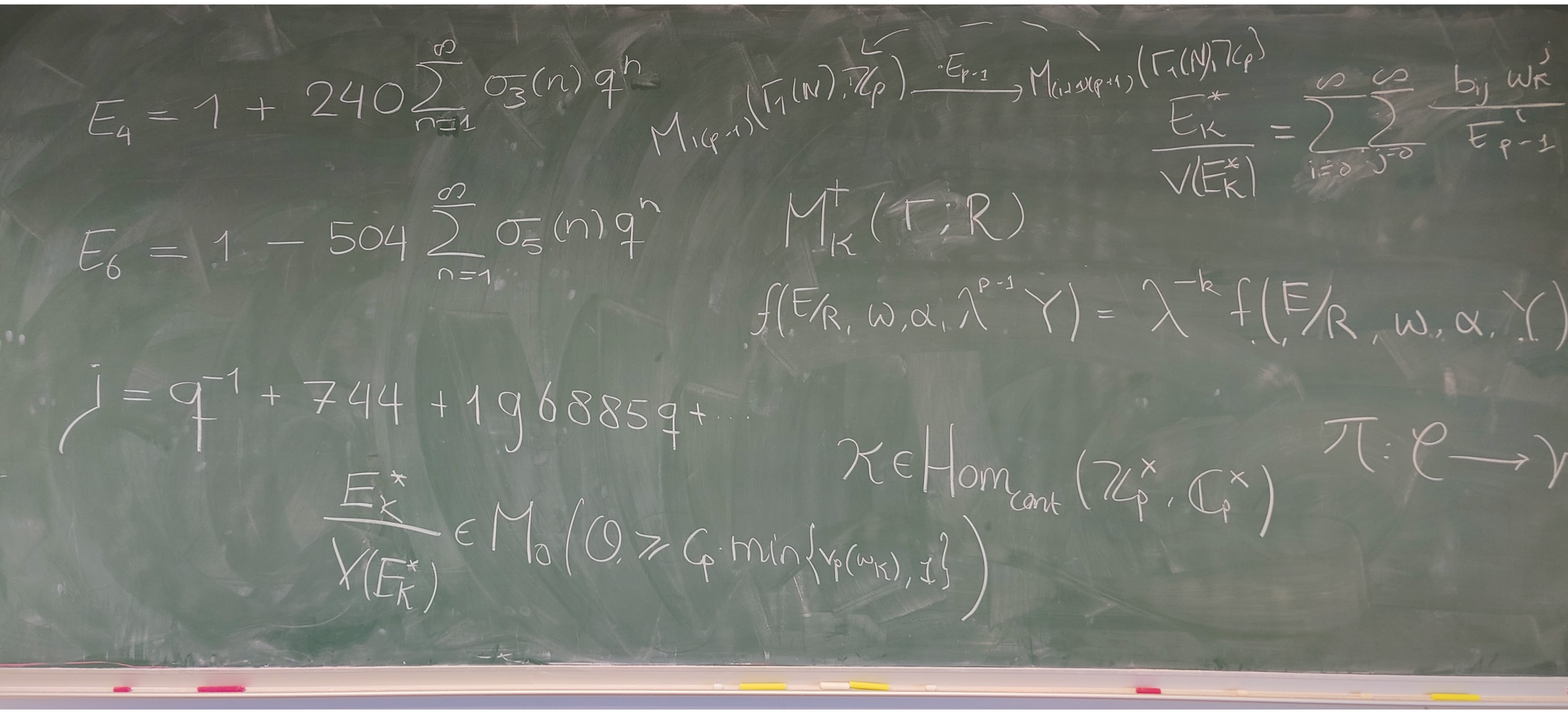
UNIVERSITY OF COPENHAGEN
FACULTY OF SCIENCE



PhD thesis

Explicit Overconvergence Rates Related to Eisenstein Series

Bryan Advocaat

Advisors: Ian Kiming, Gabor Wiese

This thesis has been submitted to the PhD School of The Faculty of Science, University of Copenhagen

Voor oma

Contents

Acknowledgements

First and foremost, I extend my gratitude to both my supervisors, Gabor Wiese and Ian Kiming. I'm extremely grateful to Gabor's invaluable advice and feedback. He was always open to questions and always devoted his time to help me, from which I have learned so many things.
Even though my interactions with Ian were mostly virtual, at 9 'o clock in the morning (when I was not fully awake), he always listened to my questions and provided useful and grounding perspectives. During my time in Copenhagen, we would have tons of interesting conversations, not only about mathematics, but also about political or philosophical issues. I'm very thankful for the privilege of having had you as my cosupervisor.

I would like to thank the members of my PhD defense committee for sacrificing their time to read and evaluate my thesis. Special thanks go to Antonella for all her contributions to the number theory group, including the Kummer Theory course which even led to a paper! I would like to thank Andrea Conti for organizing so many of the seminars in Luxembourg and Adel Bettini for taking the time to explain difficult mathematics to me. I would like to thank Robert Pollack for having me over as a guest at the Max Planck institute, and explaining a lot of new maths to me.

Without my amazing office mates, this PhD would have been a lot worse, so thank you Emiliano, Sebastiano, Antigona, Breki, Tim and my temporary office mate Viola. It was always a pleasure to be in my office, and I will miss it a lot. Tüm Türkçe dersleri için ve eğlence için teşekkür ederim!

Just as important as nice office mates, are all the kind people in the department, without whom, the days would have been so much less fun. In particular, I would like to thank my parsing friends, the teapots, Pierre, Shiwi, Tara and Valentin (even though he was a fake parser). It was my pleasure to have the opportunity to hang out with you. Y'all are a partial labyrinth in front of a bank. (7)

I would like to thank Flavio, Oskar and Pietro for all the fun and having nice relaxing (mostly) conversations almost every day. I want to thank Alisa for being always so enthusiastic (and loud) and all the climbing adventures. I want to thank Alexandre, Thilo, Francesca, Francesca and the other names that I inevitably forgot to mention for their wonderful presence in the department.

I'm also very grateful to the secretaries, especially Marie Leblanc who never got angry at me, despite informing her about everything at the absolute latest moment possible (I'm very sorry for that). I also want to thank all other people involved at the Doctoral school for making sure that everything went so smoothly.

Velen malen dank voor Anna, bij wie ik toch altijd weer eventjes m'n ei kwijt kon als ik er doorheen zat. Maar uiteraard ook bedankt voor alle gezelligheid (en alle roddels).

Ik wil mijn ouders en mijn broertje bedanken, die, ondanks dat we elkaar niet zo vaak zagen, me altijd ondersteunden en bij wie ik altijd terecht kon. Ik wil ook heel graag mijn oma bedanken, die helaas net niet heeft mee kunnen maken dat mijn scriptie af is.

Finally, I would like to thank Pierre for everything. Luxembourg has become such a wonderful place thanks to you. I had absolutely not expected this four years ago. In any timeline I would choose to do this PhD, just to meet you again.

Abstract

Abstract

In this thesis, we study explicit overconvergence rates related to Eisenstein series. We start by providing the necessary theoretical background on the theory of overconvergent modular forms. This includes the theory of Katz expansions, which is the main tool in the rest of the thesis to deduce overconvergence rates. Our main object will then be the family of modular functions $E^*_\kappa/V(E^*_\kappa)$, which is derived from the (p-stabilized) Eisenstein series. These functions appear naturally when one moves between different weights of overconvergent modular forms and hence a good understanding of them is crucial to the entire theory. In particular, their overconvergence rates show up when studying the slopes (i.e. the valuations of eigenvalues of a p-adic Hecke-operator) which have been the object of a lot of research.

The first result will then be, assuming $p \geq 5$, an explicit bound on its overconvergence rate, involving a on p-depending constant. To prove this, we introduce the notion of a "formal Katz expansion", which is an interpolation of the regular Katz expansions. A technical argument involving valuations of Vandermonde matrices will then allow us to deduce bounds on the overconvergence rates. We show that these rates are not optimal and give some improved bounds in certain specific cases. We comment on its relation to a conjecture made by Coleman, and on its connections to statements, such as the Halo conjecture. We furthermore demonstrate why it would be desirable to know the exact overconvergence rate, which prompts the remaining part of the thesis.

In the final chapter, we give a computation counterpart to these statements. We provide (again under the assumption that $p \geq 5$) two algorithms. The first one allows us to compute the Katz expansion of an overconvergent modular form (given its q-expansion as input). The second algorithm, the more important one, uses the first algorithm to compute valuations of terms appearing in the formal Katz expansion. As these valuations are key in bounding the overconvergence rates of $E^*_\kappa/V(E^*_\kappa)$, this algorithm gives us insight in these rates. Based on data obtained through this algorithm, we provide a conjecture that would directly imply an improved bound on the overconvergence rate as proved in the previous chapter. The correctness of both algorithms is proved as well.

Abstrakt

I denne afhandling studerer vi eksplicitte overkonvergensrater relaterede til Eisenstein-rækker. Vi starter med at opbygge den nødvendige, teoretiske baggrund for teorien for overkonvergente modulformer. Dette omfatter teori for Katz-udviklinger, som er hovedværktøjet til at udlede overkonvergensrater i resten af afhandlingen. Vores hovedmål er da studiet af familien af modulfunktioner $E^*_\kappa/V(E^*_\kappa)$, som er afledt af (p-stabiliserede) Eisenstein-rækker. Disse funktioner optræder naturligt, når man bevæger sig mellem forskellige vægte af overkonvergente former, og følgelig er en god forståelse af dem afgørende for hele teorien. Specielt optræder deres overkonvergensrater, når man studerer hældninger (dvs., valuationer af egenværdier af en vis p-adisk Hecke-operator), som har været mål for megen forskning.

Det første resultat er da for $p \geq 5$ en eksplicit vurdering af overkonvergensraten. Denne vurdering afhænger af en af p afhængig konstant. I beviset for dette indfører vi begrebet ”formel Katz-udvikling”, som er en interpolation af de normale Katz-udviklinger. Et teknisk argument, der involverer valuationer af Vandermonde-determinanter, tillader os da at udlede vurderinger af overkonvergensraterne. Vi viser, at disse rater ikke altid er optimale og giver optimerede vurderinger i visse, specielle tilfælde. Vi relaterer vurderingen til en formodning af Coleman og kommenterer på forbindelser til visse, andre udsagn såsom Halo-formodningen. Vi viser ydermere, hvorfor det ville være ønskværdigt at kende til den præcise overkonvergensrate, hvilket giver anledning til den sidste del af afhandlingen.

I det sidste kapitel giver vi et beregningsmæssigt modstykke til de foregående resultater. Vi udvikler to algoritmer (igen under antagelse af $p \geq 5$). Den første tillader beregning af Katz-udviklingen af en overkonvergent modulform (givet formens q-udvikling). Den anden algoritme, som er vigtigere, bruger den første til at beregne valuationen af led, der forekommer i en formel Katz-udvikling. Idet disse valuationer er nøglen til at vurdere overkonvergensrate af $E^*_\kappa/V(E^*_\kappa)$, giver denne algoritme informationer om disse rater. Pågrundlag af data opnået gennem denne algoritme opstiller vi en formodning, som direkte ville medføre en optimeret vurdering af overkonvergensraten, som givet i tidligere kapitler. Algoritmernes korrekthed bliver ogsåbevist.

1 Introduction

Modular forms have occupied, and still occupy, a central role in modern number theory. Classically, modular forms are defined as holomorphic functions on the upper half plane, exhibiting certain transformation rules with regards to a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and satisfying a growth condition. These functions turn out to have a Fourier expansion, also called q-expansion, and the Fourier coefficients contain a plethora of arithmetic information.

While the complex theory is already very rich and beautiful on its own, people found that many modular form satisfy intriguing congruences modulo (powers of) a prime p. For example, famously Lehner proved in 1949 that the j-function, a modular function of weight 0, exhibits many congruences modulo powers of $2, 3, 5, 7$ and 11 [Leh49b, Leh49a]. For example, if j is the modular j-function with q-expansion given by

$$j = q^{-1} + 744 + 196885q + ... = \sum_{n \geq -1} a_n q^n,$$

then one has, for all integers $m \geq 1$,

$$a_{2^m n} \equiv 0 \mod 2^{3m+8}.$$

One might hope for a p-adic theory of modular forms explaining this. This is indeed the case, and it was initiated by Atkin, Swinnerton-Dyer and Serre. Serre was initially motivated by the theory of p-adic zeta-functions. In particular, special values of zeta functions are constant terms of Eisenstein series. Serre then used p-adic modular forms to obtain information about p-adic zeta functions.

We will give a short description of the theory of p-adic modular forms as given by Serre, see [Ser73]. The idea of his approach is to use congruences of q-expansions. To this end, if we have a formal power series

$$f = a_0 + a_1 q + a_2 q^2 + \ldots \in \mathbb{Q}_p[[q]],$$

then we define $\nu_p(f) := \min_i \nu_p(a_i)$, where ν_p is the p-adic valuation such that $\nu_p(p) = 1$. Serre then simply defined p-adic modular forms to be limits of Cauchy sequences of q-expansions of classical modular forms with respect to this valuation. Interestingly enough, Serre shows that if classical modular forms converge p-adically, then *also* their weights will converge to a limit p-adically. In particular, Serre established that p-adic modular forms have a well-defined weight

$$\kappa \in \varprojlim_i \mathbb{Z}/(p-1)p^i\mathbb{Z} \cong \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p,$$

which is a first hint that p-adic modular forms will take more general weights than just integral weights. The space of p-adic modular forms of fixed weight will then form an infinite dimensional Banach space (with the norm induced by the valuation mentioned above). Serre's theory is elegant, and it indeed sheds light on congruences as above and on p-adic zeta functions. Nevertheless, it also has some shortcomings. Firstly, Serre's theory relies crucially on the q-expansions of modular forms, and it is difficult to apply more geometrical tools to their study. Thus, a more geometrical definition of these modular forms would be desirable. Another issue pertains to the Hecke operators. Serre showed that the space of p-adic modular forms is equipped with Hecke operators (as one would want), including a Hecke operator at the prime p, also called the Atkins U_p-operator. One would expect the eigenvalues of this operator to contain a wealth of arithmetic insights, similarly to the Fourier coefficients of classical modular forms. Unfortunately, it turns out that the U_p-operator is not compact on the space of p-adic modular forms. This complicates the study of its spectrum a lot, and one can indeed show that the spectrum is way too large to contain meaningful arithmetic information.

To remedy this, Katz has introduced the notion of overconvergent modular forms, or p-adic modular forms with a growth condition [Kat73]. Katz shows that also on this space there is a theory of Hecke operators and now the U_p-operator *is compact*, and the study of its spectrum has since begun. As overconvergent modular forms à la Katz are the main topic of this thesis, we have devoted Chapter 2 to the theoretical background regarding these. Note that the study of overconvergent modular forms has shifted a lot since Katz. In particular, Coleman has developed the theory using rigid-analytic language and used this to prove a lot of new of properties, see e.g. [Col97c]. More recently, work by Pilloni [Pil13], and Andreatta, Iovita, Stevens [AIS14], provides a more intrinsic and geometrical definition of overconvergent modular forms. While this has shown great theoretical results, it is difficult to apply to explicit questions such as studied in this thesis. As a result, we will focus mainly on Katz' theory, as it ultimately is what allows us to prove explicit results about overconvergence rates in Chapter 3 and Chapter 4.

To state the main results of this thesis, we will give a short introduction to Katz' theory (as in [Kat73]), but for more details we refer the reader to Chapter 2. Katz has interpreted classical modular forms as rules that assign values to elliptic curves together with additional data, for example a nonzero differential and a level structure. To arrive at the definition of overconvergent modular forms we fix an element $r \in \mathbb{C}_p$, which is called the overconvergence rate. Then we look again at tuples consisting of elliptic curves with additional data, but we require that our elliptic curves are "not too supersingular". To make this precise, we need to lift the Hasse invariant of elliptic curves over finite fields to characteristic 0, which is particularly easy to do when $p \geq 5$. If our overconvergence rate r is a unit, i.e. $\nu_p(r) = 0$, then one only considers ordinary elliptic curves. Consequently, overconvergent modular forms of rate r, can (a priori) only be evaluated on ordinary elliptic curves. In fact, it is a theorem that this space coincides with the space of Serre's p-adic modular forms (at least, for integral weights). If, however, we pick an r such that $\nu_p(r) > 0$, then we consider a larger class of elliptic curves, including some supersingular curves, and thus overconvergent modular forms of rate r can be evaluated beyond only ordinary elliptic curves. We will now simply define the space of overconvergent modular forms, to be those modular forms that are overconvergent for some overconvergence rate r such that $\nu_p(r) > 0$. Note that this theory only applies for integral weights (i.e. an integer $k \in \mathbb{Z}$), even though overconvergent modular forms take a weight from a much larger space. More concretely, a weight will refer to a continuous map $\kappa : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$. To a weight we will then assign the value $w_\kappa := \kappa(p+1) - 1 \in \mathbb{C}_p$. There are various ways to extend the theory of overconvergent modular forms to this full weight space, but our results are mainly concerned with weight 0, so we will not delve too deeply into this.

The aim of this thesis is to study, and to develop methods to compute overconvergence rates of explicit overconvergent modular forms, derived from the Eisenstein family. In particular, Chapter 3 provides theoretical results on these overconvergence rates, whereas Chapter 4 provides algorithms to obtain data closely connected to these rates. We consider weights κ which are trivial on the $(p-1)$st roots of unity. We denote this subspace of the weight space by $\mathcal{B}$. For a weight $\kappa \in \mathcal{B}\backslash\{1\}$, denote by ζ^* the p-adic zeta function and define the Eisenstein series, whose q-expansion is given by (if $\zeta^*(\kappa) \neq 0$)

$$E_\kappa^*(q) = 1 + \frac{2}{\zeta^*(\kappa)} \sum_{n=1}^{\infty} \sigma_n'(\kappa) q^n,$$

where

$$\sigma_n'(\kappa) := \sum_{\substack{d \mid n \\ p \nmid d}} \frac{\kappa(d)}{d}.$$

This will be an overconvergent modular form over $\mathbb{Z}_p$, of weight κ and tame level 1 (see [Col97c]). If we have an integer $k \in \mathbb{Z}$, then we have the associated weight $k : x \mapsto x^k$ (also denoted by k), and E_k^* will be the p-stabilized classical Eisenstein series. The Eisenstein series is extremely important in the classical context of overconvergent modular forms as it allows one to move between weights.
Furthermore, one can show that there is a powerseries $E(w, q) \in \mathbb{Z}_p[[q, w]]$, such that $E(w_\kappa, q) = E_\kappa^*$, for any weight $\kappa \in \mathcal{B}\backslash\{1\}$. On the space of overconvergent modular forms we have the Frobenius operator, V, which acts on q-expansions as

$$V\left(\sum a_n q^n\right) = \sum a_n q^{pn}.$$

Then one can consider the modular functions $E^*_\kappa/V(E^*_\kappa)$, for weights $\kappa \in \mathcal{B}\backslash\{1\}$, which will be overconvergent modular forms of weight 0. Similarly to before, one can show that this interpolates to a family and we finally arrive at the main object of this thesis, which is the family

$$\frac{E}{V(E)} \in 1 + \mathbb{Z}_p[[q, w]],$$

which specializes to $E^*_\kappa/V(E^*_\kappa)$ if we substitute w by w_κ. The main results in Chapter 3 and Chapter 4 of this thesis pertain to computing (bounds on) the overconvergence rates of $E^*_\kappa/V(E^*_\kappa)$.

Before we state the main results in this thesis regarding these overconvergence rates, let us digress a bit on why one would want to study this. One main motivation comes from the so-called spectral halo conjecture. Coleman and Mazur have constructed the eigencurve, a rigid analytic curve $\mathcal{C}$ with a projection to the weight space $\pi : \mathcal{C} \to \mathcal{W}$ (see [CM98]). The fibers under this projection of a weight $\kappa \in \mathcal{W}$ correspond to eigenforms of the U_p-operator with a non-zero eigenvalue. We call the p-adic valuations of these eigenvalues the *slopes* of the U_p-operator. The slopes have been studied extensively and in some cases a lot is known. In the case that the slopes are 0, we are in Hida's ordinary case, and the theory is quite well understood, see e.g. [Hid93]. The non-ordinary case seems to be more complicated, but partial results have been obtained. For example, in the case that $p = 2$, weight 0 and tame level 1, Buzzard and Calegari have computed all the slopes [BC05]. Pollack and Bergdall have given a conjectural recipe for a complete description of the slopes [BP19b, BP19a, BP22]. Another fruitful approach is to move towards the edge of the weight space, i.e. to look at the weights κ such that $|w_\kappa|$ is very close to 1. While the geometry of the eigencurve (and thus also its slopes) can potentially be very complicated, it is expected that it behaves in a much simpler way near this boundary of weight space. In particular, it is expected (and in some cases known) that if a weight κ is close enough to the boundary, the slopes appearing in this weight should be composed of a finite union of arithmetic sequences (see [AIP18]). For $p = 2$ we have the following theorem.

Theorem. *(Buzzard, Kilford [BK05]) If $\kappa \in \mathcal{B}$ is a weight such that $|w_\kappa| > 1/8$, then the slopes of the overconvergent modular forms of weight κ (and tame level 1) are given by $\{0, w_\kappa, 2w_\kappa, 3w_\kappa, \ldots\}$ and all of these slopes occur with multiplicity one.*

It is also known in the case $p = 3$.

Theorem. *(Roe [Roe14]) If $\kappa \in \mathcal{B}$ is a weight such that $|w_\kappa| > 1/3$, then the slopes of the overconvergent modular forms of weight κ (and tame level 1) are given by $\{0, 1/2 \cdot w_\kappa, w_\kappa, 3/2 \cdot w_\kappa, \ldots\}$ and all of these slopes occur with multiplicity one.*

Both theorems are proven in a similar manner that is very explicit and they make crucial use of the fact that for $p = 2$ and $p = 3$, the modular curve $X_0(p)$ has genus 0. In particular, this means that there is uniformizer f such that powers of cf, for an appropriate chosen constant c, form a Banach basis for overconvergent modular forms of tame level

1 and of fixed overconvergence rate [Loe07]. Both in [BK05] and [Roe14] an appropriate uniformizer is chosen which interacts nicely with the U_p-operator. Denote this uniformizer by y (of course, the uniformizer is not the same function for $p = 2$ and $p = 3$) and let $\kappa \in \mathcal{B}$ be a weight close enough to the boundary of the eigencurve as in the theorems above. It is then possible to show that the space of overconvergent modular forms of weight κ has a Banach space basis given by

$$B := \{V(E_\kappa^*)(cy)^i\}_{i\in\mathbb{N}},$$

where c is a constant in $\mathbb{C}_p$ with a specific valuation. To study the spectrum of the U_p-operator, we need to study its matrix with respect to this basis. In particular, define $m_{i,j} \in \mathbb{C}_p$ to be

$$U_p(V(E_\kappa^*)(cy)^j) = V(E_\kappa^*)\sum_{i=0}^{\infty} m_{i,j}(cy)^i, \tag{1.1}$$

i.e. the matrix entries of U_p. Now we can use the following property of the V-operator and the U_p-operator

$$U_p(gV(h)) = hU_p(g),$$

where h, g are overconvergent modular forms (note that this statement is straightforward to prove on q-expansions). Applying this to (1.1), we obtain

$$\sum_{i=0}^{\infty} m_{i,j}(cy)^i = \frac{E_\kappa^*}{V(E_\kappa^*)}U_p((cy)^j). \tag{1.2}$$

As the uniformizer was chosen to interact nicely with the U_p-operator, it only remains to have a good understanding of the Eisenstein term. Indeed, both in the case of $p = 2$ and $p = 3$, a study of the overconvergence rate of $E_\kappa^*/V(E_\kappa^*)$ leads to be able to deduce from (1.2) the precise shape of the Newton polygon of the characteristic powerseries of the U_p-operator and as a consequence its slopes.

While this does not easily generalize to all primes (i.e. the genus of the modular curve is not 0 in general), it does show that the exact overconvergence rates of the modular forms $E_\kappa^*/V(E_\kappa^*)$ play a crucial part in the examination of slopes.

