Projections and sensitivities of life insurance liabilities

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Abstract

The topics of this thesis are projections and sensitivities, respectively, of life insurance liabilities. The first three chapters study projections of life insurance liabilities. We consider projections of balances in with-profit life insurance, and focus on calculation of the future bonus payments of an insurance contract. The first chapter studies the inclusion of the policyholder behavior options free-policy and surrender in the projection model. The following two chapters consider the forward-backward dependence structure that arises if we let the future bonus allocation strategy depend on prospective reserves. We study both a simulation based approach and an analytical approach to handle the interdependence. The last three chapters of the thesis consider sensitivities of life insurance liabilities. Life insurance companies are exposed to systematic insurance risks if assumptions on for instance future mortality or disability rates change. We study two methods to hedge systematic insurance risks. One is natural hedging, and we describe how to construct portfolios of different insurance products where the insurance liabilities are invariant to shifts in the underlying assumptions on for instance mortality and disability rates. Next, we study de-risking strategies where we assume there exists a market for trading securities linked to for instance mortality or disability rates, and we describe the optimization problem to find the hedging strategy that minimizes systematic insurance risks. The risk margin can be considered a measure of systematic insurance risk, and lastly, we describe how to calculate the risk margin with a scenario-based model for the future Solvency Capital Requirement.

Resumé

Denne PhD-afhandling omhandler henholdsvis projektioner og følsomheder af livsforsikringshensættelser. De første tre kapitler studerer projektioner af livsforsikringshensættelser. Vi betragter projektioner af balancen for gennemsnitsrenteprodukter, og fokuserer på beregningen af fremtidige bonusbetalinger for en forsikringskontrakt. Det første kapitel studerer inklusionen af forsikringstageroptionerne fripolice og genkøb i modellen til projektion af hensættelser. De følgende to kapitler betragter den fremadrettede-bagudrettede afhængighedsstruktur der opstår hvis beslutninger om fremtidig bonus afhænger af prospektive reserver. Vi studerer både en simulationsbaseret og en analytisk tilgang til at løse problemet. De sidste tre kapitler i afhandlingen omhandler følsomheder af livsforsikringshensættelser. Livsforsikringsselskaber er eksponeret for systematisk forsikringsrisiko, hvis antagelser om for eksempel dødeligheds- eller invaliderater ændrer sig. Vi studerer to metoder til at hedge systematisk forsikringsriskiko. En er natural hedging, og vi beskriver hvordan man konstruerer porteføljer af forskellige forsikringsprodukter, hvor hensættelserne er invariante over for forskydninger i de underliggende antagelser om for eksempel dødeligheds- og invaliderater. Derudover studerer vi de-risking strategier, hvor vi antager at der eksisterer et marked for handel med derivater, der knytter sig til for eksempel dødeligheds- og invaliderater, og vi beskriver optimeringsproblemet til at finde den optimale hedgingstrategi, der minimerer systematisk forsikringsrisiko. Risikomargen kan betragtes som et mål for systematisk forsikringsrisiko, og til sidst beskriver vi hvordan man kan regne risikomargen med en scenarie-baseret model for det fremtidige solvenskapitalkrav.

Preface

This thesis was prepared in fulfillment of the requirements for the PhD degree at the Department of Mathematical Sciences, Faculty of Science, University of Copenhagen. The work was carried out under supervision of Professor Mogens Steffensen between August 2019 and April 2023 (including 9 months of maternity leave). Research presented in this PhD thesis was part of the project "ProBaBli - Projection of Balances and Benefits in Life Insurance" funded by Innovation Fund Denmark (award number 7076-00029) with investments from the software company Edlund A/S. Edlund A/S implemented the outcome of the theoretical research in software products currently used by multiple Danish insurance companies.

The PhD thesis consists of an introduction and six manuscripts written throughout my PhD studies. The manuscripts constitute independent scientific contributions, and should be read with this in mind, and therefore discrepancies in notation across chapters appear. The introduction provides an overview of the main contributions of each chapter and the relations between the chapters to give the reader the coherent story behind the studies. I take full responsibility for any mathematical or typographical error.

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List of papers

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Contents

Chapter 1

Introduction

This PhD thesis deals with aspects of projections and sensitivities of life insurance liabilities. The Chapters 2, 3, and 4 consider projections and the last three chapters (Chapters 5, 6, and 7) consider sensitivities. In the introductory chapter, we present the background on projections of life insurance liabilities and sensitivities of life insurance liabilities, respectively, that precedes the results in the six chapters. We give an overview of the chapters and describe how they are related. The introduction only contains few references to existing literature, and for additional references, we refer to the introductions of the individual chapters, where contributions from this thesis are related to existing literature.

1.1 The classical setup in multi-state life insurance

The classical multi-state setup in life insurance is the starting point of the topics covered in this thesis; projections and sensitivities of life insurance liabilities. A life insurance contract is an agreement between an insurer and an insured, where the insured agrees to pay a premium in return of insurance benefits payed by the insurer. Future benefit and premium payments link to the future (unknown) state of life of the insured, for instance *Active*, *Disabled*, or *Dead*. Therefore, for the insurer to be able to valuate the contractual payments both for accounting purposes and for determining the premium, a model for the state of the insured is needed. A tractable and therefore popular model is the multi-state Markov model, that dates back to Hoem (1969) and Norberg (1991). Here, the state of life of the insured is described by a Markovian jump process, $Z = \big(Z(t)\big)_{t\geq 0}$, on a finite dimensional state space $\mathcal{J} = \{0, 1, ..., J\}$, where each state corresponds to a biometric state of the insured. The transition probabilities of Z are given by

$$p_{ij}(t,s) = \mathbb{P}\big(Z(s) = j \;\big|\; Z(t) = i\;\big),$$

for $i, j \in \mathcal{J}$ and $t \leq s$. Usually, the distribution of Z is described through the transition intensities

$$\mu_{jk}(t) = \lim_{h \downarrow 0} \frac{p_{jk}(t, t+h)}{h},$$

and we assume the transition intensities exist and are well-defined for $j, k \in \mathcal{J}$, $j \neq k$.

Payments of the insurance contract are associated with either the sojourn in a state or the transition between states. The sojourn payments are continuous with intensity $b^j(t)$ and the payments upon transition are given by $b^{jk}(t)$ for $j, k \in \mathcal{J}, j \neq k$. Premium payments are negative, and benefit payments are positive. The stochastic payments of the insurance contract are then described by the payment process B with dynamics

$$\mathrm{d}B(t) = b^{Z(t)}(t)\mathrm{d}t + \sum_{k \neq Z(t-)} b^{Z(t-)k}(t)\mathrm{d}N^k(t),$$

where $N^k(t)$ is a counting process counting the number of jumps of Z into state k before time t.

The expected present value of future contractual insurance payments is denoted the prospective reserve and is given by

$$V^{Z(t)}(t) = \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B(s) \,\middle|\, Z(t) \right],$$

where $r(t)$ is the interest rate and n denotes termination of the insurance contract. If the interest rate is deterministic, a classical method to calculate the state-wise reserve is as the solution to Thiele's differential equation. If the interest rate is stochastic and independent of Z, a cash flow approach for computation of the reserve is efficient.

As the title indicates, this PhD thesis covers topics on projections and sensitivities of life insurance liabilities. The prospective reserve $V^{Z(t)}(t)$ is the classical example of an insurance liability, since it is the amount the insurance company should set aside today to be able to meet the expected future payments of the insurance contract.

1.2 Projections of life insurance liabilities

The topic of the Chapters 2, 3, and 4 is projections of life insurance liabilities. Here, we give an overview of projections of insurance liabilities and describe the extensions studied in each of the three chapters.

1.2.1 With-profit life insurance

We study projections of life insurance liabilities in the context of with-profit life insurance. In with-profit life insurance, valuation of insurance liabilities is performed under two bases; one is the technical basis that consists of prudent assumptions of the interest rate and transition intensities and is denoted by r^* and μ^*_{jk}, $j, k \in \mathcal{J}, j \neq k$. The other is the market basis that consists of the model of the actual future development of the interest rate and the transition intensities and is denoted by r and μ_{jk}, $j, k \in \mathcal{J}, j \neq k$. The technical basis is deterministic and modelled in advance. The model of the market basis, and therefore also valuation of insurance liabilities under the market basis, is strongly connected to present legislation. This thesis is part of the project "Projection of Balances and Benefits in Life Insurance", and the purpose of the project is to develop methods to calculate insurance liabilities to be compliant with the Solvency II regulation. Therefore, the methods presented here relate to valuation of liabilities under the market basis.

Premiums and benefits of a with-profit life insurance contract are settled according to the principle of equivalence on the technical basis. The principle of equivalence states that the expected present value of future benefits should equal the expected present value of future premiums at initialization of the insurance contract. One can state that an insurance contract is fair if it satisfies the principle of equivalence. Still, the technical basis is chosen to be on the safe side with the result that premiums are intentionally conservative and too high, which gives rise to a surplus. If the world behaves exactly according to the technical basis, no surplus is generated. Hopefully, it turns out that the prudent technical basis is in fact prudent, which results in a positive surplus. Most of the surplus belongs to the policyholders, since they pay an abundant premium, and the rest of the surplus is a payment to the insurance company for taking on risks. In the case, where the technical basis turns out not to be prudent, an example is interest rate guarantees of up to 4% on insurance contracts signed in the 1980's and 1990's, the insurance company has to cover the loss caused by increasing liabilities, since they are contractually bounded and cannot increase the premiums.

The surplus is paid back to the policyholders in terms of bonus. A typical Danish product design is to use bonus payments to buy additional benefits for the insured, which complicates the setup, since the additional benefits themselves give rise to a surplus. One of the main tasks in a projection model of life insurance liabilities is to model the future bonus payments to get a better understanding of how bonus is distributed and to be able to calculate the balance sheet accurately.

1.2.2 The balance sheet

The liabilities of an insurance contract can be decomposed in different ways. One decomposition consists of the expected present value of future guaranteed benefits (GB), the expected present value of future discretionary benefits (FDB), and the expected future profit (FP). New legislation regarding the calculation of FDB for solvency purposes has been introduced 1/1-2023, and the purpose of the project

"Projection of Balances and Benefits in Life Insurance" has, broadly speaking, been to build a model for calculation of the FDB in accordance with the new legislation.

A widely used method for calculation of the balance sheet item GB in the Danish insurance industry is to calculate the cash flow of the expected future payments guaranteed at the time of the calculation, and then discount the cash flow with the relevant risk free interest rate. The future guaranteed payments are, up to the future state of the insured, deterministic and established methods for calculation of the cash flow exist under the assumption that Z is independent of the interest rate. The new solvency legislation requires that calculation of the FDB is performed by discounting a cash flow of the future bonus payments. This introduces the need of a model for the future distribution of bonus. The distribution of bonus is denoted the dividend allocation strategy. Decisions on how to distribute bonus are part of the management actions of the insurance company, and may depend on a lot of different factors as for instance the interest rate, the balance sheet items or the financial situation of the insurance company. Therefore, a model for the cash flow of future bonus payments requires the modeling of future management actions, and since they are complex by nature and most likely depend on the interest rate, we cannot use existing methods. Instead, we turn to simulation techniques.

The idea is to simulate paths or scenarios of the financial market, project the insurance liabilities as for instance the FDB in each scenario, and then average the relevant quantities to form Monte Carlo estimates of the insurance liabilities. Our projection model of the insurance liabilities takes an economic scenario as input and models insurance liabilities within the scenario. Therefore, in this thesis, we consider the economic scenario as given, and we do not study the generation of economic scenarios. We focus on the modelling of life insurance liabilities in with-profit life insurance including bonus, in particular possibilities and limitations in the calculation of the future bonus payments and the FDB.

1.2.3 The basic projection model

The basic projection model studied in the project "Projection of Balances and Benefits in Life Insurance" is described in Bruhn and Lollike (2020), that present a basic model for projection of insurance liabilities in an economic scenario. Here, we give a brief overview of the results of Bruhn and Lollike (2020), and describe how the Chapters 2, 3, and 4 of this thesis extend the basic projection model.

To be able to project the FDB in with-profit life insurance, we need a model of the future bonus payment under the bonus scheme additional benefits, where bonus is used to buy extra insurance benefits. The model of the insurance payments consists of two payment streams, one consists of the fixed payments, and another of the benefits that are increased by bonus. A process $Q(t)$ then measures how many extra benefits that are bought for bonus. Hence, a model of $Q(t)$ enables us to model the bonus cash flow and therefore also the FDB. The evolution of $Q(t)$ is connected to the amount of bonus the insurance company distributes, which depends on the financial scenario. The future management actions describe the amount of bonus distributed, and in the projection model, we put the future management actions on mathematical formulas to be able to project the amount of bonus distributed in each scenario.

The insurance company distributes bonus from the surplus to the policyholders' savings accounts by increasing the benefits. Therefore, a natural modeling choice is that the amount of bonus and also the future management actions depend on the surplus and the savings account. The savings account, $X(t)$, and the surplus, $Y(t)$, are retrospective in the sense, that they depend on the payments and the surplus generated up till time t, whereas the prospective reserves as for instance GB and FDB depend on payments and bonus in the future. With the bonus scheme additional benefits, the process $Q(t)$ depends linearly on the savings account, $X(t)$, and therefore it makes no difference whether we model $Q(t)$ or $X(t)$.

The future bonus payments depend on $Q(t)$ or equivalently $X(t)$, and the expected future bonus payments (the bonus cash flow) depend on the state-wise projection of $Q(t)$ (or $X(t)$). Bruhn and Lollike (2020) derives a system of forward differential equations for the state-wise projections of the savings account and the surplus in the case, where future management actions in terms of the dividend allocation strategy are linear in the savings account and the surplus. This implies that if we put up rules for the future dividend allocation strategy such that it is linear in the savings account and the surplus, we are able to calculate the state-wise projection of the savings account, and therefore also the cash flow of expected future bonus payments, in each financial scenario as the solution to a system of forward differential equations. Then, in each scenario, we discount the cash flow of the future bonus payments to obtain a value for the FDB, and by averaging the resulting values, we obtain a Monte Carlo estimate of the FDB.

1.3 Overview of the Chapters 2, 3, and 4

Bruhn and Lollike (2020) provide a basic projection model, where we are able to calculate the FDB correctly under the assumption, that the future dividend allocation strategy is linear in the savings account and the surplus. This model paves the ways for various extensions. The extensions studied in this thesis are the inclusion of the policyholder behavior options surrender and conversion to free-policy (Chapter 2), and to allow for a broader range of dividend allocation strategies that depends on prospective reserves as GB and FDB (Chapters 3 and 4). Other extensions include the study of non-linear dividend allocation strategies, how to make the projection model more efficient, and the study of optimal dividend allocation strategies.

We give an overview of and describe the main contributions of each chapter related to projection of life insurance liabilities. It is not necessarily clear, how the various chapters relate to the basic projection model, and the focus in this section is to describe, how the contributions of this thesis extend the basic projection model.

1.3.1 Chapter 2: Retrospective reserves and bonus with policyholder behavior

This chapter is a direct extension of the projection model from Bruhn and Lollike (2020). The extension concerns the inclusion of the policyholder behavior options surrender and conversion to free-policy in the projection model. We describe the extension of the model in one simulated scenario of the interest rate, and therefore, throughout the manuscript, the interest rate is assumed to be deterministic.

The surrender option allows the policyholder to cancel the insurance contract and in return receive a single benefit. With the free-policy option, the policyholder cancels all future premiums, and then the future benefits are reduced by a free-policy factor. The two policyholder behavior options are modelled as random transitions of the Markov process Z by extending the state space $\mathcal{J}$.

The projection model from Bruhn and Lollike (2020) relies on linearity of the future dividend allocation strategy in the savings account and the surplus, since then the dynamics of the two retrospective accounts, the savings account and the surplus, are linear. This is the basis of the derivation of the system of differential equations for the state-wise projection of the savings account, which enables us to calculate the future bonus cash flow. The linearity of the dynamics is preserved with the introduction of the surrender option, and under the assumption of a linear dividend allocation strategy. The free-policy option conflicts with the linearity assumption, since the optimal choice of free-policy factor is non-linear in the savings account, with the result that the payment process, and therefore also the dynamics of the savings account, are non-linear in the savings account. This implies that we are not in general able to calculate the state-wise projection of the savings account with the optimal choice of free-police factor, and hence, we cannot calculate the cash flow of future bonus payments.

In the manuscript, we consider a special case, where all benefits are regulated by bonus. In this special case, we are actually able to calculate the state-wise projection of the savings account, and also calculate the bonus cash flow, with the optimal choice of free-policy factor. The same bonus cash flow arises if we use a specific approximation of the optimal free-policy factor that is in fact linear in the savings account. Therefore, based on the special case, we consider the approximation of the optimal free-policy factor a reasonable approximation, and we derive a system of differential equations for the state-wise projection of the savings account using the approximated free-policy factor outside the special case. This allows us to calculate the bonus cash flow and the FDB including policyholder behavior under the assumption that the future dividend allocation strategy is linear in the savings account and the surplus.

1.3.2 Bonus allocation dependent on FDB

Chapters 3 and 4 consider the same extension of the basic projection model. Namely the entanglement that arises if we allow the future management actions in terms of the future dividend allocation strategy to depend on the FDB. The retrospective accounts, $X(t)$ and $Y(t)$, satisfy forward stochastic differential equations. The prospective reserves as the GB and the FDB are known to be zero at termination of the insurance contract, since at termination, there are no future payments, and they satisfy backward stochastic differential equations. The backward stochastic differential equation of the FDB depend on the savings account, since the bonus payments depend on $X(t)$. Therefore, in order to calculate the FDB and the bonus cash flow, we first solve the forward problem as the solution of the forward differential equations for the state-wise projection of $X(t)$ and then, we are able to solve the backward problem and calculate the FDB and the bonus cash flow. If the future dividend allocation strategy depends on the FDB, the solution of the forward problem (calculation of $X(t)$) depends on the solution of the backward problem (calculation of the FDB), but the solution of the backward problem depends on the solution of the forward problem. It is this entanglement, the Chapters 3 and 4 untangle; Chapter 3 studies a simulation based approach, and Chapter 4 studies an analytical approach.

1.3.3 Chapter 3: An intrinsic value approach to valuation with forward-backward loops in dividend paying stocks

In this chapter, we study a solution to the forward-backward problem in a general financial market with the same dependence structure as in the projection model with a future dividend allocation strategy that depends on the FDB. The asset $S(t)$ has dynamics

$$\mathrm{d}S(t) = g\big(t, S(t), r(t), V(t)\big)\mathrm{d}t + \sigma\big(t, S(t), r(t), V(t)\big)\mathrm{d}W(t),$$

where $W(t)$ is a Brownian motion, $r(t)$ is the stochastic interest rate, and $V(t)$ is the value of an option given by

$$V(t) = \mathbb{E}\left[\int_t^T e^{-\int_t^\tau r(s)\mathrm{d}s}\phi\big(\tau, S(\tau), r(\tau), V(\tau)\big)\mathrm{d}\tau + e^{-\int_t^T r(s)\mathrm{d}s}\Phi\big(S(T)\big)\,\middle|\,\mathcal{F}_t\right].$$

The asset, the interest rate and the value of the option stipulate a forward-backward stochastic differential equation. In relation to the projection model, the asset plays the role as the retrospective accounts, $X(t)$ and $Y(t)$, and the value of the option as the FDB. Here, the feedback of $V(t)$ in the dynamics of $S(t)$ corresponds to the feedback of the FDB in the dynamics of $X(t)$ when allowing for a FDB-dependent dividend allocation strategy.

The challenging aspect of this problem is the forward-backward structure, since it is not clear, where to start the simulation. The simulation method to solve the forward-backward problem, proposed in this chapter, bases on intrinsic values of $S(t)$ and $V(t)$. The fundamental idea is to set up a deterministic as-if market of intrinsic values, and then assume a parametric relation between the intrinsic value of the option and the option value itself to use in a forward simulation of the asset. The simulation algorithm is divided in three parts. First, the forward-backward element is handled by solving a system of ordinary forward-backward differential equations in the deterministic as-if market of intrinsic values using either a perturbation or a shooting method. Second, we assume a parametric relation between the intrinsic value of the option calculated in the first step and the option value and perform a standard forward Monte Carlo simulation of the asset for a given parameter. Third, the simulation in the second step is performed a number of times for iterated determination of the parameter. We demonstrate the simulation technique in a numerical example.

1.3.4 Chapter 4: Reserve-dependent management actions in life insurance

This chapter considers the calculation of the market reserve in with-profit life insurance in the case where future management actions including the future dividend allocation strategy depend on the market reserve itself. Since the market reserve is the sum of the GB and the FDB, this is the exact same entanglement of retrospective and prospective reserves as described in Section 1.3.2 above.

In the paper, we consider the market reserve as a function of the savings account, the surplus, and the stochastic interest rate, and derive a partial differential equation (PDE) for the market reserve. If we are able to solve the PDE for all possible values of the savings account, the surplus, and the interest rate, the market reserve can be expressed in terms of the retrospective accounts. Then, the expression can be plugged into the dividend allocation strategy, which solves the forward-backward problem. Solving the PDE on the high dimensional grid consisting of all possible values of the savings account, the surplus, and the interest rate is computationally demanding if even possible, and if possible, it is not certain, that the solution fits into the linearity assumption in the basic projection model. Therefore, we study analytical solutions of the PDE in the case, where the future dividend allocation strategy is linear in the savings account, the surplus, and the market reserve. Then the state-wise market reserve, $V^j(t,x,y,r)$ has representation

$$V^j(t,x,y,r) = h_0^j(t,r) + h_1^j(t,r) \cdot x + h_2^j(t,r) \cdot y,$$

where the functions $h_i^j(t,r)$ for $j \in \mathcal{J}$ and $i = 0,1,2$ satisfy a system of PDE's, and x is the value of the savings account and y is the value of the surplus. Hence, the linearity of the future dividend allocation strategy is inherited in the expression of the market reserve, and it fits into the basic projection model. To calculate the market reserve, we still need to solve the PDE for the h-functions, but the dimension of the PDE is significantly reduced compared to the case without linearity.

If the interest rate is deterministic, the PDE for the h-functions reduces to a system of ordinary backward differential equations. This motivates an approximation of the stochastic interest rate with the deterministic forward interest rate. Based on the approximation and the assumption of linearity of the future dividend allocation strategy, we are able to calculate the market reserve as a linear function of the savings account and the surplus where the coefficients (the h-functions) are calculated by solving ordinary differential equations, which is computationally more tractable than solving PDE's.

1.4 Sensitivities of life insurance liabilities

The topic of the Chapters 5, 6, and 7 is sensitivities of life insurance liabilities. Changes in the valuation basis (r, μ) affect the life insurance liabilities, as for instance the low interest rate environment throughout the 2010's and the accelerating decrease in mortality rates in recent times. This constitutes a risk for insurance companies, since liabilities increase, and therefore, it is of interest for risk management purposes to consider sensitivities of life insurance liabilities with respect to the valuation basis.

Life insurance liabilities are exposed to broadly speaking two types of risk: financial risks and insurance risks. Financial risks in relation to life insurance liabilities regard changes in the interest rate. Studies of and methods for hedging this type of risk are well-established in the literature of financial mathematics. The focus of this thesis is insurance risks, which are usually divided in two types: unsystematic and systematic insurance risks. We model the biometric state of the insured by the Markov chain Z, and for calculation of liabilities, we take the expected value of future payments linked to Z. Unsystematic insurance risks cover the risk that the future realized payments differ from the expected value i.e. the liabilities. This type of risk is diversifiable by the law of large numbers, since for a large insurance portfolio the future payments converge towards the expected value. The distribution of Z is modelled by specifying the transition intensities $\mu_{jk}(t)$. Systematic insurance risks cover the risks that the transition intensities of the Markov model are not as expected. An example is the accelerating decrease in the mortality intensity in recent years. Life insurance liabilities are sensitive towards changes in the transition intensities, and it constitutes a large, non-diversifiable, risk for life insurance companies. The focus of the thesis is studies and hedging of systematic insurance risks.

1.5 Overview of the Chapters 5, 6, and 7

In this section, we give an overview of the last three chapters of this thesis, and describe their relations to sensitivities of life insurance liabilities. The Chapters 5 and 6 consider methods for hedging systematic insurance risks, and the last chapter of this thesis, Chapter 7, studies calculation of the risk margin with a scenario-based model for the Solvency Capital Requirement (SCR).

1.5.1 Chapter 5: Natural hedging in continuous time life insurance

This chapter studies natural hedging of life insurance liabilities. We consider the prospective reserve (or life insurance liability) as a function of time and of the valuation basis $\mu = (\mu_{ij})_{i,j \in \mathcal{J}, i \neq j}$ and study directional shifts of the valuation basis given by $\Delta\mu$. The objective is to construct a portfolio of different insurance products such that the change in the insurance liability due to the shift in the transition intensities is neutralized, i.e.

$$V^i(t, \mu + \Delta\mu) - V^i(t, \mu) = 0. \tag{1.5.1}$$

We denote this natural hedging, and it is a method for hedging systematic insurance risk since the insurance liability is immunized to changes in the transition intensities. In the paper, we find a sufficient condition for (1.5.1) to be approximately satisfied based on a first order Taylor approximation of (1.5.1), that depends on the sensitivity of the liabilities with respect to the directional shift in the transition intensities. We use directional (Gateaux) derivatives to measure the sensitivities, and derive a system of Thiele-inspired differential equations for the Gateaux derivatives of the life insurance liabilities. Furthermore, we calculate the natural hedging strategy in two numerical examples.

As a risk management tool for systematic insurance risks, natural hedging has its limitations. The insurance market is governed by supply and demand, and the insurance company cannot require that its policyholders have a specific composition of their insurance coverages such as life insurance and disability insurance to obtain a natural hedge.

1.5.2 Chapter 6: De-risking in multi-state life insurance

In this chapter, we study an alternative to natural hedging for the hedging of systematic insurance risks. We assume that the vector of transition intensities, μ, is modelled by a diffusion process, that liabilities are evaluated under a deterministic basis $\hat{\mu}$, and that there exists a market for trading two types of μ-linked securities, the de-risking option and the de-risking swap. The life insurance company is exposed to systematic insurance risk if the development of μ differs from the valuation basis, $\hat{\mu}$, and may reduce this risk by investing in μ-linked securities. We develop a model that quantifies the systematic insurance risk, and describe the optimization problem the insurance company faces to choose the amount of de-risking that minimizes risk.

The approach for hedging systematic insurance risk presented in this chapter is unrealistic in the sense that in reality there exist very few μ-linked securities and they are all linked to the mortality intensity. The purpose of the model is to quantify systematic insurance risk in a multi-state model and to study the sensitivity of the optimal choice of de-risking to various parameters of the model.

1.5.3 Chapter 7: Risk margin calculations with a scenario-based model for the Solvency Capital Requirement

The Solvency II legislation states that insurance companies should report the risk margin as part of their balance sheet. The risk margin is the amount another company would be expected to require to take over the insurance obligation for taking over systematic insurance risks. Therefore, the risk margin can be considered measure of systematic insurance risks of the life insurance liabilities.

The Solvency II legislation suggests a cost-of-capital formula to calculate the risk margin that depends on the *SCR* for all future years. Calculation of the future *SCR* using the standard model is cumbersome and computationally challenging, and therefore, the legislation suggests four approximation methods to calculate the risk margin. In the paper, we compare the approximation methods and suggest a scenario-based model for the *SCR* to use in the cost-of-capital formula. The scenario-based model bases on a stress on the transition intensities, μ^{ε}, and the *SCR* is the difference between the stressed and the unstressed liabilities

$$SCR^i(t) = V^i(t, \mu^{\varepsilon}) - V^i(t, \mu).$$

The *SCR* above is similar to the natural hedging condition in Equation (1.5.1). One connection is that the risk margin for an insurance portfolio where the natural hedging condition is satisfied should be equal to zero since systematic insurance risk is eliminated by natural hedging. Therefore, the *SCR* in the scenario-based model to use for calculation of the risk margin is also equal to zero if the natural hedging condition is satisfied. In the paper, we discuss the choice of stress on the transition intensities, and describe the calculation of the risk margin in the scenario-based model.

Chapter 2

Retrospective reserves and bonus with policyholder behavior

This chapter contains the paper *Falden and Nyegaard (2021).*

Abstract

Legislation imposes insurance companies to project their assets and liabilities in various financial scenarios. Within the setup of with-profit life insurance, we consider retrospective reserves and bonus, and we study projection of balances with and without policyholder behavior. The projection resides in a system of differential equations of the savings account and the surplus, and the policyholder behavior options surrender and conversion to free-policy are included. The inclusion results in a structure where the system of differential equations of the savings account and the surplus is non-trivial. We consider a case, where we are able to find accurate differential equations and suggest an approximation method to project the savings account and the surplus including policyholder behavior in general. To highlight the practical applications of the results in this paper, we study a numerical example.

Keywords: With-profit life insurance; Bonus; Surplus; Dividends; Projection of balances; Retrospective reserve; Policyholder behavior.

2.1 Introduction

In with-profit life insurance, prudent assumptions about the interest rate and biometric risks at initialization of an insurance contract result in a surplus emerging over time. This surplus belongs to the policyholders and must be paid back in terms of bonus. The redistribution of bonus contains certain degrees of freedom, which is part of the *Management Actions*. Furthermore, bonus must be taken into account when insurance companies determine their assets and liabilities. Legislation imposes insurance companies to project their balance sheet, and companies must be able to perform projections of assets and liabilities in a number of scenarios of the financial market. This requires a specification of the future dividend strategy and, in general, a specification of the *Future Management Actions*. Management actions may depend on the financial scenario, the present as well as the past entries of the balance sheet and their relations, and other aspects of the financial situation of the insurance company. Therefore, future management actions have a complex nature and are difficult to predict and formalize mathematically. In this paper, we model the projection of the savings account and the surplus of an insurance contract, where we assume the future dividend strategy has a simple structure. How the dividend strategy is designed in practise to fit the model is beyond the scope of this paper, but the model establishes a foundation for projecting balances in life insurance. In the projection model, biometric risks play an important role as well. We model the state of the policyholder using a Markov model, and study state-wise projections of the savings account and the surplus.

The modeling of surplus and bonus in life insurance is not new. Norberg (1999) introduces the individual surplus of a life insurance contract, and Steffensen (2006) derives differential equations for prospective reserves in the case, where dividends are linked to the surplus. In our model, we also consider dividends linked to the surplus, but distinct from Steffensen (2006), we derive differential equations for the projected savings account and surplus. Jensen and Schomacker (2015) study the valuation of an insurance contract with the bonus scheme spoken of as additional benefits, where dividends are used to buy more insurance, in a scenario-based model for the financial market. Our paper has some similarities with Jensen and Schomacker (2015) in the sense that we also study a scenario-based model with additional benefits. In Jensen and Schomacker (2015) the bonus allocation is discretized, while we allocate bonus continuously, resulting in difference equations in Jensen and Schomacker (2015) and ordinary differential equations in our model. Furthermore, we study state-wise projections of the savings account and the surplus, whereas Jensen and Schomacker (2015) study the expected savings account and the expected surplus.

Steffensen (2006) considers prospective reserves, while we focus on the savings account, which is a retrospective reserve including past bonus, and the surplus of an insurance contract. The retrospective approach without bonus is studied in Norberg (1991) and studied with bonus in Asmussen and Steffensen (2020). Bruhn and Lollike (2020) also reflect on the retrospective perspective, and study retrospective reserves with and without bonus. They model the savings account and the surplus of an insurance contract, and derive differential equations for the state-wise projections. The retrospective approach is practicable when considering projection of liabilities in various financial scenarios, since the retrospective reserves depend on the past interest rate, whereas prospective reserves depend on the unknown future interest rate.

This paper serves as an extension to Bruhn and Lollike (2020). The extension resides in the incorporation of the policyholder behavior options surrender and conversion to free-policy. Upon surrender, the policyholder receives a single payment and all future payments cancel, and with the free-policy option, all future premiums cancel and benefits are reduced by a free-policy factor. We model policyholder behavior as random transitions in the Markov model from the classical life insurance setup extended with surrender and free-policy states as studied in for instance Henriksen et al. (2014). This is in contrast to modeling rational policyholder behavior as in Steffensen (2002). Buchardt and Møller (2015) study the calculation of prospective reserves without bonus including policyholder behavior using a cash flow approach, and Buchardt, Møller, and Schmidt (2014) consider the inclusion of policyholder behavior in semi-Markov models. A general extension of the concepts to non-Markovian models is studied in Christiansen and Djehiche (2020), where in addition payments are allowed to depend on prospective reserves. In our model, payments depend on the retrospective savings account. In Ahmad, Buchardt, and Furrer (2022), they study a setup similar to ours with bonus and policyholder behavior, but they are included separately. We include policyholder behavior options in combination with bonus in our model of the retrospective savings account and surplus, and our approach is based on differential equations of the state-wise projections. Buchardt and Møller (2015) introduce the notion of modified probabilities to calculate prospective reserves including conversion to free-policy. The same modified probabilities appear in our system of differential equations for the state-wise projections of the savings account and the surplus.

We propose here a framework for the projection of liabilities in various financial scenarios with a general model of the future management actions, among these the redistribution of bonus. Furthermore, any policyholder response to the financial market and the savings account and the surplus can be implemented in our framework. Other papers derive or suggest specific rules for management and/or policyholder decision making. In both financial and actuarial literature, optimization of life insurance payments are discussed, typically from an individual point of view over the life cycle. Seminal works are Richard (1975) and Campbell (1980), but the area continues to attract interest, see for instance Chen et al. (2006), Chiappori et al. (2006), and Kraft and Steffensen (2008). Browne and Kim (1993) discuss life insurance demand from a macroeconomic perspective, and Nielsen (2005) considers optimal distribution of surplus on a corporate level. Modeling or derivation of optimal policyholder behavior is a recurrent topic in actuarial literature. De Giovanni (2010) models surrender risk adapted to the financial market, and the modeling and statistical examination of surrender on macroeconomic conditions are studied in for instance Loisel and Milhaud (2011) and Barsotti, Milhaud, and Salhi (2016). The modeling of free-policy behavior is most often assumed random and uncorrelated across the portfolio, see for instance Henriksen et al. (2014) and Buchardt and Møller (2015).

In Section 2.2, we present the general life insurance setup and the model of the savings account, the surplus, and the dividends. We define the projection of the savings account and the surplus without policyholder behavior and state the results from Bruhn and Lollike (2020) in Section 2.3. Section 2.4 extends the setup from Section 2.2 to include policyholder behavior. Section 2.5 consists of the key results in this paper. We consider the ideal free-policy factor in our retrospective setup including bonus, but this free-policy factor does not satisfy the simple structure of the model in Section 2.3. Therefore, the result concerning the projection of the savings account and the surplus in Section 2.3 does not apply with the ideal free-policy factor. We consider the case with all benefits regulated by bonus. In this case, we show that we actually can project the savings account and the surplus with the ideal free-policy factor. Furthermore, we suggest an approximation of the free-policy factor, for which the state-wise projections of the savings account and the surplus coincide with the state-wise projections using the ideal free-policy factor. This is one of the two main results of the paper. The second main result is a method to project the savings account and the surplus with the approximated free-policy factor in a general case. In Section 2.6, we present a numerical example to emphasize the practical applications of our results. Section 2.7 concludes the paper.

2.2 Life Insurance Setup

The classic multi-state setup in life insurance is taken as a starting point, and we extend this with policyholder behavior in Section 2.4. A Markov process, $Z = \big(Z(t)\big)_{t\geq 0}$, in a finite state space $\mathcal{J}^\circ = \{0, 1, ..., J-1\}$ describes the state of the holder of a life insurance contract, and payments in the contract link with sojourns in states and transitions between states. The transition probabilities of Z are

$$p_{ij}(s,t) = \mathbb{P}\big(Z(t) = j \bigm| Z(s) = i\big),$$

for $i, j \in \mathcal{J}^\circ$ and $s \leq t$. We assume that the transition intensities

$$\mu_{ij}(t) = \lim_{h\downarrow 0} \frac{1}{h} p_{ij}(t, t+h),$$

exist for $i, j \in \mathcal{J}^\circ$, $i \neq j$.

The transition probabilities satisfy the Kolmogorov's differential equations (see for instance Buchardt and Møller (2015) Proposition 4).

The processes $N^k(t)$ for $k \in \mathcal{J}^\circ$ count the number of jumps of Z into state k up to time t.

$$N^k(t) = \#\{s \in (0,t] \mid Z(s-) \neq k, Z(s) = k\},$$

where $Z(s-) = \lim_{h \downarrow 0} Z(s-h)$.

We consider with-profit life insurance products, where payments specified in the contract are based on prudent assumptions about interest rate and transition intensities. These assumptions are called the technical basis, and denoted by (r^*, μ^*_{ij}) for $i, j \in \mathcal{J}^\circ$, $i \neq j$. The market basis models the actual development of the interest rate and transition intensities of the insurance portfolio. The market basis is denoted by (r, μ_{ij}) for $i, j \in \mathcal{J}^\circ$, $i \neq j$. The market interest rate is stochastic, and practice is to simulate a number of scenarios of the interest rate and study the projection model in each scenario, as we do in the numerical simulation study in Section 2.6. Available information about the market interest rate is represented by the filtration $\mathcal{F}^r = \left(\mathcal{F}^r_t\right)_{t \geq 0}$, where $\mathcal{F}^r_t = \sigma\big(r(s) \big| 0 \leq s \leq t\big)$. We assume the market transition intensities are deterministic.

Due to the prudent technical basis, a surplus arises, which by legislation is to be paid back to the policyholders as bonus. We use the bonus scheme spoken of as additional benefits, where bonus is used to buy more insurance. This is denoted as defined contributions since premiums are fixed and benefits are increased by bonus in contrast to defined benefits, where bonus is used to lower premiums and benefits are fixed.

The accumulated payments of an insurance contract is decomposed into two payment streams; one that contains the payments not regulated by bonus, B_1, and one that contains the profile of payments regulated by bonus, B_2, as presented in Asmussen and Steffensen (2020). An example is an insurance contract consisting of a life annuity and a term insurance. Often only the life annuity is scaled by bonus and the term insurance as well as the premiums are fixed. Then the payment stream B_1 consists of the term insurance and the premiums, and the payment stream B_2 consists of the life annuity.

The dynamics of the payment streams are in the following form for $i = 1, 2$

$$\mathrm{d}B_i(t) = b_i^{Z(t)}(t)\mathrm{d}t + \sum_{k: k \neq Z(t-)} b_i^{Z(t-)k}(t)\mathrm{d}N^k(t), \tag{2.2.1}$$

where $b_i^j(t)$ denotes the payment rate during sojourn in state j and $b_i^{jk}(t)$ the single payment upon transition from state j to state k at time t. The payment functions $b_i^j(t)$ and $b_i^{jk}(t)$ are assumed to be deterministic and sufficiently regular. For notational convenience, we disregard lump sum payments at fixed time points during sojourn of states, even though it does not impose mathematical difficulties.

Definition 2.2.1. *The prospective technical reserve at time $t \leq n$ for payment stream $\mathrm{d}B_i(t)$, $i = 1, 2$ is given by*

$$V_i^{*Z(t)}(t) = \mathbb{E}^* \left[\int_t^n e^{-\int_t^s r^*(u)\mathrm{d}u} \mathrm{d}B_i(s) \,\middle|\, Z(t) \right],$$

where n denotes termination of the contract and $\mathbb{E}^*$ means that the technical transition intensities, μ^*_{jk}, $j,k \in \mathcal{J}^\circ$, $j \neq k$, are used in the distribution of Z.

Since the technical interest rate and transition intensities are determined at initialization of the insurance contract and therefore known for all $t \in [0,n]$, the prospective technical reserves are deterministic conditional on $Z(t) = j$. The principle of equivalence states that $V_1^{*Z(0)}(0) + V_2^{*Z(0)}(0) = 0$.

2.2.1 The Savings Account, the Surplus and the Dividends

Similar to Asmussen and Steffensen (2020), the surplus is returned to the insured through a dividend payment stream D. A process $Q(t)$ denotes the number of payment processes B_2 bought up to time t. Additional benefits are bought under the technical basis, and as dividends are used to buy $B_2(t)$ at the price of $V_2^*(t)$, we must have that

$$\mathrm{d}D^{Z(t)}(t) = \mathrm{d}Q(t)V_2^{*Z(t)}(t). \tag{2.2.2}$$

The policyholder experiences the total payment process with dynamics

$$\mathrm{d}B(t) = \mathrm{d}B_1(t) + Q(t-)\mathrm{d}B_2(t),$$

which is the payment process guaranteed at time t. A decreasing Q results in decreasing guaranteed benefits, which from a practical point-of-view is unreasonable. A negative value of Q results in benefit payments from the insured to the insurance company which is unrealistic. We do not require that Q is non-decreasing or that Q is non-negative in this setup in order to obtain a simple mathematical model.

The savings account of an insurance contract is denoted by $X(t)$, and it is the technical value of future payments guaranteed at time $t \geq 0$, i.e. the following relation between $X(t)$ and $Q(t)$ holds

$$X(t) = V_1^{*Z(t)}(t) + Q(t-)V_2^{*Z(t)}(t) \Leftrightarrow Q(t-) = \frac{X(t) - V_1^{*Z(t)}(t)}{V_2^{*Z(t)}(t)}.$$

The savings account is equal to zero at the beginning of the insurance contract, $X(0-) = 0$. Then by the principle of equivalence, $V_1^{*Z(0-)}(0-) + V_2^{*Z(0-)}(0-) = 0$, the initial condition $Q(0-) = 1$ holds.

Due to the relationship between X and Q, the payment process, $\mathrm{d}B(t)$, is a linear function in X

$$\mathrm{d}B(t, X(t)) = b^{Z(t)}(t, X(t))\mathrm{d}t + \sum_{k:k\neq Z(t-)} b^{Z(t-)k}(t, X(t-))\mathrm{d}N^k(t), \tag{2.2.3}$$

where

$$b^j(t,x) = b_1^j(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} b_2^j(t),$$
$$b^{jk}(t,x) = b_1^{jk}(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} b_2^{jk}(t).$$

Proposition 2.2.2. *The savings account, X, has dynamics*

$$\begin{aligned}\mathrm{d}X(t) &= r^*(t)X(t)\mathrm{d}t - \mathrm{d}B(t, X(t)) + \mathrm{d}D^{Z(t)}(t) \\ &\quad + \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t, X(t-))\big(\mathrm{d}N^k(t) - \mu^*_{Z(t-)k}(t)\mathrm{d}t\big),\end{aligned}$$

where the sum-at-risk is given by

$$R^{*jk}(t,x) = b^{jk}(t,x) + \chi^{jk}(t,x) - x,$$

and

$$\chi^{jk}(t,x) = V_1^{*k}(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} V_2^{*k}(t),$$

is the technical value of guaranteed payments after the transition from state j to state k.

Proof. See Asmussen and Steffensen (2020), Chapter 6.7. □

The surplus $Y(t)$ is the difference between past premiums less benefits over time $[0,t]$ accumulated with the market interest rate and the savings account at time t.

$$Y(t) = -\int_0^t e^{\int_s^t r(u)\mathrm{d}u} \mathrm{d}B(s, X(s)) - X(t).$$

The market interest rate over time $[0,t]$ is known at time t such that $Y(t)$ only depends on the market interest rate prior to time t, and $Y(0-) = 0$.

Proposition 2.2.3. *The surplus, Y, has dynamics*

$$\begin{aligned}\mathrm{d}Y(t) &= r(t)Y(t)\mathrm{d}t - \mathrm{d}D^{Z(t)}(t) + c^{Z(t)}(t, X(t))\mathrm{d}t \\ &\quad - \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t, X(t-))\big(\mathrm{d}N^k(t) - \mu_{Z(t-)k}(t)\mathrm{d}t\big),\end{aligned}$$

where the surplus contribution is given by

$$c^j(t,x) = (r(t) - r^*(t))x + \sum_{k:k\neq j} R^{*jk}(t,x)(\mu^*_{jk}(t) - \mu_{jk}(t)).$$

Proof. See Asmussen and Steffensen (2020), Chapter 6.7. □

We assume that the technical basis is prudent compared to the market basis such that the surplus contribution, $c^j(t,x)$, is non-negative. A prudent technical basis chosen several years ago may not be prudent today due to the current low interest rate environment and therefore the interest rate part of the surplus contribution may be negative resulting in a possibly negative surplus. In practice, a negative surplus would be covered by the equity of the insurance company, but in this setup, we allow the surplus to be negative.

The dividend payments stream, $\mathrm{d}D^{Z(t)}(t)$, describes how the surplus is returned to the insured. We assume that the dividend process is continuous and depends on the savings account and the surplus, such that the dynamics are

$$\mathrm{d}D^{Z(t)}(t) = \delta^{Z(t)}(t, X(t), Y(t))\mathrm{d}t.$$

The dynamics of the savings account and the surplus are affine if and only if the dividend process is. The main results of this paper rely on affinity in the dynamics of the savings account and the surplus, and therefore we make the assumption that the dividend process is affine in $X(t)$ and $Y(t)$

$$\delta^j(t,x,y) = \delta_0^j(t) + \delta_1^j(t) \cdot x + \delta_2^j(t) \cdot y, \tag{2.2.4}$$

for sufficiently regular and deterministic functions δ_0^j, δ_1^j and δ_2^j, $j \in \mathcal{J}^\circ$. This is a restriction in the degree of freedom in the dividend allocation strategy of the insurance companies, and therefore of the future management actions in the model. How the dividend strategy is chosen in practise to cope with our model is beyond the scope of this paper, but other papers derive specific rules for management actions and agents behavior, see for instance Nielsen (2005), Chen et al. (2006), and Kraft and Steffensen (2008). The restriction that the dividends are affine may lead to negative dividends, which results in a decreasing Q and that the insurance company lowers the guaranteed benefits. From a practical point-of-view this is unreasonable, but affine dividends turn out to be mathematical tractable, and therefore we make the assumption of affine dividends in our model. The user of the model must be aware of the possibility of negative dividends.

2.3 State-wise Projections without Policyholder Behavior

In order to satisfy legislation, insurance companies and present research focus on the projection of balances in life insurance using simulation methods. Both the savings account, X, and the surplus, Y, are entries of the balance sheet, and in order to project these, we simulate scenarios of the interest rate and study the projection of the savings account and the surplus in each scenario. To account for the biometric risks, one approach is to use simulation methods. In practice, it can be computational heavy to simulate the biometric history of an entire insurance portfolio, and therefore we study state-wise projections to eliminate the biometric part of the simulation.

Definition 2.3.1. *The state-wise projections of the savings account, X, and the surplus, Y, are*

$$\tilde{X}^j(t) = \mathbb{E}_{Z(0)}\Big[\mathbb{1}_{\{Z(t)=j\}}X(t) \,\Big|\, \mathcal{F}_t^r\Big],$$
$$\tilde{Y}^j(t) = \mathbb{E}_{Z(0)}\Big[\mathbb{1}_{\{Z(t)=j\}}Y(t) \,\Big|\, \mathcal{F}_t^r\Big],$$

for $j \in \mathcal{J}^\circ$. The subscript $Z(0)$ denotes that the expectation is the conditional expectation given $Z(0)$. The expectation is taken under the market basis conditional on $Z(0)$ and the interest rate filtration at time t. Therefore, the market interest rate is known up to and including time t, but information about the state process Z is only known at time 0.

Bruhn and Lollike (2020) derive differential equations for the state-wise projections of the savings account and the surplus from Definition 2.3.1 to use for projection in a given interest rate scenario. The theorem below states the main result of Bruhn and Lollike (2020), and the purpose of this paper is to extend these differential equations to a setup including policyholder behavior.

Lemma 2.3.2. *The dynamics of the savings account, X, from Proposition 2.2.2 and the dynamics of the surplus, Y, from Proposition 2.2.3 are in the form*

$$\begin{aligned}
\mathrm{d}X(t) &= \Big(\alpha_{0,X}^{Z(t)}(t) + \alpha_{1,X}^{Z(t)}(t)X(t) + \alpha_{2,X}^{Z(t)}(t)Y(t)\Big)\mathrm{d}t \\
&\quad + \sum_{k:k\neq Z(t-)} \Big(\lambda_{0,X}^{Z(t-)k}(t) + \lambda_{1,X}^{Z(t-)k}(t)X(t-)\Big)\mathrm{d}N^k(t), \\
\mathrm{d}Y(t) &= \Big(\alpha_{0,Y}^{Z(t)}(t) + \alpha_{1,Y}^{Z(t)}(t)X(t) + \alpha_{2,Y}^{Z(t)}(t)Y(t)\Big)\mathrm{d}t \\
&\quad + \sum_{k:k\neq Z(t-)} \Big(\lambda_{0,Y}^{Z(t-)k}(t) + \lambda_{1,Y}^{Z(t-)k}(t)X(t-)\Big)\mathrm{d}N^k(t),
\end{aligned}$$

for deterministic functions $\alpha_{i,H}^j$ and $\lambda_{i,H}^{jk}$ for $i = 0, 1, 2$, $H = X, Y$ and $j, k \in \mathcal{J}^\circ$, $j \neq k$.
See Appendix 2.A for the expressions of α and λ for the savings account and the surplus.

Theorem 2.3.3. *Let X and Y have dynamics in the form of Lemma 2.3.2. Then the state-wise projections of X and Y from Definition 2.3.1 satisfy the following* *system of ordinary differential equations*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{X}^j(t) = &\sum_{k:k\neq j}\mu_{kj}(t)\tilde{X}^k(t) - \sum_{k:k\neq j}\mu_{jk}(t)\tilde{X}^j(t) \\
&+ \alpha^j_{0,X}(t)p_{Z(0)j}(0,t) + \alpha^j_{1,X}(t)\tilde{X}^j(t) + \alpha^j_{2,X}(t)\tilde{Y}^j(t) \\
&+ \sum_{k:k\neq j}\mu_{kj}(t)\Big(\lambda^{kj}_{0,X}(t)p_{Z(0)k}(0,t) + \lambda^{kj}_{1,X}(t)\tilde{X}^k(t)\Big), \\
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Y}^j(t) = &\sum_{k:k\neq j}\mu_{kj}(t)\tilde{Y}^k(t) - \sum_{k:k\neq j}\mu_{jk}(t)\tilde{Y}^j(t) \\
&+ \alpha^j_{0,Y}(t)p_{Z(0)j}(0,t) + \alpha^j_{1,Y}(t)\tilde{X}^j(t) + \alpha^j_{2,Y}(t)\tilde{Y}^j(t) \\
&+ \sum_{k:k\neq j}\mu_{kj}(t)\Big(\lambda^{kj}_{0,Y}(t)p_{Z(0)k}(0,t) + \lambda^{kj}_{1,Y}(t)\tilde{X}^k(t)\Big),
\end{aligned}
$$

and $\tilde{X}^j(0-) = \tilde{Y}^j(0-) = 0$ *for* $j \in \mathcal{J}^\circ$.

Proof. See Bruhn and Lollike (2020). □

Kolmogorov's forward differential equations can be used to calculate the transition probabilities in Theorem 2.3.3.

2.4 Life Insurance Setup Including Policyholder Behavior

Now, we extend the setup from Section 2.2 to include policyholder behavior. We include the policyholder behavior options surrender and conversion to free-policy. Upon surrender, the policyholder receives a single payment and all future payments cancel. With the free-policy option, all future premiums cancel, and benefits are reduced by a free-policy factor, f, that depends on the time at which the policyholder goes from premium paying to free-policy. We study how the introduction of policyholder behavior affects the dynamics of the savings account, X, from Proposition 2.2.2 and the surplus, Y, from Proposition 2.2.3. The objective is to be able to perform state-wise projections of the savings account and the surplus including policyholder behavior.

Policyholder behavior is modelled in the classic way by extending the state space of the Markov chain, Z, to include surrender and free policy states as presented in Henriksen et al. (2014), and the state space of Z from Section 2.2 is extended as illustrated in Figure 2.1. We do not consider the modeling or derivation of the surrender rate and the free-policy rate. The modeling of optimal surrender rates is studied in for instance De Giovanni (2010), Loisel and Milhaud (2011), and Barsotti, Milhaud, and Salhi (2016), but little attention has been paid in existing literature to the choice of free-policy rate, which is often modelled as a deterministic intensity as in Henriksen et al. (2014) and Buchardt and Møller (2015). The extension of the

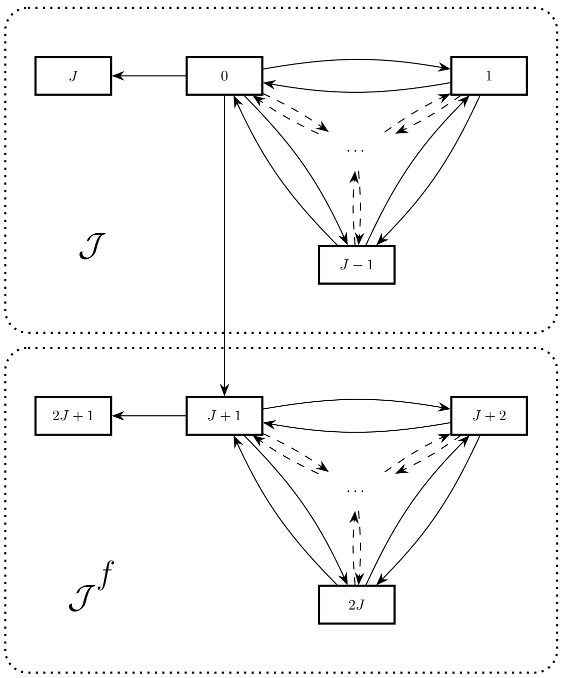

Figure 2.1: *Multi-state model including policyholder behavior options*

state space in Figure 2.1 can also be obtained as a specific case of the more general state space expansion in Christiansen and Djehiche (2020).

The state J corresponds to surrender, and we assume that surrender can only happen from state 0. The state space $\mathcal{J}^f$ denotes the free-policy states, and it is a copy of $\mathcal{J}$ in the sense that it holds the same number of states and that state $i \in \mathcal{J}^f$ corresponds to state $i - (J+1) \in \mathcal{J}$. We assume that conversion to free-policy can only occur from state 0 and that the transition intensities in $\mathcal{J}^f$ equal the transition intensities in $\mathcal{J}$. We assume throughout the rest of this paper that $Z(0) \in \mathcal{J}$. The classical 7-state model from for example Buchardt and Møller (2015) is contained in this setup, where state 0 in our model corresponds to the premium-paying active state.

In order to model payments including policyholder behavior, the payment streams from Equation (2.2.1) are decomposed in benefits, $\mathrm{d}B_i^+(t)$ and premiums, $\mathrm{d}B_i^-(t)$ for $i = 1, 2$. The sojourn payments and payments upon transition are then decomposed in b_i^{j+} and b_i^{j-}, and b_i^{jk+} and b_i^{jk-} respectively. We consider defined contributions such that the payment stream increased by bonus only contains benefits i.e. $b_2^{j-}(t) = b_2^{jk-}(t) = 0$ for all $t \geq 0$ and $j, k \in \mathcal{J}$, $j \neq k$.

The technical benefit and premium reserves respectively in the non-free-policy states,

$Z(t) \in \mathcal{J}$, are given by

$$V_i^{*Z(t)\pm}(t) = \mathbb{E}^* \left[\int_t^n e^{-\int_t^s r^*(u)\mathrm{d}u} \mathrm{d}B_i^{\pm}(s) \,\middle|\, Z(t) \right],$$

for $i = 1, 2$, and where n is termination of the insurance contract.

Defined contributions imply that

$$V_2^{*Z(t)}(t) = V_2^{*Z(t)+}(t) + V_2^{*Z(t)-}(t) = V_2^{*Z(t)+}(t), \tag{2.4.1}$$

for $Z(t) \in \mathcal{J}$.

