Projections in Life Insurance and the Equilibrium Approach to Utility Optimization

Alexander Sevel Lollike

Department of Mathematical Sciences
Faculty of Science
University of Copenhagen

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PhD Thesis by:

Alexander Sevel Lollike
Håbets Allé 71, 1.
DK-2700 Copenhagen NV
alexander.lollike@hotmail.com

Assessment committee:

Professor Mogens Bladt (chairperson)
University of Copenhagen, Denmark

Associate Professor David Ruiz Baños
University of Oslo, Norway

PhD Agatha Murgoci
Ørsted A/S, Denmark

Supervisors:

Professor Mogens Steffensen
University of Copenhagen, Denmark

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Abstract

We treat three topics in projections of multi-state life insurance contracts, and two topics in utility theory using the equilibrium approach. We derive a system of forward differential equations for the retrospective reserve of a with-profit insurance contract, where the dynamics of the reserve are affine. To reduce the sheer size of the system of differential equations required for a projection of an entire insurance business, we reduce the state space of insurance contracts through a transformation of the transition intensities and payment streams, resulting in a smaller, approximating system of differential equations. We derive a system of infinite partial differential equations for the moment-generating function of retrospective reserves with polynomial dynamics. We truncate the infinite partial differential equations to produce numerically feasible procedures, applicable for the projection of retrospective reserves. Using an equilibrium approach, we study how to dynamically approximate utility functions by polynomials so that there is a small difference in the corresponding optimal controls. Finally we derive a fixed-point equation for the equilibrium control of an investor with a prospect-theoretic utility function.

Preface

This thesis is based on five manuscripts of papers as well as already published papers and written as part of my fulfilment of the PhD degree at the Department of Mathematical Sciences, Faculty of Science, University of Copenhagen. Professor Mogens Steffensen was my PhD supervisor. The research leading to these manuscripts received funding from Innovation Fund Denmark under grant number 7076-00029, project title "ProBaBLI - Projection of Balances and Benefits in Life Insurance". The ProBaBLI project has also received an investment from the Danish software company Edlund A/S, in the form of man-hours. The man-hours provided by Edlund A/S transform the theoretical research into software products aimed at (primarily Danish) life insurance companies.

Apart from personal goals, I undertook the PhD studies with a professional goal to contribute to research in personal finance in a way that I felt was meaningful. Life insurance and optimal investment are the two topics of personal finance that I have studied. Given that both topics have an enormous influence on the quality of life of many people, even a small contribution is in my opinion worthwhile chasing.

The five manuscripts that constitute the bulk of the thesis are written as isolated scientific contributions. They fall into the two categories that make up the title of the thesis; projections in life insurance and the equilibrium approach to utility optimization. Chapter 1 and Chapter 5 are not independent scientific manuscripts, but introductions to the two categories of the thesis.

I hope that the results of the thesis will be used not only by me, but also by others who share my interests.

Disacknowledgments

I would like to disacknowledge the COVID-19 pandemic, which meant that my stay abroad at the Free University of Bozen-Bolzano came to an abrupt end after merely three weeks. In that way, the pandemic has had an influence on the research topics of the thesis. But even more so, COVID-19 affected my personal life in ways that are hopefully not evident in the thesis.

Acknowledgments

First, I want to thank my supervisor Mogens Steffensen for straightening my back. Not in a physical sense (that task still remains) but in a professional sense. He has given me an actuarial resolve that I have no doubt will benefit me for the rest of my career. In addition, I want to thank him for trusting me and giving me a large degree of freedom, and for being supportive when I needed help.

I would like to thank the people in the Actulus department in Edlund who have always made me feel welcome in their office, and enlightened me on the more technical aspects and problems of computations. In particular Kenneth Bruhn and Ida E. Andersen deserve recognition for the many hours of discussions that has let me understand some of the discrepancies between the mathematical world and the real world.

Lastly, but most importantly, I want to thank my wife, Trine, and my kids, Freja and Harald. Trine for being my apparently inexhaustible source of support and happiness. My kids for letting me be silly, and for reminding me that a job is just a job.

Alexander Sevel Lollike
Copenhagen, January 2022

An ISBN number has been added, and minor typos and errors have been corrected since the submission of the thesis.

Alexander Sevel Lollike
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List of papers

Chapter 2:
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Chapter 3:
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Summary

The raison d'être of life insurance and pension companies is to carry a risk in exchange for a premium. With the purpose of improving risk assessment, a statutory set of rules from the European Parliament require that life insurance and pension companies project their business into the future. The first part of this thesis deals with the computational challenges with the task of projection, and the second part deals with utility optimization problems.

To project an insurance contract, a decision has to be made about which path the state-process of the contract should follow. Chapters 2, 3 and 4 take as a starting point, the expected path of the state-process in the projection. The main result in Chapter 2 is a system of forward differential equations for the reserve of a with-profit insurance contract with affine dynamics of the reserve. Chapter 3 improves the computational viability of these differential equations, through a system of approximating ad hoc differential equations.

Revisiting the affine dynamics requirement on the reserve from Chapter 2, a system of infinite partial differential equations for the moment-generating function of reserves with polynomial dynamics is derived in Chapter 4. The infinite partial differential equations are truncated to produce a numerically feasible procedure.

In the last two chapters, two different utility optimization investment problems are studied in the context of equilibrium theory. How to construct a polynomial approximation of a utility function so that the resulting optimal control approximates the true control, is the topic of Chapter 6. This discussion is centered around a balance between the divergence of the polynomial and the distribution of the stochastic control variable. In Chapter 7 a fixed point equation for the equilibrium control of a utility function from prospect theory is derived by imposing a certain structure on the control.

Resumé

Eksistensberettigelsen for livsforsikrings- og pensionsselskaber er at påtage sig risiko imod en præmie. Med formålet at forbedre risikovurdering, kræver et lovpålagt regelsæt fra Europa-Parlamentet at livsforsikrings- og pensionsselskaber laver en fremregning af deres forretning. Første del af denne afhandling omhandler de beregningsmæssige udfordringer med fremregningsopgaven, og anden del omhandler investeringsproblemer med endelig tids horisont.

For at fremregne en forsikringsaftale skal der træffes en beslutning om, hvilken sti kontraktens tilstandsprocess skal følge. Kapitlerne 2, 3 og 4 bruger den forventede sti for tilstandsprocessen i fremregningen. Hovedresultatet i kapitel 2 er et system af fremadrettede differentialligninger for reserven af en forsikringskontrakt i et gennemsnitsrentemiljø som har en affin dynamik. Kapitel 3 forbedrer den beregningsmæssige anvendelighed af disse differentialligninger gennem et system af approksimerende ad hoc differentialligninger.

Kapitel 4 genbesøger kravet om affin dynamik af reserven fra kapitel 2, og der udledes et system af uendelige partielle differentialligninger for den moment-genererende funktion af reserver med polynomial dynamik. De uendelige partielle differentialligninger trunkeres for at frembringe en numerisk anvendelig procedure.

I de sidste to kapitler studeres to forskellige nytteoptimeringsproblemer inden for investering, gennem ligevægtsteori. Hvordan man konstruerer en polynomiel approksimation af en nyttefunktion så den resulterende optimale kontrol tilnærmer den sande kontrol, er emnet i kapitel 6. Denne diskussion er centreret omkring en balance mellem divergensen af polynomiet og fordelingen af den stokastiske kontrolvariabel. I kapitel 7 udledes en fikspunktsligning for ligevægtskontrollen af en nyttefunktion fra prospektteori ved at pålægge kontrollen en bestemt struktur.

Contents

Chapter 1

Introduction to Life Insurance

In this introduction we provide a non-technical background for the three chapters on life insurance topics, Chapters 2, 3 and 4, and give an overview of the underlying ideas that drive the results in each chapter. By life insurance, we mean all kinds of contracts that can be specified as payment streams that depend on the state of life of the insured, including pensions. In Chapter 5 we introduce the chapters pertaining to the equilibrium approach to utility optimization in the same manner. Since each chapter is written as an independent manuscript, many of the concepts in this introduction are repeated in the introduction of the individual chapters, but the non-technical manner of writing here hopefully serves a broader audience and provides a gentle lead-in.

1.1 Background

Life insurance is a great idea. At its core, insurance is about sharing the risks that is an inevitable part of life, and life insurance deals with the aspects pertaining to human health. One of the oldest forms of life insurance, is the product provided by the ancient roman burial clubs. For the entry cost of 100 sesterces and a jar of wine followed by monthly premiums, the burial club would cover funeral expenses and provide financial aid to bereaved[1]. While the premium currency has changed, and other coverages have been added, the fundamental idea of modern life insurance remains the same; pay a premium, avoid a risk.

Whenever there are multiple providers of life insurance that compete for customers in a free market, there is an incentive to accurately price the provided coverages. Underpricing leads to an unprofitable business and overpricing leads to a loss of customers to cheaper companies. To a large degree, this incentive drives the forefront of actuarial science. Ever more accurate but (often) more complicated models gives and edge in this competitive market.

[1] thegreatcoursesdaily.com, 2020.

Legislation is another component that drives actuarial research. Life insurance is an integral part of the welfare system of most developed countries where the inhabitants rely on its stability and resilience. For that reason, legislators have an interest to ensure that life insurance businesses are driven in a sound and prudent way. Unlike most other branches of mathematics, actuarial concepts are often embedded in the financial laws of a country. It is perhaps due to legislative differences that there is a considerable variation across otherwise similar countries in the actuarial practice and research.

The life insurance research in this thesis is to a large (but indirect) extent driven by legislation. The Solvency II Directive requires that life insurance companies in the European Union develop new and advanced types of liability-assessment tools. The purpose is to "ensure coordinated crisis prevention and management, as well as to preserve financial stability in crisis situations."[2]. Chapter 2, 3 and 4 directly contribute to specific aspects of this toolbox.

1.1.1 Multi State Life Insurance

In 1991 Tom Cochrane sang "life is a highway" in his hit song of the same name. If Cochrane had been an actuary, he might have sung "life is a J-laned highway", with a reference to the multi-state life insurance models where J states are used to represent different biometric and behavioural states of life such as 'alive','dead' and 'unemployed'. Marrying probability theory and life insurance, Markov chain models form the basis for most modern life insurance theory.

For Markov chain models, a Markov chain that lives on a state space of J states, called the state process, represents the state of life of an insured. The defining property of a Markov chain is that its future depends on its past only through the present, which is a property with considerable mathematical advantages. The state process is assumed to have transition intensities that characterize the infinitesimal probability of transition between states. These transition intensities are one of the most important ingredients in the making of a life insurance contract, as they provide information about how we can expect the state of life of the insured to develop.

On top of the foundation of randomness stemming from the Markovian state process, an insurance contract is conceived by specifying a payment process. The payment process is determined by two types of payments; payments during sojourns in states, and payments on transitions between states. In the with-profit insurance contract of consideration in this thesis, both types of payment are agreed upon at initialization of the contract, with a possibility of additional bonus benefits.

[2] eiopa.europa.eu, 2021.

Once payments and transition intensities have been settled, the insurance company needs to know how much money they should have in their bank account (figuratively), to be able to cover future liabilities. For this purpose, the prospective reserve is calculated. The prospective reserve is the expected present value of accumulated future net outflow (benefits less premiums), given some set of information available today. The equivalence principle states that on the outset of the contract, premiums and benefits should balance on average.

Similar to the prospective reserve, the retrospective reserve is defined as the expected present value of accumulated past net inflow (premiums less benefits), given some set of information available today. If all information about the past is included in the retrospective reserve, it is simply equal to the value of the account in which premiums have been deposited and benefits withdrawn. Even though it may seem like a weird construction, the retrospective reserve has its merits when it comes to projecting the insurance business.

1.1.2 With-profit Insurance

Apart from the randomness imparted by the state process, there are at least two more sources of randomness that ought to be included in the life insurance model from the insurers point of view, namely financial risk and systematic risk on transition intensities. When selling insurance contracts, insurers make assumptions about the future interest rate and transition intensities and commit to these assumptions by guaranteeing the premiums and benefits that these assumptions entail. There is a risk that these assumptions fall short, which could imply that the insurer has to increase the liabilities, but they can not increase premiums correspondingly due to the contractual obligation. Such a shortfall would have to be covered by the shareholders of the insurance company, and is an undesirable scenario.

To mitigate the risk that follows from promising a certain interest rate and set of transition intensities, insurers make conservative promises. Prudent assumptions on interest and transition intensities constitute the conservative promises, and they form the so-called technical basis or first-order basis. When the equivalence principle is fulfilled on a set of prudent assumptions about the future, a surplus or profit is expected to develop. This profit belongs in part to the insured for paying an ample premium, and when an agreement is made to repay some of the profit to the insured, we call it a with-profit insurance.

The surplus acts as a buffer to absorb risk in interest rate and transition intensities – if interest levels and transition intensities develop exactly like predicted by the prudent assumptions, no surplus is generated. In the more likely event that a surplus is generated, it can absorb fluctuations in the difference between the predicted and realised interest rate and transition intensities. Most of what remains of the surplus can be returned to the insured. We define the surplus as the accumulated (retrospective) premiums less benefits, minus the prospective first-order reserve. A complication in the with-profit insurance set-up, is that often the surplus is distributed via dividends to the insured in the form of additional benefits. These additional benefits may themselves give rise to a surplus, which again is spent on buying additional benefits.

Prior to the financial crisis of 2008, financial supervisory authorities were content with regarding surplus as a bonus, and not particularly concerned with its development, other than demanding that it be distributed fairly. In recent years the dynamics of the surplus, and how it is distributed has gained more attention. Some of the surplus is not returned to the insured, but instead given to the shareholders in the form of a so-called shareholder fee to compensate for the risk covered by the equity of the insurance company, in the event that the technical basis leads to premiums that do not cover future liabilities. The question of what the shareholder fee should be to fairly reflect the risk taken by the equity, is extremely difficult to answer. The first task in the search for an answer, is the means to actually calculate the proportions of the surplus that belong to the insured and equity respectively, as a function of the surplus distribution strategy.

1.1.3 Simulating the Future

The ability to self-simulate is the ability to imagine yourself in a scenario constructed by your thoughts. Some examples could be

> "If I plant this seed, I will get grain this summer, which I can use for food in winter."
> "If I do not have insurance and crash my car, I will be financially ruined."

It is a critical human ability, and the essence of what planning is. By self-simulating a scenario, adjusting your actions in the simulation, and then iterating, you form a plan for the future.