In the cases $p = 2$ and $p = 3$, the overconvergence rates of these modular forms was obtained using a uniformizer for $X_0(p)$, and thus crucially relies on the fact that for $p = 2, 3$ the modular curve has genus 0, which also holds for $p = 5, 7, 13$. However, this fails to be true for all other primes. We thus need to take a step away from the existence of uniformizers and use a different approach. The way this is done in this thesis is using the theory of Katz expansions. For $p \geq 5$, weight $k = 0$ and tame level 1, the Katz expansion is particularly easy to describe. Katz showed that for each $i \in \mathbb{N}_{\geq 0}$ there is a splitting

$$M_{i(p-1)}(\mathbb{Z}_p) = E_{p-1} \cdot M_{(i-1)(p-1)}(\mathbb{Z}_p) \oplus B_i(\mathbb{Z}_p), \tag{1.3}$$

where E_{p-1} is the Eisenstein series of weight $p - 1$ and level 1, normalized such that its constant coefficient is 1 (see [Kat73, Lemma 2.6.1]). Such a splitting is not unique, but once it has been chosen, Katz has shown that an overconvergent modular form of weight

0 and tame level 1 can be written uniquely as $f = \sum_{i=0}^{\infty} \frac{b_i}{E_{p-1}^i}$, where $b_i \in B_i(\mathbb{Z}_p)$, which is called *the Katz expansion of* f. However, even more is true, if $f \in M_0(1, r; \mathbb{Z}_p)$, where $r \in \mathbb{C}_p$ is the overconvergence rate, then the Katz expansion satisfies that $\nu_p(b_i) \geq i\nu_p(r)$ for all i and the difference $\nu_p(b_i) - i\nu_p(r)$ goes to infinity as i goes to infinity. Hence, we can use the Katz expansion to measure the overconvergence rate. To this end, we define a slightly different module $M_0(\mathbb{Z}_p; \geq \rho)$, for some $\rho \in \mathbb{Q}$, consisting of the overconvergent modular forms that are overconvergent for all $r \in \mathbb{C}_p$ such that $\nu_p(r) < \rho$. In terms of Katz expansions, that means exactly that $f \in M_0(\mathbb{Z}_p; \geq \rho)$ if and only if $\nu_p(b_i) \geq i\nu_p(r)$ for all i, where the b_i are the modular forms appearing in the Katz expansion of f. Note that one can easily extend all of this to more general rings. The first main result in Chapter 3 is then

Theorem A. *There is a constant* $0 < c_p < 1$ *such that the following holds. Let* $\kappa \in \mathcal{B}\backslash\{1\}$ *be a character and let* O *be the ring of integers in the extension of* $\mathbb{Q}_p$ *generated by the values of* κ.

Then
$$\frac{E_\kappa^*}{V(E_\kappa^*)} \in M_0(O, \geq c_p \cdot \min\{1, v_p(w(\kappa))\}).$$

Explicitly, we can take
$$c_p = \frac{2}{3} \cdot \left(1 - \frac{p}{(p-1)^2}\right) \cdot \frac{1}{p+1}.$$

The proof of this relies on the existence of a "formal Katz expansion". Morally, this is a Katz expansion of the entire family $E/V(E)$, but now the coefficients are not modular forms $\beta_i \in B_i(\mathbb{Z}_p)$, but they are formal sums of the form $\sum_j b_{i,j} w^j$, where b_j is in $B_i(\mathbb{Z}_p)$. If we specialize w to w_κ, for a weight $\kappa \in \mathcal{B}\backslash\{1\}$, we end up with the Katz expansion of $E_\kappa^*/V(E_\kappa^*)$. The second main result of Chapter 3, and the main ingredient in proving Theorem A, is then the following.

Theorem B. *(a) There are modular forms* $b_{ij} \in B_i(\mathbb{Z}_p)$ *for each* $i, j \in \mathbb{Z}_{\geq 0}$ *such that the following holds. If* $\kappa \in \mathcal{B}\backslash\{1\}$ *then the Katz expansion of the modular function* $\frac{E_\kappa^*}{V(E_\kappa^*)}$ *is*
$$\frac{E_\kappa^*}{V(E_\kappa^*)} = \sum_{i=0}^{\infty} \frac{\beta_i(w(\kappa))}{E_{p-1}^i}$$
where
$$\beta_i(w(\kappa)) := \sum_{j=0}^{\infty} b_{ij} w(\kappa)^j$$
for each i.
(b) There is a constant c_p *with* $0 < c_p < 1$ *such that for the modular forms* b_{ij} *in part (a) we have*
$$v_p(b_{ij}) \geq c_p i - j$$
for all i, j.

In fact, we can take the explicit constant c_p *from Theorem A.*

The idea of the proof of Theorem B is to use an explicit splitting given by Lauder to prove the existence of the formal Katz expansion. For part (b) we utilize the fact that overconvergence rates for classical weight Eisenstein series are known, see [KR21]. Hence, there are infinitely many weights to which we can specialise the formal Katz expansion, and we will know exactly the overconvergence rates in these cases. A combinatorial/linear algebra argument then allows us to "interpolate" these overconvergence rates (see Proposition 3.3) and to deduce the specific bound on the $b_{i,j}$. In the same chapter we furthermore compare this result to a conjecture Coleman has made regarding the overconvergence rate of the family $E/V(E)$, which seemed to be a too optimistic interpolation of the results known for $p = 2, 3$. We also provide better bounds on $\nu_p(b_{i,j})$ in specific cases (i.e. $p = 5, 7$). For example, we show that if $p = 5, 7$ we can improve the constant c_p and instead take the value

$$c_p = \left(1 - \frac{p}{(p-1)^2}\right) \cdot \frac{p-1}{p(p+1)} = \frac{p^2 - 3p + 1}{p(p^2 - 1)}$$

in Theorem A and Theorem B. However, these arguments are based more on happy coincidences, and similar arguments do not easily generalize to other primes.

Chapter 4 provides a computational counterpart to Chapter 3. In particular, we would like to gather computational data regarding Theorem B, specifically bounds on the $\nu_p(b_{i,j})$. One main reason is that the bounds in Chapter 4 might not be optimal. To this end, we first provide an algorithm, Algorithm 1, which computes the Katz expansion of an overconvergent modular form. More precisely, it takes as input a prime $p \geq 5$, two positive integers n and C, and a power series in $\mathbb{Z}_p[[q]]/(q^N, p^C)$. Here N is an explicit constant depending on n and should be seen as the accuracy we need for the q-expansion of some overconvergent modular form. The integer C is the p-adic accuracy. The algorithm will then have as output the first $n + 1$ terms of the Katz expansion (with regards to a fixed splitting). This algorithm is probably not novel, but we present it for completeness as it is used in the second algorithm. The idea of this first algorithm is quite easy. We have chosen a specific splitting that reduces finding the partial Katz expansion to solving a matrix equation of the form $Av = w$, where A is an upper triangular matrix with 1's on the diagonal.
The second algorithm is the more important and novel algorithm, which allows us to (most of the time) compute the valuations $\nu_p(b_{i,j})$, where the $b_{i,j}$ are as in Theorem B. More precisely, it takes as input a prime $p \geq 5$, a nonnegative integer r and a list of integral weights $L = [k_1, \ldots, k_\lambda]$, for some integer $\lambda \geq 0$. The algorithm will have as its output the values $\nu_p(b_{r,j})$ for the values $0 \leq j \leq r$, *if* the algorithm can conclude that these are the precise valuations. If not, it will give "inconclusive" as its output, and one could increase the number of weights as its input and run the algorithm again. For the full discussion on whether one will always find the valuation of a fixed $b_{i,j}$, and regarding the number of weights, see Chapter 4, in particular the discussion after Lemma 6, Remark 9 and Remark 10. The idea of this algorithm is as follows. One computes the Katz expansions, using Algorithm 1, for the modular forms $E^*_\kappa/V(E^*_\kappa)$ and for all weights in the given list. Using linear algebra over rings of the form $\mathbb{Z}/p^\lambda\mathbb{Z}$ will then provide us with the $b_{i,j}$, up to the

kernel of a Vandermonde matrix (which will fail to be invertible over the ring $\mathbb{Z}/p^\lambda\mathbb{Z}$). However, we can bound the p-adic valuations of elements in this kernel, and this will (in some certain cases) be enough to determine $\nu_p(b_{i,j})$.
We have used Algorithm 2 to compute values of $\nu_p(b_{i,j})$, for different primes p and different values of i, j. Based on the obtained data, we propose the following conjecture.

Conjecture 1. *Let the $b_{i,j}$ be as above, then we have that $\nu(b_{i,j}) \geq d_p i - j$, for all $i, j \geq 0$, where*

$$d_p = \frac{p-1}{p(p+1)}.$$

Similarly to Chapter 3, such a bound on the valuations of $\nu_p(b_{i,j})$ gives a bound on the overconvergence rate of the overconvergent modular forms $E_\kappa^*/V(E_\kappa^*)$. Indeed, assuming the conjecture, we have the following corollary.

Corollary 1.1. *Assume that Conjecture 1 holds. Let $\kappa \in \mathcal{B}\backslash\{1\}$ be a character and let $\mathcal{O}$ be the ring of integers in the extension of $\mathbb{Q}_p$ generated by the values of κ.*

Then

$$\frac{E_\kappa^*}{V(E_\kappa^*)} \in M_0(\mathcal{O}, \geq d_p \cdot \min\{1, v_p(w(\kappa))\}).$$

Thus Conjecture 1 would imply a better bound on the overconvergence rates of $E_\kappa^*/V(E_\kappa^*)$ than in Theorem A. At the end of Chapter 3, it is described why it would be desirable to know the exact overconvergence rate (see Section 3.4.3). Conjecture 1 thus points us to what this optimal overconvergence rate could be. In fact, for certain primes, we can find specific i, j such that $\nu_p(b_{i,j}) = d_p i - j$, and thus one might wonder if this is the optimal value. It would certainly be possible, but due to the limited data obtained (as Algorithm 2 gets very slow when one increases the value of p), it is unclear if we could expect this.

2 Theoretical Background

This chapter introduces the theory of (overconvergent) modular forms. We start with an examination of classical modular forms, as defined by Katz. Subsequently, we will look at the theory of overconvergent p-adic modular forms. We will describe the main result used in the other chapters, which is the explicit description of modules of overconvergence modular forms. We will also state the relation between classical modular forms, overconvergent modular forms and p-adic modular forms as initially defined by Sere. The later sections will develop a part of the theory of the Hecke operators, including Atkin's U_p-operator. We will touch on Coleman's theory of overconvergent modular forms and the study of the spectrum of the U_p-operator. Most of this text will be based on [Kat73, Gou88, Col97c].

2.1 Classical Modular Forms

In this section we will describe the space of classical modular modular forms as defined by Katz (see [Kat73]). The way Katz does this, is to define modular forms as certain 'rules' that assign to tuples consisting of an elliptic curve E (over an adequately chosen algebra R), a weight (an integer k), a level structure (see below for its exact definition), and a non-zero differential on E, an element in R, such that these 'rules' behave well. For example, they should behave well with base change of the elliptic curve, and they should be invariant with respect to the chosen differential up to an automorphy factor, which is related to the weight. Normally, classical modular forms are defined as functions on the upper half plane behaving nicely with respect to the action of a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. However, this definition does not have the required flexibility to consider modular forms over more general rings and Katz' definition works much better if one wishes to achieve this. Furthermore, overconvergent modular forms as defined by Katz, allow for a very similar definition, and thus this section can be seen as an introduction to these ideas.

2.1.1 Elliptic Curves and Level Structures

One main ingredient in the definition of modular forms is the level. As is standard, a level will simply be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. One could develop the whole theory for a large class of congruence subgroups, but we will mainly be working with the following (commonly used) congruence subgroups to simplify a lot of the exposition. For $N \in \mathbb{N}$ we have the following congruence subgroups,

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \mod N, \quad b, c \equiv 0 \mod N \right\},$$

of the matrices that reduce to the identity matrix modulo N. We have the group

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \mod N \right\},$$

which are the matrices that reduce to an upper triangular matrix modulo N, and finally we have the subgroup given by

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \mod N, \quad c \equiv 0 \mod N \right\},$$

consisting of those matrices that reduce to upper triangular matrices with 1's on the diagonal.

As we will define modular forms as rules that can be evaluated on elliptic curves with a level structure, we need to define what a level structure on an elliptic curve is. To this end, we fix a ground ring R_0. In our cases, this is often be the ring of integers in a number field, a finite field, or the ring of integers in a local field. We consider elliptic curves over R_0-algebras R and make the following definitions.

Definition 2.1. *A* $\Gamma(N)$ level structure *on an elliptic curve* E/R *is an isomorphism of finite flat group schemes (over* R*)*

$$\alpha_N : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N],$$

where $E[N]$ *is the kernel of the "multiplication by* N*" map.*
A $\Gamma_1(N)$ level structure *on an elliptic curve* E/R *is an injection of finite flat group schemes (over* R*)*

$$\alpha_N : \mu_N \hookrightarrow E[N],$$

where μ_N *is the finite flat group scheme of the* N*th roots of unity.*
A $\Gamma_0(N)$ level structure *on an elliptic curve* E/R *is a finite flat subgroup scheme* $H \subset E[N]$ *such that* $H \simeq \mu_N$*.*

When considering elliptic curves over, for example, the complex numbers, these level structures are often interpreted as follows: a $\Gamma(N)$ level structure on an elliptic curve E corresponds to two points (P, Q) generating the N-torsion subgroup plus a condition on their Weil pairing, a $\Gamma_1(N)$ level structure corresponds to a point of exact order N, and a level $\Gamma_0(N)$ structure corresponds to a cyclic subgroup of order N of $E[N]$. However, we want to define modular forms over any ring R, and in particular one might encounter supersingular elliptic curves. For example, if the base ring is chosen to be $\mathbb{F}_p$, and we want to consider a level $\Gamma_1(p)$ structure on a supersingular elliptic curve $E/\mathbb{F}_p$, then one would run into problems, as $E[p] = 0$. This can be solved by appealing to the theory of finite flat group schemes, as in the definition above. However, a deep understanding of the theory of group schemes will not be required to understand the rest of this thesis.

Remark 2.2. *Before we give the first definition of modular forms, a small remark is in place regarding the (allowed) levels. A lot of the theory of (Katz) modular forms requires the existence of a scheme representing the functor that sends an R_0-algebra, say R, to the set of isomorphism classes (appropriately defined) of all tuples $(E/R, \alpha, \omega)$ consisting of an elliptic curve over R, with a prescribed level structure α and a nonzero differential ω on E/R. In most cases, this functor is actually representable by a scheme, however in other cases (e.g. if we consider a $\Gamma(1)$ level structure) this is not true and instead one would have to use the theory of stacks. There is a possibility to circumvent the use of stacks when it comes to modular forms, namely one can first increase the level (and hence make the above functor representable by a scheme) and then take invariants. We will not be concerned with these problems, and instead refer the reader to e.g. [Kat73].*

We are now ready to define meromorphic modular forms over some ground ring R_0, of weight k and level $\Gamma_0(N)$.

Definition 2.3. *Let $k \in \mathbb{Z}$ and $N \in \mathbb{N}$. A* meromorphic modular form of level $\Gamma_0(N)$ and weight k, *over a ring R_0, is a rule f which assigns to a triple $(E/R, \omega, \alpha_N)$, where E/R is an elliptic curve over an R_0-algebra R, α_N a level $\Gamma_0(N)$ structure on E, and ω a nonzero element of $\Omega_{E/R}$, an element $f(E/R, \omega, \alpha_N) \in R$ such that*

1. *$f(E/R, \omega, \alpha_N) \in R$ only depends on the isomorphism class of $(E/R, \omega, \alpha_N)$;*
2. *if $g : R \to S$ is a map of R_0-algebras, and E'/S the base change of E/R to S, then*
$$g(f(E/R, \omega, \alpha_N)) = f(E'/S, \omega', \alpha'_N),$$
where α'_N is the induced level structure on E'/S;
3. *for any $\lambda \in R'^{\times}$ we have $f(E/R', \lambda\omega, \alpha_N) = \lambda^{-k} f(E/R', \omega, \alpha_N)$.*

One easily adapts the definition above to include other level structures. This definition, however, only allows us to define meromorphic modular forms as there is no condition on the cusps, as in the classical theory. When working over $\mathbb{C}$, this is remedied by certain growth rates along the imaginary axis, which is a more analytic notion. Or, equivalently, one asks the Fourier expansion at the cusps to be a genuine power series, instead of just a Laurent series. We would like to turn this into an algebraic condition and in order to do that, we must consider the so-called Tate curve. While there is a very rich and deep theory behind the theory, we will content ourselves with giving just the following very concrete definition.

Definition 2.4. *The Tate curve is the elliptic curve $T(q)$ over the Laurent series ring $\mathbb{Z}((q))$, given by the equation*
$$y^2 + xy = x^3 + a_4(q)x + a_6 q,$$
where
$$a_4 := -5 \sum_{n \geq 1} \frac{n^3 q^n}{1 - q^n},$$

and

$$a_6 := -\sum_{n\geq 1} \frac{(5n^3+7n^5)q^n}{12(1-q^n)}.$$

We use the fact that there is a canonically defined differential on this curve, which we will denote by ω_{can}. Furthermore, one can describe various level structures on the Tate curve. For example, we will consider $\Gamma_1(N)$ level structures on the Tate curve $T(q^N)$, and hence one has to give an exact point of order N on this curve. When we talk about the canonical level $\Gamma_1(N)$ structure, we mean the level structure corresponding to the point q on $T(q^N)$. For a more in depth discussion on the Tate curve, see for example [DR73, Chapter VII] or [Kat73, Appendix 1].
To motivate the definition, consider a meromorphic modular form f, say of level $\Gamma(1)$, weight k and over a ring R_0. Then, we can consider the Tate curve (together with its canonical differential) as an elliptic curve over the ring $\mathbb{Z}((q)) \otimes_{\mathbb{Z}} R_0$, and hence f will assign to this a value $f(T(q), \omega_{can}) \in \mathbb{Z}((q)) \otimes_{\mathbb{Z}} R_0$, which is also called the q-expansion of our meromorphic form f. Similarly to the classical analytic definition, we then say that f is a modular form if this q-expansion lies in the subring $\mathbb{Z}[[q]] \otimes_{\mathbb{Z}} R_0$, i.e. it is a genuine powerseries. We thus have the following definition of modular forms (of level $\Gamma_0(N)$).

Definition 2.5. *A modular form of level $\Gamma_0(N)$, weight k, and over a ring R_0 (containing $1/N$ and ζ_N, a primitive Nth root of unity), is a meromorphic modular form f, such that*

$$f(T(q^N), \omega_{can}, \alpha_N) \in \mathbb{Z}[[q]] \otimes_{\mathbb{Z}} R_0$$

for all level $\Gamma_0(N)$-structures on $T(q^N)$.

Notice that in this definition we require that the ring R_0 contains $1/N$ and an Nth root of unity. It is indeed a necessary requirement to talk about the q-expansion of f, as otherwise the level structures of the Tate curve are not defined over $\mathbb{Z}[[q]] \otimes R_0$. One could, however, define modular forms over any ground ring R_0, simply by defining that a meromorphic modular form f over R_0 is a modular form, if f restricted to $R_0[1/N, \zeta_N]$ is a modular form. We will denote the space of all modular forms of weight k, level $\Gamma_0(N)$, and over a ring R_0, by $M_k(\Gamma_0(N); R_0)$. Of course, one can also define modular forms for other levels in the exact same manner.

Remark 2.6. *There is a slightly different viewpoint to the definitions as given above. Instead of considering our usual triples, one could consider instead only pairs $(E/R, \alpha_N)$. Then, we could define modular forms as rules that assign to such tuples some section of the line bundle $\Omega_{E/R}^{\otimes k}$, such that these rules are invariant under the isomorphism class of the tuples, behave well with base change and satisfy our usual condition for the Tate curve. Of course, if we are given such a modular form g, then one gets a modular form f as defined earlier, simply by posing*

$$f(E/R, \omega, \alpha_N)\omega^{\otimes k} = g(E/R, \alpha_N),$$

and thus, in our case, these give the same spaces. However, combined with the existence of moduli spaces (or stacks) of the pairs $(E/R, \alpha_N)$, this (slightly) different definition gives rise to a beautiful geometric interpretation of modular forms as sections of a specific line bundle on these moduli spaces, see [Kat73].

We conclude this section with some (well-known) examples of modular forms.

Example 2.7. *Among the most well-known examples of modular forms, are of course the Eisenstein series. For $k \geq 4$, we have the modular form $E_k \in M_k(\Gamma(1); \mathbb{Q})$, whose q-expansion is given by*

$$E_k = 1 - \frac{2k}{b_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where b_k is the kth Bernoulli number and σ_{k-1} is the usual divisor function. If we fix a prime $p \geq 5$, then we could consider $E_{p-1} \in M_{p-1}(\Gamma(1); \mathbb{Q})$. In fact, one can easily check that $E_{p-1} \in M_{p-1}(\Gamma(1); \mathbb{Q} \cap \mathbb{Z}_p)$, and thus we could look at its reduction in the residue field $\mathbb{F}_p$, which will give us a modular form $\overline{E_{p-1}} \in M_{p-1}(\Gamma(1); \mathbb{F}_p)$. Note that its q-expansion will simply given by $1 \in \mathbb{F}_p[[q]]$. This form will play a key role in the overconvergent theory, see Subsection 2.2.1.

Example 2.8. *[Kat73, A1.2] To give an example that aligns more closely with the definition of modular forms given in this section, we consider a field K such that $\mathrm{char}(K) \neq 2, 3$. Any elliptic curve over K can then be written in the form*

$$E : Y^2 = 4X^3 + AX + B.$$

Any nonzero differential of E will be a multiple of the differential $\omega := \frac{dx}{y}$, and it thus suffices to define a modular form by giving its values on (E, ω). Indeed, we can define f to be the (a priori meromorphic) modular form such that $f(E, \omega) = A$, and g to be the modular form such that $g(E, \omega) = B$. A small calculation on how they scale when the differential is scaled, then shows (assuming that they are holomorphic at the cusps) that $f \in M_4(\Gamma(1); K)$ and $g \in M_6(\Gamma(1); K)$. Note that we can write the Tate curve in its short Weierstrass form as following

$$T(q) : y^2 = 4x^3 - 27c_4 x - 54c_6,$$

where

$$c_4 = \frac{48a_4 - 1}{12}$$
$$c_6 = \frac{a_6 + 1}{108},$$

and where a_4 and a_6 are as in Definition 2.4. This allows us to compute their q-expansions (preferably using a computer) and verify that f and g are indeed modular forms. Furthermore, looking at their q-expansions we conclude that $f = -E_4/12$ and $g = E_6/216$.

One could also define Hecke operators on these spaces, which are indispensable to the study of modular forms. As we will not really use the theory of Hecke operators in the setting of classical modular forms, we postpone the discussion on Hecke operators to the p-adic setting in Section 2.2.4.

2.2 Overconvergent p-adic Modular Forms

In the previous section we have defined classical modular forms. The following step is to construct a nice theory of p-adic modular forms. As explained in the introduction, Serre has initiated the study of p-adic modular forms using congruences between q-expansions of classical modular form. Katz has instead introduced the notion of overconvergent p-adic modular forms, or, as Katz described them, p-adic modular forms with a growth condition. The specific growth condition is called the overconvergence rate, and it will play a key role in the other chapters. In this section we will introduce some of the basic notions of this theory, including the required tools for the rest of this thesis.

2.2.1 The Hasse Invariant

As the subsequent sections require the notion of the Hasse invariant, we give its definition and we provide the properties required for understanding the rest of the theory. To this end, we consider an elliptic curve E over an $\mathbb{F}_p$-algebra R. On this elliptic curve we have the absolute Frobenius, and in particular we have a map $F_{abs} : \mathcal{O}_E \to \mathcal{O}_E$, where $\mathcal{O}_E$ is the structure sheaf on our elliptic curve. This map induces a linear map on the corresponding first cohomology group, $F_{abs}^* : H^1(E, \mathcal{O}_E) \to H^1(E, \mathcal{O}_E)$. If we pick a basis of $\Omega_{E/R}$, i.e. a nonzero differential on E/R, say ω, then Serre duality gives a dual basis η of $H^1(E, \mathcal{O}_E)$. We make the following definition of the Hasse invariant.

Definition 2.9. *The* Hasse invariant *of the pair* $(E/R, \omega)$ *is defined to be the value* $A(E/R, \omega) \in R$ *such that*

$$F_{abs}^*(\eta) = A(E/R, \omega) \cdot \eta.$$

To understand the choice of differential, consider an element $\lambda \in R^\times$. Then we could, instead, take the basis $\lambda\omega$ of $\Omega_{E/R}$, and its dual basis is then given by $\lambda^{-1}\eta$. As $F_{abs}^*(\lambda^{-1}\eta) = \lambda^{-p} F_{abs}^*(\eta)$, we find that $A(E/R, \lambda\omega) = \lambda^{p-1} A(E/R, \omega)$. Furthermore, one can check that $A(E/R, \omega)$ is independent of the isomorphism class and that it commutes with base change. A computation with the Tate curve, see [Kat73, Section 2], then shows that the Hasse invariant is actually a modular form of weight $p-1$ and level $\Gamma(1)$ over $\mathbb{F}_p$, i.e. $A \in M_{p-1}(\Gamma(1); \mathbb{F}_p)$, and its q-expansion is simply given by 1.
If E/K is an elliptic curve over some finite field K of characteristic p, then $A(E/K, \omega)$ is 0 if and only if E/K is supersingular, i.e. $E[p] = 0$ [Sil09, Chapter V]. Thus, the Hasse invariant allows us to distinguish between ordinary and supersingular overconvergent elliptic curves. In the theory of overconvergent modular forms, we are mainly interested in elliptic curves over the ring of integers of some extension of $\mathbb{Q}_p$, say $\mathcal{O}$. Looking at their reduction over the residue field, the Hasse invariant allows us to see if the reduction is supersingular or not. However, for the full theory we want to measure "how supersingular" these elliptic curves are. To make this precise, we consider a lift of the Hasse invariant to the characteristic 0 ring $\mathcal{O}$. For $p = 2, 3$, this is somewhat problematic. However, if we assume $p \geq 5$, this is easily achieved. Namely, we have seen in Example 2.7, that the Eisenstein series of weight

$p-1$ has a q-expansion with coefficients in $\mathbb{Q} \cap \mathbb{Z}_p$ and modulo p, it reduces exactly to 1. This is thus indeed a lift of the Hasse invariant. Note that $E_{p-1} \in M_{p-1}(\Gamma(1); \mathbb{Z}_p)$ and thus, given an elliptic curve E/R over some $\mathbb{Z}_p$-algebra R and a nonzero differential ω, we get an element $E_{p-1}(E/R, \omega) \in R$.