The duration U in the free-policy states is

$$U(t) = \inf\{s \in [0, t] \mid Z(t - s) \in \mathcal{J}\}.$$

Payments in the free-policy states equal a free-policy factor, $f \in [0, 1]$, times the benefits in the corresponding premium-paying state. We allow the free-policy factor to depend on the savings account, i.e. $f(t, X(t))$, and the benefits are reduced with the free-policy factor evaluated at the time of conversion to free-policy, $f(t - U(t), X(t - U(t)))$. We introduce the mapping of $Z(t)$ that returns the corresponding premium-paying state if $Z(t) \in \mathcal{J}^f$

$$g\big(Z(t)\big) = \mathbb{1}_{\{Z(t) \in \mathcal{J}^f\}}\big(Z(t) - (J + 1)\big).$$

Policyholder behavior is modelled solely on the market basis, and therefore $\mu_{0J}^*(t) = \mu_{0(J+1)}^*(t) = 0$ for all $t \geq 0$. The remaining transition intensities in $\mathcal{J}^f$ equal the corresponding transition intensities in $\mathcal{J}$ on the technical basis. Hence, the technical reserve in a free-policy state equals the free-policy factor times the technical benefit reserve in the corresponding premium-paying state

$$V_i^{*Z(t)}(t) = f(t - U(t), X(t - U(t)))V_i^{*g(Z(t))+}(t),$$

for $i = 1, 2$ and $Z(t) \in \mathcal{J}^f$.

The inclusion of policyholder behavior changes the payment process from Equation (2.2.3) and the sum-at-risk from Proposition 2.2.2. Now, the payment process and the sum-at-risk depend on time, the savings account, and the duration in the free-policy states.

Proposition 2.4.1. *The total payment process guaranteed at time t including policyholder behavior is*

$$\begin{aligned}
&\mathrm{d}B(t, X(t),\ U(t),\ X(t - U(t))) \\
&\qquad = b^{Z(t)}(t, X(t),\ U(t),\ X(t - U(t)))\mathrm{d}t \\
&\qquad\quad + \sum_{k: k \neq Z(t-)} b^{Z(t-)k}(t, X(t-),\ U(t),\ X((t - U(t))-))\mathrm{d}N^k(t),
\end{aligned}$$

where the continuous payment function during sojourns in states and the payment function upon transition between states are

$$
\begin{aligned}
&b^j(t,x,u,x^f) \\
&\quad = \mathbb{1}_{\{j\in\mathcal{J}\}}\Big(b_1^j(t)+\frac{x-V_1^{*j}(t)}{V_2^{*j+}(t)}b_2^j(t)\Big) \\
&\qquad + \mathbb{1}_{\{j\in\mathcal{J}^f\}}\Big(f(t-u,x^f)b_1^{g(j)+}(t)+\frac{x-V_1^{*g(j)+}(t)f(t-u,x^f)}{V_2^{*g(j)+}(t)}b_2^{g(j)+}(t)\Big),
\end{aligned}
$$

$$
\begin{aligned}
&b^{jk}(t,x,u,x^f) \\
&\quad = \mathbb{1}_{\{j,k\in\mathcal{J},\ j\neq k\}}\Big(b_1^{jk}(t)+\frac{x-V_1^{*j}(t)}{V_2^{*j+}(t)}b_2^{jk}(t)\Big) \\
&\qquad + \mathbb{1}_{\{j,k\in\mathcal{J}^f,\ j\neq k\}}f(t-u,x^f)b_1^{g(j)g(k)+}(t) \\
&\qquad + \mathbb{1}_{\{j,k\in\mathcal{J}^f,\ j\neq k\}}\frac{x-V_1^{*g(j)+}(t)f(t-u,x^f)}{V_2^{*g(j)+}(t)}b_2^{g(j)g(k)+}(t),
\end{aligned}
$$

for $j,k\in\mathcal{J}\cup\mathcal{J}^f$, $j\neq k$. We assume that there are no continuous payments in the surrender states, and that there is no payment upon transition between $\mathcal{J}$ and $\mathcal{J}^f$.

Proposition 2.4.2. *Including policyholder behavior, the sum-at-risk from Proposition 2.2.2 is*

$$
\begin{aligned}
&R^{*jk}(t,x,u,x^f) \\
&\quad = b^{jk}(t,x,u,x^f) \\
&\qquad + \mathbb{1}_{\{j,k\in\mathcal{J},\ j\neq k\}}\bigg(V_1^{*k}(t)+\frac{x-V_1^{*j}(t)}{V_2^{*j+}(t)}V_2^{*k+}(t)-x\bigg) \\
&\qquad + \mathbb{1}_{\{j,k\in\mathcal{J}^f,\ j\neq k\}}V_1^{*g(k)+}(t)f(t-u,x^f) \\
&\qquad + \mathbb{1}_{\{j,k\in\mathcal{J}^f,\ j\neq k\}}\bigg(\frac{x-V_1^{*g(j)+}(t)f(t-u,x^f)}{V_2^{*g(j)+}(t)}V_2^{*g(k)+}(t)-x\bigg) \\
&\qquad + \mathbb{1}_{\{j=0,\ k=J+1\}}\Big(V_1^{*g(k)+}(t)f(t,x)+\frac{x-V_1^{*j}(t)}{V_2^{*j+}(t)}V_2^{*g(k)+}(t)f(t,x)-x\Big).
\end{aligned}
$$

The last line corresponds to the sum-at-risk upon conversion to free-policy, where $u=0$.

Remark 2.4.3. In the last line of the sum-at-risk from Proposition 2.4.2, $g(k)=g(J+1)=0=j$, and by Equation (2.4.1), the sum-at-risk upon conversion to free-policy is

$$\big(x-V_1^{*0-}(t)\big)f(t,x)-x.$$

The dynamics of the savings account, X, and the surplus, Y, including policyholder behavior are equal to the dynamics in Proposition 2.2.2 and Proposition 2.2.3, where the payment process and the sum-at-risk are given by Proposition 2.4.1 and Proposition 2.4.2. Thus, the dynamics of the savings account are

$$\begin{aligned} \mathrm{d}X(t) = r^*(t)X(t)\mathrm{d}t - \mathrm{d}B(t, X(t), U(t), X(t-U(t))) + \delta^{Z(t)}(t, X(t), Y(t))\mathrm{d}t \\ + \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t, X(t-), U(t-), X((t-U(t-))-)) \\ \times \left(\mathrm{d}N^k(t) - \mu^*_{Z(t-)k}(t)\mathrm{d}t\right), \end{aligned} \tag{2.4.2}$$

and the dynamics of the surplus are

$$\begin{aligned} \mathrm{d}Y(t) = r(t)Y(t)\mathrm{d}t - \delta^{Z(t)}(t, X(t), Y(t))\mathrm{d}t + c^{Z(t)}(t, X(t), U(t), X(t-U(t)))\mathrm{d}t \\ - \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t, X(t-), U(t-), X((t-U(t-))-)) \\ \times \left(\mathrm{d}N^k(t) - \mu_{Z(t-)k}(t)\mathrm{d}t\right), \end{aligned} \tag{2.4.3}$$

where the surplus contribution is given by

$$c^j(t, x, u, x^f) = (r(t) - r^*(t))x + \sum_{k:k\neq j} R^{*jk}(t, x, u, x^f)(\mu^*_{jk}(t) - \mu_{jk}(t)).$$

The dividend strategy δ is given by Equation (2.2.4).

The above dynamics of the savings account and the surplus contain the free-policy factor, f, and the duration, $U(t)$, which implies that they are not in the form of Lemma 2.3.2. Therefore, Theorem 2.3.3 cannot be used to project the savings account and the surplus including policyholder behavior.

2.5 State-wise Projections Including Policyholder Behavior

In this section, the main results of the paper are presented by extending the result from Section 2.3 to include policyholder behavior. First, we describe the inclusion of policyholder behavior in the life insurance setup with bonus and the choice of free-policy factor. In general, the inclusion of the ideal choice of free-policy factor breaks the linearity assumption of Section 2.3. We consider a certain case where the linearity assumption is satisfied, and suggest an approximation of the ideal free-policy factor. The main results of this paper are that in the certain case, the state-wise projections of the savings account and the surplus with the ideal free-policy factor and the approximated free-policy factor respectively coincide, and that we extend Theorem 2.3.3 to include policyholder behavior in a general case.

2.5.1 Policyholder Behavior Including Bonus

The extension of the classic life insurance setup without bonus to include policyholder behavior is described in existing literature. See Buchardt, Møller, and Schmidt

(2014) or Buchardt and Møller (2015) for a description of this extension. Without bonus, the payment upon surrender is usually chosen to be the technical reserve in state 0, $b^{0J}(t) = V^{*0}(t)$, such that the insured receive their savings account upon surrender, the sum-at-risk upon surrender is equal to zero, and the modeling of surrender can be omitted on the technical basis. Without bonus, the technical reserve, $V^*(t)$, is the technical value of future payments guaranteed at time t, since all payments are guaranteed. In our setup with bonus, this corresponds to the savings account, $X(t)$. The payment upon surrender in the setup with bonus is equal to the savings account $X(t)$ such that bonus obtained prior to time t is included in the payment upon surrender. Then the sum-at-risk of the savings account upon surrender is equal to zero. This complies with the assumption that payments are linear in the savings account.

Without bonus, the free-policy factor is usually chosen according to the principle of equivalence such that there is no jump in the technical reserve upon conversion to free-policy, i.e.

$$f^{\circ}(t) = \frac{V^{*0}(t)}{V^{*0+}(t)},$$

where the superscript $\circ$ refer to the setup without bonus.

To resemble the setup without bonus, the ideal free-policy factor in the setup with bonus is the free-policy factor, where the sum-at-risk of the savings account upon conversion to free-policy is equal to zero, resulting in no jump in X upon conversion to free-policy. The sum-at-risk upon conversion to free-policy is given in Remark 2.4.3, and setting this equal to zero implies that

$$f(t, X(t-)) = \frac{X(t-)}{X(t-) - V_1^{*0-}(t)}. \tag{2.5.1}$$

This free-policy factor is nonlinear in the savings account, which implies that the dynamics of the savings account and the surplus from Equations (2.4.2) and (2.4.3) do not satisfy the linearity assumption in Lemma 2.3.2 with this choice of free-policy factor.

The objective when including policyholder behavior is to ensure that the savings account is unaffected when the behavior option is exercised. This is achieved when the sum-at-risk is equal to zero upon surrender and upon conversion to free-policy. In the study of prospective reserves, Christiansen and Djehiche (2020) denote this concept actuarial equivalence, and obtain adjustment factors similar to our free-policy factor, but their adjustment factors depend on the prospective reserve where our free-policy factor depends on the retrospective savings account.

Let X_{id} be the savings account and let Y_{id} be the surplus with the ideal free-policy factor from Equation (2.5.1) above. Similar to Definition 2.3.1, the state-wise projections of the savings account and the surplus are given by

$$\tilde{X}^j_{\mathrm{id}}(t) = \mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t)=j\}} X_{\mathrm{id}}(t) \;\middle|\; \mathcal{F}^r_t\right], \tag{2.5.2}$$

$$\tilde{Y}^j_{\mathrm{id}}(t) = \mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t)=j\}} Y_{\mathrm{id}}(t) \;\middle|\; \mathcal{F}^r_t\right], \tag{2.5.3}$$

for $j \in \mathcal{J} \cup \mathcal{J}^f$.

2.5.2 The Case with All Benefits Regulated by Bonus

We consider the case, where all benefits are regulated by bonus such that the payment stream not increased by bonus, B_1, only contains premiums i.e. $B_1^+ = 0$. In this case, we show that the dynamics of the savings account and the surplus with the ideal free-policy factor from Equation (2.5.1), are in the form of Lemma 2.3.2 such that Theorem 2.3.3 can be used to find differential equations for the state-wise projections of the savings account and the surplus including policyholder behavior.

In the example of an insurance contract consisting of a life annuity and a term insurance, both products are regulated by bonus in the case $B_1^+ = 0$, in contrast to the case where only the life annuity is scaled by bonus.

The assumptions of defined contributions and $B_1^+ = 0$ imply that the total payment process has dynamics

$$\mathrm{d}B_1^-(t) + Q(t-)\mathrm{d}B_2^+(t),$$

where $Q(0-) = 1$ due to the principle of equivalence.

In the continuous payment functions during sojourns in states and the payment functions upon transition between states from Proposition 2.4.1, the terms including the free-policy factor are multiplied by either b_1^{j+}, b_1^{jk+} or V_1^{*j+} for $j, k \in \mathcal{J}$. In the case $B_1^+ = 0$, these are all equal to zero and therefore the free-policy factor does not appear in the payment functions.

The continuous payment functions during sojourns in states and the payment functions upon transition between states from Proposition 2.4.1 are in this case

$$\begin{aligned} b^j(t,x) = {} & \mathbb{1}_{\{j\in\mathcal{J}\}}\left(b_1^{j-}(t) + \frac{x - V_1^{*j-}(t)}{V_2^{*j+}(t)} b_2^{j+}(t)\right) \\ & + \mathbb{1}_{\{j\in\mathcal{J}^f\}}\left(\frac{x}{V_2^{*g(j)+}(t)} b_2^{g(j)+}(t)\right), \end{aligned} \tag{2.5.4}$$

$$\begin{aligned} b^{jk}(t,x) = {} & \mathbb{1}_{\{j,k\in\mathcal{J},\ j\neq k\}}\left(b_1^{jk-}(t) + \frac{x - V_1^{*j-}(t)}{V_2^{*j+}(t)} b_2^{jk+}(t)\right) \\ & + \mathbb{1}_{\{j,k\in\mathcal{J}^f,\ j\neq k\}}\left(\frac{x}{V_2^{*g(j)+}(t)} b_2^{g(j)g(k)+}(t)\right), \end{aligned} \tag{2.5.5}$$

for $j, k \in \mathcal{J} \cup \mathcal{J}^f$.

Similar to the payment functions, the terms including the free-policy factor in the sum-at-risk from Proposition 2.4.2 are multiplied by V_1^{*j+} for $j \in \mathcal{J}$, except for the sum-at-risk upon conversion to free-policy. Thus, in the case $B_1^+ = 0$, the sum-at-risk is

$$\begin{aligned} R^{*jk}(t,x) = b^{jk}(t,x) &+ \mathbb{1}_{\{j,k\in\mathcal{J},\ j\neq k\}}\left(V_1^{*k-}(t) + \frac{x - V_1^{*j-}(t)}{V_2^{*j+}(t)}V_2^{*k+}(t) - x\right) \\ &+ \mathbb{1}_{\{j,k\in\mathcal{J}^f,\ j\neq k\}}\left(\frac{x}{V_2^{*g(j)+}(t)}V_2^{*g(k)+}(t) - x\right) \\ &+ \mathbb{1}_{\{j=0,\ k=J+1\}}\Big(\big(x - V_1^{*j-}(t)\big)f(t,x) - x\Big). \end{aligned} \tag{2.5.6}$$

With the free-policy factor from Equation (2.5.1), the last line in the sum-at-risk above is equal to zero. Therefore, in the case $B_1^+ = 0$ with the free-policy factor from Equation (2.5.1), neither the payment functions (2.5.4) and (2.5.5) nor the sum-at-risk (2.5.6) depend on the duration in the free-policy states, and they are linear in the savings account. This implies that the dynamics of $X_{\text{id}}(t)$ and $Y_{\text{id}}(t)$ are in the form of Lemma 2.3.2, leading to the result in Theorem 2.3.3. Hence, in this case, we actually have differential equations for the projected savings account and the projected surplus with the free-policy factor from Equation (2.5.1) given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{X}_{\text{id}}^j(t) = &\sum_{k:k\neq j}\mu_{kj}(t)\tilde{X}_{\text{id}}^k(t) - \sum_{k:k\neq j}\mu_{jk}(t)\tilde{X}_{\text{id}}^j(t) \\ &+ \hat{\alpha}_{0,X}^j(t)p_{Z(0)j}(0,t) + \hat{\alpha}_{1,X}^j(t)\tilde{X}_{\text{id}}^j(t) + \hat{\alpha}_{2,X}^j(t)\tilde{Y}_{\text{id}}^j(t) \\ &+ \sum_{k:k\neq j}\mu_{kj}(t)\Big(\hat{\lambda}_{0,X}^{kj}(t)p_{Z(0)k}(0,t) + \hat{\lambda}_{1,X}^{kj}(t)\tilde{X}_{\text{id}}^k(t)\Big), \end{aligned} \tag{2.5.7}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{Y}_{\text{id}}^j(t) = &\sum_{k:k\neq j}\mu_{kj}(t)\tilde{Y}_{\text{id}}^k(t) - \sum_{k:k\neq j}\mu_{jk}(t)\tilde{Y}_{\text{id}}^j(t) \\ &+ \hat{\alpha}_{0,Y}^j(t)p_{Z(0)j}(0,t) + \hat{\alpha}_{1,Y}^j(t)\tilde{X}_{\text{id}}^j(t) + \hat{\alpha}_{2,Y}^j(t)\tilde{Y}_{\text{id}}^j(t) \\ &+ \sum_{k:k\neq j}\mu_{kj}(t)\Big(\hat{\lambda}_{0,Y}^{kj}(t)p_{Z(0)k}(0,t) + \hat{\lambda}_{1,Y}^{kj}(t)\tilde{X}_{\text{id}}^k(t)\Big), \end{aligned} \tag{2.5.8}$$

and $\tilde{X}_{id}^j(0-) = \tilde{Y}_{id}^j(0-) = 0$ for $j \in \mathcal{J} \cup \mathcal{J}^f$. The expressions for $\hat{\alpha}^j$ and $\hat{\lambda}^{jk}$ are in Appendix 2.B.

We compare the differential equations of the projected savings account and the projected surplus in the case $B_1^+ = 0$ using the free-policy factor from Equation (2.5.1) with the differential equations without policyholder behavior. This comes down to a comparison of the coefficients α^j and λ^{jk} from Appendix 2.A and $\hat{\alpha}^j$ and $\hat{\lambda}^{jk}$ from Appendix 2.B. The coefficient α^j and the corresponding $\hat{\alpha}^j$ consist of the same terms, but $\hat{\alpha}^j$ is decomposed in the cases $j \in \mathcal{J}$ and $j \in \mathcal{J}^f$ in the same sense as the payment functions and the sum-at-risk from Equations (2.5.4), (2.5.5) and (2.5.6), since there are only benefits in the free-policy states. This also goes for λ^{jk} and $\hat{\lambda}^{jk}$.

Remark 2.5.1. The case $B_1^+ = B_2^+$ corresponds to the case $B_1^+ = 0$, since the total payment process when $B_1^+ = B_2^+$ is

$$\mathrm{d}B_1(t) + Q(t-)\mathrm{d}B_2^+(t) = \mathrm{d}B_1^-(t) + \underbrace{(1 + Q(t-))}_{=\tilde{Q}(t-)}\mathrm{d}B_2^+(t),$$

which has the same form as the payment process in the case $B_1^+ = 0$, but where $\tilde{Q}(0-) = 2$ since $Q(0-) = 1$ due to the principle of equivalence. When the benefits in B_1 are equal to the benefits in B_2, all benefits are regulated equally by bonus, and therefore the case $B_1^+ = B_2^+$ can be rewritten to be in the form of $B_1^+ = 0$. Hence, the results above also apply for $B_1^+ = B_2^+$.

If benefits not regulated by bonus cancel due to conversion to free-policy, $B_1^+ = 0$ after conversion to free-policy, and the result above still applies. An example is an insurance contract consisting of a life annuity and a term insurance, where the life annuity is regulated by bonus, and the term insurance cancels upon conversion to free-policy. Throughout this paper, we assume that payments in the free-policy states equal a free-policy factor times the benefits in the corresponding premium-paying state. The example does not comply with this assumption, but we can easily extend our setup to include this case.

2.5.3 Approximation of the Free-policy Factor

In the general setup, $B_1^+(t) \geq 0$ for $t \geq 0$, we cannot project the savings account and the surplus including policyholder behavior by Theorem 2.3.3, since the assumptions are violated. The dynamics of the savings account and the surplus depend on the duration in the free-policy states, U. Furthermore, the derivation of Theorem 2.3.3 relies on linearity of X and Y in the dynamics from Lemma 2.3.2, which breaks when the free-policy factor depends on the savings account. This motivates an approximation of the ideal free-policy factor from Equation (2.5.1), which does not depend on X.

Just before conversion to free-policy, the policyholder must be premium paying and active, i.e. $Z(t-) = 0$. A reasonable approximation of the free-policy factor is therefore

$$\begin{aligned}\hat{f}(t) &= \mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t-)=0\}} f(t, X(t)) \,\middle|\, \mathcal{F}_t^r\right] \\ &= \mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t-)=0\}} \frac{X(t-)}{X(t-) - V_1^{*Z(t-)-}(t)} \,\middle|\, \mathcal{F}_t^r\right].\end{aligned}$$

We have not developed methods to calculate the projection of a fraction containing the savings account, $X(t)$, in both the nominator and the denominator. Therefore, we cannot continue with the approximation above. Alternatively, the nominator and denominator in the free-policy factor can be projected separately

$$\begin{aligned}\tilde{f}(t) &= \frac{\mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t-)=0\}}X(t-)\;\middle|\;\mathcal{F}_t^r\right]}{\mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t-)=0\}}\big(X(t-)-V_1^{*Z(t-)-}(t)\big)\;\middle|\;\mathcal{F}_t^r\right]}\\ &= \frac{\tilde{X}^0(t)}{\tilde{X}^0(t)-p_{Z(0)0}(0,t)V_1^{*0-}(t)}. \end{aligned} \tag{2.5.9}$$

The above free-policy factor does not depend on the savings account, but on the state-wise projection of the savings account. This approximation of the ideal free-policy factor motivates one of the main results of this paper presented in Corollary 2.5.2 below.

Corollary 2.5.2. *Let X_{id} be the savings account and Y_{id} be the surplus modeled with the ideal free-policy factor from Equation (2.5.1), and let X_{ap} be the savings account and Y_{ap} be the surplus modeled with the approximated free-policy factor from Equation (2.5.9). The state-wise projections are given by Equations (2.5.2) and (2.5.3), and*

$$\begin{aligned}\tilde{X}_{ap}^j(t) &= \mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t)=j\}}X_{ap}(t)\;\middle|\;\mathcal{F}_t^r\right],\\ \tilde{Y}_{ap}^j(t) &= \mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t)=j\}}Y_{ap}(t)\;\middle|\;\mathcal{F}_t^r\right],\end{aligned}$$

for $j \in \mathcal{J} \cup \mathcal{J}^f$, respectively.

In the case where all benefits are regulated by bonus, $B_1^+ = 0$

$$\begin{aligned}\tilde{X}_{id}^j(t) &= \tilde{X}_{ap}^j(t),\\ \tilde{Y}_{id}^j(t) &= \tilde{Y}_{ap}^j(t),\end{aligned}$$

for $j \in \mathcal{J} \cup \mathcal{J}^f$.

Proof. Assume all benefits are regulated by bonus, $B_1^+ = 0$. The state-wise projections of the savings account and the surplus with the ideal free-policy factor satisfy the differential equations in Equations (2.5.7) and (2.5.8).

Equations (2.5.4), (2.5.5) and (2.5.6) in Section 2.5.2 state that only the sum-at-risk depends on the free-policy factor. The sum-at-risk with the approximated free-policy factor, $\tilde{f}$, is

$$
\begin{aligned}
R^{*jk}(t,x) = b^{jk}(t,x) &+ \mathbb{1}_{\{j,k\in\mathcal{J},\ j\neq k\}}\left(V_1^{*k-}(t) + \frac{x - V_1^{*j-}(t)}{V_2^{*j+}(t)}V_2^{*k+}(t) - x\right) \\
&+ \mathbb{1}_{\{j,k\in\mathcal{J}^f,\ j\neq k\}}\left(\frac{x}{V_2^{*g(j)+}(t)}V_2^{*g(k)+}(t) - x\right) \\
&+ \mathbb{1}_{\{j=0,\ k=J+1\}}\Big(\big(x - V_1^{*j-}(t)\big)\tilde{f}(t) - x\Big).
\end{aligned} \tag{2.5.10}
$$

The dynamics of X_{ap} and Y_{ap} are in the form of Equations (2.4.2) and (2.4.3) with the payment functions from Equations (2.5.4) and (2.5.5) and the sum-at-risk from Equation (2.5.10). This implies that the dynamics of X_{ap} and Y_{ap} are in the same form as in Lemma 2.3.2, since they do not depend on the duration, U, and they are linear in $X_{\text{ap}}(t)$ and $Y_{\text{ap}}(t)$.

Theorem 2.3.3 gives differential equations of the state-wise projections of the savings account and the surplus, $\tilde{X}^j_{\text{ap}}$ and $\tilde{Y}^j_{\text{ap}}$. These differential equations can be expressed in terms of $\hat{\alpha}$ and $\hat{\lambda}$ from the differential equations (2.5.7) and (2.5.8)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{X}^j_{\text{ap}}(t) = &\sum_{k:k\neq j}\mu_{kj}(t)\tilde{X}^k_{\text{ap}}(t) - \sum_{k:k\neq j}\mu_{jk}(t)\tilde{X}^j_{\text{ap}}(t) \\
&+ \hat{\alpha}^j_{0,X}(t)p_{Z(0)j}(0,t) + \hat{\alpha}^j_{1,X}(t)\tilde{X}^j_{\text{ap}}(t) + \hat{\alpha}^j_{2,X}(t)\tilde{Y}^j_{\text{ap}}(t) \\
&+ \mathbb{1}_{\{j=J+1\}}\mu^*_{0j}(t)\Big(\tilde{X}^0_{\text{ap}}(t) + \tilde{f}(t)\big(p_{Z(0)0}(0,t)V_1^{*0-}(t) - \tilde{X}^0_{\text{ap}}(t)\big)\Big) \\
&+ \sum_{k:k\neq j}\mu_{kj}(t)\Bigg(\hat{\lambda}^{kj}_{0,X}(t)p_{Z(0)k}(0,t) + \hat{\lambda}^{kj}_{1,X}(t)\tilde{X}^k_{\text{ap}}(t) \\
&\qquad - \mathbb{1}_{\{k=0,\ j=J+1\}}\Big(\tilde{X}^0_{\text{ap}}(t) + \tilde{f}(t)\big(p_{Z(0)0}(0,t)V_1^{*0-}(t) - \tilde{X}^0_{\text{ap}}(t)\big)\Big)\Bigg),
\end{aligned} \tag{2.5.11}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Y}^j_{\text{ap}}(t) = &\sum_{k:k\neq j}\mu_{kj}(t)\tilde{Y}^k_{\text{ap}}(t) - \sum_{k:k\neq j}\mu_{jk}(t)\tilde{Y}^j_{\text{ap}}(t) \\
&+ \hat{\alpha}^j_{0,Y}(t)p_{Z(0)j}(0,t) + \hat{\alpha}^j_{1,Y}(t)\tilde{X}^j_{\text{ap}}(t) + \hat{\alpha}^j_{2,Y}(t)\tilde{Y}^j_{\text{ap}}(t) \\
&- \mathbb{1}_{\{j=J+1\}}\mu^*_{0j}(t)\Big(\tilde{X}^0_{\text{ap}}(t) + \tilde{f}(t)\big(p_{Z(0)0}(0,t)V_1^{*0-}(t) - \tilde{X}^0_{\text{ap}}(t)\big)\Big) \\
&+ \sum_{k:k\neq j}\mu_{kj}(t)\Bigg(\hat{\lambda}^{kj}_{0,Y}(t)p_{Z(0)k}(0,t) + \hat{\lambda}^{kj}_{1,Y}(t)\tilde{X}^k_{\text{ap}}(t) \\
&\qquad + \mathbb{1}_{\{k=0,\ j=J+1\}}\Big(\tilde{X}^0_{\text{ap}}(t) + \tilde{f}(t)\big(p_{Z(0)0}(0,t)V_1^{*0-}(t) - \tilde{X}^0_{\text{ap}}(t)\big)\Big)\Bigg).
\end{aligned} \tag{2.5.12}
$$

By inserting the expression for $\tilde{f}$ from Equation (2.5.9), the differential equations (2.5.11) and (2.5.12) are equal to the differential equations (2.5.7) and (2.5.8).

Furthermore, the initial conditions are

$$\tilde{X}^j_{\text{id}}(0) = \tilde{Y}^j_{\text{id}}(0) = \tilde{X}^j_{\text{ap}}(0) = \tilde{Y}^j_{\text{ap}}(0) = 0,$$

for $j \in \mathcal{J} \cup \mathcal{J}^f$. This implies that

$$\begin{aligned}\tilde{X}^j_{\text{id}}(t) &= \tilde{X}^j_{\text{ap}}(t),\\ \tilde{Y}^j_{\text{id}}(t) &= \tilde{Y}^j_{\text{ap}}(t),\end{aligned}$$

for $j \in \mathcal{J} \cup \mathcal{J}^f$ as desired. □

Corollary 2.5.2 implies that in the case $B_1^+ = 0$, we can project the savings account and the surplus with the approximated free-policy factor and actually obtain the same accurate projections as with the ideal free-policy factor. Based on this result, we consider $\tilde{f}$ to be a reasonable approximation of f, that does not depend on the savings account, but instead on the projected savings account.

2.5.4 Projections with the Approximated Free-policy Factor

In the general setup, $B_1^+(t) \geq 0$ for $t \geq 0$, with the approximated free-policy factor from Equation (2.5.9), the dynamics of the savings account and the surplus are linear, but they also depend on the duration through the payment functions from Proposition 2.4.1 and the sum-of-risk from Proposition 2.4.2. Therefore, we cannot use Theorem 2.3.3 to project the savings account and the surplus. This motivates an extension of Theorem 2.3.3 including duration dependence, where linearity in the dynamics of the savings account and the surplus is preserved.

Lemma 2.5.3. *The dynamics of the savings account, X_{ap}, from Equation (2.4.2) and the dynamics of the surplus, Y_{ap}, from Equation (2.4.3), with the approximated free-policy factor, $\tilde{f}$, from Equation (2.5.9), can be written in the form*

$$\begin{aligned}
\mathrm{d}X_{ap}(t) = &\Big(\bar{\alpha}^{Z(t)}_{0,X}(t) + \bar{\alpha}^{Z(t)}_{1,X}(t)X_{ap}(t) + \bar{\alpha}^{Z(t)}_{2,X}(t)Y_{ap}(t) + \tilde{f}(t-U(t))\bar{\beta}^{Z(t)}_{0,X}(t)\Big)\mathrm{d}t\\
&+ \sum_{k:k\neq Z(t-)} \Big(\bar{\lambda}^{Z(t-)k}_{0,X}(t) + \bar{\lambda}^{Z(t-)k}_{1,X}(t)X_{ap}(t-)\\
&\qquad + \tilde{f}(t-U(t))\bar{\gamma}^{Z(t-)k}_{0,X}(t)\Big)\mathrm{d}N^k(t),\\
\mathrm{d}Y_{ap}(t) = &\Big(\bar{\alpha}^{Z(t)}_{0,Y}(t) + \bar{\alpha}^{Z(t)}_{1,Y}(t)X_{ap}(t) + \bar{\alpha}^{Z(t)}_{2,Y}(t)Y_{ap}(t) + \tilde{f}(t-U(t))\bar{\beta}^{Z(t)}_{0,Y}(t)\Big)\mathrm{d}t\\
&+ \sum_{k:k\neq Z(t-)} \Big(\bar{\lambda}^{Z(t-)k}_{0,Y}(t) + \bar{\lambda}^{Z(t-)k}_{1,Y}(t)X_{ap}(t-)\\
&\qquad + \tilde{f}(t-U(t))\bar{\gamma}^{Z(t-)k}_{0,Y}(t)\Big)\mathrm{d}N^k(t),
\end{aligned}$$

for deterministic functions $\bar{\alpha}^j_{i,H}, \bar{\beta}^j_{i,H}, \bar{\lambda}^{jk}_{i,H}, \bar{\gamma}^{jk}_{i,H}$ *for* $i = 0, 1, 2$, $H = X, Y$ *and* $j, k \in \mathcal{J} \cup \mathcal{J}^f$, $j \neq k$, *where*

$$\bar{\beta}^j_{0,X}(t) = \bar{\beta}^j_{0,Y}(t) = \bar{\gamma}^{jk}_{0,X}(t) = \bar{\gamma}^{jk}_{0,Y}(t) = 0,$$

for all $t \geq 0$ *and* $j \in \mathcal{J}$.
See Appendix 2.C for the expressions of $\bar{\alpha}, \bar{\beta}, \bar{\lambda}$ *and* $\bar{\gamma}$ *for the savings account and the surplus.*

We consider the difference between the case with all benefits regulated by bonus, $B_1^+ = 0$, with the free-policy factor from Equation (2.5.1) from Section 2.5.2 and the general case, $B_1^+ \geq 0$, with the approximated free-policy factor. This comes down to a comparison of the coefficients $\hat{\alpha}$ and $\hat{\lambda}$ from Appendix 2.B with the coefficients $\bar{\alpha}$, $\bar{\beta}$, $\bar{\lambda}$ and $\bar{\gamma}$ from Appendix 2.C. Apart from the sum-at-risk upon conversion to free-policy and the duration dependent terms, the coefficients are equal. In the first case, the sum-at-risk upon conversion to free-policy is equal to zero, while in the second case, it is added to $\bar{\lambda}$. The duration dependent terms from Propositions 2.4.1 and 2.4.2 are equal to zero in the case with all benefits regulated by bonus, while in the general case they appear in $\bar{\beta}$ and $\bar{\gamma}$.

The dynamics of the savings account and the surplus in Lemma 2.5.3 allow for an extension of the dividend strategy from Equation (2.2.4) to be duration dependent. Dividends in form

$$\delta^j(t, x, y, u) = \delta_0^j(t, u) + \delta_1^j(t) \cdot x + \delta_2^j(t) \cdot y,$$
$$\delta_0^j(t, u) = \mathbb{1}_{\{j \in \mathcal{J}\}} \delta_0^j(t) + \mathbb{1}_{\{j \in \mathcal{J}^f\}} \tilde{f}(t - u) \delta_0^j(t),$$

comply with the dynamics in Lemma 2.5.3.

Now, we extent the result of Theorem 2.3.3 to include duration dependence in the approximated free-policy factor from the dynamics of the savings account and the surplus in Lemma 2.5.3.

Theorem 2.5.4. *Let* X_{ap} *and* Y_{ap} *have dynamics in the form of Lemma 2.5.3 and* $Z(0) \in \mathcal{J}$. *The state-wise projections of the savings account and the surplus,* $\tilde{X}^j_{ap}$ *and* $\tilde{Y}^j_{ap}$, *satisfy the system of differential equations below*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \tilde{X}^j_{ap}(t) = &\sum_{k:k \neq j} \mu_{kj}(t) \tilde{X}^k_{ap}(t) - \sum_{k:k \neq j} \mu_{jk}(t) \tilde{X}^j_{ap}(t) \\
&+ \bar{\alpha}^j_{0,X}(t) p^{\tilde{f}}_{Z(0)j}(0, t) + \bar{\alpha}^j_{1,X}(t) \tilde{X}^j_{ap}(t) + \bar{\alpha}^j_{2,X}(t) \tilde{Y}^j_{ap}(t) \\
&+ \sum_{k:k \neq j} \mu_{kj}(t) \Big(\bar{\lambda}^{kj}_{0,X}(t) p^{\tilde{f}}_{Z(0)k}(0, t) + \bar{\lambda}^{kj}_{1,X}(t) \tilde{X}^k_{ap}(t) \Big) \\
&+ \bar{\beta}^j_{0,X}(t) p^{\tilde{f}}_{Z(0)j}(0, t) + \sum_{k:k \neq j} \mu_{kj}(t) \bar{\gamma}^{kj}_{0,X}(t) p^{\tilde{f}}_{Z(0)k}(0, t),
\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{Y}^j_{ap}(t) = &\sum_{k:k\neq j}\mu_{kj}(t)\tilde{Y}^k_{ap}(t) - \sum_{k:k\neq j}\mu_{jk}(t)\tilde{Y}^j_{ap}(t)\\ &+ \bar{\alpha}^j_{0,Y}(t)p^{\tilde{f}}_{Z(0)j}(0,t) + \bar{\alpha}^j_{1,Y}(t)\tilde{X}^j_{ap}(t) + \bar{\alpha}^j_{2,Y}(t)\tilde{Y}^j_{ap}(t)\\ &+ \sum_{k:k\neq j}\mu_{kj}(t)\Big(\bar{\lambda}^{kj}_{0,Y}(t)p^{\tilde{f}}_{Z(0)k}(0,t) + \bar{\lambda}^{kj}_{1,Y}(t)\tilde{X}^k_{ap}(t)\Big)\\ &+ \bar{\beta}^j_{0,Y}(t)p^{\tilde{f}}_{Z(0)j}(0,t) + \sum_{k:k\neq j}\mu_{kj}(t)\bar{\gamma}^{kj}_{0,Y}(t)p^{\tilde{f}}_{Z(0)k}(0,t),\end{aligned}$$

where $\tilde{X}^j_{ap}(0-) = \tilde{Y}^j_{ap}(0-) = 0$, $\tilde{f}$ *is the approximated free-policy factor from Equation (2.5.9), and* $p^{\tilde{f}}_{Z(0)j}(0,t)$ *are the* $\tilde{f}$*-modified probabilities*

$$p^{\tilde{f}}_{Z(0)j}(0,t) = \mathbb{E}_{Z(0)}\left[\mathbb{1}_{\{Z(t)=j\}}\tilde{f}(t-U(t))^{\mathbb{1}_{\{j\in\mathcal{J}^f\}}}\right],$$

for $Z(0)\in\mathcal{J}$, $j\in\mathcal{J}\cup\mathcal{J}^f$, *and* $t\geq 0$.

Proof. See Appendix 2.D. □

Buchardt and Møller (2015) derive forward differential equations for the same $\tilde{f}$-modified probabilities in the case where $j\in\mathcal{J}^f$. In the case where $j\in\mathcal{J}$, the $\tilde{f}$-modified probabilities are the ordinary transition probabilities that satisfy Kolmogorov's forward differential equations. Therefore, for a general $j\in\mathcal{J}\cup\mathcal{J}^f$, the $\tilde{f}$-modified probabilities satisfy the following forward differential equations

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}p^{\tilde{f}}_{Z(0)j}(0,t) = &\mathbb{1}_{\{j=J+1\}}p^{\tilde{f}}_{Z(0)0}(0,t)\mu_{0(J+1)}(t)\tilde{f}(t) - p^{\tilde{f}}_{Z(0)j}(0,t)\sum_{k:k\neq j}\mu_{jk}(t)\\ &+ \mathbb{1}_{\{j\in\mathcal{J}^f\}}\sum_{\substack{k\in\mathcal{J}^f\\ k\neq j}}p^{\tilde{f}}_{Z(0)k}(0,t)\mu_{kj}(t) + \mathbb{1}_{\{j\in\mathcal{J}\}}\sum_{\substack{k\in\mathcal{J}\\ k\neq j}}p^{\tilde{f}}_{Z(0)k}(0,t)\mu_{kj}(t).\end{aligned}$$

We consider Theorem 2.5.4 as one of the main results of the paper, since it enables us to project the savings account and the surplus in a general setup with the policyholder behavior options surrender and free-policy with the approximated free-policy factor from Equation (2.5.9). For instance in the example with an insurance contract consisting of a life annuity and a term insurance, where the life annuity is regulated by bonus and the term insurance and the premiums are fixed.

Remark 2.5.5. Let the savings account and the surplus have dynamics in the form of Lemma 2.5.3, but with a general free-policy factor, $\bar{f}$, that does not depend on the savings account. Then Theorem 2.5.4 holds with $\bar{f}$-modified probabilities.

In the Danish life insurance business, it is common to scale all benefits (both those regulated by bonus and those not regulated by bonus) with the free-policy factor

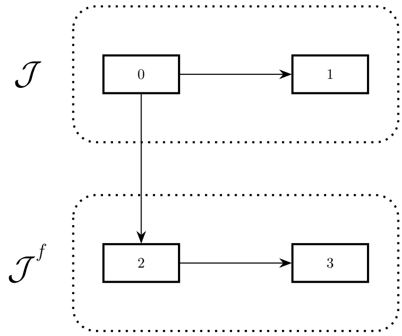

Figure 2.2: *Survival model in the numerical example*

upon conversion to free-policy. We can imagine an insurance contract where only the benefits not regulated by bonus, B_1^+, are scaled with the free-policy factor upon conversion to free-policy and where $Q(0-) = 0$. Then the free-policy factor does not depend on the savings account, and Theorem 2.5.4 applies.

2.6 Numerical Simulation Example

In this section, we emphasize the practical applications of our results in a numerical simulation example, and study the state-wise projections of the savings account and the surplus in a survival model including free-policy.

To illustrate this example, we assume the interest rate follow a Vasicek model with dynamics

$$\mathrm{d}r(t) = (\phi + \psi\ r(t))\ \mathrm{d}t + \sqrt{\theta}\ \mathrm{d}W(t),$$

where $\big(W(t)\big)_{\{t\geq 0\}}$ is a Brownian motion, see for instance Björk (2009). Any other model of the interest rate can be chosen.

The survival model including free-policy is illustrated in Figure 2.2, where state 0 corresponds to alive and state 1 corresponds to dead in the non-free-policy states and state 2 and state 3 corresponds to alive and dead, respectively, in the free-policy states. We consider an insured male at age a_0 at initialization of the insurance contract at time 0. The insurance contract consists of premiums paid continuously in state 0 until retirement age n, a term insurance not regulated by bonus payable upon dead before retirement age, and a life annuity regulated by bonus paid continuously when alive after retirement age. Hence, in this example, $B_1^+ \geq 0$ and we use Theorem

Table 2.1: *Components in the numerical example*

Component	Value
Age of policyholder, a_0	30
Age of retirement, n	65
Termination	80
Premium, $\pi(t)$	$0.3021694 \cdot \mathbb{1}_{\{a_0+t<n\}}$
Annuity, $b_2^0(t)$	$1 \cdot \mathbb{1}_{\{a_0+t\geq n\}}$
Term insurance, $b_1^{01}(t)$	$5 \cdot \mathbb{1}_{\{a_0+t<n\}}$
$Z(0)$	0
$\mu_{01}^*(t)$	$0.0005 + 10^{5.88+0.038(t+a_0)-10}$
$\mu_{02}(t)$	$0.015 \cdot \mathbb{1}_{\{a_0+t<n\}}$
$r^*(t)$	0.01
$r(0)$	0.05
ϕ	0.008127
ψ	-0.162953
θ	0.000237

2.5.4 in the projection. The payment process is

$$\begin{aligned}
&\mathrm{d}B(t,X(t)) = \\
&\quad \mathbb{1}_{\{Z(t)=0\}}\left(\left(\frac{X(t)-V_1^{*0}(t)}{V_2^{*0}(t)}b_2^0(t)-\pi(t)\right)\mathrm{d}t + b_1^{01}(t)\mathrm{d}N^1(t)\right) \\
&\quad + \mathbb{1}_{\{Z(t)=2\}}\left(\frac{X(t)-\tilde{f}(t-U(t))V_1^{*0}(t)}{V_2^{*0}(t)}b_2^0(t)\mathrm{d}t + \tilde{f}(t-U(t))b_1^{01}(t)\mathrm{d}N^3(t)\right).
\end{aligned}$$

The premium rate is determined according to the principle of equivalence on the technical basis, and we use the approximated free-policy factor from Equation (2.5.9).

Inspired by Bruhn and Lollike (2020), we choose a dividend strategy equal to

$$\begin{aligned}
&\delta^{Z(t)}(t,U(t),X(t),Y(t)) = \\
&\qquad 0.5 \cdot \big(r(t)-r^*(t)\big)^+ X(t) + 0.01 \cdot Y(t) \\
&\qquad + 0.5 \cdot \sum_{k:k\neq Z(t)} R^{*Z(t)k}(t,U(t),X(t))\big(\mu_{Z(t)k}^*(t)-\mu_{Z(t)k}(t)\big)\Big),
\end{aligned}$$

where $R^{*Z(t)k}$ is the sum-at-risk from Proposition 2.4.2 with the approximated free-policy factor. The dividend strategy resembles the surplus contribution, but with $(r(t)-r^*(t))^+$ instead of $r(t)-r^*(t)$. This is to avoid negative dividends if $r^*(t) > r(t)$. The market death intensity is the mortality benchmark from the Danish FSA from 2019. We project the savings account and the surplus in states 0 and 2, since there are no payments in the death states. The components in the projection are stated in Table 2.1.

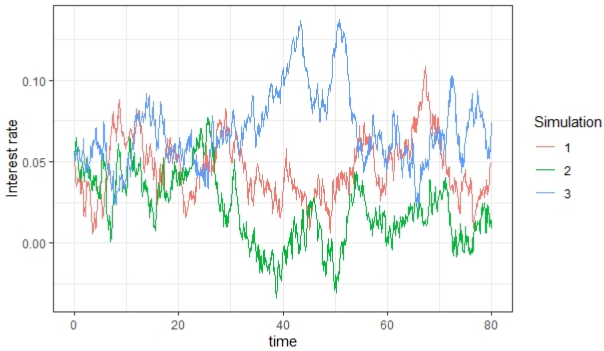

Figure 2.3: *Simulations of the interest rate in the numerical example*

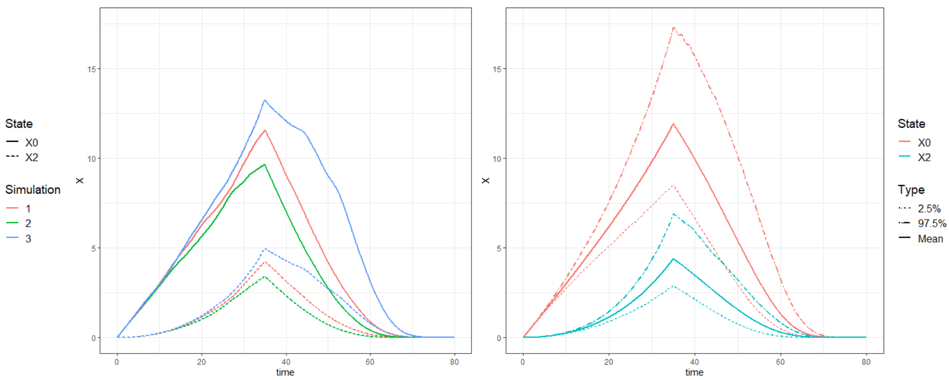

Figure 2.4: *Left: State-wise projections of the savings account in the three simulated scenarios of the interest rate. Right: The mean and confidence intervals of the projected savings account.*

Figure 2.3 illustrates three simulated paths of the interest rate, simulated with an Euler scheme based on the dynamics of the interest rate. For each path of the interest rate, we project the savings account and the surplus in state 0 and 2 using Theorem 2.5.4, and illustrate the state-wise projections in Figure 2.4 (left) and Figure 2.5 (left).

The projected savings account is larger in state 0 than in the free-policy-state, since premiums cancel upon conversion to free-policy, which lowers the savings account. The interest rate impacts the projected surplus in Figure 2.5 (left) significantly. A high (low) interest rate results in a high (low) surplus contribution, which effects the projected surplus as illustrated in simulation 3 (2). A high interest rate results in high dividends in our numerical example, and therefore the projected savings accounts are highest in simulation 3. For the effects of changing the dividend strategy, see Bruhn and Lollike (2020). With these calculations, the insurance company can

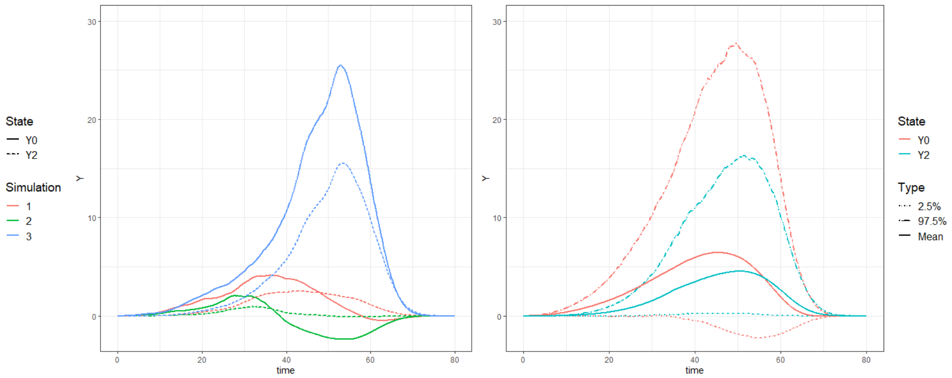

Figure 2.5: *Left: State-wise projections of the surplus in the three simulated scenarios of the interest rate. Right: The mean and confidence intervals of the projected surplus.*

monitor the development of the insurance contract in various scenarios of the interest rate, and for instance assess the effects of the chosen dividend strategy.

Based on 1000 simulations of the interest rate, we estimate the mean, the 2.5%-quantile, and the 97.5%-quantile of the projected savings account (see Figure 2.4 (right)) and the projected surplus (see Figure 2.5 (right)). This illustrates that within the Vasicek model with the chosen parameters and with the chosen dividend strategy, the 95%-confidence interval of the projected savings account is widest, when the insured retires at time 35, and the 95%-confidence interval of the projected surplus spans from -2.2 to 27.7, which indicates to the insurance company that the development of the surplus is uncertain.

The insurance company is interested in communicating the expected life annuity payment to the insured, since it is regulated by bonus, and the amount of future bonus is unknown at initialization of the insurance contract. Figure 2.6 (left) illustrates the life annuity rate in the three simulated scenarios of the interest rate conditional on the insured being alive and in the non-free-policy state at the time of the payment. In scenario 3, the savings account is higher resulting in a high life annuity. Scenario 2 has a negative surplus due to a low interest rate, which results in negative dividends with the chosen dividend strategy, and therefore the life annuity gets below 1 in this scenario. At initialization of the insurance contract, the insurance company promises the insured a life annuity of 1 given alive and non-free-policy, and hence scenario 2 is bad for the company. The projection in Figure 2.6 (left) holds information to the insurance company, that when the interest rate is low, the insurance company should react and change their dividend strategy.

Figure 2.6 (right) illustrates the expected life annuity and a 95% confidence interval of the life annuity as a function of age. The life annuity is weighted with the probability of dying and conversion to free-policy, hence it is lower than the life

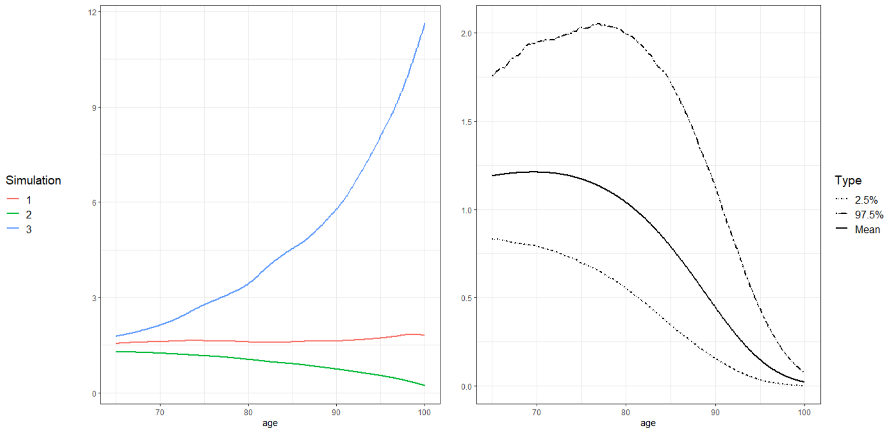

Figure 2.6: *Left: The expected life annuity in the three simulated scenarios of the interest conditional in the insured being alive and non-free-policy. Right: The expected life annuity and confidence intervals.*

annuity in Figure 2.6 (left) where we condition in being alive and non-free-policy.

2.7 Conclusion

The paper presents a method for projecting the savings account and the surplus of a life insurance contract including policyholder behavior in various financial scenarios. We present differential equations of the projected savings account and the projected surplus without policyholder behavior, which is the result of Bruhn and Lollike (2020). When including policyholder behavior, we cannot in general project the savings account and the surplus with an ideal free-policy factor using the methods from Bruhn and Lollike (2020).

In this paper, we show that in the case, where all benefits are regulated by bonus, we can actually find accurate differential equations for the state-wise projections of the savings account and the surplus with the ideal free-policy factor. We suggest an approximation to the ideal free-policy factor, and one of the main results is that in the case, where all benefits are regulated by bonus, the projections of the savings account and the surplus based on the ideal free-policy factor coincide with the projections based on the approximated free-policy factor. Therefore, we consider the approximated free-policy factor a reasonable approximation of the ideal free-policy factor.

We are able to project the savings account and the surplus with the approximated free-policy factor in a general case, and we present differential equations of the state-wise projections of the savings account and the surplus with the approximated free-policy factor. We consider this result as a key result in the projection of balances in life insurance and a good extension of Bruhn and Lollike (2020) to include policyholder behavior outside the case, where all benefits are regulated by bonus. We illustrate a numerical simulation example in three scenarios of the interest rate to highlight the practical application of our findings. This results in a projection of the savings account and the surplus for a chosen dividend strategy, which enables the insurance company to assess the effects of their chosen management actions. Furthermore, we study distributional properties of the projections.

This paper studies a simple dividend strategy which is linear in the savings account and the surplus. In order to use this model, insurance companies must choose their future dividend strategy according to this simple setup. Future research involves extending the model to include a more complex dividend strategy and allow for dependence of for instance assets and market values. Another branch is the study of how to choose an optimal dividend strategy in this multi-state setup, see for instance Nielsen (2005).

