FACULTY OF SCIENCE
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# PhD Thesis in Mathematics

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# Quasitraces, Tracial States, and Kaplansky's Conjecture

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**Abstract**

This thesis examines various notions of traces on $C^*$-algebras with a focus on quasitraces and tracial states, and on the connection between them as exemplified by Kaplansky's conjecture. Different already known characterisations of when unital $C^*$-algebras admit quasitraces and tracial states are studied, and these characterisations are used in order to construct new numerical invariants, which informally measures the failure to admit quasitraces resp. tracial states, and which have interesting asymptotic behaviour. In particular, following a construction due to Rørdam, it is shown that for any $n \in \mathbb{N}$ there exists a separable, unital, nuclear, simple $C^*$-algebra $A_n$ such that $n$ is the smallest integer satisfying that $M_n(A_n)$ is properly infinite. By taking an ultraproduct of this sequence of $C^*$-algebras without quasitraces, one obtains an example of a unital non-exact $C^*$-algebra with a quasitrace. It is unresolved whether this quasitrace is a tracial state. We discuss the possible implications for Kaplansky's conjecture, and also how the aforementioned numerical invariants may provide more information about the conjecture. Inspired by work of Robert–Rørdam, we also provide a way of viewing the existence of $C^*$-algebras with tracial states not approximable by limit tracial states by introducing the notion of almost tracial states.

Moreover, this thesis presents the original as well as an alternative proof of the result due to Haagerup that quasitraces on unital, exact $C^*$-algebras are tracial states. We emphasise the usages of the AW$^*$-completion, which gives a link between Kaplansky's original conjecture and the modern formulation, and we also emphasise when exactness is used. The alternative proof uses the result due to Haagerup–Thorbjørnsen that $C_r^*(\mathbb{F}_\infty)$ is an MF-algebra, which allows for matrix approximations of certain self-adjoint elements in $C_r^*(\mathbb{F}_\infty)$. Using a characterisation of non-traciality of unital $C^*$-algebras due to Haagerup will then allow us to prove that $A \otimes C_r^*(\mathbb{F}_\infty)$ is properly infinite, and by invoking exactness and the aforementioned self-adjoint lifts, one obtains the existence of $n \in \mathbb{N}$ for which $M_n(A)$ is properly infinite. We also show that the proof strategy does not hold universally for non-exact $C^*$-algebra by showing that, for any choice of matrix approximation of the self-adjoint elements in $C_r^*(\mathbb{F}_\infty)$, there exists a non-exact $C^*$-algebra $A$ for which the proof fails.

In the thesis, we also initiate the study of when $C^*$-algebras have the property that all quotients admit faithful tracial states. We provide a sufficient and necessary condition for when $C^*$-algebras admit faithful tracial states in terms of the existence of stable ideals by using regularity properties of Cuntz semigroups. By applying this result to the quotients, we then obtain an an equivalent formulation for all quotients to admit faithful tracial states, and we use this to determine when $C^*$-algebras are strongly quasidiagonal.

**Resumé**

Denne afhandling undersøger forskellige versioner af spor på $C^*$-algebra med et fokus på kvasispor og sportilstande og på sammenhængen mellem disse ved at studere Kaplanskys formodning. Vi kigger på allerede kendte karakteriseringer af, hvornår $C^*$-algebraer har kvasispor hhv. sportilstande, og vi benytter disse til at definere nogle nye numeriske invarianter, der uformelt sagt måler fejlen til at have kvasispor hhv. sportilstande, og som har interessante asymptotiske egenskaber. Ved at anvende Rørdams eksempel på en simpel $C^*$-algebra med en endelig og en uendelig projektion viser vi, at der for hvert $n \in \mathbb{N}$ findes en separabel, unital, nukleær og simpel $C^*$-algebra $A_n$ således at $n$ er det mindste heltal for hvilket $M_n(A_n)$ er egentlig uendelig. Ved at tage et ultraprodukt af sådan en følge af $C^*$-algebraer konstruerer vi et eksempel på en unital, ikke-eksakt $C^*$-algebra med et kvasispor. Det er uafklaret hvorvidt dette kvasispor er en sportilstand. Vi diskuterer implikationerne for Kaplanskys formodning, og hvordan de førnævnte numeriske invarianter også kan give information om formodningen. Inspireret af arbejde af Robert–Rørdam om karakterer på ultraprodukter, introducerer vi "næsten sportilstande" og anvender disse til at forklare eksistensen af ultraproduct-$C^*$-algebraer med sportilstande, som ikke er approksimable med grænsesportilstande.

Derudover præsenterer afhandlingen udvalgte dele samt et alternativt bevis af Haagerups teorem om, at kvasispor på unitale, eksakte $C^*$-algebraer er sportilstande. I gennemgangen af Haagerups oprindelige bevis lægger vi fokus på AW$^*$-fuldstændiggørelsen af unitale $C^*$-algebraer med tro kvasispor, hvilket viser sammenhængen mellem Kaplanskys oprindelige formodning og den moderne formulering, samt hvornår eksakthed præcis dukker op i beviset. Det alternative bevis, som vi præsenterer, anvender resultatet af Haagerup–Thorbjørnsen om, at $C_r^*(\mathbb{F}_\infty)$ er en MF-algebra for alle $n \in \mathbb{N}$ til at finde matrixapproksimationer af bestemte selv-adjungerede elementer i $C_r^*(\mathbb{F}_\infty)$. Ved at benytte Haagerups karakterisering af ikke at have sportilstande vil så benyttes til at bevise, at $A \otimes C_r^*(\mathbb{F}_\infty)$ er egentlig uendelig, og ved at anvende eksakthed af $A$ og betragte de førnævnte matrixapproksimationer opnår man eksistensen af et $n \in \mathbb{N}$ for hvilket $M_n(A)$ er egentlig uendelig. Vi viser også, at metoden ikke virker universelt ved for enhver matrixapproksimation af de selvadjungerede elementer i $C_r^*(\mathbb{F}_\infty)$ at konstruere en ikke-eksakt $C^*$-algebra, hvor beviset fejler.

I afhandlingen igangsætter vi undersøgelsen af, hvornår $C^*$-algebraer har den egenskab, at alle kvotienter har tro sportilstande, som vi betegner QFTS-egenskaben. Vi undersøger allerede kendt viden om, hvornår $C^*$-algebraer har separerende familier af sportilstande. Vi giver en nødvendig og tilstrækkelig betingelse for, hvornår $C^*$-algebraer har tro sportilstande i termer af eksistensen af stabilie idealer under antagelse af regularitetsegenskaber af Cuntz semigruppen. Dette resultat bliver så anvendt på kvotienter til at finde en ækvivalent formulering af, hvornår $C^*$-algebaer har QFTS-egenskaben, og vi bruger denne til at bestemme, hvornår $C^*$-algebraer er stærkt kvasidiagonale.

# Contents



The contents of Chapter 5 come from the author's single-authored preprint [45] titled *"Faithful tracial states on quotients of $C^*$-algebras"* uploaded to the arXiv e-Print archive, available at the following URL: `https://arxiv.org/abs/2111.09703v1`.

# 1 Introduction

One way of understanding the subject of operator algebras is to look at linear algebra from afar: We are not studying the properties of single matrices or operators, but we are studying the *collections* of operators as a whole. The main focus in this thesis is the study of traces, and this nicely exemplifies the differences between the interests within single operator theory and the study of operator algebras. Any first-year mathematics student knows about traces; they probably know the formula of the trace of a square matrix, they might recognize that the trace map is invariant under cyclic permutations (although they certainly would not phrase it as such) and basis changes, and they might realise that the trace of a given matrix is the sum of its eigenvalues. Similarly, a second-year mathematics student might realise that "summing the diagonal" is often ill-defined whenever one is working with infinite-dimensional matrices, and they might know of trace-class or Hilbert-Schmidt operators. While this last part does have an operator algebraic flavour, restricting oneself to look at collections of operators on a Hilbert space, the viewpoint of the student is presumably still fixated on the single operator theoretic notion: Given a matrix, what is its trace, and what does this imply for the matrix in question?

Operator algebraists deal with a different question. We look at collections of operators, and we wonder about the structure these collections may have. To name an example of what kind of questions an operator algebraist might ask, let us look at the fundamental question permeating the entire thesis: When do algebras of operators admit traces? As it is written, this question is obviously not defined rigorously: What is an operator? What is an algebra of operators? What is a trace? But the question taken at face value still provides insight into the viewpoint of operator algebraists, and the kind of questions we might be interested in asking. We are concerned with properties of structures, not specific elements, and the consequences of these properties.

Unsurprisingly, this thesis deals with traces from this point of view. That is, we shall study when operator algebras, mostly $C^*$-algebras, admit traces, and what the consequences of (non-)existence of traces are, and even what constitutes a trace — a question that might seem nonsensical to the aforementioned math students, but which is a surprisingly deep question. The reader might notice how in the sequel we switch from talking about traces to tracial states — this is a deliberate choice, which we shall return to later.

The study of traces on operator algebras is nothing new — as mentioned earlier in this introduction, the trace is an important basis-invariant map for finite-dimensional square matrices, which provides information about the eigenvalues. However old the subject might be, it is still not resolved completely when $C^*$-algebras admit tracial states; results of Haagerup [28] and Pop [54] give an (almost) algebraic characterisation of failing to admit tracial states, but before Haagerup proved his characterisation of traciality of unital, exact $C^*$-algebras, proving the existence of a tracial state often involved actually constructing a tracial state. For example, one can easily show that group-$C^*$-algebra of a discrete group admits tracial states by constructing the specific vector state $x \mapsto \langle x\delta_e, \delta_e \rangle$; here it is also worth mentioning that the question of when this tracial state is unique on reduced group-$C^*$-algebras is a surprisingly deep question. But if one is given a $C^*$-algebra, how does one know whether it admits a tracial state or not? And is it something which is humanly verifiable in some manner? Haagerup's aforementioned theorem provided a big, abstract class of $C^*$-algebras which admits a tracial state: Unital, exact $C^*$-algebras admit tracial states if and only if they are stably not properly infinite[1]. The importance of this characterisation actually goes a bit further, since it not only gives a complete classification of traciality of unital, exact $C^*$-algebras in a very natural, yet abstract manner. It is also a partial resolution to a conjecture known as Kaplansky's conjecture, which dates back to 1951 [37]: Are all so-called quasitraces, which might be seen as a formally weakened tracial state, automatically tracial states? Haagerup's theorem remains the best known answer to this conjecture and the question of abstractly classifying $C^*$-algebras with tracial states, but it is important to notice that we with this theorem are still not done — what happens for non-exact $C^*$-algebras? Is traciality of unital $C^*$-algebras completely characterised by being stably not properly infinite, as it happens for the exact case?

This thesis aims to give a mostly self-contained and historical treatment of the question of traciality of $C^*$-algebras as well as to extend the knowledge of the field by introducing new ways of analysing the problem. Outside of this introduction, the thesis is divided into four chapters.

- *Chapter 2* is an expository chapter on several different elementary topics including properly infinite projections, tracial states and quasitraces. The main point of this chapter is to examine various characterisations of admitting tracial states due to Haagerup and Pop, and for admitting quasitraces and due to Cuntz, Blackadar–Handelman, and Blackadar–Rørdam. We aim to provide proofs of all important results for the sake of being self-contained, but the proofs rarely differ much from the original sources, to which we will refer thoroughly.

- In *Chapter 3*, we reproduce and review Haagerup's result that quasitraces on unital, exact $C^*$-algebras are tracial states. While the result itself is, obviously, interesting, the proof is also delightfully illuminating to many interesting operator algebraic properties and structures. One result of interest for us will be the AW$^*$-completions of unital $C^*$-algebras with (faithful) quasitraces, which can be seen as an analogue of the W$^*$-completion of $C^*$-algebras with tracial states.

- In *Chapter 4*, we begin introducing original work. This chapter contains a new way of examining whether $C^*$-algebras admit quasitraces and/or tracial states by introducing certain numerical invariants with interesting asymptotic behaviour. Following a construction due to Rørdam [60], we shall explicitly construct a $C^*$-algebra, which admits a quasitrace, but which arises as an ultraproduct of $C^*$-algebras without quasitraces. It is still unresolved whether this quasitrace is a tracial state, and hence this might be an interesting $C^*$-algebra to study when trying to resolve Kaplansky's conjecture. We also provide an alternative proof of Haagerup's theorem with newer methods by invoking a deep result due to Haagerup–Thorbjørnsen [30] that $C^*_r(\mathbb{F}_\infty)$ is an MF-algebra. This chapter is based on joint work with my advisor, Mikael Rørdam.

- In *Chapter 5*, we introduce the study of when all quotients of a $C^*$-algebra admit *faithful* tracial states, which we denote QFTS inspired by Murphy's QTS property from [47]. We use regularity properties of the Cuntz semigroups in order to show that certain $C^*$-algebras have the QFTS property if and only if they have no stable intermediate quotients. In particular, we also find a new equivalent characterisation of when a certain class of $C^*$-algebras admit faithful tracial states, and we apply it to the question of determining strong quasidiagonality of $C^*$-algebras. This chapter is mainly a rewrite of the author's preprint [45].

[1] The reader has probably encountered the stably finite formulation, which is often accredited to Blackadar–Handelman [7] and Haagerup [28]; however, this is a strictly weaker result than what is stated above.

- Last, but certainly not least, we discuss in *Chapter 6* several open and interesting questions within the field of this thesis. Most of these questions are raised previously in the thesis, so the chapter should be seen as an itemisation of the open problems and possible generalisations we have encountered throughout the entire thesis.

## Notation and terminology

It is increasingly uncommon for two mathematical theses to agree on notation, and it is thus customary to help the reader by mentioning some of the notation.

We denote by $\mathbb{N} = \{1, 2, \ldots\}$ the natural numbers, i.e., we do not view 0 as a natural number. If we want to adjoin 0 to the collection of naturals, we shall denote this set by $\mathbb{N}_0$.

The letters $A, B$ usually denote $C^*$-algebras. By "ideal" we will, unless otherwise specified, mean "closed two-sided ideal", and ideals are usually denoted by the letters $I, J$. By "ideal", we will always mean "closed two-sided ideal", unless otherwise specified.

If $A$ is a unital $C^*$-algebra, we will most often write its unit as $1_A$; however, occasionally we do end up in situations where it is clear from context where the unit belongs, and where it would be more unreadable to adjoin the subscript — in these cases, we may just write 1 instead.

If $H$ is a Hilbert space, we denote by $\mathbb{B}(H)$ the collection of bounded operators on $H$, and by $\mathbb{K}(H)$ the collection of compact operators on $H$. We shall also work with the compacts as an abstract $C^*$-algebra, but we will use the notation $\mathbb{K}(H)$ consistently for this situation as well.

If $(A_n)_{n\in\mathbb{N}}$ is a family of $C^*$-algebras, we denote by $\ell^\infty((A_n)_{n\in\mathbb{N}})$ the product $C^*$-algebra, which in the literature is often called $\prod_{n\in\mathbb{N}} A_n$. Similarly, we write $c_0((A_n)_{n\in\mathbb{N}})$ for the $C^*$-algebra consisting of sequences converging to 0 in norm; this is in the literature often denoted by $\bigoplus_{n\in\mathbb{N}} A_n$; if we work with a finite number of $C^*$-algebras in the direct sum, and when working with elements on Hilbert spaces, we will still use this notation. If $A_n = A$ for all $n$, we write $\ell^\infty(A)$ and $c_0(A)$ for the bounded sequences in $A$, and those which tend to 0 in norm.

We shall by $\otimes$ denote the minimal or spatial tensor product, and $\otimes_{\max}$ the maximal tensor product. Moreover, throughout the entire thesis, we will let $\omega$ denote a fixed free ultrafilter on $\mathbb{N}$.

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# 2 The structure of finite and infinite $C^*$-algebras

In this chapter, we study the main objects of interest in this thesis: quasitraces and tracial states. However, in order to fully appreciate these objects, one needs to look in depth at projections of $C^*$-algebras and, in particular, properly infinite projections. Our goal is to provide an almost self-contained review of the most important known results regarding these objects including when $C^*$-algebras admit quasitraces resp. tracial states. We also aim to study one of the oldest open questions about the structures of $C^*$-algebra, namely if quasitraces are always tracial states. While the question remains unresolved in general, Haagerup answered it in the affirmative for the class of unital, exact $C^*$-algebra [28]; we provide a proof of this result in the following chapter. In general, this chapter should be viewed as an expository overview of a part of a vast field of operator algebraic theory, and the arguments presented below are all coming from the original papers, to which we shall cite thoroughly.

## 2.1 Projections in $C^*$-algebras

In the following, let $A$ be a $C^*$-algebra. A *projection* on $A$ is an element $p \in A$ satisfying $p = p^2 = p^*$. The collection of all projections on $A$ is denoted $\mathcal{P}(A)$. We say that two projections $p, q \in \mathcal{P}(A)$ are Murray-von Neumann equivalent, denoted $p \sim q$, if there exists $x \in A$ such that $p = x^*x$ and $q = xx^*$, i.e., if there exists a partial isometry $x$ such that the support projection is $p$ and the range projection is $q$. It is easily verified that this defines an equivalence relation on $\mathcal{P}(A)$.

**Example 2.1.** Two projections $p, q \in \mathbb{B}(H)$ are Murray-von Neumann equivalent if and only if $\dim(pH) = \dim(qH)$. In particular, two projections in $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ will be Murray-von Neumann equivalent if and only if their traces are equal.

Throughout this thesis, we shall compare projections belonging to different matrix algebras over a given $C^*$-algebra. Let $\mathcal{P}_n(A)$ denote the collection of projections on $M_n(A)$, and let $\mathcal{P}_\infty(A) = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n(A)$ be the collection of all projections on matrix algebras over $A$. This admits an addition given by $p \oplus q = \operatorname{diag}(p, q)$. Define on $\mathcal{P}_\infty(A)$ the order $p \precsim q$ for $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$ if there exists $v \in M_{n,m}(A)$ such that $v^*v = p$ and $vv^* \leq q$.

Intuitively, and as Example 2.1 shows, projections may have different sizes, and it is immediate that, at least for $\mathbb{B}(H)$, we can discuss finiteness of projections by looking at the dimension of the spanned subspace. However, when working with abstract $C^*$-algebras, this is not a meaningful distinction, but there is another obvious way of defining finiteness of arbitrary projections. We say that two projections $p, q \in A$ are *orthogonal*, denoted $p \perp q$, if $pq = 0$.

**Definition 2.2.** Let $p$ be a projection in $A$. We say that $p$ is:

(i) *finite* if $p \sim q \leq p$ for some projection $q \in A$ implies $p = q$.

(ii) *infinite* if $p$ is not finite. That is, there exists a proper subprojection $q < p$ for which $q \sim p$.

(iii) *properly infinite* if there exist orthogonal projections $q_1, q_2 \in \mathcal{P}(A)$ satisfying $p \sim q_1 \sim q_2$ and $q_1 + q_2 \leq p$.

We say that a unital $C^*$-algebra $A$ is *finite*, *infinite* and *properly infinite* if the unit $1_A$ is resp. finite, infinite and properly infinite. We also say that $A$ is *stably finite* or *stably not properly infinite* if $M_n(A)$ is finite resp. not properly infinite for all $n \in \mathbb{N}$.

Note that with the above definition, the zero projection is a properly infinite *and* a finite projection. The reader may find that this does not feel right — if so, the reader is welcome to add the restraint that the above definition of properly infiniteness holds for non-zero projections.

The definition of a finite and an infinite projection should remind the reader of the similar characterisations of a finite and an infinite set. For sets, obviously, the analogue notion of properly infinite will just be that of being infinite, but for projections on $C^*$-algebras, these properties are vastly different; in the example below, we mention a few examples of finite, infinite, and properly infinite $C^*$-algebras.

**Example 2.3.**
- Any commutative $C^*$-algebra is stably finite.
- Any AF-algebra, and hence any (unital) AF-embeddable $C^*$-algebra, is stably finite.
- If $H$ is a Hilbert space, then $\mathbb{B}(H)$ is properly infinite if and only if $H$ is infinite-dimensional, and it is stably finite if and only if $H$ is finite-dimensional.
- The Toeplitz algebra $\mathcal{T}$ generated by a non-unitary isometry is, by construction, infinite. However, it is not properly infinite, as it admits a finite quotient $\mathcal{T} \to C(\mathbb{T})$, see Proposition 2.10 later in the thesis. Note that $\mathcal{T}$ is not stably finite, yet is stably not properly infinite.
- The Cuntz $C^*$-algebras $\mathcal{O}_n$ generated by $n$ isometries $(s_i)_{i=1}^n$ with $\sum_{i=1}^n s_i s_i^* = 1_{\mathcal{O}_n}$ are properly infinite for any $2 \leq n \leq \infty$.
- The Cuntz-Toeplitz $C^*$-algebras $\mathcal{E}_n$ generated by $n$ isometries with mutually orthogonal range projections are properly infinite for $n \geq 2$.

Observe that all examples of finite $C^*$-algebras above are also stably finite. While it is not true that any finite $C^*$-algebra is necessarily stably finite, it is not a trivial statement to verify, and, in fact, this question is one of the focus points of this thesis.

For the remainder of this section, we shall focus on the class of properly infinite projections. First, we look at a few equivalent characterisations of properly infiniteness.

**Proposition 2.4.** *Let $p \in A$ be a non-zero projection. The following are equivalent:*

(i) *$p$ is properly infinite;*

(ii) *$p \oplus p \precsim p$;*

(iii) *There exist partial isometries $s_1, s_2 \in A$ with $s_1^* s_1 = s_2^* s_2 = p$ and $s_1 s_1^* + s_2 s_2^* \leq p$.*

(iv) *There exists a sequence of partial isometries $(s_n)_{n\in\mathbb{N}}$ such that $s_n^* s_n = p$ for each $n \in \mathbb{N}$ and such that $s_n s_n^* \perp s_m s_m^*$ for any $n \neq m$.*

(v) *There exists a unital embedding of the Cuntz-Toeplitz algebra $\mathcal{E}_2$ in the corner $C^*$-algebra $pAp$.*

(vi) *There exists a unital embedding of $\mathcal{O}_\infty$ in the corner $C^*$-algebra $pAp$.*

(vii) *The image of $p$ in any quotient of $A$ is either zero or infinite.*

*Proof.* (i)⇔(ii)⇔(iii) are immediate. Moreover, (iii) and (v) are just reformulations of one another, and so are (iv) and (vi). It is immediate that (iv)⇒(iii), and the converse is also true: If $s_1, s_2 \in A$ satisfies that $s_i^* s_i = p$ for $i = 1, 2$ and $s_1 s_1^* + s_2 s_2^* \leq p$, then the partial isometries $t_n = s_2^{n-1} s_1$ for $n \in \mathbb{N}$ satisfy the conditions of (iv). Observe that (ii)⇒(vii), and for (vii)⇒(ii) we refer to [41, Corollary 3.15]. □

The characterisation (vii) above is a generalisation of the following theorem due to Cuntz [16], which we highlight for its historical interest. In particular, it implies that for simple, unital $C^*$-algebras, being stably finite is equivalent to being stably not properly infinite.

**Theorem 2.5** (Cuntz)**.** *Let $A$ be a simple $C^*$-algebra, and let $p \in A$ be a non-zero projection. Then $p$ is properly infinite if and only if $p$ is infinite.*

It is not necessarily true that if $p, q \in A$ are projections on a $C^*$-algebra with $p$ properly infinite and $p \precsim q$, then $q$ is properly infinite. For instance, one may take any unital, properly infinite $C^*$-algebra $A$, and any unital, finite $C^*$-algebra $B$ and consider the direct sum $A \oplus B$. Then $(1_A, 0) \precsim (1_A, 1_B)$, and the right-hand side is not properly infinite, while the left-hand side is. However, we do have the following result:

**Proposition 2.6.** *Let $p, q \in A$ be non-zero projections. Suppose that $q$ belongs to the closed two-sided ideal generated by $p$, and that $p \precsim q$. If $p$ is properly infinite, then so is $q$.*

*Proof.* Since $p$ lies in the ideal generated by the projection $q$, it is an easy exercise to verify that there exists $n \in \mathbb{N}$ such that $q \precsim p \otimes 1_n$, see e.g. [63, Exercise 4.8]. Since $p$ is properly infinite, we find that $p \otimes 1_n \precsim p$, such that $q \precsim p$, and using the subequivalence $p \precsim q$, we obtain

$$q \oplus q \precsim p \oplus p \precsim p \precsim q,$$

proving that $q$ is properly infinite. □

For convenience later on, we mention a few corollaries to the above result.

**Corollary 2.7.** *Let $p, q \in A$ be projections on a $C^*$-algebra. If $p \precsim q \precsim p$ and $q$ is properly infinite, then so is $p$.*

We say that a projection $p \in A$ is *full* if $p$ does not belong to any proper ideal of $A$.

**Corollary 2.8.** *Let $q, p \in A$ be projections. If $p \precsim q$ and $p$ is a properly infinite, full projection, then $q$ is properly infinite.*

*Proof.* Since $p$ is a full projection, $q$ trivially belongs to the ideal generated by $p$, and the result now follows from Proposition 2.6. □

**Corollary 2.9.** *Let $A$ be a unital $C^*$-algebra. If $p \in A$ is a full, properly infinite projection, then $A$ is a properly infinite $C^*$-algebra.*

*Proof.* This is just an application of Corollary 2.8 on the unit $1_A$. □

Let us look at a few permanence properties of being properly infinite.

**Proposition 2.10.** *Any quotient of a unital, properly infinite $C^*$-algebra is again properly infinite.*

*Proof.* This follows immediately from Proposition 2.4(ii). □

**Proposition 2.11.** *If $(A_n)_{n\in\mathbb{N}}$ is a sequence of unital $C^*$-algebras, then the product algebra $\ell^\infty((A_n)_{n\in\mathbb{N}})$ is properly infinite if and only if each $A_n$ is properly infinite.*

*Proof.* Suppose each $A_n$ is properly infinite, then so is the product $\ell^\infty((A_n)_{n\in\mathbb{N}})$ via any unital embedding $\mathcal{E}_2 \to A_m \to \ell^\infty((A_n)_{n\in\mathbb{N}})$. Conversely, if $\ell^\infty((A_n)_{n\in\mathbb{N}})$ is properly infinite, let $p = (p_n)_{n\in\mathbb{N}}$, $q = (q_n)_{n\in\mathbb{N}} \in \ell^\infty((A_n)_{n\in\mathbb{N}})$ be projections such that $p \perp q$ and $p \sim q \sim 1$. Then, for each $n \in \mathbb{N}$, $p_n \perp q_n$ and $p_n \sim q_n \sim 1_{A_n}$, and hence each $A_n$ is properly infinite. □

**Proposition 2.12.** *Let $A$ be the inductive limit of a sequence of unital $C^*$-algebras $A_1 \to A_2 \to \cdots$ with unital connecting maps. Then $A$ is properly infinite if and only if there exists $N \in \mathbb{N}$ such that $A_n$ is properly infinite for all $n \geq N$.*

*Proof.* If any $A_n$ is properly infinite, then so is $A$ via the unital *-homomorphisms $\mathcal{E}_2 \to A_n \to A$. Conversely, suppose that $A$ is properly infinite such that there exists an embedding $\mathcal{E}_2 \to A$. Since $\mathcal{E}_2$ is semiprojective by [5, Corollary 2.24], we find that there exists $N \in \mathbb{N}$ such that $\ell^\infty((A_n)_{n \geq N})$ is properly infinite. But then, for any $n \geq N$, $A_n$ is properly infinite by Proposition 2.11. □

The following proposition is a direct way of showing that if a $C^*$-algebra admits elements that approximately satisfy the Cuntz-Toeplitz relations sufficiently well, then it also contains elements that satisfy them exactly. This is in essence Loring's notion of stable relations, see [44, Chapter 14], but the below result is an explicit way of actually constructing the Cuntz-Toeplitz pair.

**Proposition 2.13.** *Let $A$ be a unital $C^*$-algebra and suppose that $s_1, s_2 \in A$ satisfies that $\|s_i^* s_i - 1_A\| < \frac{1}{4}$ for $i = 1, 2$ and $\|s_i^* s_j\| < \frac{1}{4}$ for $i \neq j$. Then there exists $t_1, t_2 \in A$ such that $t_i^* t_i = 1_A$ for $i = 1, 2$ and $t_1 t_1^* \perp t_2 t_2^*$. In particular, $A$ is properly infinite.*

*Proof.* Since $\|s_i^* s_i - 1_A\| < \frac{1}{4} < 1$, we have that $s_i^* s_i$ is invertible in $A$. Put $u_i = s_i (s_i^* s_i)^{-1/2}$ for $i = 1, 2$ and observe that these elements are isometries, that is, $u_i^* u_i = 1_A$ for $i = 1, 2$. Moreover, an easy calculation shows that $\left\| u_j u_j^* u_k u_k^* \right\| < 1$ whenever $j \neq k$, which implies that there exists a unitary element $v \in A$ such that $v u_j u_j^* v^* \perp u_k u_k^*$. Now put $t_1 = u_1$ and $t_2 = v u_2$, then $t_i^* t_i = 1_A$ for $i = 1, 2$ and

$$t_1 t_1^* t_2 t_2^* = v u_1 u_1^* v^* u_2 u_2^* = 0.$$

As $A$ contains two isometries with orthogonal range projections, we conclude that $A$ is properly infinite. □

## 2.2 Tracial states on $C^*$-algebras

In the following we let $A$ denote a not necessarily unital $C^*$-algebra. We say that a linear functional $\rho \colon A \to \mathbb{C}$ is a *state* if it is positive with $\|\rho\| = 1$. It is an easy calculation to show that if $A$ is unital and $\rho \colon A \to \mathbb{C}$ is a positive linear functional, then $\|\rho\| = 1$ if and only if $\rho(1_A) = 1$. The collection of states on $A$ is called the state space of $A$, here denoted $\mathrm{S}(A)$, and is a convex set, and compact whenever $A$ is unital. States on $C^*$-algebras are not rare in the slightest — in fact, an application of the Gelfand-Neimark theorem tells us that for any non-zero element $a \in A$, there exists a state $\rho$ on $A$ with $\rho(a) \neq 0$. Tracial states, however, are much rarer than states. As the name suggests, a tracial state on a $C^*$-algebra $A$ is a state $\tau$ on $A$ satisfying the tracial condition $\tau(ab) = \tau(ba)$ for all $a, b \in A$. The collection of tracial states on $A$ is denoted $\mathrm{T}(A)$ and is easily seen to be a closed convex subset of $\mathrm{S}(A)$. If $A$ is unital, $\mathrm{T}(A)$ is hence a compact, convex set; in fact, it is a Choquet simplex [6, Theorem II.6.8.11]. We shall often say that a $C^*$-algebra is *tracial* if it admits a tracial state. The tracial condition may be rewritten in several equivalent manners, as the following easy proposition shows.

**Proposition 2.14.** *Let $A$ be a unital $C^*$-algebra, and let $\tau$ be a state on $A$. The following are equivalent:*

(i) *$\tau$ is a tracial state, that is, for any $a, b \in A$, we have $\tau(ab) = \tau(ba)$.*

(ii) *For any $a \in A$, we have $\tau(a^* a) = \tau(a a^*)$.*

(iii) *For any positive element $a \in A$ and unitary $u \in A$, we have $\tau(u^*au) = \tau(a)$.*

Given a tracial state $\tau$ on $A$, we define the trace-kernel ideal $I_\tau = \{a \in A \mid \tau(a^*a) = 0\}$, which, as the name suggests, is an ideal in $A$. We say that a tracial state $\tau$ on $A$ is faithful if $\tau(a) \neq 0$ for all positive non-zero $a \in A$; this is equivalent to saying that the trace-kernel ideal is trivial, that is, $I_\tau = \{0\}$. While $\mathrm{S}(A)$ is always non-empty, this could not be further from the truth for $\mathrm{T}(A)$, and we shall see many examples of non-tracial $C^*$-algebras. Informally, the existence of a tracial state on a $C^*$-algebra is often said to be a finite property; however, one should be aware that this is only partially related to the notions of finiteness of $C^*$-algebras as defined in the previous section. For example, the Toeplitz algebra $\mathcal{T}$ as defined previously is an infinite $C^*$-algebra, but it admits a tracial state via the canonical quotient $\mathcal{T} \to C(\mathbb{T})$. The existence of an infinite projection is hence not an obstruction to having a tracial state; however, the existence of a full, properly infinite projection will inhibit admitting tracial states.