The same methodology can be applied in an insurance context, where scenarios are constructed in a mathematical framework instead of in the frontal lobes of the brain. By simulating the development of the factors that influence insurance companies financially, a possible path for the future is made. Projecting the insurance business along this simulated path, gives a glimpse into the possible future. This thesis is part of the project 'Projections of Benefits and Balances in Life Insurance', which exactly refers to the crystal ball exercise of projecting the insurance business into the future. In order to project the insurance business, the management actions that influence it need to be formalized and incorporated. We might say that the management actions are a plan for the future course of action.

By projecting the insurance business for thousands of simulated scenarios and averaging the result, an expectation for the future is made. By adjusting the management actions, the expected future of the insurance business is also adjusted, and in this way a plan that produces a satisfactory expected future can be devised.

The dangers of relying on a plan devised from simulating, is that the world is not predictable and simulations are therefore inaccurate – even if the simulations could have generated the observed history. By their nature, hitherto unseen events cannot be predicted, but that does not mean that they will not happen. The author, philosopher and statistician Nassim Nicholas Taleb coined these events 'Black Swans', referring to the belief that all swans were white before black swans were discovered in Australia. How the future is simulated is therefore one of the most important aspects of projections. If we are not confident that the simulations reflect the real world, we should not be confident that our plan is going to work.

For the three chapters under the life insurance topic, we do not explicitly deal with the simulation of the financial market and transition intensities, which make up the most important factors that influence insurance companies financially. Instead, we take the financial market and transition intensities for given, and focus on the projection of the insurance business. An advantage of this approach is that we allow for any simulation input. A disadvantage is that we cannot exploit the structure that the simulation input might have.

1.2 Overview of Chapters 2, 3 and 4

In the following subsections we outline the content of Chapters 2, 3 and 4 and the ideas from which they are conceived. Calculations of future benefits and balances is the primary focus. Chapter 2 concerns the modelling of with-profit insurance, and the fundamental aspects of projecting benefits and balances. Chapter 3 is to a large degree aimed at practitioners, and deals with the modification of the model from Chapter 2, in order to make computations viable. Chapter 4 revisits and relaxes a linearity assumption of Chapter 2, in a general probability theoretic framework.

1.2.1 Discarding Information

The title of Chapter 2 is "Retrospective Reserves and Bonus". The latin word 'retrospectus' translates to "I look back at", 'prospectus' translates to "I look forward at" and 'projeter' translates to "to throw forth". After this small lesson in etymology, one might be inclined to consider the prospective reserve for projections given the similar forward-oriented nature. However, surplus accumulation is retrospective in nature – and so are the management actions that determine the redistribution of surplus.

In Chapter 2, we are looking forward at backwards-looking reserves. The difficulty in this exercise is to determine much information about the past we should include in the backwards-looking reserves. If all information about the past is included in the projected benefits and balances, the projection exercise is overwhelming as all possible paths of the state process have to be accounted for. Instead we discard all information about the policy at all points in time between initialization and the projected 'now'.

Knowing only the initial and current state of a policy, we derive a system of forward differential equations for the retrospective reserves. For this minimal amount of information to be sufficient for accurate projections, the dynamics of the retrospective reserves need to be linear. The dynamics are linear if and only if the redistribution of surplus is a linear function of the reserve.

1.2.2 Discarding More Information

The title of Chapter 3 is "Efficient Projections", and is a direct practical extension of the results from Chapter 2. Using a naïve implementation of the results from Chapter 2, a typical Danish insurance company would have to solve in the order of 10^{10} differential equations[3].

A bunch of these computations can be carried out in parallel, reducing computation time, but if the computations are performed in the cloud (which they often are), there is still an ambition to reduce the number of computations to reduce costs of computing. A simple way to reduce the number of differential equations, is to reduce the state space of the insurance contract and the number of policies. This reduced state space and insurance portfolio should represent the original state space and insurance portfolio in a way that produces little or no error when projecting the insurance business.

We present a mapping that translates payments and transition intensities of a given state space to a smaller state space, which produces no error when projecting an insurance contract without bonus. We propose to use the same mapping for insurance contracts with bonus, and study the approximation error in a numerical example.

To reduce the size of the insurance portfolio, we touch on the matter of combining policies without introducing an error in the projection.

1.2.3 Handling Non-Linear Dividends

The title of Chapter 4 is "Moment Closure for FV-processes using Moment-generating Functions". Moment closure pertains to the problem of approximating an infinite system of differential equations for the moments of a process with non-linear dynamics, with

[3] For 7 states, 250.000 policies, 1 surplus and 10.000 financial scenarios. $((7 \cdot 2 \cdot 250.000) + 1) \cdot 10.000 = 35.000.010.000$

a finite system. Instead of projecting the time-dependent moments of a process with polynomial dynamics, we propose to project the time-dependent moment-generating function of the process. To this end, we derive a system of partial differential equations involving infinite partial derivatives for the moment generating function.

Given the numerically non-viable PDE, we truncate the infinite partial differential equations to arrive at a numerically viable finite difference scheme. In the context of with-profit insurance, non-linear dynamics emerge when the redistribution of surplus is a non-linear function of the reserves, and such an example is studied numerically.

Chapter 2

Retrospective Reserves and Bonus

Abstract

Modern legislation has increased the amount of quantities that insurance companies should report in order to prove solvent as well as prudent. More of these quantities require not just simple bookkeeping but a mere projection of the future. In this paper, we provide a solid base for this crystal ball exercise as we derive differential equations for the retrospective reserves of a pension company, in a setting where the surplus and the dividends are modelled. The differential equations rely on dynamics of the stochastic reserve that are affine functions of the stochastic reserve themselves. The retrospective reserves are defined as conditional expected values, given limited information, leading to computational tractable differential equations for the reserves. We wrap up the theoretical part by suggestions for practical use in terms of considering validation of guarantees and discretionary benefits at future time points.

Keywords: Bonus, Retrospective Reserves, FMA, Dividends, With-profit Insurance.

2.1 Introduction

With-profit insurance contracts are to this day one of the most popular life insurance designs. They arose as a natural way to distribute the systematic surplus that emerges due to the prudent assumptions on which the contract is made. The redistribution of surplus is a frequent subject of discussion in the industry, and many questions have been raised in that regard, to name a few; Is it distributed fairly? How should it be invested? How is it affected by the financial market? To answer these questions we need to understand the dynamics of the surplus in a model of practical relevance.

The study of surplus and the interplay it has with other elements of an insurance contract, is not new. Norberg (1999) introduced the notion of individual surplus as well as the mean portfolio surplus. In Steffensen (2006), partial differential equations are used to describe the prospective second order reserve for various forms of bonus when the surplus is invested in a Black-Scholes market. In this paper we pay little regard to the prospective reserve, and instead we focus on the surplus and the retrospective reserve including dividends, also called the savings account. The retrospective reserve without bonus is studied by Norberg (1991), and his results form the foundation for our studies. Furthermore we do not restrict ourselves to the Black-Scholes market, but allow for an arbitrary specification of the financial market.

In the existing literature, little attention is paid to a significant retrospective element of the with-profit insurance contract: the human element. Insurance companies are governed by humans, and the decisions they make influence the portfolio of policies - in particular concerning surplus and dividends. In a with-profit insurance contract many quantities are fixed at initialisation of the policy, but the rate at which dividends are paid out is not. The insurance company has a certain degree of freedom when it comes to the distribution of surplus, and the actions that have an influence on the insurance contracts are the so-called Management Actions. By including Future Management Actions (FMAs) in the modelling of with-profit insurance contracts the human element is taken into account, which arguably is useful in its own right, but it is also required by the Solvency II Directive. As stated in Article 23 of the currently in force EU Delegated Regulation (2014),

> Assumptions about future management actions shall be realistic and include [...] an assessment of the impact of changes in the assumptions on future management actions on the value of the technical provisions.

From a mathematical point of view FMAs pose a problem as they are retrospective in nature, and may depend on the entire history of the portfolio of policies in a possibly non-linear fashion, making it difficult to calculate prospective reserves. If we want to take a glance into the crystal ball of liabilities, taking FMAs into account, we need to embrace their retrospective nature. In this paper, we do not incorporate FMAs to their full extent, but rather lay the retrospective groundwork on which models including FMAs can be built.

The contribution of the paper is comprised of two parts. The first purely mathematical contribution is the derivation of a system of differential equations that describe the expected value of a process with dynamics that depend linearly on the process itself. The second contribution is to describe a model of practical relevance where retrospective reserves can be modelled including bonus for an arbitrary financial market. Together, the two parts provide the means to project the retrospective reserves of a with-profit insurance contract.

The structure of the paper is as follows: In Section 2.2 we present a standard model for life insurance contracts, and introduce the retrospective reserve. The main mathematical result is presented in Section 2.3, where we derive a forward differential equation for a multidimensional FV-process with affine dynamics. In Section 2.4 we extend the set-up of Section 2.2 to allow for a model where surplus and dividends are considered, and crucially, identify when the main mathematical result can be applied to describe the retrospective reserves of an insurance contract. Section 5 contains a simple numerical example.

2.2 Set-up

We consider the classic multi-state life insurance set-up, comprised of a state process Z denoting the state of the policy in a finite state space $\mathcal{J} = \{0, 1, ..., J\}$. The states in $\mathcal{J}$ represent real-life states such as "alive", "dead", "disabled" et cetera. By a permutation argument, we can without loss of generality assume that $Z(0) = 0$. The filtration generated by $Z(t)$ is denoted by $\mathcal{F}_t$, and it represents all the information generated by Z up to time t. The counting process N^k defined by $N^k(t) = \#\{s; Z(s-) \neq k, Z(s) = k, s \in (0, t]\}$ describes the number of transitions into state k. The state process Z is assumed to be a continuous time Markov chain, with transition probabilities denoted by

$$p_{ij}(s,t) = \mathrm{P}(Z(t) = j | Z(s) = i),$$

for $s \leq t$. We assume that the corresponding transition intensities exist, and denote them by

$$\mu_{ij}(t) = \lim_{h \searrow 0} p_{ij}(t, t+h)/h,$$

for $i \neq j$. We assume that

$$\lim_{n \to \infty} n\mathrm{P}(N^k(t + 1/n) - N^k(t) \geq 2) = 0, \tag{2.2.1}$$

for all t, stating that two jumps cannot occur simultaneously. The predictable process $\mathbb{1}_{\{Z(t-)\neq k\}}\mu_{Z(t-)k}(t)$ is the intensity process for $N^k(t)$, i.e

$$M^k(t) := N^k(t) - \int_0^t \mathbb{1}_{\{Z(s-)\neq k\}}\mu_{Z(s-)k}(s)ds,$$

forms a martingale. The state process Z encapsulates the biometric risks involved with the insurance contract. Apart from the biometric risk, there is a financial risk connected to with-profit insurance contracts through the return on investment of the surplus. We make assumptions regarding the financial risk, by specifying the return on investment, r.

Remark 2.2.1 (Portfolio investment). In practice, the return on investment is measured on the performance of some self-financing portfolio, G, governed by insurance company. By substituting $e^{\int_s^t r(\tau)d\tau}$ with $\frac{G(t)}{G(s)}$, the results of the paper can be formulated directly in terms of return on investment in G, instead of instantaneous return r.

Together, the transition intensities and return on investment form the third order (realized) basis, which describes the actual development of the insurance portfolio. We take this third order basis as exogenously given. In practice the non-measurable elements of the third order basis are simulated. To allow for events that make it difficult to meet the obligations to the insured, a much less risky set of assumptions are used when guarantees are given. These prudent assumptions form the first order (technical) basis. Using the standard notation, a "$*$" symbolises first-order basis elements. It is precisely due to the difference between the first order basis and the realized third order basis that a surplus emerges.

In order to define an insurance contract, we introduce the payment process B, which depends on the dynamics of Z. The payment process is an $\mathcal{F}_t$-adapted process with dynamics given by

$$dB(t) = b^{Z(t)}(t)dt + \sum_{k:k\neq Z(t-)} b^{Z(t-)k}(t)dN^k(t),$$

for sufficiently regular $b^i(t)$ and $b^{jk}(t)$. The deterministic payment functions $b^j(t)$ and $b^{jk}(t)$ specify payments during sojourns in state j and on transition from state j to state k, respectively. Even though lump sum payments during sojourn in a state pose no mathematical difficulty, we assume that payments during sojourns in states are continuous for notational simplicity. Given the payment process, B, we can define the state wise prospective technical reserve as

$$V^{j*}(t) = \mathrm{E}^*\left[\int_t^n e^{-\int_t^s r^*(\tau)d\tau} dB(s)|Z(t) = j\right],$$

being the expected present value of all future payments, evaluated under the first order basis. The dynamics of the technical reserve can be found using Itô's lemma for

FV-functions. This is done in e.g Asmussen and Steffensen (2020), providing us with the following dynamics of the prospective technical reserve

$$\begin{aligned} dV^{Z(t)*}(t) &= r^*(t)V^{Z(t)*}(t)dt - b^{Z(t)}(t)dt - \sum_{k:k\neq Z(t-)} b^{Z(t-)k}(t)dN^k(t) \\ &\quad - \sum_{k:k\neq Z(t-)} \rho^{Z(t-)k}(t)dt + \sum_{k:k\neq Z(t-)} R^{Z(t-)k}(t)(dN^k(t) - \mu_{Z(t-)k}(t)dt), \end{aligned} \tag{2.2.2}$$

where ρ^{jk} is the surplus risk contribution rate for a transition from state j to state k, and R^{jk} is the so-called sum-at-risk for a transition from j to k. The sum-at-risk R^{jk} describes the required injection/withdrawal of capital on a transition from j to k, in order to meet the future liabilities of the contract in state k, evaluated under the first-order basis. The sum-at-risk is given by

$$R^{jk}(t) = b^{jk}(t) + V^{k*}(t) - V^{j*}(t).$$

As the name suggests, the surplus risk contribution rate is the contribution from the policyholder to the surplus. The surplus risk contribution rate is the premium that covers the risk carried by the insurer that can not be diversified, such as medical advancements. Naturally the surplus contribution rate is the sum-at-risk multiplied by the difference in intensity for a transition from j to k between the first-order basis and the third-order basis, i.e

$$\rho^{jk}(t) = R^{jk}(t)(\mu^*_{jk}(t) - \mu_{jk}(t)).$$

2.2.1 Retrospective Reserves

For insurance companies, reserves are key quantities of interest, as they answer the question; how much should we set aside as insurers to meet the obligations to the insured? A reserve may be either prospective or retrospective. A prospective reserve considers future payments, whereas a retrospective reserve considers past payments. One of the main contributions of Norberg (1991) is a definition of the retrospective reserve as a conditional expected value of past net inflow, in much the same manner as the prospective reserve is a conditional expected value of future net outflow. Formally Norberg (1991) defines the retrospective first order reserve, as

$$U^*_{\mathbb{E}}(t) = \mathrm{E}^*\left[\int_0^t e^{\int_s^t r^*(\tau)d\tau} d(-B(s)) \middle| \mathcal{E}_t\right],$$

for some family of σ-algrebras $\mathbb{E} = \{\mathcal{E}_t\}_{0\leq t}$, where $\mathcal{E}_t$ represents the information available at time t. It is natural to assume that $\mathcal{E}_t = \mathcal{F}_t$, implying that all information about the past is accounted for. As noted by Norberg (1991), the family of σ-algebras may be increasing, i.e $\mathcal{E}_s \subseteq \mathcal{E}_t$ for $s < t$, but it is not required. With this very general definition of the retrospective reserve, we may discard information, for instance by defining $\mathcal{E}_t = \sigma\{Z(0), Z(t)\}$. With this choice of σ-algebra we discard information about the state process in between time 0 and time t. This removal of information is useful because it is intractable to use $\mathcal{E}_t = \mathcal{F}_t$, when we want to calculate the expected value of $U^*_{\mathbb{E}}(t)$ and $\{Z(s)\}_{s\leq t}$ has not yet been realized. When t is a future point in time, it is simply too computationally demanding to take the expectation over $\mathcal{F}_t$ - all possible future paths and all possible future transition times have to be considered. We therefore let $\mathcal{E}_t = \sigma\{Z(0), Z(t)\}$, implying that we only use the state at initialization and time t to evaluate the retrospective reserve. Using this formulation of $\mathcal{E}_t$, the retrospective reserve can be interpreted as the average reserve of a group of policies that all start in $Z(0)$ and end in $Z(t)$. In the next section, we adopt the idea to discard information in order to get a definition of the expected future retrospective reserve including bonus that is computationally tractable.