Acknowledgments and declarations of interest

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2.A Additions to Lemma 2.3.2

The coefficients in the dynamics of the savings account and surplus from Lemma 2.3.2 in the setup without policyholder behavior.

$$
\begin{aligned}
\alpha^j_{0,X}(t) &= \delta^j_0(t) - b^j_1(t) - \frac{V^{*j}_1(t)}{V^{*j}_2(t)} b^j_2(t) \\
&\quad - \sum_{k:k\neq j} \left(b^{jk}_1(t) - \frac{V^{*j}_1(t)}{V^{*j}_2(t)} b^{jk}_2(t) + V^{*k}_1(t) - \frac{V^{*j}_1(t)}{V^{*j}_2(t)} V^{*k}_2(t) \right) \mu^*_{jk}(t), \\
\alpha^j_{1,X}(t) &= r^*(t) + \delta^j_1(t) - \frac{b^j_2(t)}{V^{*j}_2(t)} - \sum_{k:k\neq j} \left(\frac{b^{jk}_2(t)}{V^{*j}_2(t)} + \frac{V^{*k}_2(t)}{V^{*j}_2(t)} - 1 \right) \mu^*_{jk}(t), \\
\alpha^j_{2,X}(t) &= \delta^j_2(t), \\
\lambda^{jk}_{0,X}(t) &= V^{*k}_1(t) - \frac{V^{*j}_1(t)}{V^{*j}_2(t)} V^{*k}_2(t), \\
\lambda^{jk}_{1,X}(t) &= \frac{V^{*k}_2(t)}{V^{*j}_2(t)} - 1
\end{aligned}
$$

$$
\begin{aligned}
\alpha^j_{0,Y}(t) &= -\delta^j_0(t) + \sum_{k:k\neq j} \left(b^{jk}_1(t) - \frac{V^{*j}_1(t)}{V^{*j}_2(t)} b^{jk}_2(t) + V^{*k}_1(t) - \frac{V^{*j}_1(t)}{V^{*j}_2(t)} V^{*k}_2(t) \right) \mu^*_{jk}(t), \\
\alpha^j_{1,Y}(t) &= -\delta^j_1(t) + r(t) - r^*(t) + \sum_{k:k\neq j} \left(\frac{b^{jk}_2(t)}{V^{*j}_2(t)} + \frac{V^{*k}_2(t)}{V^{*j}_2(t)} - 1 \right) \mu^*_{jk}(t), \\
\alpha^j_{2,Y}(t) &= r(t) - \delta^j_2(t), \\
\lambda^{jk}_{0,Y}(t) &= -b^{jk}_1(t) + \frac{V^{*j}_1(t)}{V^{*j}_2(t)} b^{jk}_2(t) - V^{*k}_1(t) + \frac{V^{*j}_1(t)}{V^{*j}_2(t)} V^{*k}_2(t), \\
\lambda^{jk}_{1,Y}(t) &= -\frac{b^{jk}_2(t)}{V^{*j}_2(t)} - \frac{V^{*k}_2(t)}{V^{*j}_2(t)} + 1.
\end{aligned}
$$

2.B Addition to the case $B_1^+ = 0$-functions

The coefficients in the dynamics of the savings account and surplus from Lemma 2.3.2 in the case $B_1^+ = 0$ with the ideal free-policy factor.

$$\begin{aligned}
\hat{\alpha}^j_{0,X}(t) =& -\mathbb{1}_{\{j\in\mathcal{J}\}}\Big(b_1^{j-}(t) - \frac{V_1^{*j-}(t)}{V_2^{*j+}(t)} b_2^{j+}(t)\Big) + \delta_0^j(t) \\
&-\mathbb{1}_{\{j\in\mathcal{J}\}} \sum_{\substack{k:k\neq j\\ k\in\mathcal{J}}} \Big(b_1^{jk-}(t) + V_1^{*k-}(t) - \frac{V_1^{*j-}(t)}{V_2^{*j+}(t)}\big(b_2^{jk+}(t) + V_2^{*k+}(t)\big)\Big)\mu^*_{jk}(t), \\
\hat{\alpha}^j_{1,X}(t) =& \, r^*(t) - \mathbb{1}_{\{j\in\mathcal{J}\}}\frac{b_2^{j+}(t)}{V_2^{*j+}(t)} - \mathbb{1}_{\{j\in\mathcal{J}^f\}}\frac{b_2^{g(j)+}(t)}{V_2^{*g(j)+}(t)} + \delta_1^j(t) \\
&-\mathbb{1}_{\{j\in\mathcal{J}\}} \sum_{\substack{k:k\neq j\\ k\in\mathcal{J}}} \left(\frac{b_2^{jk+}(t) + V_2^{*k+}(t)}{V_2^{*j+}(t)} - 1\right)\mu^*_{jk}(t) \\
&-\mathbb{1}_{\{j\in\mathcal{J}^f\}} \sum_{\substack{k:k\neq j\\ k\in\mathcal{J}^f}} \left(\frac{b_2^{g(j)g(k)+}(t) + V_2^{*g(k)+}(t)}{V_2^{*g(j)+}(t)} - 1\right)\mu^*_{jk}(t), \\
\hat{\alpha}^j_{2,X}(t) =& \, \delta_2^j(t), \\
\hat{\lambda}^{jk}_{0,X}(t) =& \, \mathbb{1}_{\{j,k\in\mathcal{J},j\neq k\}}\Big(V_1^{*k-}(t) - \frac{V_1^{*j-}(t)}{V_2^{*j+}(t)}V_2^{*k+}(t)\Big), \\
\hat{\lambda}^{jk}_{1,X}(t) =& \, \mathbb{1}_{\{j,k\in\mathcal{J},j\neq k\}}\left(\frac{V_2^{*k+}(t)}{V_2^{*j+}(t)} - 1\right) + \mathbb{1}_{\{j,k\in\mathcal{J}^f,j\neq k\}}\left(\frac{V_2^{*g(k)+}(t)}{V_2^{*g(j)+}(t)} - 1\right).
\end{aligned}$$

$$\begin{aligned}
\hat{\alpha}^j_{0,Y}(t) =& -\delta_0^j(t) + \mathbb{1}_{\{j\in\mathcal{J}\}} \sum_{\substack{k:k\neq j\\ k\in\mathcal{J}}} \Big(b_1^{jk-}(t) + V_1^{*k-}(t) \\
&\qquad - \frac{V_1^{*j-}(t)}{V_2^{*j+}(t)}\big(b_2^{jk+}(t) + V_2^{*k+}(t)\big)\Big)\mu^*_{jk}(t), \\
\hat{\alpha}^j_{1,Y}(t) =& \, r(t) - r^*(t) - \delta_1^j(t) + \mathbb{1}_{\{j\in\mathcal{J}\}} \sum_{\substack{k:k\neq j\\ k\in\mathcal{J}}} \left(\frac{b_2^{jk+}(t) + V_2^{*k+}(t)}{V_2^{*j+}(t)} - 1\right)\mu^*_{jk}(t) \\
&+\mathbb{1}_{\{j\in\mathcal{J}^f\}} \sum_{\substack{k:k\neq j\\ k\in\mathcal{J}^f}} \left(\frac{b_2^{g(j)g(k)+}(t) + V_2^{*g(k)+}(t)}{V_2^{*g(j)+}(t)} - 1\right)\mu^*_{jk}(t), \\
\hat{\alpha}^j_{2,Y}(t) =& \, r(t) - \delta_2^j(t), \\
\hat{\lambda}^{jk}_{0,Y}(t) =& -\mathbb{1}_{\{j,k\in\mathcal{J},\ j\neq k\}}\Big(b_1^{jk-}(t) + V_1^{*k-}(t) - \frac{V_1^{*j-}(t)}{V_2^{*j+}(t)}\big(b_2^{jk+}(t) + V_2^{*k+}(t)\big)\Big), \\
\hat{\lambda}^{jk}_{1,Y}(t) =& -\mathbb{1}_{\{j,k\in\mathcal{J},j\neq k\}}\Big(\frac{b_2^{jk+}(t) + V_2^{*k+}(t)}{V_2^{*j+}(t)} - 1\Big) \\
&-\mathbb{1}_{\{j,k\in\mathcal{J}^f,j\neq k\}}\Big(\frac{b_2^{g(k)g(j)+}(t) + V_2^{*g(k)+}(t)}{V_2^{*g(j)+}(t)} - 1\Big).
\end{aligned}$$

2.C Additions to Lemma 2.5.3

The coefficients in the dynamics of the savings account and surplus from Lemma 2.5.3.

$$
\begin{aligned}
\bar{\alpha}^j_{0,X}(t) &= \delta^j_0(t) - \mathbb{1}_{\{j\in\mathcal{J}\}}\left(b^j_1(t) - \frac{V^{*j}_1(t)}{V^{*j+}_2(t)}b^j_2(t)\right) \\
&\quad - \mathbb{1}_{\{j\in\mathcal{J}\}}\sum_{\substack{k:k\neq j\\ k\neq J+1}}\left(b^{jk}_1(t) - \frac{V^{*j}_1(t)}{V^{*j+}_2(t)}b^{jk}_2(t) + V^{*k}_1(t) - \frac{V^{*j}_1(t)}{V^{*j+}_2(t)}V^{*k+}_2(t)\right)\mu^*_{jk}(t), \\
\bar{\alpha}^j_{1,X}(t) &= r^*(t) + \delta^j_1(t) - \mathbb{1}_{\{j\in\mathcal{J}\}}\left(\frac{b^j_2(t)}{V^{*j+}_2(t)} + \sum_{\substack{k:k\neq j\\ k\neq J+1}}\left(\frac{b^{jk}_2(t)}{V^{*j+}_2(t)} + \frac{V^{*k+}_2(t)}{V^{*j+}_2(t)} - 1\right)\mu^*_{jk}(t)\right) \\
&\quad - \mathbb{1}_{\{j\in\mathcal{J}^f\}}\left(\frac{b^{g(j)+}_2(t)}{V^{*g(j)+}_2(t)} + \sum_{k:k\neq j}\left(\frac{b^{g(j)g(k)+}_2(t)}{V^{*g(j)+}_2(t)} + \frac{V^{*g(k)+}_2(t)}{V^{*g(j)+}_2(t)} - 1\right)\mu^*_{jk}(t)\right), \\
\bar{\alpha}^j_{2,X}(t) &= \delta^j_2(t), \\
\bar{\beta}^j_{0,X}(t) &= -\mathbb{1}_{\{j\in\mathcal{J}^f\}}\left(b^{g(j)+}_1(t) - \frac{V^{*g(j)+}_1(t)}{V^{*g(j)+}_2(t)}b^{g(j)+}_2(t)\right) \\
&\quad - \mathbb{1}_{\{j\in\mathcal{J}^f\}}\sum_{k:k\neq j}\left(b^{g(j)g(k)+}_1(t) - \frac{V^{*g(j)+}_1(t)}{V^{*g(j)+}_2(t)}b^{g(j)g(k)+}_2(t)\right)\mu^*_{jk}(t) \\
&\quad - \mathbb{1}_{\{j\in\mathcal{J}^f\}}\sum_{k:k\neq j}\left(V^{*g(k)+}_1(t) - \frac{V^{*g(j)+}_1(t)}{V^{*g(j)+}_2(t)}V^{*g(k)+}_2(t)\right)\mu^*_{jk}(t), \\
\bar{\lambda}^{jk}_{0,X}(t) &= \mathbb{1}_{\{j,k\in\mathcal{J},j\neq k\}}\left(V^{*k}_1(t) - \frac{V^{*j}_1(t)}{V^{*j+}_2(t)}V^{*k+}_2(t)\right) - \mathbb{1}_{\{j=0,k=J+1\}}\tilde{f}(t)V^{*0-}_1(t), \\
\bar{\lambda}^{jk}_{1,X}(t) &= \mathbb{1}_{\{j,k\in\mathcal{J},j\neq k\}}\left(\frac{V^{*k+}_2(t)}{V^{*j+}_2(t-)} - 1\right) + \mathbb{1}_{\{j,k\in\mathcal{J}^f,j\neq k\}}\left(\frac{V^{*g(k)+}_2(t)}{V^{*g(j)+}_2(t-)} - 1\right) \\
&\quad + \mathbb{1}_{\{j=0,k=J+1\}}\left(\tilde{f}(t) - 1\right), \\
\bar{\gamma}^{jk}_{0,X}(t) &= \mathbb{1}_{\{j,k\in\mathcal{J}^f,j\neq k\}}\left(V^{*g(k)+}_1(t) - \frac{V^{*g(j)+}_1(t-)}{V^{*g(j)+}_2(t-)}V^{*g(k)+}_2(t)\right).
\end{aligned}
$$

$$
\begin{aligned}
\bar{\alpha}^j_{0,Y}(t) =& -\delta^j_0(t) + \mathbb{1}_{\{j\in\mathcal{J}\}} \sum_{\substack{k:k\neq j\\ k\neq J+1}} \Bigg(b_1^{jk}(t) - \frac{V_1^{*j}(t)}{V_2^{*j+}(t)} b_2^{jk}(t) \\
&\qquad + V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j+}(t)} V_2^{*k+}(t) \Bigg) \mu^*_{jk}(t), \\
\bar{\alpha}^j_{1,Y}(t) =& r(t) - r^*(t) - \delta^j_1(t) + \mathbb{1}_{\{j\in\mathcal{J}\}} \sum_{\substack{k:k\neq j\\ k\neq J+1}} \left(\frac{b_2^{jk}(t)}{V_2^{*j+}(t)} + \frac{V_2^{*k+}(t)}{V_2^{*j+}(t)} - 1 \right) \mu^*_{jk}(t) \\
&+ \mathbb{1}_{\{j\in\mathcal{J}^f\}} \sum_{k:k\neq j} \left(\frac{b_2^{g(j)g(k)+}(t)}{V_2^{*g(j)+}(t)} + \frac{V_2^{*g(k)+}(t)}{V_2^{*g(j)+}(t)} - 1 \right) \mu^*_{g(j)g(k)}(t), \\
\bar{\alpha}^j_{2,Y}(t) =& r(t) - \delta^j_2(t), \\
\bar{\beta}^j_{0,Y}(t) =& \mathbb{1}_{\{j\in\mathcal{J}^f\}} \sum_{k:k\neq j} \left(b_1^{g(j)g(k)+}(t) - \frac{V_1^{*g(j)+}(t)}{V_2^{*g(j)+}(t)} b_2^{g(j)g(k)+}(t) \right) \mu^*_{g(j)g(k)}(t) \\
&+ \mathbb{1}_{\{j\in\mathcal{J}^f\}} \sum_{k:k\neq j} \left(V_1^{*g(k)+}(t) - \frac{V_1^{*g(j)+}(t)}{V_2^{*g(j)+}(t)} V_2^{*g(k)+}(t) \right) \mu^*_{g(j)g(k)}(t), \\
\bar{\lambda}^{jk}_{0,Y}(t) =& -\mathbb{1}_{\{j,k\in\mathcal{J}, j\neq k\}} \left(b_1^{jk}(t) - \frac{V_1^{*j}(t)}{V_2^{*j+}(t)} b_2^{jk}(t) + V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j+}(t)} V_2^{*k+}(t) \right) \\
&+ \mathbb{1}_{\{j=0,k=J+1\}} \tilde{f}(t) V_1^{*0-}(t), \\
\bar{\lambda}^{jk}_{1,Y}(t) =& -\mathbb{1}_{\{j,k\in\mathcal{J}, j\neq k\}} \left(\frac{b_2^{jk}(t)}{V_2^{*j+}(t)} + \frac{V_2^{*k+}(t)}{V_2^{*j+}(t)} - 1 \right) - \mathbb{1}_{\{j=0,k=J+1\}} \left(\tilde{f}(t) - 1 \right) \\
&- \mathbb{1}_{\{j,k\in\mathcal{J}^f, j\neq k\}} \left(\frac{b_2^{g(j)g(k)+}(t)}{V_2^{*g(j)+}(t)} + \frac{V_2^{*g(k)+}(t)}{V_2^{*g(j)+}(t)} - 1 \right), \\
\bar{\gamma}^{jk}_{0,Y}(t) =& -\mathbb{1}_{\{j,k\in\mathcal{J}^f, j\neq k\}} \left(b_1^{g(j)g(k)+}(t) - \frac{V_1^{*g(j)+}(t)}{V_2^{*g(j)+}(t)} b_2^{g(j)g(k)+}(t) \right) \\
&- \mathbb{1}_{\{j,k\in\mathcal{J}^f, j\neq k\}} \left(V_1^{*g(k)+}(t) - \frac{V_1^{*g(j)+}(t)}{V_2^{*g(j)+}(t)} V_2^{*g(k)+}(t) \right).
\end{aligned}
$$

2.D Proof of Theorem 2.5.4

We only present the proof of the differential equation for $\tilde{X}^j$, since the differential equation for $\tilde{Y}^j$ is obtained using the same calculations. All calculations are conditioned on the interest rate filtration $\mathcal{F}_t^r$.

Due to the result in Theorem 2.3.3, it suffices to prove the result for

$$\bar{\alpha}_{0,X}^j = \bar{\alpha}_{1,X}^j = \bar{\alpha}_{2,X}^j = \bar{\lambda}_{0,X}^{jk} = \bar{\lambda}_{1,X}^{jk} = 0,$$

for all j, k, $j \neq k$.

We consider the integral equation for $\tilde{X}^j(t)$

$$\begin{aligned}\tilde{X}^j(t) &= p_{Z(0)j}(0,t)X(0)\\ &\quad + \int_0^t \sum_{g \in \mathcal{J} \cup \mathcal{J}^f} \mathbb{E}_{Z(0)}\Big[\mathbb{1}_{\{Z(s-)=g\}}\mathbb{E}_{Z(0)}\big[\mathbb{1}_{\{Z(t)=j\}}\mathrm{d}X(s) \mid Z(s-) = g\big]\Big].\end{aligned}$$

We calculate $\mathbb{E}_{Z(0)}\big[\mathbb{1}_{\{Z(t)=j\}}\mathrm{d}X(s) \mid Z(s-) = g\big]$ for both terms in the dynamics of $X(t)$ from Lemma 2.5.3.

$$\begin{aligned}&\mathbb{E}_{Z(0)}\big[\mathbb{1}_{\{Z(t)=j\}}\tilde{f}(s-U(s-))\bar{\beta}_{0,X}^{Z(s-)}(s) \mid Z(s-) = g\big]\\ &= \mathbb{1}_{\{g \in \mathcal{J}^f\}}\bar{\beta}_{0,X}^g(s)p_{gj}(s,t)\mathbb{E}_{Z(0)}\big[\tilde{f}(s-U(s-)) \mid Z(s-) = g, Z(t) = j\big]\\ &= \mathbb{1}_{\{g \in \mathcal{J}^f\}}\bar{\beta}_{0,X}^g(s)p_{gj}(s,t)\mathbb{E}_{Z(0)}\big[\tilde{f}(s-U(s-)) \mid Z(s-) = g\big]\\ &= \mathbb{1}_{\{g \in \mathcal{J}^f\}}\bar{\beta}_{0,X}^g(s)p_{gj}(s,t)\frac{p_{Z(0)g}^{\tilde{f}}(0,s)}{p_{Z(0)g}(0,s)},\end{aligned}$$

where we use that $\bar{\beta}_{0,X}^{Z(s-)}(s) = 0$ for $Z(s-) \in \mathcal{J}$ and that $U(s-)|Z(s-) = g \perp\!\!\!\perp Z(t) = j$ for $g \in \mathcal{J}^f$ and $s \leq t$. The $\tilde{f}$-modified probabilities, $p_{Z(0)g}^{\tilde{f}}(0,s)$, are defined as

$$p_{Z(0)g}^{\tilde{f}}(0,s) = \mathbb{E}_{Z(0)}\Big[\mathbb{1}_{\{Z(s)=g\}}\tilde{f}(s-U(s))^{\mathbb{1}_{\{g \in \mathcal{J}^f\}}}\Big],$$

for $Z(0) \in \mathcal{J}$ and $s \geq 0$.

$$\begin{aligned}&\mathbb{E}_{Z(0)}\big[\mathbb{1}_{\{Z(t)=j\}}\tilde{f}(s-U(s-))\bar{\gamma}_{0,X}^{Z(s-)k}(s)\mathrm{d}N^k(s) \mid Z(s-) = g\big]\\ &= \mathbb{1}_{\{g \in \mathcal{J}^f\}}\bar{\gamma}_{0,X}^{gk}(s)\frac{p_{Z(0)g}^{\tilde{f}}(0,s)}{p_{Z(0)g}(0,s)}\mu_{gk}(s)p_{kj}(s,t)\mathrm{d}s,\end{aligned}$$

where we use that $\bar{\gamma}_{0,X}^{Z(s-)k}(s) = 0$ for $Z(s-) \in \mathcal{J}$ and $U(s-)|Z(s-) = g \perp\!\!\!\perp Z(t) = j$ for $g \in \mathcal{J}^f$ and $s \leq t$ and that

$$\begin{aligned}\mathbb{E}_{Z(0)}\big[\mathbb{1}_{\{Z(t)=j\}}\mathrm{d}N^k(s) \mid Z(s-) = g\big] &= p_{gj}(s,t)\mathbb{E}_{Z(0)}\big[\mathrm{d}N^k(s) \mid Z(s-) = g, Z(t) = j\big]\\ &= \mu_{gk}(s)p_{kj}(s,t)\mathrm{d}s,\end{aligned}$$

see Norberg (1991) Equation (4.12).

Inserting in the integral equation for $\tilde{X}^j(t)$

$$\tilde{X}^j(t) = p_{Z(0)j}(0,t)X(0) + \sum_{g \in \mathcal{J}^f} \int_0^t \Big(p^{\tilde{f}}_{Z(0)g}(0,s) \Big(p_{gj}(s,t)\bar{\beta}^g_{0,X}(s) + \sum_{k:k\neq g} \mu_{gk}(s) p_{kj}(s,t) \bar{\gamma}^{gk}_{0,X}(s) \Big) \Big) \mathrm{d}s$$

We use Leibniz's rule to differentiate $\tilde{X}^j(t)$ and use that $p_{lk}(t,t) = \mathbb{1}_{\{l=k\}}$ for $l,k \in \mathcal{J} \cup \mathcal{J}^f$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{X}^j(t) = &\frac{\mathrm{d}}{\mathrm{d}t} p_{Z(0)j}(0,t)X(0) \\ &+ \mathbb{1}_{\{j\in\mathcal{J}^f\}} \bar{\beta}^j_{0,X}(t) p^{\tilde{f}}_{Z(0)j}(0,t) + \sum_{k:k\neq j} \mathbb{1}_{\{k\in\mathcal{J}^f\}} \mu_{kj}(t) \bar{\gamma}^{kj}_{0,X}(t) p^{\tilde{f}}_{Z(0)k}(0,t) \\ &+ \sum_{g\in\mathcal{J}^f} \int_0^t \Big(p^{\tilde{f}}_{Z(0)g}(0,s) \Big(\frac{\mathrm{d}}{\mathrm{d}t} p_{gj}(s,t) \bar{\beta}^g_{0,X}(s) \\ &\qquad + \sum_{k:k\neq g} \mu_{gk}(s) \frac{\mathrm{d}}{\mathrm{d}t} p_{kj}(s,t) \bar{\gamma}^{gk}_{0,X}(s) \Big) \Big) \mathrm{d}s.\end{aligned}$$

Kolmogorov's forward differential equations for the transition probabilities gives the result. □

Chapter 3

An intrinsic value approach to valuation with forward-backward loops in dividend paying stocks

This chapter contains the paper *Nyegaard, Ott, and Steffensen (2021).*

Abstract

We formulate a claim valuation problem where the dynamics of the underlying asset process contain the claim value itself. The problem is motivated here by an equity valuation of a firm, with intermediary dividend payments that depend on both the underlying, that is, the assets of the company, and the equity value itself. Since the assets are reduced by the dividend payments, the entanglement of claim, claim value, and underlying is complete and numerically challenging because it forms a forward–backward stochastic system. We propose a numerical approach based on disentanglement of the forward–backward deterministic system for the intrinsic values, a parametric assumption of the claim value in its intrinsic value, and a simulation of the stochastic elements. We illustrate the method in a numerical example where the equity value is approximated efficiently, at least for the relevant ranges of the asset value.

Keywords: Corporate finance; With-profit insurance; Forward–backward stochastic differential equations; Intrinsic value.

3.1 Introduction

We propose and demonstrate a simulation technique for claim valuation in a situation where this is fundamentally challenging, namely for non-simple contingent claims where both the claim payments and the underlying price process depend on the option value itself. A key motivating example is the equity valuation in a corporate finance framework for a dividend-paying company with a dividend strategy that depends on the equity value itself. A similarly challenging situation arises in with-profit insurance when dividends to policy holders depend on prospectively calculated liabilities themselves. The fundamental idea proposed here is to set up a deterministic as-if market of intrinsic values of both the price and the claim value process, and then assume a parametric relation between the value and the intrinsic value of the claim. From a practical point of view, the intrinsic value is particularly apt as a deterministic basis since it has a clear economic interpretation. We carry out all details and demonstrate the idea in a numerical example for the special case where the option value is approximated as a linear function of its intrinsic value.

Motivating examples, where the claim value appears in the dynamics of the underlying, can be found in both finance and insurance. In corporate finance, a main task is to calculate equity value as a claim on the assets in the presence of debt. This was first studied by Merton (1974), but the problem continues to attract interest, see for instance, Leland (1994) and Broadie and Kaya (2007). If the company pays out dividends and these dividends depend on the equity value itself, we have a situation where the contingent claim consists of equity-dependent dividends and an ultimate sharing of assets with creditors. However, since the dividends are financed by the company's assets, even the underlying asset process is influenced by the equity value. Then, we have a triangular interdependence between the underlying, the claim on the underlying, and the value of the claim. The feedback effect from the value of the claim to the underlying is non-standard, and is well-motivated by this corporate finance example, but is notoriously difficult to handle from a numerical point of view.

Another example in with-profit insurance is where a situation similar to the one in corporate finance arises for the valuation of liabilities. In with-profit insurance, the insurance company redistributes profits to the collective of policy holders in terms of dividends to policy holders but the extent and the timing are partly regulated by the financial authorities. In market valuation based accounting and solvency rules, the future redistribution strategy must be formalized in terms of so-called Future Management Actions that also include future investment decisions. It is natural to base the Future Management Actions, and thus the dividends to policy holders, on the future prospective liability value itself and, thereby, the involved forward–backward system appears again. The underlying assets of the company are reduced by dividends to policy holders and, thus, the assets, the dividends, and the liability value of policy holder dividends are completely entangled. The simpler case, where dividends depend on the assets (and other quantities that are easily calculated at every time point in a simulation), has only recently been formalized, for example, in Bruhn and Lollike (2020) and Falden and Nyegaard (2021). The Future Management Actions in with-profit insurance naturally motivate our study but we stick to the corporate finance story when we present our method and numerical example below.

Feeding back the claim value into the underlying price dynamics creates a stochastic forward–backward differential system since the known side condition of the underlying (given the claim value) exists at the initial time point, whereas the known side condition of the claim value (given the underlying) exists at the terminal time point. For example, for standard diffusive financial markets, such a situation is solvable by PDE methods where the claim value is calculated for all asset values backward. In general markets (or if one for other reasons prefers, or is forced, to simulate), the feed-back feature is challenging. Our proposal is to disentangle the problem into essentially three parts. First, the forward–backward element of the problem is handled by an iteration in a deterministic world where both asset prices and claim prices are represented by their intrinsic values. Two iterative methods are proposed here. Second, based on assuming a parametric relation between the claim value and the intrinsic value of the claim, a standard forward Monte Carlo simulation is performed for a given parameter. Third, the simulation is performed a number of times for iterated determination of the parameter value consistent with the input value (current asset price) and the ultimate output value (current claim value) of the system.

Numerical techniques in general, and simulation techniques in particular, for the valuation of contingent claims are challenged by fundamental relations between the claims themselves, claim values and underlying state processes. A canonical example is the simulation for the valuation of American options where the decision about whether to exercise, and therefore the actual claim, depends on the value itself. Thus, in that case, there is a relation between claims and claim values but claim values are (usually) not fed back into the underlying stock price dynamics. For the American option case, least-squares Monte Carlo has become a dominant numerical technique, both theoretically and practically, since Longstaff et al. (2001) introduced that idea to the domain.

The least-squares Monte Carlo is an example of a numerical method tailor-made for a specific version of a general forward–backward stochastic differential equation. Similar to that, other methods have also been proposed but they are typically constructed for certain cases with a special entanglement of the forward and backward equations. Compared to the American option valuation, we add here a layer of complexity and allow the underlying stock price dynamics to also depend on the value of the contingent claim. This completely entangles the forward and backward equations. It is outside the scope of this paper to clarify whether the least-squares Monte Carlo techniques and/or other specific approximation methods to other types of forward–backward stochastic differential equations can (be generalized to) cope with our case of value dependence in the underlying stock price. Even if they could, our approximation method based on an intrinsic value projection can be justified as a practical alternative since the intrinsic value in finance and insurance is well-understood and has its clear intuitive meaning and merits.

The entanglement of claim and claim value is standard from American option theory and is acknowledged to be numerically challenging. Therefore, upper and lower bounds have been sought and found via a dual formulation of the optimal stopping problem by, for example, Rogers (2003), Haugh and Kogan (2004), and Andersen and Broadie (2004). The upper and lower bounds obtained by Ibáñez and Velasco (2020) are particularly interesting in relation to our work as they also use the intrinsic value of the option as a state process in a recursive procedure. To calculate and simulate under the feedback effect into the asset price dynamics is new, to our knowledge, although both the corporate finance and the with-profit insurance applications and interpretations seem obvious.

After having discussed the relation to numerical techniques for American option valuation, we find it important to stress one final difference. Along with American option valuation comes the optimal stopping problem, and solving the value and the optimal stopping strategy are two sides of the same story. There is no optimization going on in our problem, which is a pure valuation problem. The computational difficulty arises purely from the specification of the dividend payment strategy and is, at least in this exposition, completely separated from any question about whether such a strategy is optimal in any sense. Therefore, our work also contains no discussion about optimal versus sub-optimal strategies such as, for example, that studied by Ibáñez and Paraskevopoulos (2010). It is, though, an interesting discussion—but also beyond our scope—to learn about which objective functions lead to equity-dependent dividends when considering dividends as an optimal control process.

The structure of the paper is as follows: in Section 3.2, we formalize and motivate the problem of the main application of equity valuation. In Section 3.3, we approximate the value of the claim that is presented by means of its intrinsic value and several iterative methods to calculate it. Section 3.4 presents the simulation part of the valuation and the second layer of iteration. A numerical study in Section 3.5 shows the quality of our method for the equity valuation.

3.2 The Problem

We consider a general financial market consisting of an (possibly stochastic) interest rate process $r = \big(r(t)\big)_{t\geq 0}$ and an asset $S = \big(S(t)\big)_{t\geq 0}$. We assume the financial market is free of arbitrage resulting in the existence of a (not necessarily unique) martingale measure $\mathbb{Q}$. We let $\big(\Omega, \mathcal{A}, \mathbb{Q}\big)$ be a complete probability space governing a 2-dimensional Brownian motion $W = \big(W_1, W_2\big)$, and denote by $\mathcal{F} = \big(\mathcal{F}_t\big)_{t\geq 0}$ the natural filtration of W. We assume that W_1 and W_2 are independent for simplicity, but the extension to correlated dynamics is straightforward.

In this model, the dynamics of the asset depend on the value process of an option derived from the asset itself. The option is a payment stream with continuous payments with a (possibly stochastic) payment rate ϕ and a lump sum payment $\Phi\big(S(T)\big)$ at time T. We denote the value of the future payments of the option at time $t \leq T$ by $V(t)$, and assume that ϕ is in the form $\phi\big(t, S(t), r(t), V(t)\big)$ for a deterministic function $\phi : \mathbb{R}_+^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$. The dynamics of the asset are of the form

$$\mathrm{d}S(t) = g\big(t, S(t), r(t), V(t)\big)\mathrm{d}t + \sigma\big(t, S(t), r(t), V(t)\big)\mathrm{d}W_1(t), \tag{3.2.1}$$

for deterministic functions $g : \mathbb{R}_+^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ and $\sigma : \mathbb{R}_+^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$. If S is the price process of an asset without a cash flow, we have $g(t, s, r, v) = r \cdot s$, since Equation (3.2.1) represents the dynamics under the martingale measure $\mathbb{Q}$. The general formulation of the function g allows the asset to have a cash flow, which depends on the interest rate and the value process of the option.

We model the interest rate with a one-factor model with dynamics

$$\mathrm{d}r(t) = b\big(t, r(t)\big)\mathrm{d}t + \gamma\big(t, r(t)\big)\mathrm{d}W_2(t), \tag{3.2.2}$$

for deterministic functions $b : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ and $\gamma : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$. The extension to multi-factor models of the interest rate is straight forward. Let B denote the money account with dynamics

$$\begin{aligned} \mathrm{d}B(t) &= r(t)B(t)\mathrm{d}t, \\ B(0) &= b_0 > 0. \end{aligned}$$

We let $P(\,\cdot\,, \bar{T})$ be the price process of a zero coupon bond with maturity $\bar{T} > T$, satisfying

$$P(t, \bar{T}) = \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^{\bar{T}} r(s)\mathrm{d}s} \;\Big|\; \mathcal{F}_t\Big] = e^{-\int_t^{\bar{T}} f(t,s)\mathrm{d}s},$$

for $t \leq \bar{T}$. The existence of the zero coupon bond enables us to find the forward rates $f(t, s)$ for $0 \leq t \leq T$ and $t \leq s \leq T$, $f(t, t) = r(t)$.

The value of the option can be represented in the following way:

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^{\tau} r(s)\mathrm{d}s}\phi\big(\tau, S(\tau), r(\tau), V(\tau)\big)\mathrm{d}\tau + e^{-\int_t^T r(s)\mathrm{d}s}\Phi\big(S(T)\big) \;\middle|\; \mathcal{F}_t\right].$$

The asset, the interest rate and the value process stipulate a forward–backward stochastic differential equation. We assume that the functions $g, \sigma, b, \gamma, \phi$ and Φ are sufficiently regular and refer to Antonelli (1993) for the existence and uniqueness of a solution. Since the functions g, b, σ and γ in the dynamics of the asset and the interest rate in Equations (3.2.1) and (3.2.2) are deterministic, $\big(S, r\big)$ is Markov.

Hence, we write

$$V(t, S(t), r(t)) = \mathbb{E}^{\mathbb{Q}}\Bigg[\int_t^T e^{-\int_t^\tau r(s)\mathrm{d}s}\phi\big(\tau, S(\tau), r(\tau), V(\tau, S(\tau), r(\tau))\big)\mathrm{d}\tau + e^{-\int_t^T r(s)\mathrm{d}s}\Phi\big(S(T)\big) \,\Bigg|\, S(t), r(t)\Bigg], \tag{3.2.3}$$

with a slight misuse of notation since V is now a function, $V : \mathbb{R}_+ \times \mathbb{R}^2 \mapsto \mathbb{R}$, and not the stochastic value process itself. Furthermore, we write

$$\mathrm{d}S(t) = g\big(t, S(t), r(t), V(t, S(t), r(t))\big)\mathrm{d}t + \sigma\big(t, S(t), r(t), V(t, S(t), r(t))\big)\mathrm{d}W_1(t). \tag{3.2.4}$$

In this model setup, there is a triangular interdependence between the underlying asset, S, the process of claims on the underlying, ϕ and Φ, and the value process, V. The key motivating example is the corporate finance example from Miller and Modigliani (1958) and Merton (1974), elaborated in Example 3.2.1 below.

Example 3.2.1. Consider a firm with a debt of K payable at time T with continuous coupon payments on the loan with continuous rate c paid until time T. The underlying assets of the firm are denoted by S. In corporate finance, a task is to calculate the equity value, V, of the firm. We assume that the firm pays out continuous dividends to its shareholders with rate δ. In the event that the firm cannot pay its debt at time T, the lending institution immediately takes over the firm. Then the equity value, V, the value of the debt, V^d, and the dynamics of the assets under the risk neutral measure $\mathbb{Q}$ are

$$\begin{aligned}
V(t, S(t), r(t)) &= \mathbb{E}^{\mathbb{Q}}\Bigg[\int_t^T e^{-\int_t^\tau r(s)\mathrm{d}s}\delta\big(\tau, S(\tau), r(\tau), V(\tau, S(\tau), r(\tau))\big)\, d\tau \\
&\qquad + e^{-\int_t^T r(s)ds} \max\big(S(T) - K, 0\big) \,\Bigg|\, S(t), r(t)\Bigg], \\
V^d(t, S(t), r(t)) &= \mathbb{E}^{\mathbb{Q}}\Bigg[\int_t^T e^{-\int_t^\tau r(s)\mathrm{d}s}c\big(\tau, S(\tau), r(\tau), V(\tau, S(\tau), r(\tau))\big)\, d\tau \\
&\qquad + e^{-\int_t^T r(s)ds} \min\big(S(T), K\big) \,\Bigg|\, S(t), r(t)\Bigg], \\
dS(t) &= r(t)S(t)\,dt - \delta\big(t, S(t), r(t), V(t, S(t), r(t))\big)\, d\,t \\
&\quad - c\big(t, S(t), r(t), V(t, S(t), r(t))\big)\, dt \\
&\quad + \sigma\big(t, S(t), r(t), V(t, S(t), r(t))\big)\, dW_1(t).
\end{aligned}$$

The equity value is the expected present value of future dividends plus the remaining part of the assets when the debt is paid at time T. The value of the debt is the expected present value of future coupon payments plus the debt payment at time

T, which is the minimum of the assets and K. Dividends and coupon payments are withdrawn from the asset. It is sufficient to model either $V(t, S(t), r(t))$ or $V^d(t, S(t), r(t)$ since

$$V(t, S(t), r(t)) + V^d(t, S(t), r(t)) = S(t).$$

As an extension to Merton (1974), this setup allows the dividends, the coupon payments and the investment strategy of the asset, σ, to depend on the equity value (or similarly the value of the debt). To be consistent with Merton (1974), and for simplicity, we disregard taxation of dividends, including possible tax benefits from paying out dividends. Adding taxation would complicate the picture further as a third party beyond debt and equity holders is entitled to a tax cash flow, which may or may not be a function of current balance scheme entries. One of the benefits of this simplicity is that we can directly compare the methods presented in Sections 3.3 and 3.4 with a relatively simple numerically exact solution in Section 3.5.

Example 3.2.2. In with-profit life insurance, payments guaranteed in the insurance contract are based on prudent assumptions regarding future interest rate and mortality. This results in a surplus which, by legislation, is to be paid back to the policyholders as a bonus. The redistribution of the bonus contains certain degrees of freedom for the insurance company and depends on their dividend strategy. References Bruhn and Lollike (2020) and Falden and Nyegaard (2021) describe a projection model of the balance sheet of a with-profit life insurance company where a bonus is used to buy more insurance (spoken of as additional benefits). The model from Bruhn and Lollike, 2020; Falden and Nyegaard, 2021 contains simplifying assumptions about the future dividend strategy, and an obvious extension of the model is to allow for a broader range of dividend strategies. A relevant extension of the model from Bruhn and Lollike, 2020; Falden and Nyegaard, 2021 is to allow the future dividend strategy of the company to depend on the market reserve of future payments and future additional benefits and, in that case, a dependence structure similar to our model setup arises.

Let B denote the payment process of both the guaranteed payments and the additional benefits of a with-profit insurance contract. The payments are linked to states of the insured as, for instance, 'Active', 'Disable', and 'Dead', usually modelled by a Markov chain $\left(Z(t)\right)_{t\geq 0}$ on a finite state space $\mathcal{J}$ with corresponding counting processes $N_{ij}(t)$, $i, j \in \mathcal{J}$, counting jumps from state i to state j. Payments are divided in continuous payments during sojourn in state i, $b^s(t, i)$, and payments upon jump from state i to state j, $b^j(t, i, j)$. The market reserve of future payments from the insurance company to the insured is

$$
\begin{aligned}
V(t) &= \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^\tau r(s)\mathrm{d}s}\, dB(\tau) \,\middle|\, \mathcal{F}(t) \right], \\
&= \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^\tau r(s)\mathrm{d}s} \Big(b^s(\tau, Z(\tau))\, d\tau \right. \\
&\qquad\qquad \left. + \sum_{j:j\neq Z(\tau-)} b^j(t, Z(\tau-), j)\, dN_{Z(\tau-)j}(\tau) \Big) \,\middle|\, \mathcal{F}(t) \right],
\end{aligned}
$$

where T is the termination of the insurance contract, r is the stochastic interest rate, and $\mathcal{F}(t)$ is the formalized information available at time t. The market reserve above has a similar representation to that of the claim value from Equation (3.2.3), although in this example, we also model payments when there are jumps in the underlying Markov process $\big(Z(t)\big)_{t\geq 0}$.

In Bruhn and Lollike (2020) and Falden and Nyegaard (2021), the authors derive the dynamics of the so-called savings account and the surplus of a with-profit insurance contract (see Equations (8) and (9) in Bruhn and Lollike (2020)), where the dividend strategy depends on the savings account and the surplus only. If we allow the dividend strategy to also depend on the market reserve, V, the dynamics of the savings account X and the surplus Y are in the form

$$
\begin{aligned}
dX(t) &= g_x\big(t, X(t), Y(t), r(t), V(t), Z(t)\big)\, dt \\
&\quad + \sigma_x\big(t, X(t), Y(t), r(t), V(t), Z(t)\big)\, dW_1(t) \\
&\quad + \sum_{j:j\neq Z(t-)} h_x\big(t, X(t-), Y(t-), V(t-), Z(t-), j\big)\, dN_{Z(t-)j}(t), \\
X(0) &= x_0 \in \mathbb{R}, \\
dY(t) &= g_y\big(t, X(t), Y(t), r(t), V(t), Z(t)\big)\, dt \\
&\quad + \sigma_y\big(t, X(t), Y(t), r(t), V(t), Z(t)\big)\, dW_1(t) \\
&\quad + \sum_{j:j\neq Z(t-)} h_y\big(t, X(t-), Y(t-), V(t-), Z(t-), k\big)\, dN_{Z(t-)j}(t), \\
Y(0) &= y_0 \in \mathbb{R}.
\end{aligned}
$$

We choose not to write out the expressions for the functions $g_x, g_y, \sigma_x, \sigma_y, h_x$, and h_y. The dynamics of the savings account and the surplus above are in the form of the dynamics of the asset from Equation (3.2.1), except that the dynamics also include a $dN_{ij}(t)$-term that models changes in the savings account and the surplus due to jumps of the state process $\big(Z(t)\big)_{t\geq 0}$. Since V appears as an argument in the coefficients of X and Y and, further, the payment coefficients in the market reserve are themselves dependent on X and Y, we find that the ultimate computational challenge of a forward–backward system is exactly the same in this example as in the previous one.

That challenge is, however, here better hidden behind generalized notation, jump risk, and a conceptual world specific to the domain of with-profit insurance.

The with-profit insurance example is here proven to be both conceptually and notationally cumbersome, and we choose not to continue with this application in focus. The structure and the idea of our approach are much easier to comprehend in a version of the problem where the jump processes and the domain specific notions of profit-sharing are peeled away. What is important is that the forward–backward dependence structure that we study in this paper arises as a natural and practically relevant extension of the model from Bruhn and Lollike (2020) and Falden and Nyegaard (2021), when allowing for dividends to depend on the market reserve.

The feed-back of the value process into the dynamics of the underlying is challenging from a simulation point of view. We disentangle the simulation of the underlying and the estimation of the value process in three parts. First, in Section 3.3, we set up a deterministic world where asset prices, processes of claims, and the value process are represented by their intrinsic values. In this deterministic as-if market, the forward–backward element of the problem is solved by an iteration procedure. Second, based on the assumption that there is a parametric relation between the value process and its intrinsic value, we perform a standard forward Monte Carlo simulation in Section 3.4 for a given parameter. Third, in Section 3.4.2, the forward Monte Carlo simulation is performed a number of times for the iterated determination of the parameter value.

There is a separate issue of existence and uniqueness to the system formalized by the feed-back construction of the problem. The question is about for which specifications of ϕ, Φ, g and σ there exist a unique solution to our value problem. While the answer to that question is of obvious relevance, our contribution is completely different. This paper contains only a financial engineering approach to the construction of an approximation technique based on the intrinsic value deterministic basis.

3.3 Intrinsic Value

3.3.1 Definition of Intrinsic Value

Our definition of intrinsic values is based on the idea of setting up a hypothetical market at a fixed time point t with an underlying process, a process of claims on the underlying, and prices as-if the market were deterministic. The hypothetical market is fixed at time t, consistent with all market prices and therefore measurable with respect to $\mathcal{F}_t$. Thereby, the interest rate r can be replaced by the forward interest rate $f(t,\cdot)$ everywhere in the hypothetical market, and the dynamics of the asset S are replaced by the dynamics of the intrinsic value of the asset S^{IV}.

The intrinsic value of the value process is defined as the value of the claim in this as-if deterministic market.

Definition 3.3.1. *We define the intrinsic value of the asset and the value process, S^{IV} and V^{IV}, at time u calculated at a fixed time $t \leq u$ by interchanging the interest rate with the forward interest rate calculated at time t and by eliminating uncertainty from the financial market from time t onwards, hence:*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} S^{\mathrm{IV}}(t,u) &= g\big(u, S^{\mathrm{IV}}(t,u), f(t,u), V^{\mathrm{IV}}(u, S(t), r(t))\big), \\ S^{\mathrm{IV}}(t,t) &= S(t), \end{aligned}$$

$$\begin{aligned} V^{\mathrm{IV}}(u, S(t), r(t)) = \int_u^T & e^{-\int_u^\tau f(t,s)\mathrm{d}s} \phi\big(\tau, S^{\mathrm{IV}}(t,\tau), f(t,\tau), V^{\mathrm{IV}}(\tau, S(t), r(t))\big)\mathrm{d}\tau \\ & + e^{-\int_u^T f(t,s)\mathrm{d}s} \Phi\big(S^{\mathrm{IV}}(t,T)\big), \end{aligned}$$

with the functions g, ϕ, and Φ as defined in the previous section.

The intrinsic value of the asset $u \mapsto S^{\mathrm{IV}}(t,u)$ and the value process $u \mapsto V^{\mathrm{IV}}(u, S(t), r(t))$ are measurable with respect to $\mathcal{F}_t$.

It is appropriate to relate the intrinsic value definition above with the conventional intuition of the intrinsic value being the option value if the option is exercised now. There, one must carefully distinguish European options from American options. For a European option, the decision to exercise now means, for example in the case of a call option, that you must decide now whether you want to buy, at maturity, the stock at strike price K. Consider the classic Black–Scholes model; since you can buy the stock today at price $S(t)$ and the strike price K at price $e^{-r(T-t)}K$, the intrinsic value of the option becomes $(S(t) - e^{-r(T-t)}K)^+$. This intrinsic value of a European call option conforms with our intrinsic value definition. In the European call case it is not an option of when to buy, only whether to buy. The intrinsic value of a European put option becomes $(e^{-r(T-t)}K - S(t))^+$.

For the American option, things are more complicated. The conventional intrinsic value of an American call (put) is $(S(t) - K)^+$ $((K - S(t))^+)$. That is the value if the option is exercised today, and exercise here means both choosing to buy and actually buying. We did not even speak of exercise timing options in our definition, but the natural generalization is to maximize the intrinsic value over exercise times in an as-if market. Then, in the Black–Scholes model with $r > 0$, our intrinsic value of the American call option becomes $(S(t) - e^{-r(T-t)}K)^+$, since it maximizes the intrinsic value to exercise at time T, even in the as-if market. Note that our intrinsic values of the European and American calls coincide but that our intrinsic value of the American call does not conform with the conventional definition. In the case of an American put, the intrinsic value is maximized by exercising now in the as-if market such that our intrinsic value becomes $(K - S(t))^+$. Note that our intrinsic values for the European and American puts do not coincide but that our intrinsic value of the American put does conform with the conventional definition. The difference between the call and the put is of course related to the fact that the American call should not be exercised prematurely, not even in the as-if market, whereas the American put should, even in the as-if market.

A different way to think of our intrinsic value is that it is the value of the option in the original stochastic market in the case where the decision about whether to buy or not to buy (in the case of a call) is made on the basis of the information one has today. Our intrinsic value is the value of the option as-if one does not learn more about the realization of S before deciding to exercise the option to buy (in the case of a call).

We now consider a decomposition of the value process in its intrinsic value and time value

$$V(t, S(t), r(t)) = V^{\mathrm{IV}}(t, S(t), r(t)) + V^{\mathrm{TV}}(t, S(t), r(t)). \tag{3.3.1}$$

Such a decomposition is standard. Following our definition of the intrinsic value, the time value represents the value added from being allowed to base optional decisions in the future on future values and not just on the current values. Thus, time value is the value of information added over time. For the conventional intrinsic value (see above), the time value can be thought of as the value of not having to exercise necessarily today but being able to time the exercise better. For plain vanilla options, this has a clear meaning. For more general options, this idea may be more difficult to generalize, whereas our definition can be directly generalized.

Obviously, one can write

$$V^{\mathrm{TV}}(t, S(t), r(t)) = \alpha(t, S(t), r(t)) V^{\mathrm{IV}}(t, S(t), r(t)),$$

for some function α. We are going to work with an approximation of the time value where we disregard some of the arguments in the function α, for example, not allowing for a stochastic α means approximating, for a deterministic and possibly parametric function α, by

$$V^{\mathrm{TV}}(t, S(t), r(t)) \approx \alpha(t) V^{\mathrm{IV}}(t, S(t), r(t)).$$

This approximation reflects the idea that options lose their time value over time. This is obviously true when the time value reflects the value added by information added in the future. However, it is obviously an approximation to assume that the function is independent of the $(S(t), r(t))$. Even simpler is the approximation

$$V^{\mathrm{TV}}(t, S(t), r(t)) \approx \alpha V^{\mathrm{IV}}(t, S(t), r(t)),$$

for a constant α. This implies

$$V(t, S(t), r(t)) \approx (1+\alpha)V^{\mathrm{IV}}(t, S(t), r(t)). \tag{3.3.2}$$

Note that

$$V(T, S(T), r(T)) = \Phi\big(S(T)\big) = V^{\mathrm{IV}}(T, S(T), r(T)).$$

Therefore, if approximated by a function α, one would prefer a function fulfilling $\alpha(T) = 0$. The approximation by a constant $\alpha \neq 0$ is also necessarily inaccurate at time T.

In the following, we approximate the value via a parametric relation between the claim value, V, and the intrinsic value of the claim value, V^{IV}. The assumption is that the value process is linear in its intrinsic value. This does not mean that we truly believe that the value is linear in the intrinsic value. This is instead a first approximation that can be intuitively thought of as being based on a first order expansion of the value around the intrinsic value. Thus, we know that we already lose accuracy at this state. Other more involved parametric forms could be proposed and the steps of our method, as explained in the next section, could be properly adapted to any parametric form. We stress that our way of working with the intrinsic value, as a key to break down the original and utterly complicated problem into a series of solvable sub-problems, does not as such depend on the special case of linearity. That is merely chosen as a simple case of demonstration, and its merits and drawbacks are made visible in Section 3.5.

Remark 3.3.2. Assume that the interest rate is deterministic and that the drift term in the dynamics of the asset and the claim processes are linear in the underlying and the value process in the sense that the functions g, ϕ and Φ are in the form

$$\begin{aligned}
g(t, s, v) &= g_0(t) + g_1(t) \cdot s + g_2(t) \cdot v, \\
\phi(t, s, v) &= \phi_0(t) + \phi_1(t) \cdot s + \phi_2(t) \cdot v, \\
\Phi(s) &= \Phi_0 + \Phi_1 \cdot s,
\end{aligned}$$

for deterministic functions g_i and ϕ_i, $i = 0, 1, 2$ and $\Phi_0, \Phi_1 \in \mathbb{R}$. Then the value process is given by

$$V(t, S(t)) = h_0(t) + h_1(t) \cdot S(t),$$

and the intrinsic value of the value process is given by

$$V^{IV}(u, S(t)) = h_0(u) + h_1(u) \cdot S^{IV}(t, u),$$

for functions h_0 and h_1 that solve a system of ordinary differential equations. Hence, in the case with a deterministic interest rate and full linearity,

$$V(t, S(t)) = V^{IV}(t, S(t)),$$

and the time value of the value process is equal to zero. See Chapter 3 in Nyegaard (2019) for the derivation of the system of ordinary differential equations for h_0 and h_1.

Appendix 3.A investigates the quality of the intrinsic value approximation for $\phi = 0$.

3.3.2 Calculation of the Intrinsic Value

In this section, we solve the forward–backward element by an iteration in a deterministic as-if market. We study how to calculate the intrinsic value of the underlying and the intrinsic value of the value process. From Definition 3.3.1, we see that in order to calculate $V^{\mathrm{IV}}(t, S(t), r(t))$, we must solve the following system of differential equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} S^{\mathrm{IV}}(t,u) &= g\big(u, S^{\mathrm{IV}}(t,u), f(t,u), V^{\mathrm{IV}}(u, S(t), r(t))\big), \\ S^{\mathrm{IV}}(t,t) &= S(t), \\ \\ \frac{\mathrm{d}}{\mathrm{d}u} V^{\mathrm{IV}}(u, S(t), r(t)) &= f(t,u) V^{\mathrm{IV}}(u, S(t), r(t)) \\ &\quad - \phi\big(u, S^{\mathrm{IV}}(t,u), f(t,u), V^{\mathrm{IV}}(u, S(t), r(t))\big), \\ V^{\mathrm{IV}}(T, S(t), r(t)) &= \Phi\big(S^{\mathrm{IV}}(t,T)\big). \end{aligned} \tag{3.3.3}$$

The intrinsic value of the underlying and the intrinsic value of the value process satisfy a deterministic forward–backward system of ordinary differential equations given by (3.3.3). We propose two iteration methods to solve the forward–backward system of ordinary differential equations. The starting point of both methods is to suppress the entanglement of S and V that prevents us from solving the system of differential equations. The iteration procedures are performed in the hypothetical market set up at fixed time $t \leq T$ and are measurable with respect to $\mathcal{F}_t$, thus the price process of the asset S, the interest rate r, and the price of the process of the zero coupon bond $P(\cdot, \bar{T})$ are known up to and including time t.

The first method is a perturbation argument, where the forward–backward nature of the equations is preserved but the equations are decoupled. The second method is a shooting method where the boundary conditions are modified but we preserve a system of coupled equations. In both methods, the modification takes place in the first iteration to trigger the iteration procedure. We describe the first and the kth iteration for $k \geq 2$ in both methods. The objective of both methods is to solve the system of differential equations in Equation (3.3.3) in order to calculate the function $u \mapsto V^{\mathrm{IV}}(u, S(t), r(t))$.

Method I: Perturbation Method

The modification in the perturbation argument is a substitution in the differential equation of $S^{\mathrm{IV}}(t,u)$ in Equation (3.3.3), where we substitute the unknown $V^{\mathrm{IV}}(u,S(t),r(t))$ with a known function, which is measurable with respect to $\mathcal{F}_t$. We denote the function $u \mapsto v(u,S(t),r(t))$.

Iteration 1 In the first iteration, the intrinsic value of the stock index satisfies the differential equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}S^{\mathrm{IV},(1)}(t,u) &= g\big(u,S^{\mathrm{IV},(1)}(t,u),f(t,u),v(u,S(t),r(t))\big),\\ S^{\mathrm{IV},(1)}(t,t) &= S(t),\end{aligned}$$

and the intrinsic value of the value process satisfies the differential equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}V^{\mathrm{IV},(1)}(u,S(t),r(t)) &= f(t,u)V^{\mathrm{IV},(1)}(u,S(t),r(t))\\ &\quad - \phi\big(u,S^{\mathrm{IV},(1)}(t,u),f(t,u),V^{\mathrm{IV},(1)}(u,S(t),r(t))\big),\\ V^{\mathrm{IV},(1)}(T,S(t),r(t)) &= \Phi\big(S^{\mathrm{IV},(1)}(t,T)\big).\end{aligned}$$

This is a solvable system of differential equations. In the numerical study in Section 3.5, we choose $v(u,S(t),r(t)) = 0$.

Iteration k We use the fact that we know the intrinsic value of the value process from the previous iteration and insert this into the differential equation of S^{IV}. The intrinsic value of the stock index satisfies the differential equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}S^{\mathrm{IV},(k)}(t,u) &= g\big(u,S^{\mathrm{IV},(k)}(t,u),f(t,u),V^{\mathrm{IV},(k-1)}(u,S(t),r(t))\big),\\ S^{\mathrm{IV},(k)}(t,t) &= S(t).\end{aligned}$$

The intrinsic value of the value process satisfies the differential equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}V^{\mathrm{IV},(k)}(u,S(t),r(t)) &= f(t,u)V^{\mathrm{IV},(k)}(u,S(t),r(t))\\ &\quad - \phi\big(u,S^{\mathrm{IV},(k)}(t,u),f(t,u),V^{\mathrm{IV},(k)}(u,S(t),r(t))\big),\\ V^{\mathrm{IV},(k)}(T,S(t),r(t)) &= \Phi\big(S^{\mathrm{IV},(k)}(t,T)\big).\end{aligned}$$

This is a solvable system of differential equations.

Stopping Criteria We suggest the stopping criteria

$$\min_{k\geq 2}\left\{\left|V^{\mathrm{IV},(k)}(t,S(t),r(t)) - V^{\mathrm{IV},(k-1)}(t,S(t),r(t))\right| < \varepsilon\right\},$$

for $\varepsilon > 0$. Another criteria is to fix the number of iterations. Let κ be the resulting number of iterations. With the perturbation method, we estimate the solution to the system of the differential equations given by Equation (3.3.3), and the resulting estimate of the intrinsic value of the value process is

$$u \mapsto V^{\mathrm{IV},(\kappa)}(u, S(t), r(t)).$$

Method II: Shooting Method

The modification in the shooting method is the assumption that we know the boundary condition at time t in the differential equation of the intrinsic value of the value process in Equation (3.3.3). We assume that $V^{\mathrm{IV}}(t, S(t), r(t)) = \tilde{V}(t, S(t), r(t))$ for a known function $\tilde{V}$, which is measurable with respect to $\mathcal{F}_t$. In the numerical study in Section 3.5, we choose $\tilde{V}(t, S(t), r(t)) = 0$.

Iteration 1 We assume that

$$V^{\mathrm{IV},(1)}(t, S(t), r(t)) = \tilde{V}(t, S(t), r(t)). \tag{3.3.4}$$

We solve the following system of forward differential equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} S^{\mathrm{IV},(1)}(t,u) &= g\big(u, S^{\mathrm{IV},(1)}(t,u), f(t,u), V^{\mathrm{IV},(1)}(u, S(t), r(t))\big), \\ S^{\mathrm{IV},(1)}(t,t) &= S(t), \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} V^{\mathrm{IV},(1)}(u, S(t), r(t)) &= f(t,u) V^{\mathrm{IV},(1)}(u, S(t), r(t)) \\ &\quad - \phi\big(u, S^{\mathrm{IV},(1)}(t,u), f(t,u), V^{\mathrm{IV},(1)}(u, S(t), r(t))\big), \\ V^{\mathrm{IV},(1)}(t, S(t), r(t)) &= \tilde{V}(t, S(t), r(t)). \end{aligned}$$

This is a solvable system of differential equations.