**Proposition 2.15.** *If a unital $C^*$-algebra $A$ admits a properly infinite, full projection, then $A$ cannot admit tracial states.*

*Proof.* If $A$ admits a properly infinite, full projection, then it follows from Corollary 2.9 that $A$ is properly infinite. Suppose that $\tau$ is a tracial state on $A$, and let $p, q \in A$ be non-zero projections with $p \perp q$ and $p \sim q \sim 1_A$. Then,

$$1 = \tau(1_A) \geq \tau(p + q) = \tau(p) + \tau(q) = \tau(1_A) + \tau(1_A) = 2,$$

which is an obvious contradiction. □

Without the assumption of fullness of the properly infinite projection, the result is false in general: If $A$ is a unital $C^*$-algebra with a tracial state and $B$ is a unital, properly infinite $C^*$-algebra, then the direct sum $A \oplus B$ will contain the tracial state arising from the quotient $A \oplus B \to A$, however the projection $(0, 1_B)$ is a properly infinite projection. If $A$ is any $C^*$-algebra with a properly infinite projection $p \in A$, then the proof of Proposition 2.15 implies that any tracial state will vanish on the properly infinite corner $pAp$ of $A$. Therefore, if a $C^*$-algebra admits a *faithful* tracial state, then we clearly cannot have properly infinite projections. However, the statement is even stronger, and the proof is just as easy.

**Proposition 2.16.** *If $A$ is a unital $C^*$-algebra with a faithful tracial state, then $A$ is stably finite.*

*Proof.* It is easily seen that the faithful tracial state $\tau$ on $A$ induces a faithful tracial state on $M_n(A)$ for all $n \in \mathbb{N}$, so it suffices to be seen that admitting a faithful tracial state implies finiteness. If $s \in A$ is an isometry, then $1_A - ss^* \geq 0$ is a positive element with

$$\tau(1_A - ss^*) = \tau(1_A) - \tau(ss^*) = \tau(1_A) - \tau(s^*s) = 0,$$

hence $ss^* = 1_A$, and $s$ is a unitary element. □

In Chapter 5, we look more closely at faithful tracial states (or, more generally, separating families of tracial states), and characterisations of when $C^*$-algebras admit faithful tracial states, and even when all quotients have faithful tracial states. In this chapter and the next, however, we are more interested in when $C^*$-algebras admit tracial states of some kind. In the following proposition, we look at some permanence properties of admitting tracial states.

**Proposition 2.17** (Permanence properties)**.** *Let $A$ be unital $C^*$-algebras in the following.*

(i) *If $I$ is an ideal in $A$, and $I$ admits a tracial state, then so does $A$.*

(ii) *If any quotient of $A$ admits a tracial state, then so does $A$.*

(iii) *If $A$,$B$ admit tracial states, then so does the minimal tensor product $A \otimes B$, and hence the full tensor product $A \otimes_{\max} B$.*

(iv) *If $A_1 \to A_2 \to \cdots$ is an inductive sequence of unital $C^*$-algebras with unital connecting maps and limit $A$, then $A$ admits a tracial state if and only if $A_n$ admits a tracial state for each $n \in \mathbb{N}$.*

The proofs are mostly obvious, but as the proof of (i) revolves around a construction we shall return to later on in the thesis, we shall briefly comment on it. The result is a generalisation of a similar well-known result for states [46, Theorem 3.3.9], and one may find a proof of this version in [66, Lemma 3.1]. The construction of the extension tracial state is fairly straightforward: Suppose that $\tau$ is a tracial state on $I$, let $(e_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for $I$, and consider the function $\tau'$ on $A$ defined by $\tau'(a) = \lim_\lambda \tau(e_\lambda a e_\lambda)$ for $a \in A$. Then $\tau'$ is clearly an extension of $\tau$, and one can verify that $\tau'$ is a tracial state. Indeed, it is the unique tracial state on $A$, which agrees with $\tau'$ on $I$ — we say that $\tau'$ is the *canonical extension* of $\tau$ to $A$.

From the tracial and linear properties of tracial states, it is immediate that $\tau([a, b]) = 0$ for any $a, b \in A$. Denote by $\overline{[A, A]}$ the norm-closure of the collection of all linear combinations of commutators in $A$. Since tracial states are assumed to be normalised, we hence find that if $A$ is a unital $C^*$-algebra with a tracial state, then $1_A \notin \overline{[A, A]}$. Using the Hahn-Banach separation theorem, it is possible to prove the converse.

**Proposition 2.18.** *A unital $C^*$-algebra $A$ admits a tracial state if and only if $1_A \notin \overline{[A, A]}$.*

*Proof.* Suppose that $1_A \notin \overline{[A, A]}$, then we may by the Hahn-Banach separation theorem find a self-adjoint, linear function $\tau\colon A \to \mathbb{C}$ such that $\tau(\overline{[A, A]}) = 0$ and $\tau(1_A) \neq 0$. Observe that $\tau$ is necessarily tracial, although it need not be a tracial state as it is not necessarily positive. By a Jordan decomposition similar to [52, Section 3.2] on the quotient space $A_{\mathrm{sa}}/A_0$ with $A_0$ being the norm-closed space of sums of self-adjoint commutators $[a^*, a]$ for $a \in A$, one may find positive, linear and tracial functions $\tau_+, \tau_-\colon A \to \mathbb{C}$ such that $\tau = \tau_+ - \tau_-$, see [17, Proposition 2.8]. After a possible normalisation of one of these traces, we hence obtain that $A$ admits a tracial state. □

By following the proof of Proposition 2.18 for some element in $A \setminus \overline{[A, A]}$, we get the following extension, which Ozawa also remarks in [51].

**Corollary 2.19.** *Let $A$ be a unital $C^*$-algebra, and assume that $a \in A$ satisfies that $a \notin \overline{[A, A]}$. Then there exists a tracial state $\tau$ on $A$ with $\tau(a) \neq 0$.*

It is natural to ask whether this result can be improved, e.g., can we for any non-tracial $C^*$-algebra express elements as *finite* sums of commutators? This is one part of the following proposition, which is a fusion of [54, Theorem 1] and [28, Lemma 2.1].

**Proposition 2.20** (Haagerup, Pop)**.** *Let $A$ be a unital $C^*$-algebra. The following are equivalent.*

(i) *$A$ admits no tracial state.*

(ii) *For any $0 < \delta < 1$ there exist $n \geq 2$ and $a_1, \dots, a_n \in A$ satisfying*

$$\sum_{i=1}^{n} a_i^* a_i = 1 \qquad \text{and} \qquad \left\| \sum_{i=1}^{n} a_i a_i^* \right\| \leq \delta.$$

(iii) *There exists $n \geq 2$ such that any element in $A$ can be written as a sum of $m$ commutators. Moreover, if $a$ is positive, then these commutators will be self-adjoint, that is, of the form $[b^*, b]$ for some $b \in A$.*

*Proof.* (i)⇒(ii): Suppose that $A$ admits no tracial state and consider the double dual $A^{**}$ of $A$. Note that $A$ embeds unitally in $A^{**}$, and that $A^{**}$ is a von Neumann algebra. In particular, $A^{**}$ cannot admit a tracial state, and as a von Neumann algebra either admits a tracial state or is properly infinite, we find that $A^{**}$ must be a properly infinite von Neumann algebra, and hence $\mathcal{O}_k$ embeds in $A^{**}$ for any $k \geq 2$. We can thus find isometries $(s_i)_{i=1}^k \subseteq A^{**}$ with mutually orthogonal range projections such that $\sum_{i=1}^k s_i s_i^* = 1$.

Now find a net $(a_i^{(\lambda)})_{\lambda \in \Lambda} \subseteq \bigoplus_{i=1}^k A_i$ converging to the $n$-tuple $(s_i)_{i=1}^k$ in the ultraweak topology. In particular,

$$\sum_{i=1}^k (a_i^{(\lambda)})^* a_i^{(\lambda)} \to \sum_{i=1}^k s_i^* s_i = k,$$
$$\sum_{i=1}^k a_i^{(\lambda)} (a_i^{(\lambda)})^* \to \sum_{i=1}^k s_i s_i^* = 1$$

in the ultraweak topology. Consequently,

$$(k, 1) \in \overline{\left\{ \left( \sum_{i=1}^n b_i^* b_i, \sum_{i=1}^n b_i b_i^* \right) \mid n \in \mathbb{N}, b_1, \ldots, b_n \in A \right\}}^{uw}.$$

Note that the pre-closed set on the right-hand side is purely expressed in terms of elements from $A$, and as the ultraweak topology on $A^{**}$ restricted to $A$ is just the weak topology, we can take the weak closure instead. However, as the set is also convex, and the weak and norm closures coincide on convex sets, we hence find that

$$(k, 1) \in \overline{\left\{ \left( \sum_{i=1}^n b_i^* b_i, \sum_{i=1}^n b_i b_i^* \right) \mid n \in \mathbb{N}, b_1, \ldots, b_n \in A \right\}}$$

in the norm topology. We can thus for any $\varepsilon > 0$ find elements $b_1, \ldots, b_n \in A$ such that

$$\left\| \sum_{i=1}^n b_i^* b_i - k \right\| < \varepsilon, \qquad \text{and} \qquad \left\| \sum_{i=1}^n b_i b_i^* - 1 \right\| < \varepsilon.$$

In particular, we obtain the inequalities

$$k - \varepsilon \leq \sum_{i=1}^n b_i^* b_i \leq k + \varepsilon,$$
$$1 - \varepsilon \leq \sum_{i=1}^n b_i b_i^* \leq 1 + \varepsilon.$$

By taking $0 < \varepsilon < 1$, we may assume that $\sum_{i=1}^n b_i^* b_i$ is invertible. Define the elements $a_i = b_i (\sum_{j=1}^n b_j^* b_j)^{-1/2}$ for $i = 1, \ldots, n$ and observe that $\sum_{i=1}^n a_i^* a_i = 1_A$, while on the other hand,

$$a_i a_i^* = b_i \left( \sum_{j=1}^n b_j^* b_j \right)^{-1} b_i^* \leq \left\| \left( \sum_{j=1}^n b_j^* b_j \right)^{-1} \right\| b_i b_i^* \leq \frac{1}{k - \varepsilon} b_i b_i^*,$$

such that

$$\left\|\sum_{i=1}^n a_i a_i^*\right\| \leq \frac{1}{k-\varepsilon}\left\|\sum_{i=1}^n b_i b_i^*\right\| \leq \frac{1+\varepsilon}{k-\varepsilon}.$$

By taking $k$ sufficiently large and $\varepsilon$ sufficiently small, we hence have found $a_1, \dots, a_n \in A$ such that $\sum_{i=1}^n a_i^* a_i = 1_A$ and $\|\sum_{i=1}^n a_i a_i^*\| \leq \delta$.

(ii)⇒(iii) Find $a_1, \dots, a_n \in A$ such that $\sum_{i=1}^n a_i^* a_i = 1_A$ and $\|\sum_{i=1}^n a_i a_i^*\| < 1$. Define the function $\varphi \colon A \to A$ by $\varphi(x) = \sum_{i=1}^n a_i x a_i^*$ for $x \in A$. Observe that $\|\varphi\| = \|\sum_{i=1}^n a_i a_i^*\| < 1$, and hence $\mathrm{id}_A - \varphi$ is an invertible element of the Banach algebra $\mathcal{B}(A)$ of bounded functions on $A$. Denote by $\psi = (\mathrm{id}_A - \varphi)^{-1}$ the inverse, then, for any $x \in A$,

$$x = (\mathrm{id}_A - \varphi) \circ \psi(x) = \psi(x) - \sum_{i=1}^n a_i \psi(x) a_i^* = \sum_{i=1}^n [a_i^* a_i \psi(a) - a_i \psi(x) a_i^*] = \sum_{i=1}^n [a_i^*, a_i \psi(x)].$$

(iii)⇒(i): This was shown in Proposition 2.18. □

**Remark 2.21.** The proof shows that the integer $n$ in (ii) works in (iii); however, it is not clear whether or not the converse holds. By following Pop's proof of the implication (i)⇒(ii) [54], one obtains that if every element, or even if just the unit $1_A$, can be expressed as the sum of $n$ elements, then one can find $n+1$ elements $a_1, \dots, a_{n+1}$ satisfying $\sum_{i=1}^{n+1} a_i^* a_i = 1_A$ and $\left\|\sum_{i=1}^{n+1} a_i a_i^*\right\| < 1$.

While the proof gives a very nice characterisation of when $C^*$-algebras admit tracial states, it is non-constructive in the sense that it does not give a way of finding the elements $a_1, \dots, a_n$ in (ii), nor does it give any way of determining the needed number of elements. In [54], Pop raised the question: Given a non-tracial $C^*$-algebra, what is the smallest number $n$ such that every element can be written as a sum of $n$ commutators? We shall, in Chapter 4, consider a related question; given a unital $C^*$-algebra $A$ and some $0 < \delta < 1$, what is the smallest integer $n$ such that property of Proposition 2.20(ii) above holds for some elements $a_1, \dots, a_n \in A$?

## 2.3 Cuntz semigroups, dimension functions and quasitraces

In the previous section, we discussed tracial states in depth; in this section, we generalise a bit further and ask whether or not one actually needs the full additivity assumption. While the question may seem somewhat arbitrary, it is a very deep question with links to many different areas of $C^*$-algebraic theory. We shall take the approach of defining the notion of quasitraces, and then show how they arise naturally by looking at the so-called Cuntz semigroups of unital $C^*$-algebras.

**Definition 2.22.** Let $A$ be a $C^*$-algebra. A 1-*quasitrace* on $A$ is a function $\tau \colon A_{\mathrm{sa}} \to \mathbb{R}$ satisfying the following:

(i) $\tau(a) \geq 0$ for all $a \geq 0$,

(ii) $\tau(\lambda a) = \lambda \tau(a)$ for all $a \in A_{\mathrm{sa}}$ and $\lambda \in \mathbb{R}$,

(iii) $\tau(a^* a) = \tau(a a^*)$ for all $a \in A$,

(iv) $\tau(a+b) = \tau(a) + \tau(b)$ for all $a, b \in A_{\mathrm{sa}}$ with $[a, b] = 0$.

If moreover $\tau$ extends to a function $\tilde{\tau} \colon M_n(A)_{\mathrm{sa}} \to \mathbb{R}$ satisfying (i)–(iv) via $\tilde{\tau}(a \otimes e_{11}) = \tau(a)$, then we call $\tau$ an $n$-*quasitrace*.

We shall always assume that quasitraces on unital $C^*$-algebras are unital, that is, $\tau(1_A) = 1$ for all $\tau \in \mathrm{QT}(A)$. It is not true that 1-quasitraces automatically are $n$-quasitraces for any $n \geq 2$ as shown by Kirchberg in the unpublished manuscript [40], but Blackadar–Handelman [7, Proposition II.4.1] showed that any 2-quasitrace is automatically an $n$-quasitrace for any $n \geq 2$. It is therefore customary to use the term quasitrace to refer to 2-quasitraces. Similarly to the case with tracial states, we shall occasionally say that a $C^*$-algebra is *quasitracial* if it admits a quasitrace. The definition of a quasitrace is formally weaker than that of a tracial state, but it is unknown whether or not they are actually the same, that is, whether or not additivity of non-commuting elements follows immediately from Definition 2.22. Tracial states and quasitraces have several similar properties, and the two properties below are generalisations of results that we have already seen to be true for tracial states.

**Proposition 2.23.** *Let $A$ be a unital $C^*$-algebra with a quasitrace $\tau$.*

(i) *The trace-kernel ideal $I_\tau = \{a \in A \mid \tau(a^*a) = 0\}$ is an ideal in $A$.*

(ii) *If $\tau$ is faithful, then $A$ is stably finite.*

Depending on the reader's philosophy of mathematics, they might be displeased with the definition of a quasitrace, because it seems as if one just removes a part of the assumption and asks *what happens now*? However, quasitraces actually appear in a quite natural manner, which we shall briefly explore in the sequel. First, we define a pair of Abelian semigroups arising from $C^*$-algebras, which are generalisations of the semigroup of Murray-von Neumann equivalence classes of projections.

Let $A$ be a separable, unital $C^*$-algebra. Let $M_\infty(A)$ be the *-algebra consisting of all finite-dimensional square matrices over entries in $A$, i.e., put $M_\infty(A) = \bigcup_{n\in\mathbb{N}} M_n(A)$. Consider the order $\preceq$ on $M_\infty(A)$ given by $a \preceq b$ if and only if there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $M_\infty(A)$ such that $v_n^* b v_n \to a$ as $n \to \infty$. We let $a \sim b$ if $a \preceq b$ and $b \preceq a$, and we say that $a$ is *Cuntz equivalent* to $B$, and we denote the equivalence class of $a$ by $\langle a \rangle$. Put $\mathrm{W}(A) = M_\infty(A)_+/{\sim}$ and equip it with the order $\langle a \rangle \leq \langle b \rangle$ if $a \preceq b$ and addition $\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$ such that $\mathrm{W}(A)$ is an ordered Abelian semigroup. We call $\mathrm{W}(A)$ the *pre-complete Cuntz semigroup*. If one applies the same construction to the stabilisation $A \otimes \mathbb{K}(H)$ instead of the pre-completion $M_\infty(A)$, we obtain the so-called *complete Cuntz semigroup*, which explicitly arises as the quotient $\mathrm{Cu}(A) = (A \otimes \mathbb{K}(H))/{\sim}$. For more information about Cuntz semigroups, we refer to e.g. [1], which also contains a lot of the results about regularity properties that we shall need later.

In general, Cuntz semigroups have bad regularity properties, and the algebraic and the order structure need not cooperate particularly well. However, whenever the $C^*$-algebra is suitably nice, we do obtain pleasant structures. In the following, we examine a few of these properties, and when they occur; we phrase the definitions in terms of ordered Abelian semigroups for the sake of generality. If $S$ is an ordered Abelian semigroup and $x, y \in S$, we write that $x <_s y$ if there exists $n \in \mathbb{N}$ such that $(n+1)x \leq ny$.

**Definition 2.24.** Let $S$ be an ordered Abelian semigroup, and let $n \in \mathbb{N}_0$. We say that $S$ has *$n$-comparison* if, for any $x, y_0, \ldots, y_n \in S$ satisfying $x <_s y_j$ for all $j = 0, \ldots, n$, we have $x \leq y_0 + \ldots + y_n$.

Clearly $n$-comparison implies $m$-comparison for $n \leq m$. Observe that 0-comparison is equivalent to the property that $(n+1)x \leq ny$ for some $n \in \mathbb{N}$ implies $x \leq y$, which is also known as *almost unperforation*. If $A$ is a $C^*$-algebra, then $\mathrm{W}(A)$ is almost unperforated if and only if $\mathrm{Cu}(A)$ is almost unperforated [1], and it is easily seen that almost unperforation passes to ideals and quotients. For our purposes, almost unperforation of the Cuntz semigroup is a desirable regularity property to have, because it gives a dichotomy of properly infiniteness: A unital $C^*$-algebra with an almost unperforated Cuntz semigroup is either properly infinite or stably not properly infinite.

**Proposition 2.25.** *Let $A$ be a unital $C^*$-algebra and suppose that $\mathrm{Cu}(A)$ is almost unperforated. If $M_n(A)$ is properly infinite for some $n \in \mathbb{N}$, then $A$ is properly infinite.*

*Proof.* Suppose that $M_n(A)$ is properly infinite for some $n \in \mathbb{N}$, then $(n+1)\langle 1_A\rangle \le 2n\langle 1_A\rangle \le n\langle 1_A\rangle$. By induction, one obtains that $(n+k)\langle 1_A\rangle \le n\langle 1_A\rangle$ for all $k \in \mathbb{N}$, and putting $k = n+2$ gives us $2(n+1)\langle 1_A\rangle \le n\langle 1_A\rangle$. Almost unperforation of $\mathrm{Cu}(A)$ implies that $2\langle 1_A\rangle \le \langle 1_A\rangle$, i.e., $1_A \oplus 1_A \precsim 1_A$, and hence $A$ is properly infinite. □

Since $\mathrm{Cu}(A)$ is almost unperforated whenever $A$ is $\mathcal{Z}$-stable by [61, Theorem 4.5], we obtain the following result

**Proposition 2.26.** *A unital $\mathcal{Z}$-stable $C^*$-algebra is either properly infinite or stably not properly infinite.*

As we shall see later in the thesis, the existence of a quasitrace on a given unital $C^*$-algebra is equivalent to being stably not properly infinite, cf. Theorem 2.32. Hence, we obtain that a unital $\mathcal{Z}$-stable $C^*$-algebra admits a quasitrace if and only if it is a properly infinite $C^*$-algebra.

A related, but weaker, regularity property of Cuntz semigroups is that of $\omega$-comparison; here we use the relation $\ll$ defined as follows: We say $s \ll t$ if, whenever $t \le \sup_n t_n$ for an increasing sequence $(t_n)_{n\in\mathbb{N}}$, there exists $n_0 \in \mathbb{N}$ such that $s \le t_{n_0}$.

**Definition 2.27.** A complete ordered Abelian semigroup $S$ has $\omega$-comparison if, for any $x, x', y_0, y_1, \ldots \in S$ with $x <_s y_j$ and $x' \ll x$, there exists $n \in \mathbb{N}$ such that $x' \le y_0 + \ldots + y_n$.

It is clear that $n$-comparison for any $n \in \mathbb{N}$ implies $\omega$-comparison. Many nice $C^*$-algebras have $\omega$-comparison, as the following result due to Robert [56] shows.

**Proposition 2.28** (Robert, 2011)**.** *If $A$ is a unital $C^*$-algebra with nuclear dimension $n$, then the Cuntz semigroup $\mathrm{Cu}(A)$ has $n$-comparison. More generally, if $A$ has nuclear dimension $\omega$, then $\mathrm{Cu}(A)$ has $\omega$-comparison.*

We refrain from defining nuclear dimension $n$ (or $\omega$) and instead refer to [56, Definition 1]. These comparison properties will be of use later on in the thesis, when we discuss under what conditions $C^*$-algebras have the property that all quotients admit faithful tracial states.

Having mentioned some regularity properties that Cuntz semigroups may possess, we now return to how quasitraces arise naturally from certain states on Cuntz semigroups.

**Definition 2.29.** Let $S$ be an ordered Abelian semigroup. A *state* on $S$ is an order-preserving map $f\colon S \to \mathbb{R}$. The collection of states on $S$ is denoted $\Sigma(S)$. If $t \in S$, then we denote by $\Sigma(S,t)$ the collection of states $f \in \Sigma(S)$ satisfying $f(t) = 1$.

**Definition 2.30.** Let $A$ be a unital $C^*$-algebra. By a *dimension function* on $A$, we mean a state $d$ on $\mathrm{Cu}(A)$ satisfying $d(\langle 1_A\rangle) = 1$.

Dimension functions are hence states on Cuntz semigroups with the obvious normalisation assumption. For each positive element $a \in M_\infty(A)_+$ and $\varepsilon > 0$, we define the $\varepsilon$-cutoff of $a$ by $(a-\varepsilon)_+ = f_\varepsilon(a)$, where the right-hand side is defined by applying the continuous functional calculus to the function $f_\varepsilon(t) = \max\{0, t-\varepsilon\}$. A dimension function $d$ on $A$ is *lower semi-continuous* if, for all $a \in M_\infty(A)_+$, $d(\langle (a-\varepsilon)_+\rangle) \to d(\langle a\rangle)$ as $\varepsilon \to 0$. We denote by

DF($A$) the dimension functions on $A$, and by LDF($A$) the lower semi-continuous dimension functions on $A$. A result due to Rørdam [58, Proposition 4.1] states that for any dimension function $d$ on $A$, there exists a lower semi-continuous dimension function $d'$ on $A$ such that $d(\langle p\rangle) = d'(\langle p\rangle)$ for all projections $p \in \mathcal{P}_\infty(A)$; and, in fact, one might realise $d'$ by the formula $d'(\langle a\rangle) = \lim_{\varepsilon\to 0} d(\langle (a-\varepsilon)_+\rangle)$. Given a quasitrace on $\tau$ on a $C^*$-algebra $A$, one can define a lower semi-continuous dimension function $d_\tau$ on $A$ via $d_\tau(\langle a\rangle) = \lim_{n\to\infty} \tau(a^{1/n})$. This correspondence is actually an equivalence as shown by Blackadar–Handelman [7, Theorem II.2.2].

**Theorem 2.31** (Blackadar–Handelman, 1982)**.** *Let $A$ be any $C^*$-algebra. Then there exists an affine bijection between* QT($A$) *and* LDF($A$).

The way one can obtain a quasitrace from a lower semi-continuous dimension function is not as straight-forward as the opposite direction, but it does give some insight in to why quasitraces are, a priori, only additive on commuting elements. We sketch the construction and refer to the original article [7] for a more rigorous treatment. Suppose that $A$ is a unital $C^*$-algebra with a lower semi-continuous dimension function $d$, and let $C_0(X)$ be some Abelian $C^*$-subalgebra of $A$. Then the dimension function $d$ defines a finitely additive measure $\mu$ on $X$ via $\mu(U) = d(f)$, where $f \in C_0(X)$ is any function for which the set-theoretical support supp($f$) is exactly $U$. The measure $\mu$ can be extended to a countably additive Borel measure on $X$, and the induced map $\tau\colon C_0(X)_+ \to \mathbb{R}$ by $\tau(f) = \int_X f \,\mathrm{d}\mu$, which defines a tracial state on $C_0(X)$, provides a quasitrace on $A$.

The above sketch, while without many details, shows that quasitraces on Abelian $C^*$-subalgebras correspond to measures; this should remind the reader of the one-to-one correspondence between $C_0(X)^*$ and the space of Radon measures on $X$. Moreover, it is obvious from this construction that quasitraces are additive on commuting elements, however it is not immediate that the extension preserves linearity of the integral.

Quasitraces are therefore not as mysterious as they might have seen when first introduced. In fact, in some cases quasitraces might be of more interest in structure theory of $C^*$-algebra, as we shall see in the following theorem. Recall from Proposition 2.15 that unital, properly infinite $C^*$-algebras cannot admit tracial states. The converse is still unresolved, but if one looks at quasitraces, we do have an equivalence.

**Theorem 2.32** (Cuntz, Blackadar–Handelman, Blackadar–Rørdam)**.** *Let $A$ be a unital $C^*$-algebra. The following are equivalent.*

(i) *$A$ admits a quasitrace.*

(ii) *$A$ has a stably finite quotient.*

(iii) *$A$ is stably not properly infinite.*

(iv) *There exists a state on* $\mathrm{K}_0(A)$.

*Proof.* (i)⇒(ii): If $A$ admits a quasitrace $\tau$, then the trace-kernel ideal $I_\tau = \{a \in A \mid \tau(a^*a) = 0\}$ is an ideal in $A$, and there exists a unique faithful quasitrace $\tilde{\tau}$ on the quotient $A/I_\tau$ defined by $\tilde{\tau} \circ \pi = \tau$, where $\pi\colon A \to A/I_\tau$ is the canonical quotient map. Since $A/I_\tau$ admits a faithful quasitrace, it is a stably finite $C^*$-algebra.

(ii)⇒(iii): Suppose that $M_n(A)$ is properly infinite for some $n \in \mathbb{N}$. Then all quotients of $M_n(A)$ will be properly infinite by Proposition 2.4, and hence $M_n(A/I)$ will be properly infinite for any ideal $I$ in $A$ by Proposition 2.10. Consequently, $A$ has no stably finite quotients.

(iii)⇒(iv): Assume that $K_0(A)$ admits no states. Let $u = [1_A]_0$ be the equivalence class of the unit $1_A$ in $K_0(A)$. First we prove the existence of $k, \ell \in \mathbb{N}$ with $k > \ell$ and $ku \leq \ell u$. Indeed, suppose that $ku \leq \ell u$ implies $k \leq \ell$. Using [26, Theorem 3.2], the function $f\colon \mathbb{Z}u \to \mathbb{R}$ by $f(nu) = n$ for $n \in \mathbb{Z}$ extends to a state on $K_0(A)$, which is a contradiction with our assumption. Therefore, there exists some $k, \ell \in \mathbb{N}$ with $k > \ell$ and $ku \leq \ell u$.

Let in the following $e_n \in M_n(A)$ be the unit in $M_n(A)$. Since $[e_n]_0 = nu$ for any $n \in \mathbb{N}$, it is immediate that the inequality $[e_k]_0 \leq [e_\ell]_0$ holds. Find $m \in \mathbb{N}$ such that $e_k \oplus e_m \precsim e_\ell \oplus e_m$ and put $n = \ell + m$ and $d = k - \ell > 0$. Then $e_{n+d} \precsim e_n$, and an iterative process will show that $e_n$ is properly infinite. Indeed, for any $r \in \mathbb{N}$, we have

$$e_{n+rd} \sim e_{n+d} \oplus e_{(r-1)d} \precsim e_n \oplus e_{(r-1)d} \sim e_{n+(r-1)d} \precsim \cdots \precsim e_n.$$

Choosing $r \in \mathbb{N}$ such that $rd \geq n$ then gives us

$$e_n \oplus e_n \sim e_{n+n} \precsim e_{n+rd} \precsim e_n,$$

which shows that $M_n(A)$ is properly infinite.

(iv)⇒(i): This result follows from the result due to Blackadar–Rørdam [9, Theorem 3.3] that any state $\varphi$ on $K_0(A)$ arises from a quasitrace $\tau$ on $A$ such that $\varphi([p]_0) = \tau(p)$ for all projections $p \in \mathcal{P}_\infty(A)$. We sketch the proof below.

Let $V(A)$ denote the subsemigroup of the (pre-completed) Cuntz semigroup $W(A)$ consisting of all equivalence classes of projections in $M_\infty(A)$. Since the order arising from the Cuntz subequivalences is a generalisation of the Murray-von Neumann subequivalences on projections, we find that $V(A)$ admits the same algebraic order as $W(A)$, and that they have the same order unit. Let $\varphi\colon K_0(A) \to \mathbb{C}$ be a state and observe it induces a state $\tilde{\varphi}$ on $V(A)$ through the map $\langle p \rangle \mapsto [p]_0$. By [9, Corollary 2.7], $\tilde{\varphi}$ extends to a dimension function $d$ on $W(A)$. Now we note that any dimension function on $W(A)$ agrees with a *lower semi-continuous* dimension function $\tilde{d}$ on projections, and that any lower semi-continuous dimension function comes from a quasitrace $\tau \in QT(A)$ by Theorem 2.31. It thus follows that $\varphi([p]_0) = \tau(p)$ for all $p \in \mathcal{P}_\infty(A)$, and this proves that all states on $K_0(A)$ arises from quasitraces. □

The equivalences of (i)⇔(ii)⇔(iii) can be seen as a generalisation of the following result for simple $C^*$-algebras from Theorem 2.5.

**Corollary 2.33** (Cuntz, 1978)**.** *A simple, unital $C^*$-algebra $A$ is stably finite if and only if it admits a quasitrace.*

As it is a curious result, we explicitly mention the following result from the proof of the implication (iii)⇒(iv) in Theorem 2.32.

**Proposition 2.34.** *Let $A$ be a unital $C^*$-algebra, and let $\langle 1_A \rangle$ denote the equivalence class of the unit $1_A$ in $\mathrm{Cu}(A)$. Then $A$ is stably not properly infinite if and only if the function $f\colon \mathbb{N}_0 \langle 1_A \rangle \to [0, \infty)$ given by $f(n \langle 1_A \rangle) = n$ is order-preserving.*

One way of informally looking at the equivalence of (i)⇔(iii) in Theorem 2.32 is by viewing the size of matrix algebras necessary to achieve properly infiniteness as a measure of the failure to admit a quasitrace. This way of thinking about the equivalence will be of use to us in Chapter 4, when we shall define a numerical invariant measuring exactly the size of matrix algebras needed in order to obtain properly infiniteness of the unit.

We have several times throughout the thesis alluded to the question of when quasitraces are tracial states. This is known as Kaplansky's conjecture [37], and, as we have also mentioned prior in this thesis, it is a deep and still unresolved problem. In the following section, we shall take a closer look at the original conjecture, and how it is equivalent to the more modern version as expressed below.

**Conjecture 2.35** (Kaplansky)**.** All quasitraces are tracial states.

As of writing, the best known answer to Kaplansky's conjecture is the following theorem due to Haagerup [28]. We shall in the following chapter give an overview of the original proof, with an emphasis on the usage of the AW$^*$-completions of unital $C^*$-algebras with quasitraces, as well as provide an alternative proof using the fact that $C_r^*(\mathbb{F}_\infty)$ is an MF-algebra [30].

**Theorem 2.36** (Haagerup)**.** *Any quasitrace on a unital, exact $C^*$-algebra is a tracial state.*

Combining this result with Theorem 2.32, we hence obtain the result that whenever $A$ is a unital, exact and stably not properly infinite $C^*$-algebra, then $A$ admits a tracial state. This fact, though often used in the stably finite formulation, provides an abstract way of guaranteeing the existence of tracial states. Moreover, in [11], Kirchberg generalised Haagerup's result to the non-unital case.

