pi t_{n-1}}, X))dt_1 \dots dt_{n-1}.$$

By Lemma A.4, we have for all $(t_1, \dots, t_{n-1}) \in [0,1]^{n-1}$, $|P_{m_n}(e^{2i\pi t_1}, \dots, e^{2i\pi t_{n-1}})| \leqslant \binom{d}{m_n} \exp(m(P(e^{2i\pi t_1}, \dots, e^{2i\pi t_{n-1}}, X)))$. Since $P_{m_n}(e^{2i\pi t_1}, \dots, e^{2i\pi t_{n-1}})$ is nonzero for almost all $(t_1, \dots, t_{n-1})$, we may write
$m(P(e^{2i\pi t_1}, \dots, e^{2i\pi t_{n-1}}, X)) \geqslant \log|P_{m_n}(e^{2i\pi t_1}, \dots, e^{2i\pi t_{n-1}})| - \log\binom{d}{m_n}$ and integrate, yielding

$$m(P) \geqslant m(P_{m_n}) - \log\binom{d}{m_n},$$

i.e., $\exp(m(P_{m_n})) \leqslant \binom{d}{m_n}\exp(m(P))$ Since P_{m_n} has degree at most $d - m_n$, the induction hypothesis gives $|a_m| \leqslant \binom{d-m_n}{m_1,\dots,m_{n-1}}\exp(m(P_{m_n}))$. Since $\binom{d}{m_n}\binom{d-m_n}{m_1,\dots,m_{n-1}} = \binom{d}{m_1,\dots,m_n}$, this concludes. ☐

Corollary A.6.

$$|m(P) - \log\|P\|| \leqslant d\log(n+1)$$

Proof. First, it is clear that $m(P) \leqslant \log(S(P)) \leqslant \log\|P\| + \log\binom{n+d}{n}$ by Lemma A.2. A basic counting argument shows that $\binom{n+d}{n} \leqslant (n+1)^d$, hence $m(P) \leqslant \log\|P\| + d\log(n+1)$.

For the other direction, write $P = \sum_{m\in\mathbb{N}^n} a_m X_1^{m_1}\dots X_n^{m_n}$. By Lemma A.5, we have for all $m \in \mathbb{N}^n$,

$$|a_m| \leqslant \binom{d}{m_1,\dots,m_n}\exp(m(P))$$

A basic counting argument shows that $\binom{d}{m_1,\dots,m_n} \leqslant (n+1)^d$, hence by taking the maximum over all coefficients,

$$\|P\| \leqslant (n+1)^d \exp(m(P)),$$

i.e. $\log\|P\| \leqslant m(P) + d\log(n+1)$, which concludes the proof. ☐

Remark A.7. If $P \in \mathbb{C}[X_1, \dots, X_n]$ is homogeneous, we may replace $n+1$ by n in the above inequality. Indeed, evaluating in $X_n = 1$ does not change $m(P)$ and $\|P\|$, so we may replace P with a polynomial in $n-1$ variables.

Lemma A.8.

$$|m_{\mathbb{S}_n}(P) - \log\|P\|| \leqslant 2d\log(n+1).$$

Proof. It is clear from the definition that $m_{\mathbb{S}_n}(P(e^{2i\pi t_1}X_1, \dots, e^{2i\pi t_n}X_n)) = m_{\mathbb{S}_n}(P)$ for all $t_1, \dots, t_n \in [0,1]$. So, we may write

$$m_{\mathbb{S}_n}(P) = \int_{[0,1]^n} m_{\mathbb{S}_n}(P(e^{2i\pi t_1}X_1, \dots, e^{2i\pi t_n}X_n))dt_1\dots dt_n.$$

By Fubini, this is equal to

$$\int_{\mathbb{S}_n}\left(\int_{[0,1]^n}(P(z_1e^{2i\pi t_1}, \dots, z_ne^{2i\pi t_n}))dt_1\dots dt_n\right)d\eta_n(\overline{z}) = \int_{\mathbb{S}_n} m(P(z_1X_1, \dots, z_nX_n))d\eta_n(\overline{z})$$

But, by Lemma A.6, we have for all $\overline{z} \in \mathbb{S}_n$:

$$|m(P(z_1X_1, \dots, z_nX_n)) - \log\|P\|| \leqslant d\log(n+1)$$

So, by integrating:

$$\left|m_{\mathbb{S}_n}(P) - \int_{\mathbb{S}_n}\log\|P(z_1X_1, \dots, z_nX_n)\|d\eta_n(\overline{z})\right| \leqslant d\log(n+1)$$

Moreover, it is clear that for all $\overline{z} \in \mathbb{S}_n$, $\|P(z_1X_1,\dots,z_nX_n)\| \leqslant \|P\|$, therefore $\int_{\mathbb{S}_n} \log\|P(z_1X_1,\dots,z_nX_n)\| d\eta_n(\overline{z}) \leqslant \log\|P\|$.

On the other hand, if P is written as $\sum_{s\in\mathbb{N}^n} a_s X_1^{s_1}\dots X_n^{s_n}$, we have for all s such that $a_s \neq 0$,

$$\begin{aligned}\int_{\mathbb{S}_n} \log\|P(z_1X_1,\dots,z_nX_n)\| d\eta_n(\overline{z}) &= \int_{\mathbb{S}_n} \left(\max_s \log|a_s z_1^{s_1}\dots z_n^{s_n}|\right) d\eta_n(\overline{z}) \\ &\geqslant \max_s \int_{\mathbb{S}_n} \log|a_s z_1^{s_1}\dots z_n^{s_n}| d\eta_n(\overline{z}) \\ &= \max_s \left(\log|a_s| + \sum_{i=1}^n s_i \int_{\mathbb{S}_n} \log|z_i| d\eta_n(\overline{z})\right) \\ &= \max_s \left(\log|a_s| + |s| \int_{\mathbb{S}_n} \log|z_1| d\eta_n(\overline{z})\right)\end{aligned}$$

$$\int_{\mathbb{S}_n} \log\|P(z_1X_1,\dots,z_nX_n)\| d\eta_n(\overline{z}) \geqslant \log\|P\| + d\int_{\mathbb{S}_n} \log|z_1| d\eta_n(\overline{z}).$$

It remains to show that $\int_{\mathbb{S}_n} \log|z_1| d\eta_n(\overline{z}) \geq -\log(n+1)$. In fact, we even know from [Lel94, Equation 2.28] that this integral evaluates to $-\frac{1}{2}\sum_{i=1}^{n-1}\frac{1}{k}$. □

Now, we move on to the mixed case. Fix a nonzero polynomial $P \in \mathbb{C}[\overline{X}_1,\dots,\overline{X}_n]$ and denote $d_i := \deg_{\overline{X}_i} P$. Our goal is to adapt the result of Lemma A.8 and find a bound for the distance between this measure and $\log\|P\|$ in terms of the d_i.

Lemma A.9. Again, let

$$S(P) := \sup\{|P(\overline{z_1},\dots,\overline{z}_n)| (\overline{z}_1,\dots,\overline{z}_n) \in \mathbb{C}^{m_1+\dots+m_n} \text{ and } |z_{i,j}| \leqslant 1 \text{ for all } i,j\}.$$

Then,

$$\|P\| \leqslant S(P) \leqslant \left(\prod_{i=1}^n \binom{m_i+d_i}{m_i}\right)\|P\|$$

Proof. This follows from the fact that the number of nonzero coefficients of P is at most $\prod_{i=1}^n \binom{m_i+d_i}{m_i}$, by the same argument as in the proof of Lemma A.2. □

Lemma A.10. Write

$$P = \sum_{(k_1,\dots,k_n)\in\mathbb{N}^{m_1}\times\dots\times\mathbb{N}^{m_n}} a_{k_1,\dots,k_n} \overline{X}_1^{k_1}\dots\overline{X}_n^{k_n},$$

where $\overline{X}_i^{k_i} := \prod_{j=1}^{m_i} X_{i,j}^{k_{i,j}}$. Then, for every $k = (k_1,\dots,k_n) \in \mathbb{N}^{m_1}\times\dots\times\mathbb{N}^{m_n}$, we have

$$|a_{k_1,\dots,k_n}| \leqslant \left(\prod_{i=1}^n \binom{d_i}{k_{i,1},\dots,k_{i,m_i}}\right)\exp(m(P)).$$

Proof. The proof is a straightforward induction on n, based on Lemma A.5. □

Corollary A.11.

$$|m(P) - \log\|P\|| \leqslant \sum_{i=1}^{n} d_i \log(m_i + 1)$$

Proof. First, it is clear that $m(P) \leqslant \log(S(P)) \leqslant \log\|P\| + \sum_{i=1}^{n} \log \binom{m_i+d_i}{m_i}$ by Lemma A.2. As in the proof of Corollary A.6, a counting argument shows that $\binom{m_i+d_i}{m_i} \leqslant (m_i+1)^{d_i}$, so $m(P) \leqslant \log\|P\| + \sum_{i=1}^{n} d_i \log(m_i+1)$.

Then, it follows from Lemma A.10 and the fact that every multinomial coefficient $\binom{d_i}{k_1,\ldots,k_{m_i}}$ is smaller or equal to $(m_i+1)^{d_i}$, that

$$\|P\| \leqslant \left(\prod_{i=1}^{n}(m_i+1)^{d_i}\right)\exp(m(P)),$$

i.e. $\log\|P\| \leqslant m(P) + \sum_{i=1}^{n} d_i \log(m_i+1)$, which concludes the proof. □

Remark A.12. If $P \in \mathbb{C}[\overline{X}_1,\ldots,\overline{X}_n]$ is homogeneous in each of the tuples $\overline{X}_i$, we may replace m_i+1 by m_i in the above inequality. Indeed, evaluating in $X_{i,m_i} = 1$ does not change $m(P)$ nor $\|P\|$, so we may replace $\overline{X}_i$ by a (m_i-1)-tuple.

We are finally able to prove the main result of this appendix.

Proof of Proposition A.1. We first prove the inequality

$$m_{\mathbb{S}_{m_1}\times\ldots\times\mathbb{S}_{m_n}} \leqslant \log\|P\| + \sum_{i=1}^{n} d_i \log(m_i+1)$$

exactly as in the proof of Corollary A.11.

For the other inequality, we again reason by induction on n. If $n=1$, the result follows directly from Lemma A.8. So, assume $n \geqslant 2$. Write $P = \sum_{k\in\mathbb{N}^{m_n}} P_k \overline{X}_n^k$, where the P_k are in $\mathbb{C}[\overline{X}_1,\ldots,\overline{X}_{n-1}]$. Let $l \in \mathbb{N}^{m_n}$ be such that $\|\mathbb{P}_l\| = \|P\|$. Then, we have

$$m_{\mathbb{S}_{m_1}\times\ldots\times\mathbb{S}_{m_n}} = \int_{\mathbb{S}_1\times\ldots\times\mathbb{S}_{m_{n-1}}} m_{S_{m_n}}(P(\overline{z}_1,\ldots,\overline{z}_{n-1},\overline{X}))d\eta_{m_1}(\overline{z}_1)\wedge\ldots\wedge d\eta_{m_{n-1}}(\overline{z_{n-1}})$$

By Lemma A.8, we have for all $\overline{z}_1,\ldots,\overline{z}_{n-1}$,

$$\begin{aligned} m_{S_{m_n}}(P(\overline{z}_1,\ldots,\overline{z}_{n-1},\overline{X})) &\geqslant \log\|P(\overline{z}_1,\ldots,\overline{z}_{n-1},\overline{X})\| - d_n\left(\log(m_n+1) + \frac{1}{2}\sum_{k=1}^{m_n-1}\frac{1}{k}\right) \\ &\geqslant \log|P_l(\overline{z}_1,\ldots,\overline{z}_{n-1})| - d_n\left(\log(m_n+1) + \frac{1}{2}\sum_{k=1}^{m_n-1}\frac{1}{k}\right), \end{aligned}$$

so by integrating, we get

$$m_{\mathbb{S}_{m_1}\times\ldots\times\mathbb{S}_{m_n}} \geqslant m_{\mathbb{S}_{m_n}}(P_l) - d_n\left(\log(m_n+1) + \frac{1}{2}\sum_{k=1}^{m_n-1}\frac{1}{k}\right).$$

But, by induction hypothesis,

$$m_{\mathbb{S}_{m_n}}(P_l) \geqslant \log\|P_l\| - \sum_{i=1}^{n-1} d_i\left(\log(m_i+1) + \frac{1}{2}\sum_{k=1}^{m_i-1}\frac{1}{k}\right),$$

where $\|P_l\| = \|P\|$ by assumption, which concludes the proof. □

References

[Ben08] Itaï Ben Yaacov. "Continuous first order logic for unbounded metric structures". In: *Journal of Mathematical Logic* (2008). DOI: 10.1142/S0219061308000737.

[Ben+24] Itaï Ben Yaacov, Pablo Destic, Ehud Hrushovski, and Michał Szachniewicz. *Globally valued fields: foundations*. 2024. arXiv: 2409.04570 [math.LO].

[BGS94] J-B Bost, Henri Gillet, and Christophe Soulé. "Heights of projective varieties and positive Green forms". In: *Journal of the American Mathematical Society* (1994), pp. 903–1027.

[BPS14] José Ignacio Burgos Gil, Patrice Philippon, and Martín Sombra. "Arithmetic geometry of toric varieties. Metrics, measures and heights". In: *Astérisque* 360 (2014), pp. vi+222. ISSN: 0303-1179,2492-5926.

[CM00] Julien Cassaigne and Vincent Maillot. "Hauteur des hypersurfaces et fonctions Zêta d'Igusa". In: *Journal of Number Theory* 83.2 (2000), pp. 226–255. ISSN: 0022-314X. DOI: https://doi.org/10.1006/jnth.1999.2490.

[Cha21] Antoine Chambert-Loir. "Arakelov Geometry, Heights, Equidistribution, and the Bogomolov Conjecture". In: *Springer International Publishing* Arakelov Geometry and Diophantine Applications (2021). DOI: 10.1007/978-3-030-57559-5_8.

[Cha17] François Charles. "Arithmetic ampleness and an arithmetic Bertini theorem". In: *Annales scientifiques de l'École Normale Supérieure* 54 (Mar. 2017). DOI: 10.24033/asens.2488.

[CM20] Huayi Chen and Atsushi Moriwaki. *Arakelov Geometry over Adelic Curves*. Springer Singapore, 2020. DOI: https://doi.org/10.1007/978-981-15-1728-0.

[CM21] Huayi Chen and Atsushi Moriwaki. *Arithmetic intersection theory over adelic curves*. 2021. arXiv: 2103.15646 [math.AG].

[CM24] Huayi Chen and Atsushi Moriwaki. *Positivity in Arakelov Geometry over Adelic Curves: Hilbert-Samuel Formula and Equidistribution Theorem*. Springer, 2024.

[DH22] Vesselin Dimitrov and Philipp Habegger. *Galois orbits of torsion points near atoral sets*. 2022. arXiv: 1909.06051 [math.NT].

[Fal91] Gerd Faltings. "Diophantine approximation on abelian varieties". In: *Annals of Mathematics* 133.3 (1991), pp. 549–576.

[Gua18a] Roberto Gualdi. "Height of cycles in toric varieties". Theses. Université de Bordeaux ; Universitat internacional de Catalunya, Sept. 2018. URL: https://theses.hal.science/tel-01931089.

[Gua18b] Roberto Gualdi. "Heights of hypersurfaces in toric varieties". In: *Algebra Number Theory* 12.10 (2018), pp. 2403–2443. ISSN: 1937-0652,1944-7833. DOI: 10.2140/ant.2018.12.2403.

[GS23] Roberto Gualdi and Martín Sombra. *Limit heights and special values of the Riemann zeta function*. 2023. arXiv: 2304.01966 [math.NT].

[GS24] Roberto Gualdi and Martín Sombra. "Heights of codimension-2 complete intersections in toric varieties". In: *Unpublished note* (2024).

[Gub16] Walter Gubler. "Forms and Currents on the Analytification of an Algebraic Variety (After Chambert-Loir and Ducros)". In: *Nonarchimedean and Tropical Geometry*. Ed. by Matthew Baker and Sam Payne. Cham: Springer International Publishing, 2016, pp. 1–30. ISBN: 978-3-319-30945-3.

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. New York: Springer-Verlag, 1977.

[Lel94] Pierre Lelong. "Mesure de Mahler et calcul de constantes universelles pour les polynômes de N variables". In: *Mathematische Annalen* 299 (1994), pp. 673–695.

[Mai00] Vincent Maillot. "Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables". In: *Mémoires de la Société Mathématique de France* (2000).

[Mor00] Atsushi Moriwaki. "Arithmetic height functions over finitely generated fields". In: *Invent. Math.* 140.1 (2000), pp. 101–142. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s002220050358.

[PP24] Fabien Pazuki and Riccardo Pengo. "On the Northcott property for special values of L-functions". In: *Revista Matemática Iberoamericana* 40.1 (2024), pp. 1–42.

[QY23] Binggang Qu and Hang Yin. *Arithmetic Demailly Approximation Theorem*. 2023. arXiv: 2208.13230 [math.NT].

[Sto21] Martino Stoffel. *Real-valued differential forms on non-archimedean abelian varieties with totally degenerate reduction*. Jan. 2021. URL: https://epub.uni-regensburg.de/44263/.

[Sza23] Michał Szachniewicz. *Existential closedness of $\overline{\mathbb{Q}}$ as a globally valued field via Arakelov geometry*. 2023. arXiv: 2306.06275 [math.LO].

[Yua08] Xinyi Yuan. "Big line bundles over arithmetic varieties". In: *Inventiones mathematicae* 173 (Sept. 2008). DOI: 10.1007/s00222-008-0127-9.

[YZ24] Xinyi Yuan and Shouwu Zhang. *Adelic line bundles on quasi-projective varieties*. 2024. arXiv: 2105.13587 [math.NT].

[Zha95a] Shouwu Zhang. "Positive Line Bundles on Arithmetic Varieties". In: *Journal of The American Mathematical Society* 8 (Jan. 1995), pp. 187–187. DOI: 10.2307/2152886.

[Zha95b] Shouwu Zhang. "Small points and adelic metrics". In: *Journal of Algebraic Geometry* 4.2 (1995), pp. 281–300.

Arakelov geometry of toric bundles: Okounkov bodies and BKK

ARAKELOV GEOMETRY OF TORIC BUNDLES: OKOUNKOV BODIES AND BKK

NUNO HULTBERG

Abstract. This article introduces the study of toric bundles and the morphisms between them from the perspective of adelic fibre bundles, as introduced by Chambert-Loir and Tschinkel. We study the Okounkov bodies and Boucksom-Chen transforms of suitable adelic line bundles on toric bundles. Finally, we prove an arithmetic analogue of a formula for intersection numbers due to Hofscheier, Khovanskii and Monin. We apply this to the study of compactifications of semiabelian varieties, whose height and successive minima we compute. This extends computations of Chambert-Loir to arbitrary toric compactifications.

1. Introduction

Let G be a semiabelian variety over a field K. A semiabelian variety is canonically an extension of an abelian variety A by a torus $\mathbb{T}$:

$$0 \longrightarrow \mathbb{T} \xrightarrow{i} G \xrightarrow{\pi} A \longrightarrow 0.$$

After possibly passing to a finite extension of K, we assume that $\mathbb{T}$ is split, i.e. $\mathbb{T} \cong \mathbb{G}_m^t$. We refer to $\mathbb{T}$ as the torus part of G and to A as its abelian quotient. Abstractly, we may view G as a $\mathbb{T}$-torsor over A. This perspective clarifies the canonical identification $\mathrm{Ext}^1_K(A, \mathbb{G}_m) \cong A^\vee(K)$ given by the Weil-Barsotti formula, see [Oor66, §III.8].

Many methods of algebraic and arithmetic geometry rely on the properness of the studied varieties. If $\mathbb{T}$ is not trivial, G is not proper. We obtain a compactification $\overline{G}$ of G by considering it as a $\mathbb{T}$-torsor and applying a toric compactification X of $\mathbb{T}$. This naturally places us in the setting of toric bundles. We concretize the constructions of [CT01] in this setting. Details of the upcoming discussion are contained in Section 3.

Let B be a variety over K and let $\mathcal{T}$ be a $\mathbb{T}$-torsor over B, for a split torus $\mathbb{T}$. Let $N = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ be the lattice of co-characters and $M = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ the lattice of characters of $\mathbb{T}$. A fan Σ in $N_\mathbb{R}$ defines a toric variety X_Σ containing $\mathbb{T}$. We define a toric bundle $\mathcal{X}_\Sigma = (\mathcal{T} \times X_\Sigma)/\mathbb{T}$ by using Zariski descent, where $x \in \mathbb{T}$ acts by (x, x^{-1}). The fibres of $\mathcal{X}_\Sigma \to B$ can be identified with X_Σ. Such an identification is canonical up to the action of $\mathbb{T}$. Furthermore, a $\mathbb{T}$-invariant Cartier divisor D on X_Σ gives rise to a $\mathbb{T}$-invariant Cartier divisor $\rho(D)$ on $\mathcal{X}_\Sigma$.

More precisely, there is a map $\rho : \mathrm{Div}_{\mathbb{T}}(X_\Sigma) \to \mathrm{Div}_{\mathbb{T}}(\mathcal{X}_\Sigma)$ which after restricting to a fibre identified with X_Σ yields the identity.

The torus bundle $\mathcal{T}$ induces a homomorphism $c : M \to \mathrm{Pic}(B)$ by sending $m \in M$ to $(\mathcal{T} \times \mathbb{G}_m)/\mathbb{T}$, where the action of $x \in \mathbb{T}$ is by $(x, \chi^{-m}(x))$. In fact, this defines an isomorphism between the group of isomorphism classes of $\mathbb{T}$-torsors and $\mathrm{Hom}(M, \mathrm{Pic}(B))$. If K is endowed with an absolute value or is a global field, we define metrized and adelic torus bundles. Their isomorphism classes will be isomorphic to $\mathrm{Hom}(M, \mathrm{Pic}^{\mathrm{metr}}(B))$ and $\mathrm{Hom}(M, \widehat{\mathrm{Pic}}(B))$ respectively. Given a metrized/adelic torus bundle $\widehat{\mathcal{T}}$ with underlying torus bundle $\mathcal{T}$, we define analogous constructions $\rho^{\mathrm{metr}} : \mathrm{Div}^{\mathrm{metr}}_{\mathbb{T}}(X_\Sigma) \to \mathrm{Div}^{\mathrm{metr}}_{\mathbb{T}}(\mathcal{X}_\Sigma)$ and $\widehat{\rho} : \widehat{\mathrm{Div}}_{\mathbb{T}}(X_\Sigma) \to \widehat{\mathrm{Div}}_{\mathbb{T}}(\mathcal{X}_\Sigma)$. We denote the homomorphism describing its isomorphism class by $\widehat{c} : M \to \widehat{\mathrm{Pic}}(B)$.