If we solve the differential equations from Equation (3.3.3) with the boundary condition in Equation (3.3.4), we obtain

$$\begin{aligned} V^{\mathrm{IV},(1)}(T, S(t), r(t)) &= \tilde{V}(t, S(t), r(t)) e^{\int_t^T f(t,s)\mathrm{d}s} \\ &\quad - \int_t^T e^{\int_\tau^T f(t,s)\mathrm{d}s} \phi\big(\tau, S^{\mathrm{IV},(1)}(t,\tau), f(t,\tau), V^{\mathrm{IV},(1)}(\tau, S(t), r(t))\big) \mathrm{d}\tau. \end{aligned}$$

The boundary condition in Equation (3.3.3) states that

$$V^{\mathrm{IV}}(T, S(t), r(t)) = \Phi\big(S^{\mathrm{IV}}(t,T)\big).$$

The difference $V^{\mathrm{IV},(1)}(T, S(t), r(t)) - \Phi\big(S^{\mathrm{IV},(1)}(t,T)\big)$ is an estimate of how wrong our assumption is that $V^{\mathrm{IV},(1)}(t, S(t), r(t)) = \tilde{V}(t, S(t), r(t))$. We use the estimate to adjust the boundary condition of the intrinsic value of the value process at time t in the next iteration, such that we, in the second iteration, assume that

$$\begin{aligned}
&V^{\mathrm{IV},(2)}(t,S(t),r(t)) \\
&= \tilde{V}(t,S(t),r(t)) - e^{-\int_t^T f(t,s)\mathrm{d}s}\Big(V^{\mathrm{IV},(1)}(T,S(t),r(t)) - \Phi\big(S^{\mathrm{IV},(1)}(t,T)\big)\Big) \\
&= \int_t^T e^{-\int_t^\tau f(t,s)\mathrm{d}s}\phi\big(\tau, S^{\mathrm{IV},(1)}(t,\tau), f(t,\tau), V^{\mathrm{IV},(1)}(\tau,S(t),r(t))\big)\mathrm{d}\tau \\
&\quad + e^{-\int_t^T f(t,s)\mathrm{d}s}\Phi\big(S^{\mathrm{IV},(1)}(t,T)\big),
\end{aligned}$$

which is the solution to the differential equation of V^{IV} from Equation (3.3.3) with the boundary condition $V^{\mathrm{IV}}(T,S(t),r(t)) = \Phi\big(S^{\mathrm{IV},(1)}(t,T)\big)$.

Iteration k In the kth iteration, we assume that

$$\begin{aligned}
&V^{\mathrm{IV},(k)}(t,S(t),r(t)) \\
&= \int_t^T e^{-\int_t^\tau f(t,s)\mathrm{d}s}\phi\big(\tau, S^{\mathrm{IV},(k-1)}(t,\tau), f(t,\tau), V^{\mathrm{IV},(k-1)}(\tau,S(t),r(t))\big)\mathrm{d}\tau \\
&\quad + e^{-\int_t^T f(t,u)\mathrm{d}u}\Phi\big(S^{\mathrm{IV},(k-1)}(t,T)\big).
\end{aligned}$$

We solve the forward system of differential equations with the boundary condition above

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}S^{\mathrm{IV},(k)}(t,u) &= g\big(u, S^{\mathrm{IV},(k)}(t,u), f(t,u), V^{\mathrm{IV},(k)}(u,S(t),r(t))\big), \\
S^{\mathrm{IV},(k)}(t,t) &= S(t),
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}V^{\mathrm{IV},(k)}(u,S(t),r(t)) &= f(t,u)V^{\mathrm{IV},(k)}(u,S(t),r(t)) \\
&\quad - \phi\big(u, S^{\mathrm{IV},(k)}(t,u), f(t,u), V^{\mathrm{IV},(k)}(u,S(t),r(t))\big).
\end{aligned}$$

Hopefully, we have that

$$\Big|V^{\mathrm{IV},(k)}(T,S(t),r(t)) - \Phi\big(S^{\mathrm{IV},(k)}(t,T)\big)\Big| < \Big|V^{\mathrm{IV},(k-1)}(T,S(t),r(t)) - \Phi\big(S^{\mathrm{IV},(k-1)}(t,T)\big)\Big|,$$

such that we in the kth iteration are closer to the true value of the intrinsic value of the value process at time T than in the previous iteration.

Stopping Criteria We suggest the stopping criteria

$$\begin{aligned}
\min_k \Bigg\{\bigg| &\Big|V^{\mathrm{IV},(k)}(T,S(t),r(t)) - \Phi\big(S^{\mathrm{IV},(k)}(t,T)\big)\Big| \\
&- \Big|V^{\mathrm{IV},(k-1)}(T,S(t),r(t)) - \Phi\big(S^{\mathrm{IV},(k-1)}(t,T)\big)\Big|\bigg| < \varepsilon\Bigg\},
\end{aligned}$$

for $\varepsilon > 0$. Another criteria is to fix the number of iterations. Let κ be the resulting number of iterations. The resulting estimate of the intrinsic value of the value process is

$$u \mapsto V^{\mathrm{IV},(\kappa)}(u, S(t), r(t)).$$

3.4 Intrinsic Value Monte Carlo

We perform a standard forward Monte Carlo simulation based on the parametric relation between the value process and the intrinsic value of the value from Equation (3.3.2) for a given parameter $\alpha \in \mathbb{R}$. The objective is to estimate the value process V from Equation (3.2.3), when the price process of the underlying asset has the dynamics in Equation (3.2.4). Solutions to the system of ordinary differential equations in the deterministic as-if market from Section 3.3.2 are used as input in the Monte Carlo simulation.

Valuation of the value process in the setup from Section 3.2 is beyond a standard Monte Carlo simulation since the asset itself depends on the value process. The intrinsic value approximation of the value process enables us to simulate the assets despite the dependence of the value process.

Our objective is to use a Monte Carlo simulation to estimate the value process at a fixed time t_0

$$\begin{aligned}\hat{V}(t_0, S(t_0), r(t_0)) = \frac{1}{N}\sum_{i=1}^{N}\Bigg(\int_{t_0}^{T} e^{-\int_{t_0}^{\tau} r_i(u)\mathrm{d}u}\phi\big(\tau, S_i(\tau), r_i(\tau), V(\tau, S_i(\tau), r_i(\tau))\big)\mathrm{d}\tau \\ + e^{-\int_{t_0}^{T} r_i(u)\mathrm{d}u}\Phi\big(S_i(T)\big)\Bigg),\end{aligned} \tag{3.4.1}$$

for independent realizations $\big(S_1(T), (r_1(u))_{u\in[t_0,T]}\big), ..., \big(S_N(T), (r_N(u))_{u\in[t_0,T]}\big)$ of the asset and the interest rate. How to simulate the asset is not obvious when its dynamics depend on the unknown value process, nor is how to calculate the continuous payments ϕ since they also depend on the unknown value process.

3.4.1 Simulation of the Asset

We divide the interval $[t_0, T]$ in M equidistant subintervals

$$\begin{aligned} t_0 < t_1 < ... < t_M = T, \\ t_j - t_{j-1} = \Delta. \end{aligned}$$

We simulate N paths of the interest rate according to a Euler scheme based on its dynamics from Equation (3.2.2)

$$r_i(t_{j+1}) = r_i(t_j) + b\big(t_j, r_i(t_j)\big)\Delta + \gamma\big(t_j, r_i(t_j)\big)\sqrt{\Delta}Y_{i,j},$$

for i.i.d. $Y_{i,j} \sim \mathcal{N}(0,1)$ for $i = 1, ..., N$ and $j = 0, ..., M-1$. The forward interest calculated at time t_j for path i is based on $r_i(t_j)$ is $u \mapsto f_i(t_j, u)$ for $u \geq t_j$. We assume that the forward interest rates are known based on the simulated interest rates.

We simulate the underlying asset according to a Euler scheme based on its dynamics from Equation (3.2.4), and simulate N paths in the grid $(t_0, t_1, ..., t_M)$. The Euler scheme is

$$\begin{aligned} S_i(t_{j+1}) = S_i(t_j) &+ g\big(t_j, S_i(t_j), r_i(t_j), V(t_j, S_i(t_j), r_i(t_j))\big)\Delta \\ &+ \sigma\big(t_j, S_i(t_j), r_i(t_j), V(t_j, S_i(t_j), r_i(t_j))\big)\sqrt{\Delta} Z_{i,j}, \end{aligned} \tag{3.4.2}$$

for i.i.d. $Z_{i,j} \sim \mathcal{N}(0,1)$ for $i = 1, ..., N$ and $j = 0, ..., M-1$. We cannot simulate from this Euler scheme since it depends on the unknown value process. Our solution is to use the intrinsic value approximation of the value process from Equation (3.3.2).

The method is an iteration procedure, where we iterate over the value of α in the intrinsic value approximation in Equation (3.3.2). We also suggested two iteration methods for calculating the intrinsic value of the value process in Section 3.3.2. We denote the iteration over α as the outer iteration and the iteration to calculate V^{IV} as the inner iteration. The outer iteration procedure is described in Section 3.4.2. For now, we assume that the value of α is fixed, and we describe the simulation of the asset for a general value of α.

Inserting the intrinsic value approximation of the value process from Equation (3.4.2) in the Euler scheme yields

$$\begin{aligned} S_i(t_{j+1}) = S_i(t_j) &+ g\big(t_j, S_i(t_j), r_i(t_j), (1+\alpha)V^{\mathrm{IV}}(t_j, S_i(t_j), r_i(t_j))\big)\Delta \\ &+ \sigma\big(t_j, S_i(t_j), r_i(t_j), (1+\alpha)V^{\mathrm{IV}}(t_j, S_i(t_j), r_i(t_j))\big)\sqrt{\Delta} Z_{i,j}. \end{aligned}$$

We see in Section 3.3.2 above that we need an iteration procedure (denoted as the inner iteration) to calculate the intrinsic value of the value process, $V^{\mathrm{IV}}(t, S(t), r(t))$, for fixed t and $S(t)$ known. This implies that, with the Euler scheme above, we need to perform the inner iteration procedure $N \times (M-1)+1$ times in order to simulate N independent realizations of $S(T)$. Therefore, we implement the possibility of calculating the intrinsic value of the value process in a looser partition of the interval $[t_0, T]$ than in the partition in the Euler scheme above. Let $L \leq M-1$ and choose

$$\begin{aligned} &t_0 = \tau_0 < \tau_1 < ... < \tau_L < T, \\ &\big(\tau_0, \tau_1, ..., \tau_L\big) \subseteq \big(t_0, t_1, ..., t_{M-1}\big). \end{aligned}$$

We define the mapping

$$h(t) = \max_{l=0,...,L} \Big\{\tau_l \;\Big|\; \tau_l \leq t\Big\}, \quad \text{for } t \in [t_0, T].$$

Then, $h(t)$ returns the largest τ_l less than t. This results in the Euler scheme

$$\begin{aligned} S_i(t_{j+1}) = S_i(t_j) &+ g\big(t_j, S_i(t_j), r_i(t_j), (1+\alpha)V^{\mathrm{IV}}(t_j, S_i(h(t_j)), r_i(h(t_j)))\big)\Delta \\ &+ \sigma\big(t_j, S_i(t_j), r_i(t_j), (1+\alpha)V^{\mathrm{IV}}(t_j, S_i(h(t_j)), r_i(h(t_j)))\big)\sqrt{\Delta} Z_{i,j}. \end{aligned} \tag{3.4.3}$$

If $t_j = \tau_l$ for some $l = 0, ..., L$, we insert the intrinsic value of the value process at time t_j calculated at time t_j, $V^{\mathrm{IV}}(t_j, S_i(t_j), r_i(t_j))$, which depends on $S_i(t_j)$ and $r_i(t_j)$. If instead $t_j \neq \tau_l$ for any $l = 0, ..., L$, we insert the intrinsic value of the value process at time t_j calculated at the largest τ_l less than t_j, $V^{\mathrm{IV}}(t_j, S_i(h(t_j)), r_i(h(t_j)))$, which does not depend on $S_i(t_j)$ and $r_i(t_j)$, since we have not updated the calculation of V^{IV} with new information about the asset and the interest rate. We need $N \times L + 1$ calculations of V^{IV} with the inner iteration procedure to simulate N independent realizations of the asset if we simulate from the Euler scheme in Equation (3.4.3).

In the case $L = 0$, we only estimate the intrinsic value of the value process once, using one of the iteration procedures (the perturbation argument or the shooting method) based on $S(t_0)$ and insert $V^{\mathrm{IV}}(t_j, S(t_0), r(t_0))$ into the Euler scheme. This is the choice of L with the lowest computation time, since we only need one inner iteration procedure.

By solving the Euler scheme, we obtain N independent paths of the asset and the intrinsic value of the value process. We calculate a Monte Carlo estimate of the value process at time t_0 as

$$\begin{aligned} &\hat{V}(t_0, S(t_0), r(t_0)) \\ &= \frac{1}{N}\sum_{i=1}^{N}\bigg(\int_{t_0}^{T} e^{-\int_{t_0}^{\tau} r_i(u)\mathrm{d}u}\phi\big(\tau, S_i(\tau), r_i(\tau), (1+\alpha)V^{\mathrm{IV}}(\tau, S_i(\tau), r_i(\tau))\big)\mathrm{d}\tau \\ &\qquad + e^{-\int_{t_0}^{T} r_i(u)\mathrm{d}u}\Phi\big(S_i(T)\big)\bigg), \end{aligned}$$

with appropriate approximations of the integrals.

3.4.2 The Choice of α-functions

In this section, we describe the outer iteration procedure used in the determination of the parameter value in the intrinsic value approximation of the value process. Simulation of the asset from the Euler scheme in Equation (3.4.3) depends on the choice of α in the intrinsic value approximation of the value process from Equation (3.3.2).

The intention is to choose α such that the approximation is exact at time t_0

$$V(t_0, S(t_0), r(t_0)) = (1+\alpha)V^{\mathrm{IV}}(t_0, S(t_0), r(t_0)).$$

With the intrinsic value Monte Carlo method described above, we need the value of α to estimate $V(t_0, S(t_0), r(t_0))$, and therefore, we iterate over the value of α. This is denoted by the outer iteration.

In the first outer iteration, we choose $\alpha = \alpha^{(\mathrm{I})}$. Based on this value of α, we simulate the asset according to the Euler scheme in Equation (3.4.3), which includes $N \times L + 1$ inner iterations of either the perturbation argument or the shooting method, and calculate a Monte Carlo estimate of the value process at time t_0:

$$\begin{aligned} &\hat{V}^{(\mathrm{I})}(t_0, S(t_0), r(t_0)) \\ &= \frac{1}{N} \sum_{i=1}^{N} \bigg(\int_{t_0}^{T} e^{-\int_{t_0}^{\tau} r_i(u)\mathrm{d}u} \phi\big(\tau, S_i(\tau), r_i(\tau), (1+\alpha^{(\mathrm{I})}) V^{\mathrm{IV}}(\tau, S_i(\tau), r_i(\tau))\big)\mathrm{d}\tau \\ &\qquad + e^{-\int_{t_0}^{T} r_i(u)\mathrm{d}u} \Phi\big(S_i(T)\big) \bigg), \end{aligned}$$

with appropriate approximations of the integrals. For the second outer iteration, we choose α such that

$$\hat{V}^{(\mathrm{I})}(t_0, S(t_0), r(t_0)) = (1 + \alpha^{(\mathrm{II})}) V^{\mathrm{IV}}(t_0, S(t_0), r(t_0)),$$

and estimate the value process at time t_0 based on this value of α. We continue the outer iteration procedure until α (or equivalently the estimate of the value process) reaches a fixed point.

3.4.3 Relations to Least-Squares Monte Carlo and Nested Simulation

Simulation of the asset from the Euler scheme from Equation (3.4.2) requires that we know or can estimate the value process at time $t_j > t_0$, $V(t_j, S_i(t_j), r_i(t_j))$. The value process from Equation (3.2.3) is a conditional expectation. The estimation of future conditional expectations also appears in the valuation of American options to establish when to exercise the option. The authors of Longstaff et al. (2001) proposed least-squares Monte Carlo for estimating future conditional expectations when valuing American options. Another solution is nested simulation.

The main difference between our intrinsic value Monte Carlo method and Longstaff et al. (2001) is that, in Longstaff et al. (2001), they are able to simulate the underlying whereas in our setup, the underlying depends on the conditional expectation since we feed back the value process into the dynamics of the underlying, which complicates the simulation. A perturbation argument, as described in Section 3.3.2, on the dynamics of the asset from Equation (3.2.4), combined with the least-squares Monte Carlo method, is an alternative to the intrinsic value Monte Carlo method in order to estimate the value process in Equation (3.2.3), but the clarification of this is beyond the scope of this paper. If the feedback of the value process into the underlying is eliminated and the dependence of the value process V only appears in the process of claims, ϕ, classical least-squares Monte Carlo can be used to estimate the value process.

3.5 Numerical Study

In this section, we illustrate our approximation methods with an example. Specifically, we consider the corporate finance example (Example 3.2.1) with the underlying assets of a firm, $S(t)$, governed by a Black–Scholes like market with dividends that depend on the value of the equity, $V(t, S(t))$, which again depends on the assets in the future. For simplicity, we choose the dividends to be linear in the equity value. In this situation, Equation (3.2.1) reduces to

$$\mathrm{d}S(t) = \big(rS(t) - \delta V(t, S(t))\big)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t), \tag{3.5.1}$$

with the constant interest rate r, the dividend yield $\delta V(t, S(t))$ for $\delta \in \mathbb{R}$, which depend on the value of the equity, and $\sigma > 0$ is the volatility. When the interest rate is constant, $f(t, u) = r$ for all $t \geq 0$ and $u \geq t$.

For the example model, the Euler scheme, Equation (3.4.2), reduces to

$$S_i(t_{j+1}) = S_i(t_j) + \big(rS_i(t_j) - \delta V(t_j, S_i(t_j))\big)\Delta + \sigma S_i(t_j)\sqrt{\Delta} Z_{i,j}, \tag{3.5.2}$$

and the Monte Carlo simulation of the option value, Equation (3.4.1), reduces to

$$\hat{V}(t_0, S(t_0)) = \frac{1}{N}\sum_{i=1}^{N}\left(\int_{t_0}^{T} e^{-(\tau - t_0)r}\delta V(\tau, S_i(\tau))\mathrm{d}\tau + e^{-(T-t_0)r}\Phi\big(S_i(T)\big)\right), \tag{3.5.3}$$

with $\Phi(s) = \big(s - K\big)^+$, where K is the debt of the firm payable at time T. Equations (3.5.2) and (3.5.3) illustrate the problem that we aim to solve. The two equations depend on each other, and the asset is governed by a forward equation while the value is governed by a backward equation. This entanglement is solved by the approximation methods described in this paper.

3.5.1 Approximation Method Details

The approximation methods consist of a combination of two iteration schemes, which we call the inner and the outer iterations. We describe the schemes separately below and note that we have sketched the numerical schemes in Appendix 3.B.

The Outer Iteration

The outer iteration is based on the intrinsic value Monte Carlo method described in Section 3.4. The method is to approximate the value $V(t_j, S_i(t_j))$ in the Euler scheme in Equation (3.5.2) by assuming that it is proportional to the intrinsic value defined in Section 3.3. The Euler scheme is then given by

$$S_i(t_{j+1}) = S_i(t_j) + \big(rS_i(t_j) - \delta(1+\alpha)V^{\mathrm{IV}}(t_j, S_i(t_j))\big)\Delta + \sigma S_i(t_j)\sqrt{\Delta}Z_{i,j}. \quad (3.5.4)$$

This way, the asset dynamics are disentangled from the value process. The proportionality factor $1+\alpha$ is then determined in an iterative way such that the value of the factor in the next iteration would make the intrinsic value approximation correct.

The calculation of the intrinsic value is the subject of the next section.

The Inner Iteration

The calculation of the intrinsic value involves solving a forward–backward system of equations. Although the equations are deterministic, this is not a trivial task in itself. In the paper, we have presented two approximation methods and, since both of the methods are iterative, we call this step the inner iteration.

In this example, the intrinsic value of the asset and the value process at a fixed time t satisfy the differential equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u}S^{\mathrm{IV}}(t,u) &= rS^{\mathrm{IV}}(t,u) - \delta V^{\mathrm{IV}}(u,S(t)), \\ S^{\mathrm{IV}}(t,t) &= S(t), \\ \frac{\mathrm{d}}{\mathrm{d}u}V^{\mathrm{IV}}(u,S(t)) &= (r-\delta)V^{\mathrm{IV}}(u,S(t)), \\ V^{\mathrm{IV}}(T,S(t)) &= \Phi\big(S^{\mathrm{IV}}(t,T)\big). \end{aligned} \quad (3.5.5)$$

In the present case, the intrinsic value can actually be calculated analytically, but this is not always the case. Instead, we use the two approximation methods described in Sections 3.3.2 and 3.3.2.

Perturbation Method In the perturbation method, the forward–backward nature of the equations is preserved, but the equations are decoupled. This is done by initially assuming that, to the lowest order, the dividend is small such that it is negligible in the differential equation of the intrinsic value of asset, which can then be used to calculate an approximation of the intrinsic value of the equity, which then in turn can be used to calculate a better approximation of the intrinsic value of the asset and so on. That is, the k'th iteration of Equation (3.5.5) can be written as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u}S^{\mathrm{IV},(k)}(t,u) &= rS^{\mathrm{IV},(k)}(t,u) - \delta V^{\mathrm{IV},(k-1)}(u,S(t)), \\ S^{\mathrm{IV},(k)}(t,t) &= S(t), \\ \frac{\mathrm{d}}{\mathrm{d}u}V^{\mathrm{IV},(k)}(u,S(t)) &= (r-\delta)V^{\mathrm{IV},(k)}(u,S(t)), \\ V^{\mathrm{IV},(k)}(T,S(t)) &= \Phi\big(S^{\mathrm{IV},(k)}(t,T)\big), \end{aligned}$$

for $V^{\mathrm{IV},(0)}(u, S(t)) = 0$ for all $u \geq t$.

Shooting Method In the shooting method, the forward–backward nature of the equations is removed while preserving a system of coupled equations. This is done by guessing an initial value of the value process and then solving the coupled forward equations. The result is checked against the actual final value of the value process, then the initial value is adjusted, and the system of equations is solved again. That is, the k'th iteration of Equation (3.5.5) can be written as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u} S^{\mathrm{IV},(k)}(t,u) &= r S^{\mathrm{IV},(k)}(t,u) - \delta V^{\mathrm{IV},(k)}(u, S(t)), \\
S^{\mathrm{IV},(k)}(t,t) &= S(t), \\
\frac{\mathrm{d}}{\mathrm{d}u} V^{\mathrm{IV},(k)}(u, S(t)) &= (r-\delta) V^{\mathrm{IV},(k)}(u, S(t)), \\
V^{\mathrm{IV},(k)}(t, S(t)) &= e^{-(r-\delta)(T-t)} \Phi\big(S^{\mathrm{IV},(k-1)}(t,T)\big),
\end{aligned}$$

for $\Phi\big(S^{\mathrm{IV},(0)}(t,T)\big) = 0$.

3.5.2 Finite Difference Method

For our example model in Equation (3.5.1), an exact numerical solution of the value, V, can be obtained by numerically solving the partial differential equation (PDE) given by the corresponding Feynmann–Kac formula

$$\frac{\partial V}{\partial t} = -rs\frac{\partial V}{\partial s} + \delta V \frac{\partial V}{\partial s} - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rV - \delta V,$$

with the boundary condition given by the payoff $V(T, s) = \Phi(s)$.

We solve the PDE on a rectangular grid for the spot and time dimensions. In the time dimension, we transform to a forward equation and use the explicit Euler method for the time stepping. For the spot dimension, we make the usual transformation into a log spot grid, $x = \ln s$, for numerical stability and choose a central difference scheme. Other numerical schemes can be used, but we remind the reader to ensure numerical stability when choosing a solution method.

3.5.3 Numerical Results

After having described the different numerical schemes, we are now ready to present the resulting values calculated by the schemes. We calculate the value as a function of the initial spot, both in units of K. The numerical parameters are described in the caption of Figure 3.1, where we show the value, $V(S_0, t_0)$, as a function of initial spot, S_0, as calculated using the approximation methods and compared to the numerically exact solution of the PDE. The left plots show the perturbation method and the right plots show the shooting method. The value is calculated for different dividend rates δ from 0.0, corresponding to no dividends (lower plots) and up to 0.5 corresponding to paying half the value in dividends (top plots). In each plot, the solid graph is the numerically exact value as calculated by the PDE while the values given by the approximation methods are calculated with either one, two, or five updates of the intrinsic value in the Monte Carlo simulation. All the plots have the same axis for better visual comparison. Let us carefully walk through what can be seen.

Let us first notice that the two approximation methods (the perturbation method and the shooting method) yield very similar results and, for this reason, we need not distinguish between these. Let us next look at the lower plots corresponding to no dividend, that is, a pure Black–Scholes market. In this case, the market is decoupled from the value and the outer iteration reduces to a Monte Carlo simulation of the Black–Scholes market, see Equation (3.5.4). For this reason, the accuracy of the approximation methods are independent of the number of updates of the intrinsic value, N_t. Lastly, as we increase the dividend ratio (middle and top plots), the market and the value are no longer decoupled and simple Monte Carlo no longer suffices. Therefore, we need the methods presented in this paper. If we turn our attention to the top plots where the dividend rate is largest, the value calculated with only one update of the intrinsic value overestimates the actual value, but as we increase the number of updates of the intrinsic value, the approximation methods converge to the exact value. Specifically, we note that the methods reach the accuracy of the Monte Carlo simulation with five updates of the intrinsic value.

An interesting feature of the intrinsic value methods is that it is directly possible to find the time value of the value process from Equation (3.3.1), which represents the value added from being allowed to base optimal decisions in the future on future values, since α is the ratio between the time value and the intrinsic value. This is shown in Figure 3.2 for $\delta = 0.5$ for both methods and with a different number of updates of the intrinsic value, N_t. We note three observations. First, the two methods yield similar α, as they should. Second, when the intrinsic value is not updated frequently enough, the approximation methods overestimate the time value in the present example. Third, α decreases as we are increasingly in-the-money. This is also what we would expect since stochasticity has less influence on the value far from the strike.

Lastly, in Figure 3.3, we show the value (top plot) and α (bottom plot) as a function of the initial spot, S_0, including the crossover from being in-the-money to being out-of-the-money at $S_0 = 1$ for different dividends. When out-of-the-money, the intrinsic value, V^{IV}, is equal to zero and the outer iteration in Equation (3.5.4) becomes independent of α and reduces to a Monte Carlo simulation in a pure Black–Scholes market for all values of the dividend rate δ. Therefore, our calculations for the three values of δ coincide when out-of-the-money. A significant part of the value is caused

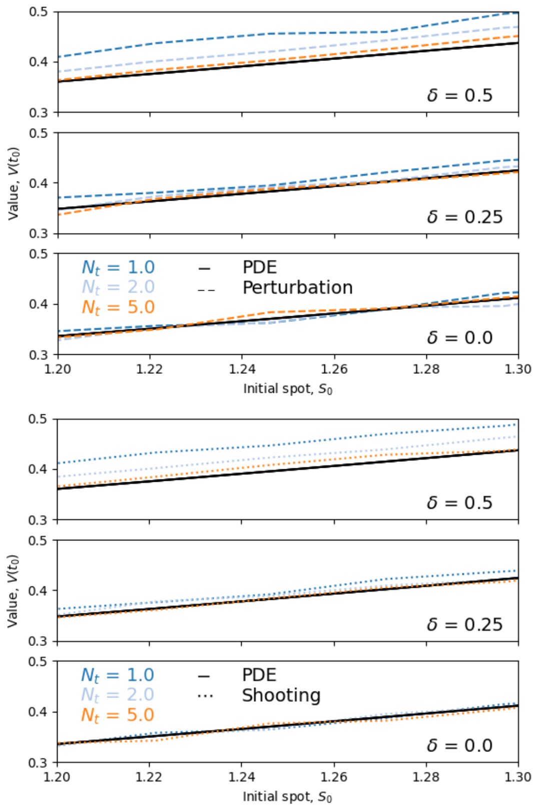

Figure 3.1: *The value as a function of the initial spot as calculated using the perturbation and shooting methods for different dividend rates, δ, and number of updates of the intrinsic value, N_t. The calculated values are compared to numerically exact values calculated by the PDE. The PDE is calculated on a grid with 10^2 spot and 10^3 time points. The intrinsic value methods are calculated with 10^4 Monte–Carlo paths that have 10^2 time points. The inner and outer iterations are stopped after 5 iterations.*

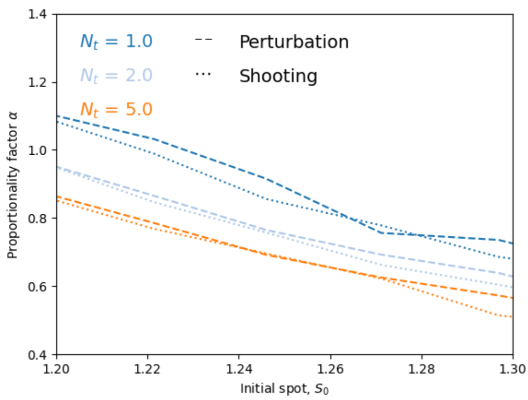

Figure 3.2: *The proportionality factor, α, as a function of the initial spot, S_0, calculated using the perturbation and shooting methods for a dividend rate of $\delta = 0.5$ and for various numbers of updates of the intrinsic value, N_t.*

by the time value when the initial spot is close to the strike, and therefore the value of α increases when the initial spot approaches the strike. The consequence is that our method is unstable close to the strike around $S_0 = 1$. When in-the-money, the method is stable and we saw already in Figure 3.1 that it converges fast and properly to the true equity value.

In this section, we have discussed the performance and quality of the approximation method proposed in this paper. We have seen that our method estimates quite accurately the equity value of a firm in a corporate finance setup with dividend payments linear in the equity value itself. This is clearly a numerically challenging problem but in our example it is possible to disentangle, without really losing accuracy, the forward–backward equation for the intrinsic value and the stochastic simulation via an assumption of the time value relation to the intrinsic value. The specific numerical illustration relies on the choice of dividend function (here linearity), the choice of parametric relation (here linearity), and terminal claim (here the standard call option payoff). Although both the parametric relation and the terminal claim are here chosen naturally, the lack of correctness when out-of-the-money and the instability at-the-money are a consequence of their relation. It is left to future works to better understand how sensitive the high quality of the approximation is to these assumptions and how the parametric relation might be improved to avoid problems when not in-the-money. However, for this particular corporate value example, one may note the following. When not in-the-money, that is, when the asset value does not exceed the value of the debt, a delicate equity valuation is predominantly relevant for areas such as credit risk assessment. Then, the firm is probably bankrupt by solvency rules that do not rely on the market valuation performed here. So, within that context, we live with our method being approximately accurate only when in-the-money. Of course, introducing premature bankruptcy in the arguments calls for introducing premature bankruptcy within the model. This is also left for future studies.

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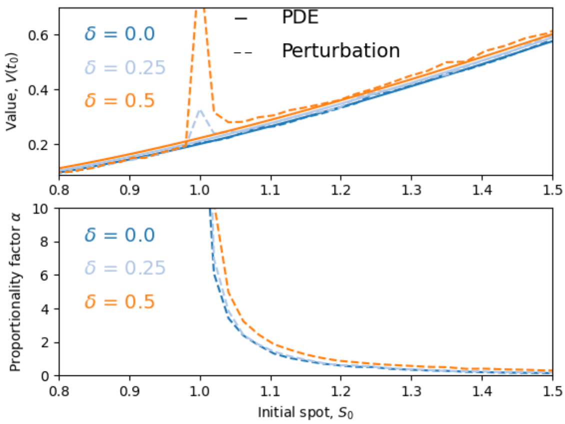

Figure 3.3: *The value and the proportionality factor, α, as a function of the initial spot, S_0, on an interval including the cross over from being out-of-the-money to being in-the-money at $S_0 = 1$ for different dividends rates, δ, and with 5 updates of the intrinsic value, $N_t = 5$.*

3.A Quality of the Intrinsic Value Approximation

We study the quality of the approximation in Equation (3.3.2) under the assumption that there is no continuous process of claims, $\phi = 0$, and that the claim Φ is infinitely differentiable. The Taylor series of $\Phi\big(S(T)\big)$ around $S^{\mathrm{IV}}(t,T)$ is

$$\Phi\big(S(T)\big) = \sum_{n=0}^{\infty} \frac{1}{n!}\Phi^{(n)}\big(S^{\mathrm{IV}}(t,T)\big)\Big(S(T) - S^{\mathrm{IV}}(t,T)\Big)^n,$$

where $\Phi^{(n)}(s) = \frac{\mathrm{d}^n}{\mathrm{d}s^n}\Phi(s)$. With the Taylor series above, we may write

$$\begin{aligned}
&V(t,S(t),r(t))\\
&= \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(s)\mathrm{d}s}\Phi\big(S(T)\big) \;\Big|\; S(t), r(t)\Big]\\
&= \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(s)\mathrm{d}s}\sum_{n=0}^{\infty} \frac{1}{n!}\Phi^{(n)}\big(S^{\mathrm{IV}}(t,T)\big)\Big(S(T) - S^{\mathrm{IV}}(t,T)\Big)^n \;\Big|\; S(t), r(t)\Big]\\
&= \Bigg(1 + \sum_{n=1}^{\infty} \frac{\Phi^{(n)}\big(S^{\mathrm{IV}}(t,T)\big)}{n!\Phi\big(S^{\mathrm{IV}}(t,T)\big)}\\
&\quad \times \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T (r(s)-f(t,s))\mathrm{d}s}\Big(S(T) - S^{\mathrm{IV}}(t,T)\Big)^n \;\Big|\; S(t), r(t)\Big]\Bigg) \times V^{\mathrm{IV}}(t,S(t),r(t)).
\end{aligned}$$

Hence, the quality of the approximation in Equation (3.3.2) relies on how well we can approximate $\alpha(t,S(t),r(t))$ with a constant α. We study the approximation analytically in the classic Black-Scholes market.

Example 3.A.1. Let the asset have dynamics

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dW(t),$$

for a deterministic interest rate $r \in \mathbb{R}$ and volatility $\sigma > 0$, and where W is a Brownian motion under the risk neutral measure $\mathbb{Q}$. Since the interest rate is deterministic, the forward interest rate is equal to r. The intrinsic value of the asset is

$$S^{IV}(t,u) = S(t)e^{r(u-t)},$$

and

$$\mathbb{E}^{\mathbb{Q}}\Big[S(T) - S^{IV}(t,T) \;\Big|\; S(t)\Big] = 0,$$
$$\mathbb{E}^{\mathbb{Q}}\Big[\big(S(T) - S^{IV}(t,T)\big)^2 \;\Big|\; S(t)\Big] = S^{IV}(t,T)^2\big(e^{\sigma^2(T-t)} - 1\big).$$

We consider a first order polynomial claim

$$\Phi(s) = a + b \cdot s \quad \Rightarrow \quad \alpha(t, S(t)) = 0.$$

Hence, for a first order polynomial claim, the approximation in Equation (3.3.2) is exact for $\alpha = 0$ in the classic Black-Scholes market.

For a second order polynomial claim, we have that

$$\Phi(s) = a + b \cdot s + c \cdot s^2 \quad \Rightarrow \quad \alpha(t, S(t)) = \frac{2 \cdot c \cdot S^{IV}(t,T)^2\big(e^{\sigma^2(T-t)} - 1\big)}{2\big(a + b \cdot S^{IV}(t,T) + c \cdot S^{IV}(t,T)^2\big)}.$$

For $a = b = 0$, the function α depends solely on time to maturity and the volatility. Therefore, if we allow for a time dependent α in the approximation in Equation (3.3.2), the approximation is exact with the claim $\Phi(s) = c \cdot s^2$.

In general, for a polynomial claim in the form $\Phi(s) = c \cdot s^M$, $M \in \mathbb{N}$, in the classic Black-Scholes market, the approximation in Equation (3.3.2) is exact if we allow for an α which depends on time.

In general, the quality of the intrinsic value approximation of the value process depends on how well $S^{\mathrm{IV}}(t,T)$ approximates $S(T)$ and the behaviour of Φ and its derivatives. The size of the terms $\mathbb{E}^{\mathbb{Q}}\Big[\big(S(T) - S^{\mathrm{IV}}(t,T)\big)^n \;\Big|\; S(t)\Big]$, $n \in \mathbb{N}$, in the expression of $\alpha(t, S(t))$ depends on the volatility function σ and a time to maturity $T - t$, since we disregard the volatility in the interval $[t,T]$ when we define $S^{\mathrm{IV}}(t,T)$.

3.B Sketch of Algorithms

3.B.1 Outer Iteration

The algorithm for the outer iteration can be sketched as follows

```
alpha[1] = alpha_0
for n_alpha = 1 : N_alpha
    S[:,1] = S_0
    V_IV[1:N_t] = CalculateIntrinsicValue(t[1:N_t], S_0)
    for i = 1 : N_MC
        for j = 1 : N_t-1
            if j in t_IV
                V_IV[j:N_t] =
                    CalculateIntrinsicValue(t[j:N_t], S[i,j])
            end
        S[i,j+1] = Euler(t[j], S[i,j], (1+alpha[n_alpha])*
            V_IV[j:N_t])
        end
    end
    V = CalculateMonteCarloValue(t[:], S[:,:])
    if V_IV[1] != 0
        alpha[n_alpha + 1] = V/V_IV[1] - 1
    else
        alpha[n_alpha + 1] = alpha[n_alpha]
    end
end
```

3.B.2 Inner Iteration

Perturbation Method

The algorithm for the perturbation method iteration can be sketched as follows

```
V_IV[:] = 0
for k = 1 : N_k
    S_IV[1] = S_0
    for j = 1 : N_t-1
        S_IV[j+1] = Euler(t[j], S_IV[j], V_IV[j])
    end
    V_IV[N_t] = Phi(S:IV[N_t])
    for j = N_t : 1
        V_IV[j-1] = BackwardEuler(t[j], V_IV[j], S_IV[j])
```

```
    end
end
```

Shooting Method

The algorithm for the shooting method iteration can be sketched as follows

```
V_IV[1] = V_0
for k = 1 : N_k
    S_IV[1] = S_0
    for j = 1 : N_t-1
        S_IV[j+1] = Euler(t[j], S_IV[j], V_IV[j])
        V_IV[j] = Euler(t[j], S_IV[j])
    end
    V_IV[1] = Discount(t[1], t[Nt], S_IV[Nt])
end
```

Chapter 4

Reserve-dependent management actions in life insurance

This chapter contains the paper *Falden and Nyegaard (2023)*.

Abstract

In a set-up of with-profit life insurance including bonus, we study the calculation of the market reserve, where Management Actions such as investment strategies and bonus allocation strategies depend on the reserve itself. Since the amount of future bonus depends on the retrospective savings account, the introduction of Management Actions that depend on the prospective market reserve results in an entanglement of retrospective and prospective reserves. We study the complications that arise due to the interdependence between retrospective and prospective reserves, and characterize the market reserve by a partial differential equation (PDE). We reduce the dimension of the PDE in the case of linearity, and furthermore, we suggest an approximation of the market reserve based on the forward rate. The quality of the approximation is studied in a numerical example.

Keywords: With-profit life insurance; Bonus; Prospective reserves; Management actions.

4.1 Introduction

In this paper, we study the calculation of the market reserve of a with-profit life insurance contract in a set-up, where the so-called Management Actions have a complex structure. The market reserve is the expected present value of future guaranteed and non-guaranteed payments from the insurer to the insured, and the Management Actions influence the payments of a life insurance contract, for instance, through the investment strategy and the bonus allocation strategy. Especially, the non-guaranteed payments are influenced by future Management Actions. The life insurance company takes many considerations into account when deciding on its Management Actions, and the decisions depend on the financial situation of the company, which is measured by the balance sheet. A fair redistribution of bonus is of great importance in with-profit life insurance, such that the policyholders who contributed to the surplus receive a reasonable amount of bonus. In order to fairly model the future bonus allocation strategy, we need a sophisticated model that takes the entire balance sheet into account. In our model, we allow the future Management Actions to depend on all balance sheet items, and the dependence on the market reserve complicates the set-up.

The modelling of bonus in with-profit life insurance is studied in Norberg (1999), Steffensen (2006) and Asmussen and Steffensen (2020). We extend the model from Asmussen and Steffensen (2020) to allow for a broader range of investment and bonus allocation strategies, and characterize the prospective market reserve within this model. The core of the model is the surplus that arises due to prudent assumptions about the interest rate and insurance risks on which payments are specified at initialization of the life insurance contract. By legislation, the surplus is to be paid back to the policyholders as bonus. We use the bonus scheme spoken of as additional benefits, where bonus is used to buy more insurance, and therefore, the savings account of the insurance contract is influenced by bonus in terms of dividend payments. This results in a link between the savings account and the guaranteed payments, which is different from the set-up in Steffensen (2006),where dividends only depend on the surplus, and guaranteed payments are not influenced by dividends. With the introduction of Management Actions that depend on the market reserve, the stochastic differential equation of the retrospective savings account and the retrospective surplus depend on the prospective market reserve. This paper studies the complications that arise due to the interdependence between retrospective and prospective reserves caused by the structure of the Management Actions. The result is a characterization of the market reserve by a partial differential equation (PDE) for a general model of the financial market with methods inspired by Steffensen (2000). We reduce the dimension of the PDE under the assumption of linearity of the dividend strategy with calculations similar to those in Steffensen (2006), and suggest an approximation of the market reserve based on the forward interest rate. The quality of the approximation is studied in a numerical example.

Christiansen, Denuit, and Dhaene (2014) study reserve-dependence in benefits and costs in a life insurance set-up without bonus, and characterize the prospective market reserve by a Thiele differential equation. The inclusion of bonus in our set-up prevents us from applying the results from Christiansen, Denuit, and Dhaene (2014). The results in this paper combine the modelling of bonus in life insurance from Asmussen and Steffensen (2020) with reserve-dependence from Christiansen, Denuit, and Dhaene (2014). Djehiche and Löfdahl (2016) study nonlinear reserve-dependence in life insurance payments in a set-up without bonus and derive a backward stochastic differential equation (BSDE) of the prospective reserve. Under the Markov assumption, Djehiche and Löfdahl (2016) derive the nonlinear Thiele's equation from the BSDE. We use a similar Markov assumption to derive the PDE of the prospective market reserve in our set-up with bonus. From a simulation point of view, the entanglement of retrospective and prospective reserves is notoriously difficult to handle, and Nyegaard, Ott, and Steffensen (2021) propose a simulation method to disentangle the problem based on intrinsic values. The derivation of the PDE of the prospective market reserve under the assumption of reserve-dependent Management Actions draws parallels to the valuation of contingent claims and option pricing of financial derivatives. Especially the valuation of American options, where the decision to exercise depends on the value of the option itself. Valuation of American options is studied in, for instance, Rogers (2003) and Haugh and Kogan (2004), but our model contains an additional layer of complexity, since the underlying savings account depends on the prospective reserve. Therefore, we cannot use existing valuation methods for American options as presented in for instance Rogers (2003) and Haugh and Kogan (2004) to calculate the prospective reserve in our model. Bruhn and Lollike (2020) and Ahmad, Buchardt, and Furrer (2022) focus on a projection model of the retrospective savings account and surplus in a setting similar to ours, but where the dividend strategy is restricted to depend on the state of the insured, the savings account and the surplus. The inclusion of the prospective market reserve in the specification of the dividend strategy makes projection of the savings account and the surplus with the methods developed in Bruhn and Lollike (2020) and Ahmad, Buchardt, and Furrer (2022) impossible.

The structure of the paper is the following. In Section 4.2, we present the set-up of with-profit life insurance including bonus, introduce a model of the financial market, define the assets and liabilities of the insurance company, and link Management Actions in terms of investments and dividends to the market reserve. Calculation of the market reserve is studied in Section 4.3, where we derive the PDE, and study the case of linearity. A numerical example in Section 4.4 emphasizes the practical applications of our result.

4.2 Reserves in Life Insurance

We introduce the set-up of with-profit life insurance including bonus from Asmussen and Steffensen (2020) in a general financial market. Two decompositions of the liabilities of the insurer are presented, and we link Management Actions in terms of dividends and the investment strategy to the liabilities.

4.2.1 Set-up

We consider the classical model of a life insurance contract, as presented in, for instance, Norberg (1991), where a Markov process $Z = \big(Z(t)\big)_{\{t\geq 0\}}$ on a finite state space $\mathcal{J}$ describes the state of the policyholder of a life insurance contract. Payments in the contract link with sojourns in states and transitions between states.

The transition probabilities of Z are given by

$$p_{ij}(s,t) = \mathbb{P}\big(Z(t) = j \;\big|\; Z(s) = i\big),$$

for $i, j \in \mathcal{J}$ and $s \leq t$. We assume that the transition intensities

$$\mu_{ij}(t) = \lim_{h\downarrow 0} \frac{1}{h} p_{ij}(t, t+h),$$

exist for $i, j \in \mathcal{J}$, $i \neq j$ and are suitably regular. The process $N^k(t)$ counts the number of jumps of Z into state $k \in \mathcal{J}$ up to and including time t

$$N^k(t) = \#\big\{s \in (0,t] \;\big|\; Z(s-) \neq k, Z(s) = k\big\},$$

where $Z(s-) = \lim_{h\downarrow 0} Z(s-h)$. Let $\mathcal{F}^Z = \big(\mathcal{F}_t^Z\big)_{t\geq 0}$ be the natural filtration generated by the state process Z.

We consider a general financial market, where the insurance company invests in a money market account governed by the interest rate r and K traded assets. The financial market is assumed to be free of arbitrage resulting in the existence of a (not necessarily unique) martingale measure $\mathbb{Q}$. All quantities in the model of the financial market are modelled directly under the martingale measure.

The interest rate is modelled as a diffusion process with dynamics

$$\mathrm{d}r(t) = \alpha_r(t, r(t))\mathrm{d}t + \sigma_r(t, r(t))\mathrm{d}W_r(t), \tag{4.2.1}$$

where W_r is a Brownian motion under the martingale $\mathbb{Q}$, and $\alpha_r : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ and $\sigma_r : [0,\infty) \times \mathbb{R} \to (0,\infty)$ are deterministic and sufficiently regular functions.

The general market consists of a money market account with dynamics

$$\mathrm{d}S_0(t) = r(t)S_0(t)\mathrm{d}t,$$

and K traded assets $S(t) = \big(S_1(t), ..., S_K(t)\big)^T$ with dynamics

$$\mathrm{d}S(t) = r(t)S(t)\mathrm{d}t + \tilde{\sigma}(t, S(t), r(t))\mathrm{d}W(t), \tag{4.2.2}$$

where $W(t) = (W_1(t), ..., W_M(t))^T$ is a M-dimensional Brownian motion under $\mathbb{Q}$ independent of $W_r(t)$, and where

$$\tilde{\sigma}(t,s,r) = \begin{pmatrix} \sigma_{11}(t,s,r)\cdot s_1 & \sigma_{12}(t,s,r)\cdot s_1 & \dots & \sigma_{1M}(t,s,r)\cdot s_1 \\ \sigma_{21}(t,s,r)\cdot s_2 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_{K1}(t,s,r)\cdot s_K & \sigma_{K2}(t,s,r)\cdot s_K & \dots & \sigma_{KM}(t,s,r)\cdot s_K \end{pmatrix},$$

for $s \in \mathbb{R}^K$ and sufficiently regular and deterministic functions $\sigma_{ij} : [0,\infty)\times\mathbb{R}^K\times\mathbb{R} \to (0,\infty)$. The natural filtration generated by the financial market is $\mathcal{F}^S = \big(\mathcal{F}_t^S\big)_{t\geq 0}$, and the combined information about the state process Z and the financial market at time t is given by $\mathcal{F}_t = \mathcal{F}_t^S \cup \mathcal{F}_t^Z$. We assume independence between the state process Z and the financial market. With this specification of the financial market, the interest rate and the traded assets, $\big(r(t), S(t)\big)$, are Markov, and the ideas presented in this paper rely on the Markov property of the financial market. Our results generalize directly to any financial market, that is Markov and independent of the state process Z.

Furthermore, we assume the existence of a suitable regular forward interest rates $u \mapsto f(t,u)$ for $t \geq 0$, which satisfies

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s r(u)\mathrm{d}u} \,\middle|\, \mathcal{F}_t^S\right] = e^{-\int_t^s f(t,u)\mathrm{d}u},$$

and $f(t,t) = r(t)$ for all $0 \leq t \leq s$. The forward interest rate $u \mapsto f(t,u)$ is measurable with respect to $\mathcal{F}_t^S$.

The insurance company invests in an account G that consists of investments in the money market account and in the traded assets. We assume that the proportion of G invested in risky asset k is given by $q_k(t)$. The account G has dynamics

$$\begin{aligned} \mathrm{d}G(t) &= \Big(1 - \sum_{k=1}^K q_k(t)\Big)G(t)\frac{\mathrm{d}S_0(t)}{S_0(t)} + \sum_{k=1}^K q_k(t)G(t)\frac{\mathrm{d}S_k(t)}{S_k(t)} \\ &= r(t)G(t)\mathrm{d}t + G(t)q(t)^T\sigma(t,S(t),r(t))\mathrm{d}W(t), \end{aligned} \tag{4.2.3}$$

where $q(t) = (q_1(t), ..., q_K(t))^T$, and

$$\sigma(t,s,r) = \begin{pmatrix} \sigma_{11}(t,s,r) & \sigma_{12}(t,s,r) & \dots & \sigma_{1M}(t,s,r) \\ \sigma_{21}(t,s,r) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_{K1}(t,s,r) & \sigma_{K2}(t,s,r) & \dots & \sigma_{KM}(t,s,r) \end{pmatrix}.$$

4.2.2 With-profit Life Insurance

In with-profit life insurance, payments specified in the insurance contract are based on prudent assumptions about insurance risks and the return in the financial market. We denote these assumptions the first-order (technical) basis. The first-order basis consists of the technical interest rate r^* and the technical transition intensities μ_{ij}^*, $i, j \in \mathcal{J}$, $i \neq j$. Assumptions about the interest rate and transition intensities on the first-order basis are prudent compared to the expectation of the actual development of the market interest rate and transition intensities. The actual future development of the market interest rate and the market transition intensities μ_{ij} is unknown and needs to be modelled. Throughout, we assume that the market transition intensities are modelled in advance, and consider μ_{ij} as externally given, which is also practise in, for instance, Danish life insurance industry. The model of the market interest rate is specified in Equation (4.2.1).

Due to the prudent first-order basis, a surplus arises which by product design is to be paid back to the policyholders in terms of bonus. The redistribution of bonus is governed by legislation (in Denmark denoted *Kontributionsbekendtgørelsen*), and life insurance companies have certain degrees of freedom in the redistribution of bonus, which is part of the Management Actions of the company. We use the bonus scheme spoken of as additional benefits where bonus is used to buy more insurance. Inspired by Asmussen and Steffensen (2020) Chapter 6, the payments of the insurance contract consist of two types of payments. The payment stream B_1 represents payments not regulated by bonus, and B_2 represents the profile of payments regulated by bonus. The payment streams contain benefits less premiums of the insurance contract

$$\mathrm{d}B_i(t) = \mathrm{d}B_i^{Z(t)}(t) + \sum_{k:k\neq Z(t-)} b_i^{Z(t-)k}(t)\mathrm{d}N^k(t),$$

and

$$\mathrm{d}B_i^j(t) = b_i^j(t)\mathrm{d}t + \Delta B_i^j(t)\mathrm{d}\epsilon_n(t),$$

for $j \in \mathcal{J}$ and where $\epsilon_n(t) = \mathbb{1}_{\{t\geq n-\}}$ is the Dirac measure, b_i^j denotes continuous payments during sojourn in state j, and b_i^{jk} denotes the single payment upon transition from state j to state k. There is a lump sum payment of size $\Delta B_i^j(n-)$ just before the contract terminates at time n. Other lump sum payments at fixed time points during sojourn in states are disregarded in this set-up. We assume that the payment functions b_i^j, b_i^{jk} and ΔB_i^j are deterministic and sufficiently regular.

The technical reserve for the payment stream B_i for $i = 1, 2$ in this set-up is the present value of future payments discounted with the technical interest rate

$$V_i^{*Z(t)}(t) = \mathbb{E}^* \left[\int_t^n e^{-\int_t^s r^*(u)\mathrm{d}u} \mathrm{d}B_i(s) \;\middle|\; Z(t) \right],$$

where $\mathbb{E}^*$ implies that we use the first-order transition intensities in the distribution of Z. See Asmussen and Steffensen (2020) Chapter 6 Section 4 for the dynamics of $V_i^{*Z(t)}(t)$.

Bonus is distributed from the insurance company to the insured through a dividend payment stream D. With the bonus scheme additional benefits, bonus is used to buy more insurance, and we denote by $Q(t)$ the number of payment processes B_2 bought up to time t. Additional benefits are bought under the technical basis, and as we then use dividends to buy $B_2(t)$ at the price of $V_2^*(t)$, we must have that

$$\mathrm{d}D(t) = V_2^{*Z(t)}(t)\mathrm{d}Q(t).$$

The payment process guaranteed the policyholder at time t is

$$\mathrm{d}B(s) = \mathrm{d}B_1(s) + Q(t)\mathrm{d}B_2(s).$$

4.2.3 Assets and Liabilities

The assets, $U(t)$, of the insurance company are given by past premiums less benefits accumulated with the capital gains from investing in G, which consists of investments in the money market account, S_0, and the risky assets, S,

$$U(t) = -\int_0^t \frac{G(t)}{G(s)} \big(\mathrm{d}B_1(s) + Q(s)\mathrm{d}B_2(s)\big), \tag{4.2.4}$$

under the assumption that $U(0) = 0$.

We consider two decompositions of the liabilities of the insurance company. One decomposition is in the savings account of the policyholder and the surplus. The savings account X of an insurance contract is the technical value of future payments guaranteed at time t, i.e.

$$X(t) = V_1^{*Z(t)}(t) + Q(t)V_2^{*Z(t)}(t).$$

The savings account $X(t)$ depends on the process $Q(t)$, which denotes the number of payment processes B_2 bought up to time t. We can express $Q(t)$ in terms of the savings account and link the payment stream experienced by the policyholder to the savings account

$$\begin{aligned}
\mathrm{d}B(t) &= \mathrm{d}B(t, X(t)) \\
&= b^{Z(t)}(t, X(t))\mathrm{d}t + \Delta B^{Z(t-)}(t, X(t-))\mathrm{d}\epsilon_n(t) \\
&\quad + \sum_{k:k\neq Z(t-)} b^{Z(t-)k}(t, X(t-))\mathrm{d}N^k(t),
\end{aligned}$$

where

$$
\begin{aligned}
b^j(t,x) &= b_1^j(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} b_2^j(t), \\
\Delta B^j(t,x) &= \Delta B_1^j(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} \Delta B_2^j(t), \\
b^{jk}(t,x) &= b_1^{jk}(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} b_2^{jk}(t).
\end{aligned}
$$

Note that since $V_i^{*j}(n-) = \Delta B_i^j(n-)$, the lump sum payment at termination of the contract is equal to the savings account, $\Delta B^j(n-,x) = x$.

The surplus Y is the difference between the assets and the savings account

$$Y(t) = U(t) - X(t). \tag{4.2.5}$$

We assume that the proportion of the account G invested in the risky asset S_k can be written in the form

$$q_k(t) = \frac{\tilde{\pi}_k(t)Y(t)}{U(t)},$$

where $\tilde{\pi}_k$ is a sufficiently regular process. The investment strategy of the insurance company is $\tilde{\pi}(t) = (\tilde{\pi}_1(t), ..., \tilde{\pi}_K(t))^T$. Hence, the proportion of G invested in the risky asset k is proportional to the surplus divided by the assets, leading to a larger investment if the surplus is large compared to the savings account.