# 3 The original proof of Haagerup's theorem

In this chapter, we shall look at Haagerup's paper [28] in which he proves that Kaplansky's conjecture holds true for exact, unital $C^*$-algebra. Our focus will, of course, be on the specific result, but special interest will also be on the theory of AW$^*$-algebras, which are highly connected to the study of quasitraces. The chapter is essentially in two parts, both firmly based around [28] — first, we look at how one can mimic certain W$^*$-algebraic methods in order to construct an AW$^*$-algebraic completion of $C^*$-algebras with (faithful) quasitraces. This is also of great interest independently of the rest of the paper, since it will illuminate how Kaplansky's conjecture is really a conjecture about the structure of II$_1$-AW$^*$-algebraic factors. Secondly, we shall use these AW$^*$-completions in order to prove Haagerup's theorem, and here the goal is primarily to showcase how the proof works, and where precisely one needs exactness in order to proceed. The reader should also be aware that we in Chapter 4 will provide an alternative proof of Haagerup's theorem, which heavily uses the machinery introduced in Section 3.1 below along with a deep result about $C_r^*(\mathbb{F}_\infty)$ being an MF-algebra due to Haagerup–Thorbjørnsen [30].

## 3.1 AW$^*$-algebras and their connections to quasitraces

The main objects of interest in Haagerup's proof are the AW$^*$-algebras. Historically, Kaplansky introduced them in [37] as an attempt to abstractly axiomatise von Neumann algebras intrinsically without a choice of a representation. The naming convention also reflects this — von Neumann algebras are often referred to as W$^*$-algebras, and the A in AW$^*$ suggestively stands for "abstract". The axiomatisation attempt failed, however, and there are many known non-von Neumann algebraic AW$^*$-algebras. However, the study of AW$^*$-algebras, while not as fashionable, is still an interesting area, and it is very tightly connected to the study of tracial states and quasitraces, as we shall see in the following section.

**Definition 3.1.** A unital $C^*$-algebra $A$ is an AW$^*$-algebra if it satisfies the following two conditions:

(i) Every maximal Abelian $C^*$-subalgebra of $A$ is generated by its projections.

(ii) Any family of mutually orthogonal projections in $A$ has a least upper bound projection in $A$.

The properties are very von Neumann algebraic, and it is clear that any von Neumann algebra will be an AW$^*$-algebra; however, the converse is not generally true. Any student of von Neumann algebras should immediately recognize the following classification of AW$^*$-factors. As is the case for von Neumann algebras, we say that an AW$^*$-algebra is a factor if its center is trivial.

**Theorem 3.2.** *Any* AW$^*$*-factor belongs to one (and only one) of the following types:*

(i) I$_n$ *if it contains a minimal projection and is of dimension* $n$ *(i.e., if it is* $M_n(\mathbb{C})$*).*

(ii) I$_\infty$ *if it contains a minimal projection and is infinite-dimensional (i.e., if it is* $\mathbb{B}(H)$ *for* $H$ *an infinite-dimensional Hilbert space).*

(iii) II$_1$ *if it contains finite non-zero projections and the unit is finite.*

(iv) II$_\infty$ *if it contains finite non-zero projections and the unit is infinite.*

(v) III$_\infty$ *if it contains no non-zero finite projections.*

It is clear that $\mathrm{I}_n$ and $\mathrm{I}_\infty$ AW*-factors are von Neumann algebras, since they are just $C^*$-algebras of the form $\mathbb{B}(H)$ for some Hilbert space $H$. Moreover, it is known that there exist AW*-factors of type $\mathrm{II}_\infty$ and $\mathrm{III}_\infty$, which are not von Neumann algebras, see for instance [65]. The case for $\mathrm{II}_1$ is unresolved and is actually the original conjecture due to Kaplansky [37].

**Conjecture 3.3** (Kaplansky)**.** All $\mathrm{II}_1$-AW*-factors are von Neumann algebras.

The relation to the more well-known version of Kaplansky's conjecture follows from the following result.

**Proposition 3.4.** *A unital* $\mathrm{II}_1$-AW*-*factor* $\mathcal{M}$ *admits a* unique *quasitrace, which is a tracial state if and only if* $\mathcal{M}$ *is a von Neumann algebra.*

For the first statement, we refer to [7], and for the second statement we refer to [74, Corollary 7]. It is immediate that if all quasitraces are tracial states, then the original version of Kaplansky's conjecture holds true. That the converse is true requires a bit more work, but one can for instance show it by invoking AW*-completions of $C^*$-algebras in a suitable manner, as we shall see later. The connection between quasitraces and AW*-algebras is further exemplified by the following factorisation result [7, Theorem I.4.1].

**Proposition 3.5** (Blackadar-Handelman)**.** *For any quasitrace* $\tau$ *on a unital* $C^*$*-algebra* $A$*, there exist a finite* AW*-*algebra* $\mathcal{M}$ *with a quasitrace* $\tilde{\tau}$ *and a unital *-homomorphism* $\varphi\colon A \to \mathcal{M}$ *such that* $\tau = \varphi \circ \tilde{\tau}$*.*

If $\tau$ is a tracial state (or, in general, a quasitrace) on a $C^*$-algebra $A$, we define the 2-norm $\|\cdot\|_{2,\tau}$ on $A$ by $\|a\|_{2,\tau} = \tau(a^*a)$ for $a \in A$. If $\tau$ is a tracial state, then the 2-norm is a seminorm, and it is a proper norm if and only if $\tau$ is faithful. The 2-norm is not an unknown structure to work with in W*-algebraic theory. For instance, suppose that $A$ is a unital $C^*$-algebra with a tracial state $\tau$ and define the tracial ultrapower $A^\omega$ of $A$ by

$$A^\omega = \ell^\infty(A)/I_\tau^\omega$$

with $I_\tau^\omega = \{(a_n)_{n\in\mathbb{N}} \in \ell^\infty(A) \mid \lim_{n\to\omega} \|a_n\|_{2,\tau} = 0\}$ being the trace-kernel ideal of the induced tracial state $\tau_\omega((a_n)_{n\in\mathbb{N}}) = \lim_{n\to\omega} \tau(a_n)$ on $\ell^\infty(A)$. Then $A^\omega$ will be a finite von Neuman algebra with a tracial state $\tau^\omega$ , and if $\tau$ is an extremal tracial state, then $A^\omega$ will be a $\mathrm{II}_1$-von Neumann algebraic factor. In the following we shall construct an AW*-analogue of this phenomenon and use it to obtain a canonical way of taking AW*-completions of unital $C^*$-algebras with (faithful) quasitraces.

Note that if $\tau$ is a quasitrace, then the induced 2-norm may not satisfy the triangle inequality; at least, it is not a priori the case. To combat this, one can take a sufficient power of the 2-norm and obtain a (semi)metric instead.

**Definition 3.6.** Let $\tau$ be a quasitrace on a unital $C^*$-algebra $A$. We denote by $d_\tau$ the semimetric on $A$ given by $d_\tau(x, y) = \|x - y\|_2^{2/3}$.

We skip the proof that this is a semimetric, and in fact a proper metric when $\tau$ is a faithful quasitrace, and refer to [28, Lemma 3.5]. The following results will show us that working with faithful and extremal quasitraces are beneficial; however, as we shall see later, these limitations will have no impact on our results in the end via the quotient of the trace-kernel ideal and by using Krein-Milman's theorem, [24, Theorem II.1.8].

**Proposition 3.7** (Lemma 3.8–3.11 in [28])**.** *Let* $\tau$ *be a faithful quasitrace on a unital* $C^*$*-algebra* $A$*. Then:*

(i) *The unit ball $(A)_1$ is closed in $d_\tau$.*

(ii) *The unit ball $(A)_1$ is complete in $d_\tau$ if and only if $A$ is a finite* AW*-*algebra and $\tau$ is a normal quasitrace.*

(iii) *If $B \subseteq A$ is a unital $C^*$-subalgebra, then $B$ is dense in $A$ in the $d_\tau$-metric if and only if the unit ball $(B)_1$ is dense in $(A)_1$ in the $d_\tau$-metric.*

We shall not prove the statements here, but refer to [28, Lemma 3.8–3.11]. Combining the results, we obtain:

**Proposition 3.8.** *Let $\mathcal{M}$ be a finite* AW*-*algebra with a faithful normal quasitrace $\tau$, and let $A$ be a $C^*$-subalgebra of $\mathcal{M}$. Then the $d_\tau$-closure of $A$ in $\mathcal{M}$ is the smallest* AW*-*algebra in $\mathcal{M}$ containing $A$.*

*Proof.* Denote by $\overline{A}^{d_\tau}$ the $d_\tau$-closure of $A$, which is a unital $C^*$-algebra, whose unit ball is the $d_\tau$-closure of the unit ball of $A$ by the $d_\tau$-version of Kaplansky's density theorem, Proposition 3.7(iii). In particular, the unit ball of $\overline{A}^{d_\tau}$ must be complete, since it is a closed subset of the complete unit ball in $\mathcal{M}$ by Proposition 3.7(ii). This proves that $\overline{A}^{d_\tau}$ is an $AW^*$-algebra, and a calculation shows that, whenever $p_\lambda \in \mathcal{P}(\overline{A}^{d_\tau})$ is a set of mutually orthogonal projections on $\overline{A}^{d_\tau}$, then the supremum $\bigvee_{\lambda\in\Lambda} p_i$ as taken in $\mathcal{M}$ belongs to $\overline{A}^{d_\tau}$ again, and hence $\overline{A}^{d_\tau}$ is an AW*-subalgebra of $\mathcal{M}$. To show that it is the smallest AW*-subalgebra of $\mathcal{M}$ containing $A$, suppose that $B$ is another AW*-subalgebra of $\mathcal{M}$ containing $A$. Then the unit ball of $B$ must be complete in the $d_\tau$-metric, and hence $B$ must be $d_\tau$-closed by Proposition 3.7, such that $\overline{A}^{d_\tau} \subseteq B$. ☐

Note the dependence of having some unital embedding in a finite AW*-algebra to begin with. We shall resolve this later on by finding a very concrete embedding into an AW*-algebra, which gives rise to a definition of the AW*-algebraic completion of unital $C^*$-algebras with (faithful, extremal) quasitraces. First, let us mention this ultraproduct construction of an embedding in an AW*-factor, which mimics how, if $A$ is a unital $C^*$-algebra with a tracial state $\tau \in \mathrm{T}(A)$, that the tracial ultrapower $A^\omega$ is a finite von Neumann algebra with a tracial state $\tau^\omega$. We only sketch the AW*-algebraic proof below.

**Proposition 3.9.** *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of unital $C^*$-algebras with quasitraces $\tau_n \in \mathrm{QT}(A_n)$ for $n \in \mathbb{N}$ and define*

$$\mathcal{I}_\omega = \left\{ (a_n)_{n\in\mathbb{N}} \in \ell^\infty((A_n)_{n\in\mathbb{N}}) \mid \lim_{n\to\omega} \tau_n(a_n^* a_n) = 0 \right\}.$$

*This is a closed two-sided ideal in $\ell^\infty((A_n)_{n\in\mathbb{N}})$, and the quotient $\ell^\infty((A_n)_{n\in\mathbb{N}})/\mathcal{I}_\omega$ is a finite* AW*-*algebra with a faithful normal quasitrace $\tau_\omega$ given by*

$$\tau_\omega([(a_n)_{n\in\mathbb{N}}]) = \lim_\omega \tau_n(a_n), \qquad (a_n)_{n\in\mathbb{N}} \in \ell^\infty((A_n)_{n\in\mathbb{N}}).$$

*Proof.* By taking the quotient $A_n \to A_n/I_{\tau_n}$, we may assume that each $\tau_n$ is a faithful quasitrace, and hence that $d_{\tau_n}$ is a metric. It is clear by construction that $\ell^\infty((A_n)_{n\in\mathbb{N}})/\mathcal{I}_\omega$ is a $C^*$-algebra with a faithful quasitrace $\tau_\omega$ as defined above. It thus follows from Proposition 3.7 that we only need to verify that the closed unit ball is closed with respect to the metric $d_{\tau_\omega}$; however, the metric $d_{\tau_\omega}$ arises as the ultralimit of the metrics $d_{\tau_n}$, and the proof now only relies on the fact that the ultraproduct of uniformly bounded, complete metric spaces is again a complete metric space, cf. [28, Lemma 4.1]. ☐

By taking $A_n = A$ and $\tau_n = \tau$ for some unital $C^*$-algebra $A$ with a quasitrace $\tau$, we hence obtain a trace-preserving, unital embedding of $A$ into a finite AW$^*$-algebra. One might now be tempted to use Proposition 3.8 on this embedding to define the AW$^*$-completion, but note that the construction is inherently dependent on the ultrafilter, whereas the AW$^*$-algebraic completion should be only dependent on the $C^*$-algebra and the quasitrace. However, this completion actually has a very concrete construction, which should remind the reader of constructing $\mathbb{R}$ by taking the Cauchy completion of $\mathbb{Q}$.

**Proposition 3.10.** *Let $(A, \tau)$ be a unital $C^*$-algebra with a faithful quasitrace $\tau$. Define*

$$\begin{aligned}\tilde{A} &= \left\{ (a_n)_{n\in\mathbb{N}} \in \ell^\infty(A) \mid (a_n)_{n\in\mathbb{N}} \text{ is } d_\tau\text{-Cauchy} \right\},\\ \tilde{I} &= \left\{ (a_n)_{n\in\mathbb{N}} \in \ell^\infty(A) \mid a_n \xrightarrow{d_\tau} 0 \right\},\\ \tilde{\tau}([(a_n)_{n\in\mathbb{N}}]_{\tilde{I}}) &= \lim_{n\to\infty} \tau(a_n).\end{aligned}$$

*Then $\tilde{I}$ is a closed two-sided ideal in $\tilde{A}$ and $\tilde{\tau}$ is a faithful, normal quasitrace on the finite* AW$^*$*-algebra $\tilde{A}/\tilde{I}$. In fact, if $(A, \tau)$ embeds into a finite* AW$^*$*-algebra $(\mathcal{M}, \tau')$ with a faithful normal quasitrace $\tau'$ in a unital and trace-preserving way, then $\tilde{A}/\tilde{I}$ is isomorphic to the smallest* AW$^*$*-algebra of $\mathcal{M}$ contained in $A$ (i.e., $\tilde{A}/\tilde{I} \cong \overline{A}^{d_{\tau'}}$). Moreover, if $\tau$ is extremal, then $\tilde{A}/\tilde{I}$ is a* II$_1$-AW$^*$*-factor.*

*Proof.* By Proposition 3.9, we may find a finite AW$^*$-algebra $\mathcal{M}$ with a faithful normal quasitrace $\tau'$ such that $A$ embeds unitally in $\mathcal{M}$ in a trace-preserving manner. Consider the closure $\overline{A}^{d_{\tau'}}$ of $A$ in $\mathcal{M}$ with respect to the metric $d_{\tau'}$ induced by the quasitrace $\tau'$. This is by Proposition 3.8 the smallest AW$^*$-subalgebra of $\mathcal{M}$ containing $A$. By the AW$^*$-version of Kaplansky's density theorem, Proposition 3.7(iii), the unit ball $(\overline{A}^{d_{\tau'}})_t$ is complete in $d_{\tau'}$ for all $t > 0$. Thus, for any $t > 0$, we can find a *-homomorphism $(\tilde{A})_t \to ((\overline{A}^{d_{\tau'}})_t$, which sends a Cauchy sequence to its $d_{\tau'}$-limit. By extending this *-homomorphism to $\tilde{A} \to \overline{A}^{d_{\tau'}}$, we hence obtain a surjective map with kernel $\tilde{I}$, and therefore $\overline{A}^{d_{\tau'}} = \tilde{A}/\tilde{I}$, which proves the first part of the proposition. We now only need to verify that $\tilde{A}/\tilde{I}$ is a II$_1$-AW$^*$-factor whenever $\tau$ is extremal. Indeed, if $\tilde{A}/\tilde{I}$ is not a factor, it contains a non-trivial central projection $p$. Define the maps $\tau_1, \tau_2 \colon (\tilde{A}/\tilde{I})_{\mathrm{sa}} \to \mathbb{R}$ by

$$\tau_1(x) = \tau'(px) \qquad \text{and} \qquad \tau_2(x) = \tau'((1-p)x),$$

for $x \in A_{\mathrm{sa}}$, where we have implicitly used the inclusion $A \hookrightarrow \tilde{A}/\tilde{I}$. Then $\tau = \tau_1 + \tau_2$, and after normalising we obtain that $\tau$ is a non-trivial convex combination of quasitraces, and thus $\tau$ is not extremal. ☐

The above proposition gives us a way of defining the *AW$^*$*-completion of a unital $C^*$-algebra with a faithful quasitrace in a manner which is only dependent on this information, e.g., it is independent of the choice of free ultrafilter seen in Proposition 3.9.

**Definition 3.11.** Let $A$ be a unital $C^*$-algebra with a faithful quasitrace $\tau$ and define $\tilde{A}$ and $\tilde{I}$ as in Proposition 3.10. We call the *AW$^*$*-algebra $\tilde{A}/\tilde{I}$ the AW$^*$*-completion* of $A$.

The AW$^*$-completion is a useful tool for resolving Kaplansky's conjecture, as the question of when quasitraces are actually tracial states is true for all unital $C^*$-algebras if and only if it holds for II$_1$-AW$^*$-factors.

**Proposition 3.12.** *Conjecture 2.35 is true for unital $C^*$-algebras if and only if it holds for all* II$_1$-AW$^*$*-factors.*

*Proof.* Suppose that all quasitraces are tracial states on $\mathrm{II}_1$-AW*-factors. Let $A$ be a unital $C^*$-algebra and $\tau \in \mathrm{QT}(A)$ be a quasitrace. Since $\mathrm{QT}(A)$ is a compact, convex set, we may approximate $\tau$ by convex combinations of extremal quasitraces by the Krein-Milman theorem [24, Theorem II.1.8]. If we were to show that all extremal quasitraces are tracial states, we would be done. So we may assume without loss of generality that $\tau$ is an extreme point in $\mathrm{QT}(A)$. Moreover, by taking the quotient of $A$ with respect to the trace-kernel ideal $I_\tau$, we may assume that $\tau$ is faithful. Since $\tau$ is a faithful, extremal quasitrace, the AW*-completion $\mathcal{M}$ of $A$ is a $\mathrm{II}_1$-AW*-factor with a unique quasitrace $\tau_0$ by Proposition 3.10, which hence must be induced by $\tau$. Since all quasitraces on $\mathrm{II}_1$-AW*-factors are tracial states, $\tau_0$ is a tracial state, hence so is $\tau$. □

### 3.2 Proving Haagerup's theorem

Having constructed the AW*-completion for unital $C^*$-algebras with faithful quasitraces above, we now proceed to proving Haagerup's theorem. Our goal is to use Proposition 3.12 and verify that, whenever $A$ is an exact, unital $C^*$-algebra with a faithful and extremal quasitrace $\tau$, then the AW*-completion $\mathcal{M}$ will admit a tracial state, and since it admits a unique quasitrace, which is furthermore an extension of $\tau$, it will necessarily be the tracial state, hence $\tau$ is a tracial state. However, it is by no means trivial to show that the $\mathrm{II}_1$-AW*-completion of $(A, \tau)$ has this property, so there is still some work to be done.

Our first goal is to show that if $A$ admits no tracial state, then $A \otimes C^*_r(\mathbb{F}_\infty)$ admits a proper isometry; in fact, as we shall see in Corollary 4.36, $A \otimes C^*_r(\mathbb{F}_\infty)$ will be properly infinite.

Let $H$ be an infinite-dimensional Hilbert space with an orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Consider the Fock space $\mathcal{F}(H) = \bigoplus_{n\in\mathbb{N}} H^{\otimes n}$, where $H^{\otimes n} = \mathbb{C}$ for $n = 0$ for notational purposes. Consider now, for any $n \in \mathbb{N}$, the creation operator $s_n$ on $\mathcal{F}(H)$ via $s_n(x) = e_n \otimes x$ for $x \in \mathcal{F}(H)$. Then $(s_n)_{n\in\mathbb{N}}$ is a family of isometries with orthogonal range projections, and $1 - \sum_{n\in\mathbb{N}} s_n s_n^*$ is the projection onto $H^{\otimes 0} \cong \mathbb{C}$. Define for each $n \in \mathbb{N}$ the element $x_n = \frac{1}{2}(s_n + s_n^*)$ and consider the $C^*$-algebra $\mathcal{V}_\infty = C^*(x_n \mid n \in \mathbb{N})$. Using Voiculescu's theory of semicircular systems, see [71], we may realise $\mathcal{V}_\infty$ as a free product $*_{n\in\mathbb{N}}(C([-1,1]), \tau)$, where $\tau$ is the tracial state on $C([-1,1])$ arising from the semicircular distribution on $[-1,1]$, that is,

$$\tau(f) = \int_{-1}^{1} \frac{2}{\pi}\sqrt{1-t^2} f(t)\,\mathrm{d}t$$

for any $f \in C([-1,1])$.

It is immediate that $\mathcal{V}_\infty$ embeds unitally in $\mathcal{O}_\infty$. Interestingly, $\mathcal{V}_\infty$ also embeds unitally in $C^*_r(\mathbb{F}_\infty)$:

**Proposition 3.13.** *There exists a unital, injective *-homomorphism $\mathcal{V}_\infty \to C^*_r(\mathbb{F}_\infty)$.*

*Proof.* Let $(u_n)_{n\in\mathbb{N}}$ be the canonical unitary generators of $C^*_r(\mathbb{F}_\infty)$. In a manner reminiscent of constructing $\mathcal{V}_\infty$, we define for each $n \in \mathbb{N}$ the elements $y_n = \frac{1}{2}(u_n + u_n^*)$. Expressed within Voiculescu's theory of free systems, the unital $C^*$-algebra $C^*(y_n \mid n \in \mathbb{N})$ is a free system $*_{n\in\mathbb{N}}(C([-1,1], \tau')$ with $\tau'$ being the tracial state on $C([-1,1])$ arising from the uniform circular distribution on $[-1,1]$, that is,

$$\tau'(f) = \int_{-1}^{1} \frac{1}{\pi\sqrt{1-t^2}} f(t)\,\mathrm{d}t$$

for $f \in C([-1,1])$. Observe that the distribution on $[-1,1]$ is not the same as the one in the free product representation of $\mathcal{V}_\infty$, which needs to be remedied. Let $f(t) = \frac{2}{\pi}\sqrt{1-t^2}$

and $g(t) = \frac{1}{\pi\sqrt{1-t^2}}$ be the probability density functions of the tracial state on $\mathcal{V}_\infty$ and of $\tau'$, respectively. Denote by $F$ and $G$ their antiderivatives and consider the homeomorphism $\Phi = F^{-1} \circ G$ on $[-1, 1]$. Then defining $z_n = \Phi(y_n)$ obtains the correct measure, and we thus find that $\mathcal{V}_\infty \cong C^*(z_n \mid n \in \mathbb{N})$ unitally embeds in $C^*_r(\mathbb{F}_\infty)$. □

Having proved that $\mathcal{V}_\infty$ embeds unitally in $C^*_r(\mathbb{F}_\infty)$, we now show that whenever a unital $C^*$-algebra $A$ admits no tracial states, then $A \otimes C^*_r(\mathbb{F}_\infty)$ is an infinite $C^*$-algebra.

**Proposition 3.14.** *If $A$ is a unital $C^*$-algebra with no tracial states, then $A \otimes \mathcal{V}_\infty$, and hence $A \otimes C^*_r(\mathbb{F}_\infty)$, contains a proper isometry.*

*Proof.* Since $A$ has no tracial state, we can find elements $a_1, \ldots, a_n \in A$ satisfying the Haagerup condition $\sum_{i=1}^n a_i^* a_i = 1$ and $\|\sum_{i=1}^n a_i a_i^*\| < 1$. With $(s_i)_{i=1}^\infty$ a sequence of isometries with mutually orthogonal range projections, consider the element $y = \sum_{i=1}^n a_i \otimes (s_i + s_i^*)$ in $A \otimes \mathcal{V}_\infty$. Decompose $y$ as $y = v + w$ with $v = \sum_{i=1}^n a_i \otimes s_i$ and $w = \sum_{i=1}^n a_i \otimes s_i^*$ with $v, w \in A \otimes \mathcal{O}_\infty$. It is immediate that $v$ is an isometry, and since

$$vv^* \leq 1_A \otimes \sum_{i=1}^n s_i s_i^*,$$

and $\sum_{i=1}^n s_i s_i^* < 1$, we obtain that $v$ is a proper isometry in $A \otimes \mathcal{O}_\infty$. In particular, we find that $y = (1 + wv^*)v$; we aim to prove that $y^*y$ is invertible, yet that $y$ is non-invertible.

Firstly, since $\|wv^*\| < 1$, it is immediate that $1 + wv^*$ is invertible, and since $v$ is a non-invertible isometry, we find that $y$ is non-invertible. Moreover, by the Haagerup condition on the family $(a_i)_{i=1}^n$ and as $s_i^* s_j = \delta_{ij}$, we find that

$$\|w\|^2 = \|ww^*\| = \left\| \sum_{i=1}^n a_i a_i^* \otimes 1_{\mathcal{O}_\infty} \right\| < 1$$

such that $\|w\| < 1$. Consider some faithful, unital *-representation of $A \otimes \mathcal{O}_\infty$ on $\mathbb{B}(H)$ for a Hilbert space $H$, then for any $\xi \in H$, we have

$$\|y\xi\| = \|(v + w)\xi\| \geq \|v\xi\| - \|w\xi\| \geq (1 - \|w\|)\|\xi\| > 0.$$

which implies that $y^*y$ is an injective, self-adjoint operator, and hence $y^*y$ is invertible. This implies that we may find an isometry $u = y(y^*y)^{-1/2} \in A \otimes \mathcal{V}_\infty$, which is the isometry arising from the polar decomposition of $y = u\,|y|$, which is not unitary by non-invertibility of $y$. □

**Remark 3.15.** As mentioned in the introduction to this section, this result can be improved to stating that $A \otimes C^*_r(\mathbb{F}_\infty)$ is properly infinite — in fact, $A$ admits no tracial state if and only if $A \otimes C^*_r(\mathbb{F}_\infty)$ is properly infinite, as we shall see in Corollary 4.36.

Now suppose $A$ is an exact, unital $C^*$-algebra and consider once again the tensor product $A \otimes C^*_r(\mathbb{F}_\infty)$. We aim to use the exactness assumption and some properties of $C^*_r(\mathbb{F}_\infty)$ to embed $A \otimes C^*_r(\mathbb{F}_\infty)$ in a finite AW*-algebra. We shall use the following fact about $C^*_r(\mathbb{F}_\infty)$ due to Wassermann [72].

**Theorem 3.16** (Wassermann, 1976)**.** *There exists an embedding $C^*_r(\mathbb{F}_\infty) \hookrightarrow \mathcal{M}^\omega$, where*

$$\mathcal{M}^\omega = \ell^\infty((M_n(\mathbb{C}))_{n \in \mathbb{N}})/J_\omega,$$

*where $J_\omega$ is the trace-kernel ideal on $\ell^\infty((M_n(\mathbb{C}))_{n\in\mathbb{N}})$ coming from the ultraproduct tracial state $\mathrm{tr}_\omega = \lim_\omega \mathrm{tr}_n$.*

Observe that $\mathcal{M}^\omega$ as defined above is a $\mathrm{II}_1$-von Neumann algebraic factor.

**Proposition 3.17.** *Let $A$ be a unital,* exact *$C^*$-algebra with a faithful quasitrace $\tau$ and let $\omega$ be a free ultrafilter. Then there exists a finite $\mathrm{AW}^*$-algebra $(\mathcal{N}, \tau')$ with a faithful normal quasitrace $\tau'$ such that $A \otimes \mathcal{M}^\omega$ embeds unitally in $\mathcal{N}$ and such that the quasitraces satisfy*

$$\tau'(a \otimes 1) = \tau(a), \qquad \text{for all } a \in A,$$
$$\tau'(1_A \otimes x) = \mathrm{tr}_\omega(x), \qquad \text{for all } x \in \mathcal{M}^\omega.$$

*In fact, $\mathcal{N}$ will be given by the (quasi)tracial ultraproduct $\ell^\infty((M_n(A))_{n\in\mathbb{N}})/I_{\tau,\omega}$ with $I_{\tau,\omega}$ being the trace-kernel ideal of the induced quasitrace $\tau' = (\tau_n)_{n\in\mathbb{N}}$ in $\ell^\infty((M_n(A))_{n\in\mathbb{N}})$.*

*Proof.* As this proposition is the only instance in Haagerup's paper where exactness of $A$ is required, we sketch the proof. Consider $\mathcal{N} = \ell^\infty((M_n(A))_{n\in\mathbb{N}})/I_{\tau,\omega}$, which is a finite $\mathrm{AW}^*$-algebra with a faithful quasitrace $\tau_\omega$, see Proposition 3.9. One can verify the constructions of two trace-preserving, unital embeddings $\pi\colon A \to \mathcal{N}$ and $\rho\colon \mathcal{M}^\omega \to \mathcal{N}$ given by

$$\pi(a) = [(a \otimes 1_n)_{n\in\mathbb{N}}], \qquad a \in A$$
$$\rho([(x_n)_{n\in\mathbb{N}}]) = [(1 \otimes x_n)_{n\in\mathbb{N}}], \qquad (x_n)_{n\in\mathbb{N}} \in \ell^\infty((M_n(\mathbb{C}))_{n\in\mathbb{N}}).$$

Since $\pi(A)$ and $\rho(\mathcal{M}^\omega)$ have commuting ranges by construction, we can define an embedding $\pi \otimes \rho\colon A \odot \mathcal{M}^\omega \to \mathcal{N}$ of the algebraic tensor product. Our goal is to extend this to an embedding $A \otimes \mathcal{M}^\omega \to \mathcal{N}$ of the spatial tensor product. In order to achieve this, we need to show that the norm induced by the map $\pi \otimes \rho$ is, in fact, the minimal tensor norm. Denote this norm by $\beta$, that is, let $\beta$ be defined by

$$\beta\left(\sum_{i=1}^{\ell} a_i \otimes z_i\right) = \left\|\sum_{i=1}^{\ell} \pi(a_i)\rho(z_i)\right\|, \qquad a_i \in A, z_i \in \mathcal{M}^\omega.$$

It is immediate that this is a $C^*$-seminorm, and one can verify that it is a proper $C^*$-norm. A calculation shows that there exists a *-homomorphism $\varphi\colon A \otimes \ell^\infty((M_n(\mathbb{C}))_{n\in\mathbb{N}}) \to \mathcal{N}$ satisfying

$$\varphi\left(\sum_{i=1}^{\ell} a_i \otimes y_i\right) = \sum_{i=1}^{\ell} \pi(a_i)\rho([y_i]), \qquad a_i \in A, y_i \in \ell^\infty((M_n(\mathbb{C}))_{n\in\mathbb{N}}).$$

Observe that $\beta(a\otimes[z]) = \|\varphi(a \otimes z)\|$ for all $a \in A$, $z \in \ell^\infty((M_n(\mathbb{C}))_{n\in\mathbb{N}})$. Another calculation shows that $A \otimes J_\omega \subseteq \ker\varphi$, and hence one obtains that the norm $\beta$ is bounded by the norm on $A \odot \mathcal{M}^\omega$ coming from the quotient $A \otimes \mathcal{M}^\omega / A \otimes J_\omega$. However, by exactness of $A$, we have an isomorphism

$$\frac{A \otimes \mathcal{M}^\omega}{A \otimes J_\omega} \cong A \otimes \mathcal{M}^\omega / J_\omega,$$

and the norm induced by the right-hand side on $A \odot \mathcal{M}^\omega$ is exactly the minimal norm. Therefore, $\beta$ is the minimal tensor norm, and hence $\pi \otimes \rho$ extends to an embedding of $A \otimes \mathcal{M}^\omega \to \mathcal{N}$. $\square$

Since $C_r^*(\mathbb{F}_\infty)$ embeds unitally in $\mathcal{M}^\omega$, we thus obtain that $A \otimes C_r^*(\mathbb{F}_\infty)$ embeds in a finite $\mathrm{AW}^*$-factor. However, this will just give us that $A$ must admit a tracial state if it admits a quasitrace, and not that a given quasitrace is necessarily a tracial state. Instead, we want to use the fact that if $\tau$ is an extremal, faithful quasitrace on $A$, then the $\mathrm{AW}^*$-completion $\mathcal{M}_\tau$ is a $\mathrm{II}_1$-$\mathrm{AW}^*$-algebra with a *unique* quasitrace, which is hence the one induced from $\tau$. In other words, we want to apply Proposition 3.14 not to $A$ but to $\mathcal{M}_\tau$. For this, we require one last embedding, the proof of which we omit, although we will emphasise that the result uses exactness and simplicity of $C_r^*(\mathbb{F}_\infty)$, see [14, Proposition 5.1.8] and [55, Theorem 2].