2.3 The Mathematical Tool

In this section we present the main result, which generalizes the result from Norberg (1991), and provides us with the tool to project retrospective reserves in a model with bonus. The value of the result becomes evident in Section 2.4, when the retrospective reserves of interest are seen to fall within the framework of Theorem 2.3.1. Let $W(t)$ be a multi-dimensional stochastic process, with dynamics given by

$$dW(t) = g^{Z(t)}(t, W(t))dt + \sum_{k:k\neq Z(t-)} h^{Z(t-)k}(t, W(t-))dN^k(t),$$

for g and h functions that are affine functions of W. Without loss of generality we assume that $W(0) = w_0$ for some deterministic but arbitrary initial vector w_0. To illustrate the central idea of this section in a simple setting, consider the case where W has dynamics

$$dW(s) = g^{Z(s)}(s)W(s)ds,$$

and say we want to calculate

$$\tilde{W}^i(t) := \mathrm{E}_0[W(t)\mathbb{1}_{\{Z(t)=i\}}] = \mathrm{E}_0[W(t)|Z(t) = i]p_{0i}(0, t),$$

where we by the subscript 0 on the expectation denote the conditional expectation given $Z(0) = 0$ and $W(0) = w_0$. With this definition of $\tilde{W}^i$ we discard information about where the policy has been in between time 0 and time t, in the same way that Norberg (1991) discards information about where the policy has been. Similar reserves where state-dependent indicator functions appear as multipliers on integrals over a payment stream, are also used by Bladt et al. (2020) to arrive at matrix formulae for higher-order moments of prospective reserves. By the tower property and Fubini's theorem,

$$\begin{aligned}
\tilde{W}^i(t) &= p_{0i}(0,t)w_0 + \int_0^t \mathrm{E}_0[\mathbb{1}_{\{Z(t)=i\}} g^{Z(s)}(s)W(s)]ds \\
&= p_{0i}(0,t)w_0 + \int_0^t \mathrm{E}_0\left[\sum_{j:j\in\mathcal{J}} \mathbb{1}_{\{Z(s)=j\}} \mathrm{E}_0[\mathbb{1}_{\{Z(t)=i\}} g^{Z(s)}(s)W(s)|Z(s)=j]\right] ds \\
&= p_{0i}(0,t)w_0 + \int_0^t \sum_{j:j\in\mathcal{J}} p_{0j}(0,s)g^j(s)\mathrm{E}_0[\mathbb{1}_{\{Z(t)=i\}} W(s)|Z(s)=j]ds.
\end{aligned}$$

By the Markov property $W(s) \perp\!\!\!\perp Z(t)|Z(s)$ for $s<t$, as $W(s)$ is $\mathcal{F}_s$-measurable, and therefore

$$\tilde{W}^i(t) = p_{0i}(0,t)w_0 + \int_0^t \sum_{j:j\in\mathcal{J}} g^j(s)\tilde{W}^j(t)p_{ji}(s,t)ds.$$

The Kolmogorov forward differential equations state that

$$\frac{d}{dt}p^*_{ji}(s,t) = \sum_{g:g\neq i} p^*_{jg}(s,t)\mu^*_{gi}(t) - \mu^*_{ig}(t)p^*_{ji}(s,t).$$

Differentiating $\tilde{W}^i(t)$ with respect to t, and using Kolmogorov's forward differential equations yields the following system of differential equations

$$\begin{aligned}
\frac{d}{dt}\tilde{W}^i(t) &= g^i(t)\tilde{W}^i(t) + \sum_{j:j\neq i}\left(\mu_{ji}(t)\tilde{W}^j(t) - \mu_{ij}(t)\tilde{W}^i(t)\right), \\
\tilde{W}^i(0) &= \mathbb{1}_{\{i=0\}}w_0.
\end{aligned}$$

It is crucial to note that this differential equation relies on the affine structure of the dynamics of W, as it allows us to write $\tilde{W}^i(t)$ as an integral over $\tilde{W}^j(s)$ for $0\le s\le t$. Using the tower property and the fact that $W(s-) \perp\!\!\!\perp Z(t)|Z(s-)$, we get the following theorem.

Theorem 2.3.1. *Let $Z(t)$ be a Markov process on the state space $\mathcal{J}$, and let $W(t)$ be a q-dimensional, $\mathcal{F}_t$-measurable process with dynamics*

$$dW(s) = g^{Z(s)}(s,W(s))ds + \sum_{k:k\neq Z(s-)} h^{Z(s-)k}(s,W(s-))dN^k(s),$$

for q-dimensional functions g and h of the form

$$\begin{aligned}
g^{Z(s)}(s,W(s)) &= g_1^{Z(s)}(s)W(s) + g_0^{Z(s)}(s), \\
h^{Z(s-)k}(s,W(s-)) &= h_1^{Z(s-)k}(s)W(s-) + h_0^{Z(s-)k}(s),
\end{aligned}$$

where g_1^j and h_1^{jk} are $q\times q$-matrices, and g_0^j and h_0^{jk} are vectors of length q. Then

$$\tilde{W}^i(t) = E_0[\mathbb{1}_{\{Z(t)=i\}}W(t)]$$

is described by the differential equation

$$\frac{d}{dt}\tilde{W}^i(t) = \sum_{j:j\neq i}\left(\mu_{ji}(t)\tilde{W}^j(t) - \mu_{ij}(t)\tilde{W}^i(t)\right) \tag{2.3.1}$$

$$+ g_1^i(t)\tilde{W}^i(t) + p_{0i}(0,t)g_0^i(t) \tag{2.3.2}$$

$$+ \sum_{j:j\neq i}\mu_{ji}(t)\left(h_1^{ji}(t)\tilde{W}^j(t) + p_{0j}(0,t)h_0^{ji}(t)\right), \tag{2.3.3}$$

$$\tilde{W}^i(0) = \mathbb{1}_{\{i=0\}}w_0. \tag{2.3.4}$$

Proof. See Appendix 2.B. □

The proof of Theorem 2.3.1 solely relies on the affine dynamics of W, and even if the dynamics directly depend on the past values of W in the following manner

$$dW(t) = \int_0^t g_\rho^{Z(s)}(s,t)W(s)dsdt,$$

there are no significant changes in the proof. The resulting differential equations for $\tilde{W}^i(t)$ become a set of more involved integro-differential equations. This extension is practically relevant for instance if the dividend depends on how the surplus and savings have evolved over the last year.

The differential equation given by (2.3.1)-(2.3.4) generalizes the differential equation for the retrospective reserve derived by Norberg (1991). In fact, for

$$g_1 = h_1 = 0,\ w_0 = 0,\ g_0(t,i) = b^i(t),\ h_0(t,j,i) = b^{ji}(t),$$

we arrive at the differential equation derived by Norberg (1991).

2.4 Set-Up Including Surplus and Dividends

In this section we extend our set-up, allowing us to accurately describe the benefits and reserves in a model where surplus and dividends are included. The ideas and notation are inspired by Asmussen and Steffensen (2020). The first order basis on which insurance contracts are signed, are a set of prudent assumptions regarding interest rates and transition intensities. Knowing that the assumptions are prudent, the insurer and insured agree that when surplus has emerged as a consequence of the realized interest and transitions, this surplus should be given back to the insured. The surplus is returned to the insured through a dividend payment stream. What the insured does with his dividend can vary, but a standard product design is to use the dividends to buy more benefits. In a sense, the dividend payment stream becomes a premium for a bonus benefit stream. We introduce the two payment streams B_1 and B_2 with dynamics

$$dB_i(t) = b_i^{Z(t)}(t)dt + \sum_{k:k\neq Z(t-)} b_i^{Z(t-)k}(t)dN^k(t).$$

The payments specified by B_1 are the benefits and premiums which are fixed, and part of the original contract. The payments of B_2 specify the profile of the payment stream that the dividend is converted into. The payment streams B_1 and B_2, have corresponding technical reserves given by

$$V_i^{j*}(t) = \mathrm{E}^* \left[\int_t^n e^{-\int_t^s r^*(\tau)d\tau} dB_i(s) \middle| Z(t) = j \right].$$

When the contract is signed, both B_1 and B_2 are agreed upon, and while there is practically no restriction on the design of B_1, B_2 should be constructed in such a way that $V_2^{j*}(t) \neq 0$ for all t and all j. This should be required simply because it does not make sense to use the dividend to buy a payment stream that has zero technical value. In order to keep track of how much dividend has been materialized into the B_2 payment stream, we introduce the process $Q(t)$ which denotes the quantity of B_2 payment stream purchased at time t. The dividends are instantaneously used to increase benefits, by buying more of the B_2 payment stream. These additional benefits are, like the fixed benefits, priced under the first order basis, which means that one unit of B_2 has a technical value of $V_2^{Z(t)*}(t)$ at time t. The total amount of accrued dividends at time t are denoted by $D(t)$, and as the dividends are used to buy B_2, we must have that

$$dD(t) = V_2^{Z(t)*}(t)dQ(t). \tag{2.4.1}$$

The payment process experienced by the policyholder, B, consists of one unit B_1 payment stream and Q units of B_2 payment stream, thus having dynamics

$$dB(t) = dB_1(t) + Q(t-)dB_2(t),$$

where the left limit version of Q is used to ensure that it is predictable. We now define the savings account as the technical value of future guaranteed payments, for a certain quantity of B_2 payment stream,

$$X(t) = V_1^{Z(t)*}(t) + Q(t)V_2^{Z(t)*}(t).$$

To fix ideas, we think of the savings account as the accumulated past benefits, premiums and dividends compounded with the first order interest, and thus it behaves just like a bank account. Norberg (1999) defines an individual surplus and derives a system of differential equations for the mean surplus, which he studies in a Markov chain environment. Our definition of the savings account resembles the individual surplus defined by Norberg (1999), but in contrast, we do not restrict ourselves to a Markov chain environment for the modelling of the realized interest, and most importantly, we furthermore allow for payments that depend on the savings account. Noting that

$$Q(t) = \frac{X(t) - V_1^{Z(t)*}(t)}{V_2^{Z(t)*}(t)},$$

we see that the payment stream experienced by the policyholder has dynamics

$$dB(t) = b^{Z(t)}(t, X(t))dt + \sum_{k:k\neq Z(t-)} b^{Z(t-)k}(t, X(t-))dN^k(t),$$

for deterministic functions b^j and b^{jk}. These payments depend linearly on the savings account, and in a setting without bonus, an expression for the corresponding retrospective reserve is derived by Christiansen et al. (2014). By the principle of equivalence

$$0 = X(0) = V_1^{0*}(0) + Q(0)V_2^{0*}(0)$$
$$\Leftrightarrow$$
$$Q(0) = -\frac{V_1^{0*}(0)}{V_2^{0*}(0)},$$

providing us with the initial condition for Q, which along with (2.4.1) fully specifies Q. Note that the principle of equivalence puts no restrictions on the form of B_1 and B_2.

Using integration by parts for FV-functions, and plugging in the dynamics of $V_1^{Z(t)*}$ and $V_2^{Z(t)*}$ given by (2.2.2), we find the dynamics of X to be

$$\begin{aligned} dX(t) &= dV_1^{Z(t)*}(t) + Q(t-)dV_2^{Z(t)*}(t) + V_2^{Z(t)*}(t)dQ(t) \\ &= r^*(t)X(t)dt + dD(t) - b^{Z(t)}(t, X(t))dt - \sum_{k:k\neq Z(t-)} b^{Z(t-)k}(t, X(t-))dN^k(t) \\ &\quad - \sum_{k:k\neq Z(t-)} \rho^{Z(t-)k}(t, X(t-))dt \\ &\quad + \sum_{k:k\neq Z(t-)} R^{Z(t-)k}(t, X(t-))dM^k(t), \end{aligned} \tag{2.4.2}$$

where

$$\rho^{jk}(t, X(t-)) = \rho_1^{jk}(t) + Q(t-)\rho_2^{jk}(t) = \rho_1^{jk}(t) + \frac{X(t-) - V_1^{j*}(t-)}{V_2^{j*}(t-)}\rho_2^{jk}(t),$$
$$R^{jk}(t, X(t-)) = R_1^{jk}(t) + Q(t-)R_2^{jk}(t) = R_1^{jk}(t) + \frac{X(t-) - V_1^{j*}(t-)}{V_2^{j*}(t-)}R_2^{jk}(t),$$

respectively can be interpreted as the surplus risk contribution and sum-at-risk for the savings account. The savings account plays a crucial role in the understanding of the with-profit insurance contract. The dynamics of X are remarkably similar to the dynamics of the prospective reserve as seen in (2.2.2). In fact, if no dividends are ever allotted, i.e. $dD(t) = V_2^{Z(t)*}(t)dQ(t) = 0$, then the dynamics of X are identical to the dynamics of the technical reserve found in (2.2.2) for an X-independent payment process B_G given by

$$dB_G(t) = dB_1(t) - \frac{V_1^{0*}(0-)}{V_2^{0*}(0-)}dB_2(t).$$

Note that the technical prospective reserve for the payment process B_G automatically fulfills the principle of equivalence, in the sense that

$$\mathrm{E}^*\left[\int_0^n e^{-\int_0^s r^*} dB_G(s) | Z(t)=0\right] = 0.$$

If no dividends are allotted, then the savings account could be seen as a prospective first order reserve for a contract with reserve dependent payments, therefore falling under the set-up of Christiansen et al. (2014). However, if dividends are allotted, they are accumulated in the savings account, implying that the value of the savings account depends on the total amount of allotted dividends, as well as how much of these dividends that have been materialized into the B_2 payment stream. In other words, when dividends are allotted, the savings account is not only dependent on the current state of the policy, but the entire history of the policy as well.