Proposition 4.2.1. *The savings account, X, and the surplus, Y, have dynamics*

$$
\begin{aligned}
\mathrm{d}X(t) &= r^*(t)X(t)\mathrm{d}t - \mathrm{d}B(t,X(t)) + \mathrm{d}D(t) \\
&\quad + \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t,X(t-))\big(\mathrm{d}N^k(t) - \mu^*_{Z(t-)k}(t)\mathrm{d}t\big),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}Y(t) &= r(t)Y(t)\mathrm{d}t + Y(t)\tilde{\pi}(t)^T\sigma(t,S(t),r(t))\mathrm{d}W(t) - \mathrm{d}D(t) + c^{Z(t)}(t,X(t))\mathrm{d}t \\
&\quad - \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t,X(t-))\big(\mathrm{d}N^k(t) - \mu_{Z(t-)k}(t)\mathrm{d}t\big),
\end{aligned}
$$

where

$$
\begin{aligned}
R^{*jk}(t,x) &= b^{jk}(t,x) + \chi^{jk}(t,x) - x, \\
\chi^{jk}(t,x) &= V_1^{*k}(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} V_2^{*k}(t), \\
c^j(t,x) &= (r(t) - r^*(t))x + \sum_{k:k\neq j} \big(\mu^*_{jk}(t) - \mu_{jk}(t)\big)R^{*jk}(t,x).
\end{aligned}
$$

Proof. See Asmussen and Steffensen (2020) Chapter 6 Section 7, for the dynamics of the savings account. For the surplus, insert the dynamics of the account G from Equation (4.2.3) and the dynamics of the savings account

$$\begin{aligned}
\mathrm{d}Y(t) = & -\mathrm{d}G(t)\int_0^t \frac{1}{G(s)}\big(\mathrm{d}B_1(s) + Q(s)\mathrm{d}B_2(s)\big) \\
& - \big(\mathrm{d}B_1(t) + Q(t)\mathrm{d}B_2(t)\big) - \mathrm{d}X(t) \\
= {} & r(t)\Big(- \int_0^t \frac{G(t)}{G(s)}\big(\mathrm{d}B_1(s) + Q(s)\mathrm{d}B_2(s)\big) - X(t)\Big)\mathrm{d}t + r(t)X(t)\mathrm{d}t \\
& + \Big(- \int_0^t \frac{G(t)}{G(s)}\big(\mathrm{d}B_1(s) + Q(s)\mathrm{d}B_2(s)\big)\Big)\frac{\tilde{\pi}(t)^T Y(t)}{U(t)}\sigma(t, S(t), r(t))\mathrm{d}W(t) \\
& - r^*(t)X(t)\mathrm{d}t - \mathrm{d}D(t) \\
& - \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t, X(t-))\big(\mathrm{d}N^k(t) - \mu^*_{Z(t-)k}(t)\mathrm{d}t\big) \\
= {} & r(t)Y(t)\mathrm{d}t + Y(t)\tilde{\pi}(t)^T\sigma(t, S(t), r(t))\mathrm{d}W(t) + c^{Z(t)}(t, X(t)\mathrm{d}t - \mathrm{d}D(t) \\
& - \sum_{k:k\neq Z(t-)} R^{*Z(t-)k}(t, X(t-))\big(\mathrm{d}N^k(t) - \mu_{Z(t-)k}(t)\mathrm{d}t\big),
\end{aligned}$$

which completes the proof. □

Based on the principle of equivalence on the technical basis, a natural constraint is that the savings account and the surplus are equal to zero at initialization of the contract i.e. $X(0-) = 0$ and $Y(0-) = 0$. This assumption implies that the savings account and the surplus are retrospective reserves. Hence, the decomposition of the liabilities into the savings account and the surplus is a decomposition based on retrospective reserves. Another decomposition of the liabilities is based on prospective reserves, and the natural constraint on the prospective reserves is that they are equal to zero at termination of the insurance contract. The prospective reserves are the market value of guaranteed payments, the market value of future bonus payments, also denoted as Future Discretionary Benefits (FDB), and future profits.

Uncertainties in future payments arise from two different types of risk. There is the risk associated with the state of the insured described by the state process Z, and the risk from investments in the risky assets. Inspired byAsmussen and Steffensen, 2020 Chapter 6 Section 3, we evaluate the risk associated with Z under the physical measure $\mathbb{P}$ due to diversification, and evaluate financial risks under the risk-neutral measure $\mathbb{Q}$ determined by the financial market. Therefore, valuation of future payments is performed under the product measure $\mathbb{P} \otimes \mathbb{Q}$.

The market value of the guaranteed payments, $V^{g,Z(t)}(t)$, is the expected present value of the future payments that are guaranteed the insured at time t

$$V^{g,Z(t)}(t) = \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u}\big(\mathrm{d}B_1(s) + Q(t)\mathrm{d}B_2(s)\big) \;\middle|\; \mathcal{F}_t\right].$$

Remark 4.2.2. We can express the market value of the guaranteed payments in terms of the savings account

$$\begin{aligned}
V^{g,Z(t)}(t) &= \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_t^n e^{-\int_t^s r(u)\,du}\big(dB_1(s) + Q(t)dB_2(s)\big) \;\middle|\; \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_t^n e^{-\int_t^s r(u)\,du}\,dB_1(s) \;\middle|\; Z(t), \mathcal{F}_t^S\right] \\
&\quad + Q(t)\cdot\mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_t^n e^{-\int_t^s r(u)\,du}\,dB_2(s) \;\middle|\; Z(t), \mathcal{F}_t^S\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\int_t^n e^{-\int_t^s f(t,u)\,du}\,dB_1(s) \;\middle|\; Z(t),\ r(t)\right] \\
&\quad + Q(t)\cdot\mathbb{E}^{\mathbb{P}}\left[\int_t^n e^{-\int_t^s f(t,u)\,du}\,dB_2(s) \;\middle|\; Z(t),\ r(t)\right],
\end{aligned}$$

since $Q(t)$ is measurable with respect to $\mathcal{F}_t$, Z is Markov, and $Q(t)$ is a function of $X(t)$. The forward rate can be inserted in the discount factor, since the state process Z and the financial market are independent, such that the market value of the payment streams dB_1 and dB_2 consists of the valuation of risks associated with Z only and can be performed under $\mathbb{P}$ independent of the financial market.

The market value of the future bonus payments (FDB) is

$$V^{b,Z(t)}(t) = \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u}\big(Q(s) - Q(t)\big)\mathrm{d}B_2(s) \;\middle|\; \mathcal{F}_t\right].$$

The market reserve is the expected present value under the market basis of future guaranteed and non-guaranteed payments, and therefore it is the sum of the market value of the guaranteed payments and the market value of the future bonus payments

$$\begin{aligned}
V^{Z(t)}(t) &= V^{g,Z(t)}(t) + V^{b,Z(t)}(t) \\
&= \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_t^n e^{-\int_t^s r\mathrm{d}u}\big(\mathrm{d}B_1(s) + Q(s)\mathrm{d}B_2(s)\big) \;\middle|\; \mathcal{F}_t\right].
\end{aligned}$$

Future profit is the difference between the assets and the market reserve, $V^{p,Z(t)}(t) = U(t) - V^{Z(t)}(t)$. The market reserve is the expected present value of future payments from the insurance company to the insured, while future profit is the expected present value of payments allotted the insurance company for taking on risks.

Note that in the first decomposition of the liabilities, the sum of the retrospective savings account and surplus is equal to the assets, and in the second decomposition, the sum of the prospective market reserve and future profit is equal to the assets. Hence, $U(t) = X(t) + Y(t) = V^{p,Z(t)}(t) + V^{Z(t)}(t)$.

4.2.4 Reserve-dependent Dividends and Investments

Calculation of the balance sheet items requires a specification of the investment strategy and the dividend payment stream. These are part of the Management Actions of the insurance company, and the determination of the investment strategy and the dividend payment stream holds certain degrees of freedom.

We assume that dividends are allocated continuously such that

$$\mathrm{d}D(t) = \tilde{\delta}(t)\mathrm{d}t,$$

where $\tilde{\delta}$ is the dividend strategy of the insurance company.

When deciding the investment strategy and the dividend allocation strategy, the insurance company considers its financial situation in terms of relations between balance sheet items. Therefore, an attractable model of the Management Actions includes the possibility that dividends and investments depend on all balance sheet items.

We consider a set-up where the investment strategy of the insurance company depends on the savings account, the surplus, the market reserve, the interest rate, and the traded assets

$$\tilde{\pi}_k(t) = \pi_k\big(t, X(t), Y(t), V^{Z(t)}(t), r(t), S(t)\big), \tag{4.2.6}$$

for deterministic and sufficiently regular functions π_k, $k = 1, ..., K$. In the same way, we allow dividends to depend on the savings account, the surplus, the market reserve, the interest rate, and the traded assets

$$\tilde{\delta}(t) = \delta^{Z(t)}\big(t, X(t), Y(t), V^{Z(t)}(t), r(t), S(t)\big), \tag{4.2.7}$$

for a deterministic and sufficiently regular function δ. Due to the relations between the balance sheet items, this specification of the investment strategy and the dividend strategy above also allow investments and dividends to depend on the assets, $U(t)$, the market value of guaranteed payments, $V^{g,Z(t)}(t)$, the market value of future bonus payments, $V^{b,Z(t)}(t)$, and future profits, $V^{p,Z(t)}(t)$. It is reasonable to assume that the dividend process depends on FDB, since it is likely that the amount of bonus depends on the reserve of future bonus. Hedging of interest rate risks in the market value of guaranteed payments, $V^{g,Z(t)}(t)$, is of great interest of the insurance company, and the general specification of the investment strategy above enables this. To find such an investment strategy, the insurance company must compute the interest rate sensitivity of $V^{g,Z(t)}(t)$ and then choose an investment strategy π_k with the same interest rate sensitivity.

With this specification of the investment strategy and the dividend strategy, there is a forward-backward entanglement of the prospective market reserve in the retrospective savings account and surplus, since the investment strategy and the dividend strategy appear in the dynamics from Proposition 4.2.1.

4.3 Calculation of the Market Reserve

The set-up with investments and dividends linked to all balance sheet items is attractable, since the Management Actions of the insurance company may depend on the entire balance sheet. Calculation of the market reserve within this set-up is complicated due to the interdependence between retrospective and prospective reserves. We characterize the market reserve by a PDE and consider the case of linearity that leads to a reduction in the dimension of the PDE.

4.3.1 PDE of the Market Reserve

Informally, $\big(X(t), Y(t), r(t), S(t), Z(t)\big)$ is seen to be Markov with the specification of the investment strategy and the dividend strategy in Equations (4.2.6) and (4.2.7), since the dynamics of the savings account and the surplus depend solely on $\big(X(t), Y(t), r(t), S(t), Z(t)\big)$. Hence, with a slight misuse of notation where V is now a function and not a stochastic process, we write the market reserve as

$$
\begin{aligned}
&V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) = \\
&\mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u}\big(\mathrm{d}B_1(s) + Q(s)\mathrm{d}B_2(s)\big) \,\middle|\, X(t), Y(t), r(t), S(t), Z(t)\right].
\end{aligned}
\tag{4.3.1}
$$

Since the savings account and the surplus depend on the stochastic interest rate and the traded assets, the market reserve also depends on $r(t)$ and $S(t)$.

Proposition 4.3.1. *Assume that $V^j(t, x, y, r, s)$ is sufficiently differentiable. Then the market reserve satisfies the following PDE*

$$
\begin{aligned}
\frac{\partial}{\partial t}V^j(t,x,y,r,s) &= rV^j(t,x,y,r,s) - b^j(t,x) - \sum_{k:k\neq j} R^{jk}(t,x,y,r,s)\mu_{jk}(t) \\
&\quad - \mathcal{D}_x V^j(t,x,y,r,s) - \mathcal{D}_y V^j(t,x,y,r,s) \\
&\quad - \mathcal{D}_r V^j(t,x,y,r,s) - \mathcal{D}_s V^j(t,x,y,r,s), \\
V^j(n-,x,y,r,s) &= x,
\end{aligned}
\tag{4.3.2}
$$

where

$$
\begin{aligned}
\mathcal{D}_x V^j(t,x,y,r,s) &= \frac{\partial}{\partial x} V^j(t,x,y,r,s)\Big(r^*(t)x - b^j(t,x) \\
&\quad + \delta(t,x,y,V^j(t,x,y,r,s),r,s) - \sum_{k:k\neq j} R^{*jk}(t,x)\mu^*_{jk}(t)\Big), \\
\mathcal{D}_y V^j(t,x,y,r,s) &= \frac{\partial}{\partial y} V^j(t,x,y,r,s)\Bigg(ry - \delta(t,x,y,V^j(t,x,y,r,s),r,s) + c^j(t,x) \\
&\quad + \sum_{k:k\neq j} R^{*jk}(t,x)\mu_{jk}(t)\Bigg) + \frac{1}{2}\frac{\partial^2}{\partial y^2} V^j(t,x,y,r,s) y^2 \sigma_y^2, \\
\mathcal{D}_r V^j(t,x,y,r,s) &= \frac{\partial}{\partial r} V^j(t,x,y,r,s)\alpha_r(t,r) + \frac{1}{2}\frac{\partial^2}{\partial r^2} V^j(t,x,y,r,s)\sigma_r^2(t,r), \\
\mathcal{D}_s V^j(t,x,y,r,s) &= \sum_{k=1}^{K} \frac{\partial}{\partial s_k} V^j(t,x,y,r,s) r s_k \\
&\quad + \frac{1}{2}\sum_{k=1}^{K}\sum_{l=1}^{K} \frac{\partial^2}{\partial s_k \partial s_l} V^j(t,x,y,r,s) s_k s_l \sum_{m=1}^{M} \sigma_{km}(t,r,s)\sigma_{lm}(t,s,r), \\
R^{jk}(t,x,y,r,s) &= b^{jk}(t,x) + V^k(t,\chi^{jk}(t,x), y - R^{*jk}(t,x), r, s) - V^j(t,x,y,r,s),
\end{aligned}
$$

and

$$
\sigma_y^2 = \pi(t,x,y,V^j(t,x,y,r,s),r,s)^T \sigma(t,s,r)\sigma(t,s,r)^T \pi(t,x,y,V^j(t,x,y,r,s),r,s).
$$

Conversely, if a function $V^j(t,x,y,r,s)$ satisfies the PDE above, it is indeed the market reserve defined in Equation (4.3.1).

Proof. See Appendix 4.A. □

The boundary condition is due to the lump sum payment at time $n-$.

Remark 4.3.2. In the Black-Scholes model of the financial market, where the interest rate is constant and deterministic, $r(t) = r \in \mathbb{R}$, and the volatility is constant, $\sigma(t,s,r) = \sigma > 0$, the state-wise market reserve is a function of the savings account and the surplus, and is independent of the traded asset, S, given the savings account and the surplus, under the condition that the dividend strategy and the investment strategy do not depend on S. The function $V^j(t,x,y)$ satisfies a PDE equal to the PDE in Proposition 4.3.1, but where $\mathcal{D}_r V^j(t,x,y,r,s) = \mathcal{D}_s V^j(t,x,y,r,s) = 0$ and $\sigma_y = \sigma\pi(t,x,y,V^j(t,x,y))$. This result also applies in the Black-Scholes model with a deterministic and time-dependent interest rate $r(t)$.

In order to calculate the market reserve, we must solve the PDE from Proposition 4.3.1 for all values of j, x, y, r and s, which is computationally demanding if even possible. One way to reduce the dimension of the PDE is to assume a more specific model for the financial market, which is the case in Remark 4.3.2. Another approach is to study the special case of linearity in the dividend strategy.

4.3.2 Linearity

The payment stream, $\mathrm{d}B(t,x)$, and the sum-at-risk, $R^{*jk}(t,x)$, are by construction linear in the savings account. Therefore, the dynamics of the savings account and the surplus from Proposition 4.2.1 are linear in the savings account, the surplus and the market reserve if and only if the investment strategy from Equation (4.2.6) and the dividend strategy from Equation (4.2.7) are linear.

Proposition 4.3.3. *Assume that the dividend strategy from Equation* (4.2.7) *is in the form*

$$\delta^j(t,x,y,v,r) = \delta_0^j(t,r) + \delta_1^j(t,r)\cdot x + \delta_2^j(t,r)\cdot y + \delta_3^j(t,r)\cdot v,$$

for deterministic functions $\delta_0^j, \delta_1^j, \delta_2^j$ *and* δ_3^j*. Then the market reserve is given by*

$$V^j(t,x,y,r) = h_0^j(t,r) + h_1^j(t,r)\cdot x + h_2^j(t,r)\cdot y, \tag{4.3.3}$$

where the functions h_0*,* h_1*, and* h_2 *satisfy the system of PDEs stated in Appendix 4.B.*

Proof. Since the function $V^j(t,x,y,r) = h_0^j(t,r) + h_1^j(t,r)\cdot x + h_2^j(t,r)\cdot y$ satisfies the PDE in Proposition 4.3.1, when $h_0^j(t,r), h_1^j(t,r)$ and $h_2^j(t,r)$ satisfy the system of PDEs in Appendix 4.B for all $j \in \mathcal{J}$, Proposition 4.3.1 gives the result. □

It is worth noticing that linearity of the dividend strategy is enough to make sure that the market reserve does not depend on the risky assets, S, when dividends are independent of S, and therefore the result applies for any choice of investment strategy. Therefore, the insurance company can choose an investment strategy that hedges interest rate risk in the market value of guaranteed payments. The existence of a solution is not certain, but if the system of PDEs has a solution, Proposition 4.3.1 gives that Equation (4.3.3) is in fact the market reserve. The linear structure of the market reserve in Equation (4.3.3) is similar to the results in Steffensen (2006), where linearity of the surplus in the dividend strategy is inherited in the prospective reserve.

The result in Proposition 4.3.3 reduces the dimension of the PDE of the market reserve compared to the case without linearity. This simplifies the calculation of the market reserve, since it is less computational heavy to solve the system of PDEs for the h functions for all values of r, compared to finding the solution to the PDE in Proposition 4.3.1 for all values of x, y, r and s.

Remark 4.3.4. In the Black-Scholes model of the financial market, still under the assumption of linearity of the dividend strategy, the market reserve has representation

$$V^j(t, x, y) = h_0^j(t) + h_1^j(t) \cdot x + h_2^j(t) \cdot y,$$

where the functions h_0, h_1 and h_2 satisfy a system of ordinary differential equations (ODEs). Hence, despite the forward-backward entanglement of the market reserve in the savings account and the surplus, the market reserve can be calculated as the solution to a system of backward ODEs in this case. The result also apply in the case, where the interest rate is time-dependent and deterministic.

From a computational point of view, it is demanding to solve PDEs, and therefore it is a desirable result that a combination of linearity of the dividend strategy and the Black-Scholes model of the financial market, reduces the dimension of the PDE from Proposition 4.3.1 in such a way that we are able to calculate the market reserve as the solution to a system of ODEs. The ODEs in Remark 4.3.4 fit into the class of Riccati equations. It is not certain that Riccati equations have solutions, but if a solution exists it is relatively easy to solve the system of ODEs numerically. The existence of solutions highly depends on the choice of the dividend strategy. With the choice in Example 4.3.5 below, we actually have an analytical solution.

Example 4.3.5. When dividends are equal to the surplus contribution from Proposition 4.2.1

$$\delta^j(t, x, y, v) = c^j(t, x),$$

the dividends are linear in the savings account and the market reserve is given by

$$V^j(t, x, y, r) = x,$$

since the functions $h_0^j(t, r) = h_2^j(t, r) = 0$ and $h_1^j(t, r) = 1$ solve the PDEs from Appendix 4.B for all $j \in \mathcal{J}$. Hence, the market reserve is equal to the savings account. In this case, the technical basis become redundant since the surplus that arise due to the prudent technical basis is immediately distributed as dividends to the savings account. In this case, the surplus, $Y(t)$, is equal to zero, and the same holds for future profits.

For the majority of dividend strategies, an analytical expression for the market reserve is difficult to obtain, and the market reserve must be calculated numerically.

4.3.3 Approximation of the Market Reserve

In general, it is computationally more demanding to solve PDEs compared to solving ODEs by numerical methods, and there exist more precise methods for solving ODEs.

Under the assumption of linearity in the dividend strategy, we are able to calculate the market reserve as the solution to a system of backwards PDEs by Proposition 4.3.3. In a Black-Scholes model of the financial market, we actually obtain a system of ODEs by Remark 4.3.4.

It may be desirable to lose some accuracy in order to decrease computation time by making approximations that result in ODEs instead of PDEs. Therefore, we aim to approximate the model with a stochastic interest rate in Equation (4.2.1) by a Black-Scholes model. To do this, we replace the stochastic interest rate with the deterministic forward interest rate. Due to linearity in the dividend strategy, the market reserve does not depend on the risky assets, S, by Proposition 4.3.3, and therefore we only approximate the stochastic interest rate. When calculating the market reserve, this corresponds to approximating the solution of the PDEs for the h functions from Proposition 4.3.3 by the solution to a system of ODEs based on the forward interest rate. We consider the approximation

$$\begin{aligned} r(t) &\approx f(0,t), \\ h_i^j(t,r) &\approx \tilde{h}_i^j(t), \end{aligned}$$

for $i = 0, 1, 2$ and $j \in \mathcal{J}$. The functions $\tilde{h}_i^j$ satisfy the system of ODEs given by the equations in Appendix 4.B, where $h_i^j(t,r)$ is replaced by $\tilde{h}_i^j(t)$, r is replaced by $f(0,t)$, and it is noted that $\frac{\partial}{\partial r}\tilde{h}_i^j(t) = 0$.

In a set-up without bonus, the market reserve is the expected present value of future payments discounted by the forward interest rate. Therefore, we consider the forward interest rate an appropriate approximation of the stochastic interest rate. Due to linearity of the dividend strategy, calculation of the market reserve does not depend on the investment strategy, and therefore the quality of the approximation does not depend on the choice of investment strategy. When we approximate the interest rate with the forward interest rate, the quality of the investment strategy decreases (for instance the investment strategy, where the insurance company hedges interest rate risk in the market value of guaranteed payments), but the examination of this is out of the scope of this paper, since our focus is the calculation of the market reserve. We investigate the quality of the approximation with the forward interest rate in a numerical example.

4.4 Numerical Study

In this section, we emphasize the practical applications of our results in a numerical example. Within a survival model with a stochastic interest rate and linearity in the dividend strategy, we solve the PDEs in Proposition 4.3.3 and compare the resulting market reserve with the solution of the ODEs obtained by approximating with the forward interest rate as described in Section 4.3.3.

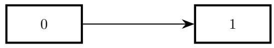

Figure 4.1: *Survival model in the numerical example*

Table 4.1: *Components in the numerical example*

Component	**Value**
Age of policyholder, a_0	65
Termination, n	45
Premium	15.22021
Annuity, $b_2^0(t)$	1
$Z(0)$	0
$\mu_{01}^*(t)$	$0.0005 + 10^{5.6+0.04(t+a_0)-10}$
$\mu_{01}(t)$	$1.1 \cdot \mu_{01}^*(t)$
$r^*(t)$	0.01
$r(0)$	0.05
ϕ	0.008127
ψ	-0.162953
θ	0.000237

The survival model is illustrated in Figure 4.1, where state 0 corresponds to alive and state 1 corresponds to dead. We consider an insured male at age a_0 at initialization of the insurance contract, and the insurance contract consists of a life annuity regulated by bonus, which is paid by a single premium of $V_2^{*0}(0)$ at time 0. Then the savings account at time 0 is equal to the single premium, $X(0) = V_2^{*0}(0)$, and $\mathrm{d}B_1 = 0$, since all payments are regulated by bonus. The payment process is

$$\mathrm{d}B(t, X(t)) = \mathbb{1}_{\{Z(t)=0\}} \frac{X(t)}{V_2^{*0}(t)} b_2^0(t)\mathrm{d}t.$$

We assume the interest rate from Equation (4.2.1) follows a Vasicek model with dynamics

$$\mathrm{d}r(t) = (\phi + \psi\, r(t))\, \mathrm{d}t + \sqrt{\theta}\, \mathrm{d}W(t), \tag{4.4.1}$$

where $\big(W(t)\big)_{\{t\geq 0\}}$ is a Brownian motion under the risk-neutral measure $\mathbb{Q}$. Let $u \mapsto f(t,u)$ be the forward interest rate calculated at time $t \geq 0$.

The components in this example are stated in Table 4.1. The parameters in the interest rate model are inspired by Falden and Nyegaard (2021), and the technical mortality rate is the same as in Bruhn and Lollike (2020). The market mortality rate in this example is chosen to be $1.1 \cdot \mu^*(t)$, such that the technical basis is prudent compared to the market basis. The premium is determined according to the principle of equivalence.

There are only dividends in state 0, since upon death all payments cancel. We assume the dividend process is in the form

$$\begin{aligned}\delta^0(t) =&\lambda_1(t)c^0(t,X(t)) + \lambda_2(t)V^{b,0}(t)\\ =&\lambda_1(t)X(t)\Big(r(t) - r^*(t) + \mu_{01}(t) - \mu_{01}^*(t)\Big)\\ &+ \lambda_2(t)\Big(V^0(t,X(t),Y(t)) - \frac{X(t)}{V_2^{*0}(t)}V_2^{g,0}(t,r(t))\Big),\end{aligned}$$

where $V_2^{g,0}(t,r) = \mathbb{E}^{\mathbb{P}}\left[\int_t^n e^{-\int_t^s f(t,u)\mathrm{d}u}\mathrm{d}B_2(s) \,\middle|\, Z(t), r(t)\right]$ and c^0 is the surplus contribution from Proposition 4.2.1.

The case where $\lambda_1(t) = 1$ and $\lambda_2(t) = 0$ corresponds to Example 4.3.5, and in this case the market reserve is equal to the savings account. It is reasonable to assume that $\lambda_1(t) \in (0,1)$, since a part of the surplus contribution is then immediately distributed as dividends. We let $\lambda_1(t) = 0.5$ and $\lambda_2(t) = 0.05$, hence half of the surplus contribution and 5 % of FDB are distributed as bonus.

In this example, the PDEs from Proposition 4.3.3 result in $h_0^0(t,r) = 0$, $h_2^0(t,r) = 0$ and

$$\begin{aligned}\frac{\partial}{\partial t}h_1^0(t,r) =& -h_1^0(t,r)^2\lambda_2(t) - \frac{b_2^0(t)}{V_2^{*0}(t)}\\ &+ h_1^0(t,r)\Bigg((1-\lambda_1(t))\Big(r - r^*(t) + \mu_{01}(t) - \mu_{01}^*(t)\Big)\\ &\qquad + \frac{b_2^0(t)}{V_2^{*0}(t)} + \frac{\lambda_2(t)V_2^{g,0}(t,r)}{V_2^{*0}(t)}\Bigg)\\ &- (\phi + \psi r)\frac{\partial}{\partial r}h_1^0(t,r) - \frac{\theta}{2}\frac{\partial^2}{\partial r^2}h_1^0(t,r),\\ h_1^0(n,r) =& 1,\end{aligned}$$

for the model with stochastic interest rate, which reduces to an ODE when inserting the forward interest rate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{h}_1^0(t) =& -\tilde{h}_1^0(t)^2\lambda_2(t) - \frac{b_2^0(t)}{V_2^{*0}(t)}\\ &+ \tilde{h}_1^0(t)\Bigg((1-\lambda_1(t))\Big(f(0,t) - r^*(t) + \mu_{01}(t) - \mu_{01}^*(t)\Big)\\ &\qquad + \frac{b_2^0(t)}{V_2^{*0}(t)} + \frac{\lambda_2(t)V_2^{g,0}(t)}{V_2^{*0}(t)}\Bigg),\\ \tilde{h}_1^0(n) =& 1.\end{aligned}$$

Table 4.2: *The market reserve at time zero*

PDE solution	ODE solution	Relative difference
13.16423	13.24555	0.00618

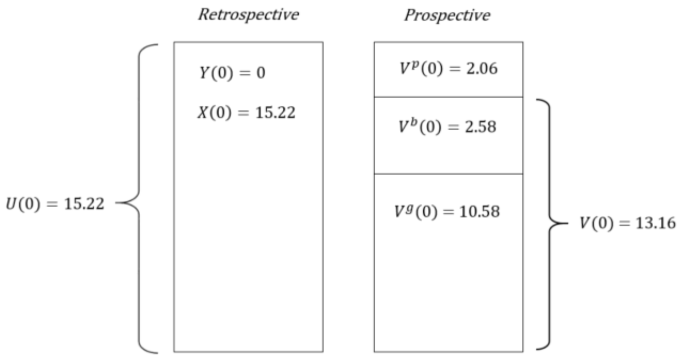

Figure 4.2: *The retrospective and the prospective decomposition of the liabilities at time 0 in the numerical example. The market reserve is calculated using the PDE method.*

The PDE for the function h_1^0 is solved numerically using the Explicit finite difference method, and the ODE for the function $\tilde{h}_1^0$ is solved numerically using the Runge Kutta forth-order method.

We calculate the market reserve at time zero by computing the function h_1^0 as the solution to the PDE and by solving the ODE for $\tilde{h}_1^0$ based on the deterministic forward interest rate. The results are presented in Table 4.2.

In this example, there is a small difference in the value of the market reserve at time zero. When we approximate using the forward interest rate, the market reserve is larger than in the model with the stochastic interest rate. Hence, the approximation method is conservative from an accounting point-of-view.

The decomposition of the liabilities based on retrospective and prospective reserves, respectively, at time 0 for this example is illustrated in Figure 4.2. The surplus is equal to zero at initialization of the contract, and therefore, the retrospective decomposition only consists of the savings account. The market value of the guaranteed payments constitutes around two thirds of the prospective decomposition, and FDB is almost equal to future profits.

In order to get a better understanding of the difference between the two methods to calculate the market reserve, we compare the function $t \mapsto \tilde{h}_1^0(t)$, which is the solution of the ODE, to the mean, the 2.5%-quantile, and the 97.5%-quantile of the stochastic

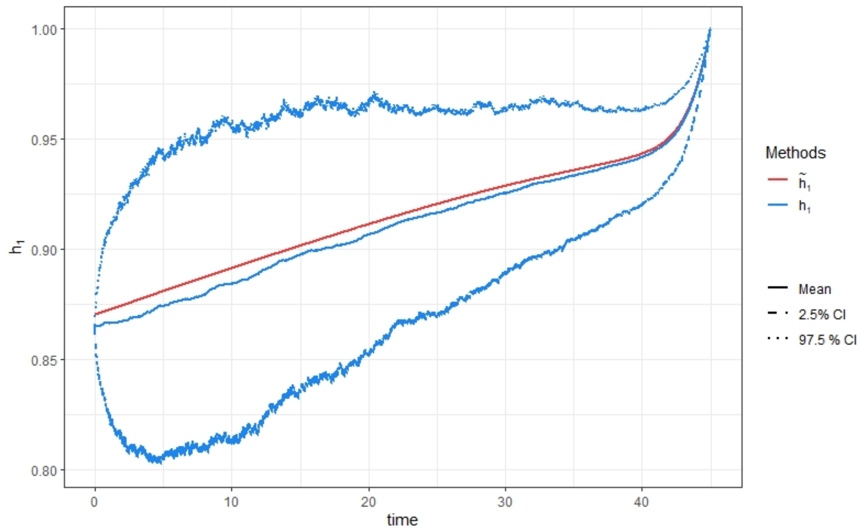

Figure 4.3: *Illustration of the functions $\tilde{h}_1$ and h_1. The red is $\tilde{h}_1$ based on the forward interest rate, the blue is h_1 based on the simulated interest rate paths, where the mean is the solid line, and the* 2.5%*-quantile and the* 97.5%*-quantile is the dashed line.*

process $t \mapsto h_1^0(t, r(t))$, since the market reserve is $V^0(t, x, r) = h_1^0(t, r) \cdot x$, and the approximated market reserve is $\tilde{V}^0(t, x) = \tilde{h}_1^0(t) \cdot x$. We compute $\mathbb{E}\left[h_1^0(t, r(t))\right]$ by simulating 1000 interest rate paths, simulated with an Euler scheme based on the dynamics of the interest rate in Equation (4.4.1), interpolate the solution to the PDE of h_1^0 over r, consider the function for each simulated interest rate path and calculate the empirical mean.

In this example, the market reserve is decreasing since benefits are paid out immediately after the premium payment at time 0. The market reserve at time t is $V(t, X(t)) = h_1^0(t, r) \cdot X(t)$, and therefore the development of the h_1^0 functions in Figure 4.3 does not have a one-to-one correspondence with the development of the market reserve. Based on the values of $\mathbb{E}\left[h_1(t, r(t))\right]$ and $\tilde{h}_1^0(t)$ in Figure 4.3, the development of the market reserve is similar to the development of the savings account, which is also a decreasing process in this example. When the contract terminates, the market reserve equals zero since there are no future payments. The payment of $X(n-)$ at termination of the insurance contract results in the boundary conditions $h_1^0(n-, r) = \tilde{h}_1^0(n-) = 1$, and is consistence with $V(n, X(n)) = 0$, since

$$X(n) = Q(n)V_2^{*0}(n) = 0.$$

The approximation $\tilde{h}_1^0(t)$ is in general larger than, but close to $\mathbb{E}\left[h_1^0(t, r(t))\right]$. Therefore based on this example, we consider the approximation with the forward interest rate reasonable, since $\tilde{h}_1^0$ is close to the estimated mean and within the 95% confidence interval of $h_1^0(t, r(t))$. The computation time for solving the ODE is significantly lower than for solving the PDE, and therefore the approximation is useful if one can accept the relative difference.

Acknowledgments and declarations of interest

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4.A Proof of Proposition 4.3.1

Construct a martingale m as

$$\begin{aligned} m(t) &= \mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\left[\int_0^n e^{-\int_0^s r(u)\mathrm{d}u}\big(\mathrm{d}B_1(s) + Q(s)\mathrm{d}B_2(s)\big) \,\middle|\, \mathcal{F}_t\right] \\ &= \int_0^t e^{-\int_0^s r(u)\mathrm{d}u}\big(\mathrm{d}B_1(s) + Q(s)\mathrm{d}B_2(s)\big) \\ &\quad + e^{-\int_0^t r(u)\mathrm{d}u} V^{Z(t)}(t, X(t), Y(t), r(t), S(t)). \end{aligned}$$

The dynamics of m are

$$\begin{aligned} \mathrm{d}m(t) = e^{-\int_0^t r(u)\mathrm{d}u}\Big(&\mathrm{d}B_1(t) + Q(t)\mathrm{d}B_2(t) + r(t)V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\mathrm{d}t \\ &+ \mathrm{d}V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\Big). \end{aligned}$$

By the multidimensional Itô formula, we have the dynamics of the market reserve

$$\begin{aligned} &\mathrm{d}V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) = \\ &\frac{\partial}{\partial t}V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\mathrm{d}t \\ &+ \mathcal{D}_x V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\mathrm{d}t \\ &+ \mathcal{D}_y V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\mathrm{d}t + \frac{\partial}{\partial y}V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \\ &\times Y(t)\pi\big(t, X(t), Y(t), V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\big)^T \sigma(t, S(t), r(t))\mathrm{d}W(t) \\ &+ \mathcal{D}_r V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\mathrm{d}t \\ &+ \frac{\partial}{\partial r}V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\sigma_r(t, r(t))\mathrm{d}W_r(t) \\ &+ \mathcal{D}_s V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\mathrm{d}t \\ &+ \sum_{k=1}^K \frac{\partial}{\partial s_k}V^{Z(t)}(t, X(t), Y(t), r(t), S(t))S_k(t)\sum_{m=1}^M \sigma_{km}(t, S(t), r(t))\mathrm{d}W_m(t) \\ &+ \sum_{k:k\neq Z(t-)}\Big(V^k(t, \chi^{Z(t-)k}(t, X(t-)), Y(t-) - R^{*Z(t-)k}(t, X(t-)), r(t), S(t)) \\ &\qquad - V^{Z(t-)}(t, X(t-), Y(t-), r(t), S(t))\Big)\mathrm{d}N^k(t). \end{aligned} \tag{4.A.1}$$

Combining this, the dynamics of $m(t)$ are

$$\begin{aligned}
\mathrm{d}m(t) = e^{-\int_0^t r(u)\mathrm{d}u}\Big(& b^{Z(t)}(t,X(t)) + r(t)V^{Z(t)}(t,X(t),Y(t),r(t),S(t)) \\
&+ \frac{\partial}{\partial t}V^{Z(t)}(t,X(t),Y(t),r(t),S(t)) + \mathcal{D}_x V^{Z(t)}(t,X(t),Y(t),r(t),S(t)) \\
&+ \mathcal{D}_y V^{Z(t)}(t,X(t),Y(t),r(t),S(t)) \\
&+ \mathcal{D}_r V^{Z(t)}(t,X(t),Y(t),r(t),S(t)) \\
&+ \mathcal{D}_s V^{Z(t)}(t,X(t),Y(t),r(t),S(t)) \\
&+ \sum_{k:k\neq Z(t)} R^{Z(t)k}(t,X(t),Y(t),r(t),S(t))\mu_{Z(t)k}(t)\Big)\mathrm{d}t \\
+ e^{-\int_0^t r(u)\mathrm{d}u}&\Delta B^{Z(t)}(t,X(t))\mathrm{d}\epsilon_n(t) + e^{-\int_0^t r(u)\mathrm{d}u}\mathrm{d}M(t),
\end{aligned}$$

where M is a martingale with dynamics

$$\begin{aligned}
&\mathrm{d}M(t) = \\
&\frac{\partial}{\partial r}V^{Z(t)}(t,X(t),Y(t),r(t),S(t))\sigma_r(t,r(t))\mathrm{d}W_r(t) \\
&+ \frac{\partial}{\partial y}V^{Z(t)}(t,X(t),Y(t),r(t),S(t)) \\
&\times Y(t)\pi\big(t,X(t),Y(t),V^{Z(t)}(t,X(t),Y(t),r(t),S(t))\big)^T\sigma(t,S(t),r(t))\mathrm{d}W(t) \\
&+ \sum_{k=1}^{K}\frac{\partial}{\partial s_k}V^{Z(t)}(t,X(t),Y(t),r(t),S(t))S_k(t)\sum_{m=1}^{M}\sigma_{km}(t,S(t),r(t))\mathrm{d}W_m(t) \\
&+ \sum_{k:k\neq Z(t-)} R^{Z(t-)k}(t,X(t-),Y(t-),r(t),S(t))\big(\mathrm{d}N^k(t) - \mu_{Z(t-)k}(t)\mathrm{d}t\big).
\end{aligned}$$

Since $e^{-\int_0^t r(u)\mathrm{d}u}\mathrm{d}M(t)$ also are the dynamics of a martingale and since $m(t)$ is a martingale, the term in front of $\mathrm{d}t$ in the dynamics of $m(t)$ must be equal to zero for all $t, X(t), Y(t), r(t)$, and $S(t)$ which results in the PDE for the market reserve. Due to the lump sum payment at time $n-$, $\Delta B(n-, X(n-)) = X(n-)$, the boundary condition of the PDE is $V^j(n,x,y,r,s) = x$.

Now, assume that a function $\bar{V}^j(t,x,y,r,s)$ satisfies the PDE in Equation (4.3.2). We show that this function is in fact the market reserve in Equation (4.3.1). Consider an investment strategy and dividend strategy given by

$$\begin{aligned}
\tilde{\pi}_k(t) &= \pi_k\big(t,X(t),Y(t),\bar{V}^{Z(t)}(t,X(t),Y(t),r(t),S(t)),r(t),S(t)\big), \\
\mathrm{d}D^{Z(t)}(t) &= \delta\big(t,X(t),Y(t),\bar{V}^{Z(t)}(t,X(t),Y(t),r(t),S(t)),r(t),S(t)\big)\mathrm{d}t,
\end{aligned}$$

for $k = 1, ..., K$.

The multidimensional Itô formula, the dynamics from Equation (4.A.1) with $\bar{V}$ inserted instead of V, and the fact that $\bar{V}$ satisfies the PDE in Equation (4.3.2) yield that

$$
\begin{aligned}
&\mathrm{d}\Big(e^{-\int_0^t r(u)\mathrm{d}u}\bar{V}^{Z(t)}(t,X(t),Y(t),r(t),S(t))\Big) \\
=& -r(t)\bar{V}^{Z(t)}(t,X(t),Y(t),r(t),S(t))\mathrm{d}t \\
&+ e^{-\int_0^t r(u)\mathrm{d}u}\mathrm{d}\bar{V}^{Z(t)}(t,X(t),Y(t),r(t),S(t)) \\
=& \, e^{-\int_0^t r(u)\mathrm{d}u}\Bigg(\sum_{k:k\neq Z(t-)} \bar{R}^{Z(t-)k}(t,X(t-),Y(t-),r(t),S(t)) \\
&\times \big(\mathrm{d}N^k(t)-\mu_{Z(t-)k}(t)\mathrm{d}t\big) - b^{Z(t)}(t,X(t))\mathrm{d}t \\
&- \sum_{k:k\neq Z(t-)} b^{Z(t-)k}(t,X(t-))\mathrm{d}N^k(t) \\
&+ \frac{\partial}{\partial y}\bar{V}^{Z(t)}(t,X(t),Y(t),r(t),S(t))Y(t) \\
&\times \pi\big(t,X(t),Y(t),\bar{V}^{Z(t)}(t,X(t),Y(t),r(t),S(t))\big)^T\sigma(t,S(t),r(t))\mathrm{d}W(t) \\
&+ \frac{\partial}{\partial r}V^{Z(t)}(t,X(t),Y(t),r(t),S(t))\sigma_r(t,r(t))\mathrm{d}W_r(t) \\
&+ \sum_{k=1}^{K}\frac{\partial}{\partial s_k}V^{Z(t)}(t,X(t),Y(t),r(t),S(t))S_k(t)\sum_{m=1}^{M}\sigma_{km}(t,S(t),r(t))\mathrm{d}W_m(t)\Bigg).
\end{aligned}
$$

Integrating over the interval $[t,n)$ and taking the $\mathbb{P}\otimes\mathbb{Q}$ expectation conditioning on $\mathcal{F}_t$ give that

$$
\begin{aligned}
&e^{-\int_0^n r(u)\mathrm{d}u}\underbrace{\bar{V}^{Z(n-)}(n,X(n-),Y(n-),r(n-),S(n-))}_{=X(n-)} \\
&- e^{-\int_0^t r(u)\mathrm{d}u}\bar{V}^{Z(t)}(t,X(t),Y(t),r(t),S(t)) \\
=& -\mathbb{E}^{\mathbb{P}\otimes\mathbb{Q}}\Bigg[\int_t^n e^{-\int_0^s r(u)\mathrm{d}u}\big(b^{Z(s)}(s,X(s))\mathrm{d}s \\
&\qquad + \sum_{k:k\neq Z(s-)} b^{Z(s-)k}(s,X(s-))\mathrm{d}N^k(s)\big)\,\Bigg|\,\mathcal{F}_t\Bigg],
\end{aligned}
$$

since the remaining terms in the dynamics of $\bar{V}^{Z(t)}(t,X(t),Y(t),r(t),S(t))$ are martingales with respect to the filtration $\mathcal{F}$. Multiplying by $-\exp(-\int_0^t r(u)\mathrm{d}u)$ and including the boundary condition at time $n-$ in the payment stream gives that $\bar{V}^j(t,x,y,r,s)$ is the market reserve.

4.B PDEs for h-functions

$$\begin{aligned}
\frac{\partial}{\partial t}h_0^j(t,r) = {} & rh_0^j(t,r) - b_1^j(t) + \frac{V_1^{*j}(t)}{V_2^{*j}(t)}b_2^j(t) - \sum_{k:k\neq j}\mu_{jk}(t) \\
& \times \Big(b_1^{jk}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}b_2^{jk}(t) + h_0^k(t,r) + h_1^k(t,r)\big(V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}V_2^{*k}(t)\big) \\
& - \; h_2^k(t,r)\big(b_1^{jk}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}b_2^{jk}(t) + V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}V_2^{*k}(t)\big) - h_0^j(t,r)\Big) \\
& - h_1^j(t,r)\Big(- b_1^j(t) + \frac{V_1^{*j}(t)}{V_2^{*j}(t)}b_2^j(t) + \delta_0^j(t) + \delta_3^j(t)h_0^j(t,r) \\
& - \sum_{k:k\neq j}\mu_{jk}^*(t)\big(b_1^{jk}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}b_2^{jk}(t) + V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}V_2^{*k}(t)\big)\Big) \\
& - h_2^j(t,r)\Big(- \delta_0^j(t) - \delta_3^j(t)h_0^j(t,r) \\
& + \sum_{k:k\neq j}\mu_{jk}^*(t)\big(b_1^{jk}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}b_2^{jk}(t) + V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)}V_2^{*k}(t)\big)\Big) \\
& - \frac{\partial}{\partial r}h_0^j(t,r)\alpha_r(t,r) - \frac{1}{2}\frac{\partial^2}{\partial r^2}h_0^j(t,r)\sigma_r^2(t,r), \\
h_0^j(n-,r) = {} & 0,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}h_1^j(t,r) = {} & rh_1^j(t,r) - \frac{1}{V_2^{*j}(t)}b_2^j(t) - \sum_{k:k\neq j}\mu_{jk}(t)\Bigg(\frac{1}{V_2^{*j}(t)}b_2^{jk}(t) - h_1^j(t,r) \\
& + h_1^k(t,r)\frac{1}{V_2^{*j}(t)}V_2^{*k}(t) - \; h_2^k(t,r)\big(\frac{1}{V_2^{*j}(t)}b_2^{jk}(t) + \frac{1}{V_2^{*j}(t)}V_2^{*k}(t) - 1\big)\Bigg) \\
& - h_1^j(t,r)\Big(r^*(t) - \frac{1}{V_2^{*j}(t)}b_2^j(t) + \delta_1^j(t) + \delta_3^j(t)h_1^j(t,r) \\
& \qquad - \sum_{k:k\neq j}\mu_{jk}^*(t)\big(\frac{1}{V_2^{*j}(t)}b_2^{jk}(t) + \frac{1}{V_2^{*j}(t)}V_2^{*k}(t) - 1\big)\Big) \\
& - h_2^j(t,r)\Big(- \delta_1^j(t) - \delta_3^j(t)h_1^j(t,r) + r - r^*(t) \\
& \qquad + \sum_{k:k\neq j}\mu_{jk}^*(t)\big(\frac{1}{V_2^{*j}(t)}b_2^{jk}(t) + \frac{1}{V_2^{*j}(t)}V_2^{*k}(t) - 1\big)\Big) \\
& - \frac{\partial}{\partial r}h_1^j(t,r)\alpha_r(t,r) - \frac{1}{2}\frac{\partial^2}{\partial r^2}h_1^j(t,r)\sigma_r^2(t,r), \\
h_1^j(n-,r) = {} & 1,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t} h_2^j(t,r) = & - \sum_{k:k\neq j} \mu_{jk}(t)\big(h_2^k(t,r) - h_2^j(t,r)\big) \\
& - h_1^j(t,r)\big(\delta_2^j(t) + \delta_3^j(t) h_2^j(t,r)\big) + h_2^j(t,r)\big(\delta_2^j(t) + \delta_3^j(t) h_2^j(t,r)\big) \\
& - \frac{\partial}{\partial t} h_2^j(t,r)\alpha_r(t,r) - \frac{1}{2}\frac{\partial^2}{\partial r^2} h_2^j(t,r)\sigma_r^2(t,r), \\
h_2^j(n-,r) = & \; 0.
\end{aligned}$$

Chapter 5

Natural hedging in continuous time life insurance

This chapter contains the paper *Nyegaard (2023)*.

ABSTRACT

Life insurance companies face several types of risks including financial risks and insurance risks. Financial risks can to a large extent be hedged by trading in the financial market, but there exists no such market for insurance risks. We suggest an alternative to hedge insurance risks. In a multi-state setup in continuous time life insurance, we describe the concept of natural hedging, which enables us to compose a portfolio of different insurance products where the liabilities are unaffected by shifts in the transition intensities. We describe how to find and how to calculate the natural hedging strategy using directional derivatives (Gateaux derivatives) to measure the sensitivity of the life insurance liabilities with respect to shifts in the transition intensities of a Markov chain governing the state of the insured. Furthermore, we implement the natural hedging strategy in two numerical examples based on the survival model and the disability model, respectively.

Keywords: Life insurance; Natural hedging; Risk management; Multi-state models.

5.1 Introduction

An important aspect of the life insurance business is to maintain a reserve, such that insurance companies are able to meet future liabilities towards its policyholders. Valuation of future liabilities is based on the choice of valuation basis, that consists of estimates of for instance future interest rates, mortality rates and disability rates. For risk management purposes, the insurance company and regulatory authorities are interested in sensitivities of the future liabilities towards the valuation basis. Broadly speaking, life insurance companies face two types of risks: Financial risks and insurance risks. Sensitivities of insurance liabilities towards financial risks such as interest rate can to a large extent be hedged by trading in the financial market. Methods for risk management of insurance risks are few, and the market for trading for instance mortality or disability linked securities is not developed. Systematic insurance risks in continuous time life insurance refer to the uncertain development of the transition intensities of a Markov chain governing the state of the insured. We use directional derivatives (Gateaux derivatives) to measure the sensitivity of life insurance liabilities with respect to shifts in the transition intensities. The liabilities of different insurance products such as life annuities and term insurances have different sensitivities towards changes in the transition intensities, and the objective of this paper is to make use of this netting effect to construct a portfolio of different insurance products, where the total liabilities of the portfolio are invariant to changes in the transition intensities, and in this way minimize systematic insurance risks for the portfolio. We denote this the natural hedging problem, and based on a first order Taylor approximation, we formulate the natural hedging strategy that solve the natural hedging problem. The natural hedging strategy is derived starting from a shift or stress of the transition intensities. In two numerical examples in the survival model and the disability model, respectively, we find the natural hedging strategy and discuss the practical applications of natural hedging.

Sensitivities of liabilities in life insurance with respect to valuation bases are studied in depth in Kalashnikov and Norberg (2003) and Christiansen (2008a). The focus of this paper is how sensitivities of liabilities with respect to transition intensities relate to natural hedging. Christiansen (2011) discusses netting effects in relation to natural hedging and different risk measures, and investigates the netting effect between two different products under a stress on the mortality intensity in two numerical examples. We discover the same effect in our first numerical example of a natural hedging strategy in the survival model, and in our second numerical example in the disability model we use the netting effect between four different insurance product to find the natural hedging strategy under which the liabilities are invariant to changes in the mortality intensity, disability intensity and the recovery intensity.

The idea of a portfolio consisting of a mixture between life annuities and death benefits to make use of the effect that the liabilities of the two products move in different directions when the mortality intensity changes is not new and studied in for instance Cox and Lin (2007) and Wang et al. (2010), and is denoted natural hedging. Lin and Tsai (2020) extend hedging of mortality risks using natural hedging strategies, and study immunization strategies that hedge changes in mortality and interest rate simultaneously. In this paper, we move away from the survival model and study natural hedging in a general multi-state model of the state of the insured, which enables us to construct natural hedging strategies against for instance disability risks and lapse risks. It is not obvious how to hedge for instance disability risks, but the model framework we present in this paper enables us to investigate if natural hedging of disability risk is possible. Levantesi and Menzietti (2017) describe the concept of natural hedging in a disability model in discrete time, and our results are a continuous time version of the natural hedging strategy described in Levantesi and Menzietti (2017). Our formulation of the natural hedging strategy applies for any choice of model for instance the survival model and the disability model.

The paper is structured as follows. In Section 5.2, we present the multi-state setup in continuous time life insurance and define the liabilities of an insurance contract. The definition of the natural hedging problem and the natural hedging strategy are described in Section 5.3, where we also discuss calculation of the natural hedging strategy using directional derivatives. Two numerical examples in the survival model and the disability model, respectively, in Section 5.4 show implementations of natural hedging strategies. We conclude the paper in Section 5.5

5.2 Setup

We assume the classical setup in life insurance, where the state of the holder of an insurance contract is modelled by a continuous time Markov chain $Z = \big(Z(t)\big)_{t\geq 0}$ on a finite dimensional state space $\mathcal{J}$. The transition probabilities of Z are

$$p_{ij}(t,s) = \mathbb{P}\big(Z(s) = j \mid Z(t) = i\big),$$

for $i, j \in \mathcal{J}$. We assume the transition intensities

$$\mu_{ij}(t) = \lim_{h\downarrow 0} \frac{p_{ij}(t,t+h)}{h},$$

exists for $i, j \in \mathcal{J}$, $i \neq j$, and let

$$\mu = (\mu_{jk})_{j,k\in\mathcal{J}, j\neq k},$$

denote the vector of transition intensities of the Markov chain Z. The number of possible transitions of Z is given by $K = \#\big\{(j,k), j\neq k \mid \mu_{jk}(t) > 0 \text{ for } t \in [0,n]\big\}$. The processes $N^k(t)$ count the number of jumps of Z into state $k \in \mathcal{J}$ up to and including time t

$$N^k(t) = \#\big\{s \in (0,t] \mid Z(s-) \neq k, Z(s) = k\big\},$$

where $Z(s-) = \lim_{h\downarrow 0} Z(s-h)$. Let $\mathcal{F} = \big(\mathcal{F}_t\big)_{t\geq 0}$ be the natural filtration generated by the state process Z.

Payments of the insurance contract link to sojourns in states and transitions between states. The payment stream has dynamics

$$\mathrm{d}B(t) = b^{Z(t)}(t)\mathrm{d}t + \sum_{k:k\neq Z(t-)} b^{Z(t-)k}(t)\mathrm{d}N^k(t),$$

where b^j and b^{jk} for $j,k \in \mathcal{J}$, $j \neq k$ are deterministic functions. Payments specified by b^j link to continuous payments during sojourn in state j and b^{jk} links to payments upon transition from state j to state k. We do not consider lump sum payments during sojourn in states, and assume that the interest rate $t \to r(t)$ is a deterministic function.

The insurance company is interested in the valuation of future payments of the insurance contract, which is the liabilities towards its policyholders. The liabilities or prospective reserve of the payment stream B is the expected present value of future payments

$$\begin{aligned} V(t) &= \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B(s) \mid \mathcal{F}_t \right] \\ &= \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B(s) \mid Z(t) \right], \end{aligned}$$

due to the Markov property of Z, and where n denotes termination of the insurance contract. Conditioning on $Z(t) = i$, we arrive at the state-wise prospective reserve

$$V^i(t,\mu) = \mathbb{E}^\mu\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B(s) \mid Z(t) = i \right], \tag{5.2.1}$$

where the superscript μ denotes that the distribution of Z has transition intensities μ. Our interest in this paper lies in the sensitivity of the liabilities with respect to the choice of transition intensities of Z, and therefore we write $V^i(t,\mu)$ to emphasize the dependence on μ. The derivative of the state-wise prospective reserve with respect to t is well known and is given by Thiele's differential equation

$$\begin{aligned} \frac{\partial}{\partial t} V^j(t,\mu) &= r(t)V^j(t,\mu) - b^j(t) - \sum_{k:k\neq j} \mu_{jk}(t) R^{jk}(t,\mu), \qquad (5.2.2) \\ V^j(n,\mu) &= 0, \end{aligned}$$

where R^{jk} is the sum-at-risk upon transition from state j to state k and is given by

$$R^{jk}(t,\mu) = b^{jk} + V^k(t,\mu) - V^j(t,\mu).$$

5.3 Natural hedging

Natural hedging exploits the fact that insurance liabilities for different insurance products may move in opposite directions in response to changes in the valuation basis. Our aim is to compose a portfolio of different insurance products for which the liabilities are unaffected by changes in the transition intensities. The concept of natural hedging is described in discrete time in the disability model without reactivation in Levantesi and Menzietti (2017), where constant shifts of the transition probabilities are studied, and we develop the concept in continuous time in a general multi-state model.