2.2.2 Overconvergent p-adic Modular Forms

We are finally ready to state the definition of overconvergent modular forms. The definition is very similar to the definition of classical modular forms as given in the previous section, but we consider slightly different tuples. We will consider the specific setting as follows. We fix a prime $p \geq 5$. Let K be a field extension of $\mathbb{Q}_p$ and denote by R_0 its ring of integers. As always, we normalize the valuation on K so that $\nu_p(p) = 1$. We fix an element $r \in R_0$, which will be our overconvergence rate. Furthermore, fix an integer $N \in \mathbb{N}$ coprime to p, which is called the tame level. We consider quadruples $(E/R, \omega, \alpha_N, Y)$ where

- R is a p-adically complete and seperated R_0-algebra,
- E/R is an elliptic curve over R,
- ω is nonzero element of $\Omega_{E/R}$,
- α_N is a level $\Gamma_1(N)$ structure,
- $Y \in R$ is an element such that $Y \cdot E_{p-1}(E, \omega) = r$.

As E_{p-1} is a lift of the Hasse invariant (at least when $p \geq 5$) such a tuple could only possibly exist if E/R is not "too supersingular". For example, if $R = \mathbb{Z}_p$ and $r = 1$, then $\nu_p(1) = 0 = \nu_p(E_{p-1}(E, \omega)) + \nu_p(Y)$. Since $Y \in \mathbb{Z}_p$ we have $\nu_p(Y) \geq 0$, and thus the existence of the quadruple $(E/R, \omega, \alpha_N, Y)$ implies that $\nu_p(E_{p-1}(E, \omega)) = 0$, i.e. E is an ordinary elliptic curve. On the other hand, if we were to take a different r with a nonzero valuation, supersingular curves may be allowed, as long as their Hasse invariant (or at least, a lift thereof) is not too large.
In any case, the Tate curve is ordinary, and hence we can always define

$$Y_{r,tate} := r \cdot E_{p-1}(Tate(q), \omega_{can})^{-1}.$$

Similarly as before, this is what allows us to interpret "holomorphicity at the cusps" as an algebraic property and we are led to the following definition.

Definition 2.10. *An* overconvergent p-adic modular form f over R_0, of weight k, level N and of overconvergence rate r *is a rule that assigns to quadruples as above an element $f(E/R, \omega, \alpha_N, Y) \in R$, which only depends on the isomorphism class of the quadruple, commutes with base change, satisfies*

$$f(E/R, \lambda\omega, \alpha_N, \lambda^{p-1}Y) = \lambda^{-k} f(E/R, \omega, \alpha_N, Y)$$

for all $\lambda \in R_0^\times$ *and such that*

$$f(Tate(q^N), \omega_{can}, \alpha_N, Y_{r,tate}) \in \mathbb{Z}[[q]] \otimes R_0[\zeta_N],$$

for all level $\Gamma_1(N)$ *structures on the Tate curve.*

Note that as $E_{p-1}(E/R, \lambda\omega) = \lambda^{1-p} E_{p-1}(E/R, \omega)$, we indeed have that

$$\lambda^{p-1} Y \cdot E_{p-1}(E/R, \lambda\omega) = Y \cdot E_{p-1}(E/R, \omega) = r.$$

The R_0 module of all such elements is denoted by $M_k(\Gamma_1(N), r; R_0)$. The module consisting of all overconvergent modular forms of some overconvergence rate r, for r not a unit in R_0, is denoted by $M_k^\dagger(\Gamma_1(N); R_0)$. Note that when we say *the* q-expansion of an overconvergent modular form, of level $\Gamma_1(N)$, we mean the q-expansion at the cusp at ∞, i.e. we take the Tate curve $T(q^N)$ together with its canonical differential and the level structure corresponding to the point q (which will have exact order N). Note that there is the q-expansion principle, which states that if two modular forms (of level $\Gamma_1(N)$) have the same q-expansion at this cusp, then they define the same form (see e.g. [DI95, 12.3]).

Remark 2.11. *There is a nice geometric interpretation to this definition. To this end, we need to take a look at the rigid analytic modular curve* $\mathcal{X} := X(\Gamma_1(N))$ *over* $\mathbb{Z}_p$. *One can show that there is a reduction map*

$$\pi : \mathcal{X}(\mathbb{C}_p) \to \mathcal{X}_{\mathbb{F}_p}(\overline{\mathbb{F}}_p).$$

Elements on the right correspond to elliptic curves over $\overline{\mathbb{F}}_p$ *(with a level structure) and its inverse under the reduction map will be isomorphic to a rigid analytic open disk in* $\mathbb{C}_p$. *In particular, we can consider* $\mathcal{X}^{ord}$, *which is the rigid analytic modular curve consisting of purely the ordinary elliptic curves. It is precisely the modular curve, without a finite number of open rigid analytic disks corresponding to the supersingular elliptic curves. If one considers modular forms of overconvergence rate* $r = 1$, *then we can only evaluate them on* $\mathcal{X}^{ord}$. *However, if we choose an* r *such that* $\nu_p(r) > 0$, *we require that we can evaluate our modular forms on a slightly larger region of the modular curve than just the ordinary part; we can go a bit deeper into the supersingular disks. This also motivates the name: the overconvergent modular forms "converge" on a slightly larger region.*

The case $r = 1$ is easily described. In fact, one re-obtains Serre's definition of p-adic modular forms.

Proposition 2.12. *[Lemma 2.7.2 in [Kat73]] Let* $N \geq 3$ *and* $k \neq 1$ *(or* $k = 1$ *and* $N \leq 11$*). Then, given a power series* $f(q) \in R_0[[q]]$, *the following are equivalent:*

1. $f(q)$ *is the* q*-expansion of a form* $f \in M_k(N, 1; R_0)$
2. *for any* $n \geq 1$ *there exists an* $m \geq 1$ *such that* $m \equiv 0 \mod p^{n-1}$ *and a classical modular form* $g_n \in M_{k+m(p-1)}(N; R_0)$ *such that* $g(q) \equiv f(q) \mod p^m$.

Note that Serre defines p-adic modular forms for a larger set of weights, namely a weight is an element in $\mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$. In Section 2.2.7 we will show one way to define overconvergent modular forms for a larger set of weights, and not just for integral weights.

2.2.3 The Katz Expansion

Now that we have defined the main object of this thesis, we would like to give a more precise description. One way of doing this, is through the so-called Katz expansion. These expansions will play a crucial role in the following chapters and provide an approach to the entire theory that is very accessible to computations. The main result that leads to the theory of Katz expansions is the following lemma.

Lemma 2.13 (Lemma 2.6.1 in [Kat73]). *Under the assumptions of N, k as in Lemma 2.12, or that $k = 0$ and $p \neq 2$, and for every $i \geq 0$, the following map admits a section*

$$M_{k+i(p-1)}(\Gamma_1(N); \mathbb{Z}_p) \xrightarrow{\cdot E_{p-1}} M_{k+(i+1)(p-1)}(\Gamma_1(N); \mathbb{Z}_p).$$

To prove this lemma, one has to show that the cokernel of the "multiplication by E_{p-1}"-map is finite and free, the proof of which relies on coholomogical methods. For each N, k (satisfying the assumptions made as above) and for each $i \geq 0$, we will then fix once and for all such a splitting. Note that in Chapter 3 and Chapter 4 we will make use of an explicit splitting, see Chapter 4, section 4.2.1 for its description. Having chosen such a splitting, we thus have, for all $i \geq 1$, the following decomposition

$$M_{k+i(p-1)}(\Gamma_1(N); \mathbb{Z}_p) \simeq E_{p-1} \cdot M_{k+(i-1)(p-1)}(\Gamma_1(N); \mathbb{Z}_p) \oplus B_i(\Gamma_1(N), k; \mathbb{Z}_p),$$

where $B_i(\Gamma_1(N), k; \mathbb{Z}_p)$ is a $\mathbb{Z}_p$-submodule of $M_{k+i(p-1)}(\Gamma_1(N); \mathbb{Z}_p)$. For $i = 0$, we set $B_0(\Gamma_1(N), k; \mathbb{Z}_p) := M_k(\Gamma_1(N); \mathbb{Z}_p)$. Moreover, if R is a $\mathbb{Z}_p$-algebra, we define $B_i(N, k; R) := B_i(N, k; \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} R$. In Chapter 3 and Chapter 4, we will be mainly interested in the case that $k = 0$ and $N = 1$, and we will simply denote these spaces by $B_i(\mathbb{Z}_p)$. It is these spaces that can be used to give an explicit expansion of overconvergent p-adic modular forms. To this end, for R the ring of integers in a field extension K of $\mathbb{Q}_p$, we denote the following R-module

$$B_k^{\text{rigid}}(\Gamma_1(N); R) := \left\{ \sum_{i=0}^{\infty} b_i : b_i \in B_i(\Gamma_1(N), k; R) \text{ for all } i \text{ and such that } \lim_{i \to \infty} b_i \to 0 \right\},$$

where the last limit is in the sense that the coefficients in the q-expansion of the b_i go to 0 with respect to the p-adic topology. As it turns out, all spaces $M_k(\Gamma_1(N), r; R)$, will then be isomorphic (but via different isomorphisms) to the ring $B_k^{\text{rigid}}(\Gamma_1(N); R)$ through the so-called Katz expansion, as explained in the following theorem.

Theorem 2.14. *Let N and k satisfy the assumptions as in Lemma 2.13. Then, for any $r \in R$ we get an isomorphism $B_k^{rigid}(\Gamma_1(N), r; R) \to M_k(\Gamma_1(N); R)$ given by the map*

$$\sum_{i \geq 0} b_i \mapsto \sum_{i \geq 0} r^i \frac{b_i}{E_{p-1}^i}.$$

Having chosen a splitting as in Lemma 2.13 and for an element $f \in M_k(\Gamma_1(N), r; R)$, the previous theorem then tells us that we can write this as

$$f = \sum_{i \geq 0} r^i \frac{b_i}{E_{p-1}^i}$$

for $b_i \in B_i(\Gamma_1(N), k; R)$, and we refer to this as the Katz expansion of f. Of course this expansion depends on the chosen splitting, but once chosen, it is unique. These Katz expansions will play a central role in the other chapters of this thesis, as they are extremely amenable to computations. In anticipation of these computations in later chapters, we consider a slightly altered module of overconvergent modular forms. If we are given an $f \in M_k^\dagger(\Gamma_1(N); R_0)$, then (as it is overconvergent of *some* rate), we can write it as

$$f = \sum_{i \geq 0} \frac{b'_i}{E_{p-1}^i},$$

where the $b'_i \in B_i(\Gamma_1(N), k; R)$. We then define

$$\nu_p(b'_i) := \min_{j \geq 0}(\nu_p(a_j(b'_i)),$$

where $a_j(b'_i)$ is the jth coefficient in the q-expansion of b'_i. Then, $f \in M_k(\Gamma_1(N), r; R)$ is precisely the statement that $\nu_p(b'_i) \geq i\nu_p(r)$ and furthermore $\nu_p(b'_i) - i\nu_p(r) \to \infty$. While the first condition could in theory be checked for any finite number of terms, the second condition cannot be checked with a computer. For this reason, we make the following definition, which will be used a lot in the later chapters.

Definition 2.15. *For any $\rho \in \mathbb{Q}$, the module $M_k(\Gamma(N), R; \geq \rho)$ is the module consisting of the overconvergent modular forms f, of weight k, level $\Gamma(N)$ and of overconvergence rate r for all $r \in R$ such that $\nu_p(r) < \rho$.*

Indeed, an element $f \in M_k^\dagger(\Gamma_1(N); R_0)$ is an element of $M_k(\Gamma(N), R; \geq \rho)$ if and only if its Katz expansion is

$$f = \sum_{i \geq 0} \frac{b'_i}{E_{p-1}^i},$$

and $\nu_p(b'_i) \geq i\rho$ for all $i \geq 0$.

Remark 2.16. *As we have noted before, the theory of overconvergent modular forms exists for a much larger set of weights, and not just the integral ones. However, these Katz expansions only exist for integral weights. This will not be a big concern for us, as the other chapters are mainly concerned with a weight 0 overconvergent modular form, and thus the theory of Katz expansions can be applied.*

2.2.4 Hecke Operators

Just as in the case of classical modular forms, the space of overconvergent modular forms is equipped with Hecke operators. For l a prime, not dividing our fixed prime p or the level N, one can define the Hecke operators T_l in a very similar fashion as one would define them in the classical case. As it allows a very nice definition in terms of elliptic curves, we show how to do this. We thus fix such a prime l and fix an elliptic curve E/R, where R is a separated and p-adically complete $\mathbb{Z}_p$-algebra. If H is a subgroup of order l of E, then we get the isogeny $\check{\pi} : E \to E/H$, and its dual isogeny $\pi : E/H \to E$. Hence, if we have a level $\Gamma_1(N)$ structure on E, i.e. an inclusion $\alpha : \mu_N \to E$, then we also get a level $\Gamma_1(N)$ level structure, α', on E/H, simply by the composition $\pi \circ \alpha$ (as $l \nmid pN$). Then we define the lth Hecke operator as

$$T_l(f)(E, \omega, \alpha, Y) := l^{k-1} \sum_{\substack{H \hookrightarrow E \\ \#H = l}} f(E/H, (\check{\pi})^* \omega, \alpha', (\check{\pi})^* Y).$$

A computation with Tate curves allows us to compute the q-expansion of $T_l(f)$, which will be similar to the classical case. In a very similar fashion, one could extend this to the case $l|N$ (but $l \nmid p$), however we have to make sure that we only sum over the subgroups H of order l that are not contained in the given level $\Gamma_1(N)$ structure on E.
In the case that $l = p$, we have the so-called Atkin-Lehner U_p-operator (or simply, the U-operator). This operator does not have as easy of a description as the other Hecke operators in the overconvergent case, so we will refrain from giving it, but see [Kat73, Section 3.11]. Hence we will simply assume its existence and provide some of its properties. Firstly, on q-expansions the U_p-operator acts very simply. Namely if $f = \sum_{i \geq 0} a_i q^i$ is the q-expansion of an overconvergent modular form, then

$$U_p(f) = \sum_{i \geq 0} a_{pn} q^n.$$

A very important property of the U-operator is that it improves the overconvergence rate.

Theorem 2.17 (Lemma 3.11.4 in [Kat73])**.** *Let $r \in R$ such that $\nu_p(r) < 1/(p+1)$. Now let $f \in M_k(\Gamma_1(N), r; R)$. Then $pU_p(f) \in M_k(\Gamma_1(N), r^p; R)$.*

Another key property of the U_p operator is that it is *compact* on the space of overconvergent modular forms. In particular, it will have a discrete spectrum and thus it makes sense to study its eigenvalues. We make the following important definition.

Definition 2.18. *If f is an overconvergent modular form that is an eigenform (or a generalized eigenform) for the Hecke operators, including U_p, then the* slope *of f is the p-adic valuation of the U_p-eigenvalue of f.*

Extensive research has been conducted on the slopes, and there are many, still open, questions regarding them. We will delve a bit deeper into this in the following sections.

2.2.5 The Canonical Subgroup

We have now both defined classical modular forms and overconvergent modular forms. In Theorem 2.12 we have seen that overconvergent modular forms of overconvergence rate $r = 1$ coincide with the p-adic modular forms as defined by Serre (at least, for integral weights). It is now also interesting to know how, or if, classical modular forms can be embedded into the space of overconvergent modular forms. If we are given as base ring R_0 some ring of integers in a finite extension of $\mathbb{Q}_p$, where $p \geq 5$, then from the definition of classical modular forms and overconvergent modular forms, we can immediately see that any classical form $f \in M_k(\Gamma_1(N); R_0)$ gives rise to an overconvergent modular form $f' \in M_k(\Gamma_1(N), r; R_0)$ for any $r \in R_0$. Indeed, if $(E/R, \omega, \alpha_N, Y)$ is a quadruple as in the definition of overconvergent modular forms, then we can simply define $f'(E/R, \omega, \alpha_N, Y) := f(E/R, \omega, \alpha_N)$. However, one can do much better, as we will see in a bit. To explain this, we need the theory of the canonical subgroup.

As above, let R be the ring of integers inside some finite field extension of $K/\mathbb{Q}_p$ and let k denote its residue field. Consider an elliptic curve E/R which is ordinary. We then get a reduction map

$$E(\overline{K})[p] \to E(\overline{k})[p]$$

and the assumption on E to be ordinary, implies that the kernel will precisely give us a finite flat subgroup scheme of rank p. So, in particular, any ordinary elliptic curve E/R automatically comes equipped with a canonical choice for a level $\Gamma_0(p)$ level structure. If, however, we pick an elliptic curve that is supersingular, then the reduction map does not provide us with such a canonical choice. Surprisingly, it is nevertheless possible to furnish certain supersingular elliptic curves with a canonical $\Gamma_0(p)$ level structure, which is called the canonical subgroup. In particular, we have the following theorem by Lubin and Katz on the existence of a canonical subgroup, see [Kat73, Lub79].

Theorem 2.19 (Lubin-Katz)**.** *Let R be a $\mathbb{Z}_p$-algebra (and assume it is complete), and let E/R be an elliptic curve. Then E has a canonical subgroup of order p if and only if*

$$\nu_p(A(E, \omega)) < \frac{p}{p+1},$$

where $A(E, \omega)$ is a lift of the Hasse invariant.

As we know that the elliptic curve E/R is supersingular precisely when $\nu(A(E, \omega)) > 0$, we can interpret this as the statement that elliptic curves that are "not too supersingular" admit a canonical subgroup. The proof of this theorem as given by Katz requires the theory of formal group laws. Furthermore, Coleman has given an explicit description of the canonical subgroup of an elliptic curve which is not too supersingular, see [Col05]. Having stated the theorem of the canonical subgroup now allows us to state when classical modular forms can be embedded into the space of overconvergent modular forms.

Theorem 2.20 (Katz)**.** *Let $N \geq 3$ and $(p, N) = 1$. Consider the inclusion*

$$M_k(\Gamma_1(N) \cap \Gamma_0(p); R) \to M_k(\Gamma_1(N), 1; R).$$

If $r \in R$ such that $\nu_p(r) < \frac{p}{p+1}$, then the above map factors through $M_k(\Gamma_1(N), r; R)$.

Note that the proof is quite simple, assuming Theorem 2.19. Let $r \in R$ such that $\nu(r) < \frac{p}{p+1}$ and let $f \in M_k(\Gamma_1(N) \cap \Gamma_0(p); R)$. Then we define its image $\tilde{f} \in M_k(\Gamma_1(N), r; R)$ as

$$\tilde{f}(E/R', \omega, \alpha_N, Y) := f(E/R', \omega, \alpha_N, C),$$

where α_N is a level $\Gamma_1(N)$ structure on E and C is its canonical subgroup (which we know to exist by Theorem 2.19). One can also prove similar statements where we allow levels of the form $\Gamma_1(N) \cap \Gamma_0(p^s)$, for $s \geq 1$, but one has to take an $r \in R$ such that $\nu_p(r) < \frac{p}{p^{s-2}(p+1)}$, see [Gou88, Theorem 2.2.7].

Example 2.21. *We will provide an example to the above theorem, which will play a crucial role in the rest of this thesis. Let $k \geq 4$, then we can consider the Eisenstein series of weight k, denoted by E_k. We normalize it, such that its q-expansion is given by*

$$E_k = 1 - \frac{2k}{b_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where b_k is the kth Bernoulli number. This will be a classical modular form, of level 1 and weight k. We then define its p-stabilization, for a prime p, as

$$E_k^*(z) := E_k(z) - p^{k-1} E_k(pz),$$

which will have as q-expansion

$$E_k^* = 1 - (1 - p^{k-1}) \frac{2k}{b_k} \sum_{n \geq 1, p \nmid n} \sigma_{k-1}(n) q^n.$$

This will now be a modular form of weight k and level $\Gamma_0(p)$. The above theorem then says that E_k^* can also be considered as an element of $M_k(1, r; \mathbb{Z}_p)$ for any $r \in \mathbb{C}_p$ such that $\nu_p(r) < \frac{p}{p+1}$. In particular, it is overconvergent.

As we have seen, classical modular forms appear in the theory of overconvergent modular forms. One could ask, whether it is possible to easily determine whether an overconvergent modular form is classical. The following theorem of Coleman shows that eigenforms of low slope are classical [Col97a].

Theorem 2.22 (Coleman)**.** *Let k be an integer and f an overconvergent modular form of weight k, level $\Gamma_1(N)$. Assume that f is an eigenform with slope $\leq k - 2$, then f is a classical modular form.*

2.2.6 Weight Space

In Katz' theory of overconvergent modular forms as we gave above, the weight is simply given by an integer $k \in \mathbb{Z}$. However, we have already noted before that Serre's theory of p-adic modular forms actually provides a weight $\kappa \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$, which is already a larger set of weights. However, it is actually possible to define p-adic overconvergent modular forms on an even larger set of weights. More precisely, the weights will consist of the following elements.

Definition 2.23. *The weight space, denoted by $\mathcal{W}$, is the set consisting of all continuous $\mathbb{C}_p^\times$-valued characters of $\mathbb{Z}_p^\times$, i.e.*

$$\mathcal{W} := \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times).$$

From now on, when we pick a weight κ, we mean an element $\kappa \in \mathcal{W}$. The classical weights can be embedded into the weight space by sending $n \in \mathbb{Z}$ to the character $x \mapsto x^n$. In general, we will simply denote integral weights by n.

Remark 2.24. *The weight space can be given the structure of a rigid analytic space over $\mathbb{Q}_p$. While this is crucial in Coleman's theory of overconvergent modular forms and to the construction of the eigenvariety (a rigid analytic variety parametrising eigenforms), we will not appeal to rigid analysis throughout this thesis, and hence we will not go into further details, but see for example [CM98].*

Note that, assuming $p \geq 3$, we have the decomposition

$$\mathbb{Z}_p^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times (1 + p\mathbb{Z}_p)\,,$$

and we write $x = \bar{x}\langle x \rangle$ for an element $x \in \mathbb{Z}_p$ corresponding to the decomposition above. For the case that $p = 2$, we instead get the following decomposition

$$\mathbb{Z}_2^\times \cong (\mathbb{Z}/4\mathbb{Z})^\times \times (1 + 4\mathbb{Z}_p)\,.$$

To facilitate notation, we will simply assume that $p \geq 3$, but most of the theory does extend to the case $p = 2$, under some slight changes.

The previous isomorphism then gives rise to the following bijection

$$\mathcal{W} \to \widehat{(\mathbb{Z}/p\mathbb{Z})}^\times \times B(1, 1^-),$$

$$\kappa \mapsto (\kappa|_{(\mathbb{Z}/p\mathbb{Z})^\times}, \kappa(p+1) - 1)$$

where $B(1, 1^-)$ denotes the open unit disk in $\mathbb{C}_p$ with radius 1. The inverse is given by $(\chi, \lambda) \mapsto \chi(\bar{x})\lambda^{\frac{\log_p(x)}{p}}$, where $\log_p$ is the p-adic logarithm. Note that this implies that we can see the weight space as $\varphi(p)$ distinct open unit disks in $\mathbb{C}_p$, parametrised by $\widehat{(\mathbb{Z}/p\mathbb{Z})}^\times$. If we consider two integral weights $n, m \in Z$, then their corresponding weights in the weight space will lie in the same disk if and only if $n \equiv m \mod p - 1$.

2.2.7 The Eisenstein Family

We have just defined the weight space on which to consider the theory of overconvergent modular forms. Nevertheless, all definitions given in the previous sections only consider integral weights. In fact, it is not easy at all to give a geometric interpretation of non-integral weight modular forms. While this has been done nowadays, see [AIS14, Pil13], Coleman approached the construction of overconvergent modular forms in a different way. The main idea is that one can rather easily construct a family of Eisenstein series over an entire disk of weight space. That is, a function on the weight space, such that if we specialize it at an integral weight (lying in a specific disk), we obtain the classical integral weight(p-stabilized) Eisenstein series. Then, one can define the weight κ overconvergent modular forms, as those (rigid analytic) functions that, if divided by an Eisenstein series of appropriate weight, are an overconvergent modular form of integral weight.
We will define the Eisenstein family for tame level $N = 1$. For a more in depth discussion of the Eisenstein series, see [CM98, Section 2.2] Furthermore, we will assume $p \geq 5$. We will work with only one unit disk of the weight space, namely the subspace $\mathcal{B} \subset \mathcal{W}$ consisting of the characters that are trivial on $(\mathbb{Z}/p\mathbb{Z})^\times$. Then for a weight $\kappa \in \mathcal{B}\backslash\{1\}$ we consider the modular form $E^*_\kappa(q)$ with q-expansion given by

$$E^*_\kappa(q) = 1 + \frac{2}{\zeta^*(\kappa)} \sum_{n=1}^{\infty} \sigma'_n(\kappa) q^n,$$

if $\zeta^*(\kappa) \neq 0$, where ζ^* is the p-adic zeta function, and where

$$\sigma'_n(\kappa) := \sum_{\substack{d|n \\ p\nmid d}} \frac{\kappa(d)}{d}.$$

We will put $E^*_\kappa = 1$ for $\kappa = 1$. Coleman then simply *defines* that a powerseries

$$f(q) = \sum_{n=0}^{\infty} a_n q^n,$$

where the a_n lie in some extension K of $\mathbb{Q}_p$, is the q-expansion of an overconvergent modular form of tame level N and weight $\kappa \in \mathcal{B}$ if $f(q)/E^*_\kappa(q)$ is the q-expansion of an overconvergent modular form of tame level 0 (as defined by Katz). One could easily generalise this to include all weights $\kappa \in \mathcal{W}$, see [CM98, Section 2.4]. Note that this is perhaps not the most elegant definition; one would like to be able to define a sheaf for a given weight κ over the modular curve whose sections will give the space of overconvergent modular forms of weight κ. Indeed, Pilloni [Pil13] and Andreatta, Iovita, Stevens [AIS14] manage to do this. Nevertheless, the Eisenstein series combined with the theory of Katz is much more approachable from a computational perspective as we will see in the following chapters. In particular, as pointed out in the introduction, and also in Coleman's and Mazur's paper on the construction of the eigencurve [CM98, Section 2.2], the Eisenstein series is deeply connected with the U_p-operator acting on overconvergent modular forms of general weight. Hence, extensive knowledge of the Eisenstein series facilitates a good control on the geometry of the eigencurve. In the same paper they write about the Eisenstein series: " It is at the root of much of the theory."