Remark 2.4.1 (Future Discretionary Benefits). The difference $dB(t) - dB_G(t)$ is precisely the instantaneous value of the additional benefits bought using the allotted dividends. Hence, the expected time-t value

$$\mathrm{FDB}^i(t) = \mathrm{E}_0\left[\int_t^n e^{-\int_t^s r} d(B(s) - B_G(s)) | Z(t) = i\right]$$

is the market value of Future Discretionary Benefits.

Given the savings account, we can readily define the surplus as

$$Y(t) = -\int_0^t e^{\int_s^t r(\tau)d\tau} dB(s) - X(t),$$

being the accumulated premiums less benefits excess over the savings account, compounded with the realized interest, r. The dynamics of Y are then

$$\begin{aligned} dY(t) &= r(t)\left(-\int_0^t e^{\int_s^t r(\tau)d\tau} dB(s)\right) - dB(t) - dX(t) \\ &= r(t)\left(Y(t) + X(t)\right)dt - dB(t) - dX(t) \\ &= r(t)Y(t)dt + dC(t) - dD(t) - \sum_{k:k\neq Z(t-)} R^{Z(t-)k}(t, X(t-))dM^k(t), \end{aligned} \tag{2.4.3}$$

for

$$dC(t) = (r(t) - r^*(t))X(t)dt + \sum_{k:k\neq Z(t)} \rho^{Z(t)k}(t, X(t))dt,$$

which we call the surplus contribution process, as it represents the contributions from the savings account to the surplus.

As stated in the introduction, management actions are one of the main motivators of this paper. The influence of management actions is present in our set-up through mainly two terms; the third order interest rate and the specification of dividends, since the management decides how to invest their assets, and how the surplus should be returned to the customers. For this reason, the dynamics of the dividend process D is a central element of a with-profit insurance contract. We assume that the dynamics of the dividend process are given by

$$dD(t) = \delta^{Z(t)}(t, X(t), Y(t))dt,$$

but do not yet impose any restrictions on the δ^j-functions. We can for suitable functions g and h, write the dynamics of X and Y as

$$dX(t) = g_x^{Z(t)}(t, X(t), Y(t))dt + \sum_{k:k\neq Z(t-)} h_x^{Z(t-)k}(t, X(t-), Y(t-))dN^k(t), \quad (2.4.4)$$

$$dY(t) = g_y^{Z(t)}(t, X(t), Y(t))dt + \sum_{k:k\neq Z(t-)} h_y^{Z(t-)k}(t, X(t-), Y(t-))dN^k(t). \quad (2.4.5)$$

It is a crucial point that when these dynamics are affine, we can apply Theorem 2.3.1. The dynamics of X and Y given by (2.4.2) and (2.4.3) are affine if and only if the dividend process is affine in X and Y, that is, if the δ^j-functions can be written as

$$\delta^j(t, x, y) = \delta_1^j(t) + \delta_2^j(t)x + \delta_3^j(t)y. \quad (2.4.6)$$

We refer to Section 2.C of the Appendix for the specification of g and h leading to the dynamics given in (2.4.2) and (2.4.3) for dividend process determined by (2.4.6). Assuming that (2.4.6) holds, is an assumption that is eligible for criticism, but also an important assumption, as Theorem 2.3.1 relies on affine dynamics. In practice, the dividend is based on more information than simply the actual value of the savings and surplus. The specification of the dynamics of D is at the heart of what a future management action is, and, as stated earlier, we do not fully incorporate these FMA's in all their generality and glory, but suffice with crude surrogates. Some of these crude surrogates can actually perform a decent job at describing real world dividend strategies, for instance by defining the dividend as some affine function of the contribution, since this by construction leads to affine dynamics for X and Y.

Apart from notational ease, the use of affine g and h functions serve to generalise the results of the paper to any FV-process with affine dynamics of the form given by (2.4.4) and (2.4.5). We could for instance easily introduce expenses affine in X and Y. Even though we work with the dynamics given by (2.4.4) and (2.4.5), we think of the g and h functions as the ones required to achieve the dynamics of (2.4.2) and (2.4.3). We are interested in the interconnected dynamics of X and Y, and therefore we introduce the two-dimensional process

$$W(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix},$$

with dynamics given by

$$dW(t) = g^{Z(t)}(t, W(t))dt + \sum_{k:k\neq Z(t-)} h^{Z(t-)k}(t, W(t-))dN^k(t),$$

for g and h functions that are affine functions of W, and determined by the dynamics of X and Y. Steffensen (2006) derives a set of differential equations that can be used to determine $\tilde{W}^i$, when investment returns are formed from assets that are traded in a Black-Scholes market.

Applying Theorem 2.3.1, gives a system of differential equations for

$$\tilde{W}^j(t) = \begin{pmatrix} \mathrm{E}_0[X(t)\mathbb{1}_{\{Z(t)=j\}}] \\ \mathrm{E}_0[Y(t)\mathbb{1}_{\{Z(t)=j\}}] \end{pmatrix} = \begin{pmatrix} \tilde{X}^j(t) \\ \tilde{Y}^j(t) \end{pmatrix},$$

described by the terms (2.3.1)-(2.3.3),

$$\begin{aligned} \frac{d}{dt}\tilde{W}^i(t) = & \sum_{j:j\neq i} \left(\mu_{ji}(t)\tilde{W}^j(t) - \mu_{ij}(t)\tilde{W}^i(t)\right) \\ & + g_1^i(t)\tilde{W}^i(t) + p_{0i}(0,t)g_0^i(t) \\ & + \sum_{j:j\neq i} \mu_{ji}(t)\left(h_1^{ji}(t)\tilde{W}^j(t) + p_{0j}(0,t)h_0^{ji}(t)\right), \end{aligned}$$

which has an intuitive interpretation. If the policy is in state i at time t, it develops with the continuous dynamics for that state, given by $g_1^i(t)W(t) + g_0^i(t)$. Due to the uncertainty involved pertaining to the state of the policy and the value of W, we have to weigh these dynamics with the probability of $Z(t) = i$, as well as the expected value of W, thus arriving at (2.3.2) as

$$\mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}}\left(g_1^i(t)W(t) + g_0^i(t)\right)\right] = g_1^i(t)\tilde{W}^i(t) + p_{0i}(0,t)g_0^i(t).$$

Similarly, we have to account for any transitions into the current state i, over the small interval $t + dt$. The infinitesimal probability of transition from j to i over an interval from t to $t + dt$ is given by $\mu_{ji}(t)$, and if such a transition was made, the savings and surplus are bumped by $h_1^{ji}(t)W(t) + h_0^{ji}(t)$. In order for a transition from j to i to be possible over the interval $t + dt$, the policy has to be in state j at time t, thus arriving at (2.3.3) as

$$\mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=j\}}\left(h_1^{ji}(t)W(t) + h_0^{ji}(t)\right)\right] = h_1^{ji}(t)\tilde{W}^j(t) + p_{0j}(0,t)h_0^{ji}(t).$$

Furthermore, when a transition from j to i is made, the savings and surplus from state j (after the bump) are transferred to the savings and surplus of state i, amounting to the term given in (2.3.1).

For dynamics of X and Y given by (2.4.2) and (2.4.3) we emphasize that if the dividend function δ is affine in X and Y, then the dynamics of X and Y are also affine in X and Y as all other terms in the dynamics are affine by construction.

Remark 2.4.2 (Non-linear dynamics). While the reach of models with affine dynamics is extensive, there are limitations to consider. It is not uncommon to have dynamics that include some non-linear term, for instance if the transition intensities are X or Y dependent. However, if the dynamics of W are not affine, we can still produce an approximation of $\tilde{W}^i$. We simply replace $W(t)$ with $\tilde{W}^{Z(t)}(t)$ in the terms of the dynamics that are not affine in $W(t)$. This idea is motivated by producing a Taylor approximation of the non-affine term.

While other quantities could be studied, the projection of expected savings and surplus provides us with useful information. One practically important quantity that can be calculated based on $\tilde{X}$ and $\tilde{Y}$ is the time 0 expected present value of future guaranteed benefits, given state i at time t, which are

$$\begin{aligned}
\mathrm{GB}^i(t) &= \mathrm{E}_0\left[\int_t^n e^{-\int_t^s r} d\left(B_1(s) + \frac{X(t) - V_1^{Z(t)*}(t)}{V_2^{Z(t)*}(t)} B_2(s)\right) \middle| Z(t) = i\right] \\
&= \mathrm{E}_0\left[\int_t^n e^{-\int_t^s r} dB_1(s) \middle| Z(t) = i\right] \\
&\quad + \frac{\mathrm{E}_0[X(t)|Z(t) = i] - V_1^{i*}(t)}{V_2^{i*}(t)} \mathrm{E}_0\left[\int_t^n e^{-\int_t^s r} dB_2(s) \middle| Z(t) = i\right] \\
&= V_1^i(t) + \frac{\tilde{X}^i(t)/p_{0i}(0,t) - V_1^{i*}(t)}{V_2^{i*}(t)} V_2^i(t).
\end{aligned}$$

The second equality follows from $X(t) \perp\!\!\!\perp B_2(s)|Z(t)$ for $s > t$. Note that GB^i is affine in $\tilde{X}^i$, and therefore it can be used as an input to the dividend function δ - for instance by letting the dividend be some percentage of the guaranteed future benefits.

Another practically important quantity that can be calculated is the present value of expected future discretionary benefits as defined in Remark 2.4.1. For $t = 0$ the quantity can be calculated as

$$\begin{aligned}
\mathrm{FDB}^0(0) &= \mathrm{E}_0\left[\int_0^n e^{-\int_0^s r} d\left(B(s) - B_G(s)\right)\right] \\
&= \mathrm{E}_0\left[\int_0^n e^{-\int_0^s r} \left(Q(s) - Q(0)\right) dB_2(s)\right] \\
&= \int_0^n e^{-\int_0^s r} \sum_{j \in \mathcal{J}} p_{0j}(0,s) \left(\frac{\tilde{X}^j(s)/p_{0j}(0,s) - V_1^{j*}(s)}{V_2^{j*}(s)} - \frac{V_1^{0*}(0)}{V_2^{0*}(0)}\right) dB_2^j(s).
\end{aligned}$$

The value of $\mathrm{FDB}^0(0)$ corresponds to the increase in benefits the policyholder experiences, as a consequence of the bonus received. This quantity can then be used for investigating contractual fairness at contract initiation as well as studying the Future Profits (FP) of the contract. In this setup, we would then define the latter as $\mathrm{FP} = X(0) + Y(0) - \mathrm{GB}^0(0) - \mathrm{FDB}^0(0)$, corresponding to all assets under management subtracted the amount designated for the policyholders.

Whereas more practically applications could easily be listed, we leave that for the interested reader.

2.5 Numerical Example

To highlight the practical usefulness of the results in the paper, we produce a numerical example and examine the effects of three different FMAs in the form of dividend strategies. We consider a two-state contract $\mathcal{J} = \{0,1\}$ with $Z(0) = 0$, where state 0 should be thought of as "alive", and state 1 should be thought of as "dead". We assume that all benefits are scaled with dividends, and therefore $dB_1 = 0$. A single premium of $V_2^{0*}(0)$ is paid by the policyholder just before time 0, such that at time 0, the policyholder can afford one unit of the B_2 payment stream i.e. $Q(0) = 1$. The payment process B_2 consists of a benefit being continuously paid out from time of death and until time n. Thereby the policyholder experiences the payment stream

$$dB(t) = \mathbb{1}_{\{Z(t)=1\}}\mathbb{1}_{\{t\leq n\}}b_2(t)\frac{X(t)}{V_2^{1^*}(t)}dt,$$

which can be thought of as an annuity for the bereaved children of the policyholder. The technical- and market-reserves are found using Thieles differential equation. We assume that $Y(0) = 0$. As $dB_1(t) = 0$ the differential equations for $\tilde{X}^i$ and $\tilde{Y}^i$ are given by

$$\begin{aligned}
\frac{d}{dt}\tilde{X}^i(t) &= g_{11}^i(t)\tilde{X}^i(t) + g_{12}^i(t)\tilde{Y}^i(t) + \sum_{j\neq i}\mu_{ji}(t)\left\{\tilde{X}^j(t) + \tilde{X}^j(t)h_{11}^{ji}(t)\right\} \\
&\quad - \mu_{ij}(t)\tilde{X}^i(t), \\
\tilde{X}^0(0) &= V_2^{0*}(0),\ \tilde{X}^1(0) = 0,
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\tilde{Y}^i(t) &= g_{21}^i(t)\tilde{X}^i(t) + g_{22}^i(t)\tilde{Y}^i(t) + \sum_{j\neq i}\mu_{ji}(t)\tilde{Y}^j(t) - \mu_{ij}(t)\tilde{Y}^i(t), \\
\tilde{Y}^0(0) &= 0,\ \tilde{Y}^1(0) = 0,
\end{aligned}$$

where the g^i and h^{ji} functions are specified in Appendix 2.C. We consider three different FMAs in the form of dividend strategies:

1) $dD(t) = 0$, i.e. zero dividends.

2) $dD(t) = 0.5dC(t)$, i.e. 50% of the contributions to the surplus are immediately returned to the policyholders.

3) $dD(t) = dC(t)$, i.e. all contributions to the surplus are immediately returned to the policyholders.