We assume the insurance company sells P different products. The payment functions in product p is $b^{j}_{(p)}(t)$ and $b^{jk}_{(p)}(t)$, $j,k \in \mathcal{J}$, $j \neq k$, and the liabilities in state i of product p is

$$V^{i}_{(p)}(t,\mu) = \mathbb{E}^{\mu}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B_{(p)}(s) \mid Z(t) = i \right],$$

where $\mathrm{d}B_{(p)}(t)$ is the payment stream for product p given by

$$\mathrm{d}B_{(p)}(t) = b^{Z(t)}_{(p)}(t)\mathrm{d}t + \sum_{k:k\neq Z(t-)} b^{Z(t-)k}_{(p)}(t)\mathrm{d}N^k(t).$$

The weight $w_{(p)}(t)$ is the proportion of the total liabilities in product p at time t, and we have the condition that the weights sum to 1. The total liability of the portfolio is then

$$V^i(t,\mu) = \sum_{p=1}^{P} w_{(p)}(t) V^{i}_{(p)}(t,\mu).$$

5.3.1 The natural hedging problem

The basis of the natural hedging problem is the change in the transition intensities. We denote the change by $\Delta\mu = (\Delta\mu_{jk})_{j,k\in\mathcal{J},j\neq k}$, such that the changed or shifted transition intensities become $\mu + \Delta\mu$.

The changes or shifts of the transition intensities may come in many different varieties. Here, we consider shifts in the form

$$\Delta\mu_{jk}(t) = \varepsilon_{jk} g_{jk}(t),$$

where the vector $g(t) = (g_{jk}(t))_{j,k\in\mathcal{J},j\neq k}$ is the direction of the shift and $\varepsilon = (\varepsilon_{jk})_{j,k\in\mathcal{J},j\neq k} \in \mathbb{R}^K$ is the magnitude of the shift. Two examples of shifts of the transition intensities are what we denote additive shifts, where the direction of the shift is a constant function, $g^{\mathrm{add}}_{jk}(t) = \mathbb{1}_{[0,n]}(t)$, and what we denote as multiplicative shifts, where the shifts are proportional to the transition intensities, $g^{\mathrm{mul}}_{jk}(t) = \mu_{jk}(t)\mathbb{1}_{[0,n]}(t)$. The shifts are restricted to the course of the insurance contract that runs from time 0 to time n. The multiplicative shifts are similar to the stresses on the transition intensities that appear in the Standard Formula of Solvency II.

The objective is to compose a portfolio of different insurance products such that the change in the liabilities due to the shift in the transition intensities is neutralized. This is a so-called natural hedging strategy against systematic insurance risks. Hence, the objective is to choose weights, $\left(w_{(1)}(t), ..., w_{(P)}(t)\right)^T$, such that

$$V^i(t, \mu + \Delta\mu) - V^i(t, \mu) = 0. \tag{5.3.1}$$

To find a sufficient condition to ensure that Equation (5.3.1) is (approximately) satisfied, we study a first order Taylor approximation. The Taylor approximation depends on the sensitivities of the liabilities with respect to the directional shift in the transition intensities, $\Delta\mu$. First, we note that the sensitivity of the liabilities with respect to a directional shift in the transition intensity μ_{jk}, $j, k \in \mathcal{J}, j \neq k$, in the direction g_{jk} can be described by the directional (Gateaux) derivative

$$\frac{\partial}{\partial \eta_{jk}} V^i(t, \mu + \eta g)\Big|_{\eta=0},$$

where $\eta = (\eta_{jk})_{j,k\in\mathcal{J}, j\neq k}$. The vector of sensitivities with respect to directional shifts is given by

$$\frac{\partial}{\partial \eta} V^i(t, \mu + \eta g)\Big|_{\eta=0} = \left(\frac{\partial}{\partial \eta_{jk}} V^i(t, \mu + \eta g)\Big|_{\eta=0}\right)_{j,k\in\mathcal{J}, j\neq k}.$$

For the shift in the direction g with magnitude ε, a first order Taylor approximation yields that

$$\begin{aligned} V^i(t, \mu + \Delta\mu) - V^i(t, \mu) &\approx \frac{\partial}{\partial \eta} V^i(t, \mu + \eta g)\Big|_{\eta=0} \cdot \varepsilon \\ &= \sum_{j\in\mathcal{J}} \sum_{k:k\neq j} \frac{\partial}{\partial \eta_{jk}} V^i(t, \mu + \eta g)\Big|_{\eta=0} \varepsilon_{jk}. \end{aligned} \tag{5.3.2}$$

In Section 5.3.2, we describe how to calculate the directional derivative above. The directional derivative of the total liabilities is given by

$$\frac{\partial}{\partial \eta} V^i(t, \mu + \eta g)\Big|_{\eta=0} = \sum_{p=1}^{P} w_{(p)}(t) \frac{\partial}{\partial \eta} V^i_{(p)}(t, \mu + \eta g)\Big|_{\eta=0}.$$

We define the natural hedging strategy as the weights that, independent of the magnitude of the shifts in the transition intensities, ensure that the right hand side of Equation (5.3.2) is equal to zero, and formalize the natural hedging problem.

Definition 5.3.1 (The natural hedging strategy). *The natural hedging strategy is* *the weights that solve*

$$\sum_{p=1}^{P} w_{(p)}(t) = 1,$$
$$\sum_{p=1}^{P} w_{(p)}(t) \sum_{j\in\mathcal{J}} \sum_{k:k\neq j} \frac{\partial}{\partial \eta_{jk}} V^i_{(p)}(t,\mu+\eta g)\Big|_{\eta=0} \varepsilon_{jk} = 0,$$

for fixed $t \in [0,n]$, $i \in \mathcal{J}$ and for all choices of magnitudes of the shifts, ε.

A sufficient condition to ensure that the natural hedging problem has a solution is that all of the directional derivatives of the total liabilities are equal to zero. This results in K equations. Combined with the condition that the weights sum to 1, we need $K+1$ unknowns to solve the system of equations, and therefore the number of insurance products, P, should be equal to $K+1$. Then, we find the weights as the solution to the following system of $K+1$ equations

$$\sum_{p=1}^{P} w_{(p)}(t) = 1,$$
$$\sum_{p=1}^{P} w_{(p)}(t) \frac{\partial}{\partial \eta_{jk}} V^i_{(p)}(t,\mu+\eta g)\Big|_{\eta=0} = 0,$$

for all $j,k \in \mathcal{J}$, $j \neq k$.

The natural hedging strategy is independent of the magnitude of the shift, but since the directional derivative (obviously) depends on the direction of the shift, the weights of the natural hedging strategy will depend on the choice of g. This implies that the total liabilities with the natural hedging strategy are insensitive to shifts of the transition intensities in only one specific direction. To compose portfolios that are insensitive to shifts of the transition intensities in any direction, the framework of directional derivatives is insufficient. Christiansen (2008a) introduces functional gradients and shows that there exist functions $h = (h_{jk})_{jk,\in\mathcal{J},j\neq k}$ (given in Theorem 4.3 in Christiansen (2008a)) such that

$$\frac{\partial}{\partial \eta_{jk}} V^i_{(p)}(t,\mu+\eta g)\Big|_{\eta=0} = \int_0^\infty h_{jk}(s) g_{jk}(s) \mathrm{d}s.$$

If we can compose a portfolio, where the functional gradients h are equal to zero, we would have a perfect hedge towards shifts in any direction, but the study of this is beyond the scope of this paper.

Remark 5.3.2. Consider a specific choice of direction, $\bar{g} = (\bar{g}_{jk})_{j,k\in\mathcal{J},j\neq k}$, of the shifts, and a specific choice of magnitude of the shifts, $\bar{\varepsilon} = (\bar{\varepsilon}_{jk})_{j,k\in\mathcal{J},j\neq k}$, such that $\Delta\mu = (\bar{\varepsilon}_{jk}\bar{g}_{jk})_{j,k\in\mathcal{J},j\neq k}$. If d of the entries in $\Delta\mu$ are equal, hence we consider the same shift in d of the K transition intensities of Z, it is sufficient to have a portfolio of $P = K - (d-1) + 1$ different insurance products to solve the natural hedging problem in Definition 5.3.1.

Levantesi and Menzietti (2017) have a similar result in discrete time in the disability model without reactivation. They need two different insurance products or lines of business to find the natural hedging strategy in the case where the transition probabilities in the disability model without recovery change of the same size, and four products of the changes in the transition probabilities are of difference size and sign. We have a general formulation of the result in Remark 5.3.2, since we do not put restrictions on the state-space of the insured and study a general multi-state model of the state of the insured.

5.3.2 Solving the natural hedging problem

Solving the natural hedging problem and finding the natural hedging strategy in Definition 5.3.1 require calculation of the directional derivative of the liabilities. The directional (Gateaux) derivative of the liabilities is studied in Christiansen (2008a). An expression for $\frac{\partial}{\partial \eta_{jk}} V^i_{(p)}(t, \mu + \eta g)\Big|_{\eta=0}$ is given in Equation (A.13) in Christiansen (2008a).

In this paper, we study a different approach for the calculation of the directional derivative of the liabilities. The approach is similar to and inspired by the methods presented in Kalashnikov and Norberg (2003), where the authors derive differential equations for the derivative of the reserve with respect to an underlying parameter θ that influences the valuation basis and the payment functions. Here, we derive a system of differential equations for $\frac{\partial}{\partial \eta_{jk}} V^i_{(p)}(t, \mu + \eta g)\Big|_{\eta=0}$ for $i \in \mathcal{J}$. Using Thiele's differential equation from Equation (5.2.2) and interchanging the order of differentiation, we obtain that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\frac{\partial}{\partial \eta_{jk}} V^i_{(p)}(t, \mu + \eta g)\Big|_{\eta=0} \\
&= \frac{\partial}{\partial \eta_{jk}}\frac{\partial}{\partial t} V^i_{(p)}(t, \mu + \eta g)\Big|_{\eta=0} \\
&= \frac{\partial}{\partial \eta_{jk}}\Big(r(t) V^i_{(p)}(t, \mu + \eta g) - b^i_{(p)}(t) - \sum_{l: l \neq i} \big(\mu_{il}(t) + \eta_{il} g_{il}(t)\big) R^{il}_{(p)}(t, \mu + \eta g)\Big)\Big|_{\eta=0} \\
&= \Bigg(r(t) \frac{\partial}{\partial \eta_{jk}} V^i_{(p)}(t, \mu + \eta g)\Big|_{\eta=0} - \mathbb{1}_{\{i=j\}} g_{jk}(t) R^{jk}_{(p)}(t, \mu) \\
&\qquad - \sum_{l: l \neq i} \mu_{il}(t) \Big(\frac{\partial}{\partial \eta_{jk}} V^l_{(p)}(t, \mu + \eta g)\Big|_{\eta=0} - \frac{\partial}{\partial \eta_{jk}} V^i_{(p)}(t, \mu + \eta g)\Big|_{\eta=0}\Big)\Bigg),
\end{aligned}
\tag{5.3.3}
$$

where

$$R^{il}_{(p)}(t,\mu) = b^{il}_{(p)}(t) + V^{l}_{(p)}(t,\mu) - V^{i}_{(p)}(t,\mu).$$

Now, we have obtained a system of differential equations for the directional derivative of the liabilities in Equation (5.3.3) above with boundary condition $\frac{\partial}{\partial \eta_{jk}} V^i_{(p)}(n, \mu + \eta g)\big|_{\eta=0} = 0$.

A numerical procedure to calculate $\frac{\partial}{\partial \eta_{jk}} V^i_{(p)}(t, \mu + \eta g)\big|_{\eta=0}$ is to solve the system of differential equations that consists of the equations in Equation (5.3.3) for all $i \in \mathcal{J}$ and Thiele's differential equation in Equation (5.2.2). This results in a system of at most $J \times P + J \times K \times P$ differential equations. Based on the calculation of $\frac{\partial}{\partial \eta_{jk}} V^i_{(p)}(t, \mu + \eta g)\big|_{\eta=0}$, we can find the natural hedging strategy as the solution to the problem in Definition 5.3.1.

5.4 Examples

Now, we calculate the natural hedging strategy in two examples, the survival model and the disability model, respectively, with two choices of directions of the shifts, g. The first choice is the additive shift, where $g^{\text{add}}_{jk}(t) = \mathbb{1}_{[0,n]}(t)$ and the shift equals $\Delta\mu^{\text{add}}_{jk}(t) = \varepsilon_{jk}\mathbb{1}_{[0,n]}(t)$ for $\varepsilon_{jk} \in \mathbb{R}$. The second choice is the multiplicative shift, where $g^{\text{mul}}_{jk}(t) = \mu_{jk}(t)\mathbb{1}_{[0,n]}(t)$ and the shift equals $\Delta\mu^{\text{mul}}_{jk}(t) = \varepsilon_{jk}\mu_{jk}(t)\mathbb{1}_{[0,n]}(t)$ for $\varepsilon_{jk} \in \mathbb{R}$.

5.4.1 Example 1: Survival model

We consider the survival model where the state space of the Markov model consists of two states, Alive and Dead, with one transition from Alive to Dead with intensity $\mu(t)$. The survival model is illustrated in Figure 5.1.

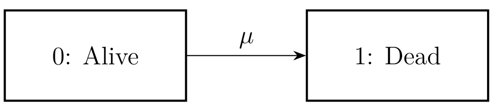

Figure 5.1: *The survival model*

The insurance company offers two insurance products, an annuity if alive after retirement (Product 1) and a term insurance paid upon death (Product 2). The non-zero benefit payment functions for the two products are

$$\begin{aligned} b^0_{(1)}(t) &= a^0_{(1)} \cdot \mathbb{1}_{\{t \geq T\}}, \\ b^{01}_{(2)}(t) &= a^{01}_{(2)}, \end{aligned}$$

where T is the time of retirement and $a^{0}_{(1)}$ and $a^{01}_{(2)}$ are constants. Both products are paid by the same continuous premium while alive before retirement with payment intensity $\pi^0(t) = \pi^0 \cdot \mathbb{1}_{\{t \leq T\}}$, where π^0 is a constant. The payment intensity is the same for both products and the size of the benefits for both products is settled according to the principle of equivalence at time 0

$$V^0_{(1)}(0, \mu) = 0, \quad V^0_{(2)}(0, \mu) = 0.$$

The natural hedging is according to Definition 5.3.1, the weights that solve the following system of equations

$$\begin{aligned} &w_1(t) + w_2(t) = 1, \\ &w_1(t) \frac{\partial}{\partial \eta} V^0_{(1)}(t, \mu + \eta g)\big|_{\eta=0} + w_2(t) \frac{\partial}{\partial \eta} V^0_{(2)}(t, \mu + \eta g)\big|_{\eta=0} = 0. \end{aligned}$$

We calculate the weights for two different choices of the direction g, namely $g^{\text{add}}(t) = \mathbb{1}_{[0,n]}(t)$ and $g^{\text{mul}}(t) = \mu(t)\mathbb{1}_{[0,n]}(t)$, numerically. We solve the system of differential equations that consists of Thiele's differential equation in Equation (5.2.2) and the differential equation of the directional derivative of the liabilities in Equation (5.3.3), and then solve the natural hedging problem above numerically. The components of the numerical example are reported in Table 5.1.

Component	**Value**
Age of policyholder at $t = 0$, x_0	30
Termination, n	80
Time of retirement, T	35
π^0	1
Annuity, $a^0_{(1)}$	4.14
Term insurance, $a^{01}_{(2)}$	60.04
r	0.02

Table 5.1: *Components in the numerical example*

The mortality intensity is given by

$$\mu(t) = 0.0005 + 10^{5.6 + 0.04 \cdot (t + x_0) - 10}.$$

The calculated weights for both choices of g are illustrated in Figure 5.2. We note that the weights at the end points at time 0 and time n are right and left limits, respectively, since the prospective reserve at time 0 and time n by construction is equal to zero.

The natural hedging strategy in this example depends on the direction of the shift, although after retirement the additive shift and the multiplicative shift result in the

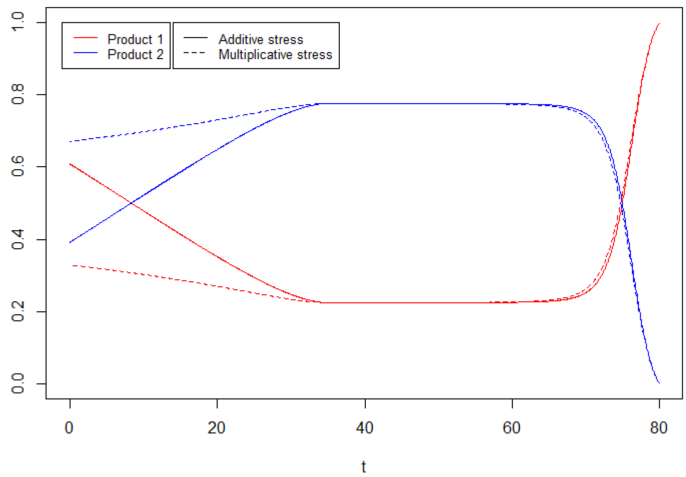

Figure 5.2: *Weights in the numerical example*

same natural hedging strategy. Before retirement, the weight for the life annuity (Product 1) is decreasing and the weight for the term insurance is increasing towards stable levels at retirement for both choices of shift, but the increase and decrease, respectively, are larger for the additive shift than the multiplicative shift.

The natural hedging strategy cannot be a perfect hedge, since it is based on a first order Taylor approximation in Equation (5.3.2). In this example, we study how good the natural hedging strategy with the multiplicative shift hedges insurance risk by applying it against a multiplicative stress on the transition intensities. We consider a stress where the mortality intensity is lowered by 10%, and measure the quality of the natural hedging strategy by comparing $V^0(t, \mu)$ and $V^0(t, \mu + \Delta\mu)$, where $\Delta\mu = -0.1\mu$, calculated with the weights from the natural hedging strategy with a multiplicative stress. The comparison is illustrated in Figure 5.3.

In Figure 5.3 (left), there is no visual difference between the total liabilities calculated with and without the stress in the mortality intensity. The difference in Figure 5.3 (right) is largest with value -0.025 close to termination of the contract. The difference between the liabilities with and without the stress is due to inaccuracy of the Taylor approximation.

To evaluate the quality of the natural hedging strategy against a multiplicative shift in the transition intensities, we compare with the change in the liabilities for each product due to the shift, which is illustrated in Figure 5.4. The change in the mortality intensity to $0.9 \cdot \mu$ decreases the liabilities of the term insurance, since with a lower mortality intensity, less people is expected to die. For the life annuity, the

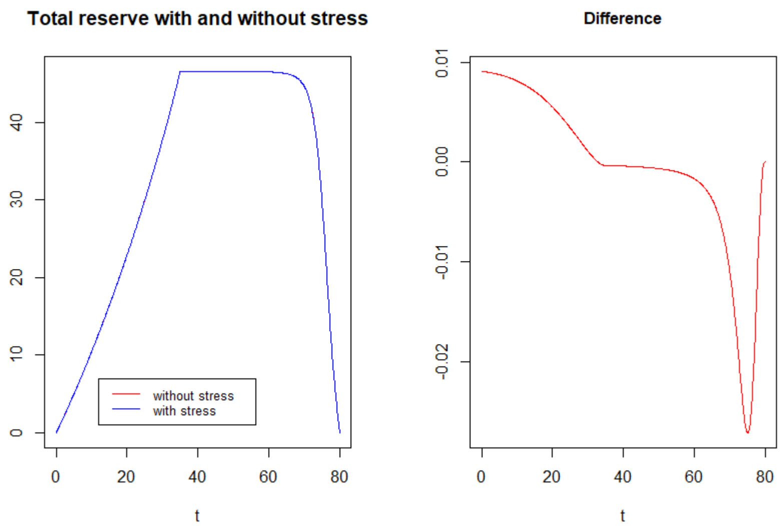

Figure 5.3: *The total liabilities with and without stress in the mortality intensity (left). The difference in the total liabilities after the stress (right).*

liabilities increase with the stress in the mortality intensity since we expect people live a longer and hence receive the life annuity for a longer period of time. The differences in the liabilities of the two products are significantly higher than the difference of the total liabilities, where we use the weights from the natural hedging strategy with the multiplicative shift.

The example illustrated here is simple in the sense that there is only one possible transition from Alive to Dead, and the liabilities for the two products, the life annuity and the term insurance, respectively, increase and decrease, when the mortality intensity decreases. Since the liabilities for the two products change in different directions, when the mortality intensity change, we expect that there exist a natural hedging strategy, where the liabilities of a portfolio with a certain combination of the two products are immune to changes in the mortality intensity. In Christiansen (2011), this is described as a netting effect in the portfolio consisting of the life annuity and the term insurance, and the example we study here is similar to Example 3.1 in Christiansen (2011).

The weights in the natural hedging strategy for both the additive and the multiplicative shift in this example are time-dependent, and insurance contracts are long-term agreements between the insured and the insurer and typically not terminable by the insurer. In order to fully implement the natural hedging strategies in this example, the insurance company must continuously cancel life annuity contracts and obtain new term insurance contracts until the time of retirement from where the weights stabilize, see Figure 5.2. In practice, this is not possible, since the insurance company

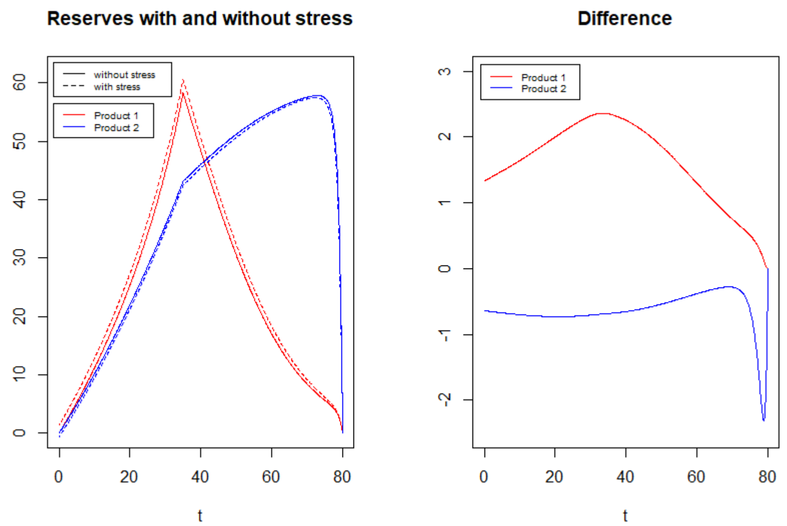

Figure 5.4: *Changes in the liabilities for Product 1 and Product 2 due to the multiplicative shift in the mortality intensity (left). The difference of the liabilities for Product 1 and for Product 2 (right).*

is contractually bounded and cannot cancel contracts, and the market for insurance is governed by supply and demand. Therefore, natural hedging strategies with time-dependent weights are not implementable in practice, but they may serve as a guideline for the insurance company of the sensitivity of the liabilities of a portfolio of insurance products towards changes in the transition intensities. The weights for the multiplicative shift are more constant than the weights for the additive shift in this example, and therefore we believe that a portfolio that consists of constant weights equal to $w_1(T)$ and $w_2(T)$, is more robust towards a multiplicative shift in the mortality intensity than towards an additive shift.

An alternative to obtain the natural hedging strategy is to let the payments of the contract, the life annuity and the term insurance in this example, be time-dependent and dependent on the weights from the natural hedging strategies such that the insured experiences the payments $w_1(t) \cdot a^0_{(1)}$ as the annuity at time t and $w_2(t) \cdot a^{01}_{(2)}$ as the payment upon death at time t. In this case, the insured must buy both the life annuity and the term insurance, and cannot buy more of for instance the life annuity since that would break the natural hedge included in the product design. It is still unrealistic that insurance companies can sell such contracts in practice, since we assume the policyholders demand stable insurance payments that do not vary over time, and since the relation between the life annuity and the term insurance is decided by the natural hedging strategies, and not necessarily the needs of the policyholder.

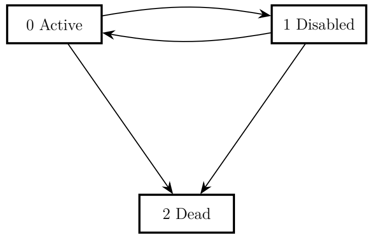

Figure 5.5: *Disability model*

5.4.2 Example 2: Disability model

In this section, we extend the model of the state space from Section 5.4.1 to the disability model illustrated in Figure 5.5.

We assume that the direction and the magnitude of the shift in the transition intensities is the same for the transitions to state 2, Dead, hence $\varepsilon_{02} = \varepsilon_{12}$ and $g_{02}(t) = g_{12}(t)$, and the objective is to find weights such that a portfolio of insurance products is invariant towards shifts in the disability intensity, μ_{01}, the recovery intensity, μ_{10}, and towards the same shift in the death intensity as both active and disabled. We find the weights for two choices of direction of the shift namely the additive shift $g_{jk}^{\text{add}}(t) = \mathbb{1}_{[0,n]}(t)$ and the multiplicative shift $g_{jk}^{\text{mull}}(t) = \mu_{jk}(t)\mathbb{1}_{[0,n]}(t)$. A sufficient condition such that the natural hedging problem has a solution, is that the number of insurance product is equal to four, $P = 4$. The insurance company offers four products, an annuity if alive after retirement (Product 1), a disability annuity as disabled (Product 2), a term insurance paid upon death as active or disabled (Product 3), and a payment upon transition to state 1, Disabled (Product 4). The non-zero benefit payment functions for the four products are

$$\begin{aligned}
b_{(1)}^{0}(t) &= a_{(1)}^{0} \cdot \mathbb{1}_{\{t \geq T\}}, \\
b_{(2)}^{1}(t) &= a_{(2)}^{1}, \\
b_{(3)}^{02}(t) &= b_{(3)}^{12}(t) = a_{(3)}^{02}, \\
b_{(4)}^{01}(t) &= a_{(4)}^{01},
\end{aligned}$$

where T is the time of retirement and $a_{(1)}^{0}$, $a_{(2)}^{1}$, $a_{(3)}^{02}$ and $a_{(4)}^{01}$ are constants. All products are paid by a continuous premium while active before retirement with payment intensity $\pi^0(t) = \pi^0 \cdot \mathbb{1}_{\{t \leq T\}}$, where π_0 is a constant. The payment intensity is the same for all products and the size of the benefits for all products is settled according to the principle of equivalence at time 0

$$V_{(1)}^{0}(0, \mu) = 0, \quad V_{(2)}^{0}(0, \mu) = 0, \quad V_{(3)}^{0}(0, \mu) = 0, \quad V_{(4)}^{0}(0, \mu) = 0.$$

The natural hedging strategy is according to Definition 5.3.1, the weights that solve

$$\sum_{p=1}^{4} w_p(t) = 1,$$
$$\sum_{p=1}^{4} w_p(t)\bigg(\frac{\partial}{\partial\eta_{01}} V^i_{(p)}(t,\mu+\eta g)\Big|_{\eta=0}\varepsilon_{01} + \frac{\partial}{\partial\eta_{10}} V^i_{(p)}(t,\mu+\eta g)\Big|_{\eta=0}\varepsilon_{10}$$
$$+ \frac{\partial}{\partial\eta_{02}} V^i_{(p)}(t,\mu+\eta g)\Big|_{\eta=0}\varepsilon_{02} + \frac{\partial}{\partial\eta_{12}} V^i_{(p)}(t,\mu+\eta g)\Big|_{\eta=0}\varepsilon_{12}\bigg) = 0.$$

where $g \in \{g^{\text{add}}, g^{\text{mul}}\}$.

Since we in this example assume that the direction of the shift in the death intensities is the same, $g_{02} = g_{12}$, and the magnitudes of the shifts are the same $\varepsilon_{02} = \varepsilon_{12}$, a solution to the natural hedging problem is the weights that solve the following four equations

$$\sum_{p=1}^{4} w_p(t) = 1,$$
$$\sum_{p=1}^{4} w_p(t)\frac{\partial}{\partial\eta_{01}} V^i_{(p)}(t,\mu+\eta g)\Big|_{\eta=0} = 0,$$
$$\sum_{p=1}^{4} w_p(t)\frac{\partial}{\partial\eta_{10}} V^i_{(p)}(t,\mu+\eta g)\Big|_{\eta=0} = 0,$$
$$\sum_{p=1}^{4} w_p(t)\bigg(\frac{\partial}{\partial\eta_{02}} V^i_{(p)}(t,\mu+\eta g)\Big|_{\eta=0} + \frac{\partial}{\partial\eta_{12}} V^i_{(p)}(t,\mu+\eta g)\Big|_{\eta=0}\bigg) = 0.$$

We calculate the weights for two different choices of the direction g, namely $g^{\text{add}}_{jk}(t) = \mathbb{1}_{[0,n]}(t)$ and $g^{\text{mul}}_{jk}(t) = \mu_{jk}(t)\mathbb{1}_{[0,n]}(t)$. The directional derivatives of the liabilities in this example is calculated numerically by solving Thiele's differential equation from Equation (5.2.2) and the differential equation for the directional derivative of the liabilities in Equation (5.3.3), and then solve the four equations above numerically. The components of the numerical example are reported in Table 5.2.

The transition intensities are given by

$$\mu_{01}(t) = 0.0004 + 10^{4.54+0.06\cdot(t+x_0)-10},$$
$$\mu_{10}(t) = 2.0058 \cdot e^{-0.117\cdot(t+x_0)},$$
$$\mu_{02}(t) = \mu_{12}(t) = 0.0005 + 10^{5.6+0.04\cdot(t+x_0)-10}.$$

The weights with the additive and multiplicative stress, respectively, are reported in Figure 5.6, and we note that the weights at the end points at time 0 and time n are

Component	Value
Age of policyholder at $t = 0$, x_0	30
Termination, n	80
Time of retirement, T	35
π^0	1
Life annuity, $a^0_{(1)}$	8.60
Disability annuity, $a^1_{(2)}$	6.03
Term insurance, $a^{02}_{(3)}$	58.13
Payment upon transition to Disabled, $a^{01}_{(4)}$	76.42
r	0.02

Table 5.2: *Components in the numerical example*

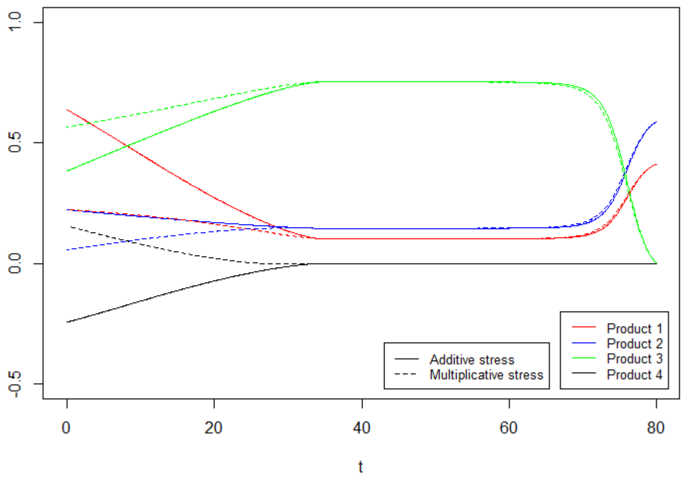

Figure 5.6: *Weights in the numerical example*

right and left limits, respectively, since the prospective reserve at time 0 and time n by construction is equal to zero.

The natural hedging strategy in this example depends on the direction of the shift, although after the time of retirement, the two choices of shift result in the same natural hedging strategy, as we also see in the example in Section 5.4.1. The main difference between the natural hedging strategies for the two choices of shift is that for the additive shift, the weight in Product 4 is negative before retirement, whereas for the multiplicative shift all weights are non-negative. Hence, to have a portfolio where the insurance liabilities are invariant to an additive shift of the transition intensities, the insurance company should buy Product 4. It is worth noticing that all weights are stable from retirement until the insured is more than 100 years old. After the time of retirement, the natural hedging strategy does not consists of Product 4, and it is sufficient with Product 1, Product 2, and Product 3 to perform the natural hedge against additive and multiplicative shifts in the transition intensities. It is possible to hedge disability risk and mortality risk simultaneously in this example, and the natural hedging strategy is the same after retirement for both choices of direction, g. The weight of the term insurance is predominant compared to the weight of the life annuity and the disability annuity, and this composition of an insurance portfolio might not meet the demand of the insured. The example illustrates that after retirement some disability risk is hedged by a combination of the three insurance products: the life annuity, the disability annuity, and the term insurance.

Christiansen (2011) studies a similar example, and investigates the netting effect between a term insurance and a disability annuity towards changes in the death intensity as both active and disabled, but the disability and the recovery intensities are fixed. Christiansen (2011) concludes that there is a netting effect and the possibility to make a natural hedge with a combination of term insurance and disability annuity towards changes in the death intensity. In our example, we study an extended model where we also allow for changes of the disability and recovery intensity, and therefore we need more products to perform the natural hedge. We see a similar netting effect as in Christiansen (2011) after retirement, where a portfolio that consists of the life annuity, the disability annuity and the term insurance hedges against shifts in all the transition intensities in the directions $g_{jk}(t) = \mathbb{1}_{[0,n]}(t)$ and $g_{jk}(t) = \mu_{jk}(t)$.

Similar to the first example in Section 5.4.1, we study the quality of the natural hedging strategy in the direction $g = \mu$ by applying it to a multiplicative stress on the transition intensities. The stress is an increase in the disability intensity, $\mu_{01}(t)$, of 10%, a decrease in the mortality intensity as both Active and Disabled, $\mu_{02}(t)$ and $\mu_{12}(t)$, of 10%, and a decrease in the rehabilitation intensity $\mu_{10}(t)$ of 10%. Since the natural hedging strategy is based on a first order Taylor approximation, we do not expect a perfect hedge. The comparison of the total liabilities calculated with the weights from the natural hedging strategy and with and without the stressed transition intensities is illustrated in Figure 5.7.

There is no visual difference between the total liabilities calculated with and without the multiplicative stress in the transition intensities in Figure 5.7 (left). The difference in Figure 5.7 (right) is (numerically) highest at initialization of the insurance contract at time $t = 0$ with value -0.35 and decreases to below 0.05 at the time of retirement. The difference in Figure 5.7 is due to inaccuracy of the Taylor approximation. Compared to the differences between the liabilities for each of the four products with and without the stress illustrated in Figure 5.8, the difference for the total liabilities is significantly lower, and we conclude again that the natural hedging strategy works

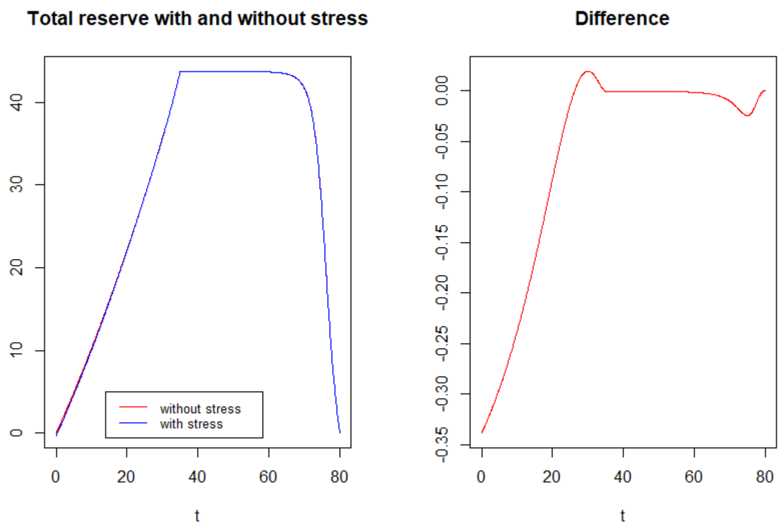

Figure 5.7: *The total liabilities with and without stress in the transition intensities (left). The difference in the total liabilities after the stress (right).*

as intended.

Compared to the previous example in Section 5.4.1, where we study the survival model, the example studied in this section is complex in the sense that it contains multiple possible transitions and the possibility to return to a state after leaving it, see Figure 5.5. It is not clear up front how a simultaneous change in the transition intensities effects the liabilities of each of the four products, and therefore not obvious whether there exists a good natural hedging strategy. The natural hedging strategy with $g(t) = \mathbb{1}_{[0,n]}(t)$ in this example contains a negative weight in Product 4 at initialization of the contract, and hence to implement this natural hedging strategy at time 0, the insurance company should buy Product 4, which is unrealistic in practice. In line with the discussion in the end of Section 5.4.1, the implementation of a natural hedging strategy with time-dependent weights is not possible in practice, even if the weights are non-negative.

In both examples, the age of the policyholders in the portfolio at initialization is fixed, and we only study one cohort of policyholders. A topic for future research is the study of netting effects or natural hedging strategies between policyholders of different ages.

5.5 Conclusion

We study natural hedging as a risk management tool in continuous time life insurance. There is an obvious netting effect between a life annuity and a term insurance in

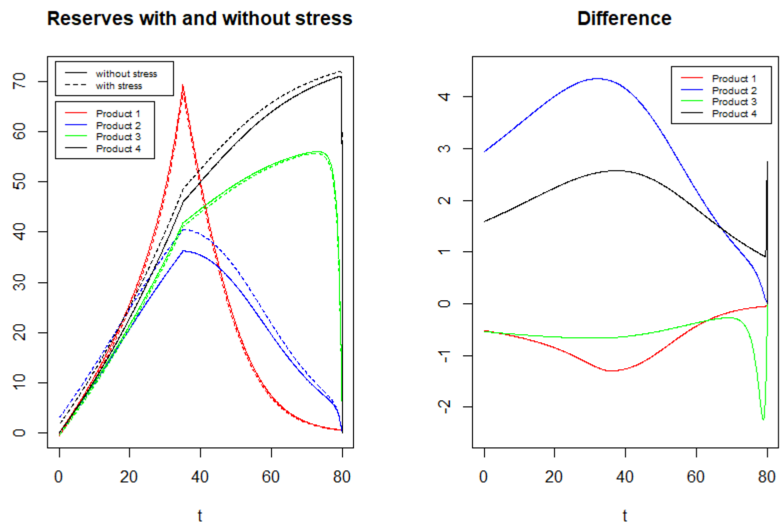

Figure 5.8: *Changes in the liabilities for Product 1, Product 2, Product 3, and Product 4 due to the multiplicative shift in the transition intensities (left). The difference of the liabilities for all products (right).*

the survival model, since when mortality decreases, the liabilities of the life annuity increase and the liabilities of the term insurance decrease and vice versa. In this paper, we search for similar netting effects in multi-state models using natural hedging by studying sensitivities of the liabilities towards directional shifts of the transition intensities. Our implementation of the natural hedging strategy in the survival model in Section 5.4.1 shows the obvious netting effect between a life annuity and a term insurance, but it also shows that the weights of the natural hedging strategy are time-dependent, and that they stabilize after retirement. The natural hedging strategy depends on the direction of the shift, but we see in this example that the natural hedging strategies against an additive and a multiplicative shift, respectively, are equal after the time of retirement. We investigate the natural hedging strategy in the disability model with reactivation in Section 5.4.2. After retirement, the weights of the natural hedging strategy are stable, and there is a netting effect between a life annuity, a disability annuity, and a term insurance, which enables natural hedging of mortality and disability risk simultaneously. Again, the natural hedging strategy depends on the direction of the shift, but also in this example, the natural hedging strategies against an additive and a multiplicative shift, respectively, are equal after the time of retirement.

Acknowledgments and declarations of interest

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Chapter 6

De-risking in multi-state life insurance

This chapter contains the paper *Levantesi, Menzietti, and Nyegaard (2023).*

Abstract

Calculation of insurance liabilities bases on assumptions of mortality rates, disability rates, etc., and insurance companies face systematic insurance risks if assumptions about these rates change. In the paper, we assume there exists a market for trading two securities linked to for instance mortality and disability rates, the de-risking option and the de-risking swap, and we describe the optimization problem to find the de-risking strategy that minimizes systematic insurance risks in a multi-state setup. We illustrate the results in two numerical examples in the survival model and in the disability model, respectively, and the results imply that systematic insurance risks decrease significantly with the use of de-risking strategies.

Keywords: Life insurance; de-risking; risk management; systematic insurance risks.

6.1 Introduction

Insurance companies are exposed to different kinds of risks, and risk management is an important aspect of the insurance business. One type of risk is financial risks, that to a large extent, can be hedged by trading in the financial market. Another type of risk is insurance risk. We refer to unsystematic insurance risks as an adverse development of the insurance portfolio and assume that unsystematic insurance risks are negligible for a large portfolio of insurance contracts. Systematic insurance risks refer to the risks that the future development of the underlying mortality intensities, disability intensities, lapse rates etc. differs from the expected development. Insurance contracts are typically long-term obligations for the insurance company, and therefore an unforeseen development of for instance the underlying mortality of an insurance portfolio may result in large losses, and in the worst-case scenario ruin, for the insurance company.

Possibilities for hedging insurance risks are few. One proposal is the so-called natural hedging, which utilizes that liabilities of different insurance products have different sensitivities towards changes in the underlying transition intensities of a Markov process describing the state of the insured. This enables the construction of portfolios of insurance products that are invariant to changes in the transition intensities. The classical example is in the survival model, where the liabilities of a life annuity increase (decrease) and the liabilities of a term insurance decrease (increase), respectively, when the death intensity decreases (increases). Therefore, we can construct a portfolio with a combination of the two products where the liabilities are immune to changes in the death intensity. This is denoted natural hedging and is studied in Cox and Lin (2007) and Wang et al. (2010). Natural hedging in a multi-state setup with the possibility to hedge for instance disability risks is studied in Levantesi and Menzietti (2017) and Nyegaard (2023). Natural hedging turns out to be an insufficient tool for risk management of systematic insurance risks since the optimal natural hedging portfolios differ a lot from the demands of the insurance market.

Another proposal is in with-profit life insurance, where systematic insurance risks, illustrated by uncertainty in the development of future transition intensities, are handled by choosing prudent transition intensities for the pricing of insurance benefits. The expected surplus is then returned to the policyholders as a bonus. For a long-term agreement, as an insurance contract typically is, what seems as safe-side transition intensities at initialization may not be on the safe side 20 or 30 years later. This is for instance the case for death rates, where longevity improvements have occurred faster than expected during the last 40 years.

What constitutes safe-side transition intensities depends on the insurance product. For a life annuity, a low death intensity is on the safe side, while for term insurance, a high death intensity is on the safe side. Insurance companies face different types of risks, and what is characterized as an adverse development of future transition intensities varies from company to company. Therefore, the demand for hedging systematic insurance risks depends on the type of business. In this paper, we consider a multi-state setup in continuous-time life insurance, where the state of the insured is modelled by a continuous-time Markov process. We model the vector of transition intensities of the Markov process, μ, by a diffusion process, and develop a model for the unfunded liabilities quantifying the systematic insurance risk of the insurance company. The unfunded liabilities describe a potential loss of an insurance company and consist of terms that are linear in the future unknown and stochastic transition intensities. Hence, it would be convenient for the insurance company if there existed a market for μ-linked securities to be able to hedge systematic insurance risks. We assume in the paper that there exists a market for trading two types of μ-linked securities, the de-risking option and the de-risking swap, and describe the optimization problem faced by the insurance company to choose the optimal amount of de-risking. The model presented here is unrealistic in the sense that very few μ-linked securities exist in the market, and in the sense that the model is very simple. The purpose of the model is to quantify systematic insurance risks in a multi-state setup and identify the kinds of hedging strategies that minimise risks. We illustrate the de-risking strategies in two numerical examples in the survival model and the disability model, respectively, and study sensitivities of the optimal choice of de-risking to the parameters of the model. The numerical examples are based on the stochastic model for the transition intensities. For the example in the survival model, we model the mortality intensity with a Feller process and use parameters from Luciano, Spreeuw, and Vigna (2008). In the disability model, we model the transition intensities from active to disabled and from active to dead, respectively, with a CIR process, and we estimate the parameters on data of a cohort of the Italian population qualified for a disability benefit paid by the Italian Government to disabled people.

The stochastic process μ is, in contrast to for instance stock prices and interest rates, not observable, and it is based on assumptions about the state space and possible transitions of the insured. There exist a lot of statistical methods to estimate μ and a derivative with μ as the underlying is special since its value depends on the data from which μ is estimated. This introduces basis risk for an insurance company buying the derivative if the portfolio of the insurance company differs from the data basis of the derivative. We disregard this kind of basis risk in our model.

Existing literature on this topic focuses on mortality-linked securities to hedge mortality risks or longevity risks in the survival model. Two examples of traded mortality-linked securities, the Swiss Re mortality bond and the EIB/BNP longevity bond, are discussed in Blake, Cairns, and Dowd (2006). In general, Blake, Cairns, and Dowd (2006) and Blake et al. (2019) discuss the concept of and the issues that arise with mortality-linked securities. Mortality-linked securities are also studied in Dahl (2004) and Lin and Cox (2005). Pricing of mortality-linked securities requires a stochastic model of the mortality intensity. Biffis (2005) studies affine models of the mortality intensity, and Luciano, Spreeuw, and Vigna (2008) model mortality intensities for dependent lives.

The aim of this paper is to go beyond the survival model and investigate how to manage systematic insurance risks in a multi-state setup by studying μ-linked securities and not only mortality-linked securities in the survival model. Disability-linked securities are studied in D'Amato, Levantesi, and Menzietti (2020) as a possibility to hedge systematic disability risks for long-term care insurance modelled in discrete time. Our formulation applies to any choice of state space of the Markov model, and we study de-risking in continuous time.

The structure of the paper is as follows. In Section 6.2, we introduce the double-stochastic multi-state Markov setup and model the assets, the liabilities, and the unfunded liabilities. Section 6.3 introduces the de-risking strategies: the de-risking option and the de-risking swap. The optimization problem is described in Section 6.4. In Section 6.5, we illustrate the de-risking strategies in two numerical examples.

6.2 Setup

6.2.1 Doubly-stochastic Markov setup

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space, and $\mathcal{J} = \{0, 1, ..., J\}$ be some finite state-space. As in Buchardt, Furrer, and Steffensen (2019), we consider a doubly-stochastic Markov setup, where the state of the holder of an insurance contract is described by a stochastic (jump) process on $(\Omega, \mathbb{F}, \mathbb{P})$ taking values in $\mathcal{J}$. The number of possible transitions in the state-space is denoted by K, and we consider a K-dimensional stochastic process $\mu(t) = (\mu_{jk})_{j,k \in \mathcal{J}, j \neq k}$ on $(\Omega, \mathbb{F}, \mathbb{P})$ with continuous sample paths taking values in $[0, \infty)^K$. The dynamics of μ are assumed to be in the form

$$\mathrm{d}\mu(t) = \alpha^{\mu}(t, \mu(t))\mathrm{d}t + \sigma^{\mu}(t, \mu(t))\mathrm{d}W(t), \tag{6.2.1}$$

where W is a P dimensional Brownian motion, $\alpha^{\mu} : [0, \infty)^{K+1} \mapsto \mathbb{R}^K$ is a deterministic and sufficiently regular function, and

$$\sigma^{\mu}(t, \mu) = \begin{pmatrix} \sigma^{\mu}_{11}(t, \mu) & \sigma^{\mu}_{12}(t, \mu) & \dots & \sigma^{\mu}_{1P}(t, \mu) \\ \sigma^{\mu}_{21}(t, \mu) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma^{\mu}_{K1}(t, \mu) & \sigma^{\mu}_{K2}(t, \mu) & \dots & \sigma^{\mu}_{KP}(t, \mu) \end{pmatrix},$$

for deterministic and sufficiently regular functions $\sigma^{\mu}_{ij} : [0, \infty)^{K+1} \mapsto [0, \infty)$. We omit the age of the insured in the notation since we assume it is a one-cohort model where the age of the insured is x_0 at time 0.

The assumption that the transition intensities are modelled by a diffusion process is in line with assumptions on the models of the mortality intensity in for instance Dahl (2004) and Jevtić, Luciano, and Vigna (2013). Luciano, Spreeuw, and Vigna (2008) has a similar model for the mortality intensity, where a jump measure in the dynamics of the mortality intensity is included. The natural filtration generated by the stochastic process μ is $\mathcal{F}^{\mu} = \left(\mathcal{F}^{\mu}_t\right)_{t \geq 0}$, where $\mathcal{F}^{\mu}_t = \sigma(\mu(s) : 0 \leq s \leq t)$, and we interpret $\mathcal{F}^{\mu}$ as all information about $\mu(t)$ for $t \in [0, \infty)$.

Similar to Buchardt, Furrer, and Steffensen (2019), we can construct a jump process $Z = \left(Z(t)\right)_{t \geq 0}$ on $(\Omega, \mathbb{F}, \mathbb{P})$ taking values in $\mathcal{J}$ with $Z(0) = 0$, where Z conditional on $\mathcal{F}^{\mu}$ is a continuous time Markov chain with transition intensities μ. We assume that

Z indicates the state (e.g. Active, Disabled or Dead) of the holder of an insurance contract who is x_0 years old at time $t = 0$. The natural filtration generated by Z is given by $\mathcal{F}^Z = \left(\mathcal{F}_t^Z\right)_{t\geq 0}$, and we interpret $\mathcal{F}^Z$ as information about $Z(s)$ for $s \in [0,t]$. There exist transition probabilities of Z conditional on μ given by

$$\mathbb{P}\big(Z(s) = j \;\big|\; \mathcal{F}_t^Z, \mathcal{F}^\mu\big) = \mathbb{P}\big(Z(s) = j \;\big|\; Z(t), \mathcal{F}^\mu\big) := p^\mu_{Z(t)j}(t,s),$$

since Z is Markov conditional on $\mathcal{F}^\mu$. The fact that Z has transition intensities μ conditional on $\mathcal{F}^\mu$ implies that

$$\mu_{ij}(t) = \lim_{h\downarrow 0} \frac{p^\mu_{ij}(t,t+h)}{h},$$

for all $t \geq 0$, and $i,j \in \mathcal{J}$, $i \neq j$. The transition probabilities conditional on μ satisfy Kolmogorov's backward and forward differential equations. We introduce the processes $N^k(t)$ that count the number of jumps of Z into state $k \in \mathcal{J}$ up to and including time t

$$N^k(t) = \#\big\{s \in (0,t] \;\big|\; Z(s-) \neq k, Z(s) = k\big\},$$

where $Z(s-) = \lim_{h\downarrow 0} Z(s-h)$. If μ is a deterministic process, this setup corresponds to the classical Markov chain setup in life insurance as described in e.g. Hoem (1969) and Norberg (1991).

6.2.2 Insurance contract

Now, we model the payments of an insurance contract. Payments link to sojourns in states and transitions between states and therefore payments depend on Z. The payment stream has dynamics

$$\mathrm{d}B(t) = b_{Z(t)}(t)\mathrm{d}t + \sum_{k:k\neq Z(t-)} b_{Z(t-)k}(t)\mathrm{d}N^k(t),$$

where b_j and b_{jk} for $j,k \in \mathcal{J}$, $j \neq k$ are deterministic functions. The payments b_j link to continuous benefits or premiums during sojourn in state j, and the payments b_{jk} link to payments upon transition from state j to state k. Benefit payments are positive and premium payments are negative.

6.2.3 Assets and liabilities

The basis of our model is an insurance company that sells insurance contracts with payments specified by $\mathrm{d}B(t)$ to a cohort of policyholders aged x_0 at time $t = 0$. The assets and liabilities of the insurance company are affected by the underlying mortality rate, disability rate etc. of the portfolio modelled by the stochastic process μ. Hence, the insurance company is exposed to systematic insurance risks if its valuation basis differs from the realized μ. We assume that the portfolio is large such that unsystematic insurance risks are negligible. Our aim is to quantify the effect of systematic insurance risks on the assets and liabilities of the portfolio under the assumption that μ is modelled by Equation (6.2.1). Insurance companies are also exposed to financial risks. Since the focus of this paper is systematic insurance risks, we make the assumption that the interest rate is a deterministic function $t \mapsto r(t)$ and that the insurance company invests in an account with interest rate $r(t)$. The interest rate $r(t)$ is also used for discounting the value of future payments.

Model of the assets

The expected assets at time t are given by the expectation of premiums minus benefits in the interval $[0, t]$ accumulated with the interest rate. We denote the expected assets at time t by $\tilde{A}(t)$

$$\tilde{A}(t) = A_0 e^{\int_0^t r(u)\mathrm{d}u} + \mathbb{E}\left[\int_0^t e^{\int_s^t r(u)\mathrm{d}u}\big(-\mathrm{d}B(s)\big)\right], \tag{6.2.2}$$

where A_0 is the initial assets. The assets depend on the stochastic process μ since the payment stream depends on Z, that in the doubly-stochastic Markov setup, depends on μ. Calculation of the expectation in Equation (6.2.2) is non-trivial, and instead of focusing on $\tilde{A}(t)$, we study the expected assets conditioned on μ

$$A(t) = A_0 e^{\int_0^t r(u)\mathrm{d}u} + \mathbb{E}\left[\int_0^t e^{\int_s^t r(u)\mathrm{d}u}\big(-\mathrm{d}B(s)\big) \,\middle|\, \mathcal{F}^{\mu}\right],$$

with the relation

$$\tilde{A}(t) = \mathbb{E}\big[A(t)\big].$$

Due to the Markov property of Z conditioned on μ and that $Z(0) = 0$, we have that

$$A(t) = A_0 e^{\int_0^t r(u)\mathrm{d}u} - \int_0^t e^{\int_s^t r(u)\mathrm{d}u} \sum_{i \in \mathcal{J}} p_{0i}^{\mu}(0, s)\big(b_i(s) + \sum_{j: j \neq i} \mu_{ij}(s) b_{ij}(s)\big)\mathrm{d}s.$$

Model of the liabilities

The liabilities are the expected present value of future payments of the insurance contract. We assume that the insurance company uses a deterministic valuation basis for the calculation of the liabilities given by assumptions on the interest rate $\hat{r}(t)$ and assumptions on the transition intensities $\hat{\mu}(t)$. We assume that $\hat{r}(t) = r(t)$ and that $\hat{\mu}(t)$ is deterministic and independent of the stochastic process μ. With deterministic transition intensities, we are in the classical Markov chain setup in life insurance. The liabilities at time t are given by

$$\mathbb{E}^{\hat{\mu}}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u}\mathrm{d}B(s) \,\middle|\, \mathcal{F}_t^Z\right] = \mathbb{E}^{\hat{\mu}}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u}\mathrm{d}B(s) \,\middle|\, Z(t)\right] := \hat{V}^{Z(t)}(t),$$

where the superscript $\hat{\mu}$ denotes that Z has transition intensities $\hat{\mu}$. The state-wise liabilities, $\hat{V}^i(t)$, where we condition on $Z(t) = i$ for $i \in \mathcal{J}$, are deterministic and satisfy Thiele's differential equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\hat{V}^i(t) &= r(t)\hat{V}^i(t) - b_i(t) - \sum_{j:j\neq i} \hat{\mu}_{ij}(t)\hat{R}^{ij}(t), \\ \hat{V}^i(n) &= 0, \end{aligned} \tag{6.2.3}$$

where $\hat{R}^{ij}$ is the sum-at-risk upon transition from state i to state j and is given by

$$\hat{R}^{ij}(t) = b_{ij}(t) + \hat{V}^j(t) - \hat{V}^i(t).$$

The liabilities at time t depend on the state of the insured at time t, $Z(t)$, and are therefore stochastic. Hence in the doubly-stochastic Markov model, the liabilities depend on the stochastic process μ. Similar to the model of the assets, we model the expected liabilities at time t

$$\tilde{V}(t) = \mathbb{E}\big[\hat{V}^{Z(t)}(t)\big] = \mathbb{E}\big[V(t)\big],$$

for

$$V(t) = \mathbb{E}\big[\hat{V}^{Z(t)}(t) \;\big|\; \mathcal{F}^\mu\;\big] = \sum_{i\in\mathcal{J}} p^\mu_{0i}(0,t)\hat{V}^i(t).$$

The unfunded liabilities

The insurance company faces a potential loss or gain if the development of μ is different from the valuation basis $\hat{\mu}$. Our aim is to quantify the loss or gain as a basis for deciding whether a de-risking strategy described in Section 6.3 is useful for the insurance company. The expected unfunded liabilities at time t are given by

$$\tilde{L}(t) = \tilde{V}(t) - \tilde{A}(t) = \mathbb{E}\big[L(t)\big],$$

for $L(t) = V(t) - A(t)$. We refer to $L(t)$ as the unfunded liabilities. If the unfunded liabilities are positive, the insurance company faces a potential loss, since the liabilities exceed the assets, and the insurance company faces a potential gain if $L(t)$ is negative. Using Kolmogorov's forward differential equations for the transition probabilities and Thiele's differential equation in Equation (6.2.3) we obtain that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}L(t) &= r(t)L(t) + \sum_{i\in\mathcal{J}}\sum_{j:j\neq i} p^\mu_{0i}(0,t)\big(\mu_{ij}(t) - \hat{\mu}_{ij}(t)\big)\hat{R}^{ij}(t), \\ &= r(t)L(t) + \sum_{i\in\mathcal{J}}\sum_{j:j\neq i} l_{ij}(t), \\ L(0) &= \hat{V}^0(0) - A_0, \end{aligned} \tag{6.2.4}$$

for $l_{ij}(t) = p^\mu_{0i}(0,t)\big(\mu_{ij}(t) - \hat{\mu}_{ij}(t)\big)\hat{R}^{ij}(t)$.