3 A Conjecture of Coleman on the Eisenstein Family

This chapter is based on [AKW22] and is joint work with Ian Kiming and Gabor Wiese.

Abstract We prove for primes $p \geq 5$ a conjecture of Coleman on the analytic continuation of the family of modular functions $\frac{E^*_\kappa}{V(E^*_\kappa)}$ derived from the family of Eisenstein series E^*_κ.

The precise, quantitative formulation of the conjecture involved a certain constant depending on p. We show by an example that the conjecture with the constant that Coleman conjectured cannot hold in general for all primes. On the other hand, the constant that we give is also shown not to be optimal in all cases.

The conjecture is motivated by its connection to certain central statements in works by Buzzard and Kilford, and by Roe, concerning the "halo" conjecture for the primes 2 and 3, respectively. We show how our results generalize those statements and comment on possible future developments.

3.1 Introduction

In what follows, p will denote a fixed prime ≥ 5. We let v_p denote the p-adic valuation of $\mathbb{C}_p$ normalized so that $v_p(p) = 1$.

The conjecture of Coleman referred to in the title is Conjecture 1.1 of Coleman's paper [Col13]. Let us briefly recall the setup as in [Col13]: let $\mathcal{W}$ be the analytic group of continuous $\mathbb{C}_p$-valued characters on $\mathbb{Z}_p^\times$ with the subgroup $\mathcal{B}$ consisting of those characters that are trivial on the $(p-1)$st roots of unity. For $\kappa \in \mathcal{B}\backslash\{1\}$ we have the family E^*_κ of Eisenstein series with q-expansions

$$E^*_\kappa(q) = 1 + \frac{2}{\zeta^*(\kappa)} \sum_{n=1}^{\infty} \left(\sum_{\substack{d \mid n \\ p \nmid d}} \kappa(d) d^{-1} \right) \cdot q^n$$

with ζ^* the p-adic zeta function on $\mathcal{W}$.

Our convention in the present paper is that $\mathbb{N} = \mathbb{Z}_{\geq 1}$. If $k \in (p-1)\mathbb{N}$ then k defines an element $\kappa \in \mathcal{B}\backslash\{1\}$ by $x \mapsto x^k$. For such k we shall abuse notation and identify the corresponding κ with k. The specialization to $\kappa = k$ of the Eisenstein family gives us the classical Eisenstein series E_k^* with q-expansion

$$E_k^*(q) = 1 + \frac{2}{(1-p^{k-1})\zeta(1-k)} \cdot \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n \\ p\nmid d}} d^{k-1} \right) q^n$$

(with ζ the Riemann zeta function) that is an Eisenstein series of weight k on $\Gamma_0(p)$.

Furthermore, we denote by E_k the standard, normalized Eisenstein series of level 1 and even integer weight $k \geq 4$.

Recall that we have a function w on $\mathcal{W}$ defined by $w(\kappa) := \kappa(1+p) - 1$. Thus, for $k \in (p-1)\mathbb{N}$ we have $w(k) = (1+p)^k - 1$.

The setting for Coleman's conjecture is as follows. Suppose that $\kappa \in \mathcal{B}\backslash\{1\}$. Let V be the p-adic Frobenius operator, acting on q-expansions as $q \mapsto q^p$. Coleman had already shown that the p-adic modular function $\frac{E_\kappa^*}{V(E_\kappa^*)}$ is defined on the ordinary locus of $X := X_1(p)$ and defines an overconvergent function, cf. p. 2946 of [Col13], or the reference on that page, or, alternatively, [Col97b, Corollary 2.1.1]. (This is a function that can also be considered when $p = 2, 3$.) Conjecture 1.1 of [Col13] is a precise prediction of how far into the supersingular region this function converges, i.e., what is its rate of overconvergence. In this formulation, the conjecture also represents a conjectural answer to a question posed in Coleman and Mazur's foundational paper [CM98] on the eigencurve, – see the remarks at the end of p. 43 of [CM98].

The following theorem proves a version of the conjecture. When we say "version", what we primarily mean is that the constant c_p appearing in the theorem is not precisely the constant that Coleman was expecting in his conjecture (for primes $p \geq 5$.) We shall comment further upon that below, but would like here to note that we do not believe that Coleman's conjecture is true with the exact value of the constant that he gave (that would correspond to being able to take $c_p = 1$ in our theorem.) We shall discuss this in detail below, and especially in section 3.4.

To formulate our main theorem, we find the following notation convenient: $f \in M_0(O, \geq \rho)$ means that f is an overconvergent function of tame level 1, defined over O that is r-overconvergent whenever $v_p(r) < \rho$. See below in section 3.2 for a few additional details on this notation.

Theorem A. *There is a constant* $0 < c_p < 1$ *such that the following holds. Let* $\kappa \in \mathcal{B}\backslash\{1\}$ *be a character and let* O *be the ring of integers in the extension of* $\mathbb{Q}_p$ *generated by the values of* κ.

Then

$$\frac{E_\kappa^*}{V(E_\kappa^*)} \in M_0(O, \geq c_p \cdot \min\{1, v_p(w(\kappa))\}).$$

Explicitly, we can take

$$c_p = \frac{2}{3} \cdot \left(1 - \frac{p}{(p-1)^2}\right) \cdot \frac{1}{p+1}.$$

We see the background and motivation for Coleman's conjecture as follows. A conjecture about the behavior of U near the boundary of weight space, the "halo" conjecture, [WXZ17, Conjecture 2.5], [LWX17, Conjecture 1.2], [BP16, Conjecture 1.2], seems to be attributed to Coleman, but has also developed from the main result of the paper [BK05] by Buzzard and Kilford that can now be seen as establishing that conjecture for the prime $p = 2$. Subsequently, Roe established the conjecture in [Roe14] for $p = 3$ by similar methods. What Coleman did in [Col13] was first to reinterpret certain central, indeed decisive, results from [BK05], [Roe14], specifically [BK05, Theorem 7], [Roe14, Theorem 4.2], as a precise statement, for $p = 2, 3$, about rates of overconvergence of the functions $\frac{E_\kappa^*}{V(E_\kappa^*)}$. Coleman then went on and conjectured in [Col13, Conjecture 1.1]) a similar and precise statement for all primes $p \geq 5$.

It will be seen that Theorem A is not asserting precisely the same as what [Col13, Conjecture 1.1] expects for primes $p \geq 5$. First, [Col13, Conjecture 1.1] is formulated from a rigid analytic perspective. Though this is unimportant as far as the substance of the statement is concerned, we shall comment briefly on it at the beginning of section 3.4.1 below. Secondly, and more importantly, the precise value that we give for the constant c_p (the reader should note that Coleman's c_p denotes something else than our c_p) is not precisely what [Col13, Conjecture 1.1] would expect for primes $p \geq 5$: though formulated in rigid analytic terms, we can reinterpret [Col13, Conjecture 1.1] as expecting the statement of Theorem A, but with the value $c_p = 1$ for the constant in the theorem. Below in section 3.4.1 we will show by means of a numerical example that we cannot take $c_p = 1$ in Theorem A. Thus, the precise formulation of [Col13, Conjecture 1.1] for primes $p \geq 5$ appears to us to have been a too optimistic extrapolation from the cases $p = 2, 3$.

On the other hand, we also do not claim optimality of the constant c_p in our Theorem A, at least not for all primes. Thus, in section 3.4.2, using certain ad hoc arguments, we will show that c_p can be improved a little bit for the cases $p = 5, 7$.

We will derive Theorem A from Theorem B below that may be of some interest in itself. It gives a "formal Katz expansion" and a lower bound for the valuation of its coefficients. Since in the theorem as well as in the proof we will be talking about Katz expansions ([Kat73, Section 2.6] or [Von21, Section 4.1]) of overconvergent p-adic modular functions, we briefly remind the reader of these: as Katz first showed, there is for each $i \in \mathbb{N}$ a direct sum decomposition

$$M_{i(p-1)}(\mathbb{Z}_p) = E_{p-1} \cdot M_{(i-1)(p-1)}(\mathbb{Z}_p) \oplus B_i(\mathbb{Z}_p).$$

of $\mathbb{Z}_p$-modules where M_k denotes modular forms of weight k on $\mathrm{SL}_2(\mathbb{Z})$ (for the proof of this, one can refer to Katz' original work, [Kat73, Proposition 2.8.1], but a simple, elementary proof is also possible by using "Victor Miller" bases in level 1, see e.g. [KR21, Section 5].)

The splitting is not unique, but we will fix a specific choice for the B_i in section 3.3 below. Katz expansions of the modular functions we will be working with then take the form $\sum_{i=0}^{\infty} \frac{b_i}{E_{p-1}^i}$ with $b_i \in B_i(\mathbb{Z}_p)$ (we put $B_0(\mathbb{Z}_p) = \mathbb{Z}_p$.)

Theorem B. *(a) There are modular forms $b_{ij} \in B_i(\mathbb{Z}_p)$ for each $i, j \in \mathbb{Z}_{\geq 0}$ such that the following holds. If $\kappa \in \mathcal{B}\backslash\{1\}$ then the Katz expansion of the modular function $\frac{E_\kappa^*}{V(E_\kappa^*)}$ is*

$$\frac{E_\kappa^*}{V(E_\kappa^*)} = \sum_{i=0}^{\infty} \frac{\beta_i(w(\kappa))}{E_{p-1}^i}$$

where

$$\beta_i(w(\kappa)) := \sum_{j=0}^{\infty} b_{ij} w(\kappa)^j$$

for each i.
(b) There is a constant c_p with $0 < c_p < 1$ such that for the modular forms b_{ij} in part (a) we have

$$v_p(b_{ij}) \geq c_p i - j$$

for all i, j.

In fact, we can take the explicit constant c_p from Theorem A.

The plan of the paper is as follows. After setting up notation and various preliminaries in the next section, in section 3.3 we first prove part (a) of Theorem B. We derive that part conveniently as an application of the existence of "Victor Miller" bases for modular forms in level 1.

The idea of proof of the more difficult part (b) of Theorem B is to utilize the fact that the paper [KR21] gives us information about rates of overconvergence of p-adic modular functions of form $\frac{E_k^*}{V(E_k^*)}$ with $k \in (p-1)\mathbb{N}$. The observation that these infinitely many "data points" imply the divisibility properties of part (b) is the technical core of the paper, and it depends on the combinatorial/linear algebra Proposition 3.3. Given that proposition, the proofs of part (b) of Theorem B and after that of Theorem A proceed along straightforward lines.

Finally, in section 3.4 we comment on Coleman's original conjecture as compared with our Theorem A as well as on the question of optimality of the constant c_p. We also show that our results can be used to generalize certain statements from the papers [BK05, Roe14] pertaining to the study of the U operator in weights κ with $0 < v_p(w(\kappa)) < 1$.

We need the condition $p \geq 5$ primarily for the usual reasons such as that E_{p-1} is a modular form, but occasionally in more general discussion and remarks the condition can be relaxed. We will indicate when that is the case.

We close the paper with some remarks about the context of this work. Our paper follows Coleman's original approach ([Col97c]) to the existence of what we now refer to as Coleman families of modular forms, which builds on the family $\frac{E_k^*}{V(E_k^*)}$ of p-adic modular functions. Whereas this approach has now been superseded by a more intrinsic, geometric definition of the notion of an overconvergent modular form of arbitrary weight, cf. the works of Pilloni [Pil13], and Andreatta, Iovita, Stevens [AIS14], we feel that Coleman's original way is still very valuable, in particular, due to its explicit nature, which we exploit and explore in this article. Especially, if one wants to do explicit, computational work, for instance with Coleman families, of which there are very few, explicit examples, the approach using Eisenstein series (see [CST98] and [Des17] for examples with small primes) might still have merit and in fact might at this point in time be the only option. In [Adv24], the first-named author applies the methods of this paper in order to describe an algorithm for computing the valuations of the $b_{i,j}$ appearing in Theorem B and used the algorithm to predict a larger constant c'_p such that $v_p(b_{ij}) \geq c'_p i - j$ for all i, j.

Finally, we would like to mention that the paper [Ye20] is concerned, as are we, with problems of extending modular forms further into the supersingular locus. At this point, though, neither do we see immediate implications of that paper for the problems we are addressing here, nor vice versa.

Acknowledgements

This work was supported by the Luxembourg National Research Fund PRIDE17/1224660/GPS.

3.2 Notation and preliminaries

Consider a finite extension K of $\mathbb{Q}_p$ with ring of integers O. By $M_k(O)$ we shall denote the O-module of weight k modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with coefficients in O.

For $r \in O$ we can talk about r-overconvergent modular forms of tame level 1. We will only be dealing with weight 0 forms, i.e., modular functions, that are holomorphic at ∞. The r-overconvergent of these with coefficients in O form an O-module that we will denote by $M_0(O, r)$.

Most of our arguments will proceed via consideration of "the" Katz expansion of such forms: for each $i \in \mathbb{N}$, there is a (non-unique) direct sum decomposition

$$M_{i(p-1)}(\mathbb{Z}_p) = E_{p-1} \cdot M_{(i-1)(p-1)}(\mathbb{Z}_p) \oplus B_i(\mathbb{Z}_p).$$

In section 3.3 we will make a specific, fixed choice of these splittings that is convenient both theoretically and computationally. For now it suffices to say that an element $f \in M_0(O, r)$ has a "Katz expansion"

$$f = \sum_{i=0}^{\infty} \frac{b_i}{E_{p-1}^i}$$

where $b_i \in B_i(O)$ satisfy $v_p(b_i) \geq i \cdot v_p(r)$ for all i, as well as $v_p(b_i) - i \cdot v_p(r) \to \infty$ for $i \to \infty$. This expansion is unique once the splittings above have been fixed. One should note that these expansions are not necessarily exactly the ones that Katz introduced in [Kat73] (the reason being that he used the geometric language and had to contend with the usual issues when the level is 1.) However, all we will be concerned with are growth properties of the valuations of the b_i and these are independent of which splitting we use. But see the more general discussion in [KR21, Section 2], for instance. We should note here that the modules B_i obviously depend on p though out of convenience we will suppress that information from our notation.

From [KR21] we shall also borrow the following notation. For a rational $\rho \in [0,1]$ let $M_0(O, \geq \rho)$ denote the O-module of forms f such that $f \in M_0(O,r)$ for some r, and such that for the coefficients b_i of the Katz expansion of f we have $v_p(b_i) \geq \rho \cdot i$ for all i.

Elementary considerations ([KR21, Proposition 2.3]) show that an element $f \in M_0(O,r)$ is in $M_0(O, \geq \rho)$ if and only if $f \in M_k(O', r')$ whenever K'/K is a finite extension with ring of integers O' and $r' \in O'$ satisfies $v_p(r') < \rho$. Again, this is the case if and only if we have $f \in M_0(O,r)$ for some r as well as $f \in M_k(O', r')$ for a sequence of finite extensions K'/K with rings of integers O' and elements r' such that $v_p(r')$ converges to ρ from below.

3.3 Proof of the main theorems

3.3.1 Existence of the "formal Katz expansion"

Proof of Theorem B, part (a). We start the proof by repeating the observation made in section 5 of [BK05] that with $w = w(\kappa)$ we have a formal power series expansion

$$\frac{E_\kappa^*}{V(E_\kappa^*)} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{nj} w^j \right) \cdot q^n \in \mathbb{Z}_p[[w,q]]$$

in the sense that if we specialize w on the right hand side to $w = w(\kappa)$ for a character $\kappa \in \mathcal{B} \backslash \{1\}$, we obtain the q-expansion of the function $\frac{E_\kappa^*}{V(E_\kappa^*)}$. (The argument at the beginning of [BK05, Section 5] is for $p = 2$, but carries over to a general prime p.)

For Katz expansions at tame level 1 it is both theoretically and computationally convenient to use the idea of Lauder [Lau11] of exploiting the existence of "Miller bases" for modular forms of level 1: Put $d_{s(p-1)} := \dim M_{s(p-1)}(\mathbb{Q}_p)$. There are splittings

$$M_{i(p-1)}(\mathbb{Z}_p) = E_{p-1} \cdot M_{(i-1)(p-1)}(\mathbb{Z}_p) \oplus B_i(\mathbb{Z}_p).$$

of $\mathbb{Z}_p$-modules where the free $\mathbb{Z}_p$-module $B_i(\mathbb{Z}_p)$ has a basis

$$\{g_{i,j} \mid d_{(i-1)(p-1)} \leq j \leq d_{i(p-1)} - 1\}$$

with the property that the q-expansion of $g_{i,j}$ starts with q^j (for $i = 0$ the definition is $g_{0,0} := 1$.) Cf. for instance [KR21, Section 5] for explicit formulas for the $g_{i,j}$. This means that the (infinite) matrix that has the coefficients of the q-expansions

$$g_{0,0}(q), \ldots, g_{i,d_{(i-1)(p-1)}}(q), \ldots, g_{i,d_{i(p-1)}-1}(q), \ldots,$$

as rows will be upper triangular with 1s in the diagonal.

Since $E_{p-1}(q) \in 1 + pq\mathbb{Z}_p[[q]]$ we also have, formally, $E_{p-1}^{-i}(q) \in 1 + pq\mathbb{Z}_p[[q]]$ for each $i \geq 0$. Thus, the matrix with rows the coefficients of the (formal) q-expansions

$$g_{0,0}(q) \cdot 1, \ldots, g_{i,d_{(i-1)(p-1)}}(q)E_{p-1}^{-i}(q), \ldots, g_{i,d_{i(p-1)}-1}(q)E_{p-1}^{-i}(q), \ldots,$$

is again upper triangular with 1s in the diagonal. It follows from these considerations that we have an isomorphism $\phi : \prod_{i\geq 0} B_i(\mathbb{Z}_p) \cong \mathbb{Z}_p[[q]]$ of $\mathbb{Z}_p$-modules given by

$$\phi((b_i)_{i\geq 0}) := \sum_{i\geq 0} b_i(q)E_{p-1}^{-i}(q).$$

In particular, for each j we have a sequence of unique elements $b_{ij} \in B_i(\mathbb{Z}_p)$, $i \geq 0$, such that

$$\sum_{n=0}^{\infty} a_{nj}q^n = \sum_{i=0}^{\infty} \frac{b_{ij}(q)}{E_{p-1}^i(q)}.$$

Define then

$$H(w,q) := \sum_{i=0}^{\infty} \frac{\sum_{j=0}^{\infty} b_{ij}(q)w^j}{E_{p-1}^i(q)}$$

as a formal power series in w and q with coefficients in $\mathbb{Z}_p$.

Consider now a character $\kappa \in \mathcal{B}\backslash\{1\}$. Let O be the ring of integers of an extension of $\mathbb{Q}_p$ large enough to contain the values of κ. Let $\mathfrak{p}$ be the maximal ideal of O and let us consider the specialization $H(w(\kappa), q)$ modulo $\mathfrak{p}^m$ for a fixed $m \in \mathbb{N}$. As $v_p(w(\kappa)) > 0$ there is $j(m) \in \mathbb{N}$ such that $w(\kappa)^j \equiv 0 \pmod{\mathfrak{p}^m}$ for $j > j(m)$. We then find in $O/\mathfrak{p}^m$:

$$\begin{aligned} H(w(\kappa), q) &= \sum_{i=0}^{\infty} \frac{\sum_{j=0}^{\infty} b_{ij}(q)w(\kappa)^j}{E_{p-1}^i(q)} \equiv \sum_{i=0}^{\infty} \frac{\sum_{j=0}^{j(m)} b_{ij}(q)w(\kappa)^j}{E_{p-1}^i(q)} \\ &= \sum_{j=0}^{j(m)} \left(\sum_{i=0}^{\infty} \frac{b_{ij}(q)}{E_{p-1}^i(q)} \right) w(\kappa)^j = \sum_{j=0}^{j(m)} \left(\sum_{n=0}^{\infty} a_{nj}q^n \right) w(\kappa)^j \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{j(m)} a_{nj}w(\kappa)^j \right) q^n \equiv \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{nj}w(\kappa)^j \right) q^n = \frac{E_\kappa^*}{V(E_\kappa^*)}(q). \end{aligned}$$

As this congruence holds for all $m \in \mathbb{N}$ we conclude that

$$\frac{E_\kappa^*}{V(E_\kappa^*)}(q) = H(w(\kappa), q) = \sum_{i=0}^{\infty} \frac{\sum_{j=0}^{\infty} b_{ij}(q)w(\kappa)^j}{E_{p-1}^i(q)}$$

in $O[[q]]$.

Now, as we remarked above the function $\frac{E_\kappa^*}{V(E_\kappa^*)}$ is an overconvergent modular function with a Katz expansion

$$\frac{E_\kappa^*}{V(E_\kappa^*)} = \sum_{i=0}^{\infty} \frac{\beta_i(\kappa)}{E_{p-1}^i}$$

where $\beta_i(\kappa) \in B_i(O)$ for all i. Then $\sum_{i=0}^{\infty} \frac{\beta_i(\kappa)(q)}{E_{p-1}^i(q)} = \sum_{i=0}^{\infty} \frac{\sum_{j=0}^{\infty} b_{ij}(q)w(\kappa)^j}{E_{p-1}^i(q)}$ in $O[[q]]$ and so $\beta_i(k)(q) = \sum_{j=0}^{\infty} b_{ij}(q)w(\kappa)^j$ for all i by the injectivity of the isomorphism ϕ above. Then $\beta_i(\kappa) = \sum_{j=0}^{\infty} b_{ij}w(\kappa)^j$ for all i by the q-expansion principle. □

As explained in the introduction above, the non-trivial part of Theorem B is part (b) that will be obtained by using information from [KR21], specifically information about the overconvergence of modular functions $\frac{E_k^*}{V(E_k^*)}$ for classical weights $k \in (p-1)\mathbb{N}$: if we combine information about the rate of overconvergence of these modular functions, cf. [KR21, Theorem A], with part (a) of Theorem B, we obtain a statement about the growth w.r.t. i of the valuations of infinite sums

$$\sum_{j=0}^{\infty} b_{ij}w^j$$

with w corresponding to such classical weights. The combinatorial and linear algebra observations of the next subsection will show that this suffices to make a statement about the valuations of the modular forms b_{ij} themselves.

3.3.2 Valuations of the inverse Vandermonde matrix

In this section, p is any prime number. Let $n \in \mathbb{N}$ and $x_0, \ldots, x_{n-1} \in \mathbb{C}_p$ be pairwise distinct. Consider the Vandermonde matrix

$$V = V(x_0, \ldots, x_{n-1}) = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{pmatrix}.$$

The following lemma appears to be well-known (see e.g. [MS58] or [Par64]), but we provide the short proof.

Lemma 3.1. *Let $0 \le i, j \le n-1$. Then the coefficient at position $(i+1, j+1)$ of the matrix $V(x_0, \ldots, x_{n-1})^{-1}$ equals*

$$(-1)^{n-1-i} \cdot \frac{s_{n-1-i}(x_0, \ldots, \hat{x}_j, \ldots, x_{n-1})}{\prod_{0 \le \ell \le n-1, \ell \ne j}(x_j - x_\ell)},$$

where $s_d(\ldots)$ is the elementary symmetric polynomial of degree $0 \le d \le n-1$ in $n-1$ variables (the hat in $\hat{x}_j$ means that the variable x_j is omitted).

Proof. We start from the formula defining the elementary symmetric polynomials in $n-1$ variables $\prod_{\ell=1}^{n-1}(T - t_\ell) = \sum_{i=0}^{n-1}(-1)^{n-1-i} \cdot s_{n-1-i}(t_1, \ldots, t_{n-1}) \cdot T^i$ and replace $(t_1, \ldots, t_{n-1})$ by $(x_0, \ldots, \hat{x}_j, \ldots, x_{n-1})$ and T by x_k for $0 \le j, k \le n-1$, leading to $\prod_{0 \le \ell \le n-1, \ell \ne j}(x_k - x_\ell) = \sum_{i=0}^{n-1}(-1)^{n-1-i} \cdot s_{n-1-i}(x_0, \ldots, \hat{x}_j, \ldots, x_{n-1}) \cdot x_k^i$, implying the claim. □

We need to study the valuations of denominators occurring in the inverse Vandermonde matrix. In the next proposition we prove a bit more than we will actually need for the proof of Theorem B. We do this in order to show that the estimates that we get from the proposition are in fact optimal.