These dividend strategies represent three levels of safety set by the management of the insurance company. Other components of the contract are seen in Table 2.1.

Component	Value
Age of policyholder, a_0	30
n	80
$\mu^*_{01}(t)$	$0.0005 + 10^{5.6+0.04\cdot(t+a_0)-10}$
$\mu_{01}(t)$	$0.9 \cdot \mu^*_{01}(t)$
$b_2(t)$	1
$r^*(t)$	0.015
$r(t)$	$0.01 + r^*(t)\frac{t}{n-a_0}$

Table 2.1: *Components of insurance contract.*

With this, we can project the statewise expected savings and surplus, and use these to calculate several quantities of interest. In Figure 2.1 we have plotted $\tilde{X}^i$, $\tilde{Y}^i$ and the expected payment stream given death, $\frac{\tilde{X}^1(t)}{p_{01}(0,t)V_2^{1*}(t)}b_2(t)$.

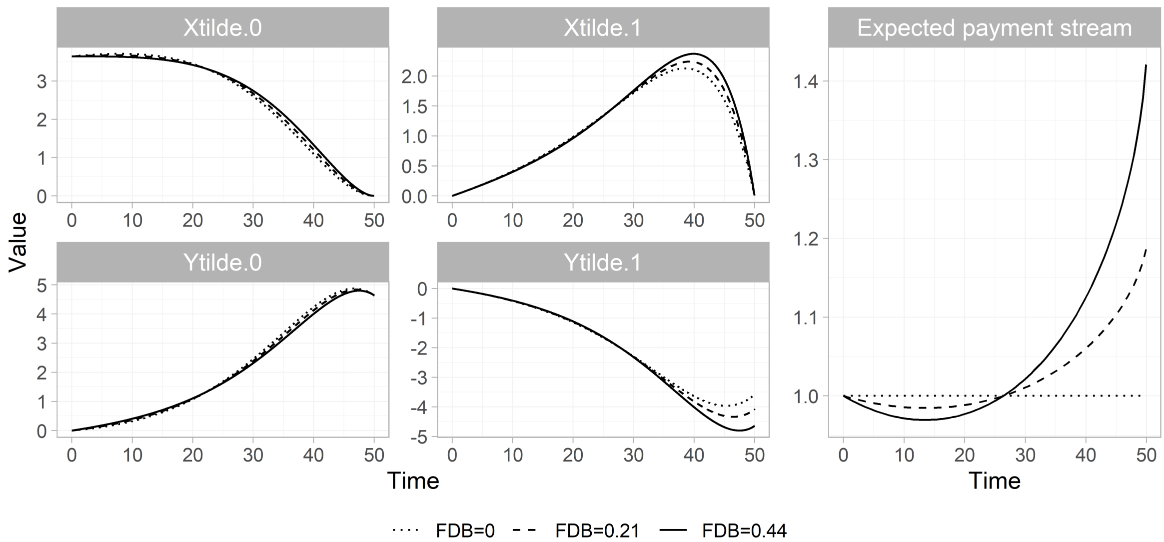

Figure 2.1: *Savings, surplus and expected payment given death for different dividend strategies. Dotted line: $\delta^j(t,x,y) = 0$. Dashed line: $\delta^j(t,x,y) = 0.5c^j(t,x,y)$. Solid line: $\delta^j(t,x,y) = c^j(t,x,y)$*

As a consequence of the third order interest being lower than the first order interest, the surplus contribution is negative for the first approximately 12 years. Using $\tilde{X}^i$ and $\tilde{Y}^i$ we calculate, the future discretionary benefits, the guaranteed benefits and the future profits, see Table 2.2.

$dD(t)$	FDB	GB	FP
$0 \cdot dC(t)$	0.00	3.20	0.44
$0.5 \cdot dC(t)$	0.21	3.20	0.23
$1 \cdot dC(t)$	0.44	3.20	0.00

Table 2.2: *FDB, GB and FP for three different dividend strategies.*

The result that more dividends allotted leads to less profit for the insurance company is no surprise with this toy-box example, as it is only meant as an illustration of the model framework and the results are easily obtained within that.

Geolocation Information

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2.A Predictable Compensator of $\mathbb{1}_{\{Z(t)=i\}}N^{jk}(s)$

In this section we consider the FV-process given by

$$\tilde{N}^{jk}_{t,i}(s) := \mathbb{1}_{\{Z(t)=i\}}N^{jk}(s)$$

for $s < t$ and fixed but arbitrary $t > 0$ and $i \in \mathcal{J}$. The stochastic process $\tilde{N}$ is adapted to the filtration given by $\tilde{\mathcal{F}}^{t,i}_s := \sigma\{\{Z(\tau)\}_{\tau \leq s}, Z(t) = i\}$. Consider now the predictable process

$$\lambda(s) := \mathbb{1}_{\{Z(t)=i\}}\mathbb{1}_{\{Z(s-)=j\}}\mu_{jk}(s)\frac{p_{ki}(s,t)}{p_{ji}(s,t)},$$

and define

$$Y_n(s) := n\mathrm{E}\left[\tilde{N}^{jk}_{t,i}(s+1/n) - \tilde{N}^{jk}_{t,i}(s)|\tilde{\mathcal{F}}^{t,i}_s\right].$$

If a few mild conditions are satisfied and

$$\lim_{n\to\infty} Y_n(s) = \lambda(s) \text{ a.s} \tag{2.A.1}$$

then, by theorem 1 of Aven (1985), $\lambda(s)$ is the predictable compensator for $\tilde{N}^{jk}_{t,i}(s)$. In order to establish (2.A.1), note that

$$\begin{aligned}\lim_{n\to\infty} Y_n(s) &= \lim_{n\to\infty}\sum_{m=1}^{\infty} nm\mathrm{P}\left(\tilde{N}^{jk}_{t,i}(s+1/n) - \tilde{N}^{jk}_{t,i}(s) = m\middle|\, \tilde{\mathcal{F}}^{t,i}_s\right)\\ &= \sum_{m=1}^{\infty} m \lim_{n\to\infty} n\mathrm{P}\left(\tilde{N}^{jk}_{t,i}(s+1/n) - \tilde{N}^{jk}_{t,i}(s) = m\middle|\, \tilde{\mathcal{F}}^{t,i}_s\right),\end{aligned}$$

as we have assumed that $\lim_{n\to\infty} Y_n(s)$ exists. The relation (2.2.1) implies that

$$\lim_{n\to\infty} n\mathrm{P}\left(\tilde{N}^{jk}_{t,i}(s+1/n)-\tilde{N}^{jk}_{t,i}(s)=m\middle|\,\tilde{\mathcal{F}}^{t,i}_s\right)=0 \text{ for } m>1.$$

and therefore

$$\begin{aligned}
\lim_{n\to\infty} Y_n(s) &= \lim_{n\to\infty} n\mathrm{P}\left(\tilde{N}^{jk}_{t,i}(s+1/n)-\tilde{N}^{jk}_{t,i}(s)=1\middle|\,\tilde{\mathcal{F}}^{t,i}_s\right)\\
&= \mathbb{1}_{\{Z(t)=i\}}\lim_{n\to\infty} n\mathrm{P}\left(N^{jk}(s+1/n)-N^{jk}(s)=1\middle|\,\tilde{\mathcal{F}}^{t,i}_s\right)\\
&= \mathbb{1}_{\{Z(t)=i\}}\lim_{n\to\infty} n\mathrm{P}\left(Z(s+1/n)=k, Z(s)=j\middle|\,\tilde{\mathcal{F}}^{t,i}_s\right)\\
&= \mathbb{1}_{\{Z(t)=i\}}\mathbb{1}_{\{Z(s)=j\}}\lim_{n\to\infty} n\mathrm{P}\left(Z(s+1/n)=k\middle|\,\tilde{\mathcal{F}}^{t,i}_s\right).
\end{aligned}$$

By the Markov property,

$$\begin{aligned}
\lim_{n\to\infty} Y_n(s) &= \mathbb{1}_{\{Z(t)=i\}}\mathbb{1}_{\{Z(s)=j\}}\lim_{n\to\infty} n\mathrm{P}\left(Z(s+1/n)=k\middle|\,Z(s)=j, Z(t)=i\right),\\
&= \mathbb{1}_{\{Z(t)=i\}}\mathbb{1}_{\{Z(s)=j\}}\lim_{n\to\infty} n\frac{p_{jk}(s,s+1/n)p_{ki}(s+1/n,t)}{p_{ji}(s,t)}\\
&= \mathbb{1}_{\{Z(t)=i\}}\mathbb{1}_{\{Z(s)=j\}}\mu^{jk}(s)\frac{p_{ki}(s,t)}{p_{ji}(s,t)}\\
&\overset{a.s}{=}\mathbb{1}_{\{Z(t)=i\}}\mathbb{1}_{\{Z(s-)=j\}}\mu^{jk}(s)\frac{p_{ki}(s,t)}{p_{ji}(s,t)}.
\end{aligned}$$

2.B Proof of Theorem 2.3.1

Proof of theorem 2.3.1. The proof consists of two steps. First, we derive an integral equation for $\tilde{W}^i(t)$. Second, we differentiate this integral equation.
Assume that $p_{0i}(0,s)>0$ for all $s>0$. The general case where some states cannot be reached by time s is considered at the end of the proof. Writing out $\tilde{W}^i(t)$,

$$\begin{aligned}
\tilde{W}^i(t) &:= \mathrm{E}_0[W(t)\mathbb{1}_{\{Z(t)=i\}}]\\
&= p_{0i}(0,t)w_0+\mathrm{E}_0\left[\int_0^t \mathbb{1}_{\{Z(t)=i\}}dW(s)\right]\\
&= p_{0i}(0,t)w_0+\mathrm{E}_0\left[\int_0^t \mathbb{1}_{\{Z(t)=i\}}g^{Z(s)}(s,W(s))ds\right]\\
&\quad+\mathrm{E}_0\left[\int_0^t \sum_{k:k\neq Z(s-)}\mathbb{1}_{\{Z(t)=i\}}h^{Z(s-)k}(s,W(s-))dN^k(s)\right].
\end{aligned}$$

Based on the calculations in Section 2.A of the Appendix, note that

$$\begin{aligned}
&\mathrm{E}_0\left[N^{jk}(s)-\int_0^s \mathbb{1}_{\{Z(\tau-)=j\}}\mu_{jk}(\tau)\frac{p_{ki}(\tau,t)}{p_{ji}(\tau,t)}d\tau\middle|\,Z(t)=i\right]\\
&=\mathrm{E}\left[\mathbb{1}_{\{Z(t)=i\}}N^{jk}(s)-\int_0^s \mathbb{1}_{\{Z(t)=i\}}\mathbb{1}_{\{Z(\tau-)=j\}}\mu_{jk}(\tau)\frac{p_{ki}(\tau,t)}{p_{ji}(\tau,t)}d\tau\middle|\,\tilde{\mathcal{F}}^{t,i}_0\right]\\
&=0.
\end{aligned}$$

As $h^{Z(s-)k}(s, W(s-))$ is predictable, we may replace the integrator $dN^k(s)$ with its predictable compensator. Using Fubini's theorem and the tower property,

$$
\begin{aligned}
&\tilde{W}^i(t) && (2.B.1)\\
&= p_{0i}(0,t)w_0 + \int_0^t \mathrm{E}_0\left[\mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}} g^{Z(s)}(s, W(s)) | Z(s)\right]\right] ds\\
&\quad + \mathrm{E}_0\left[\mathrm{E}_0\left[\int_0^t \sum_{k:k\neq Z(s-)} \mathbb{1}_{\{Z(t)=i\}} h^{Z(s-)k}(s, W(s-)) dN^k(s) | Z(t)\right]\right]\\
&= p_{0i}(0,t)w_0 + \int_0^t \sum_{j:j\in\mathcal{J}} p_{0j}(0,s)\mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}} g^j(s, W(s)) | Z(s)=j\right] ds\\
&\quad + \mathrm{E}_0\left[\int_0^t \sum_{k:k\neq Z(s-)} \mathbb{1}_{\{Z(t)=i\}} h^{Z(s-)k}(s, W(s-)) \mathbb{1}_{\{Z(s-)=j\}} \mu^{jk}(s) \frac{p_{ki}(s,t)}{p_{ji}(s,t)} ds\right]\\
&= p_{0i}(0,t)w_0 + \int_0^t \sum_{j:j\in\mathcal{J}} p_{0j}(0,s)\mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}} g^j(s, W(s)) | Z(s)=j\right] ds\\
&\quad + \int_0^t \mathrm{E}_0\left[\sum_{k:k\neq Z(s-)} \mathbb{1}_{\{Z(t)=i\}} h^{Z(s-)k}(s, W(s-))\right] \mu^{jk}(s) \frac{p_{ki}(s,t)}{p_{ji}(s,t)} ds\\
&= p_{0i}(0,t)w_0 + \int_0^t \sum_{j:j\in\mathcal{J}} p_{0j}(0,s)\mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}} g^j(s, W(s)) | Z(s)=j\right] ds && (2.B.2)\\
&\quad + \int_0^t \sum_{j:j\in\mathcal{J}} \sum_{k:k\neq j} p_{0j}(0,s)\mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}} h^{jk}(s, W(s-)) | Z(s-)=j\right] \mu^{jk}(s) \frac{p_{ki}(s,t)}{p_{ji}(s,t)} ds\\
& && (2.B.3)
\end{aligned}
$$

Since $W(s)$ is $\mathcal{F}_s$-measurable, the Markov property gives us

$$
\mathrm{E}_0[\mathbb{1}_{\{Z(t)=i\}} W(s) | Z(s)=j] = \frac{\tilde{W}^j(s)}{p_{0j}(0,s)} p_{ji}(s,t). \qquad (2.B.4)
$$