We may study the group structure on G by exhibiting it as a map of toric bundles, a notion we will introduce. Using the theory of toric bundles we can extend maps on G to maps to its compactifications. For instance, let $[n]$ denote the multiplication by n on G and G_Σ denote the compactification of G as a toric bundle associated to the fan Σ. Then, the morphism $[n]$ extends to an endomorphism $[n]_\Sigma$ of G_Σ. Furthermore, the multiplication on G extends to an action of G on G_Σ. The multiplication $[n]_\Sigma$ on G_Σ does not give rise to a polarized dynamical system unless G is a torus or an abelian variety. Instead, canonical heights are defined in terms of a toric contribution $\bar{\mathcal{L}}$ and a contribution from the abelian variety $\bar{\pi}^*\bar{\mathcal{M}}$. Here $\bar{\mathcal{M}}$ is a canonically metrized ample symmetric line bundle on the abelian variety A. The toric contribution is defined by endowing Cartier divisors of the form $\rho(D)$ with canonical metrics with respect to $[n]_\Sigma$, which we denote by $\rho(D)^{\mathrm{can}}$. We will prove that $\rho(D)^{\mathrm{can}}$ is isometric to $\widehat{\rho}(D^{\mathrm{can}})$, when endowing the line bundles on the abelian variety with their canonical metric. This framework enables us to conceptualize the computations in [Cha00], which serves as a major inspiration for this work.

1.1. Statement of results.

We say that K is a global field if it is a number field or the function field of a geometrically irreducible curve $\mathcal{S}$ over a field k. In the number field case, we denote by $\mathcal{S}$ the spectrum of the ring of integers. For details on notation we refer to Section 2.

Let $\widehat{\mathcal{T}}$ be an integrable torus bundle (cf. Definition 3.1.4) over a proper variety B over a global field K with associated homomorphism $\widehat{c} : M \to \widehat{\mathrm{Pic}}(B)$. Let $\Delta \subset M_\mathbb{R}$ be a rational polytope and Σ a complete rational fan that refines the normal fan of Δ. Denote the toric bundle defined by these data by $\mathcal{X}$. Let D be the toric Cartier divisor defined by Δ. Let $\overline{\Delta}$ denote the datum of a polytope Δ together with a collection (θ_v) of local roof functions on Δ with global roof function

θ. It gives rise to a semipositive toric Cartier divisor $\overline{D}$. We define $\rho(\Delta) := \rho(D)$ and $\widehat{\rho}(\overline{\Delta}) := \widehat{\rho}(\overline{D})$.

We briefly recall the notion of Zhang minima. Let $\bar{L}$ be an adelically metrized line bundle on a geometrically irreducible variety X of dimension d. For $\lambda \in \mathbb{R}$, define the set $X_{\overline{L}}(\lambda)$ to be the Zariski closure in X of the set $\{x \in X(\bar{K}) \mid h_{\overline{L}}(x) \leq \lambda\}$.

Definition 1.1.1. The i-th *Zhang minimum* of $\bar{L}$ is defined to be

$$\zeta_i(\overline{L}) = \inf\{\lambda \mid \dim X_{\overline{L}}(\lambda) \geq i-1\} \tag{1}$$

in [Zha95a]. The *absolute minimum* $\zeta_{\mathrm{abs}}(\overline{L})$ is defined to be the first Zhang minimum, i.e. the infimum over the λ such that $X_{\overline{L}}(\lambda) \neq \emptyset$. The *essential minimum* is the $(d+1)$-st Zhang minimum, i.e. the infimum over the λ such that $X_{\overline{L}}(\lambda) = X$.

We compute the essential and absolute minima of adelically metrized line bundles on toric bundles by studying their Okounkov body and Boucksom-Chen transform with respect to a suitable flag.

Theorem A. *Let $\overline{L}$ be an adelic line bundle on B such that $\overline{L} + \widehat{c}(m)$ is geometrically big for some $m \in \Delta$. Let Δ° denote the interior of Δ. Then, we have the following formula for the essential minimum of $\widehat{\rho}(\overline{\Delta}) + \pi^*\overline{L}$ on $\mathcal{X}$:*

$$\zeta_{\mathrm{ess}}(\widehat{\rho}(\overline{\Delta}) + \pi^*\overline{L}) = \sup_{m\in\Delta} \left\{\zeta_{\mathrm{ess}}(\overline{L} + \widehat{c}(m)) + \theta(m)\right\}.$$

If in addition $\overline{L} + \widehat{c}(m)$ is semipositive for all $m \in \Delta$,

$$\zeta_{\mathrm{abs}}(\widehat{\rho}(\overline{\Delta}) + \pi^*\overline{L}) = \inf_{m\in\Delta^\circ} \left\{\zeta_{\mathrm{abs}}(\overline{L} + \widehat{c}(m)) + \theta(m)\right\}.$$

Remark 1.1.2. One may replace $\inf_{m\in\Delta^\circ}$ by $\inf_{m\in\Delta}$ provided $\overline{L} + \widehat{c}(m)$ is geometrically big for all $m \in \Delta$.

We are able to compute the height filtration of a toric bundle in terms of data from the base variety, see Section 3.6. The height filtration has previously been computed explicitly for toric varieties in [BPS15] and in the case of flag varieties over function fields in [FLQ24]. Without such kind of additional structure, the problem of computing height filtrations seems very hard.

In addition to computing the essential and absolute minima, we seek a combinatorial formula for arithmetic intersection numbers on toric bundles inspired by [HKM21]. We will refer to this as the arithmetic bundle BKK theorem. It is a common generalization of the arithmetic BKK theorem [BPS14, Theorem 5.2.5] and the bundle BKK theorem [HKM21, Theorem 4.1]. The study of Okounkov bodies used in the proof of Theorem A suffices to obtain the arithmetic bundle BKK theorem for complete intersection cycles. To prove it for arbitrary cycles requires other methods.

Let $\widehat{\mathcal{X}}$ be an adelic integrable proper toric bundle of relative dimension t over a smooth projective base variety B of dimension g. Let $\mathcal{B}$ be a regular scheme flat and projective over $\mathcal{S}$ with $\mathcal{B}_K = B$. Let $\gamma \in \widehat{\mathrm{CH}}_i(\mathcal{B})$ and denote by $[\infty] \in \widehat{\mathrm{CH}}^1(\mathcal{B})_{\mathbb{R}}$ the class of a trivial Cartier divisor endowed with constant Green's functions at all places such that $h_{[\infty]}(x) = 1$ for all $x \in B(\bar{K})$.

Theorem B. *The intersection numbers below are well-defined and the following identity holds:*

$$i!\widehat{\rho}(\overline{\Delta})^{t+i}\pi^*\gamma = (t+i)! \int_\Delta (\widehat{c}(m) + \theta(m)[\infty])^i \gamma dm.$$

We remark that the formula can be polarized, see [Gre69, Equation 7.8], to obtain a description of the intersection number $\widehat{\rho}(D_1)\dots\widehat{\rho}(D_{t+i})\pi^*\gamma$ for integrable toric Cartier divisors $D_1, \dots, D_{t+i}$. By approximation, one can replace γ by a product $\gamma'\widehat{c}_1(L_1)\dots\widehat{c}_1(L_l)$ for integrable line bundles $L_1, \dots, L_l$ on B and $\gamma \in \widehat{\mathrm{CH}}_{i+l}(\mathcal{B})$. The condition that $\mathcal{B}$ is regular is a strong assumption. It can be omitted in the case that the toric bundle has a model over $\mathcal{B}$.

We would like to note that the bundle BKK theorem of [HKM21] is used to compute the cohomology ring of smooth toric bundles. This relies on Poincaré duality for oriented manifolds and Leray's theorem on the cohomology of fibrations. It would be interesting to find a similar description for a suitable subring of the arithmetic Chow group of a toric bundle or for some equivariant arithmetic Chow group.

Let G be a semiabelian variety over a global field K with abelian quotient A, given as a torus bundle by a map $c : M \to A^\vee(\bar{K})$. Let $\mathcal{M}$ denote an ample symmetric line bundle on A. Endowing it with the canonical metrics gives it the structure of an adelically metrized line bundle $\overline{\mathcal{M}}$. Let $\widehat{h}$ denote the Néron-Tate height on $A(\bar{K}) \otimes \mathbb{R}$. We note that it factors through the polarization $A(\bar{K}) \otimes \mathbb{R} \to A^\vee(\bar{K}) \otimes \mathbb{R}$. We denote its evaluation on $A^\vee(\bar{K}) \otimes \mathbb{R}$ by $\widehat{h}$ as well. Let $\overline{G}$ be the compactification of G with respect to a fan in $M_{\mathbb{R}}$. Let D be an ample toric Cartier divisor on X_Σ with Newton polytope $\Delta \subset M_{\mathbb{R}}$. Let $\mathcal{F}(\Delta)^i$ denote the set of i-dimensional faces of Δ.

Theorem C. *The height of a compactified semiabelian variety $\overline{G}$ can be computed as*

$$h_{\rho(D)^{\mathrm{can}} \otimes \pi^* \bar{\mathcal{M}}}(\overline{G}) = -(d+1)! \int_\Delta \widehat{h}(c(m))dm.$$

Theorem D. *The i-th successive minimum of $\bar{G}$ with respect to $\rho(D)^{\mathrm{can}} \otimes \pi^* \bar{\mathcal{M}}$ satisfies $\zeta_i(\bar{G}) = \zeta_1(\bar{G})$ for $i \leq g+1$. For $i \geq g+1$,*

$$\zeta_i(\bar{G}) = - \max_{F \in \mathcal{F}(\Delta)^{t+g+1-i}} \min_{m \in F} \widehat{h}(c(m)).$$

By specializing Δ in the above theorems, one can recover the results of Section 4 in [Cha00].

1.2. **Organization of the article.** Section 2 is devoted to establishing notations and recalling known facts on toric varieties and Okounkov bodies in algebraic and arithmetic geometry. We further explain how to associate intersection numbers to an arbitrary arithmetic cycle and a collection of integrable divisors.

In Section 3 we introduce metrized toric bundles and construct the map $\widehat{\rho}$. We proceed to study their basic properties.

In Section 4, we compute Okounkov bodies and the Boucksom-Chen transform for line bundles on toric bundles. We apply this to prove Theorem A.

We prove the arithmetic bundle BKK theorem in Section 5 by means of arithmetic convex chains.

Section 6 illustrates the results from the previous sections with examples in the realm of semiabelian varieties. In particular, we prove Theorem C and D. We then apply this to recover computations of Chambert-Loir.

2. Preliminaries

2.1. **Arakelov geometry of toric varieties.** We will assume basic familiarity with toric varieties, but still give a brief recollection on some basic facts. We then present the extension of these facts in arithmetic geometry as developed in [BPS14], which we recommend for an in-depth treatment of the Arakelov geometry of toric varieties. In addition, we introduce the original notion of an adelic polytope. This turns out to be convenient to transfer arguments from the classical theory.

Let $\mathbb{T}$ be a split torus over a field K.

Definition 2.1.1. A *toric variety* with torus $\mathbb{T}$ is a normal variety X with a dense open embedding $\mathbb{T} \subseteq X$ and an action $\mathbb{T} \times X \to X$ extending multiplication on $\mathbb{T}$.

We follow the convention to denote by N the set of cocharacters $\mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ and by M the set of characters $\mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$. They are finitely generated free abelian groups, dual to one another. Toric varieties with torus $\mathbb{T}$ are in bijection to rational fans on $N_\mathbb{R}$. We denote by X_Σ the toric variety associated to a fan Σ.

Definition 2.1.2. A *virtual support function* or *virtual polytope* with respect to a fan Σ on $N_\mathbb{R}$ is a function $|\Sigma| \to \mathbb{R}$ that is linear on each cone in Σ. The set of virtual support functions with respect to Σ is denoted by $\mathcal{P}_\Sigma$.

Virtual polytopes are generalizations of polytopes by Legendre-Fenchel duality. Consider the monoid of polytopes $\mathcal{P}_\Sigma^+$ in $M_\mathbb{R}$ whose normal fan coarsens Σ with addition given by Minkowski sum. The *normal fan* of a polytope Δ is defined by associating to each i-dimensional face F the $(n-i)$-dimensional cone

$$\left\{ n \in N_{\mathbb{R}} \,\middle|\, \forall u \in F: \ \langle n, u\rangle = \sup_{x\in\Delta} \langle n, x\rangle \right\}.$$

The normal fan is the collection of these cones.

The group completion of $\mathcal{P}_\Sigma^+$ can be identified with $\mathcal{P}_\Sigma$ through Legendre-Fenchel duality. Under this duality polytopes are identified with concave functions.

Definition 2.1.3. We denote by $\mathcal{P}$ and $\mathcal{P}^+$ the set of (virtual) polytopes with respect to any rational fan.

Definition 2.1.4. A *toric Cartier divisor* on a toric variety X is defined to be a Cartier divisor which is invariant under the action of the torus $\mu : \mathbb{T} \times X \to X$. This means that a Cartier divisor D is toric if $\mu^* D = \pi^* D$, where π denotes the projection map. Denote the set of toric Cartier divisors on a toric variety X by $\mathrm{Div}_{\mathbb{T}}(X)$. A toric $\mathbb{R}$-Cartier divisor is an element of $\mathrm{Div}_{\mathbb{T}}(X)_{\mathbb{R}} = \mathrm{Div}_{\mathbb{T}}(X) \otimes \mathbb{R}$.

Theorem 2.1.5 (Section 3.3 [Ful93])**.** There is an isomorphism $\rho : \mathcal{P}_\Sigma \to \mathrm{Div}_{\mathbb{T}}(X_\Sigma)_{\mathbb{R}}$.

Let X be a finite type scheme over a field K endowed with an absolute value. Let $\widehat{K}$ denote its completion. Then, we define X^{an} to be the analytification of $X_{\widehat{K}}$ in the sense of Berkovich, introduced in [Ber12]. Suppose $X_{\widehat{K}} = \mathrm{Spec}(A)$ is affine. Then, X^{an} as a set can be identified with

$$\{|\cdot| : A \to \mathbb{R} \text{ multiplicative seminorm extending the norm on } \widehat{K}\}.$$

For a split torus $\mathbb{T}$ we can define the tropicalization map by

$$\begin{aligned} \mathbb{T}^{\mathrm{an}} &\to N_{\mathbb{R}} \\ x &\mapsto (m \mapsto |\chi^m(x)|). \end{aligned}$$

We will not need to introduce tropicalization in the general setting.

Definition 2.1.6. Let D be a Cartier divisor on a variety X over a field K endowed with an absolute value. Then, a continuous Green's function for D is a function $g : (X \setminus \mathrm{Supp}\, D)^{\mathrm{an}} \to \mathbb{R}$ such that for each $U \subseteq X$ on which D is defined by a section f, the function $g + \log |f(x)| : (U \setminus \mathrm{Supp}\, D)^{\mathrm{an}} \to \mathbb{R}$ extends to a continuous function on U^{an}.

Definition 2.1.7. Let K be a field with an absolute value. A *metrized Cartier divisor* $\bar{D} = (D, g)$ on a proper variety X consists of a Cartier divisor D and a continuous D-Green's functions g.

If K is a global field, an *adelic Cartier divisor* $\bar{D} = (D, g_v)$ consists of a Cartier divisor and continuous Green's functions g_v for all places $v \in M_K$ such that there exists a dense open subset $U \subseteq \mathcal{S}$ and a normal proper model $(\mathcal{X}_U, D_U)$ over U such that for all $v \in U$, g_v is induced by the model. An adelic Cartier divisor is called effective if the underlying Cartier divisor is effective and $g_v \geq 0$ for all places v.

These notions have an analogue on the level of line bundles, namely metrized line bundles and adelically metrized line bundles. We denote the set of adelic Cartier divisors by $\widehat{\mathrm{Div}}(X)$.

For the purposes of arithmetic intersection this is too general. We restrict to *integrable* metrics at each place. An integrable metric is the difference of semipositive metrics. *Semipositive* metrics are limits of model/smooth metrics satisfying certain positivity conditions.

Let X be a proper variety of dimension d. Then, there is an intersection pairing defined in [Zha95b] that to $d+1$ integrable divisors $\overline{D}_0, \dots, \overline{D}_d$ associates an intersection number $\overline{D}_0 \dots \overline{D}_d \in \mathbb{R}$. This pairing factors through the group of integrable adelic $\mathbb{R}$-Cartier divisors, see [Bal22b, Section 3.2]. The group of adelic $\mathbb{R}$-Cartier divisors $\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ consists of an $\mathbb{R}$-Cartier divisor and compatible Green's functions at every place. It admits a natural surjection $\widehat{\mathrm{Div}}(X) \otimes \mathbb{R} \to \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Given an adelic $\mathbb{R}$-Cartier divisor $\overline{D}$ we denote its top intersection product $\overline{D}^{d+1}$ by $h_{\overline{D}}(X)$ and call it the height of X with respect to $\overline{D}$. This convention differs from [Cha00], where the height is normalized by a factor $\frac{1}{(\dim X+1)\deg_D(X)}$, but agrees with the convention of [Bal21].

When X is a toric variety we call a Green's function g toric if it factors through tropicalization on the underlying torus, i.e. is invariant under the action of the unit torus. We will denote the set of toric adelic divisors by $\widehat{\mathrm{Div}}_{\mathbb{T}}(X)$. The toric dictionary extends to this setting. The continuous metrics on $\Psi \in \mathcal{P}_\Sigma$ are in bijection to continuous functions ψ on $N_{\mathbb{R}}$ such that $\psi - \Psi$ is bounded. Under this bijection, semipositive metrics correspond to concave functions ψ, see [BPS14, Theorem 4.8.1].

It will be useful to relate integrable divisors directly to polytopes.

Definition 2.1.8. Let J be a finite set and V a finite-dimensional vector space. A J-metrized polytope is a polytope $\overline{\Delta} \subset V \oplus \bigoplus_{j \in J} \mathbb{R}$ that can be obtained via the following construction.

Let $(\Delta, (\theta_j)_{j\in J})$ consisting of a polytope Δ in V and for each $j \in J$, a concave function $\theta_j : \Delta \to \mathbb{R}_{\geq 0}$. The associated polytope is

$$\overline{\Delta} = \{(x, t_j) \in V \oplus \bigoplus_{j\in J} \mathbb{R} \mid x \in \Delta, 0 \leq t_j \leq \theta_j(x)\}.$$

Denote the set J-metrized polytopes by $\mathcal{P}^{J,+}$.

For an infinite set I, we define the set of I-metrized polytopes to be the filtered colimit

$$\operatorname*{colim}_{J \subset I \text{ finite}} \mathcal{P}^{J,+},$$

where the transition map for $J \subset J'$ is given by $\overline{\Delta} \mapsto \overline{\Delta} \times \prod_{j \in J' \setminus J} 0$. This is compatible with the monoid structure, hence giving a monoid of I-metrized polytopes.

We denote the group completion of $\mathcal{P}^{I,+}$ by $\mathcal{P}^I$ and call it the group of I-metrized virtual polytopes. When the set I is the set of places M_K of a global field K we will use the notation $\widehat{\mathcal{P}}^+$ and $\widehat{\mathcal{P}}$. Their elements will be referred to as (virtual) adelic polytopes.

For $I = \{*\}$, we speak simply of metrized polytopes.

Definition 2.1.9. An I-metrized polytope $\overline{\Delta}$ has an associated underlying polytope $\Delta \subset V$ and for each $i \in I$ a *local roof function* $\theta_i : \Delta \to \mathbb{R}$ satisfying $\theta_i = 0$ for almost all i such that $\overline{\Delta}$ is associated to $(\Delta, (\theta_j)_{i \in I})$ as described in Definition 2.1.8. The *global roof function* is defined as the finite sum $\theta = \sum_{i \in I} n_i \theta_i$, where n_i is a choice of weights that is clear from the context. In particular, for adelic divisors n_i will be 1 for all i if K is a function field and $\frac{[K_i:\mathbb{Q}_i]}{[K:\mathbb{Q}]}$ if K is a global field.

For each $i \in I$, we associate a local metrized polytope

$$\Delta_i = \{(x, t) \in V \oplus \mathbb{R} \mid x \in \Delta, 0 \le t \le \theta_i(x)\}.$$

We define the global polytope $\widehat{\Delta}$ to be

$$\{(x, t) \in V \oplus \mathbb{R} \mid x \in \Delta, 0 \le t \le \theta(x)\}.$$

Let us recall [BPS14, Theorem 4.8.1] using this new language. There is an isomorphism of monoids between semipositive effective metrized divisors and metrized polytopes. This globalizes to an isomorphism of monoids between semipositive effective adelic divisors and adelic polytopes. By group completion, the isomorphisms extend to isomorphisms between virtual (adelically) metrized polytopes and integrable (adelically) metrized divisors.

Model metrics will play a crucial role in the proof of the arithmetic bundle BKK-theorem.

Let K be a complete field with respect to an absolute value associated to a non-trivial discrete valuation. Let Σ be a complete rational fan on $M_{\mathbb{R}}$. Then, the set of toric models can be identified with rational fans $\widetilde{\Sigma}$ in $N_{\mathbb{R}} \oplus \mathbb{R}_{\ge 0}$ whose intersection with $N_{\mathbb{R}} \oplus 0$ is Σ, cf. [BPS14, Theorem 3.5.3]. We refer to the fan Σ^{can} consisting of the cones of the form $\sigma \oplus 0$ and $\sigma \oplus \mathbb{R}_{\ge 0}$ for $\sigma \in \Sigma$ as the *canonical (metrized) fan* associated to Σ. Denote the toric model associated to $\widetilde{\Sigma}$ by $\mathcal{X}_{\widetilde{\Sigma}}$. The following follows from [BPS14, Theorem 3.6.7].

Theorem 2.1.10. The set of semipositive effective $\mathbb{R}$-divisors on $\mathcal{X}_{\widetilde{\Sigma}}$ is in bijection to metrized polytopes whose normal fan restricted to $N_{\mathbb{R}} \oplus \mathbb{R}_{\ge 0}$ coarsens $\widetilde{\Sigma}$.

Proof. Piecewise affine concave functions ψ on $\widetilde{\Sigma} \cap (N_{\mathbb{R}} \times \{1\})$ are in bijection with semipositive $\mathbb{R}$-divisors on $\mathcal{X}_{\widetilde{\Sigma}}$ by a combination of [BPS14, Theorem 3.6.7] and

[BPS14, Theorem 3.7.3]. The divisor corresponding to ψ is effective precisely if ψ is nonnegative. Let Ψ be the recession function of ψ. We may extend it to a conical function on $\widetilde{\psi} : N_{\mathbb{R}} \oplus \mathbb{R}$ by setting $\psi(n, x) = \Psi(n)$ for $x < 0$. Legendre-Fenchel duality restricts to a correspondence between metrized polytopes whose normal fan coarsens $\widetilde{\Sigma}$ and conical functions of the form above. □

We can consider metrics on toric divisors compatible with $\widetilde{\Sigma}$ even if K does not have a discrete valuation.