Proposition 3.2. *Let $S \subseteq \mathbb{Z}_p^\times$ be a finite subset and put $n := |S|$. For $x \in S$ put*

$$v(S,x) := v_p\left(\prod_{s\in S, s\neq x}(x-s)\right) = \sum_{s\in S, s\neq x} v_p(x-s).$$

Then

$$\max_{x\in S} v(S,x) \geq \sum_{i=1}^{\infty}\left\lfloor \frac{n-1}{(p-1)p^{i-1}}\right\rfloor =: f(n).$$

Furthermore, for each $n \in \mathbb{N}$ there exists $S \subseteq \mathbb{Z}_p^\times$ with $|S| = n$ such that $\max_{x\in S} v(S,x) = f(n)$. For instance, one has equality if S consists of the first n natural numbers prime to p.

Proof. For the beginning of the argument we allow S more generally to be any non-empty finite subset of $\mathbb{Z}_p$ of cardinality n.

Let $r(S)$ be the maximal $r \in \mathbb{Z}_{\geq 0}$ such that all elements in S are congruent to each other modulo p^r. We write $S' := \{s \operatorname{div} p^{r(S)} \mid s \in S\}$ where $s \operatorname{div} p^r$ denotes the number $\sum_{i\geq r} a_i p^{i-r}$ if $s = \sum_{i\geq 0} a_i p^i$ is the standard p-adic expansion of s, i.e., with the a_i in $\{0,\ldots,p-1\}$. Thus, if s is an ordinary integer, $s \operatorname{div} p^r$ is the quotient of division with remainder of s by p^r. We observe that $|S| = |S'|$.

For any $d \in \mathbb{Z}/p\mathbb{Z}$, let $S_d = \{s \operatorname{div} p \mid s \in S, s \equiv d \pmod p\}$.

By the definition of S' and for any $d \in \mathbb{Z}/p\mathbb{Z}$, we have

$$\begin{aligned}\max_{x\in S} v(S,x) &= r(S)\cdot(|S|-1) + \max_{x\in S'} v(S',x)\\ &\geq r(S)\cdot(|S|-1) + \max_{x\in S',\ x\equiv d \pmod p} v(S',x)\\ &= r(S)\cdot(|S|-1) + (|(S')_d|-1) + \max_{x\in (S')_d} v((S')_d,x)\end{aligned}$$

because only those $s \in S'$ contribute to $\sum_{s\in S', s\neq x} v_p(x-s)$ that are congruent to x modulo p.

Now, for cardinality reasons there must exist $d \in \mathbb{Z}/p\mathbb{Z}$ such that $|(S')_d| \geq \left\lceil \frac{|S'|}{p}\right\rceil = \left\lceil \frac{|S|}{p}\right\rceil$. Applying this we obtain

$$\max_{x\in S} v(S,x) \geq r(S)\cdot(|S|-1) + \left\lceil \frac{|S|}{p}\right\rceil - 1 + \max_{x\in (S')_d} v((S')_d,x)$$

from which we can see the inequality

$$\max_{x\in S} v(S,x) \geq r(S)\cdot(|S|-1) + \sum_{i=1}^{\infty}\left(\left\lceil \frac{|S|}{p^i}\right\rceil - 1\right) \tag{3.1}$$

by induction on $|S|$: if $|S| = 1$ the statement is trivial. If $|S| > 1$ then for the induction step we use that $|(S')_d| < |S'| = |S|$ for any d, apply the induction hypothesis to $\max_{x\in(S')_d} v((S')_d, x)$, drop the term $r((S')_d)\cdot(|(S')_d| - 1)$, and use the inequality $|(S')_d| \geq \left\lceil \frac{|S|}{p} \right\rceil$.

The inequality (3.1) obviously implies the inequality

$$\max_{x\in S} v(S,x) \geq \sum_{i=1}^{\infty}\left(\left\lceil \frac{|S|}{p^i}\right\rceil - 1\right) = \sum_{i=1}^{\infty}\left\lfloor \frac{|S|-1}{p^i}\right\rfloor. \tag{3.2}$$

Let us now assume the setup of the proposition, i.e., that $S \subseteq \mathbb{Z}_p^\times$. Assume first that $r(S) = 0$ so that $S' = S$. In that case, we can improve (3.1) slightly because there now exists $d \in (\mathbb{Z}/p\mathbb{Z})^\times$ such that $|(S')_d| = |S_d| \geq \left\lceil \frac{|S|}{p-1}\right\rceil$. Then as above we have

$$\max_{x\in S} v(S,x) \geq (|S_d| - 1) + \max_{x\in S_d} v(S_d, x) \geq \left\lceil \frac{|S|}{p-1}\right\rceil - 1 + \max_{x\in S_d} v(S_d,x).$$

We now apply (3.2) to the right most term and obtain

$$\max_{x\in S} v(S,x) \geq \sum_{i=1}^{\infty}\left(\left\lceil \frac{|S|}{(p-1)p^{i-1}}\right\rceil - 1\right) = \sum_{i=1}^{\infty}\left\lfloor \frac{|S|-1}{(p-1)p^{i-1}}\right\rfloor = f(n). \tag{3.3}$$

Next we claim that this formula also holds when $r(S) \geq 1$. Indeed, applying again (3.1), we have

$$\begin{aligned}
\max_{x\in S} v(S,x) &\geq r(S)\cdot(|S|-1) + \sum_{i=1}^{\infty}\left(\left\lceil \frac{|S|}{p^i}\right\rceil - 1\right)\\
&= r(S)(|S|-1) + \sum_{i=1}^{\infty}\left(\left\lfloor \frac{|S|-1}{p^i}\right\rfloor\right) \geq \sum_{i=0}^{\infty}\left(\left\lfloor \frac{|S|-1}{p^i}\right\rfloor\right)\\
&\geq \sum_{i=0}^{\infty}\left(\left\lfloor \frac{1}{p-1}\cdot\frac{(|S|-1)}{p^i}\right\rfloor\right) = \sum_{i=1}^{\infty}\left\lfloor \frac{|S|-1}{(p-1)p^{i-1}}\right\rfloor.
\end{aligned}$$

Moreover, the above analysis shows that (3.3) is an equality if there is a sequence $d_0, d_1, \ldots \in \mathbb{Z}/p\mathbb{Z}$ (only finitely many terms matter) such that the recursively defined sets $S^{(0)} := S$ and $S^{(i+1)} = (S^{(i)})_{d_i}$ for $i \geq 0$ satisfy $r(S^{(i)}) = 0$ for all $i \geq 0$, as well as $|S^{(1)}| = \left\lceil \frac{|S|}{p-1}\right\rceil$ and $|S^{(i+1)}| = \left\lceil \frac{|S^{(i)}|}{p}\right\rceil$ for all $i \geq 1$.

If S consists of the first n natural numbers prime to p we take $d_0 = 1$ and $d_i = 0$ for $i \geq 1$ and then these conditions are actually satisfied: writing $n = (p-1)q + r$ with $q \geq 0$, $0 \leq r < p-1$ one verifies that S_1 consists of the first $\lceil \frac{n}{p-1}\rceil = q + \lceil \frac{r}{p-1}\rceil$ consecutive integers starting from 0. One also sees that if $\mathbb{N} \ni m \geq 0$ and $\Sigma = \{0, \ldots, m-1\}$ then Σ_0 consists of the first $\lceil \frac{m}{p}\rceil$ consecutive integers, starting from 0. □

Proposition 3.3. *Let $n \in \mathbb{N}$ and set $f(n) := \sum_{i=1}^{\infty} \left\lfloor \frac{n-1}{(p-1)p^{i-1}} \right\rfloor$. Let $x_0, \dots, x_{n-1} \in \mathbb{Z}_p^\times$ be any set of units such that*

$$\max_{0\le j\le n-1} v_p\left(\prod_{0\le \ell\le n-1, \ell\ne j} (x_j - x_\ell)\right) = f(n) \le (n-1)\cdot \frac{p}{(p-1)^2},$$

the existence of which is assured by Proposition 3.2.

(a) Let $V = V(x_0, \dots, x_{n-1})$ be the Vandermonde matrix. Then the p-valuation of all coefficients of V^{-1} is at least $-f(n) \ge (1-n)\cdot \frac{p}{(p-1)^2}$.

(b) If $m \in \mathbb{R}$ and if $b_0, \dots, b_{n-1} \in \mathbb{C}_p$ satisfy

$$v_p(b_0 + b_1 x_i + \cdots b_{n-1} x_i^{n-1}) \ge m$$

for all $0 \le i \le n-1$, then

$$v_p(b_j) \ge m - f(n) \ge m - (n-1)\cdot \frac{p}{(p-1)^2}$$

for all $0 \le j \le n-1$. In particular, if $m = n$ we will have

$$v_p(b_j) \ge \left(1 - \frac{p}{(p-1)^2}\right)\cdot n$$

for each j.

Proof. For the first inequality, observe that $f(n) \le \frac{n-1}{p-1}\cdot \sum_{i=0}^{\infty} \frac{1}{p^i} = (n-1)\cdot \frac{p}{(p-1)^2}$. Part (a) is a direct consequence of Lemma 3.1 and the choice of $x_0, \dots, x_{n-1} \in \mathbb{Z}_p^\times$. Part (b) follows by considering $b_0 + b_1 x_i + \cdots b_{n-1} x_i^{n-1}$ as V times the vector of the b_j. □

Remark 3.4. *As we will see immediately below, the main ingredient from this section in the proof of Theorem B is Proposition 3.3. As we also see, the essential statement of Proposition 3.3 is that we can choose units $x_0, \dots, x_{n-1} \in \mathbb{Z}_p^\times$ in such a way that we have a good upper bound for $\max_{0\le j\le n-1} v_p\left(\prod_{0\le \ell\le n-1, \ell\ne j}(x_j - x_\ell)\right)$. A choice of the units x_i is provided by the second part of Proposition 3.2. The purpose of the first part of Proposition 3.2 is to show that this upper bound is optimal.*

3.3.3 Proof of Theorem B, part (b)

Considering the modular forms $b_{ij} \in B_i(\mathbb{Z}_p)$ from part (a), we will show that $v_p(b_{ij}) \ge c_p \cdot i - j$ for all i, j where $c_p := \frac{2}{3}\cdot \left(1 - \frac{p}{(p-1)^2}\right)\cdot \frac{1}{p+1}$.

Fix $i_0 \in \mathbb{Z}_{\ge 0}$ and let us show that $v_p(b_{i_0 j} p^j) \ge c_p \cdot i_0$ for all $j \ge 0$. As $b_{i_0 j}$ has coefficients in $\mathbb{Z}_p$ we certainly have $v_p(b_{i_0 j} p^j) \ge j$, and so the claim is clear if $j \ge n$ with

$$n := \left\lceil \frac{2}{3}\cdot \frac{1}{p+1}\cdot i_0 \right\rceil$$

as we will then have $v_p(b_{i_0 j} p^j) \geq n > c_p \cdot i_0$. Thus, we must show

$$v_p(b_{i_0 j} p^j) \geq c_p \cdot i_0$$

for $j = 0, \ldots, n-1$.

Consider classical weights $k \in \mathbb{N}$ divisible by $p-1$. For such weights the corresponding point $w(k)$ in weight space is $w(k) := (1+p)^k - 1$. By part (a) of Theorem B, the i_0th coefficient in the Katz expansion of the p-adic modular function $\frac{E_k^*}{V(E_k^*)}$ is

$$\sum_{j=0}^{\infty} b_{i_0 j} w(k)^j.$$

The crucial ingredient in the proof is now the observation that we know from [KR21, Theorem A] that

$$v_p(\sum_{j=0}^{\infty} b_{i_0 j} w(k)^j) \geq \frac{2}{3} \cdot \frac{1}{p+1} \cdot i_0$$

(more precisely: [KR21, Remark 4.2] combined with the proof of [KR21, Theorem A] shows that we have $\frac{E_k^*}{V(E_k^*)} \in M_0(\mathbb{Z}_p, \geq \frac{2}{3} \cdot \frac{1}{p+1})$.)

Now, recalling again that $b_{i_0 j}$ has coefficients in $\mathbb{Z}_p$ and combining this with the fact that $w(k) \in p\mathbb{Z}$ for classical weights $k \equiv 0 \pmod{p-1}$ as above, we find from the definition of $n := \lceil \frac{2}{3} \cdot \frac{1}{p+1} \cdot i_0 \rceil$ that

$$v_p(\sum_{j=0}^{n-1} b_{i_0 j} w(k)^j) \geq \left\lceil \frac{2}{3} \cdot \frac{1}{p+1} \cdot i_0 \right\rceil = n$$

for every such classical weight k.

We write the sum on the left hand side as $\sum_{j=0}^{n-1} (b_{i_0 j} p^j) \cdot \left(\frac{w(k)}{p}\right)^j$ and notice that elementary considerations show that the numbers

$$\frac{w(k)}{p} = \frac{(1+p)^k - 1}{p}$$

are dense in $\mathbb{Z}_p$ when k ranges over the classical weights $\equiv 0 \pmod{p-1}$. Then we see that Proposition 3.3 can be applied to deduce that $v_p(b_{i_0 j} p^j) \geq \left(1 - \frac{p}{(p-1)^2}\right) \cdot n$ for $j = 0, \ldots, n-1$. Indeed, a lower bound $v_p(b_{i_0 j} p^j) \geq m$ is equivalent to having the same lower bound for the valuations of all Fourier coefficients of $b_{i_0 j} p^j$. But then we have

$$v_p(b_{i_0 j} p^j) \geq \left(1 - \frac{p}{(p-1)^2}\right) \cdot n \geq \left(1 - \frac{p}{(p-1)^2}\right) \cdot \frac{2}{3} \cdot \frac{1}{p+1} \cdot i_0 = c_p \cdot i_0$$

for $j = 0, \ldots, n-1$, and we are done.

3.3.4 Proof of Theorem A

Put $w_0 := w(\kappa)$. By part (a) of Theorem B we have a Katz expansion

$$\frac{E_\kappa^*}{V(E_\kappa^*)} = \sum_{i=0}^{\infty} \frac{\beta_i(w_0)}{E_{p-1}^i}.$$

with $\beta_i(w_0) := \sum_{j=0}^{\infty} b_{ij} w_0^j$. Referring back to the remarks of section 3.2, all we have to show is that we have $v_p(\sum_{j=0}^{\infty} b_{ij} w_0^j) \geq c_p \cdot \min\{1, v_p(w_0)\} \cdot i$ for all i. To see this, fix an i, write $\rho := c_p i$, and split the sum as

$$\sum_{j=0}^{\infty} b_{ij} w_0^j = \sum_{0 \leq j \leq \rho} b_{ij} w_0^j + \sum_{j > \rho} b_{ij} w_0^j.$$

For the terms in the first sum note that part (b) of Theorem B implies that their valuations are bounded from below by

$$c_p i - j + j v_p(w_0) = c_p i - j(1 - v_p(w_0)).$$

If now $v_p(w_0) \leq 1$ this is at least $c_p i - \rho(1 - v_p(w_0)) = \rho v_p(w_0) = c_p v_p(w_0) \cdot i$, and if $v_p(w_0) \geq 1$, this is certainly at least $c_p \cdot i$.

On the other hand, as $b_{ij} \in B_i(\mathbb{Z}_p)$ for all j, the terms in the second sum have valuations bounded from below by $j v_p(w_0) > \rho v_p(w_0) = c_p v_p(w_0) \cdot i$. We are done.

3.4 Further remarks and results

3.4.1 The original conjecture of Coleman

Coleman's conjecture [Col13, Conjecture 1.1] is formulated in rigid analytic terms as a conjecture concerning analytic continuation of $\frac{E_\kappa^*}{V(E_\kappa^*)}$ considered as a function of two variables $(P, \kappa) \in X_1(p) \times \mathcal{B}$, $\kappa \neq 1$. As a consequence of Coleman's earlier results on the nonvanishing of $\frac{E_\kappa^*}{V(E_\kappa^*)}$ on Z (cf. the remarks at the bottom of p. 2946 of [Col13]), this function is initially defined for P in the ordinary locus Z where $E_{p-1}(P)$ is a unit. Given our Theorem B, the value of the function at such a point is the value of the converging infinite sum

$$\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} b_{ij}(P) w^j \right) E_{p-1}(P)^{-i}.$$

where we have written $w := w(\kappa)$. The question is how far into the supersingular region this function extends when $v_p(w(\kappa)) < 1$. Let us give the core argument showing that the function extends under the condition $\frac{1}{c_p} v_p(E_{p-1}(P)) < v_p(w) < 1$ (for primes $p \geq 5$, [Col13, Conjecture 1.1] would say this, but with $c_p = 1$.) We give the argument assuming that P corresponds to an elliptic curve defined over the ring O_0 of integers in a finite extension of $\mathbb{Q}_p$. Let us choose an extension K of $\mathbb{Q}_p$ large enough to contain O_0 and w as well as an element α with $v_p(\alpha) = c_p$. Let O denote the ring of integers of K. We can then see that the above series converges to an element of O: Rewriting the series as

$$\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} b_{ij}(P) w^{j-c_p i} \right) \left(E_{p-1}(P)^{-1} w^{c_p} \right)^i,$$

since $v_p(E_{p-1}(P)^{-1} w^{c_p}) > 0$, we see that it suffices to show that $\sum_{j=0}^{\infty} b_{ij}(P) w^{j-c_p i}$ for any fixed $i \geq 0$ converges to an element of O. To do so, fix an $i \geq 0$ and split up this sum as

$$\sum_{0 \leq j \leq c_p i} b_{ij}(P) w^{j-c_p i} + \sum_{j > c_p i} b_{ij}(P) w^{j-c_p i}.$$

In the second sum we have $w^{j-c_p i} \in O$ for each term, and since $v_p(w) > 0$, the sum converges.

The first sum is finite, and so convergence is not an issue, but we still need to see that the sum gives an element of O. But if for $j \leq c_p i$ we define the modular form $\tilde{b}_{ij}$ to be $\tilde{b}_{ij} := \alpha^{-i} p^j b_{ij}$ then Theorem B (and the q-expansion principle) implies that $\tilde{b}_{ij}$ is a modular form defined over O so that the value $\tilde{b}_{ij}(P)$ is in O. Now,

$$b_{ij}(P) w^{j-c_p i} = \tilde{b}_{ij}(P) \cdot \alpha^i p^{-j} w^{j-c_p i},$$

and since

$$v_p(\alpha^i p^{-j} w^{j-c_p i}) = c_p i - j + (j - c_p i) v_p(w) \geq 0$$

as $j \leq c_p i$ and $v_p(w) < 1$, we are done.

We now show by a numerical example one cannot take $c_p = 1$ in Theorem A: let $p = 5$ and let χ be the Dirichlet character of conductor 5^2 given by $\chi(7) = 1$, $\chi(6) = \zeta$ with ζ a primitive 5th root of unit. Then χ can be viewed as a character on $\mathbb{Z}_5^\times$ and as such is trivial on the 4th roots of unity. Let κ be the character on $\mathbb{Z}_5^\times$ given by $\kappa(x) = x^4 \chi(x)$. Then E_κ^* is a classical Eisenstein series of weight 4 on $\Gamma_1(5^2)$ with nebentypus χ. We have $v_5(w(\kappa)) = \frac{1}{4}$, and so, if we could take $c_5 = 1$ in Theorem A we would be able to conclude (via Theorem A) that $E_\kappa^*/V(E_\kappa^*) \in M_0(O, \geq \frac{1}{4})$ with O the ring of integers of $\mathbb{Q}_5(\zeta)$. But a computation shows this not to be the case: recall that for $p = 2, 3, 5, 7, 13$ where $X_0(p)$ has genus 0, the function

$$f_p(z) := \left(\frac{\eta(pz)}{\eta(z)} \right)^{\frac{24}{p-1}}$$

with η the Dedekind eta-function is a *Hauptmodul* for $\Gamma_0(p)$, *i.e.*, a generator of the function field of $X_0(p)$. D. Loeffler has shown, cf. [Loe07, Corollary 2.2], that if c is a constant with

$v_p(c) = \frac{12}{p-1} v_p(r)$ then the powers of cf_p give an orthonormal basis for the r-overconvergent modular functions of tame level 1. Hence, if we consider the expansion

$$\frac{E_\kappa^*}{V(E_\kappa^*)} = \sum_{i=0}^{\infty} a_i f_5^i,$$

then the statement that $E_\kappa^*/V(E_\kappa^*) \in M_0(O, \geq \frac{1}{4})$ together with [Loe07, Corollary 2.2] implies $v_5(a_i) \geq \frac{3}{4} \cdot i$ for all i. But, the expansion is easy to compute from q-expansions as the q-expansion of f_5 starts with q, and one finds that $v_5(a_{10}) = 1$.

It appears to us that the precise, quantitative form of [Col13, Conjecture 1.1] for primes $p \geq 5$ resulted from an optimistic extrapolation from the cases $p = 2, 3$. Coleman proved [Col13, Conjecture 1.1] for $p = 2, 3$ as a consequence of [BK05, Theorem 7] and [Roe14, Theorem 4.2], respectively, theorems that are quite central in those papers. The primes 2 and 3 differ from primes $p \geq 5$ for all the usual reasons, but in this specific setting there are additional differences: as an inspection of the proofs of [BK05, Theorem 7] and [Roe14, Theorem 4.2] shows, the fact that the U operator at tame level 1 and for these primes enjoys particularly strong integrality properties plays a significant role in the proofs. Those stronger integrality properties fail for primes $p \geq 5$, which is also one reason why the arguments in these papers do not generalize for primes $p \geq 5$ in any straightforward manner, as far as we can see. The stronger integrality properties of U for $p = 2, 3$ can ultimately be seen to derive from the fact that the exponent $\frac{24}{p-1}$ occurring in the definition of the *Hauptmodul* f_p above is divisible by p precisely when $p \in \{2, 3\}$.

One further observation on the difference between the cases $p = 2, 3$ and $p \geq 5$ is as follows. If one considers the shape of the statements of [BK05, Theorem 7] and [Roe14, Theorem 4.2], a naive generalization to primes $p \geq 5$ would be a statement of form $v_p(b_{ij}) \geq d_p(i - j)$ in part (b) of Theorem B, with some constant d_p depending on p. Extensive numerical calculation of the $v_p(b_{ij})$, the details of which will be reported on elsewhere, strongly suggests that such a statement does not hold, but that the correct lower bound for primes $p \geq 5$ is in fact a statement of the form in part (b) of Theorem B. Again we see this difference between the cases $p = 2, 3$ and $p \geq 5$ as being connected with the above stronger integrality properties of U.

3.4.2 The constant c_p

We will now discuss the specific constant c_p that appears in Theorems A and B. In particular, we will show that it is not optimal, at least not for all primes. We show this by improving the constant in the cases $p = 5, 7$ by certain ad hoc arguments, specifically:

Proposition 3.5. *For $p = 5, 7$ we can take $c_p = \left(1 - \frac{p}{(p-1)^2}\right) \cdot \frac{p-1}{p(p+1)} = \frac{p^2-3p+1}{p(p^2-1)}$ in Theorems A and B.*

Notice first from the proof of part (b) of Theorem B that the constant appears as the product of two factors: $c_p = a_p \cdot b_p$ where $a_p := 1 - \frac{p}{(p-1)^2}$ is the constant appearing in Proposition 3.3 whereas $b_p := \frac{2}{3} \cdot \frac{1}{p+1}$ comes from results of [KR21] that imply $\frac{E_k^*}{V(E_k^*)} \in M_0(\mathbb{Z}_p, \geq b_p)$ for classical weights $k \in \mathbb{N}$ divisible by $p-1$.

Here, the constant a_p does not seem to admit any essential improvement, cf. Remark 3.4. On the other hand, the constant b_p in the above is not optimal, at least not for all primes. Let us briefly recall the origin of the constant b_p in [KR21]: the statement that we have $\frac{E_k^*}{V(E_k^*)} \in M_0(\mathbb{Z}_p, \geq b_p)$ for classical weights k divisible by $p-1$ derives from the more precise statement that $\frac{V(E_k^*)}{E_k^*} \in \frac{1}{p} M_0(\mathbb{Z}_p, \geq \frac{1}{p+1})$ ([KR21, Theorem A]); judging from numerical experiments, this latter statement actually does appear close to optimal. As arguments in [KR21] show, the statement that we have $\frac{E_k^*}{V(E_k^*)} \in M_0(\mathbb{Z}_p, \geq b_p)$ with the above value of b_p is obtained as a consequence of the more precise statement coupled with the congruence $E_{n(p-1)} \equiv E_{p-1}^n \pmod{p^2}$ (for primes $p \geq 5$, $n \in \mathbb{N}$.)

For the primes $p = 5, 7$ we can improve the constant b_p as follows.