**Proposition 3.18.** *Let $A$ be a unital, exact $C^*$-algebra with a faithful quasitrace $\tau$ and let $(\mathcal{M}, \tilde{\tau})$ be its* AW*-*completion. Then $\mathcal{M}_\tau \otimes C_r^*(\mathbb{F}_\infty)$ embeds into a finite* AW*-*algebra.*

We are now able to combine the results into the following theorem.

**Theorem 3.19** (Haagerup, 1991)**.** *Any quasitrace on a unital, exact $C^*$-algebra is a tracial state.*

*Proof.* Let $A$ be a unital, exact $C^*$-algebra, and let $\tau$ be a quasitrace on $A$. As in the proof of Proposition 3.12, we may assume that $\tau$ is faithful and extremal. Consider the AW*-algebraic completion $(\mathcal{M}_\tau, \tau')$ of $(A, \tau)$, which is a finite $\mathrm{II}_1$-$\mathrm{AW}^\infty$-factor due to faithfulness and extremality of $\tau$. Being a $\mathrm{II}_1$-AW*-factor, $\mathcal{M}_\tau$ has a unique quasitrace, which hence must be $\tau'$. In particular, $\tau'$ cannot be a tracial state, since then $\tau$ would be a tracial state. Consequently, using Proposition 3.14, we obtain that $\mathcal{M}_\tau \otimes C_r^*(\mathbb{F}_\infty)$ contains a non-unitary isometry. But this is contradictory to the fact that $\mathcal{M}_\tau \otimes C_r^*(\mathbb{F}_\infty)$ embeds in a finite AW*-algebra. We conclude that $\tau$ must be a tracial state, and this completes the proof. □

Kirchberg later generalised this result to the non-unital case [39]. Let us also mention the following corollary to Theorem 3.19, which gives a larger class of $\mathrm{II}_1$-AW*-algebraic factors which are von Neumann algebras, the proof of which boils down to proving that the unique quasitrace is, in fact, a tracial state.

**Corollary 3.20.** *Suppose that $\mathcal{M}$ is an* $\mathrm{II}_1$-AW*-*algebraic factor, which arises as the* AW*-*completion of some exact, unital $C^*$-algebra. Then $\mathcal{M}$ is a von Neumann algebra.*

# 4 Measuring (quasi)traciality via numerical invariants

In this chapter, we shall introduce a few new numerical invariants that, informally stated, measure the non-(quasi)traciality of unital $C^*$-algebras. Our goal is to study these invariants in depth, and to construct a unital $C^*$-algebra, which admits a quasitrace, but which arises as an ultraproduct of $C^*$-algebras without quasitraces. At the end of the chapter, we provide a new and, perhaps, more intuitive proof of Haagerup's theorem of quasitraces on unital, exact $C^*$-algebras being tracial states; however, we also show that the method cannot extend to non-exact $C^*$-algebras. This chapter is based on joint work with my advisor, Mikael Rørdam, which is yet to be written up as a paper.

## 4.1 Introducing the numerical invariants and elementary results

The equivalences in Theorem 2.32 and in Proposition 2.20 give us some numerical data to quantify the failure to admit quasitraces resp. tracial states — for quasitraces, how large a matrix algebra is required to obtain a properly infinite unit, and for tracial states, how large an $n$-tuple is required in Proposition 2.20(ii)? While obviously these quantities do not measure failure to admit quasitraces resp. tracial states in any rigorous manner, seeing as it is a binary statement, it does give some intuition into how the following numerical invariants should be understood.

**Definition 4.1.** Let $A$ be a unital $C^*$-algebra, and let $0 < \delta < 1$. We define

- $\mu(A)$ to be the smallest integer $n \in \mathbb{N}$ such that $M_n(A)$ is properly infinite. If no such $n$ exists, we put $\mu(A) = \infty$.
- $\nu_\delta(A)$ to be the smallest integer $n \in \mathbb{N}$ such that there exists $a_1, \dots, a_n \in A$ satisfying

$$\sum_{i=1}^n a_i^* a_i = 1_A \qquad \text{and} \qquad \left\| \sum_{i=1}^n a_i a_i^* \right\| \leq \delta.$$

  If no such $n$ exists, we put $\nu_\delta(A) = \infty$.
- $\gamma(A)$ to be the smallest integer $n \in \mathbb{N}$ such that every element in $A$ can be expressed as a sum of $n$ commutators. If no such $n$ exists, we put $\gamma(A) = \infty$.

The following proposition is a reformulation of Proposition 2.20 and Theorem 2.32.

**Proposition 4.2.** *Let $A$ be a unital $C^*$-algebra. Then $A$ admits a quasitrace if and only if $\mu(A) = \infty$, and $A$ admits a tracial state if and only if $\nu_\delta(A) = \infty$ for some (hence all) $0 < \delta < 1$, which occurs if and only if $\gamma(A) = \infty$.*

At this point it should be remarked that it is not obvious that $\mu(-)$ and $\nu_\delta(-)$ can attain any non-trivial values, or indeed if they can attain infinitely many values. Later on in the thesis, we will show that $\mu(-)$ can attain all values $n \in \mathbb{N}$, but the question for $\nu_\delta(-)$ is more difficult. Before that, however, let us note the following easy remarks about these numerical quantities. For the next proposition, we remind the reader that we for any $x \in \mathbb{R}$ denote by $\lceil x \rceil$ the ceiling-function applied to $x$, that is, $\lceil x \rceil$ is the smallest integer $n \geq x$.

**Proposition 4.3.** *For any $0 < \delta < 1$ and any unital $C^*$-algebra $A$, we have the inequality $\nu_\delta(A) \geq \lceil \frac{1}{\delta} \rceil$.*

*Proof.* The proposition is clearly true whenever $A$ admits a tracial state, so suppose that $\nu_\delta(A) = n$ and find $a_1, \dots, a_n \in A$ such that $\sum_{i=1}^n a_i^* a_i = 1_A$ and $\|\sum_{i=1}^n a_i a_i^*\| \leq \delta$. Then, for any $i = 1, \dots, n$, we have

$$\|a_i\|^2 = \|a_i a_i^*\| \leq \left\| \sum_{i=1}^n a_i a_i^* \right\| \leq \delta.$$

The triangle inequality then implies that

$$1 = \left\| \sum_{i=1}^{n} a_i^* a_i \right\| \leq \sum_{i=1}^{n} \|a_i^* a_i\| = \sum_{i=1}^{n} \|a_i\|^2 \leq n\delta,$$

which implies that $n \geq \frac{1}{\delta}$. □

It turns out that the lower bound is attained for properly infinite $C^*$-algebras.

**Proposition 4.4.** *If $A$ is properly infinite, then $\nu_\delta(A) = \lceil \frac{1}{\delta} \rceil$.*

*Proof.* All we need to show is that $\nu_\delta(A) \leq \lceil \frac{1}{\delta} \rceil$. Put $n = \lceil \frac{1}{\delta} \rceil$ and since $A$ is properly infinite, we may find $n$ isometries $s_1, \dots, s_n \in A$ with mutually orthogonal range projections. Let $a_i = \frac{1}{\sqrt{n}} s_i$, then $\sum_{i=1}^{n} a_i^* a_i = 1_A$ and

$$\left\| \sum_{i=1}^{n} a_i a_i^* \right\| \leq \frac{1}{n} \leq \delta.$$

□

In the case of $\delta = \frac{1}{2}$, we actually obtain an equivalence.

**Proposition 4.5.** *A unital $C^*$-algebra is properly infinite if and only if $\nu_{1/2}(A) = 2$.*

*Proof.* Suppose that $\nu_{1/2}(A) = 2$ and let $a_1, a_2 \in A$ be elements satisfying

$$a_1^* a_1 + a_2^* a_2 = 2 \qquad \text{and} \qquad \|a_1 a_1^* + a_2 a_2^*\| \leq 1.$$

In particular, $a_1 a_1^* + a_2 a_2^* \leq 1$, hence both $a_1$ and $a_2$ are contractions, and thus the first condition implies that they are isometries. Since $a_1 a_1^* + a_2 a_2^* \leq 1$, we obtain that $a_1 a_1^* \perp a_2 a_2^*$. We have hence shown that $a_1, a_2$ are isometries with orthogonal range projections, and $A$ is thus properly infinite. □

**Proposition 4.6.** *For any unital $C^*$-algebra, and any $n \in \mathbb{N}$ and $0 < \delta < 1$, we have $\nu_\delta(A) \leq \nu_\delta(M_n(A))n^2$. In particular, $\nu_\delta(A) \leq \mu(A)^2 \lceil \frac{1}{\delta} \rceil$.*

*Proof.* Let $a_1, \dots, a_k$ be elements in the matrix algebra $M_n(A)$ satisfying $\sum_{i=1}^{k} a_i^* a_i = 1_{M_n(A)}$ and $\left\| \sum_{i=1}^{k} a_i a_i^* \right\| \leq \delta$. By taking the sum of the diagonal elements of $a_i^* a_i$ and $a_i a_i^*$, it is easily seen that, for $b_i = \frac{1}{\sqrt{n}} a_i$,

$$\sum_{\ell=1}^{k} \sum_{i,j=1}^{n} b_\ell(i,j) = 1_A, \qquad \text{and} \qquad \left\| \sum_{\ell=1}^{k} \sum_{i,j=1}^{n} b_\ell(i,j) \right\| \leq \delta,$$

proving the inequality $\nu_\delta(A) \leq \nu_\delta(M_n(A))n^2$. For the remaining claim, note that $\mu(A) = n$ implies that $\nu_\delta(M_n(A)) = \lceil \frac{1}{\delta} \rceil$ by Proposition 4.4. □

The above proposition gives us the result that $\nu_\delta(A) = \infty$ implies that $\mu(A) = \infty$, which is a sensible reality check, since it just states that whenever a unital $C^*$-algebra admits a tracial state, then it trivially also admits a quasitrace. The proposition also gives an upper bound on $\nu_\delta(A)$ for any given $C^*$-algebra $A$. Given a unital $C^*$-algebra $A$, it is by Proposition 4.3 not possible to uniformly bound $\nu_\delta(A)$ from above for all $0 < \delta < 1$, but a reading of Pop's separate proof of Proposition 2.20, see [54, Theorem 1], gives us the following bound for what could be called the minimal $\nu$-value.

**Proposition 4.7.** *Let $A$ be a unital $C^*$-algebra. For any $0 < \delta < 1$, we have $\gamma(A) \leq \nu_\delta(A)$. Moreover, we have $\min_{0<\delta<1} \nu_\delta(A) \leq \gamma(A) + 1$.*

In the following, we consider a few permanence properties of $\mu(-)$ and $\nu_\delta(-)$.

**Proposition 4.8** (Permanence properties)**.** *Let $A$ and $B$ be unital $C^*$-algebras. Then:*

(i) $\mu(A \oplus B) = \max\{\mu(A), \mu(B)\}$.

(ii) *If $\varphi\colon A \to B$ is a unital *-homomorphism, then $\mu(\varphi(A)) \geq \mu(B)$. In particular, if $A \subseteq B$ is a $C^*$-subalgebra, then $\mu(B) \leq \mu(A)$.*

(iii) *If $I$ is a closed two-sided ideal in $A$, then $\mu(A/I) \leq \mu(A)$.*

(iv) *If $p \in A$ is a non-zero full projection, then $\mu(pAp) \geq \mu(A)$.*

(v) *If $n \in \mathbb{N}$, then $\mu(M_n(A)) = \lceil \mu(A)/n \rceil$.*

(vi) *If $A_1 \to A_2 \to \cdots$ is a sequence of unital $C^*$-algebras with unital connecting maps, then $\mu(\lim_{\to} A_n) = \limsup_{n\to\infty} \mu(A_n)$.*

(vii) *If $(A_n)_{n\in\mathbb{N}}$ is a sequence of unital $C^*$-algebras, then $\mu(\ell^\infty((A_n)_{n\in\mathbb{N}})) = \sup_{n\in\mathbb{N}} \mu(A_n)$.*

(viii) *If $\omega$ is a free ultrafilter, then $\limsup_{n\to\omega} \mu(A_n) = \mu(\ell_\omega((A_n)_{n\in\mathbb{N}}))$ for any sequence $(A_n)_{n\in\mathbb{N}}$ of unital $C^*$-algebras.*

*Proof.* The statements (i) and (v) are obvious.

(ii): A *-homomorphism $\varphi\colon A \to B$ canonically induces a *-homomorphism $\varphi^{(n)}\colon M_n(A) \to M_n(B)$ on the matrix amplifications. Applying Proposition 2.10 gives the desired result.

(iii): This is an easy application of (ii).

(iv): This follows immediately from Corollary 2.9.

(vi): Put $A = \lim_{\to} A_n$ and suppose that $\mu(A) = k$. Since $M_k(A_1) \to M_k(A_2) \to \cdots$ is an inductive system with limit $M_k(A)$, it follows from Proposition 2.12 that since $M_k(A)$ is properly infinite, then there exists $n_0 \in \mathbb{N}$ such that $M_k(A_n)$ is properly infinite for all $n \geq n_0$. In particular, $\mu(A_n) \leq \mu(A)$ for all $n \geq n_0$ and, hence, $\limsup_{n\to\infty} \mu(A_n) \leq \mu(A)$. Conversely, if $\limsup_{n\to\infty} \mu(A_n) = k$, then there exists $N \in \mathbb{N}$ such that $\mu(A_n) \leq k$ for all $n \geq N$. Once again realising $M_k(A)$ as the limit of the inductive sequence $M_k(A_1) \to M_k(A_2) \to \cdots$ allows us to use Proposition 2.12 yet again to obtain the desired inequality $\mu(A) \leq k$.

(vii): If $\mu(\ell^\infty((A_n)_{n\in\mathbb{N}})) = k$, then $\ell^\infty((M_k(A_n))_{n\in\mathbb{N}}) = M_k(\ell^\infty((A_n)_{n\in\mathbb{N}}))$ is properly infinite, and Proposition 2.11 implies that $M_k(A_n)$ is properly infinite for all $n \in \mathbb{N}$, i.e., $\sup_{n\in\mathbb{N}} \mu(A_n) \leq k$. For the other direction, note that if $\sup_{n\in\mathbb{N}} \mu(A_n) = k$, then the $C^*$-algebra $M_k(\ell^\infty((A_n)_{n\in\mathbb{N}})) = \ell^\infty((M_k(A_n))_{n\in\mathbb{N}})$ is properly infinite, hence $\mu(\ell^\infty((A_n)_{n\in\mathbb{N}})) \leq k$.

(viii): Suppose that $\mu(\ell_\omega((A_n)_{n\in\mathbb{N}})) = k$, then $M_k(\ell_\omega((A_n)_{n\in\mathbb{N}}))$ is properly infinite, and hence so is the quotient $M_k(\ell^\infty((A_n)_{n\in\mathbb{N}}))/M_k(c_\omega((A_n)_{n\in\mathbb{N}}))$ by Proposition 2.4(vii), where $c_\omega((A_n)_{n\in\mathbb{N}}) = \{(a_n)_{n\in\mathbb{N}} \in \ell^\infty((A_n)_{n\in\mathbb{N}}) \mid \lim_{n\to\omega} \|a_n\| = 0\}$. By properly infiniteness, we can find isometries $t_i \in M_k(\ell^\infty((A_n)_{n\in\mathbb{N}}))/M_k(c_\omega((A_n)_{n\in\mathbb{N}}))$ for $i = 1, 2$ with orthogonal range projections. Let $s_i(n) \in M_k(A_n)$ be lifts of $t_i$ with $\|s_i(n)\| \leq \|t_i\|$ for $i = 1, 2$ and any $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\omega} \|s_i(n)^* s_i(n) - 1\| = 0,$$
$$\limsup_{n\to\omega} \|s_1(n)s_1(n)^* s_2(n)s_2(n)^*\| = 0.$$

Consequently, we may for any $\delta > 0$ find $I_\delta \in \omega$ for whic

$$\|s_i(n)^* s_i(n) - 1\| < \delta,$$
$$\|s_1(n)s_1(n)^* s_2(n)s_2(n)^*\| < \delta,$$

for each $n \in I_\delta$. By taking $0 < \delta < \frac{1}{4}$, we may invoke Proposition 2.13 to see that $M_k(A_n)$ is properly infinite. Therefore $\mu(A_n) \leq k$ for all $n \in I_\delta$ and

$$\limsup_{n\to\omega} \mu(A_n) = \inf_{J\in\omega} \sup_{n\in J} \mu(A_n) \leq \sup_{n\in I_\delta} \mu(A_n) \leq k = \mu(\ell_\omega((A_n)_{n\in\mathbb{N}})).$$

For the other inequality, suppose that $\limsup_{n\to\omega} \mu(A_n) = k$ and find $I \in \omega$ such that $\mu(A_n) \leq k$ for all $n \in I$. For $i = 1, 2$ and $n \in I$, we let $s_i(n) \in M_k(A_n)$ be isometries with orthogonal range projections witnessing the properly infiniteness of each $A_n$, and let $s_i(n) = 0$ for $n \notin I$. Let $t_i \in M_k(\ell_\omega((A_n)_{n\in\mathbb{N}}))$ be the images of the sequences $(s_i(n))_{n\in\mathbb{N}} \in \ell^\infty((A_n)_{n\in\mathbb{N}})$, then $t_1$,$t_2$ are also isometries with orthogonal ranges, and hence $M_k(\ell_\omega((A_n)_{n\in\mathbb{N}}))$ is properly infinite, i.e., $\mu(\ell_\omega((A_n)_{n\in\mathbb{N}})) \leq k$. This completes the proof. □

**Proposition 4.9** (Permanence properties)**.** *Let $A$ and $B$ be unital $C^*$-algebras and let $0 < \delta < 1$. Then:*

(i) $\nu_\delta(A \oplus B) = \max\{\nu_\delta(A), \nu_\delta(B)\}$.

(ii) *If $\varphi\colon A \to B$ is a unital *-homomorphism, then $\nu_\delta(\varphi(A)) \geq \nu_\delta(B)$. In particular, if $A \subseteq B$ is a $C^*$-subalgebra, then $\nu_\delta(B) \leq \nu_\delta(A)$.*

(iii) *If $I$ is a closed two-sided ideal in $A$, then $\nu_\delta(A/I) \leq \nu_\delta(A)$.*

(iv) *If $A_1 \to A_2 \to \cdots$ is a sequence of unital $C^*$-algebras with unital and injective connecting maps, then $\nu_\delta(\overline{\bigcup_{n=1}^\infty A_n}) \leq \inf_{n\in\mathbb{N}} \nu_\delta(A_n)$.*

(v) *For any sequence of unital $C^*$-algebras $(A_n)_{n\in\mathbb{N}}$, we have*

$$\nu_\delta(\ell^\infty((A_n)_{n\in\mathbb{N}})) = \sup_{n\in\mathbb{N}} \nu_\delta(A_n).$$

(vi) *For any sequence of unital $C^*$-algebras $(A_n)_{n\in\mathbb{N}}$, we have*

$$\sup_{\delta'>\delta} \limsup_{n\to\omega} \nu_{\delta'}(A_n) \leq \nu_\delta(\ell_\omega((A_n)_{n\in\mathbb{N}})) \leq \limsup_{n\to\omega} \nu_\delta(A_n).$$

*Proof.* (i)–(iii) are immediate.

(iv): Recall from (ii) that as we have the inclusions $A_k \subseteq \overline{\bigcup_{n\in\mathbb{N}} A_n}$, we have the inequality $\nu_\delta(A_k) \geq \nu_\delta(\overline{\bigcup_{n\in\mathbb{N}} A_n})$, and taking the infimum of the left-hand side implies the desired inequality.

(v): First assume that $\nu_\delta(\ell^\infty((A_n)_{n\in\mathbb{N}})) = k$ and let $a_1, \ldots, a_k \in \ell^\infty((A_n)_{n\in\mathbb{N}})$ be elements such that $\sum_{i=1}^k a_i^* a_i = 1$ and $\left\|\sum_{i=1}^k a_i a_i^*\right\| \leq \delta$. Write $a_i = (a_i(n))_{n\in\mathbb{N}}$ for all $i = 1, \dots, k$ with $a_i(n) \in A_n$ for $n \in \mathbb{N}$, then it is immediate that $\sum_{i=1}^k a_i(n)^* a_i(n) = 1_{A_n}$ and

$$\left\|\sum_{i=1}^k a_i(n)a_i(n)^*\right\| \leq \sup_{m\in\mathbb{N}} \left\|\sum_{i=1}^k a_i(m)a_i(m)^*\right\| = \left\|\sum_{i=1}^k a_i a_i^*\right\| \leq \delta,$$

such that $\nu_\delta(A_n) \leq k$ for all $n \in \mathbb{N}$. Conversely, suppose that $\nu_\delta(A_n) \leq k$ for all $n \in \mathbb{N}$ and find, for each $n \in \mathbb{N}$, elements $(a_i(n))_{i=1}^k \subseteq A_n$ with $\sum_{i=1}^k a_i(n)^* a_i(n) = 1_{A_n}$ and $\left\|\sum_{i=1}^k a_i(n)a_i(n)^*\right\| \leq \delta$. Since the elements $a_i(n)$ are necessarily contractions, the element $a_i = (a_i(n))_{n\in\mathbb{N}}$ is a well-defined element in $\ell^\infty((A_n)_{n\in\mathbb{N}})$ for each $i = 1, \dots, k$, and it is easily seen that these satisfy the desired property such that $\nu_\delta(\ell^\infty((A_n)_{n\in\mathbb{N}})) \leq k$.

(vi): Suppose that $\nu_\delta(\ell_\omega((A_n)_{n\in\mathbb{N}})) = k$ and find $a_1, \dots, a_k \in \ell_\omega((A_n)_{n\in\mathbb{N}})$ such that $\sum_{i=1}^k a_i^* a_i = 1$ and $\left\|\sum_{i=1}^k a_i a_i^*\right\| \leq \delta$. Let $a_i(n) \in A_n$ be lifts of each $a_i$ for $i = 1, \dots, k$ and $n \in \mathbb{N}$. Define the set

$$I = \left\{ n \in \mathbb{N} \mid \left\| 1_{A_n} - \sum_{i=1}^k a_i(n)^* a_i(n) \right\| < 1 \right\}.$$

Since $I \in \omega$, we find that $b_n = \sum_{i=1}^k a_i(n)^* a_i(n)$ is invertible with $\left\|b_n^{-1}\right\| \leq \frac{1}{1-\|b_n\|}$ for all $n \in I$. Since $\lim_{n\to\omega} \|1 - b_n\| = 0$, we may for any $\varepsilon > 0$ find $J_\varepsilon \in \omega$ such that $\left\|b_n^{-1}\right\| \leq 1+\varepsilon$ for all $n \in J_\varepsilon$.

Put $c_i(n) = a_i(n)b_n^{-1/2}$ for all $n \in I$ and $c_i(n) = 0$ elsewhere. Then, for all $n \in I$,

$$\sum_{i=1}^k c_i(n)^* c_i(n) = \sum_{i=1}^k b_n^{-1/2} a_i(n)^* a_i(n) b_n^{-1/2} = 1_{A_n},$$

and

$$\left\|\sum_{i=1}^k c_i(n)c_i(n)^*\right\| = \left\|\sum_{i=1}^k a_i(n)b_n^{-1}a_i(n)^*\right\| \leq \left\|b_n^{-1}\right\| \left\|\sum_{i=1}^k a_i(n)a_i(n)^*\right\|.$$

We may, for any $\varepsilon > 0$, find $K_\varepsilon$ such that $\left\|\sum_{i=1}^k a_i(n)a_i(n)^*\right\| \leq (\delta+\varepsilon)$ for all $n \in K_\varepsilon$. Thus, for any $n \in X \subseteq I \cap J_\varepsilon \cap K_\varepsilon$ with $X \in \omega$, we obtain

$$\sum_{i=1}^k c_i(n)^* c_i(n) = \sum_{i=1}^k b_n^{-1/2} a_i(n)^* a_i(n) b_n^{-1/2} = 1_{A_n},$$

and

$$\left\|\sum_{i=1}^k c_i(n)c_i(n)^*\right\| = \left\|\sum_{i=1}^k a_i(n)b_n^{-1}a_i(n)^*\right\| \leq \left\|b_n^{-1}\right\| \left\|\sum_{i=1}^k a_i(n)a_i(n)^*\right\| \leq (1+\varepsilon)(\delta+\varepsilon).$$

In particular, we find that, for any $\delta' > \delta$, there exists $J \in \omega$ such that $\nu_{\delta'}(A_n) \leq k$ for all $n \in J$, and thus

$$\sup_{\delta'>\delta} \limsup_{n\to\omega} \nu_{\delta'}(A_n) \leq \nu_\delta(\ell_\omega((A_n)_{n\in\mathbb{N}}))$$

as desired. For the other inequality, suppose that $\limsup_{n\to\omega} \nu_\delta(A_n) \leq k$. Then there exists $J \in \omega$ such that $\nu_\delta(A_n) \leq k$ for all $n \in J$. Find for each $n \in J$ a $k$-tuple $(a_i(n))_{i=1}^k$ in $A_n$

such that $\sum_{i=1}^{k} a_i(n)^* a_i(n) = 1_{A_n}$ and $\left\|\sum_{i=1}^{k} a_i(n) a_i(n)^*\right\| \leq \delta$. Put $a_i(n) = 0$ for $n \notin J$. Let $a_i$ be the images of the sequences $(a_i(n))_{n\in\mathbb{N}}$ in $\ell_\omega((A_n)_{n\in\mathbb{N}})$, then $\sum_{i=1}^{k} a_i^* a_i = 1$ and

$$\left\|\sum_{i=1}^{k} a_i a_i^*\right\| = \limsup_{n\to\omega} \left\|\sum_{i=1}^{k} a_i(n) a_i(n)^*\right\| \leq \delta.$$

□

A way of understanding $\mu(-)$ and $\nu_\delta(-)$ for $0 < \delta < 1$ is by looking at universal $C^*$-algebras encoding the information of the invariants. We shall briefly mention these universal $C^*$-algebras here, as they might provide some insight into how $\mu(-)$ and $\nu_\delta(-)$ behave.

**Definition 4.10.** Let $n \in \mathbb{N}$ and $0 < \delta < 1$.

(i) We denote by $\mathcal{A}_n$ the universal $C^*$-algebra generated by elements $x_{ij}$ with $1 \leq i \leq n$ and $1 \leq j \leq n+1$ such that for the non-square matrix $s = (x_{ij})_{i,j}$ we have $s^*s = 1_{n+1}$ and $ss^* \leq 1_n$.

(ii) We define $\mathcal{D}_{n,\delta} = C^*\big(a_1, \ldots, a_n \mid \sum_{i=1}^{n} a_i^* a_i = 1,\ \|\sum_{i=1}^{n} a_i a_i^*\| \leq \delta\big)$.

The following proposition is easy to verify.

**Proposition 4.11.** *Let $A$ be a unital $C^*$-algebra.*

(i) *There exists a *-homomorphism $\mathcal{A}_n \to A$ if and only if $\mu(A) \leq n$.*

(ii) *There exists a *-homomorphism $\mathcal{D}_n \to A$ if and only if $\nu_\delta(A) \leq n$.*

Since the $C^*$-algebra $\mathcal{D}_{n,\delta}$ are obviously without tracial states for any $n \in \mathbb{N}$ and $0 < \delta < 1$, an application of Proposition 2.20 implies that for any $n \in \mathbb{N}$, $0 < \delta < 1$ and for each $0 < \delta' < 1$ there exist $m \in \mathbb{N}$ and a *-homomorphism $\mathcal{D}_{m,\delta'} \to \mathcal{D}_{n,\delta}$. However, it is not clear what the integer $m$ is, except for the trivial case with $\delta \leq \delta'$, where $m = n$ would suffice. In other words, it is unclear what $\nu_{\delta'}(\mathcal{D}_{n,\delta})$ is; it is even unclear what $\nu_\delta(\mathcal{D}_{n,\delta})$ is. It seems reasonable to believe that $\nu_\delta(\mathcal{D}_{n,\delta}) = n$, but it still remains unresolved.

### 4.2 Asymptotic behaviour of the invariants and exotic traces

In Proposition 4.8 and Proposition 4.9, we examined how the invariants $\mu(-)$ and $\nu_\delta(-)$ behave when taking (ultra)products. In this section, we shall take a closer look at how this will allow us to study (quasi)traciality of ultraproducts of $C^*$-algebras. In particular, we shall construct a $C^*$-algebra with a quasitrace, but which arises as an ultraproduct of $C^*$-algebras without quasitraces.

The study of quasitraces or tracial states on ultraproducts is nothing new, and has been studied in e.g. [4, 51], and we have already worked with them in Chapter 3. Suppose that $(A_n)_{n\in\mathbb{N}}$ is a sequence of $C^*$-algebras with (quasi)traces $\tau_n$ on each $A_n$. It is easy to see that there exists a (quasi)trace $\tau_\omega$ on the ultraproduct $\ell_\omega((A_n)_{n\in\mathbb{N}})$ given by

$$\tau_\omega(\pi_\omega((a_n)_{n\in\mathbb{N}})) = \lim_{n\to\omega} \tau_n(a_n), \qquad (a_n)_{n\in\mathbb{N}} \in \ell^\infty((A_n)_{n\in\mathbb{N}}),$$

where $\pi_\omega \colon \ell^\infty((A_n)_{n\in\mathbb{N}}) \to \ell_\omega((A_n)_{n\in\mathbb{N}})$ is the canonical quotient map. We call these quasitraces *limit traces* and denote by $\mathrm{QT}_\omega(\ell^\infty((A_n)_{n\in\mathbb{N}}))$ (or, in the case of tracial states, $\mathrm{T}_\omega(\ell^\infty((A_n)_{n\in\mathbb{N}}))$) the collection of limit quasitraces of the sequence of $C^*$-algebras $(A_n)_{n\in\mathbb{N}}$. Consider the inclusion $\mathrm{T}_\omega(\ell^\infty((A_n)_{n\in\mathbb{N}}))$ inside $\mathrm{T}(\ell_\omega((A_n)_{n\in\mathbb{N}}))$. It follows from [4, Theorem 1.3] that the inclusion is proper whenever $\ell_\omega((A_n)_{n\in\mathbb{N}})$ contains infinitely many extremal tracial states. Even moreso, the collection of limit traces need not be weak*-dense in the space of traces on the ultraproduct, as the following result due to Pedersen–Petersen [53] shows.

**Proposition 4.12** (Pedersen-Petersen, 1970). *For each $n \in \mathbb{N}$, there exists a homogeneous $C^*$-algebra $A_n$ and an element $x_n \in A_n$ for which $\tau_n(x_n) = 0$ for all $\tau_n \in \mathrm{T}(A_n)$, yet $\|x_n - \sum_{i=1}^n [a_i, b_i]\| \geq 1$ for all $a_i, b_i \in A_n$. In particular, the ultraproduct $\ell_\omega((A_n)_{n\in\mathbb{N}})$ admits a tracial state, which is not in the weak*-closure of $\mathrm{T}_\omega(\ell^\infty((A_n)_{n\in\mathbb{N}}))$.*

*Proof.* The fact that such homogeneous $C^*$-algebras exist follows from [53, Lemma 3.5]. We may assume that each $x_n$ has norm $\|x_n\| \leq 1$. Put $A = \ell^\infty((A_n)_{n\in\mathbb{N}})$. Consider the element $x = (x_n)_{n\in\mathbb{N}} \in A$, then the construction entails that $\tau(x) = 0$ for any $\tau$ in $\mathrm{T}_\omega(\ell^\infty((A_n)_{n\in\mathbb{N}}))$. On the other hand, we must have that $x \notin \overline{[A, A]}$, and by Corollary 2.19 there exists a tracial state $\tau \in \mathrm{T}(A)$ such that $\tau(x) \neq 0$. This finishes the proof. □

In the literature, there is no clear consensus on what these traces that do not arise as weak*-accumulation points of limit traces should be called; we shall call them exotic traces in this thesis[2]. If each $A_n$ is both exact and $\mathcal{Z}$-stable, then there are no such exotic traces, as Ozawa showed in [51, Theorem 8].