Definition 2.1.11. A metrized divisor is compatible with $\widetilde{\Sigma}$ if it is a difference of two divisors associated to metrized polytopes whose normal fan restricted to $N_{\mathbb{R}} \oplus \mathbb{R}_{\geq 0}$ coarsens $\widetilde{\Sigma}$. We will denote the set of metrized polytopes compatible with $\widetilde{\Sigma}$ by $\mathcal{P}_{\widetilde{\Sigma}}$. It can be identified with conical functions $\psi : N_{\mathbb{R}} \oplus \mathbb{R} \to \mathbb{R}$ that is linear on cones of $\widetilde{\Sigma}$ and such that $\psi(n, x) = \psi(n, 0)$ for $x < 0$.

Definition 2.1.12. We define an adelic fan $\widetilde{\Sigma}$ to be a collection of fans $\widetilde{\Sigma}_v$ in $N_{\mathbb{R}} \oplus \mathbb{R}_{\geq 0}$ for each $v \in M_K$ such that almost all the fans are canonical. A (virtual) adelic polytope is said to be compatible with $\widetilde{\Sigma}$ if it is compatible with each $\widetilde{\Sigma}_v$ when considered as a metrized polytope. The set of such polytopes will be denoted $\widehat{\mathcal{P}}_{\widetilde{\Sigma}}$ and $\widehat{\mathcal{P}}^+_{\widetilde{\Sigma}}$.

One can identify virtual adelic polytopes with conical functions $\psi : N_{\mathbb{R}} \oplus \bigoplus_{i \in I} \mathbb{R} \to \mathbb{R}$ that factors over $N_{\mathbb{R}} \oplus \bigoplus_{j \in J} \mathbb{R}$ for some finite $J \subset M_K$ satisfying that $\psi(n, (x_j)) = \psi(n, (\max\{0, x_j\}))$ and linear on each cone of Σ and each cone of the form $\sum_j \in J\sigma_j$, where $\sigma_j \cap \mathcal{N}_{\mathbb{R}} = \sigma$ for some fixed cone $\sigma \in \Sigma$.

The set of cones of $\widehat{\mathcal{P}}_{\widetilde{\Sigma}}$ is defined to be the union of Σ and the cones of $\widehat{\mathcal{P}}_{\widetilde{\Sigma}v}$ at each place v. Here the cones at v contained in $N_{\mathbb{R}} \times 0$ are identified with Σ.

Definition 2.1.13. Let $v \in M_K$ be a place. Denote by $\widehat{\mathcal{P}}^v$ the set of virtual adelic polytopes canonical at all places $w \neq v$, i.e. the ones that come from $\{v\}$-metrized polytopes in the colimit. Then, we say that $\overline{\Delta} \in \widehat{\mathcal{P}}^+_{\widetilde{\Sigma}}$ is v-interior if its is in the interior of $\left(\overline{\Delta} + \widehat{\mathcal{P}}^v\right) \cap \widehat{\mathcal{P}}^+_{\widetilde{\Sigma}} \subset \overline{\Delta} + \widehat{\mathcal{P}}^v$.

2.2. **Okounkov bodies and roof functions.** Let X be a projective variety over a global field K and let $\overline{D}$ be a geometrically big adelic Cartier divisor on X. The concave transform of $\overline{D}$ as defined in [BC11, Definition 1.7] is a concave function on the Okounkov body Δ associated to the underlying line bundle encoding information on the adelically metrized line bundle in terms of convex geometry. It is defined in terms of the filtered linear series associated to $\overline{D}$.

Consider the graded linear series $V^\bullet = \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}(n\overline{D}))$. Let $x \in X(\bar{K})$ be a regular point and fix an isomorphism $\mathbf{z} : \widehat{\mathcal{O}}_{X,x} \cong \bar{K}[[x_1, \ldots, x_d]]$. This induces a rank d valuation ν_n on each $H^0(X, \mathcal{O}(n\overline{D}))$ by taking the valuation on

$\bar{K}[[x_1, \dots, x_d]]$ sending $x_1^{a_1} \cdot \dots \cdot x_d^{a_d}$ to $(a_1, \dots, a_d)$ and on a linear combination of monomials the lexicographically smallest term. The Okounkov body of D is defined to be as the closure of

$$\bigcup_{n=0}^{\infty} \frac{1}{n} \nu(H^0(X, \mathcal{O}(n\overline{D}))).$$

This is a convex body and we denote it by $\Delta(D)$, where the choice of $\mathbf{z}$ is understood. It is invariant under numerical equivalence by [LM09, Proposition 4.1]. Just as easily we can define Okounkov bodies for subalgebras.

Each term in the graded algebra is endowed with a filtration by minima. More precisely, on an adelically normed vector space V we may define the F^tV to be the sub-vector-space generated by vectors of height $\leq t$, see [BC11, Definition 3.1]. We define $F^tV^\bullet = \bigoplus F^{nt}H^0(X, \mathcal{O}(n\overline{D}))$. The associated Okounkov body will be denoted by $\Delta^t(\overline{D})$. Boucksom and Chen in [BC11] define the concave transform $G_{\mathcal{L},\mathbf{z}} : \Delta(D) \to \mathbb{R}$ by $x \mapsto \inf\{t \mid x \in \Delta^t(\overline{D})\}$, which is concave and upper semicontinuous on the boundary. The hypograph of this function is called the arithmetic Okounkov body.

A common way to obtain an isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \bar{K}[[x_1, \dots, x_d]]$ is by fixing a flag

$$X_\bullet : X \supset X_1 \supset X_2 \supset \dots \supset X_d = \{x\},$$

of irreducible subvarieties X_i of codimension i that are non-singular at x. If $\mathbf{z}$ is induced by $X_\bullet$, we denote the Boucksom-Chen transform by $G_{\overline{D},X_\bullet}$.

The concave transform defined above encodes important information on the adelically metrized line bundle. For instance, Ballaÿ shows that the essential and the absolute infimum of an adelic line bundle are determined by its associated filtered linear series in the semipositive case. [QY22, Theorem 1.7] allows to remove one semipositivity assumption. Here is a version of [Bal21, Proposition 7.1] adapted accordingly together with [Bal22a, Corollary 1.2].

Theorem 2.2.1. Let $\overline{D} \in \widehat{\mathrm{Div}}_{\mathbb{R}}$ with D big. Then,

$$\zeta_{\mathrm{ess}}(\overline{D}) = \max_{\alpha \in \Delta(D)} G_{\overline{D}}(\alpha).$$

If $\overline{D}$ is semipositive,

$$\zeta_{\mathrm{abs}}(\overline{D}) = \inf_{\alpha \in \Delta(D)} G_{\overline{D}}(\alpha).$$

Furthermore, the Okounkov body and the concave transform are related to the volume and the arithmetic volume respectively. We will not discuss arithmetic volume functions as we will only use its relation to heights via the arithmetic Hilbert-Samuel theorem. The following result contains a geometric part [LM09, Theorem A] and an arithmetic one taken from [Bal21, Theorem 6.4], but which is implicit in [BC11]. We apply the normalization of arithmetic volumes in [BC11] which differs from the one in [Bal21].

Theorem 2.2.2. Let D be a big $\mathbb{R}$-Cartier divisor on X. Then,

$$\mathrm{vol}(D) = d!\,\mathrm{vol}_{\mathbb{R}^d}(\Delta(D))$$

If $\overline{D}$ is an adelic divisor whose underlying divisor is D, then

$$\widehat{\mathrm{vol}}(\overline{D}) = (d+1)! \int_{\Delta(D)} \max\{0, G_{\overline{D}}\} d\lambda$$

and

$$\widehat{\mathrm{vol}}_\chi(\overline{D}) \leq (d+1)! \int_{\Delta(D)} G_{\overline{D}} d\lambda,$$

with equality if $\inf_{\alpha\in\Delta(D)} G_{\overline{D}}(\alpha) > -\infty$.

This in turn allows a comparison with heights and intersection numbers by the (arithmetic) Hilbert-Samuel theorem (see [Deb01, Proposition 1.31] and [Mor13, Theorem 5.3.2]).

Theorem 2.2.3. If D is nef, then $\mathrm{vol}(D) = \deg(D)$. If $\overline{D}$ is semipositive, then $\widehat{\mathrm{vol}}_\chi(\overline{D}) = h_{\overline{D}}(X)$.

2.3. **(Arithmetic) Okounkov bodies of toric varieties.** There are two ways to associate a convex body Δ and a concave function θ on Δ to the datum of a semipositive toric adelic Cartier divisor $\overline{D}$. The first construction is toric by nature, it is given by the Newton polytope and the toric roof function as in [BPS14]. The second construction does not depend on the toric nature(except for the choice of a flag). It is given by the Okounkov body and the Boucksom-Chen concave transform. In the setting of toric bundles, it is important to relate these constructions to apply results on toric varieties.

The equality (up to translation) of the Newton polytope and roof function on the toric side with the Okounkov body and concave transform is mentioned in passing in [BC11, Section 4.5] using the work of [Wit14]. The necessary arguments are presented in [Bur+16, Section 5] in detail. We prove the equality of the two constructions here as it is not explicitly stated in [Bur+16].

Let us first recall the geometric statement. We want to study big toric divisors on a proper toric variety X_Σ. We may refine Σ in such a way that it defines a smooth projective variety, see [CLS10, Chapter 11]. This does not change the Newton polytope. We may then take prime toric divisors $D_1, \ldots, D_t$ such that

$$X_\bullet : X_\Sigma \supset D_1 \supset D_1 \cap D_2 \supset \cdots \supset D_1 \cap \cdots \cap D_t = \{p\}.$$

defines a flag. Let v_i denote the primitive generator of the ray corresponding to D_i. Then, the v_i form a basis of N and induce an isomorphism $\mathbb{Z}^t \cong N$. Its dual basis determines an isomorphism $M \cong \mathbb{Z}^t$.

Proposition 2.3.1 ([LM09] Proposition 6.1)**.** Let D be a toric divisor not containing any of the $D_1, \dots, D_t$ in its support. Let Δ be its Newton polytope. Then the Okounkov body of $\rho(\Delta)$ is Δ under the identification $M \cong \mathbb{Z}^t$. In particular, the valuation of the section χ^m is m.

By translating Δ, we can always ensure that the conditions in the above proposition are satisfied. We now introduce the toric roof function and prove an arithmetic analogue of the statement above. We apply [Bur+16, Proposition 5.1] in order to give an alternative, but equivalent definition of the toric roof functions.

Definition 2.3.2. Let $\overline{D}$ be a toric metrized $\mathbb{R}$-Cartier divisor over a field K with absolute value $|\cdot|_v$. Let Δ be the Newton polytope of the underlying geometric Cartier divisor. Then, the local roof function θ of $\overline{D}$ is the unique continuous function $\theta_{v,\overline{D}} : \Delta \to \mathbb{R}$ satisfying the following condition: For $m \in l\Delta$, the toric section $\chi^m \in \Gamma(X, lD)$ satisfies

$$-\log \|\chi^m\|_{v,\sup} = \theta_{\overline{D}}(m/l).$$

If $\overline{D}$ is a toric adelic $\mathbb{R}$-Cartier divisor over a global field K, we define its global roof function as a sum over local roof functions

$$\theta_{\overline{D}}(x) = \sum_{v \in M_K} n_v \theta_{\overline{D},v}(x).$$

The sum is finite since the local roof function is constantly 0 at all the places for which $\overline{D}$ carries the canonical metric.

Proposition 2.3.3. Let D be a toric divisor not containing any of the $D_1, \dots, D_t$ in its support. Let Δ be its Newton polytope. Let $\overline{D}$ be an adelic Cartier-divisor obtained by endowing D with toric metrics. Identifying the Newton polytope with the Okounkov body as in Proposition 2.3.1 the global roof function agrees with the Boucksom-Chen transform.

The main technical result we use is the orthogonality of toric sections. We recall the statement for future use.

Theorem 2.3.4 (Corollary 5.4 [Bur+16])**.** Let $\overline{D}$ be a toric adelic $\mathbb{R}$-divisor on X and $s = \sum_{\Delta \cap M} \gamma_m \chi^m \in \Gamma(X, D)$. Then,

$$\|s\|_{\sup} \geq \max_{m \in \Delta \cap M} \|\gamma_m \chi^m\|_{\sup}.$$

Proof of Proposition 2.3.3. We have that the concave transform $G_{\bar{D},X_\bullet}(m/l)$ is bounded from below by $\sum_v \theta_{\bar{D},v}(m/l)$ as χ^m provides a $\mathbb{Q}$-section with valuation

m/l and height $\theta(m/l)$. Let us now prove that $G_{\bar{D},X_\bullet}(m/l) \leq \theta(m/l)$. By definition of the Boucksom-Chen transform, for any $\epsilon > 0$, there is a sufficiently large N such that there is a section $s \in \Gamma(ND)$ of valuation mN/l generated by sections of height $< N(G_{\bar{D},X_\bullet}(m/l) + \epsilon)$. So write $s = \sum s_i$. At least one s_i will be of the form $\gamma_m \chi^m + \sum_{m' \neq m} \gamma_{m'} \chi^{m'}$. By the orthogonality of eigenspaces this vector has height $\geq N\theta(m/l)$ finishing the proof. □

We may alternatively conclude the equality of functions using the equality of their integrals provided by [Bur+16, Theorem 5.6] and Theorem 2.2.2.

2.4. **Operations on arithmetic Chow homology.** Recall that $\mathcal{S}$ denotes a geometrically irreducible projective curve over a field or the spectrum of the ring of integers in a number field. Let $\mathscr{X}$ be a flat projective scheme over $\mathcal{S}$. Then, similarly to algebraic geometry Gillet and Soulé require regularity of the scheme in order to define an intersection theory. However, it suffices to require the generic fibre $X = \mathscr{X}_K$ to be smooth in order to define arithmetic Chow groups and operational Chern classes associated to hermitian line bundles, see [GS92, Section 2.4]. The operational perspective is convenient even if one focuses on regular arithmetic varieties as it allows us to pass to fibre products, such as special fibres, that are no longer necessarily regular.

Let us give a basic recollection of arithmetic Chow groups and the intersection with hermitian line bundles. Let $\mathscr{X}$ be a flat projective scheme over $\mathcal{S}$ with smooth generic fibre. Then, we can define its arithmetic Chow groups $\widehat{\mathrm{CH}}_k(\mathscr{X})$ as equivalence classes of pairs (Z, g_Z) of a k-cycle Z on $\mathscr{X}$ and Green currents g_Z for $Z_{\mathbb{C}}$ at all archimedean places modulo rational equivalence and the image of $\partial + \bar{\partial}$. Note that there is a well-defined arithmetic degree map $\widehat{\mathrm{CH}}_0(\mathscr{X}) \to \mathbb{R}$ by the existence of a proper pushforward map, see [GS92, Paragraph 2.2.2].

Let $\overline{L}$ be a hermitian line bundle on $\mathscr{X}$. Then, we can describe the action of the first Chern class $\widehat{c}_1(\overline{L}) : \widehat{\mathrm{CH}}_k(\mathscr{X}) \to \widehat{\mathrm{CH}}_{k-1}(\mathscr{X})$ explicitly. Let $[(Z, g_Z)] \in \widehat{\mathrm{CH}}_k(\mathscr{X})$ and let s be a meromorphic section of $\overline{L}$ on Z and denote by $\omega_{\overline{L}}$ the curvature form of $\overline{L}$. Then,

$$\widehat{c}_1(\overline{L})[(Z, g_Z)] = [(\mathrm{div}(s), -\log|s|\delta_{Z(\mathbb{C})} + \omega_{\overline{L}} g_Z)].$$

This is independent of the choice of section and additive in the tensor product of hermitian line bundles. If $\overline{M}$ is a further hermitian line bundle, the Chern classes commute, i.e. $\widehat{c}_1(\overline{L})\widehat{c}_1(\overline{M}) = \widehat{c}_1(\overline{M})\widehat{c}_1(\overline{L})$ as operations on Chow groups. In particular, there is a multilinear intersection pairing

$$\widehat{\mathrm{Pic}}(\mathscr{X})^k \times \widehat{\mathrm{CH}}_k(\mathscr{X}) \to \mathbb{R}$$

$$(\overline{\mathscr{L}}_1, \ldots, \overline{\mathscr{L}}_k, [(Z, g_Z)]) \mapsto \widehat{\deg}\left(\widehat{c}_1(\overline{\mathscr{L}}_1) \cdots \widehat{c}_1(\overline{\mathscr{L}}_k)[(Z, g_Z)]\right)$$

which is symmetric in the first k entries.

Proposition 2.4.1. Suppose that $\mathscr{X}$ is regular. Then, the above intersection pairing extends uniquely to a pairing

$$\begin{aligned} \widehat{\mathrm{Pic}}^{\mathrm{int}}(X)^k \times \widehat{\mathrm{CH}}_k(\mathscr{X}) &\to \mathbb{R} \\ (\overline{L}_1, \ldots, \overline{L}_k, [(Z, g_Z)]) &\mapsto \widehat{\deg}\left(\overline{L}_1 \cdots \overline{L}_k \cdot [(Z, g_Z)]\right) \end{aligned}$$

allowing any integrable line bundle.

Proof. Suppose first that $\pi : \mathscr{X}' \to \mathscr{X}$ is a further projective model dominating $\mathscr{X}$. We let $\mathscr{Y}$ be a regular projective alteration of $\mathscr{X}'$ of generic degree d, i.e. there is a generically finite map $\phi : \mathscr{Y} \to \mathscr{X}'$ of degree d with $\mathscr{Y}$ regular. Such an alteration exists by [Jon96, Theorem 8.2].

Then, we first extend the pairing to $\widehat{\mathrm{Pic}}(\mathscr{X}')$ by setting

$$\begin{aligned} &\widehat{\mathrm{Pic}}(\mathscr{X}')^k \times \widehat{\mathrm{CH}}_k(\mathscr{X}) \to \mathbb{R} \\ &(\overline{\mathscr{L}}_1, \ldots, \overline{\mathscr{L}}_k, [(Z, g_Z)]) \mapsto \frac{1}{d}\widehat{\deg}\left(\widehat{c}_1(\phi^*\overline{\mathscr{L}}_1) \cdots \phi^*\widehat{c}_1(\overline{\mathscr{L}}_k)(\phi \circ \pi)^*[(Z, g_Z)]\right), \end{aligned}$$

where the pullback is defined in [GS90, Section 4.4]. Note that since π is projective between regular schemes, it is in particular l.c.i. and thus the pullback is defined. Furthermore, we note that the pullback along a generically finite morphism induces multiplication by the degree on $\widehat{\mathrm{CH}}_0$. This follows from the fact that $\widehat{\mathrm{CH}}_0 \cong \mathbb{R}$ via the degree map for every regular projective arithmetic variety, see [GS94, Theorem 2.1] and that if a cycle is defined on the locus where the morphism is finite the claim holds by [GS90, Theorem 4.4 ii].

Hence, the definition does not depend on the choice of alteration. We are left to show that one can extend the intersection number to integrable line bundles.

For this assume that $L, L_2, \ldots, L_k$ are line bundles on X and $[(Z, g_Z)] \in \widehat{\mathrm{CH}}_k(\mathscr{X})$ is an arithmetic cycle. Suppose that $\overline{\mathscr{L}}, \overline{\mathscr{L}}_2, \ldots, \overline{\mathscr{L}}_k$ are hermitian $\mathbb{Q}$-line bundles on a model $\mathscr{X}'$ whose generic fibres are $L, L_2, \ldots, L_k$. Suppose $\overline{\mathscr{L}}_2, \ldots, \overline{\mathscr{L}}_k$ are relatively nef at a place v and $\overline{\mathscr{L}}$ is of the form $\mathcal{O}(\mathscr{D})$ for an effective Cartier divisor supported on the special fibre at v whose associated Green's function is bounded by C. Then, we need to bound

$$\frac{1}{d}\widehat{\deg}\left(\widehat{c}_1(\phi^*\overline{\mathscr{L}})\widehat{c}_1(\overline{\phi^*\mathscr{L}_2}) \cdots \widehat{c}_1(\phi^*\overline{\mathscr{L}}_k)(\phi \circ \pi)^*[(Z, g_Z)]\right)$$

in terms of C, $L_2, \ldots, L_k$ and $[(Z, g_Z)]$. In the archimedean case this is taken to be meant as $\mathscr{D}$ is given by a non-negative smooth function bounded by C.

Suppose first that v is archimedean. Note that the curvature form $\omega(g_Z)$ is the difference of strongly positive forms. Locally, this follows from [Dem12, Section III.1.4]. We may perform a partition of unity to write $\omega(g_Z)$ as the difference of strongly positive forms $\omega^+ - \omega^-$. Let $\omega_2, \ldots, \omega_k$ be the curvature forms of

$\overline{\mathscr{L}}_2, \ldots, \overline{\mathscr{L}}_k$. Then, by assumption $\omega_2 \cdots \omega_k \omega^+$ and $\omega_2 \cdots \omega_k \omega^-$ are positive measures of mass $\leq K$. Pulling them back yields measures of mass $\leq dK$. Then,

$$|\frac{1}{d}\widehat{\deg}\left(\widehat{c}_1(\phi^*\overline{\mathscr{L}})\widehat{c}_1(\overline{\phi^*\mathscr{L}_2}) \cdots \widehat{c}_1(\phi^*\overline{\mathscr{L}}_k)(\phi \circ \pi)^*[(Z, g_Z)]\right)| \leq CK.$$

Hence, the intersection product on semipositive line bundles is continuous with respect to supremum norm at archimedean places and hence extends to integrable metrics at archimedean places.

Let us now consider the case of a finite place. Denote the inclusion of the special fibre of $\mathscr{Y}$ by $j : \mathscr{Y}_v \hookrightarrow \mathscr{Y}$ and that of $\mathscr{X}$ by $i : \mathscr{X}_v \hookrightarrow \mathscr{X}$. By assumption, $\widehat{c}_1(\phi^*\overline{\mathscr{L}})$ is represented by a cycle $\mathscr{D}$ supported on $\mathscr{Y}_v$. It therefore suffices to compute the algebraic intersection $\widehat{c}_1(j^*\phi^*\overline{\mathscr{L}}_1) \cdots \widehat{c}_1(j^*\phi^*\overline{\mathscr{L}}_{k-1}) j^*(\phi \circ \pi)^*[(Z, g_Z)] \cap \mathscr{D}$ on the special fibre by the projection formula as in [Ful98, p. 323]. Denote the base change of $\phi \circ \pi$ to the special fibre by ϖ. Note that $j^*(\phi \circ \pi)^* = \varpi^* i^*$. It suffices to write $i^*[(Z, g_Z)]$ as the difference of nef dual cycle classes on each irreducible component as nefness is preserved under pullback and under intersection with nef divisors. One can write any dual cycle class as the difference of two nef cycles by the full-dimensionality of the nef cone proven in [FL17, Lemma 3.7]. ☐

3. Toric bundles

Analogous to the toric compactification of $\mathbb{T}$-torsors in algebraic geometry we would like such a compactification in the arithmetic setting. Despite the lack of a total space, we can apply Arakelov geometry to imitate the algebro-geometric construction where toric Cartier divisors induce line bundles on the total space, cf. [CT01, Construction 2.3.5]. All torsors will be torsors in the Zariski topology.