The differential equation in Equation (6.2.4) above yields that the unfunded liabilities gain interest rate and increase or decrease with a rate, $l_{ij}(t)$, that is a probability-weighted sum of all possible transitions in the state space with terms that depends on the difference between the stochastic transition intensity, μ_{ij}, and the transition intensity from the valuation basis, $\hat{\mu}_{ij}$, times the sum-at-risk. The rate $l_{ij}(t)$ is similar to the surplus contribution rate (see e.g. (3.7) in Norberg (1999)) in with-profit life insurance, where the surplus increases due to the difference between prudent technical transition intensities used for pricing and the best estimate market transition intensities used for valuation.

The unfunded liabilities have a solution given by

$$L(t) = (\hat{V}^0(0) - A_0)e^{\int_0^t r(u)\mathrm{d}u} + \int_0^t e^{\int_s^t r(u)\mathrm{d}u} \sum_{i\in\mathcal{J}} \sum_{j:j\neq i} l_{ij}(s)\mathrm{d}s. \tag{6.2.5}$$

The representations in Equations (6.2.4) and (6.2.5) illustrate what affects the unfunded liabilities, and the effect is highest when the difference between μ and $\hat{\mu}$ is large. For instance, if the realized mortality or disability rates of the insurance portfolio differ from the rates in the valuation basis. The unfunded liabilities are stochastic since they depend on μ, and the insurance company faces a potential loss upon an adverse development of μ. Therefore, the insurance company has an interest in hedging the unfunded liabilities against systematic insurance risks.

6.3 De-risking strategies

In this section, we introduce μ-linked securities as a risk management tool for insurance companies to reduce systematic insurance risks. We assume that the insurance company can invest in K μ-linked securities each of them paying a continuous rate or cash flow of $d_{ij}\big(t, \mu_{ij}(t)\big)$ for $i, j \in \mathcal{J}$, $i \neq j$. There is a risk that the counterpart providing the de-risking defaults. This is denoted credit risks and is not studied here. D'Amato, Levantesi, and Menzietti (2020) implement the possibility that the counterpart defaults in their model as a binomial variable in discrete time.

If the insurance company invests in the securities for de-risking purposes, the unfunded liabilities including de-risking are given by

$$\begin{aligned} &L^D(t) \\ &= L(t) - \sum_{i\in\mathcal{J}} \sum_{j:j\neq i} h_{ij} D_{ij}(t), \\ &= (\hat{V}^0(0) - A_0)e^{\int_0^t r(u)\mathrm{d}u} + \int_0^t e^{\int_s^t r(u)\mathrm{d}u} \Bigg(\sum_{i\in\mathcal{J}} \sum_{j:j\neq i} \big(l_{ij}(s) - h_{ij} d_{ij}(s, \mu_{ij}(s))\big) \Bigg) \mathrm{d}s. \end{aligned}$$

where h_{ij} is the amount of μ_{ij}-linked security bought. Let

$$l_{ij}^D(t,\mu_{ij}(t)) = l_{ij}(t) - h_{ij}d_{ij}(t,\mu_{ij}(t)).$$

We define the hedging price of the μ_{ij}-linked de-risking strategy, P_{ij}, as the sum of the expected present value of the payments of the derivative and the hedging costs,

$$P_{ij} = h_{ij}\left(a_{ij} + \mathbb{E}\left[\int_0^n e^{-\int_0^t r(u)\mathrm{d}u} d_{ij}(t,\mu_{ij}(t))\mathrm{d}t\right]\right),$$

where a_{ij} is the hedging cost for the derivative with cash flow $d_{ij}(t,\mu_{ij}(t))$. The hedging costs are a risk premium on top of the expected value of the de-risking cash flow for the counterpart to take in the risk.

We consider two different types of de-risking strategies with different choices of $d_{ij}(t,\mu_{ij}(t))$. The first is a de-risking option, and the second is a de-risking swap, and we discuss the advantages and drawbacks of each type.

6.3.1 De-risking option

The insurance company is interested in hedging against a scenario where μ differs a lot from $\hat{\mu}$ since then the insurance company faces a potential loss. A possible choice of $d_{ij}(t,\mu_{ij}(t))$ is

$$d_{ij}(t,\mu_{ij}(t)) = \max\{u_{ij}(t) - \hat{\mu}_{ij}(t), 0\}\hat{R}^{ij}(t), \tag{6.3.1}$$

with a European call option structure exercised at time t with strike $\hat{\mu}_{ij}$. For this de-risking option, the rate $l_{ij}^D(t,\mu_{ij}(t))$ becomes

$$l_{ij}^D(t,\mu_{ij}(t)) = \begin{cases} (p_{0i}^{\mu}(0,t) - h_{ij})(\mu_{ij}(t) - \hat{\mu}_{ij}(t))\hat{R}^{ij}(t), & \text{if } \mu_{ij}(t) > \hat{\mu}_{ij}(t) \\ -p_{0i}^{\mu}(0,t)(\hat{\mu}_{ij}(t) - \mu_{ij}(t))\hat{R}^{ij}(t), & \text{if } \mu_{ij}(t) \leq \hat{\mu}_{ij}(t) \end{cases}$$

The rate above is always negative if $h_{ij} > p_{ij}^{\mu}(0,t)$ and if the sum-at-risk, $\hat{R}^{ij}(t)$, is positive. The sign of the sum-at-risk depends on the insurance product, and it is possible that $\hat{R}^{ij}(t)$ is positive for some $t \in [0,n]$ and negative for others. The insurance company should only choose to invest in a de-risking option with a call option structure if the sum-at-risk is positive. Otherwise, the investment increases the unfunded liabilities and introduces basis risk for the insurance company. If the sum-at-risk is negative, a European put option structure is preferred to minimize $l_{ij}^D(t,\mu_{ij}(t))$

$$d_{ij}(t,\mu_{ij}(t)) = -\max\{\hat{\mu}_{ij}(t) - \mu_{ij}(t), 0\}\hat{R}^{ij}(t). \tag{6.3.2}$$

To make a perfect hedge of the rate l_{ij} in the unfunded liabilities, the transition probabilities should be included in the de-risking cash flow. This is not possible, since we assume that $d_{ij}(t, \mu_{ij}(t))$ depends on $\mu_{ij}(t)$ and $p_{ij}^{\mu}(0, t)$ depends on other transition intensities as well.

Here, the cash flow of the de-risking option depends on the sum-at-risk of an insurance product, such that the option is designed to reduce systematic insurance risks of a specific product. Another possibility is a de-risking option where the rate only depends on the difference between the stochastic μ and $\hat{\mu}$. This introduces more basis risk for the insurance company since the option is not designed for a specific insurance product, but for the counterpart selling de-risking strategies, it is a more liquid product. In this case, the European call option structure is

$$d_{ij}(t, \mu_{ij}(t)) = \max\{\mu_{ij}(t) - \hat{\mu}_{ij}(t), 0\}, \tag{6.3.3}$$

and the European put option structure is

$$d_{ij}(t, \mu_{ij}(t)) = \max\{\hat{\mu}_{ij}(t) - \mu_{ij}(t), 0\}. \tag{6.3.4}$$

To reduce systematic insurance risks, a de-risking option is only optimal if the sum-at-risk has the same sign throughout the course of the insurance contract. In the survival model, the sum-at-risk of a life annuity is negative, and the sum-at-risk of a life insurance or term insurance is positive. A combination of the two products may have a changing sign of the sum-at-risk and requires a combination of a put and a call option to avoid basis risk for the insurance company. D'Amato, Levantesi, and Menzietti (2020) study a disability option on the transition probabilities for hedging disability risks of long-term care insurance in discrete time. For long-term care insurance products, the sum-at-risk for the transition from Active to Disabled is positive, and D'Amato, Levantesi, and Menzietti (2020) use a European call option structure on the transition probability from Active to Disabled.

We assume that the hedging costs of the de-risking option are proportional to the expected present value of the de-risking cash flow

$$a_{ij} = \delta \cdot \mathbb{E}\left[\int_0^n e^{-\int_0^t r(u)\mathrm{d}u} d_{ij}(t, \mu_{ij}(t))\mathrm{d}t\right].$$

6.3.2 De-risking swap

Inspired by D'Amato, Levantesi, and Menzietti (2020), we consider a plain vanilla de-risking swap with $\mu_{ij}(t)$ as the underlying. The cash flow of the swap is the difference between a fixed and a floating leg. We assume that the fixed leg depends on $\hat{\mu}$, and that the floating leg depends on the stochastic transition intensities μ. By buying this contract, the insurance company agrees to pay the fixed leg to the counterpart in return of the floating leg, and the hedging cash flow is the difference between the fixed and the floating leg. One possible choice of the hedging cash flow $d_{ij}(t, \mu_{ij}(t))$ is

$$d_{ij}(t, \mu_{ij}(t)) = \big(\mu_{ij}(t) - \hat{\mu}_{ij}(t)(1+\rho)\big)\hat{R}^{ij}(t), \tag{6.3.5}$$

where ρ is a fixed proportional risk premium for the counterpart to take on the risk of paying a stochastic, floating leg, $\mu_{ij}(t)\hat{R}^{ij}(t)$ is the floating leg, and $\hat{\mu}_{ij}(t)(1+\rho)\hat{R}^{ij}(t)$ is the fixed leg.

For this choice of de-risking swap, the rate $l_{ij}^D(t, \mu_{ij}(t))$ becomes

$$l_{ij}^D(t, \mu_{ij}(t)) = \big(p_{0i}^{\mu}(0,t) - h_{ij}\big)\big(\mu_{ij}(t) - \hat{\mu}_{ij}(t)\big)\hat{R}^{ij}(t) + h_{ij}\rho\hat{\mu}_{ij}(t)\hat{R}^{ij}(t).$$

The interest of the insurance company is that $l^D(t, \mu_{ij}(t))$ is low and preferably negative to keep the unfunded liabilities at a minimum. The contributions to the unfunded liabilities are stochastic since they depend on μ, and the insurance company faces the risk of adverse development of μ. The de-risking swap seek to minimize the variation of $l_{ij}^D\big(t, \mu_{ij}(t)\big)$ by interchanging the uncertainty in μ with the deterministic $\hat{\mu}$ to reduce risk for the insurance company

As with the de-risking option, the rate of de-risking swap presented in Equation (6.3.5) depends on the sum-at-risk of a specific insurance product or combination of insurance products. For the counterpart selling the de-risking strategies, a more liquid product is to let the hedging cash flow depend on the difference between the stochastic μ and $\hat{\mu}$ such that

$$d_{ij}(t, \mu_{ij}(t)) = \mu_{ij}(t) - \hat{\mu}_{ij}(t)(1+\rho), \tag{6.3.6}$$

since then, the de-risking swap does not depend on a specific type of insurance product. For this choice of de-risking swap, the contribution rate to the unfunded liabilities is

$$l_{ij}^D(t, \mu_{ij}(t)) = \big(p_{0i}^{\mu}(0,t)\hat{R}^{ij}(t) - h_{ij}\big)\big(\mu_{ij}(t) - \hat{\mu}_{ij}(t)\big) + h_{ij}\rho\hat{\mu}_{ij}(t),$$

which introduces more basis risk to the insurance company.

For the de-risking swap, we set the hedging cost to be

$$a_{ij} = -\mathbb{E}\left[\int_0^n e^{-\int_0^t r(u)\mathrm{d}u} d_{ij}(t, \mu_{ij}(t))\mathrm{d}t\right],$$

such that the hedging price is equal to zero.

6.4 The optimization problem

The insurance company faces a potential loss if the liabilities exceed the assets, i.e. the unfunded liabilities are positive. The unfunded liabilities are affected by the stochastic process μ, and if μ behaves in an adverse way, the insurance company may face a loss. Therefore, it is in the interest of the insurance company to buy de-risking strategies to minimize the risk of a potential loss and for risk management of systematic insurance risks. We assume that insurance companies can buy all the presented de-risking options from Section 6.3.1 and de-risking swaps from Section

6.3.2 in a frictionless market. The insurance company must choose the amount of de-risking to buy by choosing h_{ij} for all possible transitions. In this section, we formulate an optimization model to choose the optimal amount of de-risking for the insurance company. The formulation is inspired by Lin, MacMinn, and Tian (2015), where the authors study de-risking for defined benefit plans in the survival model, and D'Amato, Levantesi, and Menzietti (2020), where the authors formulate an optimization model to choose the amount of de-risking of disability risks in the disability model without reactivation in discrete time. Our formulation is for a general multi-state model in continuous time.

We assume that there is a capital cash flow for each transition with the rate $k_{ij}(t)$ such that the insurance company amortizes the unfunded liabilities. Hence, if the unfunded liabilities increase there is a capital injection with the change, and if the unfunded liabilities decrease there is a withdrawal with the change, such that

$$k_{ij}(t) = l_{ij}^{D}\big(t, \mu_{ij}(t)\big),$$

where we ignore that the unfunded liabilities gain interest. Let

$$k(t) = \sum_{i \in \mathcal{J}} \sum_{j : j \neq i} k_{ij}(t)$$

The total discounted costs of all the de-risking strategies are

$$\begin{aligned} TC = &\sum_{i \in \mathcal{J}} \sum_{j : j \neq i} h_{ij} a_{ij} \\ &+ \int_0^n e^{-\int_0^t r(u)\mathrm{d}u} \big(\max\{k(t), 0\}(1+\psi_1) - \max\{-k(t), 0\}(1-\psi_2)\big)\mathrm{d}t, \end{aligned} \tag{6.4.1}$$

where ψ_1 and ψ_2 are penalty factors on the capital inflow and outflow, respectively.

Inspired by Lin, MacMinn, and Tian (2015), we assume that the objective of the insurance company is to minimize its expected total costs when choosing the de-risking strategy at time 0. The constraints in the optimization problem are that the expected unfunded liabilities at termination of the contract are less than zero such that, in expectation, the assets exceed the liabilities during the course of the contract, and that the hedging price of all de-risking strategies should be lower than the assets minus the liabilities at time 0. To control the worst-case scenarios or downside risk, we impose a constraint on the conditional value-at-risk (CVaR) of the unfunded liabilities again inspired by Lin, MacMinn, and Tian (2015).

Now, we formulate the optimization problem

$$\begin{aligned} &\min_{(h_{ij})_{i,j\in\mathcal{J},i\neq j}} \mathbb{E}\big[TC\big], \\ &\text{subject to} \\ &\qquad \mathbb{E}\big[L^D(n)\big] \leq 0, \\ &\qquad CVaR_\alpha\big(L^D(n)\big) \leq \tau, \\ &\qquad \sum_{i\in\mathcal{J}} \sum_{j:j\neq i} h_{ij} P_{ij} \leq A_0 - \hat{V}^0(0). \end{aligned} \tag{6.4.2}$$

We note that the optimization problem is non-linear in h_{ij}. Calculation of the expectations and the CVaR that appear in the optimization problem requires simulation-based methods.

6.5 Numerical examples

6.5.1 The survival model

We consider a life annuity in the survival model (see Figure 6.1) which is paying a rate of b while the insured is alive and which is paid by a one-time premium, π, at initialization of the insurance contract. In this model, there is only one transition with intensity $\mu(t)$.

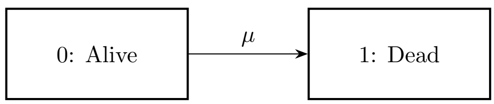


Figure 6.1: *The survival model*

The insured is x_0 years old at time $t = 0$, and inspired by Luciano, Spreeuw, and Vigna (2008), we model the mortality intensity with a Feller process, which is a non-mean-reverting modification of the traditional Cox-Ingersoll-Ross (CIR) model,

$$\mathrm{d}\mu(t) = \phi\mu(t)\mathrm{d}t + \sigma\sqrt{\mu(t)}\mathrm{d}W(t),$$

where $W(t)$ is a standard Brownian motion. This is in accordance with the model assumption in Equation (6.2.1). The parameters in the numerical example are reported in Table 6.1.

The parameters in the model for the mortality intensity is from Luciano, Spreeuw, and Vigna (2008). We assume that the deterministic valuation basis is given by $\hat{\mu}(t) = 1.05 \cdot \mathbb{E}[\mu(t)]$, and we choose the target level, τ, for the conditional value at risk as $\tau = 0.5 \cdot CVaR_\alpha\big(L(n)\big)$.

We calculate the expected total costs, $\mathbb{E}\big[TC\big]$, the expected total unfunded liabilities, $\mathbb{E}\big[L^D(n)\big]$, and the conditional value-at-risk of the unfunded liabilities,

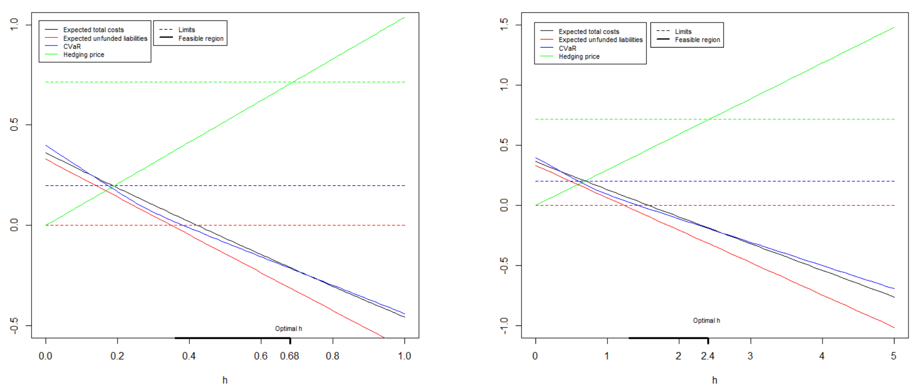
Table 6.1: *Components in numerical example*

Component	Value	Component	Value
Age of insured, x_0	68	$r(t)$	0.01
ϕ	0.0810051	$V(0)$	14.55
σ	0.0002400	$A(0) = \pi$	15.26
$\mu(0)$	0.0204276	δ	0.10
Termination, n	42	ρ	−0.03
Premium, π	15.26	ψ_1	0.10
Annuity rate, b	1	ψ_2	0.10
$Z(0)$	0	α	0.99

$CVaR_\alpha\big(L^D(n)\big)$ without de-risking and for various levels, h, of the different de-risking strategies based on 5000 simulations of $\mu(t)$.

In this example with a life annuity, the sum-at-risk from state 0 (Alive) to state 1 (Dead) is negative, and therefore, we consider the de-risking option with the European put option structure. First, we consider the four different types of de-risking described in Section 6.3: The de-risking option including the sum-at-risk (6.3.2), the de-risking option without the sum-at-risk (6.3.4), the de-risking swap with the sum-at-risk (6.3.5), and the de-risking swap without the sum-at-risk (6.3.6). In Figures 6.2 and 6.3, we illustrate the constraints from the optimization problem in Equation (6.4.2) with the standard parameters from Table 6.1 and the expected total costs as functions of the amount of de-risking, h.

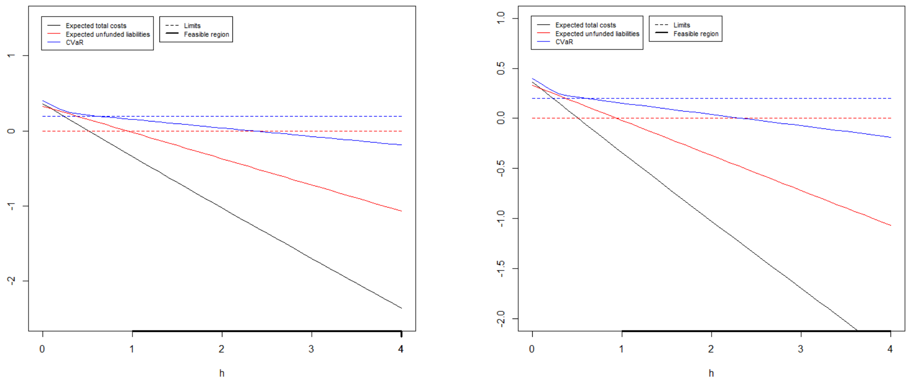
Figure 6.2: *Illustration of the optimization problem for the de-risking option*



(a) *De-risking option with the sum-at-risk* **(b)** *De-risking option without the sum-at-risk*

Figures 6.2 and 6.3 show that in all four cases the expected total costs, the expected unfunded liabilities and the CVaR are decreasing when the amount of de-risking increases. For the de-risking option, the limitation on the amount of de-risking is

Figure 6.3: *Illustration of the optimization problem for the de-risking swap*



(a) *De-risking swap with the sum-at-risk* **(b)** *De-risking swap without the sum-at-risk*

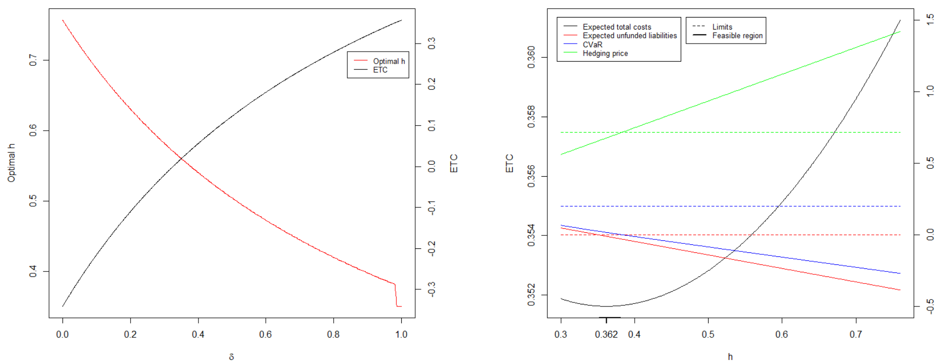
the hedging price. The expected total costs with the optimal h for the de-risking option with and without the sum-at-risk are -0.209 and -0.186, respectively, and hence the de-risking option including the sum-at-risk is better than the de-risking option without the sum-at-risk. This indicates that there is more basis risk without the sum-at-risk, as we expect. The expected total costs without de-risking are 0.3632, which is higher than including de-risking. This illustrates that the use of de-risking strategies reduces systematic insurance risks. For the de-risking swap, there is no limitation on the amount of de-risking the insurance company should buy since $\mathbb{E}\big[TC\big]$, $\mathbb{E}\big[L^D(n)\big]$, and $CVaR_\alpha\big(L^D(n)\big)$ are all decreasing in the amount of de-risking and the hedging price is equal to zero.

The de-risking option with and without the sum-at-risk behaves in the same way, which is also the case for the de-risking swap with and without the sum-at-risk. Hence, when we now study sensitivities of the optimal amount of de-risking to the parameters δ, ρ, ψ_1 and ψ_2, we only consider the de-risking option and de-risking swap with the sum-at-risk since we obtain similar results without the sum-at-risk.

The δ-parameter affects the hedging price and the total costs for the de-risking option. The insurance company can afford less de-risking when δ increases, and therefore the optimal value of h is decreasing in δ, which is illustrated in Figure 6.4(a). The expected total costs for the optimal h are increasing in δ since the hedging costs are increasing in δ, see Figure 6.4(a). Figure 6.2 shows that the expected total costs are decreasing in h making the highest feasible h optimal. For δ close to 1, the expected total costs are increasing for high h as illustrated in Figure 6.4(b) for $\delta = 0.984$ and as a result it is not the highest feasible h that is optimal.

The cash flow of the de-risking swap depends on the ρ-parameter, see Equation

Figure 6.4: *The optimal h for varying δ and the optimization problem for high δ*



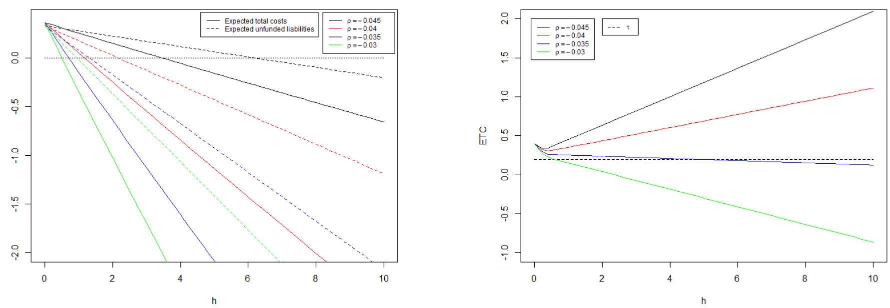
(a) *The optimal h and the expected total costs as a function of δ for the de-risking option*

(b) *Optimization problem for the de-risking option for $\delta = 0.984$*

(6.3.5). The higher ρ is, the higher is the fixed leg of the swap, and since an insurance company selling life annuities, as in this example, is at risk if the mortality intensity is low, a high fixed leg is attractable. The expected total cost, the expected unfunded liabilities and the CVaR for different values of ρ are illustrated in Figure 6.5. The expected total cost and the expected unfunded liabilities are decreasing in h for all the values of ρ considered here, and decreasing the most for high ρ's since a high fixed leg is attractable to the insurance company. For lower values for ρ than those considered here, the expected total costs and the expected unfunded liabilities are increasing in h since it is unfavourable for the insurance company to buy the de-risking swap for even lower values of ρ. The CVaR is increasing for $\rho = -0.045$ and $\rho = -0.04$ and decreasing for $\rho = -0.035$ and $\rho = -0.03$, and therefore, for the values of ρ considered here, the optimization problem only has a feasible solution for $\rho = -0.035$ and $\rho = -0.03$. Since the expected total costs are decreasing in h, it is optimal for the insurance company to buy as many de-risking swaps as possible as we also see in Figure 6.3. Hence for the de-risking swap in this example with a life annuity, it is either optimal to buy $h = \infty$ swaps or $h = 0$ depending on the value of ρ. Therefore, we only consider the de-risking option, when we study sensitivities towards ψ_1 and ψ_2.

The ψ_1 and ψ_2-parameters only affect the total costs of the insurance company (see Equation (6.4.1)), and therefore, the region of feasible h's in the optimization problem in Equation (6.4.2) does not change when ψ_1 and ψ_2 change. Hence, if the expected total cost is decreasing in h, the highest h will always be optimal. In Figure 6.6, we illustrate the optimal h, which is constant, and the expected total costs as a function of ψ in three cases: $\psi = \psi_1 = \psi_2$ (case 1), $\psi = \psi_1$ and ψ_2 is fixed (case 2), and $\psi = \psi_2$ and ψ_1 is fixed (case 3) for the de-risking option. The

Figure 6.5: *The expected total cost, the expected unfunded liabilities and the conditional value at risk for different values of ρ*



(a) *The expected total cost and the expected unfunded liabilities as a function of h for different values of ρ for the de-risking swap*

(b) *The conditional values at risk as a function of h for different values of ρ for the de-risking swap*

optimal h is constant and equal to 0.68 for all choices of ψ in all three cases, and the expected total cost is increasing in ψ, so the higher the penalty factors are, the higher are the total cost. The penalty factor ψ_1 affects the capital inflow and ψ_2 affects the capital outflow, and Figure 6.6 illustrates that the expected total costs are more sensitive to an increase in ψ_1 than to an increase in ψ_2.

6.5.2 The disability model

In this section, we study an example of the de-risking strategies in the disability model (see Figure 6.7). The disability model is more complex than the survival model from the previous example in the sense that there are two possible transitions from the initial state Active: The policyholder can become disabled or the policyholder can die. The insurance product is a payment upon disability, b, paid by a one-time premium, π, at time 0. The policyholder is x_0 years old at the initialization at time 0, and the insurance contract terminates at time n.

First, we model the stochastic process for the transition intensities in Section 6.5.2, and second, we solve the optimization problem and study sensitivities in Section 6.5.2.

Modelling intensities

Following Christiansen and Niemeyer, 2015 that study the sufficient and necessary conditions under which general transition forward rates are consistent with respect to the relevant insurance claims, we assume that $\mu_{01}(t)$, $\mu_{12}(t)$, and $(\mu_{02}(t) - \mu_{12}(t))$ are independent. Christiansen and Niemeyer, 2015 demonstrate that this assumption implies that $\mu_{02}(t)$ and $\mu_{12}(t)$ are dependent. It follows that $\mu_{01}(t)$

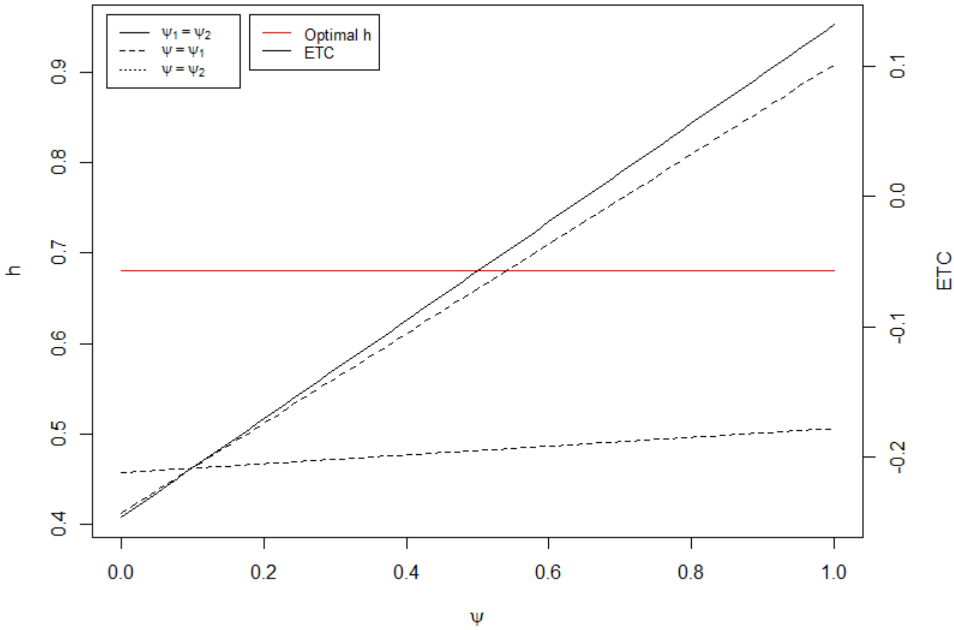


Figure 6.6: *The optimal h and the expected total costs in the three cases*

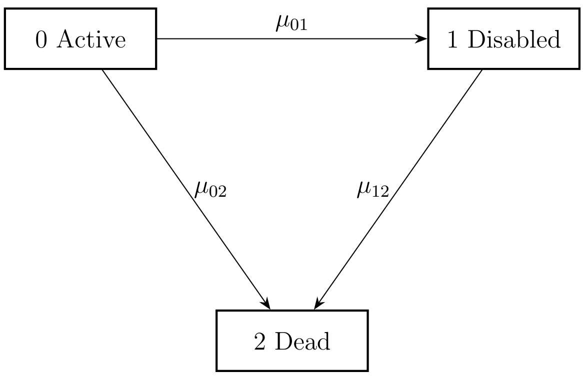


Figure 6.7: *The disability model*

can be independently estimated using a standard diffusion process (e.g., CIR), and the same can be done for $\mu_{02}(t)$ or $\mu_{12}(t)$. The difference $(\mu_{02}(t) - \mu_{12}(t))$ can be modelled as a constant, deterministic function or stochastic but independent with respect to $\mu_{02}(t)$.

We model $\mu_{01}(t)$ and $\mu_{02}(t)$ with two different time-inhomogeneous Cox-Ingersoll-Ross (CIR) processes. The CIR process has been widely used in the actuarial literature for modelling the mortality intensity (see, e.g., Dahl, 2004; Biffis, 2005; Henriksen and Møller, 2015; Zeddouk and Devolder, 2020; Huang et al., 2022). Furthermore, we assume that the difference $(\mu_{02}(t) - \mu_{12}(t))$ is a time-dependent constant.

Therefore, the estimated dynamics of the transition intensities are given by

$$\begin{aligned}
\mathrm{d}\mu_{01}(t) &= \phi_{01}\big(\beta_{01} - \mu_{01}(t)\big) + \sigma_{01}\sqrt{\mu_{01}(t)}\mathrm{d}W_1(t),\\
\mathrm{d}\mu_{02}(t) &= \phi_{02}\big(\beta_{02} - \mu_{02}(t)\big) + \sigma_{02}\sqrt{\mu_{02}(t)}\mathrm{d}W_2(t),\\
\mu_{02}(t) - \mu_{12}(t) &= \Delta(t),
\end{aligned}$$

where Δ is a deterministic function, $W_1(t)$ and $W_2(t)$ are two independent Brownian motions.

We estimate the parameters of the CIR model for $\mu_{01}(t)$ (or $\mu_{02}(t)$) from the survival probabilities that a person in state 0 at age x in the year t will remain in state 0 at age $x+n$ and year $t+n$, assuming that only the cause of decrement $j=1$ (or $j=2$) is operating, $p'_{01}(t, t+n)$ (or $p'_{02}(t, t+n)$) (to simplify notation, we have omitted the age). The procedure followed is described in Appendix 6.A.

We have calibrated the processes to the cohort of the Italian population aged $x_0 = 50$ in 2013 (the initialization time $t = 0$) and set $n = 30$. Data has been taken from Baione et al., 2016, which fitted the transition probabilities to the people qualified for a disability benefit paid by the Italian Government to disabled people, consisting of a universal cash benefit not subject to age limitations and unconnected to a means' test. The data set provides the mortality of active people, the mortality of disabled people, and the transition from active to disabled.

The values of the ϕ's, β's and σ's are reported in Table 6.2 below.

Table 6.2: *Parameter values*

Parameter	Value
ϕ_{01}	0.127580663
β_{01}	0.002728047
σ_{01}	0.027736810
$\mu_{01}(0)$	0.000721773
ϕ_{02}	0.000006236
β_{02}	2.981109000
σ_{02}	0.000854003
$\mu_{02}(0)$	0.002157350

Solving the optimization problem

The parameters in the example are reported in Table 6.3. We remind that the policyholder is 50 years old at the initialization at time 0 and holds an insurance contract paying a sum of $b = 5$ upon disability before termination at time $n = 30$ at the age of 80.

Table 6.3: *Components in numerical example*

Component	Value	Component	Value
Age of insured, x_0	50	$r(t)$	0.01
Termination, n	30	ψ_1	0.10
$Z(0)$	0	ψ_2	0.10
Annuity rate, b	5	δ	0.10
Premium, π	15.26	ρ	0.01
$V(0)$	14.55	α	0.99
$A(0) = \pi$	15.26	τ	0.13

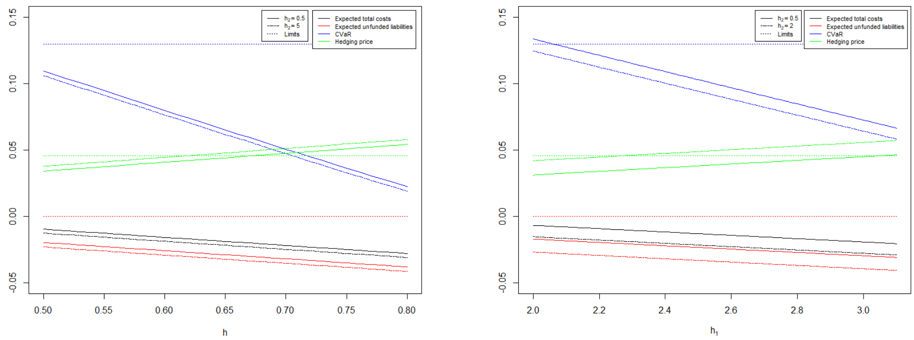
In this example, the liabilities in state 1 and state 2 are equal to zero, since the only payment is upon a transition between state 0 and 1, Therefore, the expected unfunded liabilities do not depend on the transition intensity $\mu_{12}(t)$, and the sum-at-risk $\hat{R}^{12}(t)$ is equal to zero for all t. It is only necessary to define the valuation basis by the two transition intensities $\hat{\mu}_{01}(t)$ and $\hat{\mu}_{02}(t)$. We assume that $\hat{\mu}_{01}(t) = 0.95 \cdot \mathbb{E}\big[\mu_{01}(t)\big]$ and $\hat{\mu}_{02}(t) = 1.05 \cdot \mathbb{E}\big[\mu_{02}(t)\big]$. We choose the target level for the conditional value-at-risk as $\tau = 0.5 \cdot CVaR_\alpha(L(n))$.

As in the example in Section 6.5.1, we calculate the expected total costs, the expected total unfunded liabilities, and the conditional value-at-risk of the unfunded liabilities with and without de-risking based on 5000 simulations of $\mu_{01}(t)$ and $\mu_{02}(t)$. We study the de-risking option with and without the sum-at-risk and the de-risking swap with and without the sum-at-risk. The sum-at-risk from state 0 (Active) to state 1 (Disabled) is positive, and therefore, we consider the de-risking option with the European call option structure for this transition. Contrary, the sum-at-risk from state 0 (Active) to state 2 (Dead) is negative, and therefore, we consider the de-risking option with a European put option structure for this transition. The solution to the optimization problem is the pair (h_{01}, h_{02}) that minimizes the expected total costs.

First, we consider the four different types of de-risking: The de-risking option including the sum-at-risk (6.3.1) and (6.3.2), the de-risking option without the sum-at-risk (6.3.3) and (6.3.4), the de-risking swap with the sum-at-risk (6.3.5), and the de-risking swap without the sum-at-risk (6.3.6). We consider the same type of de-risking cash flow for both the transition from Active to Disabled and from Active to Dead. In Figures 6.8 and 6.9, we illustrate the expected total costs and the constraints from the optimization problem as a function of h_{01} for two values of h_{02} for all the de-risking strategies. The optimal values of h_{01} and h_{02} for the de-risking option are reported in Table 6.4.

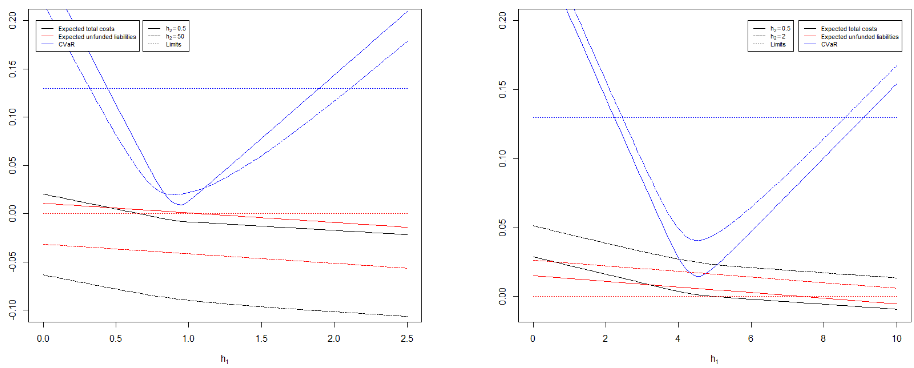
In all four cases, the expected total costs and the unfunded liabilities are decreasing

Figure 6.8: *Illustration of the optimization problem for the de-risking option*



(a) *De-risking option with the sum-at-risk* **(b)** *De-risking option without the sum-at-risk*

Figure 6.9: *Illustration of the optimization problem for the de-risking swap*



(a) *De-risking swap with the sum-at-risk* **(b)** *De-risking swap without the sum-at-risk*

Table 6.4: *Optimal amounts for the de-risking option*

	With sum-at-risk	Without sum-at-risk
h_{01}	0.6635	3.0560
h_{02}	1.0100	0.4600
$\mathbb{E}[TC]$	-0.0200	-0.0197

in the amount of de-risking. For the de-risking option, the limitation on the amount of de-risking is, as in the example in Section 6.5.1 the hedging price. The expected total costs with the optimal de-risking strategy for the de-risking option are lower with the sum-at-risk than without the sum-at-risk (see Table 6.4). This indicates that there is more basis risk without the sum-at-risk. Different from the first numerical example in the survival model, the CVaR for the de-risking swap has a parable shape (see Figure 6.9), which limits the optimal amount of h_{01}. The expected total costs decrease when we increase h_{02} for the de-risking swap with the sum-at-risk, and therefore the optimal amount of de-risking is to buy as much as possible of de-risking for the transition from Alive (0) to Dead (2) and then the amount h_{01} that minimizes the expected total costs and satisfies the criterion on the CVaR. For the de-risking swap without the sum-at-risk, the expected total costs are increasing in h_{02} since the sum-at-risk, $\hat{R}^{02}(t)$, is negative. Therefore, the optimal strategy for the de-risking swap without the sum-at-risk is $h_{02} = 0$ and the optimal amount for the transition from Active (0) to Disabled (1) is $h_{01} = 9.217$.

Now, we study sensitivities of the optimal choice of de-risking and the conditions in the optimization problem to changes in the parameters δ, ρ, ψ_1 and ψ_2. The de-risking option with and without the sum-at-risk behave in similar ways (see Figure 6.8), and therefore we only study the de-risking option including the sum-at-risk in the sensitivity analysis.

The δ-parameter affects the hedging price and the expected total costs of the de-risking option. When δ increases, the insurance company can afford less de-risking. We illustrate this in Figure 6.10, where the optimal h_{01} is decreasing in δ for fixed h_{02}. The expected total costs are increasing in δ, since the hedging costs are increasing in δ. For $\delta > 0.5$ there is no feasible values of h_{01} when fixing $h_{02} = 5$. The amount of de-risking of disability risks, h_{01}, is lower for $h_{02} = 5$ than for $h_{02} = 0.5$ since the insurance company can afford less h_{01} for high values of h_{02}.

The ρ-parameter influences the size of the fixed leg of the de-risking swap. The sum-at-risk upon transition from state Active to state Disabled is positive, and therefore a low fixed leg is favorable for the insurance company when buying a swap linked to μ_{01}, but the sum-at-risk upon transition from state Active to state Dead is negative and a high fixed leg is favorable. We illustrate the expected total costs, the expected unfunded liabilities and the CVaR for three different values of ρ for the de-risking swap with and without the sum-at-risk as a function of h_{01} for fixed values of h_{02} in Figures 6.11 and 6.12. For the de-risking swap with the sum-at-risk, the expected total costs, the unfunded liabilities, and the CVaR are increasing in ρ for $h_{02} = 0.5$ and decreasing for $h_{02} = 5$ for fixed h_{01}. A high fixed leg for the transition from state 0 to state 2 is attractable, since the sum-at-risk is negative, and therefore the expected total costs are lowest for the highest considered value of h_{02} and $h_{01} = 0$. For $\rho = 0.1$, it is optimal not to choose hedging of disability risks, $h_{01} = 0$, since the expected total costs are increasing in h_{01}. For $\rho = 0$ and $\rho = -0.1$ the boundary conditions of the optimization problem depend on the CVaR. The expected total costs and the expected unfunded liabilities are increasing in ρ for both $h_{02} = 0.5$ and $h_{02} = 2$ for the de-risking swap without the sum-at-risk. For this type of de-risking, there are no feasible solutions for $\rho = 0.1$, and the insurance company

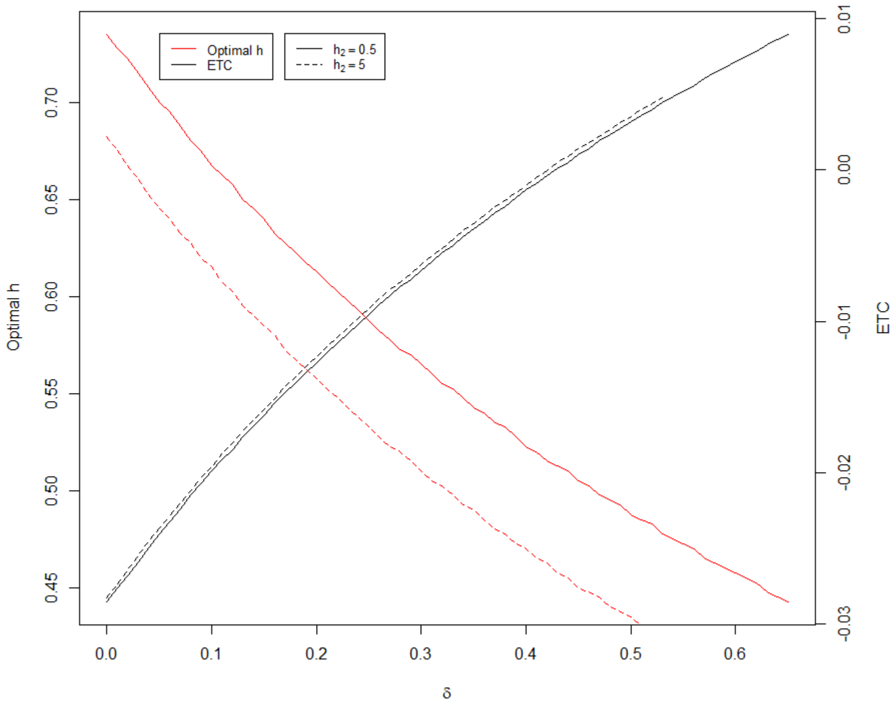
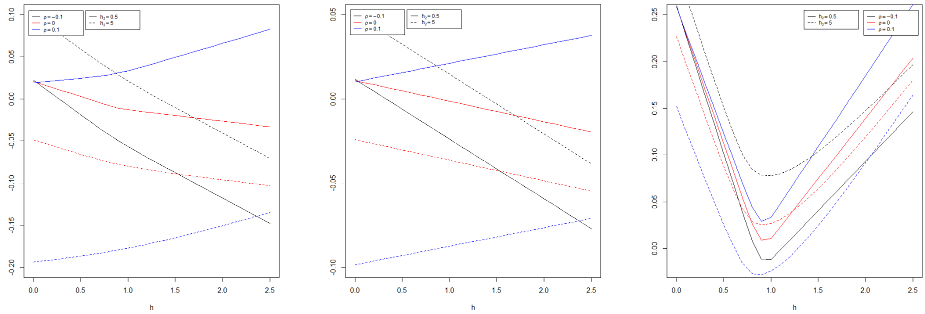


Figure 6.10: *The optimal h_{01} as a function of δ for fixed h_{02}*

is doing worse if it buys de-risking since the expected total costs are increasing in both h_{01} and h_{02}. For $\rho = -0.1$, $h_{02} > 0$ is optimal different from the case with the standard parameters in Figure 6.9 (right).

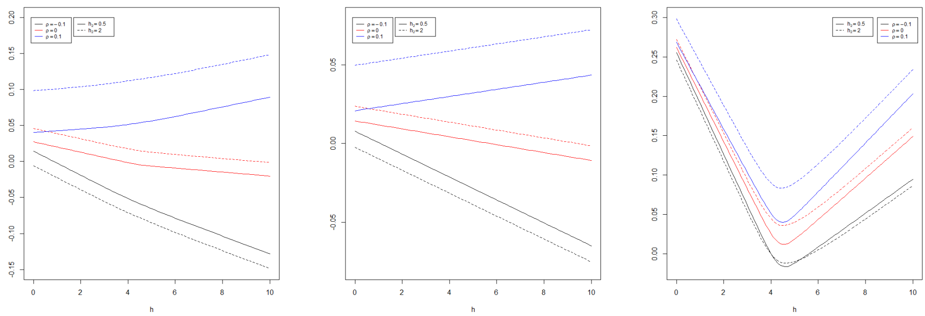
The parameters ψ_1 and ψ_2 only affect the total costs and therefore the feasible region of h_{01} and h_{02} in the optimization problem is unchanged when ψ_1 and ψ_2 vary. We illustrate the expected total costs as a function of ψ in three cases: $\psi = \psi_1 = \psi_2$, $\psi = \psi_1$ (ψ_2 fixed), and $\psi = \psi_2$ (ψ_1 fixed) for the de-risking option with the sum-at-risk and for the de-risking swap without the sum-at-risk in Figure 6.13. We do not consider the de-risking swap with the sum-at-risk, since it is optimal to buy an infinite amount of h_{02}. The expected total costs are increasing in ψ in all cases, which is obvious in the sense that the higher the penalty factors are, the higher are the total costs. The parameter ψ_1 affects the capital inflow, and ψ_2 affects the capital outflow. Figure 6.13 shows that both the de-risking option and the de-risking swap are more sensitive to an increase in the penalty factor on the capital inflow than to an increase in the penalty factor on the capital outflow.

Figure 6.11: *Varying ρ for the de-risking swap with the sum-at-risk*



(a) $\mathbb{E}[TC]$ **(b)** $\mathbb{E}[L^D(n)]$ **(c)** $CVaR_\alpha(L^D(n))$

Figure 6.12: *Varying ρ for the de-risking swap without the sum-at-risk*



(a) $\mathbb{E}[TC]$ **(b)** $\mathbb{E}[L^D(n)]$ **(c)** $CVaR_\alpha(L^D(n))$

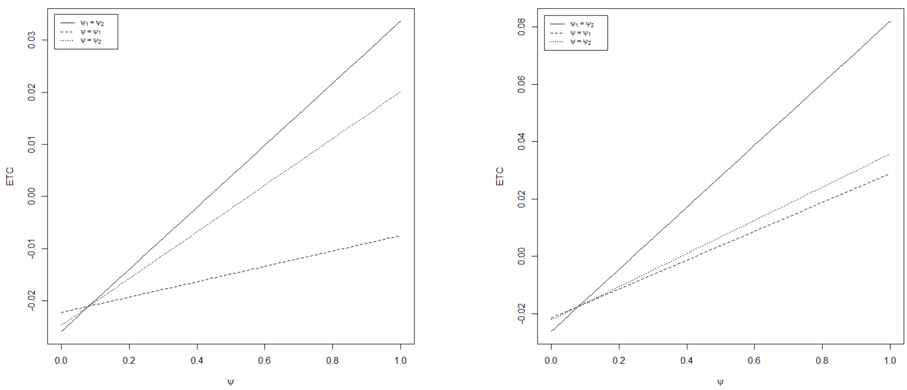
6.A Appendix

Consider the single decrement probability $p'_{ij}(t,t+1)$ with $i \neq j$, that a person in state i at a given age x in the year t, will be in the state j at age $x+1$ in the year $t+1$ (note that we omit the age for convenience). For each cause j, the collection of values $\{p'_{ij}(t,t+1)\}$ for various year t is known as the associated single-decrement table for cause j. These are annual probabilities of failure for the particular cause j, assuming that no other causes of decrement are operating.

Following Promislow (2015), the relationship between the two sets of probabilities (one set from the multiple-decrement table providing $p_{01}(t,t+1)$ and $p_{02}(t,t+1)$, and the other set from the single-decrement table providing $p'_{01}(t,t+1)$ and $p'_{02}(t,t+1)$) is as follows:

$$p_{01}(t,t+1) + p_{02}(t,t+1) = p'_{01}(t,t+1) + p'_{02}(t,t+1) - p'_{01}(t,t+1)p'_{02}(t,t+1)$$

Figure 6.13: *Varying ψ*



(a) *Varying ψ for the de-risking option with the sum-at-risk*

(b) *Varying ψ for the de-risking swap without the sum-at-risk*

Assuming a uniform distribution of failures for each part, over each year, we have:

$$p_{01}(t,t+1) = p'_{01}(t,t+1) - \frac{1}{2}p'_{01}(t,t+1)p'_{02}(t,t+1),$$

$$p_{02}(t,t+1) = p'_{02}(t,t+1) - \frac{1}{2}p'_{01}(t,t+1)p'_{02}(t,t+1).$$

Moreover, the following relation holds:

$$p_{01}(t,t+1) - p_{02}(t,t+1) = p'_{01}(t,t+1) - p'_{02}(t,t+1)$$

Therefore, we obtain the following Promislow (2015):

$$p_{01}(t,t+1) = p'_{01}(t,t+1) - \frac{1}{2}\left[p'_{01}(t,t+1)\right]^2 + \frac{1}{2}p'_{01}(t,t+1)\Delta$$

where $\Delta = p_{01}(t,t+1) - p_{02}(t,t+1)$. Finally, we get the single decrement probabilities from a multiple-decrement table with two decrements, disability (cause 1), and death (cause 2), from:

$$p'_{01}(t,t+1) = 2 + \Delta - \sqrt{(2+\Delta)^2 - 8p_{01}(t,t+1)},$$

$$p'_{02}(t,t+1) = p'_{01}(t,t+1) - \Delta.$$

Now, we can calculate the survival probabilities $p'_{00(1)}(t,t+n)$ that a person in state 0 in the year t will remain in state 0 after n years (i.e. in the year $t+n$), assuming that only the cause of decrement 1 (disability) is operating by:

$$p'_{00(1)}(t,t+n) = \prod_{k=t}^{t+n-1}\left[1 - p'_{01}(k,k+1)\right]$$

Similarly, we calculate the survival probabilities $p'_{00(2)}(t,t+n)$ that a person in state 0 in the year t will remain in state 0 after n years, assuming that only the cause of decrement 2 (death) is operating by:

$$p'_{00(2)}(t,t+n) = \prod_{k=t}^{t+n-1} \left[1 - p'_{02}(k,k+1)\right]$$

In the CIR case, we have for $j = 1, 2$:

$$p'_{00(j)}(t,t+n) = e^{\eta_{0j}(n)+\theta_{0j}(n)\mu_{0j}(t)},$$

where:

$$\eta_{0j}(n) = -\frac{2\phi_{0j}\beta_{0j}}{\sigma_{0j}^2} \ln\left(\frac{f_{0j} + g_{0j}e^{c_{0j}n}}{c_{0j}}\right) + \frac{\phi_{0j}\beta_{0j}}{f_{0j}} n,$$

$$\theta_{0j}(n) = \frac{1 - e^{c_{0j}n}}{f_{0j} + g_{0j}e^{c_{0j}n}},$$

with

$$c_{0j} = -\sqrt{\phi_{0j}^2 + 2\sigma_{0j}^2},$$

$$f_{0j} = \frac{c_{0j} - \phi_{0j}}{2},$$

$$g_{0j} = \frac{c_{0j} + \phi_{0j}}{2}.$$

Chapter 7

Risk margin calculations with a scenario-based model for the Solvency Capital Requirement

This chapter contains the paper *Nyegaard and Steffensen (2023)*.

ABSTRACT

We propose a scenario-based model for the Solvency Capital Requirement (SCR) to use for the calculation of the risk margin. The suggested cost-of-capital approach in the Solvency II directive for the calculation of the risk margin is cumbersome if the SCR is calculated using the standard model, resulting in approximation methods for the risk margin. The scenario-based model for the SCR bases on a bad- or worst-case scenario, and results in an accurate calculation of the risk margin. We discover similarities between the risk margin approximations from the Solvency II legislation and the risk margin from the scenario-based model and propose approximations of the SCR and the risk margin within the scenario-based model. We illustrate the SCR and the risk margin calculations in a numerical example.

Keywords: Life insurance; Risk margin; SCR; Solvency II; Stress scenarios.

7.1 Introduction

According to the Solvency II directive, life insurance companies must divide liabilities into the best estimate and the risk margin. The best estimate is according to European Commission, 2008 article 77(2) *the probability-weighted average of future cash-flows, taking account of the time value of money, using the relevant risk-free interest rate term structure*, and calculation of the best estimate shall be based on *applicable and relevant actuarial and statistical methods.* There exist well-established methods for the calculation of the best estimate. The risk margin is described in European Commission, 2008 article 77(3) and is the amount another company would be expected to require to take over the insurance obligations. The suggested method in the Solvency II directive for the calculation of the risk margin is the cost-of-capital approach described in European Commission, 2014 article 37, which depends on the future Solvency Capital Requirement (SCR). Usually, the SCR is calculated using the standard formula described in European Commission, 2008 Annex IV. In practice, calculation of SCR with the standard model for all future time points is cumbersome and computationally challenging; therefore, approximation methods for the risk margin are needed. Four approximation methods to calculate the risk margin are suggested in Guidelines on the valuation of technical provisions EIOPA, 2015 article 1.114.