**Theorem 4.13** (Ozawa, 2013). *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of exact and $\mathcal{Z}$-stable $C^*$-algebras. Then $\mathrm{T}_\omega(\ell^\infty((A_n)_{n\in\mathbb{N}}))$ is weak*-dense in $\mathrm{T}(\ell_\omega((A_n)_{n\in\mathbb{N}}))$.*

The proof of the theorem uses the Haagerup–Thorbjørnsen MF-embeddability of $\mathcal{V}_\infty$ [30], which we shall use later in the chapter, and then approximates the elements $\frac{1}{2}(s_j + s_j^*)$ in the $C^*$-algebra $\ell^\infty((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})/c_0((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})$ by sequences of matrices, which requires exactness. In Theorem 4.35, we shall use this machinery in order to provide a different proof of Haagerup's theorem of quasitraces being tracial states on unital, exact $C^*$-algebras.

We now sketch a construction of a (non-simple) unital $C^*$-algebra $B$ with the property that $B$ is not properly infinite, yet $M_2(B)$ is properly infinite; that is, $\mu(B) = 2$. The construction follows the structure of [60], and in that paper Rørdam provided an example of such a $C^*$-algebra within the class of separable, unital, simple, nuclear $C^*$-algebras satisfying the Universal Coefficient Theorem of KK-theory. The construction of this $C^*$-algebra is quite technical, but the main ideas can easily be used in order to construct a non-simple example.

Let $X$ be a compact Hausdorff space. It is well-known that any projection $p \in \mathcal{P}_\infty(C(X))$ corresponds to a vector bundle $\xi_p$ over $X$. Consider the sphere $X = S^2$ and let $p \in M_2(C(S^2))$ be the Bott projection, which corresponds to a vector bundle $\zeta$ with Euler class $e(\zeta) \neq 0$. Let $Z = \ell^\infty((S^2)_{n\in\mathbb{N}})$ be the infinite product of countably many copies of $S^2$, and put $A = C(Z) \otimes \mathbb{K}(H) \cong C(Z, \mathbb{K}(H))$. For any $n \in \mathbb{N}$, we define (with a suitable identification of $M_2(\mathbb{C})$ inside $\mathbb{K}(H)$) the projection $p_n \in \mathcal{P}(A)$ by $p_n(x) = p(x_n)$ for $x = (x_1, x_2, \ldots) \in Z$, that is, $p_n$ is the Bott projection on the $n$th copy of $S^2$ in $Z$. Let $\zeta_n$ be a vector bundle corresponding to $p_n$. Denote by $\theta$ the trivial vector bundle, which corresponds to a constant 1-dimensional projection in $A$; one can easily verify, see [60, Lemma 3.1] that for any $n \in \mathbb{N}$ there exists a complex line bundle $\eta_n$ over $(S^2)^n$ such that $\zeta_n \oplus \zeta_n \cong \theta \oplus \eta_n$, which, when expressed in terms of projections, implies that $g \precsim p_n \oplus p_n$.

Consider the multiplier algebra $\mathcal{M}(A)$. Since $A$ is a stable $C^*$-algebra, the properly infinite $C^*$-algebra $\mathbb{B}(H)$ embeds unitally in $\mathcal{M}(A)$, and hence $\mathcal{M}(A)$ is properly infinite. We can thus find a sequence of isometries $(S_n)_{n\in\mathbb{N}}$ in $\mathcal{M}(A)$ such that $1_{\mathcal{M}(A)} = \sum_{n=1}^\infty S_n S_n^*$ with the sum being strictly convergent. With this, we define for a sequence of projections $(q_n)_{n\in\mathbb{N}}$ in $A$ (or $\mathcal{M}(A)$) the projection $\bigoplus_{n\in\mathbb{N}} q_n = \sum_{n=1}^\infty S_n q_n S_n^*$; observe that if $(q_n)_{n=1}^N$ is a finite family of projections in $A$, then $q_1 \oplus \ldots \oplus q_N = \sum_{n=1}^N S_n q_n S_n^*$ belongs to $A$ and is Murray-von

[2]The reader should be immensely careful when reading papers on this subject. For instance, in [4], what we call limit traces, they call *trivial* traces, and any trace on ultraproducts arising as a weak*-accumulation point of trivial traces is called locally trivial in that paper.

Neumann equivalent to the usual definition of direct sums of projections.

With $p_n$ as defined previously arising from the Bott projection in $M_2(C(S^2))$, we consider the projection $Q = \bigoplus_{n\in\mathbb{N}} p_n$ in $\mathcal{M}(A)$. Our goal is to show that $Q$ is not a properly infinite projection, yet that $Q \oplus Q$ is properly infinite, and hence that $\mu(Q\mathcal{M}(A)Q) = 2$.

**Lemma 4.14.** *Let $g$ and $Q$ be projections on $A$ as defined above. Then $g \not\precsim Q$, and $Q$ is not properly infinite.*

*Proof.* We first show that if $g \precsim Q = \bigoplus_{n=1}^{\infty} p_n$, then there exists some $N \in \mathbb{N}$ such that $g \precsim \bigoplus_{n=1}^{N} p_n$. Suppose that $g \precsim Q$ and let $v \in \mathcal{M}(A)$ be a partial isometry with $g = vv^*$ and $Q \leq v^*v$. Observe that we by definition may write $Q = \sum_{n=1}^{\infty} p'_n$ with $(p'_n)_{n\in\mathbb{N}}$ being a family of mutually orthogonal projections in $A$ satisfying $p_n \sim p'_n$, and where the sum is strictly convergent. Since $gv = v$, we find that $v \in A$ and, by strict convergence, there exists $N \in \mathbb{N}$ such that

$$\left\| v - v\sum_{n=1}^{N} p'_n \right\| < \frac{1}{2}.$$

Put $x = v\sum_{n=1}^{N} p'_n \in A$. Then $xx^* \leq g$ and $x^*x \leq \sum_{n=1}^{N} p'_n$. Moreover, we find that $\|xx^* - g\| < 1$, and hence that $xx^*$ is invertible in the unital $C^*$-algebra $gAg$. Put $u = x^*(xx^*)^{-1/2}$, then $u^*u = g$ and

$$uu^* = x^*(xx^*)x \leq \sum_{n=1}^{N} p'_n \sim \bigoplus_{n=1}^{N} p_n.$$

In particular, we obtain that $g \precsim \bigoplus_{n=1}^{N} p_n$. Rewriting this in terms of vector bundles implies that there exists a vector bundle $\eta$ for which $\theta \oplus \eta \cong \bigoplus_{n=1}^{N} \zeta_n$; however, this is impossible by multiplicity of the Euler class as $e(\theta) = 0$ and $e(\zeta_n) \neq 0$ for all $n \in \mathbb{N}$. We hence conclude that $g \not\precsim Q$.

Finally, let us show that this implies that $Q$ is not properly infinite. Since $Q$ can be shown to be a full projection, $g$ belongs to the ideal generated by $Q$. Consequently, the proof of Proposition 2.6 implies that if $Q$ were a properly infinite projection, then $g \precsim Q$, which we just disproved. We conclude that $Q$ is not a properly infinite projection. □

We now turn our attention to verifying that $Q \oplus Q$ is properly infinite. The following lemma was mentioned earlier in the construction and is effectively an exercise in understanding the vector bundles over product of spheres — we refer to [60, Lemma 3.1]

**Lemma 4.15.** *For any $n \in \mathbb{N}$, we have $g \precsim p_n \oplus p_n$.*

The next lemma provides a characterisation of properly infiniteness of projections in any multiplier algebra of a stable $C^*$-algebra.

**Lemma 4.16.** *Let $B$ be a stable algebra and $P \in \mathcal{M}(B)$ be any non-zero projection. Then $1_{\mathcal{M}(B)} \precsim P$ if and only if $P$ is properly infinite and full in $\mathcal{M}(B)$.*

*Proof.* Suppose first that $1_{\mathcal{M}(B)} \precsim P$. Note that this implies that $P$ is necessarily full, as the ideal in $\mathcal{M}(B)$ generated by $P$ contains $1_{\mathcal{M}(B)}$. To prove properly infiniteness of $P$, we use that $\mathcal{M}(B)$ is properly infinite due to stableness of $B$ and invoke the following inequalities:

$$P \oplus P \precsim 1_{\mathcal{M}(B)} \oplus 1_{\mathcal{M}(B)} \sim 1_{\mathcal{M}(B)} \precsim P.$$

□

With these two lemmas, we are able to prove that $Q \oplus Q$ is properly infinite.

**Proposition 4.17.** *The projection $Q \oplus Q$ is properly infinite in $\mathcal{M}(A)$.*

*Proof.* One can verify that we can write the unit in $\mathcal{M}(A)$ as $1_{\mathcal{M}(A)} \sim \bigoplus_{n\in\mathbb{N}} g$. With this decomposition, Lemma 4.15 and the definition of $Q$ implies that

$$1_{\mathcal{M}(A)} \sim \bigoplus_{n=1}^{\infty} g \precsim \bigoplus_{n=1}^{\infty} (p_n \oplus p_n) \sim Q \oplus Q.$$

It now follows from Lemma 4.16 that $Q \oplus Q$ is a properly infinite, full projection in $\mathcal{M}(B)$. □

We have now shown that the corner $Q\mathcal{M}(A)Q$ of the multiplier algebra of $A$ satisfies that $\mu(Q\mathcal{M}(A)Q) = 2$. It is not immediate that $Q$ is a finite projection, but since $Q$ is not properly infinite, we may by Proposition 2.4(vii) find a quotient $Q\mathcal{M}(A)Q/I$ on which the image of $Q$ is finite, and where the image of $Q \oplus Q$ will remain properly infinite, hence constructing a finite $C^*$-algebra whose matrix amplification is properly infinite. By taking the limit of a suitable inductive sequence $\mathcal{M}(A) \to \mathcal{M}(A) \to \cdots$, one obtains an example of a *simple* $C^*$-algebra $B$ with $\mu(B) = 2$. While this would suit our needs for showing that $\mu(-)$ attains all possible values, it is an explicitly non-exact $C^*$-algebra, but a further refinement of the construction does yield a separable, unital, nuclear $C^*$-algebra in the UCT-class, and it has many projections in a specific manner: We say that a $C^*$-algebra has *small projections* or (SP) if any hereditary $C^*$-subalgebra admits a non-zero projection.

**Theorem 4.18** (Rørdam, 2003)**.** *There exists a separable, unital, simple, nuclear $C^*$-algebra $A$ in the UCT-class with $\mu(A) \geq 2$, and which satisfies (SP).*

The construction may be found in [60], and the proof of the (SP) property may be found in [62, Proposition 5.5]. By taking corners of this $C^*$-algebra in a careful way, we shall obtain a sequence of $C^*$-algebras $(A_n)_{n\in\mathbb{N}}$ with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$ and $\mu(A_n) \to \infty$ as $n \to \infty$. The following lemma is an easy application of Glimm's lemma [57, Proposition 3.10], and it shows that admitting (SP) gives a way of subdividing any non-zero projection.

**Lemma 4.19.** *Let $A$ be a unital, infinite-dimensional, simple $C^*$-algebra with property (SP). Then for each non-zero projection $p \in A$ and for each $n \in \mathbb{N}$, there exists a non-zero projection $q \in A$ with $q \otimes 1_n \precsim p$.*

*Proof.* By replacing $A$ with $pAp$, we may assume that $p = 1_A$ is the unit on $A$. By Glimm's lemma, see [57, Proposition 3.10], we may by infinite-dimensionality of $A$ find a *-homomorphism $\varphi\colon C_0((0,1]) \otimes M_n(\mathbb{C}) \to A$. Denote by $\iota \in C_0((0,1])$ the identity function $\iota(t) = t$ for $t \in (0,1]$, and let $(e_{ij})_{i,j=1}^n \subseteq M_n(\mathbb{C})$ be the canonical matrix units. Put $a_i = \varphi(\iota \otimes e_{ii})$. Since $A$ has property (SP), we may find a non-zero projection $q$ in the hereditary $C^*$-subalgebra $\overline{a_1 A a_1}$ of $A$.

Let $z_i = \varphi(\iota \otimes e_{i1})$ and observe that $|z_i|^2 = a_1$ for $i = 1, \ldots, n$. Consider the polar decompositions $z_i = v_i |z_i|$ with $v_i \in A^{**}$ partial isometries. It follows from [6, Proposition III.5.2.16] that $w_i = v_i q$ are partial isometries in $A$ with $w_i^* w_i = q$ for all $i = 1, \ldots, n$. Moreover, we find that $w_i w_i^* = v_i q v_i^* \in \overline{a_i A a_i}$. Put $q_i = w_i w_i^*$ for $i = 1, \ldots, n$ and observe that $q_1, \ldots, q_n$ is a set of mutually orthogonal projections in $A$, and that they are all Murray-von Neumann equivalent to $q$. Consequently, we obtain

$$q \otimes 1_n \sim q_1 + \ldots + q_n \leq 1_A,$$

which finalises the proof. □

**Theorem 4.20.** *There exists a sequence of separable, unital, simple, nuclear $C^*$-algebras $(A_n)_{n\in\mathbb{N}}$ with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$ and $\mu(A_n) \to \infty$ as $n \to \infty$.*

*Proof.* Take $A$ to be the separable, unital, simple, nuclear $C^*$-algebra with (SP) and $\mu(A) \geq 2$ from Theorem 4.18. Let $n \in \mathbb{N}$ be arbitrary. By Lemma 4.19 we may find a non-zero projection $p$ in $A$ such that $p \otimes 1_n \precsim 1_A$. Define $q = p \otimes 1_n$ and observe that $q$ is Murray-von Neumann equivalent to a non-zero projection $q' \in A$. Since $A$ is simple, $q'$ must be a full projection, and using the permanence properties of $\mu(-)$ from Proposition 4.8, we obtain the following inequality:

$$\lceil \frac{\mu(pAp)}{n} \rceil = \mu(M_n(pAp)) = \mu(q'Aq') \geq \mu(A) \geq 2.$$

In particular, we obtain that $\mu(pAp) \geq n$. All that remains to be seen is that $\mu(pAp) < \infty$. However, since $p$ is a full projection, it follows from [63, Exercise 4.8] that there exists $N \in \mathbb{N}$ such that $1_n \precsim p \otimes 1_N$ and, consequently, $1_n \otimes 1_2 \precsim p \otimes 1_N \otimes 1_2$. But since the left-hand side is a properly infinite, full projection, the right-hand side is properly infinite by Proposition 2.8 and, thus, $\mu(pAp) \leq 2N < \infty$. This completes the proof. ☐

By taking the ultraproduct of such a sequence of $C^*$-algebras, we obtain the following theorem from Proposition 4.8(vii).

**Theorem 4.21.** *There exists a sequence of separable, unital, simple, nuclear $C^*$-algebras $(A_n)_{n\in\mathbb{N}}$ with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$ such that $\mu(\ell_\omega((A_n)_{n\in\mathbb{N}})) = \infty$. In particular, the ultraproduct $\ell_\omega((A_n)_{n\in\mathbb{N}})$ admits a quasitrace, yet no $A_n$ admit quasitraces.*

It is unknown whether or not the quasitrace on $\ell_\omega((A_n)_{n\in\mathbb{N}})$ is a tracial state or not. We cannot use Haagerup's theorem to establish this, as the ultraproduct is non-exact.

**Proposition 4.22.** *The ultraproduct $\ell_\omega((A_n)_{n\in\mathbb{N}})$ as constructed in Theorem 4.21 is a non-exact, unital $C^*$-algebra.*

*Proof.* By Glimm's lemma, we may embed $M_n(\mathbb{C})$ in $A_n$ (non-unitally) for all $n \in \mathbb{N}$, and hence the $C^*$-algebra $\ell_\omega((M_n(\mathbb{C})_{n\in\mathbb{N}})$ embeds in $\ell_\omega((A_n)_{n\in\mathbb{N}})$. Since $\ell_\omega((M_n(\mathbb{C})_{n\in\mathbb{N}})$ is a non-exact $C^*$-algebra, this proves that the ultraproduct $\ell_\omega((A_n)_{n\in\mathbb{N}})$ is non-exact. ☐

It follows from the asymptotic behaviour of the $\mu$-invariant that the ultraproduct is stably not properly infinite. In fact, it is easily seen to be a stably finite $C^*$-algebra.

**Proposition 4.23.** *The ultraproduct $\ell_\omega((A_n)_{n\in\mathbb{N}})$ as constructed in Theorem 4.21 is stably finite.*

*Proof.* By construction, we find that, for any $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $M_k(A_n)$ is finite for all $n \geq N$. It thus suffices to show that the ultraproduct is finite, as the general case follows from taking matrix algebras. Hence the proposition effectively revolves around showing that an ultraproduct of finite $C^*$-algebras is once again finite. Suppose that $s \in \ell_\omega((A_n)_{n\in\mathbb{N}})$ is an isometry, that is, $s^*s = 1$. Let $(t_n)_{n\in\mathbb{N}} \in \ell^\infty((A_n)_{n\in\mathbb{N}})$ be any lift with $\|t_n\| \leq 1$. Then

$$\lim_{n\to\omega} \|t_n^* t_n - 1_{A_n}\| = \|s^*s - 1\| = 0.$$

In particular, we may find $I \in \omega$ for which $\|t_n^* t_n - 1_{A_n}\| < 1$ for all $n \in I$, such that the element $t_n^* t_n \in A_n$ is invertible for all $n \in I$. Let $u_n = t_n(t_n^* t_n)^{-1/2}$, then $u_n$ is an isometry in $A_n$, and finiteness of $A_n$ implies that $u_n u_n^* = 1$. Since $\|u_n - t_n\| \to 0$ as $n \to \omega$, we hence obtain that

$$\|ss^* - 1\| = \lim_{n\to\omega} \|u_n u_n^* - 1\| = 0,$$

proving that $s$ is an isometry. ☐

**Remark 4.24.** In [60, Corollary 7.4], Rørdam remarks an interestingly related property of the $C^*$-algebra $A$ from Theorem 4.18. Both the product $\ell^\infty(A)$ and the ultrapower $A_\omega$ has some quasitrace defined on some algebraic ideal, yet no such quasitrace exists on $A$. Our result is hence an improvement on this corollary.

By applying Proposition 4.8(iv) on the sequence $(A_n)_{n\in\mathbb{N}}$ of $C^*$-algebras from Theorem 4.21, we easily obtain the following corollary stating that $\mu(-)$ attains all possible values.

**Corollary 4.25.** *For all $n \in \mathbb{N}$, there exists a separable, unital, simple, nuclear $C^*$-algebra with $\mu(A) = n$.*

*Proof.* It suffices to show that, for any $n \in \mathbb{N}$ and any $N \geq n^2$, there exists $k$ such that $\lceil \frac{N}{k} \rceil = n$. Indeed, if this holds, then we may find a unital $C^*$-algebra $A_N$ with $\mu(A_N) = N \geq n^2$ and find that

$$\mu(M_k(A_N)) = \lceil \frac{\mu(A_N)}{k} \rceil = \lceil \frac{N}{k} \rceil = n.$$

Write $N = n^2 + pn + q$ for some $p \in \mathbb{N}_0$ and $0 \leq q < n$, then choosing $k = n + p + 1$ gives us

$$\frac{N}{k} = \frac{n^2 + pn + q}{n + p + 1} = (n-1) + \frac{q + p + 1}{n + p + 1},$$

and hence $\lceil \frac{N}{k} \rceil = n$. □

While it is relatively easy with the right tools to show that the function $\mu(-)$ attains all possible values, it is unclear whether it holds for $\nu_\delta(-)$ for some $\delta$. It is not difficult to see that $\nu_\delta(-)$ attains infinitely many values for some $0 < \delta < 1$ if and only if it does so for all $0 < \delta < 1$, but the proof becomes a bit more convoluted than expected due to the inequalities in Proposition 4.9(v).

**Proposition 4.26.** *If $\nu_\delta(-)$ attains infinitely many values for some $0 < \delta < 1$, then $\nu_{\delta'}(-)$ attains infinitely many values for all $0 < \delta' < 1$.*

*Proof.* Observe that $\nu_\delta(-)$ admits infinitely many values if and only if there exists a sequence of unital $C^*$-algebras $(A_n)_{n\in\mathbb{N}}$ with $\limsup_{n\to\omega} \nu_\delta(A_n) = \infty$ and $\nu_\delta(A_n) < \infty$ for all $n \in \mathbb{N}$. Let $(A_n)_{n\in\mathbb{N}}$ be such a sequence and consider the ultraproduct $A = \ell_\omega((A_n)_{n\in\mathbb{N}})$. We claim that $\nu_\delta(A) = \infty$. Indeed, we may use Proposition 4.9(vi) to see that

$$\nu_{\delta'}(A) \geq \sup_{\delta'' > \delta'} \limsup_{n\to\omega} \nu_{\delta'}(A_n).$$

In particular, choosing some $0 < \delta' < \delta < 1$ implies

$$\nu_{\delta'}(A) \geq \sup_{\delta'' > \delta'} \limsup_{n\to\omega} \nu_{\delta'}(A_n) \geq \limsup_{n\to\omega} \nu_\delta(A_n) = \infty.$$

Therefore, $A$ admits a tracial state, and hence $\nu_{\delta'}(A) = \infty$ for any $0 < \delta' < 1$, and thus in particular for $\delta' = \delta$. Let $0 < \delta' < 1$ be arbitrary. Note that as $\nu_\delta(A_n) < \infty$ for any $n \in \mathbb{N}$, we also obtain that $\nu_{\delta'}(A_n) < \infty$ for all $n \in \mathbb{N}$. Moreover, using Proposition 4.9(v) once more gives us

$$\limsup_{n\to\omega} \nu_{\delta'}(A_n) \geq \nu_{\delta'}(A) = \infty.$$

This completes the proof. □

Note that the proof does not entail that if $\nu_\delta(-)$ attains all possible values for some $\delta$, then it is true for any $\delta$. Moreover, it is not immediate whether or not $\nu_\delta(-)$ actually attains infinitely many values or not. For the case of $\delta = \frac{1}{2}$, we know from Proposition 4.5 that $\nu_{1/2}(A) = 2$ if and only if $A$ is properly infinite, and we know that $\nu_{1/2}(A) = \infty$ if and only if $A$ is tracial. Proving that there exists some $C^*$-algebra $A$ with $2 < \nu_{1/2}(A) < \infty$ requires the fact that not being properly infinite is not equivalent to being stably not properly infinite.

**Proposition 4.27.** *There exist $2 < n \leq 8$ and a separable, unital, nuclear, simple, UCT $C^*$-algebra $A$ with $\nu_{1/2}(A) = n$.*

*Proof.* Let $A$ be the $C^*$-algebra constructed by Rørdam as in Theorem 4.18, then $A$ is not properly infinite, hence $\nu_{1/2}(A) > 2$. On the other hand, from Proposition 4.6, we know that $\nu_{1/2}(A) \leq 2\mu(A)^2 = 8$. □

The interest in what values $\nu_\delta(-)$ can attain is due to the close relationship to Kaplansky's conjecture. If Kaplansky's conjecture is true, then the ultraproduct $C^*$-algebra $A$ from Theorem 3.19 will have a tracial state, hence $\nu_\delta(A) = \infty$ for all $0 < \delta < 1$, and an application of Proposition 4.26 implies that $\nu_\delta(-)$ will attain infinitely many values for some (hence all) $0 < \delta < 1$. If one therefore were to show that $\nu_\delta(-)$ does not attain infinitely many values, the quasitrace on the ultraproduct $C^*$-algebra $A$ would provide a concrete counterexample to Kaplansky's conjecture. Note, however, that Kaplansky's conjecture is not formally equivalent to the question of $\nu_\delta(-)$ attaining infinitely many values. It may for instance be the case that the ultraproduct $A$ from Theorem 4.21 has $\nu_\delta(A) = \infty$, yet that it admits a quasitrace, which is not a tracial state.

The possibility of exotic traces on ultraproducts may not be that mysterious — a rule of thumb for ultraproducts, which is substantiated by applications of e.g. Kirchberg's $\varepsilon$-test [42, Lemma 3.1], states that whenever $C^*$-algebras have properties that almost hold in some sense, then the ultraproduct will have that property. As we have seen above, not all tracial states on ultraproducts need be approximable by limit tracial states; however, they do arise as limits of certain "almost tracial states" in a particular way. The following discussion is an elaboration on how it might be possible for the $C^*$-algebra constructed in Theorem 4.21 to admit an exotic trace, even though no $C^*$-algebras in the product admits even quasitraces. The following results are in essence variations of similar results for characters as seen in [57, Section 8].

**Definition 4.28.** Let $A$ be a unital $C^*$-algebra, let $n \in \mathbb{N}$ and let $\varepsilon > 0$. We say that $A$ has *$(n, \varepsilon)$-almost tracial state* if for any $n$-tuple of contractions $x_1, \dots, x_n \in A$ there exists a state $\rho$ on $A$ such that $|\rho(x_i^* x_i - x_i x_i^*)| \leq \varepsilon$ for all $i = 1, \dots, n$.

Our main goal is to state that whenever an ultraproduct of unital $C^*$-algebra admits a tracial state, this tracial state exactly arises as the limit of almost tracial states. First, however, we note the following lemma, which shall be of use later.

**Lemma 4.29.** *A unital $C^*$-algebra $A$ admits a tracial state if and only if $A$ has $(n, \varepsilon)$-almost tracial states for all $n \in \mathbb{N}$ and $\varepsilon > 0$.*

*Proof.* Any tracial state is trivially an $(n, \varepsilon)$-almost tracial state, so we focus on the other direction. Suppose that $A$ has $(n, \varepsilon)$-almost tracial states for all $n \in \mathbb{N}$ and $\varepsilon > 0$. Then we may, for any finite subset $F$ of the closed unit ball $(A)_1$ of $A$ and $\varepsilon > 0$, find a state $\rho_{(F,\varepsilon)}$ for which $|\rho(x^*x - xx^*)| \leq \varepsilon$ for all $x \in F$. Let $\rho$ be an accumulation point in the state space $\mathrm{S}(A)$ of the net $(\rho_{(F,\varepsilon)})_{(F,\varepsilon)}$ with the order $(F, \varepsilon) \preceq (F', \varepsilon')$ whenever $F \subseteq F'$ and $\varepsilon \geq \varepsilon'$. Let $a \in A$ be any contraction, and let $\varepsilon > 0$ be arbitrary, and consider the set

$$\{\rho_{(F,\varepsilon')} \mid a \in F \subseteq (A)_1 \text{ finite set}, \ \varepsilon \geq \varepsilon'\},$$

whose closure, by construction of $\rho$, must contain $\rho$. Since $\left|\rho_{(F,\varepsilon')}(x^*x - xx^*)\right| \leq \varepsilon$ for any finite subset $F \subseteq (A)_1$ with $x \in F$ and any $\varepsilon' \leq \varepsilon$, we conclude that $|\rho(x^*x - xx^*)| \leq \varepsilon$. As $x \in A$ was an arbitrary contraction and $\varepsilon > 0$ was an arbitrary tolerance, we conclude that $\rho$ is a tracial state. □

Before we prove the connection between tracial states on ultraproducts and the existence of almost tracial states on the sequence, we prove a proposition showing that, while examples of exotic tracial states exist, there are no such thing as exotic states, that is, the limit states are weak*-dense in the state space on the ultraproduct.

**Lemma 4.30.** *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of unital $C^*$-algebras. Then the set $\mathrm{S}_\omega((A_n)_{n\in\mathbb{N}})$ of limit states on $\ell_\omega((A_n)_{n\in\mathbb{N}})$ is weak*-dense in the state space $\mathrm{S}(\ell_\omega(A_n)_{n\in\mathbb{N}})$ on the ultraproduct.*

*Proof.* Since $\mathrm{S}_\omega(\ell^\infty((A_n)_{n\in\mathbb{N}}))$ is easily seen to be convex, the result follows by the Krein-Milman theorem [24, Theorem II.1.8] if we show that all pure states on $\mathrm{S}(\ell_\omega((A_n)_{n\in\mathbb{N}}))$ belong to the weak*-closure of $\mathrm{S}_\omega(\ell^\infty((A_n)_{n\in\mathbb{N}}))$.

Let $\rho \in \mathrm{S}(\ell_\omega((A_n)_{n\in\mathbb{N}}))$ be a pure state, let $F \subseteq \ell_\omega((A_n)_{n\in\mathbb{N}})$ be a finite subset, and let $\varepsilon > 0$ be an arbitrary tolerance. By excision of pure states [11, Lemma 2.14], there exists a positive element $h \in \ell_\omega((A_n)_{n\in\mathbb{N}})$ with $\|h\| = 1$ for which

$$\left\|h^{1/2}xh^{1/2} - \rho(x)h\right\| < \varepsilon$$

for all $x \in F$. Let $(h_n)_{n\in\mathbb{N}} \in \ell^\infty((A_n)_{n\in\mathbb{N}})$ be a lift of $h$, where each $h_n$ is a positive contraction. Since $\|h\| = 1$, we may assume that $\|h_n\| > 0$ for each $n \in \mathbb{N}$. Find for each $n \in \mathbb{N}$ a state $\sigma_n \in \mathrm{S}(A_n)$ with $\sigma_n(h_n) = \|h_n\|$, and define the state $\mu_n \in \mathrm{S}(A_n)$ by $\mu_n(x) = \frac{1}{\|h_n\|}\sigma_n(h_n^{1/2}xh_n^{1/2})$. Observe that this is a well-defined state, as it is clearly linear, and as $\|\mu_n(1_{A_n})\| = 1$ by construction. Put $\sigma = \lim_\omega \sigma_n$ and $\mu = \lim_\omega \mu_n$. Note that $\sigma(h) = 1$, and hence

$$\begin{aligned} \varepsilon &> \left|\sigma(h^{1/2}xh^{1/2} - \rho(x)h)\right| \\ &= \left|\sigma(h^{1/2}xh^{1/2}) - \rho(x)\sigma(h)\right| \\ &= |\mu(x) - \rho(x)| \end{aligned}$$

for all $x \in F$. This completes the proof. □

**Proposition 4.31.** *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of unital $C^*$-algebras, and let $\ell_\omega((A_n)_{n\in\mathbb{N}})$ be the ultraproduct of the sequence. Define for each $n \in \mathbb{N}$ and $\varepsilon > 0$ the set*

$$I_{n,\varepsilon} = \{k \in \mathbb{N} \mid A_k \textit{ has } (n,\varepsilon)\textit{-almost tracial states}\}.$$

*Then $\ell_\omega((A_n)_{n\in\mathbb{N}})$ admits a tracial state if and only if $I_{n,\varepsilon} \in \omega$ for all $n \in \mathbb{N}$ and $\varepsilon > 0$.*

*Proof.* Suppose that the ultraproduct $\ell_\omega((A_n)_{n\in\mathbb{N}})$ admits a tracial state and let $n \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. Consider the complement

$$\mathbb{N} \setminus I_{n,\varepsilon} = \{k \in \mathbb{N} \mid A_k \text{ has no } (n,\varepsilon)\text{-almost tracial states}\}.$$

We shall prove that $\mathbb{N} \setminus I_{n,\varepsilon} \notin \omega$, and then it follows from maximality of the ultrafilter $\omega$ that the complement $I_{n,\varepsilon}$ belongs to $\omega$.

Suppose that $\mathbb{N} \setminus I_{n,\varepsilon} \in \omega$. For each $k \in \mathbb{N} \setminus I_{n,\varepsilon}$ find contractions $x_1(k), \ldots, x_n(k) \in A_k$

such that there is no state $\rho$ on $A_k$ for which $|\rho(x_i(k)^*x_i(k) - x_i(k)x_i(k)^*)| \leq \varepsilon$ holds for all $i = 1, \ldots, n$. For any $k \in I_{n,\varepsilon}$, we choose arbitrary contractions $x_i(k) \in A_k$ for $i = 1, \ldots, n$. Let, for each $i = 1, \ldots, n$, the element $x_i \in \ell_\omega((A_k)_{k\in\mathbb{N}})$ be the image of the sequence $(x_i(k))_{k\in\mathbb{N}}$. Since $\ell_\omega((A_k)_{k\in\mathbb{N}})$ admits a tracial state $\tau$, it follows from Lemma 4.30 that we may find a limit state $\rho = \lim_\omega \rho_n \in \mathrm{S}_\omega(\ell^\infty((A_n)_{n\in\mathbb{N}}))$ with $\rho_n \in \mathrm{S}(A_n)$ for each $n \in \mathbb{N}$, such that

$$|\rho(x_i^*x_i) - \tau(x_i^*x_i)| \leq \frac{\varepsilon}{2}, \qquad \text{and} \qquad |\rho(x_ix_i^*) - \tau(x_ix_i^*)| \leq \frac{\varepsilon}{2}$$

holds for all $i = 1, \ldots, n$. In particular, we may find $I \in \omega$ such that

$$|\rho_k(x_i(k)^*x_i(k)) - \rho_k(x_i(k)x_i(k)^*| \leq \varepsilon$$

holds for all $k \in I$. In particular, for any $k \in I$, $A_k$ has $(N, \varepsilon)$-almost tracial states, which implies that $I_{n,\varepsilon} \in \omega$, contradicting the assumption that $\mathbb{N} \setminus I_{n,\varepsilon} \in \omega$.