Using that g and h are affine in W, and plugging (2.B.4) into (2.B.2)-(2.B.3) gives

$$
\begin{aligned}
\tilde{W}^i(t) &= p_{0i}(0,t)w_0 + \int_0^t \sum_{j:j\in\mathcal{J}} p_{0j}(0,s)\left(g_1^j(s)\frac{\tilde{W}^j(s)}{p_{0j}(0,s)} + g_0^j(s)\right)p_{ji}(s,t)ds \\
&\quad + \int_0^t \sum_{j:j\in\mathcal{J}} p_{0j}(0,s)\left(\sum_{k:k\neq j}\mu_{jk}(t)p_{ki}(s,t)\left(h_1^{jk}(s)\frac{\tilde{W}^j(s)}{p_{0j}(0,s)} + h_0^{jk}(s)\right)\right)ds \\
&= p_{0i}(0,t)w_0 + \int_0^t \sum_{j:j\in\mathcal{J}} p_{ji}(s,t)g_1^j(s)\tilde{W}^j(s)ds \\
&\quad + \int_0^t \sum_{j:j\in\mathcal{J}}\sum_{k:k\neq j}\mu_{jk}(t)p_{ki}(s,t)\tilde{W}^j(s)h_1^{jk}(s)ds \\
&\quad + \int_0^t \sum_{j:j\in\mathcal{J}} p_{0j}(0,s)g_0^j(s)p_{ji}(s,t)ds \\
&\quad + \int_0^t \sum_{j:j\in\mathcal{J}} p_{0j}(0,s)\sum_{k:k\neq j}\mu_{jk}(t)p_{ki}(s,t)h_0^{jk}(s)ds.
\end{aligned}
$$

Differentiating with respect to t gives

$$
\begin{aligned}
\frac{d}{dt}\tilde{W}^i(t) &= w_0\left(\sum_{k:k\neq i} p_{0k}(0,t)\mu_{ki}(t) - \mu_{ik}(t)p_{0i}(0,t)\right) \\
&\quad + g_1^i(t)\tilde{W}^i(t) + p_{0i}(0,t)g_0^i(t) \\
&\quad + \sum_{k:k\neq i}\mu_{ki}(t)\left(h_1^{ki}(t)\tilde{W}^k(t) + p_{0k}(0,t)h_0^{ki}(t)\right) \\
&\quad + \int_0^t \frac{\partial}{\partial t}\sum_{j:j\in\mathcal{J}} p_{ji}(s,t)g_1^j(s)\tilde{W}^j(s)ds \\
&\quad + \int_0^t \frac{\partial}{\partial t}\sum_{j:j\in\mathcal{J}}\sum_{k:k\neq j}\mu_{jk}(t)p_{ki}(s,t)h_1^{jk}(s)\tilde{W}^j(s)ds \\
&\quad + \int_0^t \frac{\partial}{\partial t}\sum_{j:j\in\mathcal{J}} p_{0j}(0,s)g_0^j(s)p_{ji}(s,t)ds \\
&\quad + \int_0^t \frac{\partial}{\partial t}\sum_{j:j\in\mathcal{J}} p_{0j}(0,s)\sum_{k:k\neq j}\mu_{jk}(t)p_{ki}(s,t)h_0^{jk}(s)ds.
\end{aligned}
$$

Using the Kolmogorov forward differential equations and recognizing $\tilde{W}^k$ and $\tilde{W}^i$, we arrive at

$$\begin{aligned}\frac{d}{dt}\tilde{W}^i(t) = g_1^i(t)\tilde{W}^i(t) + p_{0i}(0,t)g_0^i(t) \\ + \sum_{k:k\neq i} \mu_{ki}(t)\left(h_1^{ki}(t)\tilde{W}^k(t) + p_{0k}(0,t)h_0^{ki}(t)\right) \\ + \sum_{k:k\neq i}\left(\mu_{ki}(t)\tilde{W}^k(t) - \mu_{ik}(t)\tilde{W}^i(t)\right).\end{aligned}$$

Combined with the initial condition

$$\tilde{W}^i(0) = \mathrm{E}_0[\mathbb{1}_{\{Z(0)=i\}}W(0)] = \mathbb{1}_{\{i=0\}}w_0,$$

we get the differential equations given by (2.3.1)-(2.3.4). For the case where some state, q, cannot be reached before time s for $s > 0$, the product of intensities for all paths from $Z(0)$ into that state must be zero for all τ when $\tau \leq s$, whereby $\tilde{W}^q(s) = 0$ and therefore the differential equations still hold. Thus the proof is complete. □

2.C Dynamics of X and Y

The dynamics of X are found in (2.4.2), and given by

$$\begin{aligned}dX(t) = r^*(t)X(t)dt + \delta^{Z(t)}(t,X(t),Y(t))dt - b^{Z(t)}(t,X(t))dt \\ - \sum_{k:k\neq Z(t-)} b^{Z(t-)k}(t,X(t-))dN^k(t) \\ - \sum_{k:k\neq Z(t-)} \rho^{Z(t-)k}(t,X(t-))dt \\ + \sum_{k:k\neq Z(t-)} R^{Z(t-)k}(t,X(t-))dM^k(t),\end{aligned}$$

and the dynamics of Y are found in (2.4.3), and given by

$$\begin{aligned}dY(t) =& r(t)Y(t)dt + dC(t) - \delta^{Z(t)}(t,X(t),Y(t))dt \\ &- \sum_{k:k\neq Z(t-)} R^{Z(t-)k}(t,X(t-))dM^k(t).\end{aligned}$$

Assuming that the dividend functions δ^j are affine, such that Theorem 2.3.1 can be applied, implies that

$$\delta^j(t,x,y) = \delta_1^j(t) + \delta_2^j(t)x + \delta_3^j(t)y.$$

We are interested in the specification of g_1, g_0, h_1 and h_0 for which the differential equation

$$
\begin{aligned}
\frac{d}{dt}\tilde{W}^i(t) = &\sum_{j:j\neq i}\Big(\mu_{ji}(t)\tilde{W}^j(t) - \mu_{ij}(t)\tilde{W}^i(t)\Big)\\
&+ g_1^i(t)\tilde{W}^i(t) + p_{0i}(0,t)g_0^i(t)\\
&+ \sum_{j:j\neq i}\mu_{ji}(t)\Big(h_1^{ji}(t)\tilde{W}^j(t) + p_{0j}(0,t)h_0^{ji}(t)\Big)\\
\tilde{W}^i(0) = &\mathbb{1}_{\{i=0\}}W(0),
\end{aligned}
$$

determines

$$
\tilde{W}^j(t) := \begin{pmatrix}\tilde{X}^j(t)\\ \tilde{Y}^j(t)\end{pmatrix} = \begin{pmatrix}\mathrm{E}[X(t)\mathbb{1}_{\{Z(t)=j\}}]\\ \mathrm{E}[Y(t)\mathbb{1}_{\{Z(t)=j\}}]\end{pmatrix}.
$$

The functions g_1, g_0, h_1 and h_0 are in the form

$$
\begin{aligned}
g_1^j(t) &= \begin{pmatrix} g_{11}^j(t) & g_{12}^j(t)\\ g_{21}^j(t) & g_{22}^j(t)\end{pmatrix},\\
h_1^{jk}(t) &= \begin{pmatrix} h_{11}^{jk}(t) & h_{12}^{jk}(t)\\ h_{21}^{jk}(t) & h_{22}^{jk}(t)\end{pmatrix},\\
g_0^j(t) &= \begin{pmatrix} g_{x0}^j(t)\\ g_{y0}^j(t)\end{pmatrix},\\
h_0^{jk}(t) &= \begin{pmatrix} h_{x0}^{jk}(t)\\ h_{y0}^{jk}(t)\end{pmatrix}.
\end{aligned}
$$

We want to find the twelve g and h functions that describe the dynamics of X and Y. We separate the dynamics of X into the terms that are linear in X, linear in Y and those that are neither,

$$
\begin{aligned}
dX(t) = X(t-)\Big\{ & r^*(t)dt + \delta_2^{Z(t)}(t)dt \\
& + \frac{1}{V_2^{Z(t-)*}(t-)}\Big(- b_2^{Z(t)}(t)dt - \sum_{k:k\neq Z(t-)} b_2^{Z(t-)k}(t)dN^k(t) \\
& - \sum_{k:k\neq Z(t-)} \rho_2^{Z(t-)k}(t)dt + \sum_{k:k\neq Z(t-)} R_2^{Z(t-)k}(t)dN^k(t) \\
& - \sum_{k:k\neq Z(t-)} R_2^{Z(t-)k}(t)\mu^{Z(t-)k}(t)dt\Big)\Big\} \\
& + Y(t)\delta_3^{Z(t)}(t)dt \\
& + \frac{V_1^{Z(t-)*}(t-)}{V_2^{Z(t-)*}(t-)}\Big(b_2^{Z(t)}(t)dt + \sum_{k:k\neq Z(t-)} b_2^{Z(t-)k}(t)dN^k(t) \\
& + \sum_{k:k\neq Z(t-)} \rho_2^{Z(t-)k}(t)dt - \sum_{k:k\neq Z(t-)} R_2^{Z(t-)k}(t)dN^k(t) \\
& + \sum_{k:k\neq Z(t-)} R_2^{Z(t-)k}(t)\mu^{Z(t-)k}(t)dt\Big) \\
& + \delta_1^{Z(t)}(t)dt - b_1^{Z(t)}(t)dt - \sum_{k:k\neq Z(t-)} b_1^{Z(t-)k}(t)dN^k(t) \\
& - \sum_{k:k\neq Z(t-)} \rho_1^{Z(t-)k}(t)dt + \sum_{k:k\neq Z(t-)} R_1^{Z(t-)k}(t)dN^k(t) \\
& - \sum_{k:k\neq Z(t-)} R_1^{Z(t-)k}(t)\mu^{Z(t-)k}(t)dt.
\end{aligned}
$$

These terms are then further separated into those that relate to the discrete and continuous dynamics of X, providing us with $g_{11}^j, g_{12}^j, g_{x2}^j, h_{11}^{jk}, h_{12}^{jk}$ and h_{x2}^{jk}.

$$
\begin{aligned}
g_{11}^j(t) &= r^*(t) + \delta_2^j(t) \\
&\quad + \frac{1}{V_2^{j*}(t)}\Big(- b_2^j(t) - \sum_{k:k\neq j} \rho_2^{jk}(t) - \sum_{k:k\neq j} R_2^{jk}(t)\mu^{jk}(t)\Big), \\
g_{12}^j(t) &= \delta_3^j(t), \\
g_{x0}^j(t) &= \delta_1^j(t) - b_1^j(t) - \sum_{k:k\neq j} \rho_1^{jk}(t) - \sum_{k:k\neq j} R_1^{jk}(t)\mu^{jk}(t) \\
&\quad + \frac{V_1^{j*}(t)}{V_2^{j*}(t)}\Big(b_2^j(t) + \sum_{k:k\neq j} \rho_2^{jk}(t) + \sum_{k:k\neq j} R_2^{jk}(t)\mu^{jk}(t)\Big), \\
h_{11}^{jk}(t) &= \frac{1}{V_2^{j*}(t)}\Big(\sum_{k:k\neq j} R_2^{jk}(t) - \sum_{k:k\neq j} b_2^{jk}(t)\Big), \\
h_{12}^{jk}(t) &= 0,
\end{aligned}
$$

$$h_{x0}^{jk}(t) = \sum_{k:k\neq j} R_1^{jk}(t) - \sum_{k:k\neq j} b_1^{jk}(t) + \frac{V_1^{j*}(t)}{V_2^{j*}(t)}\bigg(\sum_{k:k\neq j} R_2^{jk}(t) - \sum_{k:k\neq j} b_2^{jk}(t)\bigg).$$

We carry out the same procedure for the dynamics of Y

$$\begin{aligned}
dY(t) = r(t)Y(t)dt + (r(t) - r^*(t))X(t)dt + \sum_{k:k\neq Z(t-)} \rho^{Z(t-)k}(t, X(t))dt \\
- \delta_1^{Z(t)}(t) - \delta_2^{Z(t)}(t)X(t) - \delta_3^{Z(t)}(t)Y(t) \\
- \sum_{k:k\neq Z(t-)} R^{Z(t-)k}(t, X(t-))dN^k(t) \\
+ \sum_{k:k\neq Z(t-)} R^{Z(t-)k}(t, X(t-))\mu^{Z(t-)k}(t)dt \\
= X(t)\bigg\{ r(t)dt - r^*(t)dt - \delta_2^{Z(t)}(t)dt \\
+ \frac{1}{V_2^{Z(t-)*}(t)}\bigg(\sum_{k:k\neq Z(t-)} \rho_2^{Z(t-)k}(t)dt - \sum_{k:k\neq Z(t-)} R_2^{Z(t-)k}(t)dN^k(t) \\
+ \sum_{k:k\neq Z(t-)} R_2^{Z(t-)k}(t)\mu^{Z(t-)k}(t)dt\bigg)\bigg\} \\
+ Y(t)(r(t)dt - \delta_3^{Z(t)}(t)dt) \\
- \frac{V_1^{Z(t-)*}(t-)}{V_2^{Z(t-)*}(t-)}\bigg(\sum_{k:k\neq Z(t-)} \rho_2^{Z(t-)k}(t)dt - \sum_{k:k\neq Z(t-)} R_2^{Z(t-)k}(t)dN^k(t) \\
+ \sum_{k:k\neq Z(t-)} R_2^{Z(t-)k}(t)\mu^{Z(t-)k}(t)dt\bigg) \\
- \delta_1^{Z(t)}(t) + \sum_{k:k\neq Z(t-)} \rho_1^{Z(t-)k}(t)dt - \sum_{k:k\neq Z(t-)} R_1^{Z(t-)k}(t)dN^k(t) \\
+ \sum_{k:k\neq Z(t-)} R_1^{Z(t-)k}(t)\mu^{Z(t-)k}(t)dt.
\end{aligned}$$

Once again, we separate into the terms that are continuous and discrete providing us with $g_{21}^j, g_{22}^j, g_{y2}^j, h_{21}^{jk}, h_{22}^{jk}$ and h_{y2}^{jk}

$$\begin{aligned}
g_{21}^j(t) &= r(t) - r^*(t) - \delta_2^j(t) \\
&\quad + \frac{1}{V_2^{j*}(t)}\bigg(\sum_{k:k\neq j} \rho_2^{jk}(t) + \sum_{k:k\neq j} R_2^{jk}(t)\mu^{jk}(t)\bigg), \\
g_{22}^j(t) &= r(t) - \delta_3^j(t), \\
g_{y0}^j(t) &= -\frac{V_1^{j*}(t)}{V_2^{j*}(t)}\bigg(\sum_{k:k\neq j} \rho_2^{jk}(t) + \sum_{k:k\neq j} R_2^{jk}(t)\mu^{jk}(t)\bigg) \\
&\quad - \delta_1^j(t) + \sum_{k:k\neq j)} \rho_1^{jk}(t) + \sum_{k:k\neq j} R_1^{jk}(t)\mu^{jk}(t),
\end{aligned}$$

$$
\begin{aligned}
h_{21}^{jk}(t) &= -\frac{1}{V_2^{j*}(t)} \sum_{k:k\neq j} R_2^{jk}(t),\\
h_{22}^{jk}(t) &= 0,\\
h_{y0}^{jk}(t) &= \frac{V_1^{j*}(t)}{V_2^{j*}(t)} \sum_{k:k\neq j} R_2^{jk}(t) - \sum_{k:k\neq j} R_1^{jk}(t).
\end{aligned}
$$

We have essentially partitioned the dynamics of X and Y into twelve elements, and each of these elements have an interpretational value which is straightforward to deduce.