3.1. **Categories of torus bundles.** We can use characters to understand torus bundles more closely. More precisely, let $\mathcal{P}ic(B)$ be the category of $\mathbb{G}_m$-torsors on B with morphisms given by isomorphisms. It naturally has the structure of a symmetric monoidal category given by the tensor product. Given a $\mathbb{T}$-torsor, every character $\mathbb{T} \to \mathbb{G}_m$ gives rise to an element in $\mathcal{P}ic(B)$. We define the symmetric monoidal category $\mathcal{M}$ of characters by setting its elements to be $M = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ with no non-trivial morphisms and monoidal structure given by addition. We will call a category associated to finitely generated free abelian groups in this way *lattice category*. Let $\mathrm{Bun}_{\mathbb{T}}(B)$ denote the symmetric monoidal category of $\mathbb{T}$-torsors over B[1]. There is a monoidal equivalence of categories

$$\mathrm{Bun}_{\mathbb{T}}(B) \to \mathrm{Fun}^{\otimes}(\mathcal{M}, \mathcal{P}ic(B)).$$

[1]This notation is reminiscent of the B-valued points of the stack Bun_G studied among others in the context of the geometric Langlands program as in [BD91]. While they consider the stack $\mathrm{Bun}_G(X)$ for a curve X, we consider the stack $\mathrm{Bun}_G(*)$.

Here $\mathrm{Fun}^{\otimes}(\mathcal{M}, \mathcal{P}ic(B))$ denotes the category of monoidal functors with the monoidal structure given by the tensor product on $\mathcal{P}ic(B)$ and monoidal natural transformations as morphisms. This equivalence identifies $\mathbb{T}$-torsors with linear maps $M \to \mathrm{Pic}(B)$. This expresses that the category of $\mathbb{T}$-torsors is in equivalence to collections of line bundles $(\mathcal{T}(m))_{m\in M}$ on B indexed by M with compatible identifications $\mathcal{T}(m_1+m_2) \cong \mathcal{T}(m_1)\otimes\mathcal{T}(m_2)$. The morphisms between such maps are collections of isomorphisms of $\mathbb{G}_m$-bundles indexed over M compatible with the tensor product.

We redefine torus bundles to be monoidal functors from a lattice category $\mathcal{M}$ to $\mathcal{P}ic(B)$. Over a field K endowed with an absolute value, we let $\mathcal{P}ic^{\mathrm{metr}}(B)$ denote the category of metrized line bundles. Over a global field, we let $\widehat{\mathcal{P}ic}(B)$ denote the category of adelically metrized line bundles.

Definition 3.1.1. Let B be a variety over a field K endowed with an absolute value. We define a *metrized torus bundle* to be a symmetric monoidal functor $\mathcal{T}^{\mathrm{metr}} : \mathcal{M} \to \mathcal{P}ic^{\mathrm{metr}}(B)$ from a lattice category $\mathcal{M}$. If K is a global field we define an adelic torus bundle as a symmetric monoidal functor $\widehat{\mathcal{T}} : \mathcal{M} \to \widehat{\mathcal{P}ic}(B)$.

We will extend the meaning of qualifiers from the case of line bundles to torus bundles. We call a metrized/adelic torus bundle integrable if its image consists of integrable line bundles. We call it flat if its image consists of flat line bundles. Recall that a line bundle is flat if both it and its dual are semipositive. We call a torus bundle algebraic if the line bundles in the image are algebraic, i.e. the metric is induced by a $\mathbb{Q}$-line bundle on a model of B. We say a torus bundle has model metrics if the line bundles in the image have model metrics, i.e. the metric is induced by a line bundle on a model of B.

Torus bundles whose image consists of semipositive line bundles are automatically flat. We can identify isomorphism classes of $\mathbb{T}$-torsors with linear maps $M \to \mathrm{Pic}(B)$. The same principle holds for metrized $\mathbb{T}$-bundles.

Remark 3.1.2. We will use adelic notation such as $\widehat{\mathcal{T}}$ as stand-in for the local equivalent $\mathcal{T}^{\mathrm{metr}}$. This does not lead to confusion since all local considerations easily globalize. We will use the word metrized, both in the local context as well as for adelically metrized.

Definition 3.1.3. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be torus bundles on B for split tori $\mathbb{T}_1$ and $\mathbb{T}_2$ respectively. Let $\phi : \mathbb{T}_1 \to \mathbb{T}_2$ be a group homomorphism. A morphism $f : \mathcal{T}_1 \to \mathcal{T}_2$ over B is ϕ-equivariant if the diagram

$$\begin{array}{ccc} \mathbb{T}_1 \times \mathcal{T}_1 & \xrightarrow{\phi \times f} & \mathbb{T}_2 \times \mathcal{T}_2 \\ \mu \downarrow & & \downarrow \mu \\ \mathcal{T}_1 & \xrightarrow[f]{} & \mathcal{T}_2 \end{array}$$

commutes. We say that f is equivariant if it is ϕ-equivariant for some ϕ.

We can now define categories of torus bundles from the functor perspective.

Definition 3.1.4. We define categories of torus bundles over a base B with the additional flexibility of varying the torus. Morphisms from a torus bundle $\mathcal{T} : \mathcal{M} \to \mathcal{P}ic(B)$ to $\mathcal{T}' : \mathcal{M}' \to \mathcal{P}ic(B)$ are given by a morphism of tori encoded by a monoidal functor $\alpha : \mathcal{M}' \to \mathcal{M}$ and an equivariant morphism between the torus bundles encoded by a monoidal natural transformation $F : \mathcal{T} \circ \alpha \to \mathcal{T}'$. This defines a category $\mathrm{TorBun}(B)$. Applying the same approach to the metrized setting yields categories $\mathrm{TorBun}^{\mathrm{metr}}(B)$ and $\widehat{\mathrm{TorBun}}(B)$.

Proposition 3.1.5. For fixed $\alpha : \mathcal{M}' \to \mathcal{M}$, the set of equivariant morphisms is in bijection to the sections of $(\mathcal{T} \circ \alpha) \otimes (\mathcal{T}')^{\vee}$.

The above observation shows that if $\mathcal{T}$ and $\mathcal{T}'$ are metrized and $\mathcal{T} \to \mathcal{T}'$ is a morphism of the underlying torus bundles, there is a function on B^{an} giving the norm of the morphism. Only morphisms of constant norm 1 are considered to be morphisms of metrized torus bundles.

Example 3.1.6. For any (metrized) torus bundle $\mathcal{T}$ and integer n, there is an n-th power map $\mathcal{T} \to \mathcal{T}^{\otimes n}$, where $\mathcal{T}^{\otimes n}$ is defined by $\mathcal{T}^{\otimes n}(m) = \mathcal{T}(m)^{\otimes n}$. It is defined by the map $n : \mathcal{M} \to \mathcal{M}$ and the collection of isomorphisms $\mathcal{T}(nm) \to \mathcal{T}(m)^{\otimes n}$ that is inherent in the definition of the monoidal functor $\mathcal{T}$. More generally if $A : M' \to M$ we define the torus bundle $\mathcal{T}^{\otimes A}$ by $\mathcal{T}^{\otimes A}(m') = \mathcal{T}(A \cdot m')$

We would now like to have a notion of morphism of torus bundles that allows for a change in base scheme. Let $f : B' \to B$ be a morphism of schemes and $\mathcal{T}$ be a (possibly metrized) torus bundle over B. Then, we may define the pullback $f^*\mathcal{T} = f^* \circ \mathcal{T}$ as the composition of $\mathcal{T}$ and the pullback functor on line bundles, cf. [CT01, Proposition 1.2.4].

Definition 3.1.7. A morphism of torus bundles $f = (f_t, f_b) : (\mathcal{T}_1 \to B_1) \to (\mathcal{T}_2 \to B_2)$ is the datum of a map $f_b : B_1 \to B_2$ and an equivariant map $f_t : \mathcal{T}_1 \to f_b^*\mathcal{T}_2$. This defines categories TorBun, $\mathrm{TorBun}^{\mathrm{metr}}$ and $\widehat{\mathrm{TorBun}}$.

Example 3.1.8. This setup provides a suitable setting for the proof of the Weil-Barsotti formula. Let G be a semiabelian variety with a split torus part $\mathbb{T}$ and abelian quotient A with quotient map π. The addition on G has to be a morphism of toric bundles $f : G \times G \to G$ with underlying map of base spaces $f_b = m_A : A \times A \to A$, the addition on A, and map on tori $m_{\mathbb{T}} : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$. Let $\mathcal{T}$ be a torus bundle over A. Then, an equivariant morphism $\mathcal{T} \times \mathcal{T} \to \mathcal{T}$ of the form exists if and only if $m^*\mathcal{T} \cong \mathcal{T} \boxtimes \mathcal{T}$. This is precisely the case when the image of $\mathcal{T}$ consists of algebraically trivial line bundles. The map thus defined is unique, once a rigidification $\mathbb{T} \cong \pi^{-1}(e)$ is chosen. One can easily check that this morphism yields a group structure.

Definition 3.1.9. A *dynamical torus bundle* is a tuple $(B, f, \mathcal{T}, f^*\mathcal{T} \cong \mathcal{T}^{\otimes r})$ consisting of a proper base variety B, an endomorphism $f : B \to B$, a torus bundle $\mathcal{T}$ and an isomorphism $f^*\mathcal{T} \cong \mathcal{T}^{\otimes r}$ for some $r > 1$. This data gives rise to canonical metrics on $\mathcal{T}$. Over a global field it gives rise to an adelic torus bundle. Its isomorphism class does not depend on the chosen isomorphism.

Remark 3.1.10. One could more generally define a dynamical torus bundle to be a tuple $(B, f, \mathcal{T}, f^*\mathcal{T} \cong \mathcal{T}^{\otimes A})$ consisting of a proper base variety B, an endomorphism $f : B \to B$, a torus bundle $\mathcal{T}$ and an isomorphism $f^*\mathcal{T} \cong \mathcal{T} \circ A$ for some endomorphism $A : M \to M$ which is diagonalizable over $\mathbb{R}$ and whose eigenvalues are real numbers > 1.

3.2. Toric bundles and their morphisms.

Definition 3.2.1. A *toric bundle* over a scheme B is a scheme $\mathcal{X}$ over B with the action of a torus $\mathbb{T}$ such that, Zariski locally on $U \subseteq B$, $\mathcal{X}$ is $\mathbb{T}$-equivariantly of the form $X_\Sigma \times U$ for a fixed fan Σ in $N_{\mathbb{R}}$.

There is an underlying $\mathbb{T}$-torsor $\mathcal{T}$ such that $\mathcal{X}$ is of the form $\mathcal{X}_\Sigma = (\mathcal{T} \times X_\Sigma)/\mathbb{T}$, where $x \in \mathbb{T}$ acts by (x, x^{-1}). The quotient $(\mathcal{T} \times X_\Sigma)/\mathbb{T}$ can be defined using Zariski descent, c.f. [CT01, Construction 2.1.2]. We will call Σ the underlying fan of $\mathcal{X}$.

Remark 3.2.2. It is useful to consider less restrictive definitions in other contexts. For instance the only model of a toric variety over a DVR that is a toric bundle is the canonical model.

We define a class of maps of toric bundles respecting the toric structure. We first give a definition for a fixed base variety B inspired by the case of toric varieties.

Definition 3.2.3. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be toric bundles on B for split tori $\mathbb{T}_1$ and $\mathbb{T}_2$ respectively. Let $\phi : \mathbb{T}_1 \to \mathbb{T}_2$ be a group homomorphism. A morphism $f : \mathcal{T}_1 \to \mathcal{T}_2$ over B is ϕ-equivariant if the diagram

$$\begin{array}{ccc} \mathbb{T}_1 \times \mathcal{X}_1 & \xrightarrow{\phi \times f} & \mathbb{T}_2 \times \mathcal{X}_2 \\ \mu \downarrow & & \downarrow \mu \\ \mathcal{X}_1 & \xrightarrow[f]{} & \mathcal{X}_2 \end{array}$$

commutes. We say that f is equivariant if it is ϕ-equivariant for some ϕ. It will be called *non-degenerate* if its fiberwise image intersects the open dense torus on all fibers and *degenerate* otherwise.

Example 3.2.4. Let X_Σ be a toric variety. The $\mathbb{T}$-orbits on X_Σ are in bijection to the cones of Σ. The closure $V(\sigma)$ of a $\mathbb{T}$-orbit corresponding to a cone $\sigma \in \Sigma$ is itself naturally a toric variety though for a quotient $\mathbb{T}'$ of the original torus $\mathbb{T}$. Its character lattice is identified with $M \cap \sigma^\perp$. The fan defining $V(\sigma)$ is denoted $\Sigma(\sigma)$. Given a toric bundle of the form $\mathcal{X}_\Sigma$, there is a closed subvariety $\mathcal{V}(\sigma)$ given by $(\mathcal{T} \times \mathcal{V}(\sigma))/\mathbb{T}$. This is again a toric bundle. The isomorphism class of its underlying torus bundle is given by $M \cap \sigma^\perp \to M \to \mathrm{Pic}(B)$.

A morphism of this form is called a closed embedding of toric bundles.

Let us classify non-degenerate equivariant morphisms of toric bundles using combinatorial data. We can apply the theory of torus bundles since a morphism of toric bundles $\mathcal{X}_1 \to \mathcal{X}_2$ restricts to a map of the underlying torus bundle $\mathcal{T}_1 \to \mathcal{T}_2$. Note that a toric bundle is defined by the data of a torus bundle together with a fan Σ on $N_\mathbb{R}$.

Proposition 3.2.5. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be toric bundles defined by the data of underlying torus bundles $\mathcal{T}_i : \mathcal{M}_i \to \mathcal{P}ic(B)$ and fans Σ_i. Let $\phi : \mathbb{T}_1 \to \mathbb{T}_2$ be a group homomorphism such that for every cone $\sigma_1 \in \Sigma_1$ there exists a cone $\sigma_2 \in \Sigma_2$ such that $H(\phi)(\sigma_1) \subseteq \sigma_2$. Then the non-degenerate ϕ-equivariant morphisms from $\mathcal{X}_1$ to $\mathcal{X}_2$ are in bijection with the ϕ-equivariant maps of torus bundles $\mathcal{T}_1 \to \mathcal{T}_2$. If the condition on the fan is not satisfied there are no ϕ-equivariant morphisms.

In particular, ϕ-equivariant maps are in bijection to sections of the $\mathbb{T}_2$-torsor defined by

$$\mathcal{T}_2^\vee \otimes (\mathcal{T}_1 \circ H^\vee(\phi)) : \mathcal{M}_2 \to \mathcal{P}ic(B).$$

Proof. This is a combination of Proposition 3.2.5 and [BPS14, Theorem 3.2.4]. □

Example 3.2.6. Degenerate equivariant morphisms are not as easy to describe. For instance, there is no description of equivariant morphisms as the composition of non-degenerate morphisms and the inclusion of a closed orbit as in the non-relative setting.

An example of a degenerate morphism, where this property fails can be given as follows. Pick as base variety a toric variety B for the torus $\mathbb{T}$. We consider the trivial toric bundles $\mathbb{T} \times B$ and $B \times B$. Then, $(m, \mathrm{id}) : \mathbb{T} \times B \to B \times B$ defines a degenerate morphism if $B \neq \mathbb{T}$. More precisely, the image of (m, id) restricted to fibre over $b \in B$ lands in the orbit of b.

Another property that is not inherited from the absolute case where $B = *$ is that in order to specify an equivariant morphism $X_{\Sigma_1} \to X_{\Sigma_2}$ it suffices to give a map of tori $\phi : \mathbb{T}_1 \to \mathbb{T}_2$ respecting the fans and a ϕ-equivariant map $\mathcal{T}_1 \to X_{\Sigma_2}$. The failure of this property is illustrated by the fact that the map $(m, \mathrm{id}) : \mathbb{T} \times B \to B \times B$ does not extend to a map $B \times B \to B \times B$ for $B \neq \mathbb{T}$.

The above example motivates the following definition.

Definition 3.2.7. An equivariant morphism is called a *morphism of toric bundles* if it is the composition of a non-degenerate equivariant morphism followed by a closed embedding of toric bundles.

We would like to allow for changes of the base variety. For this note that the base change of a toric bundle is the toric bundle associated to the same fan Σ and the pullback torus bundle. We will use also use the word pullback and related notation in the context of toric bundles.

Definition 3.2.8. A morphism of toric bundles $f = (f_t, f_b) : (\mathcal{X}_1 \to B_1) \to (\mathcal{X}_2 \to B_2)$ is the datum of a map $f_b : B_1 \to B_2$ and a morphism of toric bundles $f_t : \mathcal{X}_1 \to f_b^* \mathcal{X}_2$.

Example 3.2.9. Let $\mathcal{T}$ be a torus bundle and Σ a fan on the co-character space of its torus. Let $\mathcal{X}$ be the toric bundle associated to this data and denote by $\mathcal{X}^{\otimes n}$ the toric bundle associated to $\mathcal{T}^{\otimes n}$ and Σ. Then, the map of torus bundles $[n] : \mathcal{T} \to \mathcal{T}^{\otimes n}$ extends to a non-degenerate map of toric bundles $[n] : \mathcal{X} \to \mathcal{X}^{\otimes n}$.

Example 3.2.10. Let us extend Example 3.1.8. From Proposition 3.2.5 it easily follows that the multiplication on G extends to actions on its compactifications as toric bundles. We note that multiplication by n on semiabelian varieties can be seen as the unique toric morphism with underlying map $f_b = [n] : A \to A$ and map on tori $[n] : \mathbb{T} \to \mathbb{T}$ respecting the rigidification. Again we may apply Theorem 3.2.5 to obtain that it extends to any compactification as a morphism of toric bundles. We remark that homomorphisms of semiabelian varieties can be viewed as morphisms of toric bundles and that they extend to morphisms of compactifications if this is true on the toric part.

3.3. **Divisors on toric bundles.** Although we are mainly interested in the case of torus invariant Cartier divisors on toric bundles it does not add much complexity to consider more general $\mathbb{T}$-linearized sheaves. They are convenient as contrary to Cartier divisors one can always pull them back.

Definition 3.3.1. A $\mathbb{T}$-linearized quasicoherent sheaf $\mathcal{F}$ on a toric variety X consists of a quasicoherent sheaf $\mathcal{H}$ on X together with an isomorphism $m^*\mathcal{H} \cong pr_2^*\mathcal{H}$, where m denotes the action $\mathbb{T} \times X \to X$. The category of $\mathbb{T}$-linearized sheaves is denoted by $\mathrm{QCoh}_{\mathbb{T}}(X)$.

A toric Cartier divisor D gives rise to a $\mathbb{T}$-linearized line bundle $\mathcal{O}(D)$ with the $\mathbb{T}$-linearization given by the morphism $m^*\mathcal{O}(D) \to pr_2^*\mathcal{O}(D)$ being given by sending m^*s_D to $pr_2^*s_D$ where s_D is the distinguished rational section. If a toric Cartier divisor additionally is endowed with a toric metric, it follows that the isomorphism $m^*\mathcal{O}(D) \to pr_2^*\mathcal{O}(D)$ is in fact an isometry over the preimage of $0 \times X^{\mathrm{an}} \subseteq N_{\mathbb{R}} \times X^{\mathrm{an}}$ in $(\mathbb{T} \times X)^{\mathrm{an}}$.

We fix the following setup. Let X be a variety with an action of $\mathbb{T}$ and let $\mathcal{T}$ be a $\mathbb{T}$-bundle on B. Let $\mathcal{X} = (\mathcal{T} \times X)/\mathbb{T}$. We call $\mathcal{X} \to B$ a $\mathbb{T}$-fibre bundles. A morphism of torus fibre bundles is a map induced by a map $\mathcal{T}_1 \to \mathcal{T}_2$ of torus bundles and an equivariant map of varieties $X \to Y$ with torus actions. We obtain the notion of (adelically) metrized torus fibre bundles by adding metrics to the underlying torus bundle.

Proposition 3.3.2. There is a unique functor

$$\rho : \mathrm{QCoh}_{\mathbb{T}}(X) \to \mathrm{QCoh}(\mathcal{X})$$

compatible with tensor products satisfying the following conditions:

1. If $B = \{*\}$ and $\mathcal{T}$ is the trivial bundle, the map is the identity.
2. It commutes with pullback for morphisms of torus fibre bundles, i.e. if $\mathcal{F} : \mathcal{X}_1 \to \mathcal{X}_2$ is a morphism with underlying morphism of varieties with torus action $f : X_1 \to X_2$, then $\mathcal{F}^*\rho(\mathcal{H}) = \rho(f^*\mathcal{H})$.

Proof. Locally on a Zariski open $U \subseteq B$ we may choose a trivialization of $\mathcal{T}$. This may be seen as a collection of sections $s_m \in \Gamma(\mathcal{T}(m))$ satisfying $s_{m_1+m_2} = s_{m_1} \otimes s_{m_2}$. Since Cartier divisors are a local notion we may assume w.l.o.g. that $U = B$. The trivialization of $\mathcal{T}$ induces a trivialization $s : \mathcal{X} \to X_\Sigma$. In this situation we set ρ to be s^*. The ambiguity in this definition is resolved by applying the $\mathbb{T}$-linearization. The constructed map is the unique map satisfying the imposed conditions. □

Remark 3.3.3. One may define $\rho : \mathrm{Div}_{\mathbb{T}}(X_\Sigma) \to \mathrm{Div}_{\mathbb{T}}(\mathcal{X}_\Sigma)$ on the level of Cartier divisors. The divisors D and D' being linearly equivalent does not imply that $\rho(D)$ and $\rho(D')$ are linearly equivalent.

In the metrized setting it is natural to replace $\mathbb{T}$ by the elements of unit norm and $\mathcal{T}$ by the part of the torus bundle of unit norm. This is dictated to us since we want compatibility of the two constructions in the case of model metrics. We will avoid the use of quotients due to their technical difficulty in algebraic and nonarchimedean geometry.

We denote by $\mathrm{Vect}_{\mathbb{T}}(X)$ the category of $\mathbb{T}$-linearized vector bundles on X. We denote by $\mathrm{Vect}_{\mathbb{T}}^{\mathrm{metr}}(X)$ the category of metrized vector bundles with a $\mathbb{T}$-linearization which is an isometry over the preimage of $0 \times X^{\mathrm{an}} \subseteq N_{\mathbb{R}} \times X^{\mathrm{an}}$. Let $\widehat{\mathrm{Vect}}_{\mathbb{T}}(X)$ be the category of $\mathbb{T}$-linearized adelic vector bundles, i.e. the category of $\mathbb{T}$-linearized vector bundles V with metrics at all places such that the $\mathbb{T}$-linearization is an isometry over the preimage of $0 \times X^{\mathrm{an}}$ and there is a dense open $U \subseteq \mathcal{S}$ such that the metrics over U are defined by a model V_U.