Now we assume that $I_{n,\varepsilon} \in \omega$ for all $n \in \mathbb{N}$ and $\varepsilon > 0$, and we aim to prove that the ultraproduct $\ell_\omega((A_k)_{k\in\mathbb{N}})$ admits a tracial state. By Lemma 4.29, it suffices to show that $\ell_\omega((A_k)_{k\in\mathbb{N}})$ has $(n, \varepsilon)$-almost tracial states for all $n \in \mathbb{N}$ and $\varepsilon > 0$.

Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. Let $x_1, \ldots, x_n \in \ell_\omega((A_k)_{k\in\mathbb{N}})$ be contractions, and find for each $k \in I_{n,\varepsilon}$ contractive lifts $x_1(k), \ldots, x_n(k) \in A_k$; for any $k \notin I_{n,\varepsilon}$, find some arbitrary lifts. For each $k \in I_{n,\varepsilon}$, we may find a state $\rho_k$ on $A_k$ for which

$$|\rho_k(x_i(k)^*x_i(k) - x_i(k)x_i(k)^*)| \leq \varepsilon$$

for all $i = 1, \ldots, n$. Choose for any $k \notin I_{n,\varepsilon}$ some arbitrary state $\rho_k$, and let $\rho = \lim_\omega \rho_k$ be the induced state on the ultraproduct $\ell_\omega((A_k)_{k\in\mathbb{N}})$. Then

$$|\rho(x_i^*x_i - x_ix_i^*)| \leq \varepsilon$$

for all $i = 1, \ldots, n$. This proves that $\ell_\omega((A_k)_{k\in\mathbb{N}})$ has $(n, \varepsilon)$-almost tracial states for all $n \in \mathbb{N}$ and $\varepsilon > 0$, and hence that $\ell_\omega((A_k)_{k\in\mathbb{N}})$ admits a tracial state by Lemma 4.29. □

Both the notion of $(n, \varepsilon)$-almost tracial states and the $\nu_\delta(-)$ invariant informally measure the failure to admitting tracial states. It should hence be the case that there is a connection between the two, and the following proposition gives us a link.

**Proposition 4.32.** *Let $A$ be a unital $C^*$-algebra, and let $0 < \delta < 1$ be arbitrary.*

(i) *For each $m \in \mathbb{N}$, there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that if $A$ admits $(n, \varepsilon)$-almost tracial states, then $\nu_\delta(A) \geq m$. In fact, one can take $n = m$ and $0 < \varepsilon < (1 - \delta)/n$.*

(ii) *For each $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that whenever $\nu_\delta(A) \geq m$, then $A$ has $(n, \varepsilon)$-almost tracial states.*

*Proof.* (i): We prove the negation. Find by definition of $\nu_\delta(A) \geq m$ elements $a_1, \ldots, a_m \in A$ such that $\sum_{i=1}^m a_i^*a_i = 1_A$ and $\|\sum_{i=1}^m a_ia_i^*\| \leq \delta$. Then, for any state $\rho$ on $A$, we have

$$\sum_{i=1}^m |\rho(x_i^*x_i - x_ix_i^*)| \geq 1 - \delta.$$

In particular, we obtain that, for any $\rho \in \mathrm{S}(A)$,

$$\max_{1\leq i\leq m} |\rho(x_i^*x_i - x_ix_i^*)| \geq \frac{1 - \delta}{m}.$$

Therefore, $A$ does not have $(n,\varepsilon)$-almost tracial state whenever $n = m$ and $\varepsilon < \frac{1-\delta}{n}$.

(ii): Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. Suppose that for all $m \in \mathbb{N}$ there exists some unital $C^*$-algebra $A_m$ with no $(n,\varepsilon)$-almost tracial states, yet which satisfies $\nu_\delta(A_m) \geq m$. Let $(A_m)_{m\in\mathbb{N}}$ be a sequence of such unital $C^*$-algebras and consider the ultraproduct $A = \ell_\omega((A_m)_{m\in\mathbb{N}})$. Then $A$ admits a tracial state by Proposition 4.9(vi), hence it follows from Proposition 4.31 that

$$\{m \in \mathbb{N} \mid A_m \text{ has } (n,\varepsilon)\text{-almost tracial states}\} \in \omega.$$

In particular, there exists $m \in \mathbb{N}$ such that $A_m$ has $(n,\varepsilon)$-almost tracial states. However, this is a contradiction with the assumption. We thus conclude that there exists some $m \in \mathbb{N}$ for which $\nu_\delta(A) \geq m$ implies the existence of $(n,\varepsilon)$-almost tracial states on $A$. ☐

Interestingly, the proof of (i) above is constructive in the sense that it proves the existence of $(n,\varepsilon)$-almost tracial states for an explicit pair $(n,\varepsilon)$ in terms of $m$ and $\delta$, whereas the proof of (ii) gives no indication of the lower bound achieved on $\nu_\delta(-)$ when admitting $(n,\varepsilon)$-almost tracial states. In particular, note that it is not even implied that $\nu_\delta(-)$ need attain infinitely many values — it might be the case that if a unital $C^*$-algebra $A$ has $(n,\varepsilon)$-almost tracial states for some specific $n \in \mathbb{N}$ and $\varepsilon > 0$, then $A$ admits a tracial state.

Another way of looking at Kaplansky's conjecture using the invariants $\mu(-)$ and $\nu_\delta(-)$ is by asking whether the inequalities of Proposition 4.6 can be inverted in some way such that $\mu(-)$ is bounded by $\nu_\delta(-)$ for some $0 < \delta < 1$. That is, does there for some $0 < \delta < 1$ exist a function $F_\delta \colon \mathbb{N} \to [0,\infty)$ with $F_\delta(n) \to \infty$ as $n \to \infty$ such that $\mu(A) \leq F_\delta(\nu_\delta(A))$ for all unital $C^*$-algebras $A$. A priori, this will only entail that any quasitracial $C^*$-algebra will admit a tracial state, but by Proposition 3.12, it would actually provide a proof of Kaplansky's conjecture in the unital case.

**Proposition 4.33.** *Kaplansky's conjecture is true for unital $C^*$-algebras if and only if there exists a function $F_\delta$ as described above.*

*Proof.* The existence of $F_\delta$ in particular shows that whenever $\tau$ is a *unique* quasitrace on a unital $C^*$-algebra, then $\tau$ is necessarily a tracial state. In particular, all quasitraces on $\mathrm{II}_1$-AW*-factors are tracial states, and thus Kaplansky's conjecture is true by Proposition 3.12. Conversely, if Kaplansky's conjecture is true, then all quasitraces are tracial states. Define the function $F_\delta \colon \mathbb{N} \to [0,\infty)$ by $F_\delta(k) = \sup\{\mu(A) \mid \nu_\delta(A) \leq k\}$. A priori, it is not clear that this function is well-defined in the sense that $F_\delta(k) < \infty$ for all $k \in \mathbb{N}$. Suppose that $F_\delta(k) = \infty$ for some $k \in \mathbb{N}$ and find a sequence of $C^*$-algebras $(A_n)_{n\in\mathbb{N}}$ for which $\mu(A_n) \to \infty$ as $n \to \infty$, but where $\nu_\delta(A) \leq k$. Then, by applying Proposition 4.8(viii) and Proposition 4.9(vi), we obtain that $\mu(\ell_\omega((A_n)_{n\in\mathbb{N}})) = \infty$ yet $\nu_\delta(\ell_\infty((A_n)_{n\in\mathbb{N}})) \leq k < \infty$. But then the ultraproduct admits a quasitrace, which is not a tracial state, and this contradicts Kaplansky's conjecture. Hence $F_\delta$ above is well-defined, and it is immediate from its construction that $\mu(A) \leq F_\delta(\nu_\delta(A))$ for all unital $C^*$-algebras $A$. ☐

We may summarise the above discussion in the following theorem. The reader should be aware of the implication (iv)⇒(iii), which is not necessarily an equivalence.

**Theorem 4.34.** *The implications (i)⇔(ii)⇔(iii)⇐(iv)⇔(v) below hold:*

(i) *There exists a sequence of unital $C^*$-algebras $(A_n)_{n\in\mathbb{N}}$ without tracial states such that the ultraproduct $\ell_\infty((A_n)_{n\in\mathbb{N}})$ admits a tracial state.*

(ii) *The invariant $\nu_\delta(-)$ attains infinitely many values for some $0 < \delta < 1$.*

(iii) *The invariant* $\nu_\delta(-)$ *attains infinitely many values for all* $0 < \delta < 1$.

(iv) *There exists a function* $F_\delta\colon \mathbb{N} \to [0, \infty)$ *with* $F_\delta(n) \to \infty$ *as* $n \to \infty$ *such that* $\mu(A) \leq F_\delta(\nu_\delta(A))$ *for all unital* $C^*$*-algebras* $A$.

(v) *Kaplansky's conjecture is true for the class of unital* $C^*$*-algebras.*

### 4.3 An alternative proof of Haagerup's theorem

The following result is an alternative proof of Haagerup's theorem that quasitraces on unital, exact $C^*$-algebras are tracial states. We have already covered Haagerup's proof earlier in this thesis, and we shall apply the AW$^*$-completion in this proof as well, however the use of exactness arguably appears in a much more intuitive fashion. Instead of proving an embedding of $A \otimes C_r^*(\mathbb{F}_\infty))$ in a finite AW$^*$-algebra and later extending it to $\mathcal{M}_\tau \otimes C_r^*(\mathbb{F}_\infty)$ for $\mathcal{M}_\tau$ the AW$^*$-completion of $A$ with respect to a faithful quasitrace $\tau$, one uses the fact that $C_r^*(\mathbb{F}_\infty)$ is an MF-algebra [30, Corollary 8.4] in order to approximate certain elements almost satisfying the Cuntz relation by matrices over $A$. This alternative proof is inspired by ideas from Haagerup–Thorbjørnsen, see [29, Section 9] and [30, Corollary 9.3], but does not (explicitly) use the theory of random matrices.

**Theorem 4.35.** *If* $A$ *is a unital, exact* $C^*$*-algebra, then* $\mu(A) < \infty$ *if and only if* $\nu(A) < \infty$. *In particular, all quasitraces on unital, exact* $C^*$*-algebras are tracial states.*

*Proof.* Suppose that $A$ admits no tracial states. Let $0 < \delta < 1$ be arbitrary and find by Proposition 2.20 elements $a_1, \dots, a_n \in A$ such that $\sum_{i=1}^n a_i^* a_i = 1_A$ and $\|\sum_{i=1}^n a_i a_i^*\| \leq \delta$. Consider the Cuntz algebra $\mathcal{O}_{2n}$ on $2n$ generators $s_1, \dots, s_{2n}$ and define the following four elements:

$$u_1 = \sum_{i=1}^n a_i \otimes s_i, \qquad v_1 = \sum_{i=1}^n a_i \otimes s_i^*,$$
$$u_2 = \sum_{i=1}^n a_i \otimes s_{i+n}, \qquad v_2 = \sum_{i=1}^n a_i \otimes s_{i+n}^*.$$

Using the properties of $a_1, \dots, a_n$, it is immediate that $u_i^* u_i = 1$ and $\|v_i\| = \sqrt{\|v_i^* v_i\|} = \sqrt{\delta}$ for $i = 1, 2$. Put

$$w_1 = u_1 + v_1 = \sum_{i=1}^n a_i \otimes (s_i + s_i^*),$$
$$w_2 = u_2 + v_2 = \sum_{i=1}^n a_i \otimes (s_{i+n} + s_{i+n}^*),$$

and note that $w_1, w_2 \in A \otimes \mathcal{V}_\infty$ and, since $\mathcal{V}_\infty$ embeds unitally in $C_r^*(\mathbb{F}_\infty)$ by Proposition 3.13, we find that $w_1, w_2 \in A \otimes C_r^*(\mathbb{F}_\infty)$. We claim that these elements are almost isometries with almost orthogonal range projections in such a way that we may apply Proposition 2.13. First of all, we see that

$$w_i^* w_i = u_i^* u_i + u_i^* v_i + v_i^* u_i + v_i^* v_i \leq 1 + \delta + 2\sqrt{\delta},$$

hence $\left\|w_i^* w_i - 1_{A \otimes C_r^*(\mathbb{F}_\infty)}\right\| \leq \delta + 2\sqrt{\delta}$. Moreover, a similar calculation and an application of the fact that $s_i^* s_j = \delta_{ij}$ gives us that

$$w_2^* w_1 = u_2^* u_1 + u_2^* v_1 + v_2^* u_1 + v_2^* v_1 \leq \delta + 2\sqrt{\delta},$$

and thus we find that

$$\begin{aligned}\left\|w_i^*w_i - 1_{A\otimes C_r^*(\mathbb{F}_\infty)}\right\| &\le \delta + 2\sqrt{\delta},\\ \|w_2^*w_1\| &\le \delta + 2\sqrt{\delta}.\end{aligned}$$

Since $C_r^*(\mathbb{F}_\infty)$ is an MF-algebra by [30, Corollary 8.4], we have an embedding

$$C_r^*(\mathbb{F}_\infty) \hookrightarrow \ell^\infty((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})/c_0((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}}),$$

and hence $s_i+s_i^* \in \ell^\infty((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})/c_0((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})$. Find $(c_i(k))_{k\in\mathbb{N}} \in \ell^\infty((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})$ for $i = 1,\ldots,2n$ such that $\pi((c_i(k))_{k\in\mathbb{N}}) = s_i + s_i^*$, where

$$\pi\colon \ell^\infty((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}}) \to \ell^\infty((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})/c_0((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})$$

is the canonical quotient map.

Define for each $k\in\mathbb{N}$ the elements $\tilde{w}_1(k), \tilde{w}_2(k) \in A\otimes M_{n_k}(\mathbb{C}) \cong M_{n_k}(A)$ by

$$\begin{aligned}\tilde{w}_1(k) &= \sum_{i=1}^n a_i \otimes c_i(k),\\ \tilde{w}_2(k) &= \sum_{i=1}^n a_i \otimes c_{i+n}(k).\end{aligned}$$

Using exactness of $A$, we have an isomorphism

$$A\otimes \ell^\infty((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})/c_0((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}}) \cong \frac{A\otimes \ell^\infty((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})}{A\otimes c_0((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})},$$

and hence $(\tilde{w}_j(k))_{k\in\mathbb{N}} \in \ell^\infty((M_{n_k}(A))_{k\in\mathbb{N}})$ are lifts of the elements $w_j$ in the tensor product $A\otimes \ell^\infty((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})/c_0((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})$. We thus obtain for $j = 1,2$ that

$$\begin{aligned}\limsup_{k\to\infty} \|\tilde{w}_j(k)^*\tilde{w}_j(k) - 1\| &= \left\|w_j^*w_j - 1\right\| \le \delta + 2\sqrt{\delta},\\ \limsup_{k\to\infty} \|\tilde{w}_2(k)^*\tilde{w}_1(k)\| &= \|w_2^*w_1\| \le \delta + 2\sqrt{\delta}.\end{aligned}$$

By taking $\delta > 0$ such that $\delta + 2\sqrt{\delta} < \frac{1}{4}$, we may find $k\in\mathbb{N}$ such that

$$\begin{aligned}\|\tilde{w}_j(k)^*\tilde{w}_j(k) - 1\| &< \frac{1}{4},\\ \|\tilde{w}_2(k)^*\tilde{w}_1(k)\| &< \frac{1}{4}.\end{aligned}$$

We may now invoke Proposition 2.13 to find isometries $t_1, t_2 \in M_{n_k}(A)$ such that $t_j^*t_j = 1$ for $j = 1,2$ and $t_2t_2^* \perp t_1t_1^*$. But this implies that $M_{n_k}(A)$ is properly infinite, which proves that $\mu(A) < \infty$. By (the proof of) Proposition 3.12, we conclude that all quasitraces on unital, exact $C^*$-algebras are tracial states. □

In the proof above, before we invoke exactness of $A$ in order to obtain matrix approximations for the elements $w_1, w_2$, we actually obtain the following theorem.

**Corollary 4.36** (Haagerup)**.** *Let $A$ be a unital $C^*$-algebra. Then $A$ admits no tracial state if and only if $A\otimes C_r^*(\mathbb{F}_\infty)$ is properly infinite.*

*Proof.* If $A \otimes C_r^*(\mathbb{F}_\infty)$ is properly infinite, then $A$ clearly cannot admit tracial states as $A$ embeds unitally in $A \otimes C_r^*(\mathbb{F}_\infty)$. Conversely, suppose that $A$ admits no tracial state. The proof of Theorem 4.35 provides elements $w_1, w_2 \in A \otimes C_r^*(\mathbb{F}_\infty)$ for which

$$\|w_i^* w_j - \delta_{ij} 1\| < \frac{1}{4},$$

and hence $A \otimes C_r^*(\mathbb{F}_\infty)$ will be properly infinite by Proposition 2.13. □

One might ask whether similar results hold for other $C^*$-algebras than $C_r^*(\mathbb{F}_\infty)$, that is, for what $C^*$-algebras $B$ is it true that non-traciality of $A$ implies that $A \otimes B$ is properly infinite. This question remains open and is probably quite hard; for instance, if one considers $B = \mathcal{Z}$ the Jiang-Su algebra, then $A \otimes \mathcal{Z}$ is either properly infinite or contains a quasitrace, see Proposition 2.26. Therefore, if it were true that $A$ non-tracial implies that $A \otimes \mathcal{Z}$ is properly infinite, then Kaplansky's conjecture would be answered in the affirmative for unital $C^*$-algebras.

The use of exactness in Theorem 4.35 is very explicit in that we take elements from $A \otimes \ell^\infty((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})/c_0((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})$ and approximate them by elements in the $C^*$-subalgebra $A \otimes \ell^\infty((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})$ of $\ell^\infty((M_{n_k}(A))_{k\in\mathbb{N}})$. However, it raises the question of whether there exists lifts $c_i(k) \in M_{n_k}(\mathbb{C})$ of $s_i + s_i^*$ for which the argument holds *without* resorting to exactness in $A$. The wish is unfortunately not granted as the following proposition, which is a variation of [29, Proposition 4.9], shows. First, we provide a quick lemma, which is contained in the proof of [29, Lemma 4.8].

**Lemma 4.37.** *Let $n \in \mathbb{N}$ and let $x = (x_{ij}), y = (y_{ij}) \in M_n(\mathbb{C})$ be matrices. Then*

$$\|x \otimes y\| \geq \operatorname{tr}(x^t y).$$

*Proof.* Let $(e_i)_{i=1}^n$ be an orthonormal basis for $\mathbb{C}^n$, and let $e = \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes e_i$ be a unit vector in $\mathbb{C}^n \otimes \mathbb{C}^n$. From the definition of the operator norm, we obtain that $\|x \otimes y\| \geq |\langle (x \otimes y)e, e\rangle|$, and a series of easy calculations gives us that

$$\begin{aligned}
\langle (x \otimes y)e, e\rangle &= \frac{1}{n} \sum_{i,j=1}^n \langle (x \otimes y)(e_i \otimes e_i), e_j \otimes e_j\rangle \\
&= \frac{1}{n} \sum_{i,j=1}^n \langle x e_i, e_j\rangle \langle y e_i, e_j\rangle \\
&= \frac{1}{n} \sum_{i,j=1}^n x_{ij} y_{ij} \\
&= \operatorname{tr}(x^t y)
\end{aligned}$$

as desired. □

**Proposition 4.38.** *Let $(c_j(k))_{k\in\mathbb{N}} \in \ell^\infty((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})$ for $j \in \mathbb{N}$ be some self-adjoint lifts of the elements*

$$x_j = \frac{s_j + s_j^*}{2} \in \mathcal{V}_\infty \subseteq \ell^\infty((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})/c_0((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})$$

*for $j \in \mathbb{N}$ and let $\delta > 0$. Then for all $R > 0$ there exist $n \in \mathbb{N}$, a unital $C^*$-algebra $A$ and elements $a_1, \ldots, a_n \in A$ with $\sum_{i=1}^n a_i^* a_i = 1_A$ and $\|\sum_{i=1}^n a_i a_i^*\| \leq \delta$ with the following* *property: If one defines the elements*

$$w_1(k) = \sum_{i=1}^{n} a_i \otimes c_i(k),$$
$$w_2(k) = \sum_{i=1}^{n} a_i \otimes c_{i+n}(k),$$

*for $k \in \mathbb{N}$, then we may find $k_0 \in \mathbb{N}$ such that $\|w_1(k)\| \geq R$.*

*Proof.* Find an infinite-dimensional Hilbert space $H$ such that $(c_j(k))_{k\in\mathbb{N}} \in \mathbb{B}(H)$ for each $j \in \mathbb{N}$, and let $(s_i)_{i\in\mathbb{N}}$ be a sequence of isometries with mutually orthogonal range projections witnessing the properly infiniteness of $\mathbb{B}(H)$. Let $N \in \mathbb{N}$ be arbitrary. Consider the polar decomposition $c_i(k) = u_i(k)\,|c_i(k)|$ for each $i, k \in \mathbb{N}$. Find $m \in \mathbb{N}$, and $\lambda_1, \lambda_2 > 0$ for which we obtain the following relations:

$$N\lambda_1 + m\lambda_2 = 1 \qquad \text{and} \qquad N\lambda_1 + \lambda_2 \leq \delta.$$

This can be done virtually explicitly: Let $\lambda_1 = \frac{\delta}{N+1}$ and find $\lambda_2 > 0$ such that

$$\delta\left(1 - \frac{1}{N+1}\right) \geq \lambda_2 \qquad \text{and} \qquad \frac{1 - \frac{N\delta}{N+1}}{\lambda_2} \in \mathbb{N}.$$

From these quantities, define the elements $a_1, \ldots, a_{N+m} \in \mathbb{B}(H)$ by

$$a_i = \sqrt{\lambda_1}(\overline{u_i}(k))_{k\in\mathbb{N}}, \qquad \text{for } 1 \leq i \leq N,$$
$$a_{i+m} = \sqrt{\lambda_2} s_i, \qquad \text{for } N+1 \leq i \leq N+m.$$

Here $\overline{u}$ means the matrix constructed from $u \in M_k(\mathbb{C})$ with all its entries complex conjugated.

The constraints imposed on $m$, $\lambda_1$ and $\lambda_2$ are exactly those to guarantee that the family $(a_i)_{i=1}^{N+m}$ satisfies the properties of Proposition 2.20, that is,

$$\sum_{i=1}^{N+m} a_i^* a_i = 1, \qquad \text{and} \qquad \left\|\sum_{i=1}^{N+m} a_i a_i^*\right\| \leq \delta,$$

which follows from the fact that

$$\sum_{i=1}^{N+m} a_i^* a_i = N\lambda_1 + m\lambda_2, \qquad \text{and} \qquad \left\|\sum_{i=1}^{N+m} a_i a_i^*\right\| \leq N\lambda_1 + \lambda_2.$$

Let $A = C^*((a_i)_{i=1}^{N+m})$ be the $C^*$-algebra generated by this family. Then a series of calculations and an application of Lemma 4.37 give us the following inequalities:

$$\begin{aligned}
\|w_1(k)\| &\geq \left\| \sum_{i=1}^{N} \sqrt{\lambda_1} (\overline{u}_i(\ell))_{\ell\in\mathbb{N}} \otimes c_i(k) \right\| \\
&= \sup_{\ell\in\mathbb{N}} \left\| \sum_{i=1}^{N} \sqrt{\lambda_1} \overline{u}_i(\ell) \otimes c_i(k) \right\| \\
&\geq \sqrt{\lambda_1} \left\| \sum_{i=1}^{N} \overline{u}_i(k) \otimes c_i(k) \right\| \\
&\geq \sqrt{\lambda_1} \sum_{i=1}^{N} \mathrm{tr}_{n_k}(\overline{u}_i(k)^t c_i(k)) \\
&\geq \sqrt{\lambda_1} \sum_{i=1}^{N} \mathrm{tr}_{n_k}(\overline{u}_i(k)^t u_i(k) \, |c_i(k)|) \\
&= \sqrt{\lambda_1} \sum_{i=1}^{N} \mathrm{tr}_{n_k}(|c_i(k)|).
\end{aligned}$$

Since $(c_i(k))_{k\in\mathbb{N}}$ are lifts of the elements $x_i = \frac{1}{2}(s_i + s_i^*)$ for all $i \in \mathbb{N}$, we find from the embedding of $C_r^*(\mathbb{F}_\infty)$ in $\ell^\infty((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})/c_0((M_{n_k}(\mathbb{C}))_{k\in\mathbb{N}})$ that

$$\lim_{k\to\infty} \mathrm{tr}_{n_k}(|c_i(k)|) = \tau(|x_i|)$$

with $\tau$ being the canonical tracial state on $C_r^*(\mathbb{F}_\infty)$. By considering the $C^*$-subalgebra generated by the unitary element $u_i \in C_r^*(\mathbb{F}_\infty)$, we may realise that

$$\tau(|x_i|) = \int_{\mathbb{T}} |f| \, \mathrm{d}\mu$$

with $f$ being the function for which $f(u_i) = x_i$ for $i \in \mathbb{N}$, where $(u_i)_{i\in\mathbb{N}}$ are the canonical unitary generators of $C_r^*(\mathbb{F}_\infty)$, and where $\mu$ is the Haar measure on the spectrum $\sigma(u_i) = \mathbb{T}$. In particular, one may calculate that

$$\xi = \int_{\mathbb{T}} |f| \, \mathrm{d}\mu > 0.$$

Combining everything together, we obtain that

$$\limsup_{k\to\infty} \|w_1(k)\| \geq \sqrt{\lambda_1} N \xi$$

with the right-hand side being a divergent sequence in $N$, since

$$\sqrt{\lambda_1} N \xi = \frac{N\sqrt{\delta}}{\sqrt{N+1}} \xi \to \infty$$

as $N \to \infty$. In particular, we may find some $N \in \mathbb{N}$ and $k \in \mathbb{N}$ for which $\|w_1(k)\| \geq R$. This completes the proof. □

Note that the argument deals with a universal choice of the lifts. It is hence not necessarily true that there exists any unital non-exact $C^*$-algebra $A$ and elements $a_1, \dots, a_n \in A$ with $\sum_{i=1}^n a_i^* a_i = 1$ and $\|\sum_{i=1}^n a_i a_i^*\| \leq \delta$ for which *no* lift of $x_j = \frac{1}{2}(s_j + s_j^*)$ will satisfy the arguments of Theorem 4.35. What we have shown is that there exists no universal lifts of the elements $x_j$ for which the arguments hold for *any* unital $C^*$-algebra. Moreover, one could have hoped that the $C^*$-algebra $A$ constructed might have some interesting properties to analyse with regards to the Kaplansky conjecture. Unfortunately, since $A$ admits Cuntz isometries by construction, $A$ is a properly infinite $C^*$-algebra.

# 5 Faithful tracial states and the QFTS property

Up until this point, one of the main points of the thesis has been answering the question of when $C^*$-algebras admit tracial states. This chapter is, in principle, not different in that regard, but we are now interested in having more structure to the tracial states. Earlier, we mentioned how traciality of $C^*$-algebras is often regarded as a finite property of $C^*$-algebras, and how this is somewhat a misnomer as the infinite yet tracial Toeplitz algebra $\mathcal{T}$ shows — the existence of a tracial state does not inhibit infiniteness of the unit, nor the existence of some properly infinite projection. However, Proposition 2.16 shows that the existence of a *faithful* tracial state implies stably finiteness, and would thus hinder these possible infinities from appearing. In this chapter, we shall study the phenomenon of when $C^*$-algebras has the property that they admit faithful tracial states, and more specifically when they have the deeply finite property of having faithful tracial states on all quotients.

The remainder of this chapter is essentially a rewrite of the author's preprint [45] with a few adjustments. We begin the chapter with a quick overview of (strong) quasidiagonality, which is what introduced the author to study the notion of admitting faithful tracial states on the quotients. The point of this chapter is to give a brief expository review of some cherry-picked results, with an emphasis on the finiteness properties of (strong) quasidiagonality. Afterwards, we review known results regarding existence of separating families of tracial states, which is a generalisation of admitting a faithful tracial state. Later on, and inspired by the idea of irreducible *-representations not intersecting the compact operators non-trivially, we examine when all quotients of a $C^*$-algebras admit faithful tracial states, which we coin the QFTS property. We prove that, under certain conditions, this property is equivalent to having no stable intermediate quotients. This can be seen as a converse to the well-known result that admitting stable ideals (or, in general, stable $C^*$-subalgebras) is an obstruction to admitting a faithful tracial state.

## 5.1 An overview of (strong) quasidiagonality

Originally, quasidiagonality was a property introduced by Halmos in [32] as a way of generalising block-diagonal operators, but it has been of huge importance within $C^*$-algebraic theory and especially within Elliott's classification program. In this section, we shall take a look at quasidiagonality, the related concept of strong quasidiagonality, and when $C^*$-algebras have these properties. The section is meant as an exposition to a rich field of operator algebras, and it gives a concrete reason as to why one might be interested in studying when $C^*$-algebras admit faithful tracial states, which is the main focus of this chapter. We shall only provide some proofs; many of the details may be found in [14, Chapter 7].

**Definition 5.1.** Let $A$ be a $C^*$-algebra, and let $\pi\colon A \to \mathbb{B}(H)$ be a *-representation. We say that $\pi$ is a *quasidiagonal representation*, or that $\pi(A)$ is a *quasidiagonal set of operators*, if for each finite set $F \subseteq \pi(A)$, for each finite set $V \subseteq H$, and for each $\varepsilon > 0$ there exists a finite-rank projection $P \in \mathbb{B}(H)$ for which

$$\|[P, T]\| < \varepsilon, \qquad \text{and} \qquad \|Pv - v\| < \varepsilon$$

for all $T \in F$ and $v \in V$.

The notion of quasidiagonality of operators is dependent on the choice of concrete representation, and it turns out that it is not a $C^*$-algebraic property; in [31, Example 7], Hadwin constructs two *-isomorphic, concretely represented $C^*$-algebras, one of which is a quasidiagonal set of operators, but the other set is not. This shows that the existence of some faithful, quasidiagonal *-representation does not imply that all faithful, quasidiagonal

*-representations need to be quasidiagonal. The representation theoretic definition of being quasidiagonal remedies this by assuming the existence of some faithful, quasidiagonal *-representation.

**Definition 5.2.** We say that a $C^*$-algebra is *quasidiagonal* if it admits a faithful, quasidiagonal *-representation. Moreover, we say that a $C^*$-algebra is *strongly quasidiagonal* if all its *-representations are quasidiagonal.

As mentioned above, one might have non-quasidiagonal, faithful *-representations on quasidiagonal $C^*$-algebras. The following theorem shows that the obstruction is the possibility of the *-representation intersecting the compact operators non-trivially. Terminologically, we say that a *-representation $\pi\colon A \to \mathbb{B}(H)$ is *essential* if $\pi(A) \cap \mathbb{K}(H) = \{0\}$. We also mention a characterisation of quasidiagonality, which is used briefly later, and which shows how quasidiagonality is inherently an approximation property of $C^*$-algebras similar to e.g. nuclearity, although the two are not logically related.

**Theorem 5.3.** *Let $A$ be a $C^*$-algebra. The following are equivalent.*

(i) *$A$ is a quasidiagonal $C^*$-algebra.*

(ii) *All faithful and essential *-representations of $A$ are quasidiagonal.*

(iii) *$A$ has the following approximation property: There exists a net of c.c.p. maps $\varphi_\lambda\colon A \to M_{k_\lambda}(\mathbb{C})$ for which*

$$\|\varphi_\lambda(ab) - \varphi_\lambda(a)\varphi_\lambda(b)\| \to 0 \qquad \text{and} \qquad \|\varphi_\lambda(a)\| \to \|a\|,$$

*for all $a, b \in A$.*

For a proof, we refer to [14, Theorem 7.2.5].

One way that quasidiagonality fits well within the subject of this thesis is exemplified by the following proposition, which shows that quasidiagonality implies stably finiteness, and we even obtain a tracial state in the unital case.

**Proposition 5.4.** *Let $A$ be a quasidiagonal $C^*$-algebra. Then $M_n(A)$ is finite for all $n \in \mathbb{N}$. If moreover $A$ is unital, then it admits a tracial state.*

*Proof.* We first show that $M_n(A)$ is finite for all $n \in \mathbb{N}$. As $A$ being quasidiagonal implies that any matrix algebra over $A$ as well as the unitisation of $A$ is quasidiagonal, it suffices to show that a unital, quasidiagonal $C^*$-algebra $A$ is always finite. Suppose that $s \in A$ is a proper isometry, and let $\varphi_\lambda\colon A \to M_{k_\lambda}(\mathbb{C})$ be asymptotically multiplicative and asymptotically isometric u.c.p. maps from Theorem 5.3(iii). Then $\varphi_\lambda(s^*s) \to 1$, yet $\varphi_\lambda(ss^*) \not\to 1$, which is impossible.