Proposition 3.6. *If $p \in \{5, 7\}$ and $k \in \mathbb{N}$ is divisible by $p-1$ then*

$$\frac{E_k^*}{V(E_k^*)} \in M_0\left(\mathbb{Z}_p, \geq \frac{p-1}{p(p+1)}\right).$$

The proof of Proposition 3.5 now consists of repeating the proof of part (b) of Theorem B by using Proposition 3.6 as input.

The proof of Proposition 3.6 runs along the same general lines of reasoning as were employed in [KR21], see for instance the proof of [KR21, Theorem B].

The essential point is a consideration of the rate of overconvergence of the p-adic modular functions $e_n^* := \frac{E_{n(p-1)}^*}{E_{p-1}^n}$ for $n \in \mathbb{N}$. For these we have the following that we will also formulate for the functions $e_n := \frac{E_{n(p-1)}}{E_{p-1}^n}$ as the proof is the same. By a 1-unit in a ring $M_0(O, r)$ we mean an element of form $1 + af$ where $f \in M_0(O, r)$ and $a \in O$ is a constant with $v_p(a) > 0$. A 1-unit is thus invertible in the ring $M_0(O, r)$.

Proposition 3.7. *Let $p \in \{5, 7\}$. For $n \in \mathbb{N}$ we have*

$$e_n, e_n^* \in M_0\left(\mathbb{Z}_p, \geq \frac{p-1}{p+1}\right).$$

As a consequence, e_n, e_n^* are 1-units in $M_0(O, r)$ whenever O is the ring of integers of any sufficiently large, finite extension $K/\mathbb{Q}_p$, and $r \in O$ satisfies $v_p(r) < \frac{p-1}{p+1}$.

Proof. The argument is the same for e_n and e_n^*, so let us just consider e_n^*. For the first statement, considering the Katz expansion

$$e_n^* = 1 + \sum_{i=1}^{\infty} \frac{b_i}{E_{p-1}^i}$$

of e_n^* where $b_i \in B_i(\mathbb{Z}_p)$ and the $B_i(\mathbb{Z}_p)$ as above, we must show that

$$v_p(b_i) \geq \frac{p-1}{p+1} \cdot i$$

for all i.

Since $e_n^* \in \frac{1}{p} \cdot M_0(\mathbb{Z}_p, \geq \frac{p}{p+1})$ by [KR21, Theorem C] we have $v_p(b_i) \geq -1 + \frac{p}{p+1} \cdot i$ for all i. Thus, the desired inequality is seen to hold for $i \geq p+1$ as we then have $\frac{p-1}{p+1} \cdot i \leq -1 + \frac{p}{p+1} \cdot i$.

Secondly, by [KR21, Lemma 3.11] we have the congruence $e_n^* \equiv 1 \pmod{p^2}$ of q-expansions, and by [KR21, Proposition 2.5] this implies $v_p(b_i) \geq 2$ for all i. This again implies the desired when $p = 5$ and $i = 1, 2, 3$, and when $p = 7$ and $i = 1, 2$.

To deal with the remaining cases, as we noted above in section 3.3.1, for any $p \geq 5$ the rank of the $\mathbb{Z}_p$-module $B_i(\mathbb{Z}_p)$ equals $d_{i(p-1)} - d_{(i-1)(p-1)}$ where d_k denotes the dimension of the space of modular forms of weight k on $\mathrm{SL}_2(\mathbb{Z})$.

Consider then $p = 5$. We then have $b_4 = b_5 = 0$ because $d_{12} = d_{16} = d_{20} = 2$ so that $B_4 = B_5 = 0$. Thus, the desired inequality also holds for $i = 4, 5$ and hence for all i.

Consider then $p = 7$. In this case we have $B_3 = B_5 = B_7 = 0$ because $d_{12} = d_{18} = 2$, $d_{24} = d_{30} = 3$, and $d_{36} = d_{42} = 4$. Hence $b_3 = b_5 = b_7 = 0$, and we only need to verify the inequality for $i = 4, 6$. But we have $v_7(b_4) \geq \frac{7}{8} \cdot 4 - 1 = \frac{5}{2}$, and since $b_4 \in \mathbb{Z}_7$ this implies $v_7(b_4) \geq 3 = \frac{6}{8} \cdot 4$. Similarly, $v_7(b_6) \geq \frac{7}{8} \cdot 6 - 1 = \frac{17}{4}$ whence $v_7(b_6) \geq 5 > \frac{6}{8} \cdot 6$. Thus, the desired inequality holds also for $i = 4, 6$ and so for all i.

Suppose now that $K/\mathbb{Q}_p$ is a finite extension, that O is the ring of integers of K, and that $r \in O$ has $v_p(r) < \frac{p-1}{p+1}$. Assume that K is large enough that there exists $a \in O$ with $0 < v_p(a) \leq \frac{1}{2} \cdot (\frac{p-1}{p+1} - v_p(r))$. Defining $b_i' := a^{-1} b_i$ for $i \geq 1$ with the b_i from the Katz expansion of e_n^* above, we then find

$$v_p(b_i') - i v_p(r) \geq (i - \frac{1}{2}) \cdot (\frac{p-1}{p+1} - v_p(r))$$

which shows that $v_p(b_i') - i v_p(r) \geq 0$ for $i \geq 1$ as well as $v_p(b_i') - i v_p(r) \to \infty$ for $i \to \infty$. But then

$$f := \sum_{i=1}^{\infty} \frac{b_i'}{E_{p-1}^i}$$

defines an element of $M_0(O, r)$, and as $e_n^* = 1 + a \cdot f$ with $v_p(a) > 0$ we see that e_n^* is a 1-unit in $M_0(O, r)$. □

Proof of Proposition 3.6. Let p be 5 or 7, let $k \in \mathbb{N}$ be divisible by $p - 1$, and put $n := k/(p-1)$.

Suppose that $K/\mathbb{Q}_p$ is a finite extension, that O is the ring of integers of K, and that $r \in O$ is such that $v_p(r^p) < \frac{p-1}{p+1}$. Suppose further that K is large enough so that the second part of Proposition 3.7 applies, i.e., so that e_n^* is a 1-unit in $M_0(O, r^p)$. As the Frobenius operator maps $M_0(O, r^p)$ to $M_0(O, r)$, we can conclude that

$$V(e_n^*) = \frac{V(E_k^*)}{V(E_{p-1})^n}$$

is a 1-unit in $M_0(O,r)$. Now, as $v_p(r) < \frac{1}{p+1}$, the "Coleman–Wan theorem", [Wan98, Lemma 2.1], tells us that the function $\frac{E_{p-1}}{V(E_{p-1})}$ is a 1-unit in $M_0(O,r)$. In particular, we have then that

$$\frac{E_k^*}{V(E_k^*)} = \frac{e_n^*}{V(e_n^*)} \cdot \left(\frac{E_{p-1}}{V(E_{p-1})}\right)^n \in M_0(O,r).$$

As we can choose a sequence of extensions $K/\mathbb{Q}_p$ such that the valuations $v_p(r)$ of the elements r converge to $\frac{p-1}{p(p+1)}$ from below, the proposition follows from the remarks at the end of section 3.2. □

Numerical experimentation suggests that Proposition 3.7 continues to hold for some primes $p > 7$, perhaps for all, though we do not have an explanation at this point.

3.4.3 The action of U in weight κ

The family $\frac{E_\kappa^*}{V(E_\kappa^*)}$ of functions occurs prominently in Coleman's seminal work [Col97c] as a tool that enables one to relay the study of the U operator in general weights back to weight 0. For this to work, some information about the analytical properties of the family is necessary. In the papers [BK05] and [Roe14] concerning the primes 2 and 3, respectively, very detailed information about the family was obtained and used to prove the "halo" conjecture in those cases. We will show here that our results permit us to generalize a certain aspect of the analysis of these papers. It would be possible to formulate this more generally for arbitrary primes $p \geq 5$, but for simplicity we will restrict ourselves to "genus zero primes", i.e., where $X_0(p)$ has genus zero.

These primes are $p = 2, 3, 5, 7, 13$. For these primes, instead of the formal Katz expansion of $\frac{E_\kappa^*}{V(E_\kappa^*)}$ of Theorem B one can consider a formal expansion

$$\frac{E_\kappa^*}{V(E_\kappa^*)} = \sum_{i,j\geq 0}^{\infty} a_{ij} w^j t^i$$

where $w = w(\kappa)$, $\kappa \in \mathcal{B}\backslash\{1\}$, and where for t we can take $t = f_p = \left(\frac{\eta(pz)}{\eta(z)}\right)^{\frac{24}{p-1}}$ the standard *Hauptmodul*, or, alternatively, for $p = 2, 3$ we can follow the papers [BK05, Roe14] and take for t a certain uniformizer of $X_0(4)$ (when $p = 2$) or $X_0(9)$ (when $p = 3$.) In all cases, we will have the coefficients a_{ij} in $\mathbb{Z}_p$ and the expansion has the advantage of being easy to compute for a given κ because the q-expansion of t will begin with q.

This formal expansion is a central object of study of the papers [BK05, Roe14] because it gives us information about the action of the U operator on weight κ overconvergent modular forms: for $0 \leq r < \frac{p}{p+1}$, by choosing $c \in O_{\mathbb{C}_p}$ with a specific absolute value, depending on r, one has $V(E_\kappa^*)(ct)^i$, $i = 0, 1, 2, \ldots$ as an orthonormal basis for the Banach space of r-overconvergent modular forms of weight κ. If we choose $t = f_p$, then according to [Loe07, Corollary 2.2] we should choose c with $v_p(c) = \frac{12r}{p-1}$; for the other choices of t, see for instance the discussion on pp. 614–615 of [BK05]. The action of U on this basis can be described via the above expansion of $\frac{E_\kappa^*}{V(E_\kappa^*)}$: if we write $U(ct)^i = \sum_j m_{ij}(ct)^j$ (which is of course independent of κ) then the (infinite) matrix giving the action of U on the basis element $V(E_\kappa^*)(ct)^i$ is given by the product

$$\frac{E_\kappa^*}{V(E_\kappa^*)} \cdot \sum_j m_{ij} \cdot V(E_\kappa^*)(ct)^j$$

as is seen by applying the identity $U(V(F)G) = FU(G)$ ("Coleman's trick".)

A crucial part of the papers [BK05] ($p = 2$) and [Roe14] ($p = 3$) consists in showing that when the factor $\frac{E_\kappa^*}{V(E_\kappa^*)}$ of the above matrix is properly "rescaled" as a function of t then modulo the maximal ideal of $O_{\mathbb{C}_p}$ it becomes independent of κ when $v_p(w(\kappa))$ is in a certain interval. Let us explain this in detail. Suppose that we have established a lower bound of the form

$$v_p(a_{ij}) \geq \alpha i - \beta j$$

with certain positive constants α and β. Write $w = w(\kappa)$ and define the power series $g_\kappa(x)$ such that

$$\frac{E_\kappa^*}{V(E_\kappa^*)} = g_\kappa(w^\gamma t)$$

where

$$\gamma := \frac{\alpha}{\beta}.$$

We can then see that the coefficients of g_κ are integral and the reduction $\bar{g}_\kappa$ of g_κ modulo the maximal ideal of $O_{\mathbb{C}_p}$ is independent of κ when $0 < v_p(w(\kappa)) < \beta$: writing $g_\kappa(x) = \sum_{n=0}^{\infty} c_n t^n$ we have

$$c_n = \sum_j a_{nj} w^{j-\gamma n}.$$

Assume then $0 < v_p(w) < \beta$. We can then see that each term $a_{nj}w^{j-\gamma n}$ is integral and in the maximal ideal when $j \neq \gamma n$: for $j \geq \gamma n$ this is clear as the a_{nj} are integral. Suppose then that $j < \gamma n$. Then, using $\gamma\beta = \alpha$, we have

$$v_p(a_{nj}w^{j-n\gamma}) \geq \alpha n - \beta j + (j - n\gamma)v_p(w) = (\gamma n - j)(\beta - v_p(w)) > 0.$$

In the paper [BK05] where $p = 2$ the above lower bound for the valuations of the a_{ij} was proved with $\alpha = \beta = 3$, cf. [BK05, Theorem 7] (note that their a_{ij} would be our a_{ji}.) Thus $\gamma = 1$, and they were able to conclude that $\bar{g}_\kappa$ is independent of κ when $w = w(\kappa)$ satisfies $0 < v_p(w) < 3$ as well as $\bar{c}_n = \bar{a}_{n,n}$. Similarly, for $p = 3$ the paper [Roe14] established a lower bound with $\alpha = \beta = 1$ with analogous conclusions for $\bar{g}_\kappa$.

For the primes $p = 5, 7, 13$ we choose $t = f_p$ in the above, and then arguments completely similar to those that proved part (b) of Theorem B (working with expansions in t rather than formal Katz expansions) will show that one has

$$v_p(a_{ij}) \geq d_p i - j$$

for all i, j where

$$d_p = \frac{12}{p-1} \cdot c_p$$

with c_p from Theorem B, or alternatively for $p = 5, 7$ from Proposition 3.5. Here, the factor $\frac{12}{p-1}$ once again comes from [Loe07, Corollary 2.2]. We can then conclude that we have $\frac{E_\kappa^*}{V(E_\kappa^*)} = g_\kappa(w^{d_p} t)$ for a power series g_κ with integral coefficients whose reduction $\bar{g}_\kappa$ is independent of κ when $0 < v_p(w(\kappa)) < 1$.

This statement is of course quite uninteresting unless the constant d_p is optimal as otherwise the reduction $\bar{g}_\kappa$ will just be the constant 1. However, numerical calculations, at this point mostly for $p = 5$, strongly suggests the possibility of identifying the optimal constant and perhaps even the non-trivial reduction $\bar{g}_\kappa$. This will be reported on in detail elsewhere.

Computations on Overconvergence Rates Related to the Eisenstein Family]Computations on Overconvergence Rates Related to the Eisenstein Family

4 Computations on Overconvergence Rates Related to the Eisenstein Family

This chapter has appeared as [Adv24].

Abstract We provide for primes $p \geq 5$ a method to compute valuations appearing in the "formal" Katz expansion of the family $\frac{E_\kappa^*}{V(E_\kappa^*)}$ derived from the family of Eisenstein series E_κ^*. We will describe two algorithms: the first one to compute the Katz expansion of an overconvergent modular form and the second one, which uses the first algorithm, to compute valuations appearing in the "formal" Katz expansion. Based on data obtained using these algorithms we make a precise conjecture about a constant appearing in the overconvergence rates related to the classical Eisenstein series at level p. The study of these overconvergence rates of the members of this family go back to a conjecture of Coleman.

4.1 Introduction

In the last couple of years many developments have been made in the field of overconvergent modular forms. This paper adds to the computational aspect of overconvergent modular forms. In particular, in this paper we will provide two algorithms to obtain computational data regarding the overconvergence rates of $\frac{E_\kappa^*}{V(E_\kappa^*)}$, the ($p$-stabilized) Eisenstein series of weight κ divided by its image under the Frobenius operator (see below for the precise definitions). These modular functions play a crucial role in the theory, as they allow one to jump between two different weights. Previously, theoretical computations have been done for the overconvergence rates of $\frac{E_\kappa^*}{V(E_\kappa^*)}$ for the primes $p = 2, 3$ where the modular curve $X_0(p)$ has genus 0, and these results have been applied to the description of slopes of overconvergent modular forms near the boundary of weight space (see [BC05] and [Roe14]). In [AKW22, Theorem B] the notion of a 'formal Katz expansion' is introduced, which works for all primes $p \geq 5$, independent of the genus of $X_0(p)$. The novelty of this paper is that it provides an algorithm to compute valuations appearing in the formal Katz expansion related to $\frac{E_\kappa^*}{V(E_\kappa^*)}$. These valuations then give information regarding the overconvergent rates of these modular functions. One reason for being interested in these rates is, for example, that they appear when looking at the U-operator in nonzero weight. Based on data obtained from this algorithm, we formulate a conjecture. Recently, a geometric definition of an overconvergent modular form has been given by the works of Pilloni [Pil13] and Andreatta, Iovita, Stevens [AIS14] and this has provided a lot of progress in the field. Nevertheless, the original definitions developed by Katz and Coleman still lend themselves extremely well to computational approaches.

To state our conjecture precisely, we will start by introducing the necessary terminology. Throughout, p will denote a prime ≥ 5, and ν_p (or simply ν if there is no confusion about the prime) will denote the p-adic valuation of $\mathbb{C}_p$, normalized such that $\nu_p(p) = 1$. We let $\mathcal{W}$, called the weight space, be the group $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$, i.e. the continuous characters of $\mathbb{Z}_p^\times$ with values in $\mathbb{C}_p$. We denote by $\mathcal{B}$ the subspace of these characters which are trivial when restricted to the $(p-1)$st roots of unity. If we denote by $\mathcal{D}$ the open disk of radius 1 around the origin in $\mathbb{C}_p$ then we can identify $\mathcal{W}$ with $\mathcal{D}$ by sending an element $\kappa \in \mathcal{W}$ to the element $w_\kappa := \kappa(p+1) - 1 \in \mathcal{D}$. A positive integer k corresponds to the weight given by the character $x \mapsto x^k$, and it is precisely a weight in $\mathcal{B}$ if k is divisible by $p-1$. We will denote classical weights just by an integer k instead of by their corresponding character.

Then, for a weight $\kappa \in \mathcal{B}\backslash\{1\}$ we have a family interpolating the classical Eisenstein series whose q-expansions are given by

$$E_\kappa^*(q) = 1 + \frac{2}{\zeta^*(\kappa)} \sum_{n=1}^{\infty} \left(\sum_{\substack{d \mid n \\ p \nmid d}} \kappa(d) d^{-1} \right) \cdot q^n,$$

where ζ^* is the p-adic zeta function on $\mathcal{W}$. It has been observed (see for example [BK05, Section 5]) that there exists a power series, $E/VE \in \mathbb{Z}_p[[q, w]]$ such that if we are given a weight $\kappa \in \mathcal{B}\backslash\{1\}$, then we have that $(E/VE)(w_\kappa) = E_\kappa^*/V(E_\kappa^*)$, where V is the p-adic Frobenius operator, acting on q-expansions as $q \mapsto q^p$.

Our goal will then be to deduce information about the overconvergence rate of $E_\kappa^*/V(E_\kappa^*)$. To describe the overcovergence rate we use the notion of a Katz expansion. We will give a short description of it, for more details see Section 4.2. Katz showed that for each $i \in \mathbb{N}_{\geq 0}$ there is a splitting

$$M_{i(p-1)}(\mathbb{Z}_p) = E_{p-1} M_{(i-1)(p-1)}(\mathbb{Z}_p) \oplus B_i(\mathbb{Z}_p), \tag{4.1}$$

where E_{p-1} is the Eisenstein series of weight $p-1$ and level 1, normalized such that its constant coefficient is 1 (see [Kat73, Lemma 2.6.1]). Such a splitting is not unique, but once it has been chosen, Katz has shown that an overconvergent modular form of weight 0 can be written uniquely as $f = \sum_{i=0}^{\infty} \frac{b_i}{E_{p-1}^i}$, where $b_i \in B_i(\mathbb{Z}_p)$, which is called its Katz expansion, and the values $\nu_p(b_i) := \inf_n(\nu_p(a_n(b_i)))$ can be used to measure the overconvergence rate of f. From now on, we will fix a splitting described by Lauder (see [Lau11, Section 3] and [Ste07, Lemma 2.20]), which is particularly easy to compute with. For the explicit description of this see Section 4.2. It is shown in [AKW22] that there exist modular forms $b_{i,j} \in \mathbb{Z}_p[[q]]$ for all $i, j \geq 0$ such that if we define $\beta_i(w) := \sum_{j=0}^{\infty} b_{ij} w^j$ and if $\kappa \in \mathcal{B}\backslash\{1\}$, then the Katz expansion of $E^*_\kappa / V(E^*_\kappa)$ is given by $\sum_{i=0}^{\infty} \frac{\beta_i(w(\kappa))}{E^i_{p-1}}$. In [AKW22, Section 3.3], it is proven that $\nu_p(b_{ij}) \geq c_p i - j$, where the c_p is an explicit constant depending only on p. This can be used to give explicit overconvergence rates for $E^*_\kappa / V(E^*_\kappa)$ for weights $\kappa \in \mathcal{B}\backslash\{1\}$. This c_p, however, does not seem to be optimal, in the following sense. Denote by δ_p the following quantity

$$\delta_p := \inf\left\{ \frac{\nu_p(b_{ij}) + j}{i} \middle| i \in \mathbb{Z}_{>0}, j \in \mathbb{Z}_{\geq 0} \right\}.$$

So in particular, it is known by [AKW22, Theorem B], that $\delta_p \geq c_p$. The main purpose of this paper will then be to provide an algorithm to compute the values of $\nu_p(b_{ij})$ and to compute approximations of the constant δ_p for different primes p. In particular, we conjecture the following.

Conjecture 1. *Let the $b_{i,j}$ be as above, then we have that $\nu(b_{i,j}) \geq d_p i - j$, for all $i, j \geq 0$, where*

$$d_p = \frac{p-1}{p(p+1)}.$$

Hence we conjecture that $\delta_p \geq \frac{p-1}{p(p+1)}$. Note that we do not conjecture that equality holds, but computations for low primes do give explicit values i and j for which we find $\frac{\nu_p(b_{ij})+j}{i} = d_p$, and hence in this case our conjecture would imply $d_p = \delta_p$. Assuming the conjecture, we have the following corollary.

Corollary 4.1. *Assume that Conjecture 1 holds. Let $\kappa \in \mathcal{B}\backslash\{1\}$ be a character and let $\mathcal{O}$ be the ring of integers in the extension of $\mathbb{Q}_p$ generated by the values of κ.*

Then

$$\frac{E^*_\kappa}{V(E^*_\kappa)} \in M_0(\mathcal{O}, \geq d_p \cdot \min\{1, v_p(w(\kappa))\}).$$

See Section 4.2 for the precise definition of $M_0(\mathcal{O}, \geq d_p \cdot \min\{1, v_p(w(\kappa))\})$. The proof that the conjecture implies Corollary 4.1 can be found in [AKW22, Proof 3.4]. The motivation for considering the value δ_p is because of a conjecture Coleman made regarding the overconvergence rate of $E/V(E)$. His conjecture seems to be too optimistic and in [AKW22, Section 4.1.2] a counterexample is given, and a slightly different overconvergence rate from Coleman is proven. However, this overconvergence rate seems to be not 'optimal', in the sense that the constant c_p is strictly smaller than the conjectured value δ_p. For $p = 2, 3$, information about the precise overconvergence rates of E/VE is used to obtain information about the geometry of the eigencurve near the boundary. In particular, it is shown that, close enough to the boundary, the eigencurve is a countable disjoint union of annuli. This provides information regarding the slopes close enough to the boundary, see [BC05] and [Roe14].

The precise value of d_p in Conjecture 1 is based on data obtained using a method, which we will describe in this paper. We will start by giving the necessary theoretical background regarding (formal) Katz expansions. After this, we will provide two algorithms. The first algorithm, Algorithm 1, will take as input a prime $p \geq 5$, two positive integers n and C, and a power series in $\mathbb{Z}_p[[q]]/(q^N, p^C)$ (where N is an explicit constant depending on n), and will output the first $n+1$ terms of the Katz expansion, with respect to an explicit splitting of Equation 4.1, which is particularly useful for computational methods. The second algorithm, Algorithm 2, uses Algorithm 1. It takes as input a prime $p \geq 5$, a non-negative integer r and a list of integral weights $L = [\kappa_1, \ldots, \kappa_\lambda]$, for some integer $\lambda \geq 0$. The output will be the values $\nu_p(b_{r,j})$ for $0 \leq j \leq r$, provided these can be determined exactly. Note that the output allows us to tell if they are indeed exact, or if we cannot conclude the value of some $\nu_p(b_{r,j})$. Increasing the number of weights in the input allows us (modulo some technicalities, see the discussion after Algorithm 2) in general to get a conclusive value for $\nu_p(b_{r,j})$ for fixed $0 \leq j \leq r$.
In the final section we provide data obtained using Algorithm 2. In particular, our data shows that for our obtained value we have the bound $\nu_p(b_{i,j}) \geq d_p i - j$ for $d_p = \frac{p-1}{p(p+1)}$. We indeed know from [AKW22] that a lower bound of this form exists, but the constant d_p differs from the proven constant.

Note that for the primes p such that $X_0(p)$ has genus 0 (i.e. $p \in \{2, 3, 5, 7, 13\}$) we have the so called hauptmodul, defined by

$$f_p(z) := \left(\frac{\Delta(pz)}{\Delta(z)}\right)^{\frac{1}{p-1}},$$

where Δ is the normalized cuspform of level 1 and weight 12. This function will generate the function field of $X_0(p)$ and can be used to measure the overconvergence rates of overconvergent modular forms (see [Loe07, Corollary 2]). In the cases $p = 2$ and $p = 3$, a result of Buzzard and Calegari [BC05] (for $p = 2$) and Roe [Roe14] (for $p = 3$), shows that we can write $E/V(E)$ as a power series in $\mathbb{Z}_p[[w, f]]$, and if we write $E/V(E) = \sum_{i,j \geq 0} a_{i,j} f^i w^j$ then we have a lower bound for $\nu_p(a_{i,j})$ which in both their cases is linear in $i - j$ and hence these theorems give information about, for example, the overconvergence rates of Eisenstein series. However, for all other primes from the ones mentioned above, $X_0(p)$ will be of genus strictly higher than 0 and hence there will not be a single function which can measure the overconvergence rate. So, a different method is needed to explore the overconvergence rates of the family $E/V(E)$ and one option for this is to instead use Katz expansions. In [AKW22, Theorem B] the related notion of a formal Katz expansion was introduced, in order to deduce overconvergence rates about the family $E/V(E)$ for *all* primes $p \geq 5$. This article the gives a computational approach to the theory of formal Katz expansions and give computational bounds on the overconvergence rates of the forms $E^*_\kappa/V(E^*_\kappa)$ for all primes $p \geq 5$.