Proposition 3.3.4. Let B be a variety over a valued field K and $\mathcal{X}$ be a metrized toric bundle with fan Σ and underlying metrized torus bundle $\mathcal{T}^{\mathrm{metr}}$ over B. Let X_Σ be the toric variety associated to Σ. Then, there is a unique monoidal functor

$$\rho^{\mathrm{metr}} : \mathrm{Vect}_{\mathbb{T}}^{\mathrm{metr}}(X_\Sigma) \to \mathrm{Vect}_{\mathbb{T}}^{\mathrm{metr}}(\mathcal{X}_\Sigma)$$

satisfying the following conditions:

(1) If $B = \{*\}$ and $\mathcal{T}^{\mathrm{metr}}$ is the trivial bundle, the map is the identity.
(2) It commutes with pullback for morphisms of metrized torus fibre bundles, i.e. if $\mathcal{F} : \mathcal{X}_1 \to \mathcal{X}_2$ is a morphism with underlying morphism of varieties with torus action $f : X_1 \to X_2$, then $\mathcal{F}^* \rho^{\mathrm{metr}}(\mathcal{H}) = \rho^{\mathrm{metr}}(f^* \mathcal{H})$.
(3) Let $\phi : \mathcal{X}_1 \to \mathcal{X}_2$ be a morphism of metrized torus fibre bundles. Then, there is a homomorphism $|f| : M_2 \to \mathrm{Cont}(B^{\mathrm{an}}, \mathbb{R})$ associated to the induced map $f : \mathcal{T}_1 \to \mathcal{T}_2$ giving the norm of f. Then, it holds that $f^* \circ \rho^{\mathrm{metr}}_{\widehat{\mathcal{T}}_2(-|f|)} = \rho^{\mathrm{metr}}_{\widehat{\mathcal{T}}} \circ \phi^*$. The subscript emphasizes the dependence on the metric of the underlying torus bundle.

The homomorphism ρ^{metr} automatically restricts to ρ on underlying vector bundles.

Proof. We cannot copy the proof from the non-metrized verbatim since we cannot assume the existence of local trivializing sections of constant norm 1. We can, however, trivialize the underlying torus bundle and correct for the non-triviality of the norm.

Let s be a local trivialization of $\mathcal{T}$. Then, there is a continuous map $|s| : B^{an} \to \mathbb{T}^{trop}$ given by the metrics. This allows us to write $\mathcal{X} \cong B \times X_\Sigma$. We define the vector bundle by pullback. We proceed to define the metric.

Under the isomorphism $m^*V \simeq pr^*V$ for $m, pr : \mathbb{T} \times X_\Sigma \to X_\Sigma$ we have the following relationship on norms. Suppose v is a nowhere vanishing section of $m^*V \simeq pr^*V$ over some open $U \subset \mathbb{T} \times X_\Sigma$. Then, the quotient of norms $\delta_v = \frac{|v|_{m^*V}}{|v|_{pr^*V}}$ factors over $\mathbb{T}^{trop} \times X_\Sigma$ by the assumption that the linearization be an isometry over $0 \in \mathbb{T}^{trop}$.

Let $v \in V(U)$ for $U \subseteq X_\Sigma$ be a nowhere vanishing section. We define the norm of $pr^*v \in \rho(V)(B \times U)$ to be $|v(u)| \cdot \delta_v(|s|, u)$. It is easy to see that it doesn't depend on the choice of section. This construction defines the unique functor satisfying the conditions on ρ^{metr}. $\square$

Proposition 3.3.5. Let B be a variety over a global field K and $\mathcal{X}$ be an adelic toric bundle with fan Σ and underlying adelic torus bundle $\widehat{\mathcal{T}}$ over B. Let X_Σ be the toric variety associated to Σ. Then, there is a unique homomorphism

$$\widehat{\rho} : \widehat{\mathrm{Vect}}_{\mathbb{T}}(X_\Sigma) \to \widehat{\mathrm{Vect}}(\mathcal{X}_\Sigma)$$

which restricts to ρ^{metr} at each place.

Proof. We need to show metrics defined by ρ^{metr} glue together to yield an adelic metric. Let $\overline{V}$ be an adelic vector bundle. Let $U \subset \mathcal{S}$ be an open dense subset on which both $\widehat{\mathcal{T}}$ and $\overline{V}$ have a model. By assumption, there is a model $\mathcal{B}$ over U and a $\mathbb{T}$-torsor $\widehat{\mathcal{T}}_U$ defining the metrics of $\widehat{\mathcal{T}}$ at almost all places. Then, $\rho_{\widehat{\mathcal{T}}_U}(\overline{V}_U)$ defines the metrics at all places included in U. Thus one obtains an adelic vector bundle. □

We note that one may analogously define homomorphisms for Cartier divisors $\rho : \mathrm{Div}_{\mathbb{T}}(X_\Sigma) \to \mathrm{Div}_{\mathbb{T}}(X_\Sigma)$, $\rho^{\mathrm{metr}} : \mathrm{Div}^{\mathrm{metr}}_{\mathbb{T}}(X_\Sigma) \to \mathrm{Div}^{\mathrm{metr}}_{\mathbb{T}}(X_\Sigma)$ and $\widehat{\rho} : \widehat{\mathrm{Div}}_{\mathbb{T}}(X_\Sigma) \to \widehat{\mathrm{Div}}_{\mathbb{T}}(X_\Sigma)$. The rest of the section is devoted to study their positivity properties.

Lemma 3.3.6. Let $\overline{D}$ be a semipositive toric metrized divisor on X_Σ with underlying divisor D whose Newton polytope is Δ. Let $\overline{L}$ be a metrized line bundle on B such that $\pi^*\overline{L} + \mathcal{T}^{\mathrm{metr}}(m)$ is semipositive for all $m \in \Delta \cap M_{\mathbb{Q}}$. Then, the line bundle $\pi^*\overline{L} + \rho^{\mathrm{metr}}(\overline{D})$ is semipositive.

Proof. Denote the local roof function of $\overline{D}$ by θ and let $\overline{L}(m) := \pi^*\overline{L} + \mathcal{T}^{\mathrm{metr}}(m) + \theta(m)$. For $m \in \Delta \cap M_{\mathbb{Q}}$, the line bundle $\widehat{\mathcal{T}}(m) + \theta(m)$ on B parametrizes χ^m-equivariant sections of $\rho^{\mathrm{metr}}(\overline{D})$. We approximate the metric on $\pi^*\overline{L} + \rho^{\mathrm{metr}}(\overline{D})$ by the maximum over the semipositive metrics defined on the χ^m-invariant $\mathbb{Q}$-sections $\overline{L}(m)$. The semipositivity of $\pi^*\overline{L} + \rho^{\mathrm{metr}}(\overline{D})$ follows immediately.

Recall that given a globally generated line bundle L on X, a norm $\|\cdot\|$ on $H^0(X, L)$ induces a metric on L by the quotient norm under the restriction map $H^0(X, L) \to H^0(x, L^{an}|_x)$ for $x \in X^{an}$. For two metrics on a line bundle we define their distance to be the supremum norm of the function

$$\log \frac{|\cdot|'}{|\cdot|}.$$

Let $l > 1$ be a number such that $\log |l\Delta \cap M|/l < \epsilon$ and the metric induced by the supremum norm on $\Gamma(lD)$ differs from the norm on $\overline{D}$ by at most ϵ. Such an l exists by the characterization of semipositive metrics on semiample line bundles in [CM18, Section 3.3] in the non-archimedean and [Mor15] in the archimedean case. On $\Gamma(lD)$ we know the following inequality of norms from [Bur+16, Corollary 5.4]. For $s = \sum_{m \in \frac{1}{l}\Delta \cap M} \gamma_m \chi^m \in \Gamma(lD)$, we have

$$\max_{m \in \frac{1}{l}\Delta \cap M} \|\gamma_m \chi^m\|_{\sup} \le \|s\|_{\sup} \le |l\Delta \cap M| \max_{m \in \frac{1}{l}\Delta \cap M} \|\gamma_m \chi^m\|_{\sup}.$$

The expression $\max_{m\in\frac{1}{l}\Delta\cap M}\|\gamma_m\chi^m\|_{\sup}$ defines a norm on $\Gamma(lD)$ and by taking the quotient norm a norm on D. The distance of this norm to that on $\overline{D}$ is at most 2ϵ.

We now define a norm on $\rho(D)+\pi^*L$ approximating the norm on $\widehat{\rho}(\overline{D})+\pi^*\overline{L}$. Locally on B, we can decompose sections of $l(\rho(D)+\pi^*L)$ into eigenspaces, i.e. sections will be of the form $\sum_{m\in\frac{1}{l}\Delta\cap M}\gamma_m$ for sections $\gamma_m\in\Gamma(U,lL(m))$. We metrize $l(\rho(D)+\pi^*L)$ by taking the maximum of the norms on $lL(m)$. The induced metric on $\rho(D)+\pi^*L$ differs at most by 2ϵ from the metric on $\pi^*\overline{L}+\rho^{\mathrm{metr}}(\overline{D})$. Furthermore it is semipositive since it is the maximum of semipositive metrics. □

Corollary 3.3.7. If $\mathcal{T}^{\mathrm{metr}}$ is integrable, then ρ^{metr} preserves integrability. If $\mathcal{T}^{\mathrm{metr}}$ is flat, then ρ^{metr} preserves semipositivity.

Proof. Let $\overline{D}$ be a semipositive toric divisor. We need to show that there exists a semipositive line bundle $\overline{L}$ on the base B such that $\widehat{\rho}(\overline{D})+\pi^*\overline{L}$ is semipositive. For this it suffices to find a semipositive $\overline{L}$ such that $\widehat{\mathcal{T}}(m)+\pi^*\overline{L}$ is semipositive for all $m\in\Delta$. We can replace the condition for all $m\in\Delta$ by a condition for finitely many points since semipositive divisors form a convex cone. For any single m, this is possible by the assumption that $\widehat{\mathcal{T}}$ be integrable. Summing up over semipositive line bundles yields a single $\overline{L}$ for all $m\in\Delta$. The second claim is immediate from Lemma 3.3.6. □

Example 3.3.8. We freely use the notation of Example 3.2.9. Let $(B,f,\mathcal{T},f^*\mathcal{T}\cong\mathcal{T}^{\otimes r})$ be a dynamical torus bundle and let $\widehat{\mathcal{T}}$ be $\mathcal{T}$ endowed with its canonical metric. Then, for any fan Σ there is an endomorphism of the toric bundle associated to $\mathcal{T}$ and Σ given by

$$\mathcal{F}:\mathcal{X}\xrightarrow{\square^r}\mathcal{X}^{\otimes r}\cong f^*\mathcal{X}\to\mathcal{X}.$$

There is a canonical isomorphism $\mathcal{F}^*\rho(D)\cong r\rho(D)$ that induces a canonical metric on $\rho(D)$ which we will denote by $\overline{\rho(D)}$. There is an isometry $\overline{\rho(D)}\cong\widehat{\rho}_{\widehat{\mathcal{T}}}(D)$. This, in particular, applies to multiplication by r on semiabelian varieties.

3.4. **Global sections.** We would now like to study the global sections of vector bundles constructed in the previous section. Let X be a variety with an action of $\mathbb{T}$ and let $\mathcal{T}$ be a $\mathbb{T}$-bundle on B. Let $\mathcal{X}=(\mathcal{T}\times X)/\mathbb{T}$ be the corresponding torus fibre bundle. Let $V\in\mathrm{Vect}_{\mathbb{T}}(X)$ be a $\mathbb{T}$-linearized vector bundle. Note that the pushforward π_*V along a $\mathbb{T}$-equivariant morphism is naturally endowed with a $\mathbb{T}$-linearization. After recalling [KL84, Proposition 1.8] in our notation, we prove an adaptation to the metrized setting.

Lemma 3.4.1. Let $\mathcal{F}$ be a $\mathbb{T}$-linearized quasi-coherent sheaf on X. Let $\pi:X\to Y$ be a $\mathbb{T}$-equivariant morphism of $\mathbb{T}$-varieties. Let $\mathcal{T}$ be a $\mathbb{T}$-bundle on B and

$\Pi : \mathcal{X} \to \mathcal{Y}$ be the induced map on torus fibre bundles. Then,

$$\Pi_*\rho(\mathcal{F}) \cong \rho(\pi_*\mathcal{F}).$$

In particular, if X is a toric variety and $Y = *$ the pushforward of $\mathcal{O}(\rho(\Delta))$ can be decomposed into eigenspaces as

$$\pi_*\mathcal{O}(\rho(\Delta)) \cong \bigoplus_{m \in \Delta \cap M} \mathcal{T}(m).$$

Proof. Assume first that $\mathcal{T}$ is trivial. In this case the assertion follows from flat base change. The identification given by flat base change does not depend on the choice of trivialization of $\mathcal{T}$. Hence, the isomorphism globalizes. ☐

Suppose that $Y = *$ and $\overline{V}$ is a metrized vector bundle on a proper X. Suppose further that $\mathcal{T}$ is given a metric structure. Then, the $\pi_*V = H^0(X, V)$ has the structure of a $\mathbb{T}$-linearized metrized vector bundle with norm given by the supremum norm. On the other hand, $\Pi_*\rho(V)$ carries a metric given by the fibrewise supremum norm.

Proposition 3.4.2. The isomorphism

$$\Pi_*\rho^{\mathrm{metr}}(\overline{V}) \cong \rho^{\mathrm{metr}}(H^0(X, V)_{\sup}).$$

is an isometry.

Proof. Suppose that $\mathcal{T}$ is trivial and the trivialization is a normed trivialization over $b \in B^{\mathrm{an}}$. Then by construction, the metrized vector bundle $\rho^{\mathrm{metr}}(\overline{V})$ restricted to $\Pi^{-1}(b)$ is precisely $\overline{V}$. Hereby the claim follows. ☐

Lemma 3.4.3. The eigenspaces $\mathcal{T}(m)$ of the metrized vector bundle $\pi_*\mathcal{O}(\rho^{\mathrm{metr}}(\overline{\Delta}))$ are orthogonal to one another. Each eigenspace $\mathcal{T}(m)$ carries the metric $\mathcal{T}^{\mathrm{metr}}(m) + \theta_v(m)$.

Proof. The claim follows by Proposition 3.4.2 and a study of global sections on toric varieties. The orthogonality claim follows immediately from Theorem 2.3.4. The claim on the metrics on each eigenspace follows by the definition of the local roof function in Definition 2.3.2. ☐

3.5. **Arithmetic intersections.** This section is devoted to a generalization of [Cha00, Proposition 4.1]. We work on an integrable smooth projective toric bundle $\mathcal{X}$ with underlying adelic torus bundle $\widehat{\mathcal{T}} \to B$ with B smooth projective.

Let $\widetilde{\Sigma}$ be a smooth projective adelic fan with recession fan Σ. Let τ_i denote the rays of $\widetilde{\Sigma}$ and let h_i denote the dual polytopes. For notation on adelic polytopes, we refer to Section 2.1.

Proposition 3.5.1. Let $\tau_1, \dots, \tau_r$ be rays in $\widetilde{\Sigma}$ not spanning a cone. Let $\mathcal{X}_\Sigma$ be a toric bundle whose underlying torus bundle is endowed with an adelic metric $\widehat{\mathcal{T}}$ with model metrics at finite places and a smooth metric at archimedean places. Let $\gamma \in \widehat{\mathrm{CH}}_r(\widehat{\mathcal{X}})$ be an arithmetic cycle on a regular model $\widehat{\mathcal{X}}$ of $\mathcal{X}_\Sigma$. Then,

$$\widehat{\deg}(\widehat{\rho}(h_1) \cdots \widehat{\rho}(h_r)\gamma) = 0.$$

Proof. Note first that if $\tau_1, \dots, \tau_r$ are not supported at one place the conclusion holds since one may find a regular scheme $\mathcal{Y}$ on which the $\widehat{\rho}(h_i)$ can be defined in such a way that they are still supported at the same places. On $\mathcal{Y}$ the intersection of the $\widehat{\rho}(h_i)$ is empty. Then, the conclusion follows from [GS92, Theorem 3 (2)].

Now consider the case that all τ_i are supported at a non-archimedean place v. Observe that the adelic torus bundle is induced by the pullback of an adelic torus bundle on a regular projective scheme $\mathcal{Z}$ by projectivity of B. Note that the fan $\widetilde{\Sigma}_v \subset N_\mathbb{R} \times \mathbb{R}_{\geq 0}$ not only defines a model, but also a smooth toric variety of dimension $t+1$. Denote the monomial corresponding to the dual of $\mathbb{R}_{\geq 0}$ by T. The model $\mathscr{X}_{\widetilde{\Sigma}_v}$ over R_v is given as the closed subscheme of $X_{\widetilde{\Sigma}_v, R_v}$ cut out by $T = \pi$.

The torus bundle $\widehat{\mathcal{T}} \oplus \mathbb{G}_m$ on $\mathcal{Z}$ and the fan $\widetilde{\Sigma}$ define a toric bundle over $\mathcal{B}$. Since a smooth scheme over a regular base is regular the toric bundle $\mathcal{X}_{\widetilde{\Sigma}_v}$ is in fact regular and the Cartier divisors $\widehat{\rho}(h_i)$ are defined by pullback of $\rho(h_i)$ from there. The intersection of the $\rho(h_i)$ in $\mathcal{X}_{\widetilde{\Sigma}_v}$ is empty. Hence, the claim follows again by [GS92, Theorem 3 (2)].

It remains to deal with the case that the τ_i are supported at an archimedean place. We allow ourselves to work on the toric bundle $\mathcal{X}_\Sigma$.

Let us prove the vanishing of the current $g_{\widehat{\rho}(h_1)}\omega(\widehat{\rho}(h_2)) \cdots \omega(\widehat{\rho}(h_r))$. Note that the Green's functions of the h_i are given by piecewise linear functions $g_i : N_\mathbb{R} \to \mathbb{R}$. On the preimage of an open subset $U \subset N_\mathbb{R}$ where g_i vanishes the metric is flat. In fact, the standard section of $\rho(D_i)$ is flat on this locus. The empty intersection condition in the combinatorial setting implies that $\prod g_i = 0$. Hence, $g_{\widehat{\rho}(\overline{D}_0)}$ vanishes on the support of $\omega(\widehat{\rho}(\overline{D}_1)) \cdots \omega(\widehat{\rho}(\overline{D}_t))$. We note that by approximation the intersection number is precisely the integral of $g_{\widehat{\rho}(h_1)}\omega(\widehat{\rho}(h_2)) \cdots \omega(\widehat{\rho}(h_r))$ against the curvature form of γ. This finishes the proof. □

For the n-th power map $\square^n : \mathcal{X} \to \mathcal{X}^{\otimes n}$ introduced in Example 3.2.9 we have $(\square^n)^*\widehat{\rho}_{\mathcal{X}^{\otimes n}}(\overline{D}) = \widehat{\rho}_\mathcal{X}([n]^*\overline{D})$. Moreover, for any toric $\mathbb{Q}$-Cartier divisor $\overline{D}$ on a model $X_{\widetilde{\Sigma}}$, there is a unique $\mathbb{Q}$-Cartier divisor $\overline{D}'$ on the same model such that $[n]^*\overline{D}' = \overline{D}$. This implies that the above proposition applies more generally to algebraic torus bundles because $(\square^n)^*\widehat{\rho}_{\mathcal{X}^{\otimes n}}(\overline{D})$ is algebraically metrized. Hence the conclusion stays valid for algebraically metrized torus bundles by passing to $\mathcal{X}^{\otimes n}$.

We need an approximation result in order to apply this for integrable torus bundles that are not necessarily algebraic.

Proposition 3.5.2. Let $\tau_1, \dots, \tau_r$ be rays in $\widetilde{\Sigma}$ not spanning a cone. Let $\mathcal{X}_\Sigma$ be a toric bundle whose underlying torus bundle is endowed with integrable adelic metrics $\widehat{\mathcal{T}}$. Let $\gamma \in \widehat{\mathrm{CH}}_r(\widehat{\mathcal{X}})$ be an arithmetic cycle on a regular model $\widehat{\mathcal{X}}$ of $\mathcal{X}_\Sigma$. Then,

$$\widehat{\deg}(\widehat{\rho}(h_1)\cdots\widehat{\rho}(h_r)\gamma) = 0.$$

Proof. We approximate $\widehat{\mathcal{T}}$ by algebraic metrics $\widehat{\mathcal{T}}_k$. Pick a basis $m_1, \dots, m_t$ of M. For each m_i we can write $\widehat{\mathcal{T}}(m_i)$ as the difference of limits algebraic semipositive metrics, i.e. there are semipositive algebraic line bundles $L^+_{k,i}$ and $L^-_{k,i}$ such that for $k \to \infty$ the metrics converge uniformly to semipositive line bundles L^+_i and L^-_i such that $\widehat{\mathcal{T}}(m_i) = L^+_i - L^-_i$. Extending the map $m_i \mapsto L^+_{k,i} - L^-_{k,i}$ by linearity to M allows us to approximate $\widehat{\mathcal{T}}$ by algebraic metrics $\widehat{\mathcal{T}}_k$.

Note that for C big enough the line bundle $C\sum_i \left(L^+_{k,i} + L^-_{k,i}\right) + \widehat{\rho}_{\widehat{\mathcal{T}}_k}(\overline{D}_j)$ is semipositive for all $j = 0, \dots, t$ and all k. We are done once we show that $\widehat{\rho}_{\widehat{\mathcal{T}}_k}(\overline{D})$ converges uniformly to $\widehat{\rho}_{\widehat{\mathcal{T}}}(\overline{D})$ for $\overline{D} = \overline{D}_j$ for some j. Fix a norm $\|\cdot\|$ on $N_\mathbb{R}$. Since the Green's function g of $\overline{D}$ on $N_\mathbb{R}$ is piecewise linear on finitely many polyhedra it is Lipschitz continuous with some Lipschitz constant K. Varying the metric on the torus bundle by vector $v(b)$ in $N_\mathbb{R}$ of length at most ϵ at a $b \in B^{an}$ translates the Green's function by $v(b)$ on the fibre at b. By the Lipschitz continuity, we obtain $\|g(x) - g_j(x-v)\|_{\sup} \le K\|v\|$. The uniform convergence of line bundles $\widehat{\mathcal{T}}_k(m) \to \widehat{\mathcal{T}}(m)$ implies that the difference of $\widehat{\mathcal{T}}_k$ and $\widehat{\mathcal{T}}$ given by $v_k : B^{an} \to N_\mathbb{R}$ converges uniformly to 0. ☐

3.6. Successive minima of toric bundles.

Let X be a proper variety and let $\overline{L}$ be an adelic line bundle on X whose underlying line bundle is big. For a real number λ, the set $X_{\overline{L}}(\lambda)$ denotes the Zariski closure of

$$\{x \in X(\bar{K}) \mid h_{\overline{\mathcal{L}}}(x) \le \lambda\}.$$

Since X is a noetherian topological space the filtration given by varying λ has only finitely many stages. It yields the so-called height filtration

$$X_0 = \emptyset \subsetneq X_1 \subsetneq \cdots \subsetneq X_r = X.$$

Often, one considers the sub-filtration consisting of the X_i such that $\dim X_i > \dim X_{i-1}$. This filtration is closely related to the notion of Zhang minima. Let $\mathcal{X}$ be a proper toric bundle and $\pi^*\overline{\mathcal{M}} + \widehat{\rho}(\overline{D})$ be an adelic line bundle on $\mathcal{X}$ whose underlying line bundle is big. Then, there are other natural filtrations.