Chapter 3

Efficient Projections

Abstract

We consider projections of multi-state with-profit life insurance policies where dividends are assumed to have a certain structure, giving access to a system of differential equations that provide insurance quantities of interest. These differential equations are computationally impractical in situations where insurance risk and financial risk are not independent, which are situations of particular interest due to the possibly severe combined effect on the insurance portfolio. We propose to use a smaller system of approximating differential equations, by reducing the state-space of the insurance policy and the number of policies.

Keywords: With-Profit Insurance, Multi State Life Insurance, Lumping, Projections.

Statements and Declarations

The research leading to these results received funding from Innovation Fund Denmark under grant number 7076-00029, project title "ProBaBLI - Projection of Balances and Benefits in Life Insurance". One of the authors declare a competing interest through the employment at Edlund A/S that sells the software used for the numerical study.

3.1 Introduction

We define lumping in an insurance context, and propose to use lumping as a method to reduce the state-space of a multi-state insurance policy, which also reduces the number of calculations for a projection of a portfolio of with-profit insurance policies in a simulated financial market. In line with the idea of lumping, we study how policies can be combined into so-called modelpoints. We study the approximation error introduced by lumping in a numerical example, where the calculations are performed on industry applied software.

It is natural for insurance providers to incorporate a safety margin in their predictions about elements that influence their liabilities. For that reason, insurance companies charge an ample premium for their products with guaranteed benefits, but they make an obligation to pay some of the hereby generated surplus back to the policyholder in the form of dividends. An ordinary scheme is to use these dividends to buy additional benefits. When such an agreement is made between the policyholder and the insurance company, we call it an additional benefit with-profit life insurance policy. Along with an interest for the principles of fairness governing the distribution of surplus, the with-profit insurance policy and the dynamics of the surplus have been studied since the first half of the twentieth century. This paper builds on the modern approaches as seen in Møller and Steffensen (2007), where the influence of financial risk on with-profit insurance policies is studied in a Black-Scholes market.

There are many valuable insights to gain by studying the with-profit insurance in a Black-Scholes market where closed form solutions are available. However, the complexity needed by the life insurance industry to satisfy high demands on the solvency of their business speaks in favor of using a simulation approach for the estimation of liabilities. By calculating and averaging insurance quantities of interest for thousands of simulated risk factors such as the short rate and transition intensities, Monte Carlo estimates for the expectation of the insurance quantities are formed. A simulation approach allows for great model complexity which helps to make the model realistic. Separately and together, the paths of a simulated financial market and their influence on the assets and liabilities of the insurance company can be analysed to provide valuable information about financial risks. This information can aide the management of the insurance company to construct reasonable investment strategies.

Modern legislation demands that insurance companies project their business into the future, in order to prove that they are solvent and prudent. By actually making a forecast of the business, otherwise inaccessible awareness about the influence of management actions is gained. In order to form these projections, realistic developments of the financial markets and transition intensities are essential, advocating for a simulation approach.

In Jensen (2016) the with-profit life insurance policy is studied for an arbitrary simulated financial market, conditional on the state of the policy staying in its initial state. Also for a simulated financial market, Jensen and Schomacker (2015) study the with-profit insurance policy in a framework that allows for portfolio wide projections, for a dividend strategy that may depend on the financial markets only. In a setting with financial risk, Bacinello et al. (2018) introduce systematic insurance risk via a stochastic mortality intensity which is assumed to be independent of the financial market.

The outset of this paper is provided by Bruhn and Lollike (2021), who derive a system of differential equations that describe the projections of the surplus and savings of an additional benefits with-profit life insurance policy where both financial markets and transition intensities are simulated and possibly dependent. The differential equations from Bruhn and Lollike (2021) demand a dividend strategy that is affine in the reserves that are projected. An equivalent system of differential equations are provided by Ahmad et al. (2021), who also present a class of dividend strategies that lead to particularly simple projections in the likes of Jensen and Schomacker (2015).

Just as simulation of the financial market is important to understand financial risks, simulation of transition intensities is important to understand biometric risks, and possibly how the two interact. Even with the ever increasing amount of computational power at disposal of actuaries, the differential equations provided by Bruhn and Lollike (2021) are infeasible for projecting entire portfolios of insurance policies, in particular when both investment returns and transition intensities are stochastic. In this paper we improve the feasibility through three contributions;

a) We provide a method to translate payments and transitions for an insurance policy to another but similar insurance policy that lives on a smaller state space. We denote this operation lumping.

b) Via lumping, we provide a system of differential equations smaller than the one suggested by Bruhn and Lollike (2021) for projecting a portfolio of with-profit insurance policies where some intensities are stochastic.

c) In the spirit of reducing the number of differential equations needed for projecting an insurance portfolio, we specify how policies can be combined into so-called modelpoints to further reduce the system of differential equations.

Contribution a) is purely theoretical, but provides us with a tool that we need in order to formalise our main contribution, namely contribution b). Both contribution b) and c) yield linear improvements in the computational run-time of projections, and are in that regard equally useful. By reducing the number of differential equations while maintaining a projection that incorporates stochastic intensities, we are striking a balance between the speed and accuracy of the projection, resulting in efficient projections.

The paper is structured as follows. In Section 3.2 we establish the involved framework of the with-profit insurance policy with stochastic transition intensities. In Section 3.3 we define the cash flow preserving lumping, and present some of its properties. Section 3.4 is the cornerstone of the paper, as it is here we present contribution b) and establish when the smaller system of differential equations describe an aggregated version of the original system of differential equations. Section 3.5 introduces a method to combine policies and determines when this can be done without any loss of accuracy. Finally, we conduct a numerical study in Section 3.6, investigating the approximation error introduced by lumping for a simple insurance policy, with calculations performed on industry applied software.

3.2 Set-up

3.2.1 State Process and Financial Market

Let the process $Z(t)$, denote the time-t state of a policy on a finite state space $\mathcal{J} = 0, ..., J$. The initial state of the policy is assumed to be known. Let

$$N^k(t) := \#\{s \in (0,t] : Z(s-) \neq k, Z(s) = k\},$$

be the counting process expressing the number of jumps of Z into state k. The information about Z available at time t is represented by the filtration $\mathcal{F}_t^Z$, generated by Z.

The with-profit insurance policy is through the surplus influenced by a financial market consisting of a short rate and a vector of tradable assets. We do not impose any restrictions on the dynamics of the financial market, allowing for great model complexity which is the significant strength of the simulation approach. The information about the financial market available at time t is denoted $\mathcal{F}_t^S$.

As stated in the introduction, we want Z to be a stochastic Markov process with stochastic intensities, often called a doubly stochastic Markov setting. The formal arguments to construct such a Z process, start with the continuous sample path stochastic intensities μ_{ij}. Using stochastic intensities, we employ Kolmogorovs forward differential equations,

$$\frac{d}{dt}p_{ji}(s,t) = \sum_{k \neq i} p_{jk}(s,t)\mu_{ki}(t) - p_{ji}(s,t)\mu_{ik}(t), \qquad p_{ji}(s,s) = \mathbb{1}_{\{i=j\}},$$

to construct the functions $p_{ij}(s,t)$, and one can show that these functions satisfy

$$p_{ij}(s,t) = P(Z(t) = j \mid Z(s) = i),$$

where Z is a stochastic process with intensities μ_{ij}. See Section 2.1 of Buchardt et al. (2019) and their references to Jacobsen (2006) for the details of this construction. The stochastic intensities form the best guess on the intensities of the state process and are called the third-order intensities.

In relation to transitions, an insurance company faces two kinds of risk; unsystematic and systematic. Unsystematic risk is the risk involved with transitions between states, and with how closely the portfolio of policies behave like they are expected to, given that the third-order intensities are the actual intensities of the state process. This type of risk is not particularly worrisome, as a large portfolio of homogeneous policies will behave like they are expected to, according to the law of large numbers. The systematic risk is a more serious manner, as it concerns whether or not the transition intensities actually describe the state process of the policies. If the wrong intensities are used for the entire portfolio of insurance policies, it can have a disastrous impact on the insurance company. It is therefore imperative that we understand the systematic risk, and this insight can be gained by producing thousands of projections of the insurance portfolio with simulated stochastic transition intensities.

There are multiple drivers of the systematic risk that influence the insurance provider through stochastic transition intensities, for example medical advancements, pandemics or political actions. Some of these transition intensities change due to biometric factors purely, while others change due to differences in behaviour. Classically, a multi-state policy has two options that can be exercised as long as the policy is premium paying. The policyholder can choose to cease his premium, or to cancel his policy all together. Ceasing premium payments is referred to as converting to free-policy. The free-policy option has been studied in Buchardt and Møller (2015) and for a with-profit policy in Falden and Nyegaard (2021) and Ahmad et al. (2021). Upon cancellation of the policy, the reserve is paid out to the policyholder, and this option is known as the surrender option.

It is not unrealistic that some behavioural intensities depend on the financial market, such as for instance the free-policy transition intensity – due to unemployment during times of recession, there may be a higher number of policies that choose to exercise their free-policy option. This particular type of market dependent systematic insurance risk is related to the behaviour of the policyholder. The behavioural intensities are of particular interest due to possibly hazardous combinations of effects. Imagine for instance, in times of recession when the insurance company is losing money on the financial markets, that the policyholders also have a higher tendency to surrender their policy or exercise their free-policy option. Is such a combined effect of a recession better or worse for the insurance company than if there is no market dependence? And how should the management incorporate this information in their actions? In order to answer these types of questions, we need to be able to study projections for state processes that depend on the financial market.

We assume that the intensities are $\mathcal{F}^S$-adapted, because we want the stochastic element of the intensities to be driven by the financial market. Even for stochastic intensities that do not depend on the financial market, we can define $\mathcal{F}^S$ as the σ-algebra generated by an artificial market containing information about transition intensities as well as the financial market.

In practice, the simulation of the financial market involves all sorts of assets, but for the theoretical purposes of this paper, we use the short rate $r(t)$ as a proxy for the return of a self-financing portfolio of tradable assets to ease readability. We use the notation

$$\mathrm{E}_0[\,\cdot\,] = \mathrm{E}[\,\cdot\, \mid \mathcal{F}_0^Z],$$

representing the expectation given the information about Z available at time zero, which is the mean value of interest concerning projections.

3.2.2 With-profit Insurance

In this subsection we introduce the surplus and the savings account and specify how the two parts combine to create a with-profit insurance policy.

The Savings Account

The ideas and notation of this subsection are inspired by Møller and Steffensen (2007) and Asmussen and Steffensen (2020). The with-profit life insurance policy consists of a combination of two payment streams B_1 and B_2, with dynamics

$$dB_i(t) = b_i^{Z(t)}(t)dt + \sum_{k:k\neq Z(t-)} b_i^{Z(t-)k}(t)dN^k(t), \qquad i = 1, 2.$$

The deterministic payment functions $b_i^j(t)$ and $b_i^{jk}(t)$ specify payments during sojourns in state j and on transition from state j to state k, respectively. At initialisation of the policy, the insured and insurance company agree on the B_1 payment stream that is determined to be fair under the principle of equivalence, for a set of prudent assumptions regarding interest and transition intensities. These prudent assumptions are called the first-order or technical basis and they are independent of the financial market. The first-order basis consists of an interest rate $r^*(t)$, and a set of deterministic transition intensities $\mu_{ij}^*(t)$. An asterisk is used to indicate first-order elements such as the prospective technical reserves

$$V_i^{Z(t)*} = \mathrm{E}^*\left[\int_t^n e^{-\int_t^s r^*(v)dv} dB_i(s) \mid Z(t)\right].$$

The dynamics of $V_i^{Z(t)*}$ depend on the financial market through the state process Z, but for any fixed j we can calculate V_i^{j*} without any information about the financial market using Thieles differential equation.

In contrast to the prudent first-order basis, the policy also has a best guess on the interest rate $r(t)$ and transition intensities $\mu_{ij}(t)$ called the third-order basis or market basis. It is exactly by simulating the third-order basis we can generate a realistic evolution of the future, but we have to do this thousands of times to achieve the stable averages that are our Monte Carlo estimators. Liabilities that are calculated for a simulated third-order basis, is often referred to as stochastic liabilities, and Monte Carlo estimates present a powerful tool to assess liabilities for models where analytical solutions are unavailable.

If the first-order basis truly is prudent, a surplus is expected to develop. This surplus belongs in part to the equity of the insurance company for taking a risk in guaranteeing a payment stream, but mostly to the policyholder who financed the surplus. The surplus is paid back to the policyholder in the form of dividends. The bonus scheme determines how the dividends are turned into actual payments for the policyholder, and we are fixed on the additional benefits bonus scheme. We denote by $D(t)$ the accumulated dividends at time t and assume that it consists purely of continuous payments that instantaneously are spent on buying more of the B_2 payment stream. The additional payment streams are also priced under the first-order basis, and, introducing the process $Q(t)$ to denote the quantity of B_2 payment streams held by the policyholder at time t, we have that

$$dD(t) = dQ(t)V_2^{Z(t)*}(t),$$

with the convention $Q(0) = 0$. The policyholder experiences the combined payment process

$$dB(t) = dB_1(t) + Q(t-)dB_2(t).$$

The payment stream B_1 is fixed, and can be thought of as containing premiums only, but can in general contain both premiums and benefits. The payment stream B_2 consists of bonus benefits, and to avoid a scenario where additional benefits has no value, we require for B_2 that it does not contain any negative benefits (premiums), and that $V_2^{k*}(t) \neq 0$ for all $(t,k) \in (0,n] \times \mathcal{J}$. We can now construct the savings account of the policy as

$$X(t) = V_1^{Z(t)*}(t) + Q(t)V_2^{Z(t)*}(t),$$

which corresponds to the present value of all future benefits less premiums guaranteed at time t, or equivalently, the present value of all past premiums and dividends less benefits compounded with the first-order interest. Using integration by parts we find the dynamics of X

$$\begin{aligned}dX(t) =& dV_1^{Z(t)*}(t) + Q(t-)dV_2^{Z(t)*}(t) + V_2^{Z(t)*}(t)dQ(t)\\ =& r^*(t)X(t)dt + dD(t) - b^{Z(t)}(t, X(t))dt\\ &- \sum_{k\neq Z(t-)} b^{Z(t-)k}(t, X(t-))dN^k(t) - \sum_{k:k\neq Z(t-)} \rho^{Z(t)k}(t, X(t))dt\\ &+ \sum_{k:k\neq Z(t-)} R^{Z(t-)k}(t, X(t-))dM^k(t),\end{aligned}$$

where

$$\begin{aligned}M^k(t) =& N^k(t) - \int_0^t \mathbb{1}_{\{Z(s-)\neq k\}}\mu_{Z(s-)k}(s)ds,\\ b^j(t,x) =& b_1^j(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} b_2^j(t),\\ b^{jk}(t,x) =& b_1^{jk}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} b_2^{jk}(t),\\ \rho^{jk}(t,x) =& \rho_1^{jk}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} \rho_2^{jk}(t),\\ R^{jk}(t,x) =& R_1^{jk}(t) + \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} R_2^{jk}(t),\end{aligned}$$

for R_i^{jk} being the sum-at-risk on transition from j to k, given by

$$R_i^{jk}(t) = b_i^{jk}(t) + V_i^{k*}(t) - V_i^{j*}(t),$$

and ρ^{jk} being the surplus risk contribution on transition from state j to k, given by

$$\rho_i^{jk}(t) = R_i^{jk}(t)(\mu_{jk}^*(t) - \mu_{jk}(t)).$$

Note that $dX(t)$ is affine in $X(t)$ if and only if $dD(t)$ is affine in $X(t)$. How the savings account evolves, depends on the specification of the B_1 and B_2 payment stream, but also on the amount of dividends received from the surplus.