This paper proposes a scenario-based model for the SCR to use in the cost-of-capital formula for the risk margin as an alternative to the approximation methods suggested in Guidelines on the valuation of technical provisions EIOPA, 2015. The scenario-based model for the SCR is based on a stress of the transition intensities of the Markov chain governing the state of the insured within the classical multi-state setup in continuous time life insurance. The SCR is the difference between the best estimate liabilities calculated with the stressed and the unstressed transition intensities. The resulting risk margin calculated with the SCR in the scenario-based model resembles a duration of a payment stream that depends on the sums-at-risk calculated with the stressed transition intensities. This is a crucial observation since the third approximation method of the risk margin from the guideline also resembles a duration, but the duration of the original insurance payments. Furthermore, we derive that the second and third approximation methods of the risk margin from the guideline result in the same approximation of the risk margin. The SCR in the scenario-based model has a representation that contains a mix of valuation with and without stress. This leads to two approximations of the SCR. We compare the approximations of the SCRs and the resulting risk margins. We calculate the SCR and the risk margin in the scenario-based model in a numerical example, where we compare the resulting risk margin to the approximations from the guideline and compare approximations of the SCR and approximations of the risk margin.

Throughout, we are going to study SCRs, corresponding risk margins, and approximations of these. The construction of the bad-case scenario underlying the calculation of the SCRs partly determines the structure of both the accurate expressions and their approximations. The construction of bad- and worst-case scenarios both for SCRs and for safe-side valuation in with-profit life insurance has been intensively studied in the actuarial literature.

The classic idea of with-profit life insurance is to set a first-order scenario such that the first-order reserve and the first-order equivalence premium are prudent; see Norberg (1985, 1999) and Ramlau-Hansen (1988). More recent studies look more closely into the set from which the bad- or worst-case scenarios are drawn, driven by the idea that not only first-order reserves and premiums but also probabilistic *SCR*s may be based on these scenarios. Christiansen and Denuit (2010) compare situations where not only the death intensity but instead its integral or derivative stem from a certain set. Christiansen (2010) realize the need for simultaneous calculation of worst-case intensities and reserves. Christiansen and Steffensen (2013) generalize to a situation where the set from which worst-case intensities is not rectangular due to the dependence structure in the intensities. In continuation of that work, Christiansen et al. (2016) unveil a forward-backward problem if not only reserves but expected reserves are in focus. In contrast to all these references, our discussion of the scenarios is not driven by the probabilistic set from which the real-world scenario is drawn. Rather, it is driven by the mathematical structure of the *SCR* and risk margin formulas these scenarios give rise to. This gives rise to new ideas about how to construct these scenarios, and some comparisons are made.

Connected with the literature on the construction of worst-case scenarios is also the literature on the sensitivity of reserves with respect to the intensity assumptions; see Kalashnikov and Norberg (2003), Christiansen (2008a) and Christiansen (2008b). Finally, worth mentioning here is related work on immunization and natural hedging techniques; see Christiansen (2011), Levantesi and Menzietti (2017), and Nyegaard (2023). Although the purpose of that work is substantially different from ours, some mathematical techniques and patterns of thinking are common.

The structure of the paper is as follows. In Section 7.2, we describe the multi-state setup in continuous time life insurance. The formula for the risk margin from the guideline and the four approximation methods are studied in Section 7.3. We present the scenario-based model for the *SCR*, and study approximations of the *SCR* and the choice of stress in Section 7.4. Section 7.5 studies the risk margin calculated with the *SCR* from the scenario-based model. A numerical example in Section 7.6 shows an implementation of the scenario-based model to calculate the *SCR* and the risk margin.

7.2 Setup

We consider the classical multi-state setup in life insurance. A continuous time Markov chain $Z = \big(Z(t)\big)_{t\geq 0}$ on a finite-dimensional state space $\mathcal{J} = \{1, 2, ..., J\}$ models the state of the policyholder, corresponding to for instance 'Active', 'Disabled', or 'Dead'. The transition probabilities of Z are

$$p_{ij}(t,s) = \mathbb{P}\big(Z(s) = j \bigm| Z(t) = i\big),$$

for $i, j \in \mathcal{J}$. We assume the transition intensities,

$$\mu_{ij}(t) = \lim_{h \downarrow 0} \frac{p_{ij}(t, t+h)}{h},$$

exist for $i, j \in \mathcal{J}$, $i \neq j$. The counting processes $N^k(t)$ denote the number of jumps of Z into state $k \in \mathcal{J}$ up to and including time t, i.e.,

$$N^k(t) = \#\{s \in (0, t] \mid Z(s-) \neq k, Z(s) = k\},$$

where $Z(s-) = \lim_{h \downarrow 0} Z(s-h)$. We let $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ be the natural filtration generated by the state process Z.

The insurer and the insured agree on premium payments and benefits of the insurance contract, and the payments link to sojourns in states and transitions between states. The payment stream has dynamics

$$\mathrm{d}B(t) = b^{Z(t)}(t)\mathrm{d}t + \Delta B^{Z(t)}(t)\mathrm{d}\varepsilon_n(t) + \sum_{k: k \neq Z(t-)} b^{Z(t-)k}(t)\mathrm{d}N^k(t),$$

where $b^j, \Delta B^j(t)$ and b^{jk} for $j, k \in \mathcal{J}$, $j \neq k$ are deterministic functions and denote benefits minus premiums. Payments specified by b^j link to continuous payments during sojourn in state j, the payment $\Delta B^j(t)$ is a lump sum payment upon termination of the contract at time n, and b^{jk} links to payments upon transition from state j to state k. We assume that the instantaneous forward interest rate is given by $r(t)$ with a slight misuse of notation since $r(t)$ in Equation (7.3.1) in the section below is the risk-free interest rate for the maturity of t years.

The expected present value of future payments of the insurance contract is given by

$$\begin{aligned} V(t) &= \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B(s) \,\middle|\, \mathcal{F}_t \right] \\ &= \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B(s) \,\middle|\, Z(t) \right], \end{aligned}$$

where we use that Z is Markov. We denote by V the prospective reserve of the insurance contract, and conditioning on $Z(t) = i$, we arrive at the state-wise prospective reserve

$$V^i(t) = \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B(s) \,\middle|\, Z(t) = i \right]. \tag{7.2.1}$$

A different representation of the prospective reserve is as the present value of the (conditional expected) cash flow of the payment stream B

$$\begin{aligned} V^i(t) = &\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \sum_{j \in \mathcal{J}} p_{ij}(t, s)\Big(b^j(s) + \sum_{k: k \neq j} b^{jk}(s)\mu_{jk}(s) \Big)\mathrm{d}s \\ &+ e^{-\int_t^n r(u)\mathrm{d}u} \sum_{j \in \mathcal{J}} p_{ij}(t, n)\Delta B^j(n). \end{aligned} \tag{7.2.2}$$

As previously stated, the best estimate of the insurance liabilities is according to the Solvency II directive *the probability-weighted average of future cash flows, taking account of the time value of money, using the relevant risk-free interest rate term structure.* The cash flow in Equation (7.2.2) is a probability-weighted average of the future payments in each state, and it is discounted with the interest rate $r(t)$. If we assume that $r(t)$ is the relevant risk-free interest rate term structure, the prospective reserve $V^{Z(t)}(t)$ can be considered the best estimate of the liabilities. Calculation of the prospective reserve is, for instance, obtained by numerical computation of the cash flow or as the solution to Thiele's differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}V^j(t) = r(t)V^j(t) - b^j(t) - \sum_{k:k\neq j} \mu_{jk}(t)R^{jk}(t), \qquad (7.2.3)$$
$$V^j(n-) = \Delta B^j(n),$$

where R^{jk} is the sum at risk upon transition from state j to state k given by

$$R^{jk}(t) = b^{jk}(t) + V^k(t) - V^j(t).$$

7.3 Risk margin from the guideline

A formula to calculate the risk margin is described in European Commission, 2014 article 37, and it states that the risk margin should be equal to

$$\mathrm{RM} = CoC \cdot \sum_{t\geq 0} \frac{SCR(t)}{\big(1 + r(t+1)\big)^{t+1}}, \qquad (7.3.1)$$

where CoC is the Cost-of-Capital rate, $\sum_{t\geq 0}$ is the sum of all integers greater or equal to zero, and $r(t+1)$ denotes the basic risk-free interest rate for the maturity of $t+1$ years. Hence, calculating risk margins requires the calculation of $SCR(t)$ for all integer $t \geq 0$. Usually, the current SCR calculated at time $t = 0$ is determined using the standard formula described in European Commission, 2008 Annex IV. However, calculating the SCR with the standard model is cumbersome and computationally complex. Therefore, in practice, the calculation of SCR with the standard model for all future time points is unrealistic, and alternative methods to calculate the risk margin are needed. Guidelines on the valuation of technical provisions EIOPA, 2015 article 1.114 suggests four approximation methods to calculate the risk margin, and in this section, we describe and compare the four methods. An alternative to approximate the risk margin is to model the SCR with a different model than the standard model. This is the topic of Section 7.4.

7.3.1 Continuous version of the risk margin

We model the payments of an insurance contract for one insured, and the prospective reserve or best estimate for this policyholder at time t is given by $V^{Z(t)}(t)$. Hence, the liabilities of the insurance contract depend on the state of the insured. It is likely to assume that future *SCR*s for this insurance contract also depend on the future state of the insured and are stochastic variables. A reasonable estimate of future *SCR*s is to take the mean value, which results in a slightly modified formula for the risk margin from Equation (7.3.1),

$$\mathrm{RM} = CoC \cdot \mathbb{E}\left[\sum_{t \geq 0} \frac{SCR(t)}{\big(1 + r(t+1)\big)^{t+1}} \right]. \tag{7.3.2}$$

The SCR at time t in the formula above corresponds to the time interval $[t, t+1]$. or as the SCR per year at time t. Instead, we may think of $SCR(t)$ as the SCR per year at time t, that is, as the SCR intensity at time t. If we think of the SCR as an intensity, a more correct definition of the risk margin in Equation (7.3.2) is the integral

$$\mathrm{RM} = CoC \cdot \mathbb{E}\left[\int_0^n e^{-\int_0^t r(u)\mathrm{d}u} SCR(t)\mathrm{d}t \right], \tag{7.3.3}$$

where $t \to r(t)$ is now the instantaneous forward interest rate. We denote the definition of the risk margin in Equation (7.3.3) as the continuous version of the risk margin. Throughout the rest of the paper, we use the definition (7.3.3) to compare different methods to calculate the risk margin.

The risk margin in Equation (7.3.3) above is the risk margin at time $t = 0$, and does not say anything about the future risk margin for $t > 0$. Therefore, we also study the risk margin runoff given by

$$\begin{aligned}
\mathrm{RM}(t) &= CoC \cdot \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} SCR^{Z(s)}(s)\mathrm{d}s \right] \\
&= CoC \cdot \sum_{j \in \mathcal{J}} p_{Z(0)j}(0,t) \cdot \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} SCR^{Z(s)}(s)\mathrm{d}s \,\middle|\, Z(t) = j \right].
\end{aligned}$$

7.3.2 Approximation methods from the guideline

The Guidelines on the valuation of technical provisions EIOPA, 2015 article 1.114 suggest four approximation methods to approximate the SCR to calculate the risk margin. In addition, the guideline EIOPA, 2015 states that the approximation methods are hierarchical. In this section, we describe the approximation methods from the guideline and calculate the resulting risk margin approximations with the continuous definition of the risk margin in Equation (7.3.3).

Approximation 1

The first approximation method from the guideline concerns approximating the modules and sub-modules from the standard model when calculating future SCR to insert in the formula for the risk margin. This method requires specific knowledge about the insurance company's business, so we do not look further into it.

Approximation 2

The second method from the guideline assumes that the ratio between SCR at time 0 and future SCRs equals the ratio between the best estimate at time 0 and the future best estimate. We assume that the SCR at time 0 is known and calculated with, for instance, the standard formula and the approximations concerning future values of the SCR. We interpret the best estimate as the prospective reserve of the insurance liabilities from Equation (7.2.1). Hence, the relation between the SCR and the prospective reserve is assumed to be

$$\frac{SCR(t)}{SCR(0)} = \frac{V^{Z(t)}(t)}{V^{Z(0)}(0)}.$$

The resulting approximation of the risk margin with the second method is

$$\begin{aligned}
\mathrm{RM}^{(2)} &= CoC \cdot \frac{SCR(0)}{V^{Z(0)}(0)} \mathbb{E}\left[\int_0^n e^{-\int_0^t r(u)\mathrm{d}u} V^{Z(t)}(t)\mathrm{d}t\right] \\
&= CoC \cdot \frac{SCR(0)}{V^{Z(0)}(0)} \mathbb{E}\left[\int_0^n e^{-\int_0^t r(u)\mathrm{d}u} \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B(s) \;\middle|\; Z(t)\right]\mathrm{d}t\right] \\
&= CoC \cdot \frac{SCR(0)}{V^{Z(0)}(0)} \mathbb{E}\left[\int_0^n \int_t^n e^{-\int_0^s r(u)\mathrm{d}u} \mathrm{d}B(s)\mathrm{d}t\right] \\
&= CoC \cdot \frac{SCR(0)}{V^{Z(0)}(0)} \mathbb{E}\left[\int_0^n e^{-\int_0^s r(u)\mathrm{d}u} \int_0^s \mathrm{d}t\mathrm{d}B(s)\right] \\
&= CoC \cdot \frac{SCR(0)}{V^{Z(0)}(0)} \mathbb{E}\left[\int_0^n t e^{-\int_0^t r(u)\mathrm{d}u} \mathrm{d}B(t)\right]. \qquad (7.3.4)
\end{aligned}$$

We used here the risk margin definition from Equation (7.3.3), inserted the prospective reserve from Equation (7.2.1), and interchange the order of integration. The risk margin runoff with this approximation is

$$\mathrm{RM}^{(2)}(t) = CoC \cdot \frac{SCR(0)}{V^{Z(0)}(0)} \mathbb{E}\left[\int_t^n (s-t) e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B(s)\right].$$

Approximation 3

The third method uses the modified duration of the insurance liabilities as a proportionality factor to approximate the discounted sum of all future SCRs in the definition of the risk margin in Equation (7.3.1). The risk margin with the third approximation method is

$$\mathrm{RM}^{(3)} = CoC \cdot SCR(0) \cdot D(0), \qquad (7.3.5)$$

where $D(0)$ is the modified duration of the insurance liabilities at time 0. A natural question is how to define the modified duration of the insurance liabilities. The definition presented here is inspired by Møller and Steffensen (2007), where the duration of a bond issued at time τ_0 with payments $c_1, ..., c_n$ at given times $\tau_1, ..., \tau_n$ is defined by

$$D_{\text{bond}} = \frac{\sum_{i=1}^{n} \tau_i e^{-y \cdot \tau_i} c_i}{PV}. \tag{7.3.6}$$

Here, y is the yield to maturity, and PV is the present value of the bond at time τ_0 given by

$$PV = \sum_{i=1}^{n} e^{-y \cdot \tau_i} c_i.$$

The idea is to generalize the definition of the duration of a bond above to cover insurance liabilities. First, for the denominator of Equation (7.3.6), we notice that the present value of future insurance payments is given by the prospective reserve from Equation (7.2.1). The future payments are weighed with the time until the payment occurs and discounted to time zero in the nominator of Equation (7.3.6). Regarding insurance payments, future payments of the insurance contract are stochastic since they link to the insured's state. We consider the expected present value of future payments weighed with the time they occur conditioned on the present state of the insured. The nominator of Equation (7.3.6) generalized to insurance payments is $\mathbb{E}\big[\int_t^n (s-t)e^{-\int_t^s r(u)\mathrm{d}u}\mathrm{d}B(s)|Z(t)\big]$, and we define the duration of the insurance liabilities as

$$\begin{aligned} D^{Z(t)}(t) &= \frac{\mathbb{E}\Big[\int_t^n (s-t)e^{-\int_t^s r(u)\mathrm{d}u}\mathrm{d}B(s) \,\Big|\, Z(t)\Big]}{V^{Z(t)}(t)} \\ &= \frac{\mathbb{E}\Big[\int_t^n (s-t)e^{-\int_t^s r(u)\mathrm{d}u}\mathrm{d}B(s) \,\Big|\, Z(t)\Big]}{\mathbb{E}\Big[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u}\mathrm{d}B(s) \,\Big|\, Z(t)\Big]}. \end{aligned}$$

This implies that the risk margin with the third approximation method is

$$\mathrm{RM}^{(3)} = CoC \cdot \frac{SCR(0)}{V^{Z(0)}(0)}\mathbb{E}\left[\int_0^n t e^{-\int_0^t r(u)\mathrm{d}u}\mathrm{d}B(t)\right].$$

This is equal to the approximation of the risk margin with the second method, and therefore the risk margin runoff for this method is the same as for the second method, $\mathrm{RM}^{(3)}(t) = \mathrm{RM}^{(2)}(t)$. Another definition of the modified duration of the insurance liabilities in the third method would lead to a different approximation of the risk margin.

Approximation 4

In the fourth method from the guideline, the risk margin is approximated as a percentage of the best estimate

$$\mathrm{RM}^{(4)} = q \cdot V^{Z(0)}(0), \tag{7.3.7}$$

for $q \in (0,1)$. The formula for the risk margin in Equation (7.3.1) depends on the future *SCR* in each year, and it is difficult to believe that the complexity in that formula is captured well as a percentage of the best estimate or prospective reserve at time 0. The four approximation methods from the guideline lead to two different approximations for the risk margin in Equations (7.3.4) and (7.3.7) since the first method requires specific knowledge of the business of the insurance company, and since the second and third approximation method result in the same expression for the approximated risk margin. In Section 7.4 below, we describe a scenario-based model for the *SCR*, and within this model, we calculate the resulting risk margin in Section 7.5 and compare it to the risk margin calculated with the approximation methods from the guideline. A numerical comparison is performed in Section 7.6.

7.4 Scenario-based model for the *SCR*

Valuation of insurance liabilities in the multi-state setup described in Section 7.2 is based on best estimates of the valuation basis that consists of the market interest rate and the transition intensities in the Markov model. The foundation of our scenario-based model is a valuation basis, as if we are in a bad or worst-case scenario. Then the *SCR* is the difference between the liabilities calculated with the bad scenario and with the best estimate valuation basis.

The risk margin corresponds to the amount another company would require to take over the insurance obligations as a payment to take over risks associated with managing the insurance portfolio. In the case, where another company takes over the insurance obligations, they take over the liabilities of the portfolio and the corresponding assets. The assumptions under which the risk margin should be calculated are described in article 38 in the delegated regulation European Commission, 2014. Paragraph *h* in article 38 states that *the assets are selected in such a way that they minimise the Solvency Capital Requirement for market risk that the reference undertaking is exposed to.* Hence, when calculating the risk margin, it is assumed that the insurance company that takes over the insurance liabilities invests the corresponding assets in such a way that they minimize the market risk. Therefore, when we choose the stressed valuation basis in the scenario-based model for the *SCR*, we disregard stresses on market risks and focus on stresses on biometric risks modelled as stresses on the transition intensities.

We assume that the stressed transition intensities in the scenario-based model are given by $\left(\mu^{\varepsilon}_{jk}(t)\right)_{j,k\in\mathcal{J},j\neq k}$, and define the difference between the stressed intensities and the original intensities as

$$\Delta\mu_{jk}(t) = \mu^{\varepsilon}_{jk}(t) - \mu_{jk}(t).$$

Specific choices of stress are discussed in Section 7.4.3.

We denote the prospective reserve calculated with the stressed transition intensities by $V^{Z(t)}_{\varepsilon}(t)$ and have that

$$V^{Z(t)}_{\varepsilon}(t) = \mathbb{E}^{\varepsilon}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u}\mathrm{d}B(s) \;\middle|\; Z(t)\right],$$

where the superscript ε denotes that the distribution of Z has transition intensities $\left(\mu^{\varepsilon}_{jk}(t)\right)_{j,k\in\mathcal{J},j\neq k}$.

The *SCR* in the scenario-based model, $U^j(t)$, is assumed to be the difference between the liabilities calculated with the stressed intensities and the best estimate,

$$U^{Z(t)}(t) = V^{Z(t)}_{\varepsilon}(t) - V^{Z(t)}(t). \tag{7.4.1}$$

From the stress scenario, one may want or even require that $V^j_{\varepsilon}(t) \geq V^j(t)$, to ensure a positive *SCR*. Such a requirement corresponds to the similar requirement of a first-order prudent valuation basis. In that context, the topic was studied by Christiansen (2010). He showed that a sufficient condition for this is that

$$\mathrm{sign}(\Delta\mu_{jk}(t)) = \mathrm{sign}(R^{jk}_{\varepsilon}(t)), \tag{7.4.2}$$

where $R^{jk}_{\varepsilon}(t)$ is the sum-at-risk calculated with the stressed transition intensities and is given by

$$R^{jk}_{\varepsilon}(t) = b^{jk}(t) + V^k_{\varepsilon}(t) - V^j_{\varepsilon}(t).$$

We assume that the condition in Equation (7.4.2) is met.

The scenario-based model aims to model the future *SCR* so that we can calculate the risk margin in Equation (7.3.3) without using approximation methods. Therefore, we study a different representation of the *SCR*, $U^{Z(t)}(t)$, in the scenario-based model. Using Thiele's differential equation for the state-wise prospective reserve from Equation (7.2.3), we obtain a differential equation for the state-wise *SCR*, $U^j(t)$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}U^j(t) &= \frac{\mathrm{d}}{\mathrm{d}t}V^j_{\varepsilon}(t) - \frac{\mathrm{d}}{\mathrm{d}t}V^j(t)\\
&= r(t)V^j_{\varepsilon}(t) - b^j(t) - \sum_{k:k\neq j}\mu^{\varepsilon}_{jk}(t)R^{jk}_{\varepsilon}(t)\\
&\quad - \Big(r(t)V^j(t) - b^j(t) - \sum_{k:k\neq j}\mu_{jk}(t)R^{jk}(t)\Big).
\end{aligned}$$

Collecting terms yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}U^j(t) = r(t)U^j(t) - \sum_{k:k\neq j} \mu_{jk}(t)\Big(\frac{\Delta\mu_{jk}(t)}{\mu_{jk}(t)} R^{jk}_\varepsilon(t) + U^k(t) - U^j(t)\Big),$$
$$U^j(n) = 0. \tag{7.4.3}$$

Hence, the *SCR* in the scenario-based model satisfies Thiele's differential equation and has the representation

$$U^{Z(t)}(t) = \mathbb{E}\bigg[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u}\mathrm{d}B^\varepsilon(s) \,\bigg|\, Z(t)\bigg], \tag{7.4.4}$$

for

$$\mathrm{d}B^\varepsilon(t) = \sum_{k:k\neq Z(t-)} \frac{\Delta\mu_{Z(t-)k}(t)}{\mu_{Z(t-)k}(t)} R^{Z(t-)k}_\varepsilon(t)\mathrm{d}N^k(t). \tag{7.4.5}$$

The *SCR* is the expected present value of a payment stream consisting of payments upon jumps between states given by the stressed sum-at-risk weighted with the relative change in the transition intensities due to the stress scenario. We use this representation of the *SCR* to calculate the risk margin in Section 7.5, but first, we study approximations of the *SCR* that follows in continuation of the representation in Equation (7.4.4).

7.4.1 Approximations of the *SCR*

The expression in Equation (7.4.4) contains a mix of valuation with and without stress. The payment process $B^\varepsilon(t)$ is based on valuation under stress since it contains the stressed sum-at-risk, but the conditional expectation is unstressed. This is not necessarily a problem, but the observation leads to two approximations of $U^{Z(t)}(t)$, each based on full valuation without or with stress. They are, respectively,

$$\tilde{U}^{Z(t)}(t) = \mathbb{E}\bigg[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u}\mathrm{d}\tilde{B}^\varepsilon(s) \,\bigg|\, Z(t)\bigg], \tag{7.4.6}$$

$$\hat{U}^{Z(t)}(t) = \mathbb{E}^\varepsilon\bigg[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u}\mathrm{d}B^\varepsilon(s) \,\bigg|\, Z(t)\bigg], \tag{7.4.7}$$

where

$$\mathrm{d}\tilde{B}^\varepsilon(t) = \sum_{k:k\neq Z(t-)} \frac{\Delta\mu_{Z(t-)k}(t)}{\mu_{Z(t-)k}(t)} R^{Z(t-)k}(t)\mathrm{d}N^k(t).$$

The relations between the true *SCR* from the scenario-based model, $U^{Z(t)}(t)$, and the approximations cannot be deduced from the representations above. Instead, one may use that $U^i(t)$ satisfies the differential equation in Equation (7.4.3) and the approximations $\tilde{U}^i(t)$ and $\hat{U}^i(t)$ from Equations (7.4.6) and (7.4.7) satisfy similar differential equations. This results in the following representation of the differences

$$\begin{aligned}
&U^i(t) - \tilde{U}^i(t) \\
&= \int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \sum_{j\in\mathcal{J}} p_{ij}(t,s) \sum_{k:k\neq j} \Delta\mu_{jk}(s)\Big(U^k(s) - U^j(s)\Big)\mathrm{d}s, \qquad (7.4.8)\\
&\hat{U}^i(t) - U^i(t) \\
&= \int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \sum_{j\in\mathcal{J}} p_{ij}(t,s) \sum_{k:k\neq j} \Bigg(\frac{\big(\Delta\mu_{jk}(s)\big)^2}{\mu_{jk}(s)} R_\varepsilon^{jk}(s) \\
&\qquad\qquad + \Delta\mu_{jk}(s)\big(\hat{U}^k(s) - \hat{U}^j(s)\big)\Bigg)\mathrm{d}s. \qquad (7.4.9)
\end{aligned}$$

The differences depend on the choice of stress-scenario through $\Delta\mu_{jk}(t)$, $R_\varepsilon^{jk}(t)$, $U^j(t)$, and $\hat{U}^j(t)$. Under the assumption in Equation (7.4.2), we cannot, in general, make conclusions about the relation between $U^{Z(t)}(t)$, $\tilde{U}^{Z(t)}(t)$ and $\hat{U}^{Z(t)}(t)$, but in the survival model, we have a result about the relations reported in Example 7.4.1 below.

Example 7.4.1. In the survival model consisting of the states Active (0) and Dead (1) and one transition intensity $\mu(t)$ from state 0 to state 1, the differences between the *SCR* and the approximations in Equations (7.4.8) and (7.4.9) reduce to

$$\begin{aligned}
U^0(t) - \tilde{U}^0(t) &= \int_t^n e^{-\int_t^s r(u)+\mu(u)\mathrm{d}u} \Delta\mu(s)\Big(- \big(V_\varepsilon^0(s) - V^0(s)\big)\Big)\mathrm{d}s, \qquad (7.4.10)\\
\hat{U}^0(t) - U^0(t) &= \int_t^n \big(e^{-\int_t^s r(u)+\mu^\varepsilon(u)\mathrm{d}u}\mu^\varepsilon(s) \\
&\qquad - e^{-\int_t^s r(u)+\mu(u)\mathrm{d}u}\mu(s)\big) R_\varepsilon(s)\frac{\Delta\mu(s)}{\mu(s)}\mathrm{d}s.
\end{aligned}$$

Here it is only relevant to consider $U^0(t)$ since $U^1(t) = 0$. If the stressed sum at risk has the same sign for all $t \in [0,n]$ and under the assumption in Equation (7.4.2), Equation (7.4.10) states that

$$\begin{aligned}
&U^0(t) \leq \hat{U}^0(t) \quad \text{and} \quad U^0(t) \leq \tilde{U}^0(t) \quad \text{if} \quad \operatorname{sign}(R_\varepsilon(t)) = 1 \text{ for all } t \in [0,n],\\
&U^0(t) \geq \hat{U}^0(t) \quad \text{and} \quad U^0(t) \geq \tilde{U}^0(t) \quad \text{if} \quad \operatorname{sign}(R_\varepsilon(t)) = -1 \text{ for all } t \in [0,n].
\end{aligned} \qquad (7.4.11)$$

Similarly, there is a relation between the two approximations, $\hat{U}^0(t)$ and $\tilde{U}^0(t)$, given by

$$\begin{aligned}
\hat{U}^0(t) - \tilde{U}^0(t) = &\int_t^n e^{-\int_t^s r(u)+\mu^\varepsilon(u)\mathrm{d}u}\mu^\varepsilon(s)\frac{\Delta\mu(s)}{\mu(s)}R_\varepsilon(s)\mathrm{d}s \\
&- \int_t^n e^{-\int_t^s r(u)+\mu(u)\mathrm{d}u}\Delta\mu(s)R(s)\mathrm{d}s.
\end{aligned}$$

Examples of contracts in the survival model, where the stressed sum at risk has the same sign throughout the contract and therefore fits into the framework of Example 7.4.1, are, for instance, a life annuity, a term insurance or a pure endowment.

7.4.2 Comparisons to with-profit life insurance

The *SCR* in the scenario-based model studied here consists of the difference between the insurance liabilities evaluated under two bases; an original and a stressed basis. This draws parallels to with-profit life insurance, where a set of technical (prudent) transition intensities, $\{\mu^*_{jk}(t)\}_{(j,k),j\neq k}$, and the technical (prudent) interest rate r^* are chosen for pricing insurance contracts, and the market transition intensities, $\{\mu^m_{jk}(t)\}_{(j,k),j\neq k}$, and the interest rate r^m are used for valuation of the liabilities. The purpose of the technical valuation basis is to ensure that premiums are on the safe side. A possible surplus is then paid back to the policyholders as a bonus. The level of the difference between μ^ε and μ, and μ^m and μ^*, and the motive for choosing the stressed intensities in the scenario-based model and the technical transition intensities in with-profit life insurance, respectively, might differ. Still, the mathematics turn out to have a lot of similarities.

Calculation of the expectation in Equation (7.4.4) yields that

$$U^{Z(t)}(t) = \int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \sum_{j\in\mathcal{J}} p_{Z(t)j}(t,s) \sum_{k:k\neq j} \big(\mu^\varepsilon_{jk}(s) - \mu_{jk}(s)\big) R^{jk}_\varepsilon(s)\mathrm{d}s, \quad (7.4.12)$$

The individual bonus potential, V^{IB}, of a with-profit life insurance contract from Chapter 2 of Møller and Steffensen (2007) is given by

$$V^{\mathrm{IB}}(t) = \int_t^n e^{-\int_t^s r^m(u)\mathrm{d}u} \sum_{j\in\mathcal{J}} p_{Z(t)j}(t,s) \Bigg(\big(r^m(s) - r^*(s)\big) V^{*j}(s) + \sum_{k:k\neq j} \big(\mu^*_{jk}(s) - \mu^m_{jk}(s)\big) R^{*jk}(s) \Bigg)\mathrm{d}s,$$

where V^* is the expected present value of future payments under the technical basis and R^{*jk} is the sum at risk on the technical basis. The last part of the expression above resembles the *SCR* from the scenario-based model in Equation (7.4.12) in the sense that it is the difference between the technical transition intensities and the market transition intensities times the sum at risk on the technical basis.

The surplus contribution process from Asmussen and Steffensen (2020) (Chapter 6, Proposition 4.3) in with-profit life insurance, including bonus, is equal to

$$c^j(t) = \big(r^m(t) - r^*(t)\big) V^{*j}(t) + \sum_{k:k\neq j} \big(\mu^*_{jk}(t) - \mu^m_{jk}(t)\big) R^{*jk}(t).$$

The last part of the surplus contribution rate above resembles the individual bonus potential and the *SCR* from the scenario-based model.

It holds that for the surplus contribution rate, the individual bonus, and the *SCR* from the scenario-based model to be positive, the prudent transition intensities (or the stressed transition intensities in the scenario-based model) must be chosen in such a way that the sign of the technical (stressed) intensities minus $\{\mu_{jk}(t)\}_{(j,k),j\neq k}$ is equal to the sign of the technical (stressed) sums-at-risk, which is precisely the condition in Equation (7.4.2). Therefore, the choice of stress in the scenario-based model is closely connected to the choice of a prudent basis for the transition intensities in with-profit life insurance.

7.4.3 The choice of stress

In this section, we present different choices of the stressed transition intensities, $\big(\mu^{\varepsilon}_{jk}(t)\big)_{j,k\in\mathcal{J},j\neq k}$. A criterion for the stress to be successful is that the *SCR* in the scenario-based model is positive. A sufficient condition for $U^{Z(t)}(t)$ to be positive is that the sign of the change in the transition intensities due to the stress equals the sign of the stressed sum-at-risk, see Equation (7.4.2).

One choice presented in Christiansen (2010) is where the stressed intensities are chosen from a confidence band around $\mu_{jk}(t)$ given by $l_{jk}(t) \leq \mu_{jk}(t) \leq u_{jk}(t)$. The stressed intensities are then given by

$$\mu^{\varepsilon,cb}_{jk}(t) = \begin{cases} l_{jk}(t) & \text{if } R^{jk}_{\varepsilon}(t) < 0, \\ u_{jk}(t) & \text{if } R^{jk}_{\varepsilon}(t) \geq 0. \end{cases}$$

The consequence is that, in general, the functions $\mu^{\varepsilon,cb}_{jk}(t)$ and $R^{jk}_{\varepsilon}(t)$ have to be determined simultaneously. The stressed sums-at-risk can be calculated as the solution to a system of Thiele-inspired differential equations presented in Christiansen (2010), Equation (3.8). This choice of stress satisfies the condition in Equation (7.4.2).

A different choice is a parametric stress on the transition intensities given by

$$\mu^{\varepsilon,pa}_{jk}(t) = \big(1+\beta_{jk}(t)\big)\mu_{jk}(t) + \varepsilon_{jk}(t), \tag{7.4.13}$$

for $j,k\in\mathcal{J}$, $j\neq k$, where $\varepsilon_{jk}(t)$ is a stress on the level, and $\beta_{jk}(t)$ a stress on the trend of the transition intensities. The *SCR* in the scenario-based model with this choice of stress is given by

$$\begin{aligned} U^{Z(t)}(t) = \mathbb{E}\bigg[\int_t^n & e^{-\int_t^s r(u)\mathrm{d}u} \sum_{j\in\mathcal{J}} \mathbb{1}_{\{Z(s-)=j\}} \\ & \times \sum_{k:k\neq j} \Big(\beta_{jk}(s) + \frac{\varepsilon_{jk}(s)}{\mu_{jk}(s)}\Big) R^{jk}_{\varepsilon}(s)\mathrm{d}N^k(s) \,\bigg|\, Z(t)\bigg]. \end{aligned}$$

The risk of loss due to changes in the level and the trend of mortality and disability rates is included in the life risk module of the standard formula for the *SCR* described in European Commission, 2008, Article 105. Therefore, parametric stress seems like a reasonable choice of stress in a stress-based model of the *SCR*. This choice of stress does not necessarily satisfy the condition in Equation (7.4.2), but $\varepsilon_{jk}(t)$ and $\beta_{jk}(t)$ can be chosen such that Equation (7.4.2) holds. Otherwise, the *SCR*, $U^{Z(t)}(t)$, can still be positive even though the condition in Equation (7.4.2) is not satisfied.

Remark 7.4.2. For the parametric stress in Equation (7.4.13), the approximation based on the valuation without stress, $\tilde{U}^{Z(t)}(t)$ in Equation (7.4.6), has a connection to the directional (Gateaux) derivative of the value of the original payment process, $\mathrm{d}B(t)$. If the stresses are constant, i.e. $\varepsilon_{jk}(t) = \varepsilon_{jk} \in \mathbb{R}$ and $\beta_{jk}(t) = \beta_{jk} \in \mathbb{R}$, it holds that the approximation from Equation (7.4.6) is equal to

$$\mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}\tilde{B}^{\varepsilon}(s) \;\middle|\; Z(t) \right] =$$
$$\sum_{j\in\mathcal{J}} \sum_{k:k\neq j} \left(\beta_{jk} \frac{\partial}{\partial \eta_{jk}} V^{Z(t)}(t, \mu + \eta\mu) + \varepsilon_{jk} \frac{\partial}{\partial \eta_{jk}} V^{Z(t)}(t, \mu + \eta e) \right),$$

where $V^{Z(t)}(t, \mu)$ is the prospective reserve from Equation (7.2.1) and $\eta = (\eta_{jk})_{j,k\in\mathcal{J},j\neq k}$ and $e = (\mathbb{1}_{jk})_{j,k\in\mathcal{J},j\neq k}$. Hence, the approximation $U^{Z(t)}(t) \approx \tilde{U}^{Z(t)}(t)$ can be considered as a sum of the first order Taylor approximations in the directions μ and e. A Thiele-inspired differential equation for the directional derivative is studied in Nyegaard (2023).

One specific choice of parametric stress is where the level of prudence depends on the stressed sum-at-risk. We work with a stress on the trend, β_{jk}, and set $\varepsilon_{jk} = 0$. We design the stress such that

$$\beta_{jk}(t) = cR_{\varepsilon}^{jk}(t), \tag{7.4.14}$$

for a constant $c \geq 0$. Since the intensities are positive, we immediately see that the choice of stress in Equation (7.4.14) fulfills the condition in Equation (7.4.2). The stress and the stressed sums-at-risk have to be determined simultaneously. This feature is a consequence of β_{jk} depending on the stress through $R_{\varepsilon}^{jk}(t)$.

With the particular stress in Equation (7.4.14), we can calculate the *SCR* via the representation in Equation (7.4.4) based on the payment process

$$\mathrm{d}B^{\varepsilon}(t) = c \sum_{k:k\neq Z(t-)} (R_{\varepsilon}^{Z(t-)k}(t))^2 \mathrm{d}N^k(t), \tag{7.4.15}$$

and obtain that

$$U^{Z(t)}(t) = c \cdot \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \sum_{k:k\neq Z(s-)} (R_{\varepsilon}^{Z(s-)k}(s))^2 \mathrm{d}N^k(s) \;\middle|\; Z(t) \right].$$

Similarly to Section 7.4.1, this representation is a mix of valuations with and without stress since the expectation is unstressed. Still, the payment stream $\mathrm{d}B^{\varepsilon}(t)$

depends on the stressed sums-at-risk. Following Section 7.4.1, we now have two approximations, one based on full valuation under stress and another based on valuation without stress corresponding to

$$\tilde{\tilde{U}}^{Z(t)}(t) = c \cdot \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \sum_{k:k\neq Z(s-)} (R^{Z(s-)k}(s))^2 \mathrm{d}N^k(s) \,\middle|\, Z(t) \right], \quad (7.4.16)$$

$$\hat{U}^{Z(t)}(t) = c \cdot \mathbb{E}^{\varepsilon}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \sum_{k:k\neq Z(s-)} (R_{\varepsilon}^{Z(s-)k}(s))^2 \mathrm{d}N^k(s) \,\middle|\, Z(t) \right], \quad (7.4.17)$$

where Equation (7.4.17) corresponds to Equation (7.4.7). The approximation in Equation (7.4.16) does not correspond to the approximation $\tilde{U}^{Z(t)}(t)$ in Equation (7.4.6) since the payment stream $\mathrm{d}B^{\varepsilon}(t)$ from Equation (7.4.15) consists of both the stressed sum-at-risk from the stressed intensities (Equation (7.4.14)) and from the original payment stream $\mathrm{d}B^{\varepsilon}(t)$ in Equation (7.4.5), and here we approximate them both with the unstressed sum-at-risk. Therefore, we denote the approximation by a double tilde.

It is a delicate observation that these two approximations are also well-known under different names. They are, namely,

$$\tilde{\tilde{U}}^{Z(t)}(t) = c\mathbb{V}\left[\int_t^n e^{-\int_t^s \frac{1}{2}r(u)\mathrm{d}u} \mathrm{d}B(s) \,\middle|\, Z(t) \right],$$

$$\hat{U}^{Z(t)}(t) = c\mathbb{V}^{\varepsilon}\left[\int_t^n e^{-\int_t^s \frac{1}{2}r(u)\mathrm{d}u} \mathrm{d}B(s) \,\middle|\, Z(t) \right],$$

where $\mathbb{V}$ and $\mathbb{V}^{\varepsilon}$ denote the conditional variance under the two measures without and with stress, respectively. Note that we must halve the interest rate in these expressions to make the representations work. It is a striking feature of the stress design in Equation (7.4.14) that the *SCR* relates to the variances of the original payment streams under the two valuation measures.

It is possible to determine a stress scenario for which the approximation based on the unstressed measure, $\tilde{\tilde{U}}^{Z(t)}(t)$, is not an approximation but the true solvency capital requirements. It is straightforward to see that the variance based on the unstressed measure occurs if

$$\beta_{jk}(t) = c\frac{R^{jk}(t)}{R_{\varepsilon}^{jk}(t)} R^{jk}(t).$$

Also, the condition in Equation (7.4.2) is fulfilled for this stress. We already knew, though, that the SCR, i.e., the difference between the unstressed and the stressed reserves, is positive since the variance is always positive.

7.5 Risk margin in the scenario-based model

In this section, we calculate the risk margin using the *SCR* from the scenario-based model in Section 7.4. We use the expression for the *SCR* in Equation (7.4.4) and insert in the continuous definition for the risk margin from Equation (7.3.3), and denote the risk margin in the scenario-based model by $\mathrm{RM}^{\mathrm{sce}}$

$$\begin{aligned}
\mathrm{RM}^{\mathrm{sce}} &= CoC \cdot \mathbb{E}\left[\int_0^n e^{-\int_0^t r(u)\mathrm{d}u} U^{Z(t)}(t)\mathrm{d}t\right] \\
&= CoC \cdot \mathbb{E}\left[\int_0^n e^{-\int_0^t r(u)\mathrm{d}u} \mathbb{E}\left[\int_t^n e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B^{\varepsilon}(s) \,\middle|\, Z(t)\right]\mathrm{d}t\right] \\
&= CoC \cdot \mathbb{E}\left[\int_0^n t e^{-\int_0^t r(u)\mathrm{d}u} \mathrm{d}B^{\varepsilon}(t)\right], \qquad (7.5.1)
\end{aligned}$$

where we use calculations similar to those in Equation (7.3.4). The risk margin resembles the duration of the modified payment stream $\mathrm{d}B^{\varepsilon}(t)$ that depends on the stressed sums-at-risk.

The risk margin $\mathrm{RM}^{\mathrm{sce}}$ is the risk margin at time 0. The risk margin runoff in the scenario-based model is given by

$$\mathrm{RM}^{\mathrm{sce}}(t) = CoC \cdot \sum_{j \in \mathcal{J}} p_{Z(0)j}(0,t) \cdot \mathbb{E}\left[\int_t^n (s-t) e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B^{\varepsilon}(s) \,\middle|\, Z(t) = j\right].$$

The scenario-based model proposed here is simple, and it is unrealistic that financial authorities would accept it. Still, we consider it a way to gain intuition about the analytical calculation of risk margins without approximating the *SCR*. The expression for the risk margin in the scenario-based model in Equation (7.5.1) is determined without approximation and under the assumption that the *SCR* is the difference between stressed insurance liabilities and the best estimate insurance liabilities. We consider $\mathrm{RM}^{\mathrm{sce}}$ an excellent alternative to the approximations of the risk margin from the guideline since we can calculate the risk margin without approximations. Below, we compare the scenario-based model's risk margin with the guideline's approximated risk margins.

7.5.1 Comparison with approximations from the guideline

The risk margin from the scenario-based model resembles the risk margin with approximation methods two and three from the guideline in Equation (7.3.4) in the sense that it resembles a duration. However, it is not the duration of the insurance payments but the duration of a different payment stream that depends on the sum-at-risk calculated with the stressed transition intensities. We denote the duration of the payment process $\mathrm{d}B^{\varepsilon}(t)$ in Equation (7.4.5) by D^{ε} and it is given by

$$D^{\varepsilon} = \frac{\mathbb{E}\left[\int_0^n t e^{-\int_0^t r(u)\mathrm{d}u} \mathrm{d}B^{\varepsilon}(t)\right]}{\mathbb{E}\left[\int_0^n e^{-\int_0^t r(u)\mathrm{d}u} \mathrm{d}B^{\varepsilon}(t)\right]}.$$

We denote the SCR from the scenario-based model by $SCR_\varepsilon(t)$ and note that $SCR_\varepsilon(t) = U^{Z(t)}(t)$. Then, the risk margin with the scenario-based model for the future SCR is

$$\mathrm{RM}^{\mathrm{sce}} = CoC \cdot SCR_\varepsilon(0) \cdot D^\varepsilon,$$

which is similar to the risk margin calculated with the third approximation method from the guideline from Equation (7.3.5), although with the duration of a different payment stream as the proportionality factor. This indicates that a different duration in the third approximation method for the risk margin from the guideline may result in a more accurate risk margin since $\mathrm{RM}^{\mathrm{sce}}$ is calculated without approximations.

In the second approximation method from the guideline, it is assumed that the ratio between the best estimate and the SCR is fixed such that

$$\frac{SCR(t)}{SCR(0)} = \frac{V^{Z(t)}(t)}{V^{Z(0)}(0)}.$$

With the scenario-based model, it holds, without approximation, that

$$\frac{SCR_\varepsilon(t)}{SCR_\varepsilon(0)} = \frac{U^{Z(t)}(t)}{U^{Z(0)}(0)},$$

since the SCR is simply the expected present value of the payment stream $\mathrm{d}B^\varepsilon(t)$, which is denoted by $U^{Z(t)}(t)$.

7.5.2 Risk margin with approximations of the sum-at-risk

The SCR from the scenario-based model, $U^{Z(t)}(t)$ from Equation (7.4.4), consists of a mix of valuation with and without stress. This leads to two approximations of the SCR: One based on valuation without stress, $\tilde{U}^{Z(t)}(t)$ from Equation (7.4.16), and one based on full valuation under stress, $\hat{U}^{Z(t)}(t)$ from Equation (7.4.17). Two approximations of the risk margin then follow immediately from (7.5.1). These approximations are based on valuation without and with stress, respectively, and are

$$\tilde{\mathrm{RM}}^{\mathrm{sce}} = CoC \cdot \mathbb{E}\left[\int_0^n t e^{-\int_0^t r(u)\mathrm{d}u} \mathrm{d}\tilde{B}^\varepsilon(t)\right], \tag{7.5.2}$$

$$\hat{\mathrm{RM}}^{\mathrm{sce}} = CoC \cdot \mathbb{E}^\varepsilon\left[\int_0^n t e^{-\int_0^t r(u)\mathrm{d}u} \mathrm{d}B^\varepsilon(t)\right], \tag{7.5.3}$$

and the runoffs of the risk margin approximations are

$$\tilde{\mathrm{RM}}^{\mathrm{sce}}(t) = CoC \cdot \mathbb{E}\left[\int_t^n (s-t) e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}\tilde{B}^\varepsilon(s)\right],$$

$$\hat{\mathrm{RM}}^{\mathrm{sce}}(t) = CoC \cdot \mathbb{E}^\varepsilon\left[\int_t^n (s-t) e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B^\varepsilon(s)\right].$$

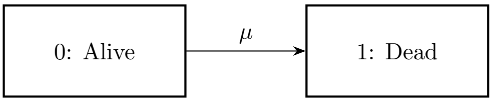


Figure 7.1: *The survival model*

Table 7.1: *Components in numerical example*

Component	Value
Age of insured, a_0	30
Termination, n	40
Interest rate $r(t)$	0.01
Pure endowment, ΔB	2
Term insurance, b	1.5

The same relations as in Example 7.4.1 in the survival model hold between $\mathrm{RM}^{\mathrm{sce}}$ and $\tilde{\mathrm{RM}}^{\mathrm{sce}}$, but since the expectation in $\hat{\mathrm{RM}}^{\mathrm{sce}}$ (see Equation (7.5.3)) is taken under the stressed measure, we cannot conclude that the relation between $\mathrm{RM}^{\mathrm{sce}}$ and $\hat{\mathrm{RM}}^{\mathrm{sce}}$ is the same as the relation between $U^{Z(t)}(t)$ and $\hat{U}^{Z(t)}(t)$. We illustrate the relation between the approximations and the resulting risk margins in a numerical example in Section 7.6.

7.6 Numerical example

We consider the survival model illustrated in Figure 7.1. The transition intensity between state 0 and state 1 is given by

$$\mu(t) = 0.0005 + 10^{5.6+0.04*(t+a0)-10},$$

where a_0 is the age of the insured at time $t = 0$. The insurance company offers two products: A pure endowment paying ΔB if the insured is alive at time n and a term insurance paying b upon death before time n.

The choice of stress is inspired by Christiansen (2010) and chosen as the worst-case stress within the boundaries

$$0.8 \cdot \mu(t) = l(t) \leq \mu(t) \leq u(t) = 1.2 \cdot \mu(t),$$

hence

$$\mu^{\varepsilon}(t) = \begin{cases} l(t) & \text{if } R^{\varepsilon}(t) < 0, \\ u(t) & \text{if } R^{\varepsilon}(t) \geq 0. \end{cases}$$

This implies that we simultaneously determine the stress and the stressed sum at risk. The components in the numerical example are given in Table 7.1.

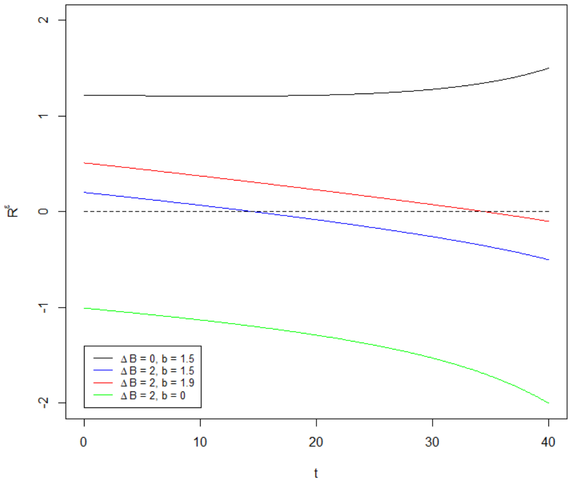


Figure 7.2: *Stressed sum at risk for four combinations of the two products*

7.6.1 Approximations of the *SCR*

First, we calculate the approximations of the SCR from Section 7.4.1 for different combinations of the two products: the pure endowment and the term insurance.

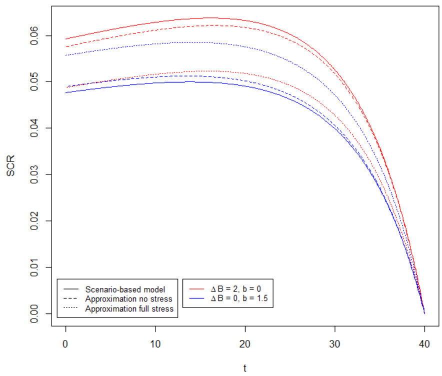
In Figure 7.2, the stressed sum at risk for the term insurance is positive for all t, it is negative for all t for the pure endowment, and for the products that consist of a combination of the term insurance and the pure endowment, the stressed sum at risk changes sign.

The SCR from the scenario-based model in Equation (7.4.4) and the approximations of the SCR in Equations (7.4.6) and (7.4.7) are plotted in Figure 7.3. The relation between the approximations and the true SCR from Equation (7.4.11) holds for the pure endowment and the term insurance since the stressed sum at risk has the same sign throughout the contract, which is also illustrated in Figure 7.3(a). For the products that consist of a combination of the pure endowment and the term insurance, where the stressed sum at risk changes sign, the order of $U^0(t)$, $\tilde{U}^0(t)$ and $\hat{U}^0(t)$ is not the same as illustrated in Figure 7.3(b). For the product where the sum upon death is $b = 1.5$, the true SCR, $U^0(t)$, is the highest, and for $b = 1.9$, the approximation based on the full stress is the highest.

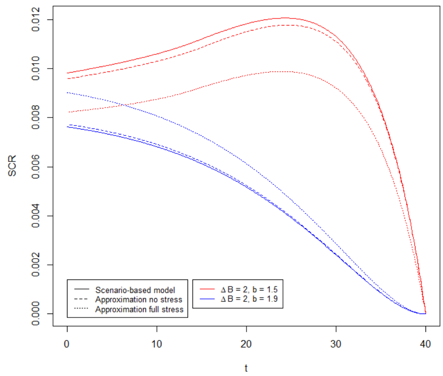
7.6.2 Calculation of the risk margin

We calculate the risk margin and the risk margin runoff based on the true SCR from the scenario-based model, the approximations of the SCR, and with the

Figure 7.3: *SCR and approximations from the scenario-based model*



(a) *The true SCR, $U^0(t)$, and the approximations, $\hat{U}^0(t)$ and $\tilde{U}^0(t)$, for the pure endowment and the term insurance.*



(b) *The true SCR, $U^0(t)$, and the approximations, $\hat{U}^0(t)$ and $\tilde{U}^0(t)$, for combinations of the pure endowment and the term insurance.*

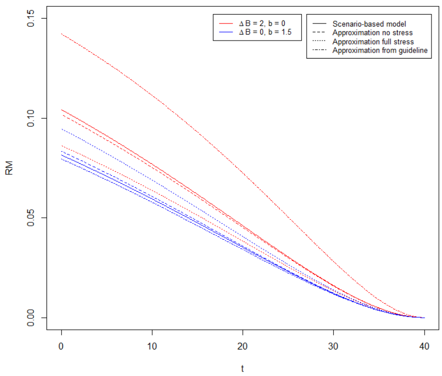
approximation method from the guideline i.e.

$$
\begin{aligned}
\mathrm{RM}^{\mathrm{sce}}(t) &= CoC \cdot p_{00}(0,t) \cdot \mathbb{E}\left[\int_t^n (s-t) e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B^{\varepsilon}(s) \;\middle|\; Z(t)=0 \right], \\
\tilde{\mathrm{RM}}^{\mathrm{sce}}(t) &= CoC \cdot p_{00}(0,t) \cdot \mathbb{E}\left[\int_s^n (s-t) e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}\tilde{B}^{\varepsilon}(s) \;\middle|\; Z(t)=0 \right], \\
\hat{\mathrm{RM}}^{\mathrm{sce}}(t) &= CoC \cdot p_{00}^{\mu^{\varepsilon}}(0,t) \cdot \mathbb{E}^{\varepsilon}\left[\int_t^n (s-t) e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B^{\varepsilon}(s) \;\middle|\; Z(t)=0 \right], \\
\mathrm{RM}^{(2)}(t) &= \mathrm{RM}^{(3)}(t) = \\
CoC &\cdot p_{00}(0,t) \cdot \frac{SCR(t)}{V^0(t)} \mathbb{E}\left[\int_0^n (s-t) e^{-\int_t^s r(u)\mathrm{d}u} \mathrm{d}B(s) \;\middle|\; Z(t)=0 \right],
\end{aligned}
$$

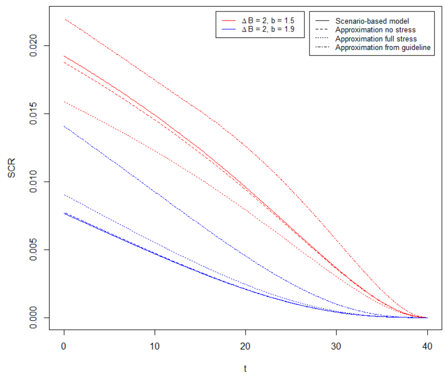
where we assume that $SCR(t) = U^0(t)$ (SCR from scenario-based model) and $CoC = 0.06$.

Figure 7.4 illustrates the risk margin runoffs. The risk margin approximation from the guideline, $\mathrm{RM}^{(2)}$, is either higher or lower than all the other risk margins. The order of the SCR and its approximations from the scenario-based model are kept when calculating the risk margin, hence if $U^0(t) > \tilde{U}^0(t) > \hat{U}^0(t)$, then it holds that $\mathrm{RM}^{\mathrm{sce}}(t) > \tilde{\mathrm{RM}}^{\mathrm{sce}}(t) > \hat{\mathrm{RM}}^{\mathrm{sce}}(t)$.

Figure 7.4: *Risk margin runoffs*



(a) *The risk margin runoff calculated with the true SCR and the approximations for the pure endowment and the term insurance.*



(b) *The risk margin runoff calculated with the true SCR and the approximations for combinations of the pure endowment and the term insurance.*

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