Lemma 3.6.1. The set $\mathcal{X}_\lambda = \overline{\{x \in \mathcal{X}(\bar{K}) \mid h_{\overline{\mathcal{L}}}(x) \leq \lambda\}}$ is invariant under the torus action.

Proof. It is enough to prove that $\mathcal{X}_\lambda(\bar{K})$ is stable under the action of $\mathbb{T}(\bar{K})$ as reduced schemes are determined by their $\bar{K}$-points. The height is invariant under the action of torsion points of $\mathbb{T}$. This is checked fibrewise. Torsion points of $\mathbb{T}$ tropicalize to 0 at all places. Hence, their action does not affect the tropicalization of points in the fibre X_Σ and thus their height.

Since torsion points are Zariski dense in $\mathbb{T}$, it follows that $\mathcal{X}_\lambda$ contains the $\mathbb{T}(\bar{K})$-orbit of any $x \in \mathcal{X}(\bar{K})$. In particular, we may view $\mathcal{X}_\lambda(\bar{K})$ as the closure of the $\mathbb{T}(\bar{K})$-invariant set $\mathbb{T}(\bar{K})\{x \in \mathcal{X}(\bar{K}) \mid h_{\overline{\mathcal{L}}}(x) \leq \lambda\}$. The claim now follows from the fact that the action map $\mathbb{T}(\bar{K}) \times \mathcal{X}(\bar{K}) \to \mathcal{X}(\bar{K})$ is continuous. □

It is natural to consider $\mathcal{T}_\lambda = \overline{\{x \in \mathcal{T}(\bar{K}) \mid h_{\overline{\mathcal{L}}}(x) \leq \lambda\}}$ in $\mathcal{T}$, where $\mathcal{T}$ denotes the underlying torus bundle. Then, the resulting filtration

$$\mathcal{T}_0 = \emptyset \subsetneq \mathcal{T}_1 \subsetneq \cdots \subsetneq \mathcal{T}_r = \mathcal{T}$$

is the pullback of a filtration on B which we call the toric filtration. The following equality holds

$$\mathcal{T}_\lambda = \pi^{-1}(\overline{\{b \in B(\bar{K}) \mid \exists x \in \pi^{-1}(b), h_{\overline{\mathcal{L}}}(x) \leq \lambda\}}).$$

One can recover the height filtration on $\mathcal{X}$ from filtrations on B associated to each cone on the underlying fan Σ. For every cone $\sigma \in \Sigma$ we denote the associated closed toric subbundle by $\mathcal{X}_\sigma$ and the associated torus bundle by $\mathcal{T}_\sigma$. We introduce the filtrations

$$B_{\sigma,\lambda} = \overline{\{b \in B(\bar{K}) \mid \exists x \in \pi^{-1}(b) \cap \mathcal{X}_\sigma, h_{\overline{\mathcal{L}}}(x) \leq \lambda\}}.$$

Then we obtain the height filtration on $\mathcal{X}$ as

$$\mathcal{X}_\lambda = \bigcup_{\sigma \in \Sigma} \pi^{-1}(B_{\sigma,\lambda}) \cap \mathcal{X}_\sigma.$$

The last step remaining is to understand the filtrations $B_{\sigma,\lambda}$. We have additional tools available to study fibres of toric bundles since they are toric varieties. For a point $b \in B(\bar{K})$, we may consider the fibre $\mathcal{X}_b$. Let $s : X \cong \mathcal{X}_b$ be a trivialization. Here X is a toric variety with torus $\mathbb{T} \subset X$.

Let $\overline{D}$ be a toric metrized Cartier divisor on X. Then, $\rho^{\mathrm{metr}}_{\mathcal{X}_b}(\overline{D})$ is the translate of $\overline{D}$ by $|s|^{-1} \in \mathbb{T}^{trop}$, when we trivialize the torsor by s. In particular, if $\overline{D}$ is semipositive the local roof function $\theta(\widehat{\rho}(\overline{D}))$ is $\theta(\overline{D}) - \log|s|$ by [BPS14, Proposition 2.3.3], where $\log|s|$ is the linear function $M \to \mathbb{R}$ defined by the norm of s.

If $\overline{D}$ is a semipositive adelic toric divisor its global roof function is $\theta(D)(m) + h_{\widehat{\mathcal{T}}(m)}(b)$. This allows us to apply the results of [BPS15] to fibers of π. We introduce the function h_Δ on B given by $h_\Delta(b) = \max_{m\in\Delta} \theta(b, m)$. The study of the toric filtration boils down to the study of h_Δ and the filtration by $\overline{\{b \in B(\bar{K}) \mid h_\Delta(b) \leq \lambda\}}$.

4. Okounkov bodies of toric bundles and applications

Let B be a variety over K of dimension g. Let $\mathcal{X}$ be a toric bundle over B with underlying torus bundle $\mathcal{T}$ and fan Σ. Let Δ be a polytope whose normal fan coarsens Σ and $\rho(\Delta)$ the corresponding line bundle on $\mathcal{X}$. Let L be a line bundle on B. We next compute the Okounkov body of $\rho(\Delta) + \pi^*L$.

In order to obtain a more convenient flag we refine Σ such that it defines a smooth projective variety, see [CLS10, Chapter 11]. Since the fibres X_Σ are smooth we can take prime toric divisors $D_1, \dots, D_t$ such that

$$X_\bullet : X_\Sigma \supset D_1 \supset D_1 \cap D_2 \supset \cdots \supset D_1 \cap \cdots \cap D_t = \{p\}.$$

defines a flag. We obtain a partial flag on $\mathcal{X}$:

$$\rho(X_\bullet) : \mathcal{X} \supset \rho(D_1) \supset \rho(D_1) \cap \rho(D_2) \supset \cdots \supset \rho(D_1) \cap \cdots \cap \rho(D_t) \cong B.$$

Given a flag $B_\bullet$ on B, we extend $\rho(X_\bullet)$ to a flag $\mathcal{X}_\bullet$ on $\mathcal{X}_\Sigma$. We now compute the Okounkov body of $\rho(\Delta) + \pi^*L$ with respect to $\mathcal{X}_\bullet$.

By translating Δ, we can always ensure that the conditions in Proposition 2.3.1 are satisfied. In the toric bundle setting this requires to change the line bundle L.

Theorem 4.0.1. The Okounkov body of $\mathcal{X}$ with respect to $\rho(\Delta) + \pi^*L$ is given by the closure of

$$\{(m, x) \mid m \in \Delta, x \in \Delta_{B_\bullet}(L + \mathcal{T}(m))\}.$$

For $m \in \Delta$ such that $L + \mathcal{T}(m)$ is big the fibre is given by $\Delta_{B_\bullet}(L + \mathcal{T}(m))$.

Proof. Just as for complete flags one can associate a valuation $\nu_{\rho(X_\bullet)}$ to the partial flag $\rho(X_\bullet)$ as the first t entries of the valuation associated to $\mathcal{X}_\bullet$. By definition, the Okounkov body of $\rho(\Delta) + \pi^*L$ is the closure of the family of Okounkov bodies over $m \in \Delta \cap M_\mathbb{Q}$ of the linear series

$$\operatorname{im}\left(\{s \in H^0(\mathcal{X}, n(\pi^*L + \rho(\Delta))) \mid \nu_{\rho(X_\bullet)}(s) \geq nm\} \xrightarrow{\alpha} H^0(X, n(L + \mathcal{T}(m)))\right).$$

We can understand this by applying Lemma 3.4.1. For $m \notin \Delta$ the image will be 0 since then the domain of the map is 0. On the other hand if $m \in \Delta \cap M_\mathbb{Q}$, the χ^m-equivariant sections are in bijection with $H^0(B, L + \mathcal{T}(m))$. All non-zero χ^m-equivariant sections s satisfy $\nu_{\rho(X_\bullet)}(s) = m$. Therefore, the map α is surjective. Hence, the map of Okounkov bodies $\Delta_{\mathcal{X}_\bullet}(\pi^*L + \rho(\Delta)))$ has fibres containing $\Delta_{B_\bullet}(L + \mathcal{T}(m))$. For fixed $m \in \Delta$ such that $L + \mathcal{T}(m)$ is big, the fibre of the closure of $\{(m, x) \mid m \in \Delta, x \in \Delta_{B_\bullet}(L + \mathcal{T}(m))\}$ doesn't contain points outside $\Delta_{B_\bullet}(L + \mathcal{T}(m))$ by the convexity of Okounkov bodies as a function of the line bundle, see [LM09, Section 4.2]. □

4.1. **Arithmetic version.** Suppose that in addition to the setup we have an adelic structure $\widehat{\mathcal{T}}$ on $\mathcal{T}$. Denote by $\widehat{c}: M \to \widehat{\mathrm{Pic}}(B)$ the homomorphism describing the isomorphism class of $\widehat{\mathcal{T}}$. Suppose further that we have a semipositive toric adelic metric $\overline{D}$ with roof function θ on the divisor associated to Δ and that $\overline{L}$ is adelically metrized.

Theorem 4.1.1. The Boucksom-Chen transform $G_{\pi^*\overline{L}+\widehat{\rho}(\overline{D}),\mathcal{X}_\bullet}(m,x)$ on

$$\{(m,x) \mid m \in \Delta, x \in \Delta_{B_\bullet}(L+\mathcal{T}(m))\}$$

is given by $\theta(m) + G_{\overline{L}+\widehat{\mathcal{T}}(m),B_\bullet}(x)$ when $L+\mathcal{T}(m)$ is big.

Proof. The vector space

$$\mathrm{im}\left(\{s \in H^0(\mathcal{X}, n(\pi^*\overline{L}+\widehat{\rho}(\overline{D}))) \mid \nu_{\rho(X_\bullet)}(s) \geq nm\} \xrightarrow{\alpha} H^0(B, n(L+\mathcal{T}(m)))\right).$$

can be endowed with quotient norms at each, where the left hand side endowed with the supremum norm. The arithmetic Okounkov body of $\overline{L}+\widehat{\rho}(\overline{D})$ is the closure of the arithmetic Okounkov bodies for all the filtered graded linear series. By Lemma 3.4.3, the quotient norm for α agrees with the supremum norm on $H^0(X, n(\overline{L}+\widehat{\mathcal{T}}(m)))$ twisted by $\theta_v(m)$.

By the convexity of arithmetic Okounkov bodies in families the theorem follows. □

Remark 4.1.2. Due to the work of Sombra and Ballaÿ one will be able to deduce equidistribution of small points on some line bundles of the form $\widehat{\rho}(\overline{\Delta}) + \pi^*\overline{L}$ as they can phrase their sufficient condition on equidistribution in terms of the Boucksom-Chen transform, see [BS24, Section 1.4].

4.2. **Application to successive minima.** In this section, we prove Theorem A by studying the Boucksom-Chen transform.

Theorem A. *Let $\overline{L}$ be an adelic line bundle on B such that $\overline{L}+\widehat{c}(m)$ is geometrically big for some $m \in \Delta$. Let Δ° denote the interior of Δ. Then, we have the following formula for the essential minimum of $\widehat{\rho}(\overline{\Delta})+\pi^*\overline{L}$ on $\mathcal{X}$:*

$$\zeta_{\mathrm{ess}}(\widehat{\rho}(\overline{\Delta})+\pi^*\overline{L}) = \sup_{m\in\Delta}\left\{\zeta_{\mathrm{ess}}(\overline{L}+\widehat{c}(m)) + \theta(m)\right\}.$$

If in addition $\overline{L}+\widehat{c}(m)$ is semipositive for all $m \in \Delta$,

$$\zeta_{\mathrm{abs}}(\widehat{\rho}(\overline{\Delta})+\pi^*\overline{L}) = \inf_{m\in\Delta^\circ}\left\{\zeta_{\mathrm{abs}}(\overline{L}+\widehat{c}(m)) + \theta(m)\right\}.$$

Proof. If m is such that $\overline{L}+\widehat{c}(m)$ is geometrically big, this holds in a neighborhood of m. It follows by Theorem 4.0.1 that $\widehat{\rho}(\overline{\Delta})+\pi^*\overline{L}$ is geometrically big. We can therefore apply Theorem 2.2.1 to compute the essential minimum in terms of the Boucksom-Chen transform. Recall that the Okounkov body of $\pi^*\overline{L}+\widehat{\rho}(\overline{D})$

maps to Δ. Over $m \in \Delta$, the fibre can be identified with $\Delta_{B_\bullet}(L + \mathcal{T}(m))$ and the restriction of the Boucksom-Chen transform $G_{\pi^*\overline{L}+\widehat{\rho}(\overline{D}),\mathcal{X}_\bullet}$ is given by $\theta(m) + G_{\overline{L}+\widehat{\mathcal{T}}(m),B_\bullet}(x)$. Hence, over each such fibre the maximum of the Boucksom-Chen function is $\zeta_{\mathrm{ess}}(\overline{L} + \widehat{c}(m)) + \theta(m)$ by Theorem 2.2.1.

Let Δ^{big} denote the locus of $m \in \Delta$ such that $L + c(m)$ is big. By concavity and upper semicontinuity of the Boucksom-Chen transform and concavity of the essential minimum it follows that

$$\begin{aligned}\zeta_{\mathrm{ess}}(\widehat{\rho}(\overline{\Delta}) + \pi^*\overline{L}) &= \max_{(m,x)\in\Delta_{\mathcal{X}_\bullet}(\pi^*L+\rho(D))} G_{\pi^*\overline{L}+\widehat{\rho}(\overline{D}),\mathcal{X}_\bullet}(m,x)\\ &= \sup_{m\in\Delta^{\mathrm{big}}} \zeta_{\mathrm{ess}}(\overline{L} + \widehat{c}(m)) + \theta(m)\\ &= \sup_{m\in\Delta} \zeta_{\mathrm{ess}}(\overline{L} + \widehat{c}(m)) + \theta(m).\end{aligned}$$

We now assume that in addition $\overline{L} + \widehat{c}(m)$ is semipositive. In order for $L + c(m)$ to admit semipositive metrics it has to be nef. Hence, the Hilbert-Samuel theorem [Deb01, Proposition 1.31] holds. Adding an arbitrarily small multiple of a big nef line bundle yields a big nef divisor. Hence, $L + c(m)$ is pseudoeffective for all $m \in \Delta$ and big for all m in the interior Δ°.

By the upper semi-continuity of the Boucksom-Chen transform and applying Theorem 2.2.1 to each fibre, we obtain

$$\begin{aligned}\zeta_{\mathrm{abs}}(\widehat{\rho}(\overline{\Delta}) + \pi^*\overline{L}) &= \inf_{(m,x)\in\Delta_{\mathcal{X}_\bullet}(\pi^*L+\rho(D))} G_{\pi^*\overline{L}+\widehat{\rho}(\overline{D}),\mathcal{X}_\bullet}(m,x)\\ &= \inf_{m\in\Delta^\circ} \zeta_{\mathrm{abs}}(\overline{L} + \widehat{c}(m)) + \theta(m).\end{aligned}$$

□

5. Arithmetic bundle BKK

The purpose of this section is to prove the arithmetic bundle BKK theorem stated in the introduction. Its proof will follow the outline of the proof of [HKM21, Theorem 4.1]. Let us swiftly recall the statement of the theorem.

Let $\widehat{\mathcal{X}}$ be an adelic integrable proper toric bundle with underlying fan Σ of relative dimension t over a smooth projective base variety B of dimension g. Let $\mathcal{B}$ be a regular scheme flat and projective over $\mathcal{S}$ with $\mathcal{B}_K = B$. Let $\gamma \in \widehat{\mathrm{CH}}_i(\mathcal{B})$ and denote by $[\infty] \in \widehat{\mathrm{CH}}^1(\mathcal{B})_{\mathbb{R}}$ a trivial Cartier divisor endowed with constant Green's functions at all places such that $h_{[\infty]}(x) = 1$ for all $x \in B(\bar{K})$. Let $\overline{\Delta} \in \widehat{\mathcal{P}}^+_\Sigma$ be an adelic polytope.

Theorem B. *The intersection numbers below are well-defined and the following identity holds:*

$$i!\widehat{\rho}(\overline{\Delta})^{t+i}\pi^*\gamma = (t+i)!\int_{\Delta}(\widehat{c}(m)+\theta(m)[\infty])^i\gamma dm.$$

Remark 5.0.1. The regularity assumption on the model $\mathcal{B}$ is needed in order to apply Proposition 2.4.1 to show that there is a well defined intersection number. However, in other settings where the intersection number is defined such as when the torus bundle has a model over $\mathcal{B}$, there is no need for a regularity assumption on $\mathcal{B}$ and the same proof applies. A similar result should hold for general arithmetic operational cohomology classes. For the product of first Chern classes of integrable line bundles the result follows from Section 4.

We introduce shorthand notations for use in the course of the proof. Let

$$\widehat{I}_\gamma : \widehat{\mathcal{P}}^+ \to \mathbb{R},\ \overline{\Delta} \mapsto \int_{\Delta}(\widehat{c}(m)+\theta(m)[\infty])^i \cdot \gamma dm$$

and

$$\widehat{F}_\gamma : \widehat{\mathcal{P}} \to \mathbb{R},\ \overline{\Delta} \mapsto \widehat{\deg}(\widehat{\rho}(\overline{\Delta})^{t+i} \cdot \pi^*\gamma).$$

The function $\widehat{F}_\gamma$ is well-defined as the intersection number does not change under birational modification. We eventually extend $\widehat{I}_\gamma$ to the space of all virtual polytopes. Using the introduced shorthand, the theorem is stated below.

Theorem 5.0.2. The polynomials $\widehat{I}_\gamma$ and $\widehat{F}_\gamma$ satisfy

$$(t+i)! \cdot \widehat{I}_\gamma(\overline{\Delta}) = i! \cdot \widehat{F}_\gamma(\overline{\Delta}).$$

In particular, the polarizations of $\widehat{I}_\gamma$ and $\widehat{F}_\gamma$ are proportional multilinear forms, i.e. for any $\overline{\Delta}_1, ..., \overline{\Delta}_{t+i} \in \widehat{\mathcal{P}}_\Sigma$

$$(t+i)! \cdot \widehat{I}_\gamma(\overline{\Delta}_1, ..., \overline{\Delta}_{t+i}) = i! \cdot \widehat{F}_\gamma(\overline{\Delta}_1, ..., \overline{\Delta}_{t+i}).$$

Remark 5.0.3. The above intersection numbers are shown to be well-defined in Section 2.4. This is the case since one can define $\pi^*\gamma$ on a regular model given by the toric bundle $\mathcal{X}_\Sigma$ for an algebraic torus bundle over $\mathcal{B}$. Since this is smooth over $\mathcal{B}$ the model in particular is regular. In fact, by approximation the formula also holds for expressions γ of the form $\widehat{c}_1(\mathcal{L}_1)\dots\widehat{c}_1(\mathcal{L}_r)\gamma'$ for integrable adelic line bundles $\mathcal{L}_1, \dots, \mathcal{L}_r$ on B and $\gamma' \in \widehat{\mathrm{CH}}_{i+r}(\mathcal{B})$.

This closely resembles the BKK-type theorem in [HKM21]. Let us recall their statement for context. Let $\mathcal{X} \to B$ be a toric bundle over a smooth compact oriented $\mathbb{R}$-manifold B with smooth underlying fan Σ. Analogous to our situation there are maps $c : M \to H^2(B,\mathbb{Z})$ and $\rho : \mathcal{P}_\Sigma \to H^2(B,\mathbb{Z})$. Furthermore, the toric bundle $\mathcal{X}$ has an induced orientation. Hence, the top cohomology groups will be identified with $\mathbb{R}$.

Theorem 5.0.4 (Theorem 4.1 [HKM21])**.** Let $\gamma \in H^{k-2i}(B, \mathbb{R})$. Then, the cup product satisfies

$$i!\rho(\Delta)^{t+i}\pi^*\gamma = (t+i)! \int_\Delta c(m)^i \cdot \gamma dm.$$

5.1. **Overview of proof.** Following the outline in [HKM21] we prove Theorem 5.0.2 by first showing that the two functions are polynomials and comparing their partial derivatives. The novel notion of adelic polytopes allows us to transfer many ideas from the classical setting. We obtain that $\widehat{I}_\gamma$ is a homogeneous polynomial function in Corollary 5.2.10. It is clear, that the same holds for $\widehat{F}_\gamma$. For $i > 0$, it suffices to show

$$i!\partial_1^{k_1} \ldots \partial_r^{k_r} \widehat{F}_\gamma(\overline{\Delta}) = (t+i)!\partial_1^{k_1} \ldots \partial_r^{k_r} \widehat{I}_\gamma(\overline{\Delta})$$

for partial derivatives along the rays of $\widetilde{\Sigma}$ of total degree $k_1 + \cdots + k_r = t+1$. In the case of multiplicity $\mu = k_1 + \cdots + k_r - r$ of the partial derivative being 0, the comparison is done by direct calculation. We perform induction on $i + \mu$.

For $i = 0$, we observe that

$$t! \cdot \widehat{I}_\gamma(\overline{\Delta}) = t! \cdot \mathrm{vol}(\Delta) \deg(\gamma) = \deg(\Delta)\widehat{\deg}(\gamma) = \widehat{\deg}(\widehat{\rho}(\overline{\Delta})^t \cdot \pi^*\gamma) = \widehat{F}_\gamma(\overline{\Delta}).$$

This is just the classical BKK theorem except for the second to last equality which follows from a projection formula that can be deduced adhoc.

The cycle γ can be represented as a sum of closed points and a measure on $\mathcal{B}$. By linearity, assume first that γ is represented by a closed point. The cycle $\pi^*\gamma$ is given as the fibre over γ. The restriction of $\rho(\Delta)$ to this fibre has degree $\deg(\Delta)$. Hence, the claim holds in this case. Now suppose γ has only an archimedean part ω. Then $\rho(\Delta)^t \cdot \pi^*\gamma = \int_{\mathcal{X}(\mathbb{C})} c_1(\widehat{\rho}(\Delta))^t \pi^*\omega = \int_{B(\mathbb{C})} \left(\int_{\pi^{-1}(b)} c_1(\widehat{\rho}(\Delta))^t|_{\pi^{-1}(b)} \right) \omega(b) = \deg(\Delta) \deg \gamma$.

5.2. **Arithmetic convex chains.** The goal of this section is to prove that $\widehat{I}_\gamma(\overline{\Delta})$ is a polynomial on the space of arithmetic virtual polytopes. This requires an extension of the ideas in [PK92] to the arithmetic setting.