Now suppose that $A$ is a unital, quasidiagonal $C^*$-algebra, and let $\varphi_\lambda\colon A \to M_{k_\lambda}(\mathbb{C})$ be asymptotically multiplicative and asymptotically isometric u.c.p. maps witnessing quasidiagonality. Let $\mathrm{tr}_{k_\lambda}$ be the (unique) tracial state on $M_{k_\lambda}(\mathbb{C})$, and consider the family of states $(\mathrm{tr}_{k_\lambda} \circ \varphi_\lambda)_{\lambda\in\Lambda}$ on $A$. Since $A$ is unital, the state space $\mathrm{S}(A)$ is compact, and any accumulation point of this family will provide a tracial state on $A$. □

To get a feeling for (strong) quasidiagonality of $C^*$-algebras, we shall take a closer look at quasidiagonality of group-$C^*$-algebras. The study was initiated by Rosenberg in an appendix to Hadwin's aforementioned paper [31], where he showed that quasidiagonality of the reduced group-$C^*$-algebra implies amenability of the group. A group $G$ is said to be *amenable* if there exists a left-$G$-invariant state on $\ell^\infty(G)$; see also [14, Theorem 2.6.8] for a few of the approximately $10^{10^{10}}$ different characterisations of amenability.

**Theorem 5.5** (Rosenberg)**.** *Let $G$ be a discrete group. If $C_r^*(G)$ is quasidiagonal, then $G$ is amenable.*

There are several proofs of this fact; the original may be found in the appendix to [31], and a more modern version dealing with amenable tracial states, which are actually those constructed in Proposition 5.4, might be found in [14, Corollary 7.1.17]. Rosenberg's theorem implies specifically that quasidiagonality and stably finiteness are not equivalent properties — for instance, the free group $\mathbb{F}_n$ on $n$ generators for any $2 \leq n \leq \infty$ are non-amenable, so $C_r^*(\mathbb{F}_n)$ are not quasidiagonal $C^*$-algebras, however they are stably finite as they admit faithful tracial states cf. Proposition 2.16. A famous conjecture due to Blackadar–Kirchberg [8] states that this is impossible in the nuclear case:

**Conjecture 5.6** (Blackadar–Kirchberg)**.** A separable, nuclear $C^*$-algebra is stably finite if and only if it is quasidiagonal.

In the appendix in which Rosenberg proved his theorem, Theorem 5.5, he conjectured that the converse should hold, and hence that amenability of a group $G$ should be equivalent with the reduced group-$C^*$-algebra $C_r^*(G)$ being quasidiagonal. In [69], Tikuisis, White, and Winter proved this by showing that a large class of $C^*$-algebras are in fact quasidiagonal, which also improved the result of the Elliott classification program [23]. We mention the following refinement due to Gabe [25]; see also [67] for an elegant proof of the result. Before we mention the result, let us define two properties of tracial states related to both amenability and quasidiagonality.

**Definition 5.7.** Let $A$ be a unital, separable $C^*$-algebra, and let $\tau$ be a tracial state on $A$. We say that:

- $\tau$ is *amenable* if there exists a sequence $\varphi_n\colon A \to M_{k_n}(\mathbb{C})$ of u.c.p. maps satisfying

$$\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\|_2 \to 0 \qquad \text{and} \qquad \mathrm{tr}_{k_n}(\varphi_n(a)) \to \tau(a),$$

  for all $a, b \in A$, where $\|\cdot\|_2$ is the 2-norm induced on $A$ by the tracial state $\tau$.

- $\tau$ is *quasidiagonal* if there exists a sequence $\varphi_n\colon A \to M_{k_n}(\mathbb{C})$ of u.c.p. maps satisfying

$$\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0 \qquad \text{and} \qquad \mathrm{tr}_{k_n}(\varphi_n(a)) \to \tau(a),$$

  for all $a, b \in A$.

It is immediate that all quasidiagonal tracial states are amenable, but it is unknown whether the converse holds. The converse to Rosenberg's theorem can be seen as a partial answer to this question for faithful tracial states in the exact, unital, separable case.

**Theorem 5.8** (Tikuisis–White–Winter, Gabe, Schafhauser)**.** *Suppose that $A$ is a separable, unital and exact $C^*$-algebra satisfying the UCT. Then all faithful, amenable tracial states on $A$ are quasidiagonal.*

The reader should be aware that the original theorem by Tikuisis–White–Winter was in the nuclear case, and any tracial state on a nuclear $C^*$-algebra is automatically amenable by [14, Proposition 6.3.4], hence this assumption is vacuous in the original formulation. The following proposition due to Gabe [25, Proposition 3.4], with a weaker formulation in the original Tikuisis–White–Winter paper [69, Proposition 1.4], provides the last piece of machinery in using Theorem 5.8 to obtain quasidiagonality.

**Proposition 5.9.** *If $A$ is a separable, unital, and exact $C^*$-algebra with a faithful, quasidiagonal tracial state, then $A$ is quasidiagonal.*

Let us look at some implications, namely how the theorem proves the converse to Rosenberg's theorem, and that it partially resolves Blackadar–Kirchberg's conjecture by showing it in the affirmative for simple $C^*$-algebras satisfying the UCT [69, Corollary 6.1].

**Corollary 5.10** (Tikuisis–White–Winter)**.** *Conjecture 5.6 holds for the class of simple $C^*$-algebras in the UCT class.*

For the unital case, the proof is quite straightforward: If $A$ is a separable, simple, nuclear and stably finite $C^*$-algebra, then $A$ admits a tracial state by Theorem 3.19, and simplicity implies that this is necessarily a faithful tracial state. This tracial state is hence quasidiagonal by Theorem 5.8, which implies that $A$ is quasidiagonal by Proposition 5.9.

The converse to Rosenberg's theorem is also quite straightforward, when one has the right tools.

**Corollary 5.11** (Tikuisis–White–Winter)**.** *If $G$ is a discrete, amenable group, then $C_r^*(G)$ is a quasidiagonal $C^*$-algebra.*

*Proof.* We prove the case for countable groups — the uncountable case follows by noting that any group can be realised as an inductive limit of its countable subgroups. Since $G$ is amenable, $C_r^*(G)$ is nuclear and UCT, cf. [14, Theorem 2.6.8] and [70]. Moreover, the $C_r^*(G)$ canonically admits a faithful tracial state, which is quasidiagonal by Theorem 5.8, hence $C_r^*(G)$ is quasidiagonal by Proposition 5.9. □

It should be noted that quasidiagonality of the full group-$C^*$-algebra $C^*(G)$ does not imply that $G$ is amenable; for example, $C^*(\mathbb{F}_n)$ is quasidiagonal for any $2 \leq n \leq \infty$, [14, Theorem 7.4.1], yet $\mathbb{F}_n$ is non-amenable for any $n$. If $C^*(G)$ is strongly quasidiagonal, then all quotients will be quasidiagonal, so in particular the reduced group-$C^*$-algebra $C_r^*(G)$, and hence strong quasidiagonality implies amenability. The converse is false, as the following example known as the Lamplighter group shows.

**Proposition 5.12** (Carrión–Dadarlat–Eckhardt)**.** *Consider the group $G = \bigotimes_{\mathbb{Z}} \mathbb{Z}_2 \rtimes_\alpha \mathbb{Z}$ with the action $\alpha$ being the shift action. Then $G$ is an amenable group such that $C^*(G)$ has an infinite quotient. In particular, $C^*(G)$ is not strongly quasidiagonal.*

*Proof.* Embed $\bigotimes_{\mathbb{Z}} \mathbb{Z}_2 \otimes_{\rtimes} \mathbb{Z}$ in $\ell^2(\mathbb{Z})$ via $\bigotimes_{\mathbb{Z}} \mathbb{Z}_2 \supseteq (t_j)_{j \in \mathbb{Z}} \mapsto \mathrm{diag}(\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots)$, where

$$\lambda_i = \begin{cases} 1 & \text{if } t_i = e \\ -1 & \text{if } t_i \neq e \end{cases},$$

and with the action $\mathbb{Z}$-being the shift $S$ on $\mathbb{B}(\ell^2(\mathbb{Z}))$. Let $(\delta_n)_{n \in \mathbb{N}}$ be the canonical basis of $\ell^2(\mathbb{Z})$. Consider the element $T \in C^*(G)$ given by $T = \mathrm{diag}(\xi_i)$ with $\xi_i = 1$ for $i > 0$ and $\xi_i = -1$ for $i \leq 0$, and let $P = \frac{1}{2}(I + T)$. Then the element $V = SP$ is defined by

$$V\delta_n = \begin{cases} \delta_{n+1} & \text{if } n > 0 \\ 0 & \text{if } n \leq 0 \end{cases}.$$

It is now an easy calculation to verify that $V^*V - VV^* > 0$, and hence $V^*V$ is equivalent to a proper subprojection of itself, that is, $V^*V$ is an infinite projection. We conclude that $C^*(G)$ has an infinite quotient, and hence it cannot be a strongly quasidiagonal $C^*$-algebra. □

A generalisation of the above result may be found in [15, Theorem 3.4]. In both the proof of Proposition 5.12 and in the generalisations, the structure of the proof is similar: Construct a representation of the group with an infinite projection. However, it is important to note that if $A$ is a $C^*$-algebra for which all quotients are stably finite or even quasidiagonal, then

$A$ is not necessarily strongly quasidiagonal [13, Example 20], so the converse strategy does not hold in general. Once again, the issue is that while a given quotient may be a quasidiagonal $C^*$-algebra, not all faithful, quasidiagonal *-representations need be quasidiagonal. To put it more mathematically: Suppose that $A$ is a $C^*$-algebra, and let $\pi\colon A \to \mathbb{B}(H)$ be a *-representation for which $\pi(A)$ is a quasidiagonal $C^*$-algebra. Then $\pi(A)$ need not be a quasidiagonal set of operators — it may be that there exists a faithful *-representation $\rho\colon \pi(A) \to \mathbb{B}(H')$ for which $\rho(\pi(A))$ is a quasidiagonal set of operator. Hence $\rho \circ \pi$ is a quasidiagonal *-representation of $A$, but $\pi$ need not be! The issue once again boils down to the fact that $\pi(A)$ might intersect non-trivially with the compact operators on $H$. Hence if all *-representations are assumed to be essential, this obstruction would not be possible, and the $C^*$-algebra $A$ would be strongly quasidiagonal by Theorem 5.3. Since a $C^*$-algebra is strongly quasidiagonal if and only if all irreducible *-representations are quasidiagonal by [31, Proposition 5], one only needs to verify that all irreducible *-representations are essential, e.g. by showing that all images of irreducible *-representations, also known as the primitive quotients, cannot intersect non-trivially with the compacts. We shall see later in the thesis, c.f. Theorem 5.47, how this might work in practice, although the way of guaranteeing quasidiagonality of the quotients is an issue due to the UCT assumption of the Tikuisis–White–Winter theorem.

Let us lastly mention the known results about strong quasidiagonality of discrete groups. First, we introduce some terminology. Let $\mathcal{Z}(G)$ be the center of $G$ and recursively define $\mathcal{Z}_n(G)$ for $n \geq 2$ via $\mathcal{Z}_n(G)/\mathcal{Z}_{n-1}(G) = \mathcal{Z}(G/\mathcal{Z}_{n-1}(G))$ with $\mathcal{Z}_1(G) = \mathcal{Z}(G)$. If there exists $n \in \mathbb{N}$ such that $\mathcal{Z}_n(G) = G$, then we say that $G$ is *nilpotent*. We furthermore say that $G$ is *virtually nilpotent* if it contains a nilpotent subgroup of finite index. It turns out that virtually nilpotent groups are always strongly quasidiagonal [22]. In fact, one may even take inductive limits of such groups, since strong quasidiagonality is preserved under inductive limits.

**Theorem 5.13** (Eckhardt–Gillaspy–McKenney)**.** *If $G$ is a inductive limit of virtually nilpotent, discrete groups, then $C^*(G)$ is strongly quasidiagonal.*

The converse does not hold — prior to the above result, Eckhardt constructed a certain semi-direct product $\mathbb{Z}^3 \rtimes_\alpha \mathbb{Z}^2$, which is strongly quasidiagonal, yet which is not virtually nilpotent [20].

We end this section with a brief comment on the quasidiagonality of just-infinite $C^*$-algebras. In [27], Grigorchuk, Musat and Rørdam introduced the notion of just-infiniteness for $C^*$-algebra analogously to the group theoretical property of the same name. We say that a $C^*$-algebra $A$ is *just-infinite* if it is infinite-dimensional and all its proper quotients are finite-dimensional. While these have a certain almost finite-dimensional flavour, they can admit quite exotic behaviour; for example, any infinite-dimensional and simple $C^*$-algebra is trivially just-infinite, e.g., the Cuntz algebra $\mathcal{O}_2$. Li initiated in [43] the analysis of quasidiagonality of just-infinite $C^*$-algebras and proved, among other things, that quasidiagonality and inner quasidiagonality coincide among separable just-infinite $C^*$-algebra. This proof uses the classification of just-infinite $C^*$-algebras of [27, Theorem 3.10], but there is an even stronger and more elementary proof without resorting to this classification as seen below.

**Theorem 5.14.** *A just-infinite $C^*$-algebra is quasidiagonal if and only if it is strongly quasidiagonal.*

*Proof.* It is clear that strongly quasidiagonal $C^*$-algebras are quasidiagonal, so suppose that $A$ is a quasidiagonal $C^*$-algebra. Observe that all non-faithful *-representations of $A$ are quasidiagonal since all the proper quotients of $A$ are finite-dimensional. Hence we only need to prove quasidiagonality of faithful irreducible *-representations. So assume that $\pi\colon A \to \mathbb{B}(H)$ is a faithful, irreducible *-representation, then there are two possibilities: Either $\pi$ is essential, or $\mathbb{K}(H)$ is an ideal in $\pi(A)$. If $\pi$ is essential, then as $A$ is quasidiagonal and $\pi$ is faithful, $\pi$ is quasidiagonal. So suppose that $\mathbb{K}(H)$ is an ideal in $\pi(A)$. Then either $\pi(A) \cong \mathbb{K}(H)$, or $\pi(A)$ is an extension of the AF-algebra $\mathbb{K}(H)$ and the finite-dimensional $C^*$-algebra $\pi(A)/\mathbb{K}(H)$. Consequently, $\pi(A)$ is an AF-algebra and, thus, strongly quasidiagonal. Hence $A$ is strongly quasidiagonal (and, in fact, an AF-algebra). □

**Corollary 5.15.** *Let $G$ be a discrete group such that $C^*(G)$ is just-infinite. Then $C^*(G)$ is strongly quasidiagonal.*

*Proof.* Consider the canonical surjection $C^*(G) \to C^*_r(G)$. As $C^*(G)$ is just-infinite, we must either have that this is an isomorphism or that $C^*_r(G)$ is finite-dimensional (in which case the surjection is also an isomorphism). In particular, $G$ is amenable and hence $C^*_r(G)$ is quasidiagonal by Corollary 5.11. Therefore, by Theorem 5.14, $C^*(G)$ is strongly quasidiagonal. □

Following the terminology of [27, Theorem 3.10], any just-infinite $C^*$-algebra of type ($\gamma$) is residually-finite dimensional and, hence, strongly quasidiagonal by Theorem 5.14. Since type ($\alpha$) consists of all simple and infinite-dimensional $C^*$-algebras, there are plenty of quasidiagonal and non-quasidiagonal examples in this type. It is unknown which just-infinite $C^*$-algebras of type ($\beta$) are quasidiagonal, but the following trivial observation, which is contained in the proof of Theorem 5.14, provides a subclass of quasidiagonal just-infinite $C^*$-algebras.

**Proposition 5.16.** *If $A$ is a non-simple just-infinite $C^*$-algebra which has a non-essential irreducible *-representation, then $A$ is of type ($\beta$) and (strongly) quasidiagonal.*

It is worth noting that, since separable just-infinite $C^*$-algebras are primitive [27, Lemma 3.2], the existence of a faithful irreducible *-representation is guaranteed. Hence one only needs to examine this representation to determine strong quasidiagonality of just-infinite $C^*$-algebras: If it is non-essential *or* it is essential *and* quasidiagonal, the $C^*$-algebra is strongly quasidiagonal. We can summarise this in the following corollary, which gives the quasidiagonality dichotomy of just-infinite $C^*$-algebras.

**Corollary 5.17.** *Let $A$ be a separable just-infinite $C^*$-algebra with a faithful irreducible *-representation $\pi\colon A \to \mathbb{B}(H)$. Then $A$ is strongly quasidiagonal if and only if $\pi$ is quasidiagonal.*

## 5.2 Separating families of tracial states on $C^*$-algebras

This section provides an overview of the mostly known results regarding separating families of tracial states. We say that $A$ has a *separating family of tracial states* if, for any positive non-zero $a \in A$, there exists $\tau \in \mathrm{T}(A)$ such that $\tau(a) \neq 0$ or, equivalently, if $\bigcap_{\tau \in \mathrm{T}(A)} I_\tau = \{0\}$. In general, admitting a faithful tracial state clearly implies having a separating family of tracial states, and the two properties are equivalent for separable, unital $C^*$-algebras:

**Proposition 5.18.** *Let $A$ be a separable, unital $C^*$-algebra. The following are equivalent.*

(i) *$A$ admits a faithful tracial state,*

(ii) *$A$ has a separating family of tracial states,*

(iii) *each non-zero ideal in $A$ admits a non-zero bounded positive trace.*

*Proof.* The implications (i)⇒(ii)⇒(iii) are trivial. In order to prove (ii)⇒(i), we first show that, for any non-zero $a \in A_+$, there exists a tracial state $\tau \in \mathrm{T}(A)$ such that $\|a + I_\tau\| > \|a\|/2$. We may, without loss of generality, assume that $\|a\| = 1$. Let $g\colon [0,1] \to [0,1]$ be the piecewise linear continuous function which is zero on $[0, 1/2]$ with $g(1) = 1$. By the continuous functional calculus, $g(a)$ is a non-zero, positive element in $A$ and, by assumption, there exists a tracial state $\tau \in \mathrm{T}(A)$ such that $\tau(g(a)) \neq 0$. It then follows that $g(a+I_\tau) = g(a)+I_\tau \neq 0$, and as $g$ is zero on $[0, 1/2]$, we hence conclude that $\|a + I_\tau\| > 1/2$. We shall now use this fact in showing that $A$ admits a faithful tracial state. By separability, there exists a countable norm-dense sequence $(a_n)_{n\in\mathbb{N}}$ of positive elements in $A$. By the above, we can, for each $n \in \mathbb{N}$, find a tracial state $\tau_n$ for which $\|a_n + I_{\tau_n}\| > \|a_n\|/2$. Let $\tau = \sum_{n\in\mathbb{N}} \tau_n/2^n$. It is easily verified that $\tau$ is a tracial state, and that $I_\tau = \bigcap_{n\in\mathbb{N}} I_{\tau_n}$. By this construction, it follows that, for any $n \in \mathbb{N}$,

$$\|a_n + I_\tau\| \geq \|a_n + I_{\tau_n}\| \geq \frac{\|a_n\|}{2},$$

and hence by continuity $\|a + I_\tau\| \geq \|a\|/2$ for all positive $a \in A$. This implies that $\tau(a) \neq 0$ for all positive non-zero $a \in A$, i.e., $\tau$ is faithful.

We now prove the remaining implication (iii)⇒(ii). Consider the ideal $I_0 = \bigcap_{\tau\in\mathrm{T}(A)} I_\tau$ and observe that this is non-zero if and only if (ii) is false. For the sake of reaching a contradiction, suppose that $I_0$ is non-zero. By assumption, there exists a non-zero bounded positive trace on $I_0$, which can be extended to all of $A$ by Proposition 2.17(i). Upon normalising to ensure unitality of this trace, we can assume that it is a tracial state; call it $\tau_0$. But since $\tau_0$ is non-zero on $I_0$, we reach the contradiction that $I_0$ is not contained in $I_{\tau_0}$. □

There is a purely algebraic reformulation of admitting a separating family of tracial states due to Cuntz–Pedersen [17, Theorem 3.4].

**Proposition 5.19** (Cuntz–Pedersen, 1979). *A $C^*$-algebra $A$ has a separating family of tracial states if and only if, for any $x_1, \ldots, x_n \in A$ with $\sum_{i=1}^n x_i x_i^* \leq \sum_{i=1}^n x_i^* x_i$, we have the equality $\sum_{i=1}^n x_i x_i^* = \sum_{i=1}^n x_i^* x_i$.*

Despite the visual appeal of this characterisation, it is very difficult to verify it for some given $C^*$-algebras and, in general, proving the existence of separating families of tracial states is not an easy task. We mention a few classes below, which are mostly trivial facts and easy to verify, with the added comment that (v) is a combination of Theorem 2.32 and Theorem 3.19 to prove the existence of a tracial state, and then simplicity will guarantee faithfulness of this tracial state, as its trace-kernel ideal must be the zero $C^*$-algebra.

**Example 5.20.** The following classes of $C^*$-algebras all admit separating families of tracial states.

(i) Finite-dimensional $C^*$-algebras,

(ii) Abelian $C^*$-algebras,

(iii) Residually finite dimensional $C^*$-algebras,

(iv) $C_r^*(G)$ for any discrete group $G$,

(v) Simple, unital, stably not properly infinite, exact $C^*$-algebras.

Note that while the reduced group $C^*$-algebras of discrete groups always admit faithful tracial states, the same does not hold for full group-$C^*$-algebras. Bekka showed in [3, Corollary 5] that the $C^*$-algebras $C^*(\mathrm{SL}_n(\mathbb{Z}))$ do not admit faithful tracial states for any $n \geq 3$.

**Proposition 5.21.** *Let $A$ and $B$ be unital $C^*$-algebras.*

(i) *If $A$ has a separating family of tracial states, then so does any $C^*$-subalgebra of $A$.*

(ii) *The minimal tensor product $A \otimes B$ has a separating family of tracial states if and only if both $A$ and $B$ do.*

(iii) *$A$ has a separating family of tracial states if and only if $M_n(A)$ does for some (hence all) $n \in \mathbb{N}$.*

(iv) *If $A = A_1 \oplus \ldots \oplus A_n$ is a finite direct sum of $C^*$-algebras each with separating families of tracial states, then $A$ has a separating family of tracial states.*

(v) *If $I$ is an ideal in $A$ and $I$ contains a separating family of tracial states, then their canonical tracial extensions to $A$ form a separating family of tracial states if and only if $I$ is an essential ideal in $A$. In particular, if any $C^*$-algebra $I$ has a separating family of tracial states, then so does the multiplier algebra $\mathcal{M}(I)$.*

(vi) *If $I$ is an ideal in $A$ with a separating family of tracial states, and if $A/I$ has a separating family of tracial states, then so does $A$. In other words, admitting separating families of tracial states is preserved by taking extensions.*

*Proof.* (i): This is immediate.

(ii): Since $A$ and $B$ can be realised as $C^*$-subalgebras of $A \otimes B$, one direction is immediate by the use of (i). So assume that both $A$ and $B$ have a separating family of tracial states. Suppose, for the sake of reaching a contradiction, that the ideal

$$I = \bigcap_{\tau \in \mathrm{T}(A \otimes B)} I_\tau$$

is non-zero. By Kirchberg's slice lemma, see [59, Lemma 4.1.9], we find a non-zero element $z \in A \otimes B$ with $z^*z \in I$ such that $zz^* = a \otimes b$ for some positive $a \in A$ and $b \in B$. Observe that this implies $z \in I$. As $A$ and $B$ both have separating family of tracial states, we may find tracial states $\tau_A$ and $\tau_B$ on $A$ and $B$, respectively, such that $\tau_A(a) \neq 0$ and $\tau_B(b) \neq 0$. But then we clearly reach a contradiction, as

$$(\tau_A \otimes \tau_B)(z^*z) = (\tau_A \otimes \tau_B)(zz^*) = (\tau_A \otimes \tau_B)(a \otimes b) = \tau_A(a)\tau_B(b) \neq 0,$$

which would imply that $z \notin I_\tau$, contradicting the construction of $z$.

(iii): This follows from (ii), since $M_n(A) \cong M_n(\mathbb{C}) \otimes A$ and $M_n(\mathbb{C})$ admits a faithful tracial state for every $n \in \mathbb{N}$.

(iv): This is obvious.

(v): Suppose that $I$ is an essential ideal in $A$. If $\tau \in \mathrm{T}(I)$ is a tracial state, we denote by $\tau'$ the canonically extended tracial state on $A$. Assume that $a \in A$ satisfies that $\tau'(a^*a) = 0$ for all $\tau' \in \mathrm{T}(A)$. In particular, for any $b \in I$, we find that

$$\tau((ba)^*(ba)) \leq \|b^*b\|\,\tau'(a^*a) = 0,$$

for all $\tau \in \mathrm{T}(I)$. Since $I$ has a separating family of tracial states, this means that $ba = 0$ for all $b \in I$ or, equivalently, that $Ia = 0$. As $I$ is an essential ideal in $A$, we get that $a = 0$ as desired. Conversely, suppose that the canonical extensions of tracial states of $I$ on $A$ are separating. Let $a \in A^+ \setminus \{0\}$ be arbitrary and suppose that $Ia = aI = 0$. By assumption, there exists a tracial state $\tau$ on $I$ for which the canonical extension $\tau'$ on $A$ satisfies that $\tau'(a) \neq 0$. However, this contradicts the assumption that $a$ is orthogonal to $I$, since

$$\tau'(a) = \lim_i \tau(ae_\lambda) = 0,$$

where $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $I$, see Proposition 2.17.

(vi): Let $a \in A^+ \setminus \{0\}$ be arbitrary and let $\pi\colon A \to A/I$ denote the quotient map. If $a \in I$, then there exists a tracial state $\tau$ on $I$ such that $\tau(a) > 0$, which extends canonically to a tracial state on $A$. On the other hand, if $a \notin I$, then $\pi(a) \in (A/I)^+ \setminus \{0\}$ and, by assumption, there exists a tracial state $\tau'$ on $A/I$ for which $\tau'(\pi(a)) > 0$. This proves that $A$ has a separating family of tracial states. □

The following proposition is an easy extension of the case with faithful tracial states, see Proposition 2.16, so we skip the proof.

**Proposition 5.22.** *Let $A$ be a $C^*$-algebra with a separating family of tracial states. Then (the unitisation of) $M_n(A)$ contains no proper isometries for any $n \in \mathbb{N}$, and $A$ has no stable $C^*$-subalgebra.*

We now give some counterexamples to other possible permanence properties.

**Example 5.23.** Admitting a separating family of tracial states does not pass to inductive limits.

*Proof.* Consider the unitization $\tilde{\mathbb{K}}$ of the compact operators. This is a unital AF-algebra, hence it is the inductive limit of finite-dimensional $C^*$-algebras admitting faithful tracial states, but it does not admit a faithful tracial state itself, since it contains $\mathbb{K}$ as an ideal. □

**Example 5.24.** Admitting a separating family of tracial states does not pass to quotients.

*Proof.* Any separable, unital $C^*$-algebra can be realised as a quotient of the residually finite-dimensional $C^*$-algebra $C^*(\mathbb{F}_\infty)$. □

It does not hold in general that admitting separating families of tracial states passes to *maximal* tensor products. By following the proof of [42, Proposition 3.13] in the case $D = A \otimes_{\max} B$, one obtains the next proposition.

**Proposition 5.25.** *Let $A$ and $B$ be unital $C^*$-algebras. If $A \otimes_{\max} B$ admits a separating family of tracial states, then $A \otimes_{\max} B = A \otimes B$.*

We can use this proposition to show that Proposition 5.21(ii) fails for *maximal* tensor products. It is well-known, see e.g. [68], that $C_r^*(\mathbb{F}_2) \otimes_{\max} C_r^*(\mathbb{F}_2) \neq C_r^*(\mathbb{F}_2) \otimes C_r^*(\mathbb{F}_2)$. By Proposition 5.25, the maximal tensor product does not admit a separating family of tracial states. On the other hand, $C_r^*(\mathbb{F}_2)$ admits a faithful tracial state being the reduced group-$C^*$-algebra of a discrete group. Hence the maximal tensor product of two $C^*$-algebras admitting separating families of tracial states need not admit a separating family of tracial states.

Another interesting usage of Proposition 5.25 is its relation to the Connes embedding problem, which is how it originally appears in [38]. One equivalent formulation of the Connes embedding problem, see [14, Theorem 13.3.1], is that $C^*(\mathbb{F}_\infty)\otimes_{\max}C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty)\otimes C^*(\mathbb{F}_\infty)$. If the maximal tensor product admitted a faithful tracial state, this equality would be true by Proposition 5.25; see also [14, Exercise 13.3.1-4]. By the announced negative answer to the Connes embedding problem [36], we would thus be able to conclude that $C^*(\mathbb{F}_\infty \times \mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$ does not admit a faithful tracial state.

Lastly, we examine some equivalent notions of having separating families of tracial states or admitting a faithful tracial state by using von Neumann terminology.

**Proposition 5.26.** *Let $A$ be a unital $C^*$-algebra.*

(i) *$A$ admits a separating family of tracial states if and only if $A$ unitally embeds into a finite von Neumann algebra.*

(ii) *$A$ admits a faithful tracial state if and only if $A$ unitally embeds into a $\mathrm{II}_1$-factor.*

*Proof.* (i): Every finite von Neumann algebra has a separating family of tracial states by [6, III.2.5.8], so we only need to prove the "only if" direction. For any tracial state $\tau \in \mathrm{T}(A)$, we can consider the GNS-representation $\pi_\tau \colon A \to \mathbb{B}(H_\tau)$. Since $\bigcap_{\tau\in\mathrm{T}(A)} I_\tau = \{0\}$, it follows that the product $\pi := \bigoplus_{\tau\in\mathrm{T}(A)} \pi_\tau \colon A \to \mathbb{B}(H)$ is injective. Moreover,

$$A \cong \pi(A) \subseteq \bigoplus_{\tau\in\mathrm{T}(A)} \pi_\tau(A)'',$$

and the right-hand side is a finite von Neumann algebra.

(ii): A $\mathrm{II}_1$-factor immediately admits a faithful tracial state so to prove the other direction, let us assume that $A$ admits a faithful tracial state $\tau$. Then $A$ unitally embeds via the GNS representation of $\tau$ into a finite von Neumann algebra $M$ with a normal faithful tracial state. The claim now follows from the well-known fact that any von Neumann algebra with a normal faithful tracial state embeds into a $\mathrm{II}_1$-factor, see e.g. the proof of [48, Theorem A.1] □

### 5.3 Faithful tracial states on quotients

In [47], Murphy initiated the study of $C^*$-algebras whose quotients all admit tracial states. He coined this notion QTS for *quotient tracial states* and examples of $C^*$-algebras with the QTS property include unital strongly quasidiagonal $C^*$-algebras and group-$C^*$-algebras of amenable groups. In this section, we shall consider a stronger condition, namely that all quotients of the $C^*$-algebra admit *faithful* tracial states. Continuing the terminology introduced by Murphy, we shall call this the *QFTS property.* Let us look at a few examples of $C^*$-algebras with the QFTS property.

**Example 5.27.** The following $C^*$-algebras have the QFTS property:

(i) Unital $C^*$-algebras with the QTS property and $T_1$ primitive ideal space,

(ii) $C^*(G)$ for virtually nilpotent groups $G$,

(iii) Subhomogeneous $C^*$-algebras.

*Proof.* (i): Suppose $A$ is a unital $C^*$-algebra with the QTS property, and suppose that the primitive ideal space $\mathrm{Prim}(A)$ is $T_1$ or, equivalently, that all primitive quotients are simple. Since $A$ has the QTS property as $G$ is amenable, all primitive quotients will therefore admit faithful tracial states. Now, if $I$ is any ideal in $A$, then $I$ is equal to the intersection of all primitive ideals containing it, and we get an embedding

$$A/I \hookrightarrow \prod_{I\subseteq J\in\mathrm{Prim}(A)} A/J,$$

and we have just proved that the right-hand side admits faithful tracial states.

(ii): If $G$ is virtually nilpotent, then $C^*(G)$ has a $T_1$ primitive ideal space by [19, Corollary 3.2], and moreover $C^*(G)$ has the QTS property. The result now follows from (i).