Acknowledgments

This work was supported by the Luxembourg National Research Fund PRIDE17/1224660/GPS.

4.2 Theoretical Background

We will start by exposing the theory of overconvergent modular forms à la Katz. While nowadays there exists a more geometric and intrinsic definition of an overconvergent modular form by the works of Pilloni [Pil13] and Andreatta, Iovita, Stevens [AIS14], the theory by Katz has the advantage that it is very explicit and allows one to do explicit computations with them. Because of this, we will only focus on the definition provided by Katz. We fix a prime $p \geq 5$ and for each integer k we let $M_k(\mathbb{Z}_p)$ denote the space of weight k modular forms of level 1 with coefficients in $\mathbb{Z}_p$: $(M_k(1) \cap \mathbb{Z}[[q]]) \otimes \mathbb{Z}_p$. We let E_{p-1} be the classical Eisenstein series of weight $p-1$, normalized such that it has constant coefficient 1. From [Kat73, Lemma 2.6.1], we know that there is a (non-canonical) splitting

$$M_{k+i(p-1)}(\mathbb{Z}_p) = E_{p-1} \cdot M_{k+(i-1)(p-1)}(\mathbb{Z}_p) \oplus B_i(\mathbb{Z}_p),$$

where the $B_i(\mathbb{Z}_p)$ are free $\mathbb{Z}_p$-submodules of $M_{k+i(p-1)}(\mathbb{Z}_p)$. Katz then shows that, given such a splitting, for $\rho \in \mathbb{R}$, the ρ-overconvergent modular forms of weight 0 and tame level 1 can be written as a series of the form

$$\sum_{i=0}^{\infty} \frac{b_i}{E_{p-1}^i},$$

where $b_i \in B_i(\mathbb{Z}_p)$, $v_p(b_i) \geq i\rho$ and $v_p(b_i) - i\rho \to \infty$ for $i \to \infty$. Sometimes, these are called r-overconvergent modular forms, where $r \in \mathbb{C}_p$ such that $\nu(r) = \rho$. We denote the space of the ρ-overconvergent modular forms by $M_0(\mathbb{Z}_p, \rho)$. If we have an overconvergent modular form, $f \in M_0(\mathbb{Z}_p, \geq \rho)$, then it can be written as such a series, and we refer to it as the Katz-expansion (even though such a series, of course, depends on the chosen splitting and hence the Katz-expansion is only unique after fixing a splitting). We shall also use the following notation. If we have a rational $\rho \in [0, 1]$, we let $M_0(\mathbb{Z}_p, \geq \rho)$ be the $\mathbb{Z}_p$-module of forms f such that$f \in M_0(\mathbb{Z}_p, \rho')$, for some $\rho' \in \mathbb{R}$, and $v_p(b_i) \geq i\rho$, for all the coefficients b_i of the Katz expansion of f. Note that the whole discussion above carries through if we use the ring of integers $\mathcal{O}$ of some finite extension $K/\mathbb{Q}_p$, and we can define in a similar fashion the modules $M_0(\mathcal{O}, \rho)$ and $M_0(\mathcal{O}, \geq \rho)$.

4.2.1 Lauder's splitting

Lauder has given an explicit splitting based on the existence of a Miller basis (see [Lau11, Section 3] and [Ste07, Lemma 2.20]), which is easy to compute with, and we will give a short description of this. While it is possible to work with higher levels, we will not use this. As we work mainly over $\mathbb{Z}_p$, we will suppress this from our notation and denote these spaces by B_i; if we need to work over another ring we will write it explicitly. To describe the spaces B_i, we start by defining some auxiliary functions. For n a non-negative integer we define

$$d_n := \left\lfloor \frac{n}{12} \right\rfloor + \begin{cases} 1 & n \not\equiv 2 \mod 12 \\ 0 & n \equiv 2 \mod 12, \end{cases}$$

that is, d_n is the dimension of the classical space of modular forms of weight n and level 1. We also have the following function

$$\epsilon(k) := \begin{cases} 0 & k \equiv 0 \mod 4 \\ 1 & k \equiv 2 \mod 4. \end{cases}$$

Then, for a fixed $i \geq 0$ and $j \geq 0$ we define

$$g_{i,j} := \Delta^j E_4^a E_6^{\epsilon(i(p-1))}, \tag{4.2}$$

where

$$a = \frac{i(p-1) - 12j - 6\epsilon(i(p-1))}{4},$$

and Δ is the normalized weight 12 cusp form. Note that for $j = 0, \ldots, d_n - 1$ the numbers a are nonnegative integers and the $g_{i,j}$ are weight $i(p-1)$ modular forms (of level 1) and the q-expansion of $g_{i,j}$ starts with q^j. Then we put $B_0(\mathbb{Z}_p) := \mathbb{Z}_p$ and for $i > 0$ we let $B_i(\mathbb{Z}_p)$ be the free Z_p-module spanned by

$$\mathcal{B}_i := \{g_{i,j} | d_{(i-1)(p-1)} \leq j \leq d_{i(p-1)} - 1\}.$$

The spaces B_i then give a splitting as in (4.1). Note that if we fix a $j \geq 0$, then there is a unique $i \geq 0$ such that $g_{i,j} \in \mathcal{B}_i$; we can find it by picking the unique i such that $d_{(i-1)(p-1)} \leq j \leq d_{i(p-1)} - 1$, we will denote this element by i_j. If we write g_j we mean the element $g_{i_j,j}$. Note that for any $j \geq 0$ there exists a g_j, but, depending on the prime, the $\mathcal{B}_i$ might be empty for certain i. For example, if $p = 5$ then we have that $\mathcal{B}_i = \emptyset$ unless i is a multiple of 3.

The main use of this specific splitting is that the (infinite) matrix whose rows contain the coefficients of the q-expansions of

$$g_{0,0}, \ldots, g_{i,d_{(i-1)(p-1)}}, \ldots, g_{i,d_{i(p-1)-1}},$$

is upper triangular with 1's on the diagonal. As the q-expansion of E_{p-1} is in $1 + p\mathbb{Z}_p[[q]]$, also the q-expansion of E_{p-1}^{-i} will be in $1 + p\mathbb{Z}_p[[q]]$, and thus the (infinite) matrix whose rows contain the coefficients of the q-expansions of

$$g_{0,0}, \ldots, g_{i,d_{(i-1)(p-1)}} E_{p-1}^{-i}, \ldots, g_{i,d_{i(p-1)-1}} E_{p-1}^{-i}, \ldots$$

will also be upper triangular, with 1's on the diagonal. This implies that we have an isomorphism $\phi : \prod_{i\geq 0} B_i(\mathbb{Z}_p) \to \mathbb{Z}_p[[q]]$ of $\mathbb{Z}_p$-modules given by

$$\phi((b_i)_{i\geq 0}) := \sum_{i\geq 0} b_i(q) E_{p-1}^{-i}(q).$$

In particular, if we are given the q-expansion, say up to $a_N(f)q^N$, of an overconvergent modular form f this can be turned into an algorithm to compute its Katz expansion (up to some precision), which is the inverse of the map ϕ.

4.2.2 The Formal Katz Expansion

Our first goal will be to compute the valuations appearing in the 'formal Katz expansion' of a family of overconvergent modular forms related to the Eisenstein series. We consider weights, i.e. characters $\kappa : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$. As we have the decomposition $\mathbb{Z}_p^\times \backsimeq (\mathbb{Z}/p\mathbb{Z})^\times \times 1 + p\mathbb{Z}_p$ (as $p \geq 5$), we can consider the characters restricted to $(\mathbb{Z}/p\mathbb{Z})^\times$. We will only consider the characters that are trivial on the $(p-1)$-st roots of unity. We denote this space by $\mathcal{B}$ (this weight space can be given a rigid analytic structure, but we will not need this). The weight space can be identified with the unit disk $\mathcal{W}$ inside $\mathbb{C}_p$, via $\kappa \mapsto \kappa(p+1)-1$. An integral weight $k \in \mathbb{Z}$ will be identified with the character $x \mapsto x^k$. For a given weight κ, we have the Eisenstein series of weight κ with q-expansion given by

$$E_\kappa^* = 1 + \frac{2}{\zeta^*(\kappa)} \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n \\ p\nmid n}} \kappa(d) d^{-1} \right) q^n$$

(note that we remove the Euler factor at p). Here $\zeta^*(\kappa)$ is the p-adic zeta function. It is known (see [Col97b, Corollary 2.1.1] or [Col97c, Corollary B4.1.2]) that $E_\kappa^*/V(E_\kappa^*)$ is overconvergent, where V is the operator acting on the q-expansion by $q \mapsto q^p$. From [AKW22, Theorem B], we have the following result.

Theorem B. *(a) There are modular forms $b_{ij} \in B_i(\mathbb{Z}_p)$ for each $i, j \in \mathbb{Z}_{\geq 0}$ such that the following holds. If $\kappa \in \mathcal{B}\backslash\{1\}$ then the Katz expansion of the modular function $\frac{E_\kappa^*}{V(E_\kappa^*)}$ is*

$$\frac{E_\kappa^*}{V(E_\kappa^*)} = \sum_{i=0}^{\infty} \frac{\beta_i(w(\kappa))}{E_{p-1}^i}$$

where

$$\beta_i(w(\kappa)) := \sum_{j=0}^{\infty} b_{ij} w(\kappa)^j$$

for each i.

(b) There is a constant c_p with $0 < c_p < 1$ such that for the modular forms b_{ij} in part (a) we have

$$v_p(b_{ij}) \geq c_p i - j$$

for all i, j.

In fact, we can take

$$c_p = \frac{2}{3}\left(1 - \frac{p}{(p-1)^2}\right)\frac{1}{p+1}.$$

This constant does not seem to be optimal, in the sense that there seems to be a maximal constant, $\delta_p > c_p$, such that in the theorem above we would have $v_p(b_{ij}) \geq \delta_p i - j$ for all i, j. The reason why we would want to know this optimal value, goes back to the action of the U-operator on weight κ overconvergent modular forms. A deep understanding of the forms $\frac{E_\kappa^*}{V(E_\kappa^*)}$, and in particular of their overconvergence rates, can be used to deduce information about this action. However, this would require us to identify the exact constant δ_p, see [AKW22, Section 4.3] for the exact details. Hence, we wish to compute $\nu(b_{i,j})$ and to compare it with the values ensured by the lower bound in the above theorem, in order to gain information about the possible value of δ_p. The main idea for an algorithm to compute these valuations is using the existence of a formal Katz expansion, as in statement (a) of Theorem B and to compute the Katz expansion for enough classical weights to deduce information about this formal Katz expansion. In the next section we will describe two algorithms, the first one will compute the Katz expansion of any weight 0 overconvergent modular form, and the second algorithm will return (modulo some technicalities) the valuations $\nu(b_{i,j})$.

4.3 The Algorithms

Recall that we have an isomorphism $\psi : \mathbb{Z}_p[[q]] \to \prod_{i\geq 0} B_i(\mathbb{Z}_p)$, the inverse of the map ϕ introduced in the previous section. Note that this map attaches to a power series (in particular to an overconvergent modular form of weight 0 and with coefficients in $\mathbb{Z}_p$) its Katz expansion. Our first algorithm will have as its goal to compute this Katz expansion, with a modular form (or a power series) as its input. We have to be careful with the precisions we choose for this. Fix an integer $n \geq 0$ and set $N := d_{n(p-1)}$. Then, for $m > n$, if $g \in B_m(\mathbb{Z}_p)$ we have that $g \equiv 0 \mod q^N$, so that ϕ descends to a map: $\phi_n : \prod_{i=0}^n B_i(\mathbb{Z}_p) \to \mathbb{Z}_p[[q]]/(q^N)$. We have the following:

Lemma 4.2. *The map*

$$\phi_n : \prod_{i=0}^{n} B_i(\mathbb{Z}_p) \to \mathbb{Z}_p[[q]]/(q^N), \qquad (b_i)_{i=0}^n \mapsto \sum_{i=0}^{n} \frac{b_i}{E_{p-1}^i}, \tag{4.3}$$

is an isomorphism.

Proof. If we are given an element $f \in \mathbb{Z}_p[[q]]/(q^N)$ and we want to find an inverse, then we have to solve the following matrix system

$$Mx = B,$$

where M is the $N \times N$ matrix whose jth column consists of the coefficients of the q-expansion of $g_j/E_{p-1}^{i_j}$, and where B is the column vector consisting of the coefficients of f. As M is lower triangular, with 1's on the diagonal, the system will have a unique solution. The preimage of f is then given by $(b_i)_{i=0}^n$, where $b_i = x_{d_{(i-1)(p-1)}} g_{d_{(i-1)(p-1)}} + \ldots + x_{d_{i(p-1)-1}} g_{d_{i(p-1)-1}}$, which is uniquely determined and hence ϕ_n is an isomorphism. □

We denote the inverse of the map given in (4.3) by

$$\psi_n : \mathbb{Z}_p[[q]]/(q^N) \to \prod_{i=0}^{n} B_i(\mathbb{Z}_p).$$

The following algorithm computes this map to any given p-adic precision, and hence can be seen as computing the partial Katz expansion of a given overconvergent modular form.

Algorithm 1. *Given a prime $p \geq 5$, positive integers n and C, and a power series f in $\mathbb{Z}[[q]]/(q^N, p^C)$, where $N = d_{n(p-1)}$, this algorithm returns $\psi_n(f)$, as an $(n+1)$-tuple, with p-adic precision C.*

1. *Dimension of $\mathcal{B}_i(\mathbb{Z}_p)$*: Compute the values i_j for the values $j = 0, \ldots, N-1$.

2. *Basis of $\mathcal{B}_i(\mathbb{Z}_p)$*: Compute the q-expansions of the forms $g_j \in \mathcal{B}_{i_j}$ for $j = 0, \ldots, N-1$ up till q^{N-1} with coefficients in $\mathbb{Z}$, using Equation (4.2), after having computed the q-expansions of E_4, E_6 and Δ up to q^N. We normalize E_4 and E_6 to have constant coefficient 1.

3. *Coefficient matrices*: Create the $N \times N$ matrix M which has as jth column the coefficients of the q-expansion $g_j E_{p-1}^{-i_j}$ up till q^{N-1}. This matrix will be lower triangular with 1s on the diagonal.

4. *Katz expansion*: Create the column vector B, containing the coefficients $a_0(f), \ldots, a_{N-1}(f)$. Solve the equation $Mx = B$ for x over $\mathbb{Z}_p/(p^C)$. Let the solution be given by $x = (x_0, \ldots, x_{N-1})^T$. Return the tuple $(f_0, \ldots, f_n)$, where

$$f_i = x_{d_{(i-1)(p-1)}} g_{d_{(i-1)(p-1)}} + \ldots + x_{d_{i(p-1)-1}} g_{d_{i(p-1)-1}}.$$

The correctness of the algorithm is a consequence of Lemma 4.2. Note that steps (1)-(3) only depend on p and n, and thus, if working with a fixed prime and precision, the matrix

M and the pairs (i,j) found in Step (1) can be saved and only Step (4) has to be executed. For the second algorithm, we investigate the formal Katz expansion of the family $E^*_\kappa/V(E^*_\kappa)$ as in Theorem B. The theorem asserts the existence of modular forms $b_{ij} \in B_i(\mathbb{Z}_p)$ for each $i,j \in \mathbb{Z}_{\geq 0}$. The following algorithm allows us to compute $\nu(b_{i,j})$, given enough weights, and otherwise it will return 'inconclusive', and one will have to adapt the input.

Algorithm 2. *Given a prime $p \geq 5$, a nonnegative integer r and a list L of integral weights $\kappa_1, \dots \kappa_\lambda$, (so $\lambda = \#L$) this algorithm will output the values $\nu(b_{r,j})$, for $0 \leq j \leq r$, if this can be determined exactly, and otherwise it will return 'inconclusive' if the choice of weights does not allow us to conclude.*

1. *Construct the Eisenstein series*: First construct the Eisenstein series $E^*_{\kappa_i}$ for the weights in the given list, and then construct $E^*_{\kappa_i}/V(E^*_{\kappa_i})$, as elements of $(\mathbb{Z}/p^\lambda\mathbb{Z})[[q]]/(q^N)$, where $N := d_{r(p-1)}$.

2. *Katz expansions*: Use Algorithm 1 to compute for all $i = 1, \dots, \lambda$ the Katz expansions of $E^*_{\kappa_i}/V(E^*_{\kappa_i})$ up to the rth term, say $\beta_r^{(i)}$. This means that we need a precision of $n = r$ and $C = \lambda$ in Algorithm 1.

3. *Construct the Vandermonde matrix*: Construct the Vandermonde matrix

$$V := \begin{bmatrix} 1 & w_1 & \dots & w_1^{\lambda-1} \\ \vdots & & \ddots & \vdots \\ 1 & w_\lambda & \dots & w_\lambda^{\lambda-1} \end{bmatrix},$$

   over $\mathbb{Z}/p^\lambda\mathbb{Z}$ and where $w_i = (p+1)^{\kappa_i} - 1$. Compute a set of generators, denoted by $\mathcal{V}$, for the (left) kernel. Note that the kernel is a subgroup of $(\mathbb{Z}/p^\lambda\mathbb{Z})^\lambda$ and $\mathcal{V}$ will be a set of generators for this kernel as a $\mathbb{Z}$-module. One can compute such a set of generators by, for instance, computing the Smith normal form of V. Then, for every $1 \leq i \leq \lambda$ compute $\gamma_i := \min\{\nu(v_i)) | v \in \mathcal{V}\}$, where v_i denotes the ith component of the vector v.

4. *Solve linear systems*: Define $S := \lceil r(p-1)/12 \rceil$. For $0 \leq i \leq S$, compute the column vector θ_i, which has as lth entry the ith coefficient of $\beta_r^{(l)}$, over $\mathbb{Z}/p^\lambda\mathbb{Z}$, for $0 \leq l \leq \lambda$. Compute the $S+1$ solutions x_i of the matrix equations $Vx_i = \theta_i$, over $\mathbb{Z}/(p^\lambda)$ (up to an element in the kernel of V).

5. *Find the minimum valuation*: For all $1 \leq j \leq r$ compute for $0 \leq i \leq S$ the minimum of the values $\nu((x_i)_j)$, say $\alpha_{r,j}$, and return the list $[\alpha'_{r,1}, \dots, \alpha'_{r,r}]$, where $\alpha'_{r,j} = \alpha_{r,j}$ if $\alpha_{r,j} < \gamma_j$, and '$\alpha'_{r,j}$ is inconclusive' if $\alpha_{r,j} \geq \gamma_j$.

To prove the correctness of Algorithm 2, we rely on Theorem B, in particular on the existence of the forms $b_{i,j}$. Hence, if the Katz expansion of $E^*_{\kappa_i}/V(E^*_{\kappa_i})$ is given by $\sum_{j\geq 0}\beta_j^{(i)}/E_{p-1}^j$, we get the following equation

$$\beta_j^{(i)} = \sum_{l=0}^{\infty} b_{j,l} w_i^l.$$

Now, $\nu(w_i) \geq 1$ and $\nu(b_{i,j}) \geq 0$, so reducing modulo p^λ then gives us

$$\beta_j^{(i)} \equiv \sum_{l=0}^{\lambda-1} b_{j,l} w_i^l \mod p^\lambda.$$

As we get this for every weight $\kappa_1, \ldots, \kappa_\lambda$ we obtain, for any $\mu \in \mathbb{N}$, the following matrix equation over $\mathbb{Z}/p^\lambda\mathbb{Z}$:

$$\begin{bmatrix} 1 & w_1 & \ldots & w_1^{\lambda-1} \\ \vdots & & \ddots & \vdots \\ 1 & w_\lambda & \ldots & w_\lambda^{\lambda-1} \end{bmatrix} \begin{bmatrix} a_\mu(b_{r,0}) \\ \vdots \\ a_\mu(b_{r,\lambda-1}) \end{bmatrix} = \begin{bmatrix} a_\mu(\beta_r^{(1)}) \\ \vdots \\ a_\mu(\beta_r^{(\lambda)}) \end{bmatrix},$$

where a_μ denotes the μ-th Fourier coefficient in the q-expansion. Note that the right hand side is known, as these are the coefficients appearing in the Katz expansion, which we can compute using Algorithm 1. We can solve this linear system, giving us a solution for $a_\mu(b_{r,0}), \ldots, a_\mu(b_{r,\lambda})$. Note that we know such a solution exists, as the forms $b_{i,j}$ exist, but this matrix equation might not have a unique solution, as V (the Vandermonde matrix) is in general not invertible over $\mathbb{Z}/p^\lambda\mathbb{Z}$. However, we can find a set of generators, $\mathcal{V}$, for the kernel of this matrix. If we let γ_i denote the minimum of the valuations of the ith entries of these generators, then we do know that if the valuation of the solution we find, say $\nu(a_\mu(b_{r,i}))$ has valuation less than γ_i, then this is the same valuation for any other solution, as any two solutions will differ from each other by an element in the kernel. Indeed, if w is any element in the kernel, then we can write it as $w = \sum_{v\in\mathcal{V}} \alpha_v v$, where the $\alpha_v \in \mathbb{Z}$. If we then consider the ith component of w, we see that $\nu(w_i) = \nu(\sum_{v_i\in\mathcal{V}} \alpha_v v_i) \geq \gamma_i$. Note that, while not necessary for the validity of the algorithm, it is true that for any two sets of generators, the values γ_i will be the same. If we were to have another set of generators, $\mathcal{V}'$, and $\gamma_i' < \gamma_i$ for some i, then there is an element $v' \in \mathcal{V}'$ such that $\nu(v_i') < \gamma_i$. But $\mathcal{V}$ generates the kernel, so v' can be written as a $\mathbb{Z}$-linear combination of elements in $\mathcal{V}$, and arguing as above, we find that $\nu(v_i') \geq \gamma_i$, contradicting our assumption and showing that we must have $\gamma_i' \geq \gamma_i$. Changing the roles of $\mathcal{V}'$ and $\mathcal{V}$ the same argument then implies that we must have $\gamma_i = \gamma_i'$.

To conclude that $\nu(b_{r,i})$ is the minimum of $\nu(a_\mu(b_{r,i}))$, we apply the following lemma:

Lemma 4.3. *If $f \in M_n(\mathbb{Z})$ and $\nu(a_i(f)) \geq b$ for $i = 0, \ldots, \lceil n/12 \rceil$, then $\nu(f) \geq b$.*

Proof. We will use Sturm's theorem, which says that if $\mathfrak{m}$ is a prime ideal of the ring of integers $\mathcal{O}$ of a number field K, Γ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of index m and

$f \in M_k(\Gamma, \mathcal{O})$ such that

$$\mathrm{ord}_q(f \bmod \mathfrak{m}) > km/12,$$

then $f \equiv 0 \bmod \mathfrak{m}$ [Ste07]. We will apply Sturm's theorem with $K = \mathbb{Q}$, $\mathcal{O} = \mathbb{Z}$ and $\mathfrak{m} = (p)$ and use induction to prove the lemma. The case that $b = 1$ immediately follows from Sturm's theorem. Then, for the general case, the induction hypothesis implies that $p^{b-1}f \in M_k(\mathcal{O})$ and we apply Sturm's theorem to $p^{b-1}f$, which implies that $p^{b-1}f \equiv 0 \bmod p$, and hence $\nu(f) \geq b$. □

Hence, for a fixed $0 \leq l \leq r$, if we know $\nu(a_\mu(b_{r,l}))$ for $0 \leq \mu \leq \lceil r(p-1)/12 \rceil$, then we know $\nu(b_{r,l})$. To apply this in the algorithm, we need $\beta_r^{(i)}$, the rth term of the Katz expansion of $E^*_{\kappa_i}/V(E^*_{\kappa_i})$, up till a precision of $S := \lceil (p-1) \cdot r/12 \rceil$. However, we remark that $d_{r(p-1)} \geq \lceil r(p-1)/12 \rceil$ and hence Algorithm 1 returns the Katz expansions with sufficient precision.
As noted before, the algorithm only finds $\nu(b_{r,j})$ if it is less than the γ_j, since $b_{r,j}$ is only found up to an element in the kernel of V. The following lemma shows that we can make γ_j arbitrarily high by increasing the number of weights.

Lemma 4.4. *For any $i \in \mathbb{Z}_{\geq 1}$ and $\gamma \in \mathbb{Z}_{\geq 0}$, there exist $n \in \mathbb{Z}$ and $w_1, \dots, w_n \in \mathbb{Z}_p$ with $\nu_p(w_j) = 1$ for all $1 \leq j \leq n$, such that for any $\alpha \in \ker(\overline{V})$, with*

$$\overline{V} = \begin{bmatrix} 1 & w_1 & \dots & w_1^{n-1} \\ \vdots & & \ddots & \vdots \\ 1 & w_n & \dots & w_n^{n-1} \end{bmatrix} \in M_{n\times n}(\mathbb{Z}/p^n\mathbb{Z}),$$

we have $\nu_p(\alpha_i) \geq \gamma$, (where α_i is the ith component of α).