Definition 5.2.1. Let V be a finite dimensional real vector space. Then, a convex chain on V is a function $\alpha : V \to \mathbb{Z}$ of the form $\alpha = \sum_{i=1}^k n_i \mathbb{1}_{A_i}$ for polytopes $A_i \in \mathcal{P}^+$ and $n_i \in \mathbb{Z}$. This forms an algebra $Z(V)$ under the usual addition and multiplication given by the convolution product.

Definition 5.2.2. A *finitely additive measure* on $\mathcal{P}^+$ is a map $\phi : \mathcal{P}^+ \to M$ to an abelian group M satisfying the following property:

If $A_1, \dots, A_N \in \mathcal{P}^+$ are such that $\cup_{i=1}^N A_i \in \mathcal{P}^+$, then the following inclusion-exclusion relation holds:

$$\phi(\bigcup_{i=1}^N A_i) = \sum_i \phi(A_i) - \sum_{i<j} \phi(A_i \cap A_j) + \dots$$

The empty set satisfies $F(\emptyset) = 0$.

Definition 5.2.3. (1) A map $p : N \to M$ of abelian groups is called a polynomial of degree $\leq k$ if one of the following two conditions holds:
   (a) $k = 0$ and p is constant, i.e. $p(N) = m \in M$
   (b) $k \geq 1$ and for any $a \in N$, the map $p_a : N \to M$, $p_a : x \mapsto p(x+a) - p(x)$, is a polynomial of degree $\leq k-1$.

(2) A measure $\phi : \mathcal{P}^+ \to M$ is polynomial of degree $\leq k$ if for each $A \in \mathcal{P}^+$ the function $\phi(A+v) : V \to M$ is polynomial of degree $\leq k$.

Remark 5.2.4. The notion of convex chains allows for a reinterpretation of the notions of measure. Namely, a finitely additive measure is an arbitrary homomorphism of additive groups $\phi : Z(V) \to M$. If the measure is polynomial of degree $\leq k$, this extends to translation of functions in $Z(V)$. The remark justifies calling $\mathcal{P}^+ \to Z(V)$ the universal measure. This is discussed below [PK92, Definition 2.8].

Let $\tau_v : Z(V) \to Z(V)$ be the translation by a vector defined by $\tau_v \alpha(x) = \alpha(x - v)$ for $\alpha \in Z(V)$. Let $J_k \subset Z(V)$ denote the subgroup generated by chains of the form

$$(\tau_{v_1} - 1) \circ \dots \circ (\tau_{v_k} - 1)(\alpha)$$

for all $v_1, \dots, v_k \in V$. The subgroup $J_k \subset Z(V)$ is an ideal. The map $\mathcal{P}^+ \to Z(V)/J_{k+1}$ is the universal polynomial measure of degree $\leq k$, i.e. all polynomial measures of degree $\leq k$ factor uniquely through a homomorphism $Z(V)/J_{k+1} \to M$.

The degree of a convex chain $\alpha = \sum_{i=1}^k n_i \mathbb{1}_{A_i}$ for $A_i \in \mathcal{P}^+$ is defined to be $\sum_{i=1}^k n_i$. This is well-defined by [PK92, Proposition/Definition 2.1]. Let $\mathscr{L} \subset Z(V)$ denote the ideal of degree 0 chains.

Theorem 5.2.5 (Theorem 2.3 [PK92])**.** For $k \geq 1$,

$$\mathscr{L}^{\dim V + k} \subset J_k.$$

This theorem that any polynomial measure on V of degree $\leq k$ restricted to the group of virtual polytopes $\mathcal{P}$ is a polynomial of degree $\leq \dim V + k$ by [PK92, Corollary 2.5].

Let us now introduce the arithmetic analogues of $\mathcal{P}$. Due to the view towards toric varieties, we denote $\dim V = t$. We freely use the notation of Definition 2.1.8 and

Definition 5.2.6. An I-metrized convex chain on V is a function $\alpha : V \oplus \bigoplus_{i\in I} \mathbb{R} \to \mathbb{Z}$ of the form $\alpha = \sum_{i=1}^{k} n_i \mathbb{1}_{A_i}$ for I-metrized polytopes $A_i \in \mathcal{P}^{I+}$ and $n_i \in \mathbb{Z}$. This forms an algebra $Z^I(V)$ under the usual addition and multiplication given by the convolution product. For $I = \emptyset$, we recover the algebra of convex chains $Z(V)$.

Let $*$ denote a one element set. Then, the addition $\bigoplus_{i\in I} \mathbb{R} \to \mathbb{R}$ via the push-forward from [PK92, Proposition/Definition 2.2] induces a ring homomorphism $Z^I(V) \to Z^*(V)$. We note that the algebra of metrized convex chains $Z^*(V)$ can be is a subalgebra of $Z(V \oplus \mathbb{R})$. The I-metrized convex chains are a subalgebra of $\bigcup_{J\subseteq I,\text{finite}} Z(V \oplus \bigoplus_{j\in J} \mathbb{R})$.

A polynomial map from an $\mathbb{R}$-vector W space to $\mathbb{R}$ will be called a polynomial function if it is continuous on every finite dimensional subvector space. A polynomial function $f : W \to \mathbb{R}$ is said to be homogeneous of degree k if for $\lambda \in \mathbb{R}$ and $w \in W$ the equality $f(\lambda w) = \lambda^k f(w)$ holds.

Definition 5.2.7. Let f be a polynomial function of degree $\leq k$ on V. Then, we denote by I_f the map $\mathcal{P} \to \mathbb{R}$ extended from $\Delta \mapsto \int_\Delta f$. It is a degree $\leq k$ measure on the space of polytopes.

The statement of [HKM21, Theorem 5.5] is an easy corollary of Theorem 5.2.5 and summarizes the discussion in a convenient way.

Theorem 5.2.8 (Theorem 5.5 [HKM21])**.** If $f : V \to \mathbb{R}$ is a homogeneous polynomial function of degree k, then the function $I_f : \mathcal{P}^+ \to \mathbb{R}; (\Delta) \mapsto I_f(\Delta) = \int_\Delta f(x)d\mu$ admits a unique extension to a homogeneous polynomial function of degree $t + k$ on $\mathcal{P}$.

We will apply this to prove an arithmetic variant. In the proof of Theorem 5.0.2 it will be applied for the function $M_\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $(m, x) \mapsto (\widehat{c}(m) + x[\infty])^i\gamma$.

Theorem 5.2.9. If $f : V \times \mathbb{R} \to \mathbb{R}$ is a homogeneous polynomial function of total degree k, then the function

$$\widehat{I}_f : \widehat{\mathcal{P}}^+ \to \mathbb{R},\ \overline{\Delta} \mapsto \int_\Delta f(m, \theta(m))dm$$

admits a unique extension to a homogeneous polynomial function of degree $t + k$ on $\widehat{\mathcal{P}}$. Here θ denotes the global roof function of $\overline{\Delta}$.

Proof. We reduce the statement to [HKM21, Theorem 5.5]. For this note that the partial derivative f' with respect to the last variable is a polynomial of degree $k-1$. Assume that $\theta_v(x) \geq 0$ for all $x \in \Delta$ and all places $v \in M_K$. Then, we have that $\widehat{I}_f(\overline{\Delta}) = I_f(\Delta) + I_{f'}(\widehat{\Delta})$ by the fundamental theorem of calculus. The first term is known to be a polynomial of degree $t + k$ by [HKM21, Theorem 5.5].

The second term is a degree $\leq k-1$ measure on $Z^*(V)$. Hence, it gives a degree $\leq k+t$ polynomial on $Z^*(V)$. Since the map $Z^{M_K}(V) \to Z^*(V)$ is a ring homomorphism, it follows that we obtain a degree $t+k$ polynomial on virtual adelic polytopes. □

Corollary 5.2.10. The function

$$\widehat{I}_\gamma : \widehat{\mathcal{P}}^+ \to \mathbb{R}, \ \overline{\Delta} \mapsto \int_\Delta (\widehat{c}(m) + \theta(m)[\infty])^i \gamma dm$$

admits a unique extension to a homogeneous polynomial function of degree $t+i$ on $\widehat{\mathcal{P}}$.

Proof. We apply Theorem 5.2.9 to the function $M_\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $(m,x) \mapsto (\widehat{c}(m) + x[\infty])^i\gamma$. □

5.3. **Differentiation of $\widehat{I}_\gamma$.** We compute the $(t+1)$-st partial derivatives of $\widehat{I}_\gamma$ of multiplicity 0 for a preferred basis of the space of adelic polytopes. We compute more generally the derivatives of functions of the form $\widehat{I}_f$ introduced in Theorem 5.2.9 for smooth f on $V \oplus \mathbb{R}$. Denote the restriction of f to $V \times 0$ by f as well and f' the partial derivative along the $\mathbb{R}$-summand. We note by the proof of Theorem 5.2.9 that $\widehat{I}_f(\overline{\Delta}) = I_f(\Delta) + I_{f'}(\widehat{\Delta})$. We compute the differentials of each summand separately.

Let $\widetilde{\Sigma}$ be a simplicial adelic fan with recession fan Σ. The vector space of virtual adelic polytopes compatible with $\widetilde{\Sigma}$, $\widehat{\mathcal{P}}_{\widetilde{\Sigma}}$, has a distinguished basis provided by the virtual polytopes corresponding to the rays of $\widetilde{\Sigma}$. We say that a set of rays spans a cone if there is a place at which they span a cone. Since $\widetilde{\Sigma}$ is simplicial, there is a unique virtual polytope h whose support function is 1 at the primitive generator of a chosen ray τ and 0 on all other rays. We refer to h as the polytope dual to τ. We will be working with a finite set of rays $\tau_1, \ldots, \tau_s$. We denote its primitive generators by $e_1, \ldots, e_s$ and its dual polytopes by $h_1, \ldots, h_s$. We denote the partial derivative in the direction of h_i by ∂_i.

We recall first the classical case. Let $f : V \to \mathbb{R}$ be a continuous function and Σ be a complete fan on V. Let Δ be a polytope in the interior of $\mathcal{P}^+_\Sigma$.

Lemma 5.3.1 (Lemma 6.1 [HKM21])**.** Let $\tau_1, \ldots \tau_t$ be rays spanning a maximal cone dual to a vertex $A \in \Delta$. Then, we have

$$\partial_1 \ldots \partial_t(I_f)(\Delta) = f(A) \cdot |\det(e_1, \ldots, e_t)|.$$

Note that in the reference f is assumed smooth. This is, however, not used in the proof. An adelic analogue of [HKM21, Lemma 6.1] is given below.

Proposition 5.3.2. Let $\tau_1, \ldots \tau_{t+1}$ be rays spanning a maximal cone σ in $\widetilde{\Sigma}$ at a place v and $\overline{\Delta} \in \widehat{\mathcal{P}}^+_{\widetilde{\Sigma}}$ a v-interior polytope. Suppose that σ is dual to the vertex $(A, \theta_v(A)) \in \Delta_v$. Then, we have

$$\partial_1 \ldots \partial_{t+1}(I_f)(\widehat{\Delta}) = f(A, \theta(A)) \cdot |\det(e_1, \ldots, e_{t+1})|.$$

Proof. Extend the function $\theta - \theta_v$ to a continuous function $\mathcal{E}$ on V. Define $\widetilde{f}(v,t) = f(v, t + \mathcal{E}(v))$. Then one easily sees that $\partial_1 \ldots \partial_{t+1}(I_f)(\widehat{\Delta}) = \partial_1 \ldots \partial_{t+1}(I_{\widetilde{f}})(\Delta_v)$. We can then apply [HKM21, Lemma 6.1] to find that the derivative is

$$\widetilde{f}(A, \theta_v(A)) \cdot |\det(e_1, \ldots, e_{t+1})| = f(A, \theta(A)) \cdot |\det(e_1, \ldots, e_{t+1})|.$$

□

If $\overline{\Delta}$ is a polytope in $\widehat{\mathcal{P}}^+_{\widetilde{\Sigma}}$ and $\tau_1, \ldots, \tau_r$ do not span a cone in $\widetilde{\Sigma}$, then we have

$$\partial_1^{k_1} \ldots \partial_r^{k_r}(I_f)(\overline{\Delta}) = 0$$

for any tuple of $k_i \geq 1$.

Corollary 5.3.3. Let $\tau_1, \ldots \tau_{t+1}$ be rays spanning a maximal cone σ in $\widetilde{\Sigma}$ at a place v and $\overline{\Delta} \in \widehat{\mathcal{P}}^+_{\widetilde{\Sigma}}$ a v-interior polytope. Suppose that σ is dual to the vertex $(A, \theta_v(A)) \in \overline{\Delta}_v$. Then, we have

$$\partial_1 \ldots \partial_{t+1}(\widehat{I}_\gamma)(\overline{\Delta}) = i\widehat{c}(A)^{i-1} \cdot [\infty] \cdot \gamma \cdot |\det(e_{i_1}, \ldots, e_{i_r})|.$$

Proof. Recall the decomposition $\widehat{I}_f(\overline{\Delta}) = I_f(\Delta) + I_{f'}(\widehat{\Delta})$. We observe that

$$\partial_1 \ldots \partial_{t+1} I_f(\Delta) = 0$$

since at least one of the rays does not lie in $V \times 0$. Note that $\widehat{I}_\gamma = \widehat{I}_f$ for $f(m,t) = (\widehat{c}(m) + t[\infty])^i \gamma$.

We compute $f'(m,t) = i(\widehat{c}(x) + \theta[\infty])^{i-1}[\infty]\gamma = i\widehat{c}(x)^{i-1}[\infty]\gamma$. The last equality follows since $[\infty]^2 = 0$ in the Chow ring. □

If $\overline{\Delta}$ is a polytope in $\widehat{\mathcal{P}}^+_{\widetilde{\Sigma}}$ and $\tau_1, \ldots, \tau_r$ do not span a cone in $\widetilde{\Sigma}$, then we have

$$\partial_1^{k_1} \ldots \partial_r^{k_r}(\widehat{I}_\gamma)(\overline{\Delta}) = 0$$

for any tuple of $k_i \geq 1$.

5.4. **Differentiation of $\widehat{F}_\gamma$.** Let us consider first the squarefree case.

Lemma 5.4.1. Let $\tau_1, \ldots \tau_{t+1}$ be rays spanning a maximal cone σ in $\widetilde{\Sigma}$ at a place v and $\overline{\Delta} \in \widehat{\mathcal{P}}^+_{\widetilde{\Sigma}}$ a v-interior polytope. Suppose that σ is dual to the vertex $(A, \theta_v(A)) \in \overline{\Delta}_v$. Then,

$$\partial_1 \ldots \partial_{t+1}(\widehat{F}_\gamma)(\overline{\Delta}) = \frac{(t+i)!}{i!} \cdot i\widehat{c}(A)^{i-1}[\infty]\gamma \cdot |\det(e_{i_1}, \ldots, e_{i_r})|.$$

If $\overline{\Delta}$ is a polytope in $\widehat{\mathcal{P}}^{+}_{\widetilde{\Sigma}}$ and $\tau_1, \ldots, \tau_r$ do not span a cone in $\widetilde{\Sigma}$, then we have

$$\partial_1^{k_1} \ldots \partial_r^{k_r}(\widehat{F}_\gamma)(\overline{\Delta}) = 0$$

for any tuple of $k_i \geq 1$.

Proof. For the vanishing result it is clearly sufficient to consider the case $k_i = 1$ for all i. Denote by $\widehat{D}_i$ the divisor associated to h_i on $\mathcal{X}_{\widetilde{\Sigma}}$.

We expand the polynomial F_γ at Δ in order to compute the derivative.

$$\widehat{F}_\gamma(\overline{\Delta} + \sum_i \lambda_i \widehat{\rho}(\widehat{D}_i)) = (\widehat{\rho}(\overline{\Delta}) + \sum_i \lambda_i \widehat{\rho}(\widehat{D}_i))^{t+i} \pi^*(\gamma)$$
$$= \sum_{\alpha_0 + \cdots + \alpha_s = t+i} \binom{t+i}{\alpha_0, \ldots, \alpha_s} \widehat{\rho}(\overline{\Delta})^{\alpha_0} \widehat{\rho}(\widehat{D}_1)^{\alpha_1} \cdots \widehat{\rho}(\widehat{D}_r)^{\alpha_r} \pi^*(\gamma) \lambda_1^{\alpha_1} \cdots \lambda_r^{\alpha_r}.$$

The expression $\partial_1 \ldots \partial_r(\widehat{F}_\gamma)(\Delta)$ shows up as the coefficient of $\lambda_1 \cdots \lambda_r$. Since the intersection of the $\widehat{D}_i$ is empty it follows by Proposition 3.5.2 that the arithmetic intersection also vanishes.

We proceed to the case, where $\tau_1, \ldots \tau_{t+1}$ span a cone. We have

$$\partial_1 \ldots \partial_{t+1} F_\gamma(\Delta) = \frac{(t+i)!}{i!} \widehat{\rho}(\overline{\Delta})^i \widehat{\rho}(\widehat{D}_1) \cdots \widehat{\rho}(\widehat{D}_{t+1}) \pi^*(\gamma).$$

Let $\widetilde{\Delta}$ be the virtual adelic polytope $\overline{\Delta} - (A, \theta_v(A))$. Then,

$$h_{\widetilde{\Delta}}(e_1) = \cdots = h_{\widetilde{\Delta}}(e_{t+1}) = 0$$

since the vertex of $\widetilde{\Delta}$ corresponding to A is sent to the origin. Therefore, $\widetilde{\Delta}$ is the linear combination of rays not belonging to σ. In particular,

$$\widehat{\rho}(\widetilde{\Delta}) \widehat{\rho}(\widehat{D}_1) \cdots \widehat{\rho}(\widehat{D}_{t+1}) = 0.$$

We compute

$$\begin{aligned} &\widehat{\rho}(\overline{\Delta})^i \widehat{\rho}(\widehat{D}_1) \cdots \widehat{\rho}(\widehat{D}_{t+1}) \pi^*(\gamma) \\ =&\widehat{\rho}(\widetilde{\Delta} + (A, \theta_v(A))^i \widehat{\rho}(\widehat{D}_1) \cdots \widehat{\rho}(\widehat{D}_{t+1}) \pi^*(\gamma) \\ =&\widehat{\rho}((A, \theta_v(A)))^i \widehat{\rho}(\widehat{D}_1) \cdots \widehat{\rho}(\widehat{D}_{t+1}) \pi^*(\gamma). \end{aligned}$$

We now apply that $\widehat{\rho}(A, \theta_v(A)) = \pi^* \widehat{c}(A, \theta_v(A))$.

We apply an explicit projection formula. We restrict to the case of model metrics by approximation. The intersection of $\widehat{D}_1, \ldots, \widehat{D}_{t+1}$ on X_Σ is given as the $|\det(e_{i_1}, \ldots, e_{i_t})|$-multiple of a closed point at the place v. In particular, $\widehat{\rho}(\widehat{D}_1) \ldots \widehat{\rho}(\widehat{D}_{t+1})$ is the multiple of a horizontal cycle mapping to the special fibre at v of $\mathcal{B}$. Hence,

$$\widehat{\rho}(\widehat{D}_1) \cdots \widehat{\rho}(\widehat{D}_{t+1}) \pi^*(\widehat{c}(A, \theta_v(A))^i \gamma)$$

is represented by the multiple of a cycle that maps one to one to the intersection of $\widehat{c}(A, \theta_v(A))^i\gamma$ with the special fibre at a place. The archimedean case follows by the projection formula for differential forms. □

Corollary 5.4.2. For any $i \leq g+1$ and $\gamma \in \widehat{CH}^{g+1-i}(A)$ and any squarefree differential monomial ∂_I of order $t+1$, we have

$$(t+i)! \cdot \partial_I \widehat{I}_\gamma(\overline{\Delta}) = i! \cdot \partial_I \widehat{F}_\gamma(\overline{\Delta}).$$

We now need to consider partial derivatives which are not squarefree. We treat them by induction on the multiplicity. The multiplicity of a multiset is the difference of the cardinality of the multiset and the cardinality of the underlying set.

Let I be a multiset of rays in $\widetilde{\Sigma}$. As induction hypothesis we assume that $(t+i)! \cdot \partial_I \widehat{I}_\gamma(\Delta) = i! \cdot \partial_I \widehat{F}_\gamma(\Delta)$ for all differential monomials ∂_I of multiplicity $\mu - 1 \geq 0$. We need to show the same equality for multiplicity μ. Let $\tau_1, \dots, \tau_r$ be rays in $\widetilde{\Sigma}$ forming a cone. By relabeling, we may assume $\partial_I = \partial_1^{k_1} \dots \partial_r^{k_r}$ and $k_1 > 1$. We may restrict to the case where $\tau_1, \dots, \tau_r$ form a cone in $\widetilde{\Sigma}$.

We express ∂_1 in terms of a Lie derivative L_v for $v \in M_\mathbb{R} \times \mathbb{R}$ and other partial derivatives. As $e_1, \dots, e_r$ generate a cone in the simplicial fan $\widetilde{\Sigma}$ they can be completed to a basis $e_1, \dots, e_{t+1}$ of $N_\mathbb{R} \oplus \mathbb{R}$. The first vector of the dual basis satisfies $\langle v, e_1 \rangle = 1$ and $\langle v, e_j \rangle = 0$ for $j = 2, \dots, r$. The vector v is of the form $\sum_{i=1}^s \langle v, e_i \rangle h_i$ for further prime virtual polytopes $h_{r+1}, \dots, h_s$. We conclude that $L_v = \sum_{i=1}^s \langle v, e_i \rangle \partial_i$ and thus $\partial_1 = L_v - \sum_{j>r} \langle v, e_i \rangle \partial_j$. We get:

$$\begin{aligned}\partial_I = \partial_1^{k_1} \dots \partial_r^{k_r} &= \left(L_v - \sum_{j>r} \langle v, e_i \rangle \partial_j \right) \partial_1^{k_1 - 1} \dots \partial_r^{k_r} \\ &= L_v \partial_1^{k_1-1} \dots \partial_r^{k_r} - \sum_{j>r} \langle v, e_i \rangle \partial_1^{k_1-1} \dots \partial_r^{k_r} \partial_j.\end{aligned}$$

We may apply the induction hypothesis to the second term.

We use induction on i to show the following statement analogous to [HKM21, Lemma 6.5].

Lemma 5.4.3. In the above situation, we have $(t+i)! \cdot L_v \widehat{I}_\gamma(\overline{\Delta}) = i! \cdot L_v \widehat{F}_\gamma(\overline{\Delta})$.