(iii): Any quotient of a subhomogeneous $C^*$-algebra is again subhomogeneous, so it suffices to show that unital subhomogeneous $C^*$-algebras admit faithful tracial states, which follows from Example 5.20. □

It is immediate that having the QFTS property implies admitting a faithful tracial state, and the converse fails in general as shown in Example 5.24. One way of viewing the QFTS property is by the fact that the ideals are completely characterised by the tracial states.

**Proposition 5.28.** *Let $A$ be a separable, unital $C^*$-algebra. Then $A$ has the QFTS property if and only if all ideals of $A$ can be realised as the trace-kernel of a tracial state on $A$.*

*Proof.* Suppose that $A$ has the QFTS property and let $I$ be an ideal in $A$. Denote by $\pi\colon A \to A/I$ the quotient mapping. By the QFTS property, $A/I$ admits a faithful tracial state $\tilde{\tau}$, which, in turn, induces a tracial state $\tau = \tilde{\tau} \circ \pi$ on $A$. It is clear that $I \subseteq I_\tau$, so suppose that $a \in I_\tau$. Then

$$0 = \tau(a^*a) = \tilde{\tau}(\pi(a^*a)) = \tilde{\tau}(\pi(a)^*\pi(a)),$$

and by faithfulness of $\tilde{\tau}$, we see that $\pi(a) = 0$ and, hence, that $a \in I$.

Now suppose that, for any ideal $I$ in $A$, there exists a tracial state $\tau \in \mathrm{T}(A)$ such that $I_\tau = I$. We can hence induce a tracial state $\tilde{\tau}$ on $A/I$ via $\tilde{\tau}(\pi(a)) = \tau(a)$, where $\pi\colon A \to A/I_\tau$ is the quotient map. It is easily verified that $\tilde{\tau}$ is a faithful tracial state, which completes the proof. □

Using the equivalences of Proposition 5.18 and the fact that stable $C^*$-algebras cannot admit bounded traces, we obtain the following.

**Proposition 5.29.** *A separable, unital $C^*$-algebra $A$ has the QFTS property if and only if every intermediate quotient[3] admits a tracial state. In particular, if $A$ has the QFTS property, then $A$ has no stable intermediate quotient and no properly infinite quotients.*

One goal of this chapter is to provide a converse to the latter part of Proposition 5.29 and, consequently, give an equivalent reformulation of the QFTS property for some classes of $C^*$-algebras. We shall attack this problem in two different manners: One by using the connections between dimension functions on Cuntz semigroups and quasitraces, or tracial states in this case as we only deal with exact $C^*$-algebras, and another by using a result on stability of hereditary $C^*$-subalgebras in [33].

We first use the connection between dimension functions and tracial states on exact, unital $C^*$-algebras as given in Theorem 2.31. Our starting point is to note that separating families of tracial states and lower semi-continuous dimension functions are equivalent conditions.

**Proposition 5.30.** *Let $A$ be a unital, exact $C^*$-algebra. Then $A$ has a separating family of tracial states if and only if* $\mathrm{Cu}(A)$ *has a separating family of lower semi-continuous dimension functions.*

*Proof.* By a separating family of lower semi-continuous dimension functions we mean that, for any $a \in (A \otimes \mathbb{K}(H))_+$, there exists a lower semi-continuous dimension function $d \in \mathrm{LDF}(A)$ for which $d(\langle a \rangle) \neq 0$. It follows from Theorem 3.19 and Theorem 2.31 that we have an affine bijection between $\mathrm{T}(A)$ and $\mathrm{LDF}(A)$. The statement of the proposition is merely rewriting the notion of separating family of tracial states via this association. □

[3]Recall that an *intermediate quotient* of $A$ is a $C^*$-algebra of the form $I/J$, where $J \subseteq I \subseteq A$ are ideals in $A$. In other words, any intermediate quotient can be realised as the ideal of a quotient of $A$.

Proposition 5.30 entails that understanding the structure of $\mathrm{LDF}(A)$ will help in understanding when $C^*$-algebras admit faithful tracial states. The following proposition due to Goodearl and Handelman [26, Proposition 4.2] gives a characterisation of when certain states exist on ordered Abelian semigroups. For the statement, recall that an element $t$ on an ordered Abelian semigroup is *properly infinite* if $2t \leq t$.

**Proposition 5.31** (Goodearl–Handelman, 1972)**.** *Let $S$ be an ordered Abelian semigroup with a distinguished order unit $u$ and assume that no multiple of $u$ is properly infinite. Let $t \in S$ be arbitrary and set*

$$\alpha_* = \sup\{k/\ell \mid k, \ell \in \mathbb{N} \text{ and } ku \leq \ell t\},$$
$$\alpha^* = \inf\{k/\ell \mid k, \ell \in \mathbb{N} \text{ and } \ell t \leq ku\}.$$

*Then $0 \leq \alpha_* \leq \alpha^*$ and there exists $d \in \Sigma(S, u)$ with $d(t) = \alpha$ if and only if $\alpha_* \leq \alpha \leq \alpha^*$.*

From this we immediately get the following corollary.

**Corollary 5.32.** *Let $S$ be an ordered Abelian semigroup with a distinguished order unit, and let $t \in S$. Then $d(t) = 0$ for all $d \in \Sigma(S, u)$ if and only if*

$$\alpha^* = \inf\{k/\ell \mid k, \ell \in \mathbb{N} \text{ and } \ell t \leq ku\} = 0.$$

Observe that while this corollary seems almost directly applicable to Proposition 5.30, there is an important subtlety, namely that in that proposition we are interested in separating families of *lower semi-continuous* dimension functions. To remedy this, we need to look at some of the regularity properties of Cuntz semigroups that we examined earlier in Chapter 2. For example, assuming almost unperforation allows us to expand on Corollary 5.32.

**Lemma 5.33.** *Let $S$ be an ordered Abelian semigroup with a distinguished order unit $u$, and suppose that $S$ is almost unperforated. Let $t \in S$. The following are equivalent:*

(i) *$d(t) = 0$ for all $d \in \Sigma(S, u)$,*

(ii) *$\inf\{k/\ell \mid k, \ell \in \mathbb{N}$ and $\ell t \leq ku\} = 0$,*

(iii) *There exists $k \in \mathbb{N}$ such that $\ell t \leq ku$ for all $\ell \in \mathbb{N}$,*

(iv) *$\ell t \leq u$ for all $\ell \in \mathbb{N}$.*

*Proof.* It is clear that (iv)⇒(iii)⇒(ii) and (ii)⇔(i) is the content of Corollary 5.32, so let us prove (i)⇒(iv). Suppose $d(t) = 0$ for all $d \in \Sigma(S, u)$. Since $d(\ell t) = 0$ for all $\ell \in \mathbb{N}$ and $d(u) = 1$, whenever $d \in \Sigma(S, u)$, and as $u$ is an order unit for $S$, we find, cf. [50], that there exists $n \in \mathbb{N}$ such that $(n + 1)\ell t < nu$, which implies $\ell t \leq u$ by almost unperforation of $S$. □

Note that this proposition needs almost unperforation to be true. Consider the ordered Abelian semigroup $S = \{0, 1, \infty\}$ equipped with $1 + 1 = \infty$ and the usual ordering. Observe that $S$ has 1-comparison, but it is not almost unperforated. Note that $u = 1$ is an order unit, and that $\ell t \leq 2u = \infty$ for all $t \in S$ and $\ell \in \mathbb{N}$, such that (i) above holds. However, (iv) is false, since $\infty \not\leq 1$.

An interesting related property to the equivalent conditions of Lemma 5.33 is the notion of $\beta$-comparison introduced by Bosa-Petzka in [12].

**Definition 5.34.** Let $S$ be an ordered Abelian semigroup. Define for each $x, y \in S$ the numerical quantity

$$\beta(x, y) = \inf\{k/\ell \mid k, \ell \in \mathbb{N} \text{ and } \ell x \leq ky\}.$$

We say that $S$ has *$\beta$-comparison* if, whenever $x, y \in S$ satisfies $\beta(x, y) = 0$, then $x \leq y$.

It holds in general that $\beta$-comparison implies $\omega$-comparison, but the above example shows that they are not equivalent properties for ordered Abelian semigroups. For simple $C^*$-algebras, however, they are equivalent by the arguments in [12, Section 5]. Note that almost unperforation implies $\beta$-comparison; the easiest way of seeing this is to note that $\beta(x,y) < 1$ if and only if $x <_s y$.

**Lemma 5.35.** *Let $S$ be an Abelian semigroup with a distinguished order unit $u$ and let $t \in S$ be arbitrary. Suppose that $S$ has $\beta$-comparison. Then (i)–(iv) in Lemma 5.33 are equivalent.*

*Proof.* We only need to prove (ii)$\Rightarrow$(iv). It is easily seen that, if $\beta(x,y) = 0$, then $\beta(\ell x, y) = 0$ for all $\ell \in \mathbb{N}$. Observe that (ii) is equivalent to $\beta(t,u) = 0$, and that this implies $\beta(\ell t, u) = 0$ for all $\ell \in \mathbb{N}$. Since $S$ has $\beta$-comparison, we conclude that $\ell t \leq u$ for any $\ell \in \mathbb{N}$. □

The next proposition follows immediately from [57, Lemma 2.4(ii)], and we extend the result in the subsequent lemma assuming stable rank one.

**Proposition 5.36.** *Let $A$ be a unital $C^*$-algebra and assume that $\ell \langle a \rangle \leq \langle 1_A \rangle$ for some $\ell \in \mathbb{N}$ and $a \in A$. Then, for any $\varepsilon > 0$, there exists mutually orthogonal, mutually equivalent positive elements $e_1, \dots, e_\ell \in A$ such that $e_i \sim (a-\varepsilon)_+$ for all $i = 1, \dots, \ell$.*

**Lemma 5.37.** *Let $A$ be a unital $C^*$-algebra with stable rank one and assume that there exists $a \in A$ such that $\ell \langle a \rangle \leq \langle 1_A \rangle$ for all $\ell \in \mathbb{N}$. Then, for any $\varepsilon > 0$, there exists a sequence of mutually orthogonal, mutually equivalent positive elements $(e_\ell)_{\ell \in \mathbb{N}} \subseteq A$ such that $e_\ell \sim (a-\varepsilon)_+$ for all $\ell \in \mathbb{N}$.*

*Proof.* Suppose that $\ell \langle a \rangle \leq \langle 1_A \rangle$ for all $\ell \in \mathbb{N}$. Let $n \in \mathbb{N}$ be arbitrary, then we can use Proposition 5.36 to construct $n$ pairwise orthogonal and pairwise equivalent positive elements $e_1, \dots, e_n$ such that $e_i \sim (a-\varepsilon)_+$ for some $\varepsilon > 0$. Let $b = \sum_{i=1}^n e_i$ and define for each $\delta > 0$ the function

$$h_\delta(t) = \begin{cases} \frac{\delta - t}{\delta} & \text{if } 0 \leq t \leq \delta \\ 0 & \text{if } t \geq \delta \end{cases}.$$

Observe that $(b-\delta)_+ \perp h_\delta(b)$ and that $b + h_\delta(b)$ is invertible. Let $\delta_n > 0$ be arbitrary, then we obtain the following inequalities:

$$n \langle (a-\varepsilon)_+ \rangle + \langle h_{\delta_n}(b) \rangle \geq \langle b \rangle + \langle h_{\delta_n}(b) \rangle \geq \langle u \rangle \geq (n+1) \langle a \rangle = n \langle a \rangle + \langle a \rangle .$$

By the cancellation theorem of [64, Theorem 4.3], we find that $\langle a \rangle \leq \langle h_{\delta_n}(b) \rangle$. Using [57, Lemma 2.4], we construct $c \in \overline{h_{\delta_n}(b) A h_{\delta_n}(b)}$ such that $c \sim (a-(\varepsilon+\delta_n))_+$. In particular, we find that $c \perp (e_i - \delta_n)_+$ for all $i = 1, \dots, n$, and that, for any $i = 1, \dots, n$,

$$c \sim (a-(\varepsilon+\delta_n))_+ = ((a-\varepsilon)_+ - \delta_n)_+ \sim (e_i - \delta_n)_+.$$

We have hence shown that, if we can construct $n$ pairwise orthogonal and pairwise equivalent positive elements $e_1, \dots, e_n$, then we can, for any $\delta > 0$, construct a positive element $e_{n+1} \in A$ such that $e_{n+1} \perp (e_i - \delta)_+$ and $e_{n+1} \sim (e_i - \delta)_+$ for any $i = 1, \dots, n$. Now we use this process inductively. Start with any positive element $e_1 \in A$ and let $\delta_1 > 0$. Use the above method to construct an element $e_2 \in A$ with the properties that $e_2 \perp (e_1 - \delta_1)_+$ and $e_2 \sim (e_1 - \delta_1)_+$. Now let $\delta_2 > \delta_1$ and construct $e_3 \in A$ such that $e_3 \sim (e_2 - \delta_2)_+ \sim (e_1 - (\delta_1 + \delta_2))_+$ and that these three elements are mutually orthogonal. Continuing this process, we can thus for any increasing sequence $0 < \delta_1 < \delta_2 < \cdots$ construct a sequence of positive elements $(e_n)_{n \in \mathbb{N}} \subseteq A$ with the property that, for any $m > n$,

$$e_m \sim \left( e_n - \sum_{k=n}^{m-1} \delta_k \right)_+ ,$$

and that these are orthogonal. Since we have no restrictions on the choice of the sequence $(\delta_n)_{n\in\mathbb{N}}$, we may take it to be any sequence such that the series $\delta = \sum_{n=1}^{\infty}\delta_n$ converges with limit $\delta < \|e_1\|$. Define $e_n' = (e_n - \sum_{k=n}^{\infty}\delta_k))_+$ for each $n \in \mathbb{N}$, then we see for any $m > n$ that

$$e_n' = \left(e_n - \sum_{k=n}^{\infty}\delta_k\right)_+ \sim \left(e_n - \left[\sum_{k=n}^{m-1}\delta_k + \sum_{k=m}^{\infty}\delta_k\right]\right)_+ \sim \left(e_m - \sum_{k=m}^{\infty}\delta_k\right)_+ = e_m'.$$

The proof of mutual orthogonality goes in a similar fashion. We have thus proved the existence of a sequence of pairwise orthogonal and pairwise equivalent elements in $A$. □

**Theorem 5.38.** *Let $A$ be a separable, unital, exact $C^*$-algebra with stable rank one. Assume that $\mathrm{Cu}(A)$ is almost unperforated. The following conditions are equivalent:*

(i) *$A$ admits no faithful tracial states.*

(ii) *$A$ contains a non-zero stable $C^*$-subalgebra.*

(iii) *$A$ contains a non-zero stable hereditary $C^*$-subalgebra.*

*Proof.* The equivalence (ii)⇔(iii) holds by [35]. We saw that (ii)⇒(i) in Proposition 5.22, so we only need to show (i)⇒(ii). By Proposition 5.30, there exists some $\langle a\rangle \in \mathrm{Cu}(A)$ such that $d(\langle a\rangle) = 0$ for all $d \in \mathrm{LDF}(A)$. Since $\mathrm{Cu}(A)$ is assumed to be almost unperforated, the set $\mathrm{LDF}(A)$ is dense in $\mathrm{DF}(A)$ in the topology of pointwise convergence [18, Theorem 3.3], and hence $d(\langle a\rangle) = 0$ for all $d \in \mathrm{DF}(A)$. Combining Corollary 5.32 and Lemma 5.33 implies that $\ell\langle a\rangle \leq \langle 1_A\rangle$ for all $\ell \in \mathbb{N}$. By Lemma 5.37, this implies the existence of a sequence of mutually orthogonal, mutually equivalent positive elements $(e_\ell)_{\ell\in\mathbb{N}}$ in $A$. It then follows from [35] that the hereditary $C^*$-algebra generated by $\{e_\ell\}_{\ell\in\mathbb{N}}$ is a stable $C^*$-subalgebra of $A$. □

Note that Theorem 5.38 provides a converse result to Proposition 5.22. It is, hence, of independent interest, but one can also use the theorem to give an equivalent formulation of QFTS. We shall, however, use a different method, with which we can avoid the assumption of stable rank one; do note, however, that this assumption would make the assumption of no properly infinite unital quotients trivially true in the results to come.

We shall in the sequel use the following result due to Hirshberg–Rørdam–Winter, [33, Theorem 3.6], which is reformulated using Theorem 3.19.

**Theorem 5.39** (Hirshberg–Rørdam–Winter, 2007)**.** *Let $A$ be a separable, unital, exact $C^*$-algebra for which $\mathrm{Cu}(A)$ has $\omega$-comparison. Let $B \subseteq A$ be a hereditary $C^*$-subalgebra. Then $B$ is stable if and only if $B$ admits no non-zero tracial states and no quotient of $B$ is unital.*

**Proposition 5.40.** *Let $A$ be a separable, unital, exact $C^*$-algebra such that $\mathrm{Cu}(A)$ is almost unperforated. If $A$ has no faithful tracial state, then either $A$ has a stable ideal or a unital, properly infinite quotient (or both).*

*Proof.* Suppose that $A$ has no faithful tracial state. By Proposition 5.18 there exists an ideal $I$ in $A$ with no tracial state. Suppose that $I$ is not stable, then Theorem 5.39 implies that $I$ has a unital quotient $I/J$. Observe that if $I/J$ had a tracial state, then so would $I$ via the quotient map $I \to I/J$. Since $I/J$ is an exact, unital $C^*$-algebra without tracial states, hence without quasitraces, there exists $n \in \mathbb{N}$ such that $M_n(I/J)$ is properly infinite by Theorem 2.32. It follows from Proposition 2.25 that $I/J$ is properly infinite. Now observe that we, from unitality of $I/J$, obtain an isomorphism $A/J \cong A/I \oplus I/J$, and hence $I/J$ is a unital and properly infinite quotient of $A$. □

We cannot immediately turn Proposition 5.40 into an equivalent reformulation of admitting a faithful tracial state, since admitting a faithful tracial state does not inhibit the existence of properly infinite quotients; for example, the full group-$C^*$-algebra $C^*(\mathbb{F}_2)$ admits a faithful tracial state as it is residually finite-dimensional, but any separable, unital $C^*$-algebra can be realised as a quotient of it. One way of salvaging this is to assume some more properties on $A$, e.g., QTS or stable rank one, which would give a necessary condition for having a faithful tracial state.

**Corollary 5.41.** *Suppose $A$ is a separable, unital, exact $C^*$-algebra such that* $\mathrm{Cu}(A)$ *is almost unperforated. Assume that $A$ has no properly infinite quotients (e.g., $A$ has the QTS property or has stable rank one). Then $A$ admits a faithful tracial state if and only if $A$ has no stable ideal.*

*Proof.* Apply Proposition 5.40 and note that $A$ cannot have any properly infinite quotients, since it either has QTS or stable rank one. □

Since almost unperforation of Cuntz semigroups is easily seen to pass to quotients, we can also apply Proposition 5.40 to the quotients and obtain the following result.

**Theorem 5.42.** *Let $A$ be a separable, unital, exact $C^*$-algebra satisfying that* $\mathrm{Cu}(A)$ *is almost unperforated. Then $A$ has the QFTS property if and only if $A$ has no stable intermediate quotients and no unital, properly infinite quotients.*

We now look at some ways of how one may apply the above theorem. Firstly, as mentioned previously, by assuming the QTS property one may disregard the possibility of unital, properly infinite quotients; since we are interested in the existence of *faithful* tracial states on the quotients, this is clearly not a big assumption. In general, unital, nuclear $C^*$-algebras have the QTS property if and only if they are hypertracial, cf. [2]. Assuming almost unperforation on the level of Cuntz semigroups might seem like a strong assumption, but we can luckily invoke various results to weaken this to just $\omega$-comparison on all quotients which, by Robert [56], is provided given finite nuclear dimension.

**Corollary 5.43.** *Let $A$ be a separable, unital $C^*$-algebra with finite nuclear dimension and QTS. Then $A$ has the QFTS property if and only if $A$ has no stable intermediate quotient.*

*Proof.* One direction is immediate, so assume that $A/I$ is a quotient of $A$ without a faithful tracial state. Since $A/I$ is unital, it holds by Proposition 5.21 that $A/I \otimes \mathcal{Z}$ has no faithful tracial state. Since $A/I \otimes \mathcal{Z}$ has QTS and an almost unperforated Cuntz semigroup, it follows from Proposition 5.40 that $A/I \otimes \mathcal{Z}$ has a stable ideal. We thus find an ideal $J$ in $A/I$ (i.e., an intermediate quotient of $A$) such that $J \otimes \mathcal{Z}$ is stable. Since the Cuntz semigroup of $J \otimes \mathcal{Z}$ is almost unperforated, it follows from [33] that $J \otimes \mathcal{Z}$ has no bounded traces and no unital quotients. In particular, neither does $J$, since any such would be easily extended from $J$ to $J \otimes \mathcal{Z}$. Since $A$ has finite nuclear dimension, which is preserved by taking ideals and quotients, $J$ has finite nuclear dimension and thus $\omega$-comparison by [56, Theorem 1]. A direct application of [49, Proposition 4.7] then proves that $J$ is stable, which finalises the proof. □

It is not immediate how one can use Corollary 5.43 to prove that specific $C^*$-algebras have the QFTS property, since proving the lack of stable intermediate quotients is a difficult task. However, we can use the result to prove a dichotomy for certain classes of $C^*$-algebras.

**Example 5.44.** Let $X$ be a compact Hausdorff space with finite covering dimension and consider an action $\alpha$ of $\mathbb{Z}$ on $C(X)$. Since $C(X)$ is Abelian, it follows from [2, Proposition 3.7] that the crossed product $C(X) \rtimes_\alpha \mathbb{Z}$ has the QTS property. Moreover, by [34], $C(X) \rtimes_\alpha \mathbb{Z}$ has finite nuclear dimension. It thus follows from Corollary 5.43 that $C(X) \rtimes_\alpha \mathbb{Z}$ has the

QFTS property if and only if it has no stable intermediate quotients. In particular, this result holds for groups $G \rtimes_\alpha \mathbb{Z}$ whenever $G$ is an Abelian group with finite-dimensional Pontryagin dual.

Recall that if $A$ is a unital $C^*$-algebra with a non-unitary isometry $v$, then we can embed the Toeplitz algebra $\mathcal{T} = C^*(v)$ in $A$, and we can identify $\mathbb{K}$ as an ideal in $\mathcal{T}$. Therefore, any infinite, unital $C^*$-algebra contains a stable $C^*$-subalgebra. By Corollary 5.43, we may extend this to prove the existence of stable *ideals* of certain $C^*$-algebras and their quotients.

**Example 5.45.** Consider the Lamplighter group $G = \mathbb{Z}_2 \wr \mathbb{Z}$ from Proposition 5.12, which is an amenable group and whose $C^*$-algebra has finite nuclear dimension by [34]. The $C^*$-algebra $C^*(G)$ admits an infinite quotient by Proposition 5.12 such that it does not have the QFTS property, and Corollary 5.43 then implies the existence of a stable intermediate quotient of $C^*(G)$.

By the exact same line of reasoning as in Corollary 5.43, we obtain the following equivalent reformulation of when certain $C^*$-algebras have faithful tracial states.

**Corollary 5.46.** *Let $A$ be a separable and unital $C^*$-algebra. Assume that* $\mathrm{Cu}(A)$ *has $\omega$-comparison (e.g., $A$ has finite nuclear dimension), and that $A$ has no properly infinite quotients (e.g., $A$ has the QTS property or stable rank one). Then $A$ admits a faithful tracial state if and only if $A$ has no stable ideals.*

Blackadar showed in [10] that an AF-algebra is stable if and only if no ideals admit a bounded trace. Since being an AF-algebra is preserved by taking ideals, it thus follows from Proposition 5.18 that an AF-algebra admits a faithful tracial state if and only if it admits no stable ideals. As AF-algebras have finite nuclear dimension (in fact, a $C^*$-algebra has zero nuclear dimension if and only if it is an AF-algebra [73]) and stable rank one, Corollary 5.46 can be seen as a generalisation of this result.

Another possible use of the QFTS property is to prove strong quasidiagonality of $C^*$-algebras under the assumption that the UCT problem is true. As mentioned earlier, the obstruction to residually quasidiagonal being equivalent to strong quasidiagonality revolves around the possibility of irreducible *-representations having non-trivial intersection with the compact operators. By assuming the QFTS property, we can avoid this issue, and the difficulties now revolve around when $C^*$-algebras have quasidiagonal quotients. One way of achieving this is by the following, which also uses the tracial states, but with the caveat that the $C^*$-algebras need to be residually UCT.

**Theorem 5.47.** *Let $A$ be a separable, nuclear, quasidiagonal $C^*$-algebra. Suppose that $A$ has the QFTS property, and that all quotients of $A$ satisfy the UCT. Then $A$ is strongly quasidiagonal.*

*Proof.* It suffices to show that all irreducible *-representations of $A$ are quasidiagonal Let $\pi\colon A \to \mathbb{B}(H)$ be an irreducible *-representation of $A$ with kernel $I$, and let $\tilde{\pi}\colon A/I \to \mathbb{B}(H)$ be the corresponding faithful *-representation on the quotient. Obviously, $\pi$ and $\tilde{\pi}$ have the same image, and, by assumption, there exists a faithful tracial state on $A/I$, hence also on $\tilde{\pi}(A/I)$. Suppose that $\tilde{\pi}$ is not essential, that is, suppose that $\mathbb{K}(H) \cap \tilde{\pi}(A/I) \neq \{0\}$. As $\tilde{\pi}$ is an irreducible *-representation, this implies that $\mathbb{K}(H)$ is an ideal in $\tilde{\pi}(A/I)$. However, this is impossible, as we would then obtain a bounded trace on $\mathbb{K}(H)$. Therefore, $\tilde{\pi}$ is a faithful and essential *-representation. Since $A$ is assumed to have the QFTS property, there exists a faithful tracial state $\tau$ on $A/I$, which is quasidiagonal by [25, Theorem 4.1]. Therefore, $\tilde{\tau}$ is a faithful quasidiagonal tracial state on $A/I$, and $A/I$ is therefore a quasidiagonal $C^*$-algebra by [69, Proposition 1.4]. Since $\tilde{\pi}$ is a faithful, essential *-representation of a quasidiagonal

$C^*$-algebra, it is a quasidiagonal *-representation. Consequently, $\pi$ is a quasidiagonal *-representation, and as $\pi$ was an arbitrary irreducible *-representation of $A$, we conclude that $A$ is strongly quasidiagonal, which completes the proof. $\square$

The assumption of all quotients satisfying the UCT is a strong assumption in the sense of being difficult to check, and even for relatively well-known classes of $C^*$-algebras it is unclear when this occurs. In [21] it is proven that it holds true for (primitive) quotients of $C^*(G)$, whenever $G$ is finitely generated and nilpotent. However, it is seemingly very difficult to extend these results, and the authors of [22] argue that resolving it for virtually nilpotent groups might not be any easier than resolving the UCT-problem for nuclear $C^*$-algebras. If all nuclear $C^*$-algebras were shown to be in the UCT-class, then the assumption of all quotients satisfying the UCT would, of course, be vacuous in Theorem 5.47 for nuclear $C^*$-algebras, but [25, Proposition 5.1] would also resolve this. In particular, we can for group-$C^*$-algebras achieve the following, which modulo the UCT assumption includes all previously known examples of strongly quasidiagonal groups.

**Corollary 5.48.** *Let $G$ be a countable, amenable group for which $C^*(G)$ has the QFTS property. If the UCT conjecture holds true, then $C^*(G)$ is strongly quasidiagonal.*

If the UCT assumption were to be either vacuous or being true for quotients of group-$C^*$-algebras, then (inductive limits of) amenable groups with the QFTS property would include the known examples of strongly quasidiagonal $C^*$-algebras such as the virtually nilpotent ones from Theorem 5.13 and Eckhardt's non-virtually nilpotent example [20].

Using the result on QFTS $C^*$-algebra from Corollary 5.43, we get the following result.

**Corollary 5.49.** *Let $A$ be a separable, unital, quasidiagonal, QTS $C^*$-algebra with finite nuclear dimension, and assume all quotients of $A$ satisfy the UCT. If $A$ has no stable intermediate quotients, then $A$ is strongly quasidiagonal.*

Proving that a given $C^*$-algebra is not strongly quasidiagonal often revolves around constructing a quotient with an infinite projection, as exemplified in Proposition 5.12 and, more generally, [15, Section 3]. Since the existence of an infinite projection in the quotient would imply the existence of a stable $C^*$-*subalgebra* on the quotient, one can hence view Corollary 5.49 as a partial converse result to this method.

# 6 Further research

As with any PhD-thesis, there are many unresolved questions scattered throughout this thesis, and, in earnest, we raise more questions than we actually succeed in answering. The all-important questions permeating the entire thesis are, obviously, still unresolved, but we also introduced new terminology and new results, which only lead to even more unknowns. This chapter is in some sense an overview of some of the most important questions in the field, as well as the new questions that this thesis has raised.

The absolute main question is whether or not Kaplansky's conjecture, Conjecture 2.35, holds true or not. If Kaplansky's conjecture were proven in the affirmative, it would have loads of implications — the state space on $\mathrm{K}_0(A)$ for unital $C^*$-algebras is completely determined by the Choquet simplex $\mathrm{T}(A)$ of tracial states, and $C^*$-algebras will have tracial states if and only if they are stably not properly infinite, cf. Theorem 2.32. If it turns out to be false, then it raises the question what separates when $C^*$-algebras have quasitraces and tracial states, i.e., what characterises when $C^*$-algebras admit tracial states?

In the discussion succeeding Theorem 4.34, it is mentioned how the invariant $\nu_\delta(-)$ is still poorly understood, and many of the famous questions regarding traciality of $C^*$-algebras such as Kaplansky's conjecture potentially revolves around understanding the range of this invariant better. One particularly immediate question is how $\nu_\delta(M_n(A))$ and $\nu_\delta(A)$ are connected for some unital $C^*$-algebra $A$ and some $0 < \delta < 1$. It is immediate via the unital embedding $A \to M_n(A)$ that $\nu_\delta(M_n(A)) \leq \nu_\delta(A)$, and Proposition 4.6 gives the inequality $\nu_\delta(A) \leq \nu_\delta(M_n(A))n^2$. The two quantities are thus asymptotically closely related, but for the $\mu(-)$ invariant one obtains $\mu(M_n(A)) = \lceil \mu(A)/n \rceil$. This relation, along with $\mu(pAp) \geq \mu(A)$ for any full projection $p \in A$, are the cornerstones of constructing the sequence of $C^*$-algebras $(A_n)_{n\in\mathbb{N}}$ for which $\mu(A_n) \to \infty$ with $\mu(A_n) < \infty$ for all $n$; the matrix algebra relation in particular entails that one obtains *strictly* larger $\mu$-values by taking the corners of the (SP) $C^*$-algebra. If one could obtain a similar way of *strictly* decreasing $\nu_\delta$-values by taking matrix algebras, one might be able to prove that the ultraproduct as constructed in Theorem 4.21 admits a tracial state. One immediate observation in this regard is that if one considers Rørdam's $C^*$-algebra $A$ with $\mu(A) = n \geq 2$, then $\nu_{1/2}(A) > 2$ yet $\nu_{1/2}(M_n(A)) = 2$, which does provide evidence towards the $\nu_\delta$-value decreasing in some sense when taking matrix algebras. More concretely, it would be interesting if the ultraproduct $A$ from Theorem 4.21 admits a tracial state or not.

Another unresolved problem regarding $\nu_\delta(-)$ is how $\nu_\delta(A)$ and $\nu_{\delta'}(A)$ relates to one another for $0 < \delta, \delta' < 1$. It is immediate that $\nu_\delta(A) \leq \nu_{\delta'}(A)$ whenever $\delta \geq \delta'$ and that one is finite if and only if the other is, but can we express $\nu_\delta(A)$ in terms of $\nu_{\delta'}(A)$? Using the language of universal $C^*$-algebras, what is $\nu_{\delta'}(\mathcal{D}_{n,\delta})$ for $n \in \mathbb{N}$ and $0 < \delta, \delta' < 1$? Or even more elementary, what is $\nu_\delta(\mathcal{D}_{n,\delta})$?

Regarding QFTS, all known strongly quasidiagonal groups have the QFTS property, and as stated in Corollary 5.48 the converse is true modulo the UCT question. Is there a way of proving this without assuming the UCT, that is, can one prove that amenable groups $G$ for which $C^*(G)$ satisfies the QFTS property are strongly quasidiagonal? It seems unreasonable that QFTS should characterise strong quasidiagonality of groups, so is there a way of constructing a strongly quasidiagonal group $G$ such that $C^*(G)$ admits a quotient without a faithful tracial state?

# 7 References

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