Proof. Given $w_1, \dots, w_n \in \mathbb{Z}_p$ with $\nu_p(w_j) = 1$ for all $1 \leq j \leq n$, we denote

$$V := \begin{bmatrix} 1 & w_1 & \dots & w_1^{n-1} \\ \vdots & & \ddots & \vdots \\ 1 & w_n & \dots & w_n^{n-1} \end{bmatrix} \in M_{n\times n}(\mathbb{Q}_p),$$

and we denote by $\overline{V}$ the matrix as in the statement of the lemma (i.e. the matrix V reduced modulo p^n). Note that since the w_j are distinct, V will be invertible over $\mathbb{Q}_p$. In particular, if we have an element $\alpha \in \ker(\overline{V})$, we can lift this to a vector $\tilde{\alpha} \in \mathbb{Z}_p^n$, such that $\nu_p(\alpha_i) = \nu_p(\tilde{a}_i)$, and thus

$$V\tilde{\alpha} = \begin{bmatrix} p^n b_1 \\ \vdots \\ p^n b_n \end{bmatrix},$$

where the $b_1, \dots, b_n \in \mathbb{Z}_p$. This implies

$$\tilde{\alpha} = V^{-1} \begin{bmatrix} p^n b_1 \\ \vdots \\ p^n b_n \end{bmatrix},$$

and hence

$$\nu_p(\alpha_i) = \nu_p(\tilde{a}_i) \geq n + \min_{1 \leq j \leq n} \nu_p((V^{-1})_{i,j}).$$

Hence, it remains to bound the valuations appearing in the inverse of the Vandermonde matrix V. From [AKW22, Lemma 3.1] we know that the coefficient of V^{-1} at position (i, j) is given by

$$(-1)^{n-i} \cdot \frac{s_{n-i}(w_1, \ldots, \hat{w}_j, \ldots, w_n)}{\prod_{0 \leq \ell \leq n, \ell \neq j}(w_j - w_\ell)},$$

where $s_d(\ldots)$ is the elementary symmetric polynomial of degree d in $n-1$ variables. As all the w_j have valuation 1, we find that $\nu_p(s_{n-i}(w_1, \ldots, \hat{w}_j, \ldots, w_n)) \geq n - i$. As for the denominator, we have

$$\max_{1 \leq j \leq n} \nu_p \left(\prod_{\substack{1 \leq \ell \leq n \\ \ell \neq j}} (w_l - w_j) \right) = \max_{1 \leq j \leq n} \nu_p \left(p^n \prod_{\substack{1 \leq \ell \leq n \\ \ell \neq j}} \left(\frac{w_l}{p} - \frac{w_j}{p} \right) \right) \geq n - 1 + f(n), \tag{4.4}$$

where

$$f(n) := \sum_{i=1}^{\infty} \left\lfloor \frac{n-1}{(p-1)p^{i-1}} \right\rfloor.$$

The last inequality in (4.4) is [AKW22, Proposition 3.2] and becomes an equality if the w_j are chosen correctly (compare with [AKW22, Proof 3.3]). We note that

$$f(n) \leq \frac{n-1}{p-1} \sum_{i=0}^{\infty} \frac{1}{p^i} = (n-1)\frac{p}{(p-1)^2},$$

and thus, assuming the w_j are chosen such that we have equality in (4.4), putting everything together we find

$$\min_{1 \leq j \leq n} \nu_p((V^{-1})_{i,j}) \geq n - i - \left(n - 1 + (n-1)\frac{p}{(p-1)^2} \right) = 1 - i - (n-1)\frac{p}{(p-1)^2}.$$

We conclude that

$$\nu_p(\alpha_i) \geq n + 1 - i - (n-1)\frac{p}{(p-1)^2} = n \cdot \left(1 - \frac{p}{(p-1)^2} \right) - i + 1 + \frac{p}{(p-1)^2}, \tag{4.5}$$

but, as $\frac{p}{(p-1)^2} < 1$ for $p \geq 5$, the right hand side can be made arbitrarily high by increasing n.

□

Note that the proof requires us to choose the weights correctly. To show that we can indeed do this, we have the following result.

Lemma 4.5. *If $a, b \in \mathbb{N}$ and $p \geq 3$ prime, then we have*

$$\nu_p\left(\frac{(1+p)^a - (1+p)^b}{p}\right) = \nu_p(a-b).$$

Proof. Assume without loss of generality that $a \geq b$, then

$$\frac{(1+p)^a - (1+p)^b}{p} = \frac{(1+p)^a(1-(1+p)^{b-a})}{p}$$

and $\nu_p((1+p)^a) = 0$, so it suffices to show that for any $c \in \mathbb{N}_{\geq 1}$ we have

$$\nu_p\left((1+p)^c - 1\right) = \nu_p(c) + 1.$$

We have

$$(1+p)^c - 1 = cp + \binom{c}{2}p^2 + \binom{c}{3}p^3 + \ldots + p^c. \tag{4.6}$$

If $n \geq 2$, then

$$\nu_p\left(\binom{c}{n}\right) = \nu_p\left(\frac{c}{n}\right) + \nu_p\left(\binom{c-1}{n-1}\right) \geq \nu_p(c) - \nu(n),$$

as $\binom{c-1}{n-1}$ is a positive integer. Furthermore, we have $\nu_p(n) \leq n-2$ and hence

$$\nu_p\left(\binom{c}{n}\right) > \nu_p(c) + 1 - n.$$

This shows that if we take the valuation of the right hand side of(4.6) we end up with

$$\nu_p\left((1+p)^c - 1\right) = \nu_p(cp) = \nu_p(c) + 1.$$

□

In particular, if we let Γ be the set containing the first n natural numbers prime to p, then we have that

$$\max_{x\in\Gamma} \nu_p\left(\prod_{s\in\Gamma, s\neq x}(x-s)\right) = f(n),$$

as explained in [AKW22, Lemma 3.1]. But now Lemma 4.5, shows that if we take the classical weights, $\{k_s : x \mapsto x^{s(p-1)} | s \in \Gamma\}$ with the corresponding $\{w_s = (p+1)^{s(p-1)} - 1 | s \in \Gamma\}$, then

$$f(n) = \max_{x\in\Gamma} \nu_p\left(\prod_{s\in\Gamma, s\neq x}(x-s)\right) = \max_{x\in\Gamma} \nu_p\left(\prod_{s\in\Gamma, s\neq x}\left(\frac{w_x}{p} - \frac{w_s}{p}\right)\right)$$

where the second equality follows from Lemma 4.5, as it implies that for all $x, s \in \Gamma$ we have the equality $\nu_p(x-s) = \nu_p((w_x - w_s)/p)$. Thus Lemma 4.4 applies to this choice of weights. In particular, if Algorithm 2 returns 'inconclusive', one can increase the number of weights given as the input as above, which will increase the values of the γ_j. Hence, as long as the $b_{i,j}$ are non-zero, there will be a number of weights such that $\gamma_j > b_{i,j}$ and thus we can determine $\nu(b_{i,j})$ exactly (at least, theoretically). However, if for some i and j we have that $b_{i,j} = 0$, then we cannot use this algorithm to determine $v(b_{i,j})$ (as it will be infinite). Computations so far seem to suggest that if $i, j \neq 0$ and $\mathcal{B}_i \neq \emptyset$, then $b_{i,j} \neq 0$, but a proof does not seem available at the time, nor are we sure to even expect that this is the case. However, if $j = 0$ we do have the following result.

Proposition 4.6. *Let the $b_{i,j}$ be as in Theorem B. Then $b_{0,0} = 1$ and $b_{i,0} = 0$ for $i > 0$.*

To prove this, we need the following lemma.

Lemma 4.7. *Let R be a commutative ring and $f \in 1 + xyR[[x, y]]$. Then f is invertible and $f^{-1} \in 1 + xyR[[x, y]]$.*

Proof. If we consider f as a power series in the variable y and with coefficients power series in x, say $f = 1 + a_1(x)y + a_2(x)y^2 + \ldots$, then the inverse is given by $f^{-1} = 1 + b_1(x)y + b_2(x)y^2 + \ldots$, where $b_k(x) = -\sum_{i=1}^{k} a_i b_{k-i}$, where we set $b_0 = 1$. Induction then shows that $b_i(x) \in xR[[x]]$ and hence $f^{-1} \in 1 + xyR[[x, y]]$. □

Now we can prove Proposition 4.6.

Proof. We first note that the isomorphism ϕ descends to an isomorphism

$$\tilde{\phi} : \prod_{i \geq 0} B_i(\mathbb{Z}/p^n\mathbb{Z}) \to (\mathbb{Z}/p^n\mathbb{Z})\,[[q]],$$

for any $n \in \mathbb{N}$, which follows from the same argument that ϕ is an isomorphism. Now, let κ be an integral weight such that $w := w_\kappa$ satisfies $\nu(w) \geq n$ (one can pick for example the integral weight corresponding to $k = (p-1)p^{n+1}$). Then we know that $E_\kappa^* \in 1 + wq\mathbb{Z}_p[[w, q]]$, see for example [BK05, Section 5]. Note that they are only interested in the case $p = 2$, but the proof holds for all primes p, as they only use the invertibility of the p-adic L-function appearing in the Eisenstein series, and the fact that the weight $\kappa : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ can be expressed as a power series in the variable w. Consequently, also $V(E_\kappa^*) \in 1 + wq\mathbb{Z}_p[[w, q]]$, and Lemma 4.7 then implies that $\frac{E_\kappa^*}{V(E_\kappa^*)} \in 1 + wq\mathbb{Z}_p[[w, q]]$. Looking at the Katz expansion

$$\frac{E_\kappa^*}{V(E_\kappa^*)} = \sum_{i=0}^{\infty} \frac{\sum_{j=0}^{\infty} b_{ij} w^j}{E_{p-1}^i},$$

and reducing modulo p^n we get

$$1 \equiv \sum_{i=0}^{\infty} \frac{b_{i,0}}{E_{p-1}^i} \mod p^n,$$

as $\nu(w) \geq n$. This is a congruence of power series. Note that for a fixed N, there are only finitely many i such that $b_{i,0}$ has a non-zero coefficient of q^N and hence the infinite sum makes sense. As ϕ_n is an isomorphism, we get that $b_{0,0} \equiv 1 \mod p^n$ and $b_{i,0} \equiv 0 \mod p^n$ for $i > 0$, since $b_{0,0} \in \mathbb{Z}_p$ and $b_{i,0}$ has no constant term for $i > 0$. But this holds for any n and we conclude. □

While this shows that, in theory, we can keep increasing the number of weights as input in Algorithm 2 and get the exact values $\nu(b_{i,j})$, this might not be feasible in real life. In particular, increasing the number of weights, also increases the p-adic accuracy with which we need to work and the more accuracy we need, the slower the algorithm will be. If we know a priori an upper bound on the value of $\nu(b_{i,j})$, then we could in practice use the above argument to give an explicit set of weights which would guarantee Algorithm 2 to return $\nu(b_{i,j})$ as output, indeed Equation 4.5 gives an explicit bound. However, in practice we do not have such an upper bound. However, in the case we are interested in, this turns out to not be as much of a problem, see Remark 4.9 for more details on this.

4.4 Observations

In this section we will present data, which is obtained using Algorithm 2. As already stated in the introduction, we define $\delta_p \in \mathbb{R}$ as follows:

$$\delta_p := \inf \left\{ \frac{\nu_p(b_{ij}) + j}{i} \middle| i \in \mathbb{Z}_{>0}, j \in \mathbb{Z}_{\geq 0} \right\}.$$

Our main objective will be to provide an upper bound for the value δ_p obtained from computations and use this to formulate a precise conjecture on a lower bound for δ_p. As we can only compute finitely many values for $\nu(b_{i,j})$, we can only obtain an upper bound. From Theorem B we know that there exists a constant c_p such that $\delta_p \geq c_p$. The main intent for Algorithm 2 is to see whether we expect $c_p = \delta_p$ or whether we expect c_p to be strictly smaller than δ_p. To be more precise, assume we have a set of tuples $(i, j) \in \mathbb{N}_{\geq 1} \times \mathbb{N}_{\geq 0}$, say S, for which we have computed $\nu(b_{i,j})$ (so S is necessarily finite), then an upper bound for δ_p can be given by computing

$$d'_p := \min_{(i,j) \in S} \left\{ \frac{\nu(b_{i,j}) + j}{i} \right\}. \tag{4.7}$$

Remark 4.8. *Note that for a fixed i we do not need to know the values $\nu(b_{i,j})$ for all j. More precisely, if we have already found an upper bound for δ_p, say d', then we only need to know the values $\nu(b_{i,j})$ for $j \leq d'i$. Indeed, as $\nu(b_{i,j}) \geq 0$ (since the $b_{i,j}$ have coefficients in $\mathbb{Z}_p$), we have that if $j \geq d'i$, then $\frac{\nu(b_{i,j})+j}{i} \geq d'$ and hence computing $\nu(b_{i,j})$ for larger values of j will have no impact on an upper bound of d_p.*

Remark 4.9. *Similarly to the remark above, if we already found an upper bound for δ_p, say d', and we want to compute $\nu(b_{i,j})$ for some given i, j, then we are only interested in* *its value if it is strictly less than $d'i - j$. This means that the number of weights we need as input for Algorithm 2 needs only to be enough to guarantee that $\gamma_j > d'i - j$. We can either use the arguments from the previous section to obtain precisely which list of weights would suffice, or we can compute the Vandermonde matrix V in Step (3) of Algorithm 2 for a list of weights and compute the γ_j, and keep adding more weights to the list, until we find $\gamma_j > d'i - j$, and then run the whole of Algorithm 2 for these weights. To add an idea on the size of weights needed, for $p = 7$, $j = 10$, and as a list of weights simply the first 20 multiples of $p - 1 = 6$ (which might very well not satisfy the lower bound given in 4.5), we obtain $\gamma_{10} = 9$. If we instead consider the first 50 multiples, we obtain $\gamma_{10} = 36$, and for the first 100 multiples we obtain $\gamma_{10} = 81$.*

We ran Algorithm 2 for the primes $p \in \{5, 7, 11, 13, 17, 37\}$ for different ranges of i. Once we find a new upper bound d'_p, we input enough weights so that we get exact values of $\nu(b_{i,j})$ for $j = 1, \dots, \lceil d'_p i \rceil$, as the values $\nu(b_{ij})$ for $j \geq d'_p i$ will not influence the minimum as in (4.7), as explained in the remark above. In the following table we present the values for the upper bounds we have found. We include up to the bound on i we computed $\nu(b_{i,j})$ and we also include the first value of i for which the value d'_p is attained.

prime	$i \leq$	d'_p	attained at $i =$
5	599	2/15	30
7	502	3/28	56
11	312	5/66	132
13	288	6/91	182
17	248	1/18	18
37	130	1/38	38

For a more visual representation, we also include the following plot for $p = 11$.

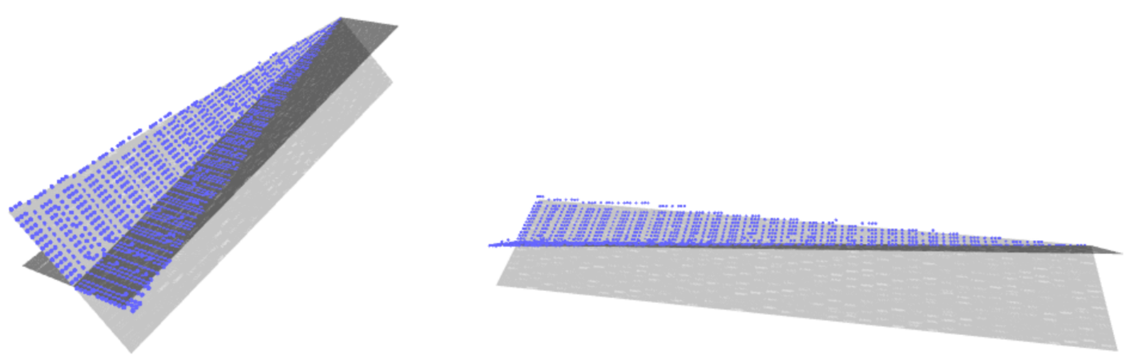


Figure 4.1: The values $(i, j, \nu(b_{i,j}))$ for $1 \leq i \leq 172$ and $1 \leq j \leq 5/66 \cdot i$ and $p = 11$. The black plane is given by $z = 0$ and the grey plane is given by $5/66 \cdot i - j = z$.

Note that there are some points (i, j) missing for which Algorithm 2 returned inconclusive. In particular, this happens for certain values for which $\mathcal{B}_i = \emptyset$, e.g. for $i = 7$, and hence $b_{i,j} = 0$ for these i.

Based on this data we formulated Conjecture 1. We would like to make a few remarks about this. First, we note that we only found an upper bound in agreement with our conjectured value of d_p in the cases $p = 5, 7, 11, 13$. In the cases $p = 17$ and $p = 37$ we find a value $1/(p+1)$ as an upper bound instead, strictly larger than our conjectured value for d_p. However, due to the nature of how we compute an upper bound, see (4.7), we can only get a value i in the denominator if we have computed $\nu(b_{i,j})$ for a multiple of i. In particular, for $p = 17$, we expect to find $d'_p = 8/153$, which means that we can only find this value if we compute $\nu(b_{i,j})$ for i a multiple of 153. Similarly, for $p = 13$, we need to compute $\nu(b_{i,j})$ where i is a multiple of 703. As the computation time increases as we increase i, we have not been able to compute $\nu(b_{i,j})$ for these values of i.

Secondly, assuming that $p = 5$ or $p = 7$, and $k \in \mathbb{N}$ divisible by $p - 1$, then we know that

$$\frac{E_k^*}{V(E_k^*)} \in M_0\left(\mathbb{Z}_p, \geq \frac{p-1}{p(p+1)}\right),$$

see [AKW22, Proposition 4.2]. In particular, we see that for these primes, our conjectured value for d_p precisely agrees with the overconvergence rate of Eisenstein series with classical weights. However, the proof of this is highly specific for $p = 5, 7$, and for larger primes than this, only strictly higher overconvergent rates are proven. Furthermore, we are not sure if it is possible to theoretically prove Conjecture 1 using the overconvergent rates for the Eisenstein series with only classical weights.

Bibliography

[Adv24] Bryan Advocaat. Computations on overconvergence rates related to the Eisenstein family. *J. Number Theory*, 259:112–130, 2024.

[AIP18] Fabrizio Andreatta, Adrian Iovita, and Vincent Pilloni. Le halo spectral. *Ann. Sci. Éc. Norm. Supér. (4)*, 51(3):603–655, 2018.

[AIS14] Fabrizio Andreatta, Adrian Iovita, and Glenn Stevens. Overconvergent modular sheaves and modular forms for $\mathbf{GL}_{2/F}$. *Israel J. Math.*, 201(1):299–359, 2014.

[AKW22] Bryan W. Advocaat, Ian Kiming, and Gabor Wiese. A conjecture of Coleman on the Eisenstein family, 2022.

[BC05] Kevin Buzzard and Frank Calegari. Slopes of overconvergent 2-adic modular forms. *Compos. Math.*, 141(3):591–604, 2005.

[BK05] Kevin Buzzard and L. J. P. Kilford. The 2-adic eigencurve at the boundary of weight space. *Compos. Math.*, 141(3):605–619, 2005.

[BP16] John Bergdall and Robert Pollack. Arithmetic properties of Fredholm series for p-adic modular forms. *Proc. Lond. Math. Soc. (3)*, 113(4):419–444, 2016.

[BP19a] John Bergdall and Robert Pollack. Slopes of modular forms and the ghost conjecture. *Int. Math. Res. Not. IMRN*, (4):1125–1144, 2019.

[BP19b] John Bergdall and Robert Pollack. Slopes of modular forms and the ghost conjecture, II. *Trans. Amer. Math. Soc.*, 372(1):357–388, 2019.

[BP22] John Bergdall and Robert Pollack. Slopes of modular forms and reducible Galois representations, an oversight in the ghost conjecture. *Proc. Amer. Math. Soc. Ser. B*, 9:432–444, 2022.

[CM98] R. Coleman and B. Mazur. The eigencurve. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 1–113. Cambridge Univ. Press, Cambridge, 1998.

[Col97a] Robert F. Coleman. Classical and overconvergent modular forms of higher level. *J. Théor. Nombres Bordeaux*, 9(2):395–403, 1997.

[Col97b] Robert F. Coleman. On the coefficients of the characteristic series of the U-operator. volume 94, pages 11129–11132. 1997. Elliptic curves and modular forms (Washington, DC, 1996).

[Col97c] Robert F. Coleman. p-adic Banach spaces and families of modular forms. *Invent. Math.*, 127(3):417–479, 1997.

[Col05] Robert F. Coleman. The canonical subgroup of E is $\operatorname{Spec} R[x]/(x^p + \frac{p}{E_{p-1}(E,\omega)}x)$. *Asian J. Math.*, 9(2):257–260, 2005.

[Col13] Robert F. Coleman. The Eisenstein family. *Proc. Amer. Math. Soc.*, 141(9):2945–2950, 2013.

[CST98] R. Coleman, G. Stevens, and J. Teitelbaum. Numerical experiments on families of p-adic modular forms. In *Computational perspectives on number theory (Chicago, IL, 1995)*, volume 7 of *AMS/IP Stud. Adv. Math.*, pages 143–158. Amer. Math. Soc., Providence, RI, 1998.

[Des17] D. Destefano. Investigating slopes of overconvergent forms, 2017.

[DI95] Fred Diamond and John Im. Modular forms and modular curves. In *Seminar on Fermat's Last Theorem (Toronto, ON, 1993–1994)*, volume 17 of *CMS Conf. Proc.*, pages 39–133. Amer. Math. Soc., Providence, RI, 1995.

[DR73] P. Deligne and M. Rapoport. Les schémas de modules de courbes elliptiques. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, volume Vol. 349 of *Lecture Notes in Math.*, pages 143–316. Springer, Berlin-New York, 1973.

[Gou88] Fernando Q. Gouvêa. *Arithmetic of p-adic modular forms*, volume 1304 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.

[Hid93] Haruzo Hida. *Elementary theory of L-functions and Eisenstein series*, volume 26 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1993.

[Kat73] Nicholas M. Katz. p-adic properties of modular schemes and modular forms. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, Lecture Notes in Math., Vol. 350, pages 69–190. Springer, Berlin, 1973.

[KR21] Ian Kiming and Nadim Rustom. Eisenstein series, p-adic modular functions, and overconvergence. *Res. Number Theory*, 7(4):Paper No. 65, 33, 2021.

[Lau11] Alan G. B. Lauder. Computations with classical and p-adic modular forms. *LMS J. Comput. Math.*, 14:214–231, 2011.

[Leh49a] Joseph Lehner. Divisibility properties of the Fourier coefficients of the modular invariant $j(\tau)$. *Amer. J. Math.*, 71:136–148, 1949.

[Leh49b] Joseph Lehner. Further congruence properties of the Fourier coefficients of the modular invariant $j(\tau)$. *Amer. J. Math.*, 71:373–386, 1949.

[Loe07] David Loeffler. Spectral expansions of overconvergent modular functions. *Int. Math. Res. Not. IMRN*, (16):Art. ID rnm050, 17, 2007.

[Lub79] Jonathan Lubin. Canonical subgroups of formal groups. *Trans. Amer. Math. Soc.*, 251:103–127, 1979.

[LWX17] Ruochuan Liu, Daqing Wan, and Liang Xiao. The eigencurve over the boundary of weight space. *Duke Math. J.*, 166(9):1739–1787, 2017.

[MS58] N. Macon and A. Spitzbart. Inverses of Vandermonde matrices. *Amer. Math. Monthly*, 65:95–100, 1958.

[Par64] F. D. Parker. Mathematical Notes: Inverses of Vandermonde Matrices. *Amer. Math. Monthly*, 71(4):410–411, 1964.

[Pil13] Vincent Pilloni. Overconvergent modular forms. *Ann. Inst. Fourier (Grenoble)*, 63(1):219–239, 2013.

[Roe14] David Roe. The 3-adic eigencurve at the boundary of weight space. *Int. J. Number Theory*, 10(7):1791–1806, 2014.

[Ser73] Jean-Pierre Serre. Formes modulaires et fonctions zêta p-adiques. In *Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, volume Vol. 350 of *Lecture Notes in Math.*, pages 191–268. Springer, Berlin-New York, 1973.

[Sil09] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009.

[Ste07] William Stein. *Modular forms, a computational approach*, volume 79 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007. With an appendix by Paul E. Gunnells.

[Von21] Jan Vonk. Overconvergent modular forms and their explicit arithmetic. *Bull. Amer. Math. Soc. (N.S.)*, 58(3):313–356, 2021.

[Wan98] Daqing Wan. Dimension variation of classical and p-adic modular forms. *Invent. Math.*, 133(2):449–463, 1998.

[WXZ17] Daqing Wan, Liang Xiao, and Jun Zhang. Slopes of eigencurves over boundary disks. *Math. Ann.*, 369(1-2):487–537, 2017.

[Ye20] Lynnelle Ye. A modular proof of the properness of the coleman-mazur eigencurve, 2020.