Proof. We prove the statement by two direct computations. Write $v = (x, t)$.

$$\begin{aligned} L_v I_\gamma(\overline{\Delta}) &= \partial_s|_{s=0} \int_{\Delta+sx} \widehat{c}(m+\theta(x)+sm)^i \gamma dm \\ &= \partial_s|_{s=0} \int_{\Delta} \widehat{c}(m+sv)^i \gamma dm \\ &= \partial_s|_{s=0} \int_{\Delta} \sum_{j=0}^{i} \binom{i}{j} s^j \widehat{c}(v)^j \widehat{c}(m)^{i-j} \gamma dm \\ &= i \int_{\Delta} \widehat{c}(v)\widehat{c}(m)^{i-1}\gamma dm = i\widehat{I}_{\widehat{c}(v)\gamma}(\overline{\Delta}). \end{aligned}$$

On the other hand,

$$\begin{aligned} L_v \widehat{F}_\gamma(\overline{\Delta}) &= \partial_s|_{s=0} \widehat{\rho}(\overline{\Delta}+sv)^{t+i}\gamma = \partial_s|_{s=0} \sum_{j=0}^{t+i} \binom{t+i}{j} s^j \pi^*\widehat{c}(v)^j \widehat{\rho}(\overline{\Delta})^{t+i-j}\pi^*\gamma \\ &= (t+i)\widehat{\rho}(\overline{\Delta})^{t+i-1}\pi^*(\widehat{c}(v)\gamma) \\ &= (t+i)F_{\widehat{c}(v)\gamma}(\overline{\Delta}). \end{aligned}$$

By induction hypothesis,

$$(t+i-1)!\widehat{I}_{\widehat{c}(v)\gamma}(\overline{\Delta}) = (i-1)!\widehat{F}_{\widehat{c}(v)\gamma}(\overline{\Delta}).$$

Hence,

$$(t+i)!L_v\widehat{I}_\gamma(\overline{\Delta}) = i(t+i)!\widehat{I}_{\widehat{c}(v)\gamma}(\overline{\Delta}) = (t+i)i!\widehat{F}_{\widehat{c}(v)\gamma}(\overline{\Delta}) = i!L_v\widehat{F}_\gamma(\overline{\Delta}).$$

□

This finishes the proof.

6. Semiabelian varieties

In this section we study the semiabelian varieties and their compactifications by applying the methods developed earlier in the article. This allows us to recover Chambert-Loir's computation of the height and the absolute minimum in [Cha00].

The first result is the calculation of the Okounkov body and Boucksom-Chen transform for semiabelian varieties. Let us recall the setup in the introduction.

Let G be a semiabelian variety over a global field K with abelian quotient A and split torus part $\mathbb{T}$. Suppose the isomorphism class as a torus bundle is given by a map $c : M \to A^\vee(\bar{K})$. Let $\mathcal{M}$ denote an ample symmetric line bundle on A. Endowing it with canonical metrics gives it the structure of an adelically metrized line bundle $\overline{\mathcal{M}}$. Let $\widehat{h}$ denote the Néron-Tate height with respect to $\mathcal{M}$ on $A(\bar{K}) \otimes \mathbb{R}$ as well as the induced map on $A^\vee(\bar{K}) \otimes \mathbb{R}$ along the polarization $A(\bar{K}) \otimes \mathbb{R} \to A^\vee(\bar{K}) \otimes \mathbb{R}$. Let $\overline{G}$ be the compactification of G with respect to a fan

Σ in $N_{\mathbb{R}}$. Let D be an ample toric Cartier divisor on X_Σ with Newton polytope $\Delta \subset M_{\mathbb{R}}$. After refining Σ, we assume it defines a smooth projective variety. Let $\overline{G}_\bullet$ be a flag on $\overline{G}$ of the type considered in Section 4 consisting of a toric part $X_\bullet$ and an abelian part $A_\bullet$.

Lemma 6.0.1. The Okounkov body of $\overline{G}$ with respect to $\pi^*\mathcal{M}+\rho(D)$ decomposes as a product $\Delta \times \Delta_{A_\bullet}(\mathcal{M})$. The Boucksom-Chen transform satisfies

$$G_{\overline{G}_\bullet,\pi^*\overline{\mathcal{M}}+\rho(D)^{\mathrm{can}}}(m,x) = -\widehat{h}(c(m)).$$

Proof. By Theorem 4.0.1, the fibers of the Okounkov body over $m \in \Delta$ are given by the Okounkov body $\Delta_{A_\bullet}(\mathcal{M}+\mathcal{Q})$ for some numerically trivial line bundle $\mathcal{Q}$ on A. By [LM09, Proposition 4.1], there is an equality of Okounkov bodies $\Delta_{A_\bullet}(\mathcal{M}+\mathcal{Q}) = \Delta_{A_\bullet}(\mathcal{M})$.

In order to compute the Boucksom-Chen transform, we need to compute the Boucksom-Chen transform $G_{A_\bullet,\overline{\mathcal{M}}+\overline{\mathcal{Q}}}(x)$ for a numerically trivial line bundle $\mathcal{Q}$ with its canonical metric. We note that $\overline{\mathcal{M}}+\overline{\mathcal{Q}}$ is semipositive and that $\zeta_{\mathrm{ess}}(\overline{\mathcal{M}}+\overline{\mathcal{Q}}) = \zeta_{\mathrm{abs}}(\overline{\mathcal{M}}+\overline{\mathcal{Q}})$, since heights with respect to $\overline{\mathcal{M}}+\overline{\mathcal{Q}}$ are invariant under translation by the Zariski dense set of torsion points of A. By the results in Section 2.2, the Boucksom-Chen transform is constant of value

$$\frac{\widehat{\deg}((\overline{\mathcal{M}}+\overline{\mathcal{Q}})^{g+1})}{(g+1)\deg((\mathcal{M}+\mathcal{Q})^g)}.$$

Suppose that $\mathcal{Q}$ is the image of $q \in A(\bar{K})$ under the polarization morphism $A \to A^\vee$. The intersection number $\widehat{\deg}((\overline{\mathcal{M}}+\overline{\mathcal{Q}})^{g+1})$ is calculated in [Cha00, Théorème 2.5] to be $-\frac{2\deg(\mathcal{M}^g)}{g}\widehat{h}_{\mathcal{M}}(q)$. This implies

$$\frac{\widehat{\deg}((\overline{\mathcal{M}}+\overline{\mathcal{Q}})^{g+1})}{(g+1)\deg((\mathcal{M}+\mathcal{Q})^g)} = -\widehat{h}(q) = -\widehat{h}(\mathcal{Q}).$$

□

This can be applied to compute the height of a semiabelian variety.

Theorem C. *The height of a compactified semiabelian variety $\overline{G}$ can be computed as*

$$h_{\rho(D)^{\mathrm{can}}\otimes\pi^*\bar{\mathcal{M}}}(\overline{G}) = -(d+1)!\int_\Delta \widehat{h}(c(m))dm.$$

Proof. By applying the results of Section 2.2 and Lemma 6.0.1, we see that

$$
\begin{aligned}
&h_{\rho(D)^{\mathrm{can}}\otimes\pi^*\bar{\mathcal{M}}}(\overline{G})\\
=&(d+1)!\int_{\Delta(\pi^*\mathcal{M}+\rho(D))} G_{\overline{G}_\bullet,\pi^*\overline{\mathcal{M}}+\rho(D)^{\mathrm{can}}}(x)dx\\
=&-(d+1)!\int_{\Delta\times\Delta_{A_\bullet}(\mathcal{M})}\widehat{h}(c(m))d(m,x)\\
=&-\frac{(d+1)!\deg(\mathcal{M})}{g!}\int_{\Delta\times\Delta_{A_\bullet}(\mathcal{M})}\widehat{h}(c(m))dm\\
=&-\frac{(d+1)!\deg(\mathcal{M})}{g!\operatorname{vol}(\mathcal{M})}\int_{\Delta}\widehat{h}(c(m))dm\\
=&-(d+1)!\int_{\Delta}\widehat{h}(c(m))dm.
\end{aligned}
$$

□

We proceed to study the successive minima of $\overline{G}$. Let $\mathcal{F}(\Delta)^i$ denote the set of i-dimensional faces of Δ.

Theorem D. *The i-th successive minimum of $\bar{G}$ with respect to $\rho(D)^{\mathrm{can}}\otimes\pi^*\bar{\mathcal{M}}$ satisfies $\zeta_i(\bar{G})=\zeta_1(\bar{G})$ for $i\leq g+1$. For $i\geq g+1$,*

$$\zeta_i(\bar{G})=-\max_{F\in\mathcal{F}(\Delta)^{t+g+1-i}}\min_{m\in F}\widehat{h}(c(m)).$$

Proof. By Section 3.6 it suffices to study successive minima on each locally closed subvariety of $\overline{G}$ corresponding to a torus orbit. Since the functions of the form h_Δ are invariant under the action of torsion points on A each subvariety each such filtration of A has only one step. In particular, the first step in the height filtration maps surjectively onto A and hence $\zeta_i(\bar{G})=\zeta_1(\bar{G})$ for $i\leq g+1$.

We start out by noting that the formula holds for ζ_{ess} by applying Lemma 6.0.1 and Theorem 2.2.1. By this discussion it suffices to show that the essential minimum of the restriction of $\pi^*\overline{\mathcal{M}}+\rho(D)^{\mathrm{can}}$ to the closed toric subbundle $\mathcal{V}(\sigma)$ given by the cone σ is $-\min_{m\in F_\sigma}\widehat{h}(c(m))$, where F_σ is the face dual to σ. For this it suffices to understand the Boucksom-Chen transform on each closed toric subbundle.

Let us understand the restriction of the $\mathbb{T}$-Cartier divisor D on X to the closure of a $\mathbb{T}$-orbit $V(\sigma)$. By [BPS14, Proposition 3.4.11], its restriction is given by the divisor on $V(\sigma)$ corresponding to the face F_σ dual to σ. The restriction of $\rho(D)$ to the toric subbundle is given precisely by $\rho(F_\sigma)$. Hence, the Okounkov body and the Boucksom-Chen transform of the restriction are precisely the restriction to the preimage of F_σ and the claim follows. Suppose otherwise that $m_\sigma\in F_\sigma$. Then, the Cartier divisor associated to $F_\sigma-m_\sigma$ intersects $V(\sigma)$ properly. Hence, the restriction of $\rho(D)$ to $\mathcal{V}(\sigma)$ is given by $\rho(F_\sigma - m_\sigma) + \pi^* c(m_\sigma)$. Hence, the Okounkov body and the Boucksom-Chen transform of the restriction are (up to translation) the restriction to the preimage of F_σ and the claim follows. □

We may now apply the above for the specific compactification studied in [Cha00]. Let us recall their setup.

Let G be a semiabelian variety with torus $\mathbb{G}_m^t$ and associated isomorphisms $N \cong \mathbb{Z}^t$ and $M \cong \mathbb{Z}^t$. Let A denote the abelian quotient of G and $\mathcal{M}$ an ample symmetric line bundle on A inducing a polarization morphism $\phi : A \to A^\vee$. Let $e_1, \ldots, e_t$ denote the standard basis of M. Let $q_1, \ldots, q_t \in A(\bar{K})$ satisfying $c(e_i) = \phi(q_i)$. The compactification is taken to be $\mathbb{G}_m^t \subset \mathbb{P}^n$. If Δ^t denotes the unit simplex, we let $\Delta = (t+1)\Delta^t - \sum_i e_i$. The line bundle on $\overline{G}$ that Chambert-Loir denotes as $\mathcal{L}$ is $\rho(\Delta)$.

Lemma 6.0.2 (Lemma 4.5 [Cha00])**.** Let $q = \sum_i q_i$. Then, one has

$$\zeta_{\text{abs}}(\rho(\Delta)^{\text{can}} + \pi^*\overline{\mathcal{M}}) = -\max\{\widehat{h}(q), \max_i \widehat{h}(q - (t+1)q_i)\}.$$

Proof. By Theorem D, the absolute minimum is given by the negative of $\max \widehat{h}(c(m))$ as m ranges over the vertices of Δ. On the vertices $v_0, \ldots, v_t$ of Δ, one has $c(v_0) = \phi(-q)$ and $c(v_i) = \phi((t+1)q_i - q)$ for $i = 1, \ldots, t$. □

Lemma 6.0.3. The height of $\overline{G}$ can be computed as

$$h_{\rho(D)^{\text{can}} \otimes \pi^* \bar{\mathcal{M}}}(\overline{G}) = -\frac{(d+1)\deg(\mathcal{M})}{(t+1)(t+2)} \left(\widehat{h}(q) + \sum_{i=1}^{t} \widehat{h}(q - (t+1)q_i) \right).$$

Proof. By Theorem C we are reduced to computing the integral $\int_\Delta \widehat{h}(c(m))$.

Let H be a symmetric multilinear form in r entries on $\mathbb{R}^t$ and Δ a polytope spanned by $v_0, \ldots, v_t$. Then, it is proven in [LA01] that

$$\int_\Delta H(m, \ldots, m) = \frac{\text{vol}(\Delta)}{\binom{t+r}{r}} \sum_{0 \le i_1 \le \cdots \le i_r \le t} H(v_{i_1}, \ldots, v_{i_r}).$$

Let us specialize to the case that $r = 2$ and $\sum_{i=0}^t v_i = 0$. Then we obtain

$$\int_\Delta H(m, \ldots, m) = \frac{\text{vol}(\Delta)}{(t+1)(t+2)} \sum_{i=0}^{t} H(v_i, v_i)$$

by subtracting $\frac{\text{vol}(\Delta)}{2\binom{t+r}{r}} H(v_0 + \cdots + v_t, v_0 + \cdots + v_t) = 0$. We apply this to Δ as in the setup and H the polarization of $\widehat{h}(c(m))$. We lastly need the geometric identity from Theorem 2.2.2. □

References

[Bal21] François Ballaÿ. "Successive minima and asymptotic slopes in Arakelov geometry". In: *Compositio Mathematica* 157.6 (2021), pp. 1302–1339. DOI: 10.1112/S0010437X21007156.

[Bal22a] François Ballaÿ. "Arithmetic Okounkov bodies and positivity of adelic Cartier divisors". working paper or preprint. Mar. 2022. URL: https://hal.science/hal-03606531.

[Bal22b] François Ballaÿ. "Nakai–Moishezon Criterion for Adelic -Cartier Divisors". In: *International Mathematics Research Notices* 2023.12 (May 2022), pp. 10532–10555. ISSN: 1073-7928. DOI: 10.1093/imrn/rnac140. eprint: https://academic.oup.com/imrn/article-pdf/2023/12/10532/50645397/rnac140.pdf. URL: https://doi.org/10.1093/imrn/rnac140.

[BC11] Sébastien Boucksom and Huayi Chen. "Okounkov bodies of filtered linear series". In: *Compositio Mathematica* 147.4 (2011), pp. 1205–1229. DOI: 10.1112/S0010437X11005355.

[BD91] Alexander Beilinson and Vladimir Drinfeld. *Quantization of Hitchin's integrable system and Hecke Eigensheaves*. 1991. URL: https://math.uchicago.edu/~drinfeld/langlands/QuantizationHitchin.pdf.

[Ber12] Vladimir G Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*. 33. American Mathematical Soc., 2012.

[BPS14] José Ignacio Burgos Gil, Patrice Philippon and Martín Sombra. "Arithmetic geometry of toric varieties. Metrics, measures and heights". In: *Astérisque* 360 (2014), pp. vi+222. ISSN: 0303-1179,2492-5926.

[BPS15] José Ignacio Burgos Gil, Patrice Philippon and Martín Sombra. "Successive minima of toric height functions". In: *Ann. Inst. Fourier (Grenoble)* 65.5 (2015), pp. 2145–2197. ISSN: 0373-0956,1777-5310. URL: http://aif.cedram.org/item?id=AIF_2015__65_5_2145_0.

[BS24] François Ballaÿ and Martín Sombra. *Approximation of adelic divisors and equidistribution of small points*. 2024. arXiv: 2407.14978 [math.NT]. URL: https://arxiv.org/abs/2407.14978.

[Bur+16] José Ignacio Burgos Gil et al. "Arithmetic positivity on toric varieties". In: *J. Algebraic Geom.* 25.2 (2016), pp. 201–272. ISSN: 1056-3911,1534-7486. DOI: 10.1090/jag/643. URL: https://doi.org/10.1090/jag/643.

[Cha00] Antoine Chambert-Loir. "Points de petite hauteur sur les variétés semi-abéliennes". In: *Annales Scientifiques de l'École Normale Supérieure* 33.6 (2000), pp. 789–821. ISSN: 0012-9593. DOI: https://doi.org/10.1016/S0012-9593(00)01053-3. URL: https://www.sciencedirect.com/science/article/pii/S0012959300010533.

[CLS10] David Cox, John Little and Hal Schenck. *Toric Varieties*. 2010.

[CM18] Huayi Chen and Atsushi Moriwaki. "Extension property of semipositive invertible sheaves over a non-archimedean field". In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 18.1 (2018), pp. 241–282. ISSN: 0391-173X,2036-2145.

[CT01] Antoine Chambert-Loir and Yuri Tschinkel. "Torseurs arithmétiques et espaces fibrés". In: *Rational points on algebraic varieties*. Vol. 199. Progr. Math. Birkhäuser, Basel, 2001, pp. 37–70. ISBN: 3-7643-6612-5.

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001, pp. xiv+233. ISBN: 0-387-95227-6. DOI: 10.1007/978-1-4757-5406-3. URL: https://doi.org/10.1007/978-1-4757-5406-3.

[Dem12] J.P. Demailly. *Complex Analytic and Differential Geometry*. Université de Grenoble I, 2012. URL: https://www-fourier.ujf-grenoble.fr/~Demailly/manuscripts/agbook.pdf.

[FL17] Mihai Fulger and Brian Lehmann. "Positive cones of dual cycle classes". In: *Algebr. Geom.* 4.1 (2017), pp. 1–28. ISSN: 2313-1691,2214-2584. DOI: 10.14231/AG-2017-001. URL: https://doi.org/10.14231/AG-2017-001.

[FLQ24] Yangyu Fan, Wenbin Luo and Binggang Qu. *Height Filtrations and Base Loci on Flag Bundles over a Curve*. 2024. arXiv: 2403.06808 [math.NT]. URL: https://arxiv.org/abs/2403.06808.

[Ful93] William Fulton. *Introduction to toric varieties*. Vol. 131. Annals of Mathematics Studies. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993, pp. xii+157. ISBN: 0-691-00049-2. DOI: 10.1515/9781400882526. URL: https://doi.org/10.1515/9781400882526.

[Ful98] William Fulton. *Intersection Theory*. Springer New York, NY, 1998.

[Gre69] Marvin J. Greenberg. *Lectures on forms in many variables*. W. A. Benjamin, Inc., New York-Amsterdam, 1969, pp. v+167.

[GS90] Henri Gillet and Christophe Soulé. "Arithmetic intersection theory". In: *Inst. Hautes Études Sci. Publ. Math.* 72 (1990), pp. 93–174. ISSN: 0073-8301,1618-1913. URL: http://www.numdam.org/item?id=PMIHES_1990__72__93_0.

[GS92] Henri Gillet and Christophe Soulé. "An arithmetic Riemann-Roch theorem". In: *Invent. Math.* 110.3 (1992), pp. 473–543. ISSN: 0020-9910,1432-1297. DOI: 10.1007/BF01231343. URL: https://doi.org/10.1007/BF01231343.

[GS94] H. Gillet and C. Soulé. "Arithmetic analogs of the standard conjectures". In: *Motives (Seattle, WA, 1991)*. Vol. 55, Part 1. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1994, pp. 129–140. ISBN: 0-8218-1636-5. DOI: 10.1090/pspum/055.1/1265527. URL: https://doi.org/10.1090/pspum/055.1/1265527.

[HKM21] Johannes Hofscheier, Askold Khovanskii and Leonid Monin. *Cohomology rings of toric bundles and the ring of conditions*. 2021. arXiv: 2006.12043 [math.AG].

[Jon96] A. J. de Jong. "Smoothness, semi-stability and alterations". In: *Inst. Hautes Études Sci. Publ. Math.* 83 (1996), pp. 51–93. ISSN: 0073-8301,1618-1913. URL: http://www.numdam.org/item?id=PMIHES_1996__83__51_0.

[KL84] F. Knop and H. Lange. "Commutative algebraic groups and intersections of quadrics". In: *Math. Ann.* 267.4 (1984), pp. 555–571. ISSN: 0025-5831,1432-1807. DOI: 10.1007/BF01455973. URL: https://doi.org/10.1007/BF01455973.

[LA01] Jean B. Lasserre and Konstantin E. Avrachenkov. "The Multi-Dimensional Version of $\int_a^b x^p dx$". In: *The American Mathematical Monthly* 108.2 (2001), pp. 151–154. DOI: 10.1080/00029890.2001.11919735. eprint: https://doi.org/10.1080/00029890.2001.11919735. URL: https://doi.org/10.1080/00029890.2001.11919735.

[LM09] Robert Lazarsfeld and Mircea Mustaţă. "Convex bodies associated to linear series". In: *Ann. Sci. Éc. Norm. Supér. (4)* 42.5 (2009), pp. 783–835. ISSN: 0012-9593,1873-2151. DOI: 10.24033/asens.2109. URL: https://doi.org/10.24033/asens.2109.

[Mor13] Atsushi Moriwaki. "Adelic Divisors on Arithmetic Varieties". In: *Memoirs of the American Mathematical Society* 242 (Feb. 2013). DOI: 10.1090/memo/1144.

[Mor15] Atsushi Moriwaki. "Semiample invertible sheaves with semipositive continuous hermitian metrics". In: *Algebra Number Theory* 9.2 (2015), pp. 503–509. ISSN: 1937-0652,1944-7833. DOI: 10.2140/ant.2015.9.503. URL: https://doi.org/10.2140/ant.2015.9.503.

[Oor66] F. Oort. *Commutative group schemes*. Vol. 15. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1966, vi+133 pp. (not consecutively paged).

[PK92] A. V. Pukhlikov and A. G. Khovanskiĭ. "Finitely additive measures of virtual polyhedra". In: *Algebra i Analiz* 4.2 (1992), pp. 161–185. ISSN: 0234-0852.

[QY22] Binggang Qu and Hang Yin. "Arithmetic Demailly Approximation Theorem". In: *arXiv:2208.13230* (2022).

[Wit14] David Witt Nyström. "Transforming metrics on a line bundle to the Okounkov body". In: *Ann. Sci. Éc. Norm. Supér. (4)* 47.6 (2014), pp. 1111–1161. ISSN: 0012-9593,1873-2151. DOI: 10.24033/asens.2235. URL: https://doi.org/10.24033/asens.2235.

[Zha95a] Shouwu Zhang. "Positive Line Bundles on Arithmetic Varieties". In: *Journal of The American Mathematical Society - J AMER MATH SOC* 8 (Jan. 1995), pp. 187–187. DOI: 10.2307/2152886.

[Zha95b] Shouwu Zhang. "Small points and adelic metrics". In: *J. Algebraic Geom.* 4.2 (1995), pp. 281–300. ISSN: 1056-3911,1534-7486.

Nuno Hultberg. University of Copenhagen, Institute of Mathematics, Universitetsparken 5, 2100 Copenhagen, Denmark; ORCiD: orcid.org/0000-0003-0097-0499

Email address: nh@math.ku.dk