The Surplus

The surplus is formed by the safety margin built into the first-order basis in the form of prudent assumptions on interest and transition intensities. Dividends are financed by the surplus, making it a major component of the with-profit insurance policy. The surplus for a group of N policies, indexed by prescript l, is defined as

$$Y(t) = -\left(\sum_{l=1}^N \int_0^t e^{\int_s^t r(\tau)d\tau} d({}_lB_1(s) + {}_lQ(s)\,{}_lB_s(s)) + {}_lX(t)\right),$$

corresponding to the sum over benefits less premiums compounded with the short rate for all policies in the group. This definition of surplus for a group of policies, is simply a sum over the individual surplus as defined by Bruhn and Lollike (2021) and Asmussen and Steffensen (2020). The dynamics of Y are

$$\begin{aligned} dY(t) =& r(t)Y(t)dt \\ &+ \sum_{l=1}^{N} \left(d\,{}_lC(t) - d\,{}_lD(t) - \sum_{k:k\neq_l Z(t)} {}_lR^{l\,Z(t-)k}(t, {}_lX(t-))d\,{}_lM^k(t) \right), \end{aligned} \tag{3.2.1}$$

where

$$d\,{}_lC(t) = (r(t) - r^*(t))\,{}_lX(t)dt + \sum_{k\neq_l Z(t)} {}_l\rho^{l\,Z(t)k}(t, {}_lX(t))dt,$$

is the surplus contribution rate for policy l, characterizing the payment stream from policy l to the surplus.

Projecting The Savings Account and Surplus

Our goal is to calculate expected future values of the savings account and surplus which are retrospective reserves. Specifically, we want to calculate the probability weighted state-wise savings account, often simply called the state-wise savings account, defined as

$${}_l\tilde{X}^i(t) := E_0[{}_lX(t)\mathbb{1}_{\{_lZ(t)=i\}}],$$

giving access to several quantities of interest such as the future discretionary benefits

$${}_lFDB := \mathrm{E}_0\left[\int_0^n e^{-\int_0^s r(\nu)d\nu}\,{}_lQ(s)dB_2(s)\right],$$

and the future guaranteed reserves

$${}_lGY^i(t) := \mathrm{E}_0\left[\int_t^n e^{\int_t^s r(\nu)d\nu} d\left({}_lB_1(s) + {}_lQ(t)\,{}_lB_2(s)\right)\middle|\,{}_lZ(t) = i\right].$$

Bruhn and Lollike (2021) consider a one-policy set-up and produce a system of forward differential equations for the state-wise retrospective reserves, when the dividends are affine in X and Y. Throughout this paper we assume that the dynamics of the dividend process are affine in X and Y, as the linear relation is a necessity for the projections to be described by the differential equations for the one-policy set-up from Bruhn and Lollike (2021). A straightforward extension of the one-policy set-up is to consider the projection of

$$\tilde{W}^{ij}(t) = \begin{pmatrix} \mathrm{E}_0[{}_1X(t)\mathbb{1}_{\{_1Z(t)=i\}}\mathbb{1}_{\{_2Z(t)=j\}}] \\ \mathrm{E}_0[{}_2X(t)\mathbb{1}_{\{_1Z(t)=i\}}\mathbb{1}_{\{_2Z(t)=j\}}] \\ \mathrm{E}_0[Y(t)\mathbb{1}_{\{_1Z(t)=i\}}\mathbb{1}_{\{_2Z(t)=j\}}] \end{pmatrix},$$

which can be calculated using the same forward differential equations as in the one-policy case. However, increasing the number of policies quickly renders the method impractical, as a system of more than $(\#\mathcal{J})^N$ differential equations has to be solved. For a 7-state model, 8 policies already requires a system of more than 5 million differential equations. The problem lies in the fact that the savings accounts are dependent through the common surplus Y. Fortunately, there are tools at the disposal of actuaries to mitigate the influence of unsystematic risk, simply by increasing the size of the portfolio of homogeneous policies. Based on the law of large numbers, we can average out the unsystematic risk carried by the surplus, if the portfolio of insurance policies is sufficiently large. This method is also implemented by Ahmad et al. (2021), using the same argumentation as Norberg (1991) and Møller and Steffensen (2007). To simplify notation, we define ${}_l p_i(t) = {}_l p_{l_0 i}(0,t)$, where l_0 denotes the state of policy l at time 0. By replacing the stochastic processes in the dynamics (3.2.1) with their state-wise expected values, we get the risk-diversified surplus with dynamics

$$\begin{aligned} d\tilde{Y}(t) =& r(t)\tilde{Y}(t)dt + (r(t) - r^*(t)) \sum_{l=1}^{N} \sum_{i\in\mathcal{J}} {}_l\tilde{X}^i(t)dt \\ &+ \sum_{k:k\neq i} {}_l p_i(t)\, {}_l\rho^{ik} \left(t, {}_l\tilde{X}^i(t)/\, {}_l p_i(t)\right) dt \\ &- \sum_{l=1}^{N} \sum_{i\in\mathcal{J}} {}_l p_i(t)({}_l\delta_0^i(t) + {}_l\delta_2^i(t)\tilde{Y}(t)) + {}_l\delta_1^i(t)\, {}_l\tilde{X}^i(t)dt. \end{aligned}$$

Using the risk-diversified surplus, the individual savings accounts are, conditional on $\mathcal{F}^S$, independent of each other. We emphasize that it is only the unsystematic insurance risk that is diversified – there is still a component of systematic insurance risk influencing the surplus through the market-dependent transition intensities. By using the risk-diversified surplus, we only need the dividends to be affine in X, but for the sake of coherence with the non-risk-diversified surplus, we assume that the dynamics of the dividend is given by

$$d\, {}_l D(t) = ({}_l\delta_0^{{}_lZ(t)}(t) + {}_l\delta_1^{{}_lZ(t)}(t)\, {}_l X(t) + {}_l\delta_2^{{}_lZ(t)}(t)\tilde{Y}(t))dt. \tag{3.2.2}$$

Projecting a portfolio of independent policies is computationally much less demanding than projecting a portfolio of policies that depend on the states of all other policies, but it is still not feasible. As it forms the benchmark for projecting a portfolio of with-profit insurance policies, we state the main theorem of Bruhn and Lollike (2021) in a setting with the risk-diversified surplus and N policies.

Theorem 3.2.1. *For a portfolio of with-profit insurance policies indexed by l with dividend processes affine in the savings and surplus, the state-wise savings accounts*

$${}_l\tilde{X}^i(t) = E_0[\mathbb{1}_{\{{}_lZ(t)=i\}}\, {}_l X(t)],$$

and the risk-diversified surplus $\tilde{Y}$, satisfy the system of differential equations

$$
\begin{aligned}
\frac{d}{dt}\,{}_{l}\tilde{X}^{j}(t) = &\sum_{i:i\neq j}\Big({}_{l}\mu_{ij}(t)\,{}_{l}\tilde{X}^{i}(t) - {}_{l}\mu_{ji}(t)\,{}_{l}\tilde{X}^{j}(t)\Big)\\
&+ {}_{l}g_{1}^{j}(t)\,{}_{l}\tilde{X}^{j}(t) + {}_{l}p_{j}(t)\,{}_{l}g_{0}^{j}(t) + {}_{l}p_{j}(t)\,{}_{l}\delta_{2}^{j}(t)\tilde{Y}(t)\\
&+ \sum_{i:i\neq j}{}_{l}\mu_{ij}(t)\Big({}_{l}h_{1}^{ij}(t)\,{}_{l}\tilde{X}^{i}(t) + {}_{l}p_{i}(t)\,{}_{l}h_{0}^{ij}(t)\Big),\\
\frac{d}{dt}\tilde{Y}(t) =& r(t)\tilde{Y}(t) - \sum_{l=1}^{N}\sum_{j\in\mathcal{J}}{}_{l}p_{j}(t)\,{}_{l}\delta_{0}^{j}(t)\\
&- \tilde{Y}(t)\sum_{l=1}^{N}\sum_{j\in\mathcal{J}}{}_{l}p_{j}(t)\,{}_{l}\delta_{2}^{j}(t)\\
&+ \sum_{l=1}^{N}\sum_{j\in\mathcal{J}}{}_{l}p_{j}(t)\,{}_{l}c_{0}^{j}(t) + {}_{l}\tilde{X}^{j}(t)\,{}_{l}c_{1}^{j}(t),\\
\frac{d}{dt}\,{}_{l}p_{j}(t) = &\sum_{i\neq j}{}_{l}\mu_{ij}(t)\,{}_{l}p_{i}(t) - {}_{l}\mu_{ji}(t)\,{}_{l}p_{j}(t),\\
{}_{l}\tilde{X}^{i}(0) =& \mathbb{1}_{\{i=l_0\}}\,{}_{l}x_{0},\\
\tilde{Y}(0) =& y_{0},\\
{}_{l}p_{j}(0) =& \mathbb{1}_{\{j=l_0\}},
\end{aligned}
$$

where

$$
\begin{aligned}
{}_{l}g_{0}^{j}(t) =& {}_{l}\delta_{0}^{j}(t) - {}_{l}b_{1}^{j}(t) - \sum_{k:k\neq j}{}_{l}R_{1}^{jk}(t)\,{}_{l}\mu_{jk}^{*}(t)\\
&+ \frac{{}_{l}V_{1}^{j*}(t)}{{}_{l}V_{2}^{j*}(t)}\left({}_{l}b_{2}^{j}(t) + \sum_{k:k\neq j}{}_{l}R_{2}^{jk}(t)\,{}_{l}\mu_{jk}^{*}(t)\right)\\
{}_{l}g_{1}^{j}(t) =& r^{*}(t) + {}_{l}\delta_{1}^{j}(t) - \frac{1}{{}_{l}V_{2}^{j*}(t)}\left({}_{l}b_{2}^{j}(t) + \sum_{k:k\neq j}{}_{l}R_{2}^{jk}(t)\,{}_{l}\mu_{jk}^{*}(t)\right)\\
{}_{l}h_{0}^{jk}(t) =& {}_{l}V_{1}^{k*}(t) - \frac{{}_{l}V_{1}^{j*}(t)\,{}_{l}V_{2}^{k*}(t)}{{}_{l}V_{2}^{j*}(t)}\\
{}_{l}h_{1}^{jk}(t) =& \frac{{}_{l}V_{2}^{k*}(t)}{{}_{l}V_{2}^{j*}(t)} - 1\\
{}_{l}c_{0}^{j}(t) =& \sum_{k:k\neq j}{}_{l}\rho_{1}^{jk}(t) - \frac{{}_{l}V_{1}^{j*}(t)}{{}_{l}V_{2}^{j*}(t)}\sum_{k:k\neq j}{}_{l}\rho_{2}^{jk}(t)\\
{}_{l}c_{1}^{j}(t) =& (r(t) - r^{*}(t)) - {}_{l}\delta_{1}^{j}(t) + \frac{1}{{}_{l}V_{2}^{j*}(t)}\sum_{k:k\neq j}{}_{l}\rho_{2}^{jk}(t).
\end{aligned}
$$

See Bruhn and Lollike (2021) for the proof, where obvious small modifications are relevant when projecting the risk-diversified surplus. In practice this system of $2 \times N \times \#\mathcal{J} + 1$ differential equations has to be solved for thousands of realisations of the financial market in order to form a stable Monte Carlo estimator. Note that the calculation of ${}_l p_k(t)$ needs to be done only once if the transition intensities are independent of the financial market. If however, at least one intensity depends on $\mathcal{F}_t^S$, then ${}_l p_k(t)$ has to be calculated for all k for each financial scenario. By means of diversification we have greatly reduced the number of differential equations, but even so, the computational task of projecting the portfolio of policies is still overwhelming in practice due to the very large number of financial scenarios. To further reduce the system, we can either reduce the number of states or the number of policies. In the next section we propose a method for approximating reserves and benefits by reducing the state space of the policy, revealing a gateway to reducing the number of differential equations for the with-profit insurance policy.

To ease readability, we consider a single policy portfolio in the remainder of the paper, except from Section 3.5.

3.3 Reducing the With-Profit Policy

In this section we look into the elements that make up a with-profit insurance policy, and present a way to modify these elements such that they describe a different, but similar insurance policy on a smaller state-space. The motivation for this reduction of the state-space is to reduce the number of differential equations from Theorem 3.2.1.

Fundamentally, there are three parts of any multi-state insurance policy; a third-order basis, a first-order basis and a payment process. The payment process accounts for payments between the savings account and the policyholder as well as payments between the savings account and the surplus. In addition, the initial distribution on the state space of the state process is relevant, and assumed to be known. We want to translate each of these parts to corresponding versions that describe a policy on a smaller state space.

Reducing the number of states of a Markov process to produce another Markov process is known within the statistical litterature as lumping. Originating in the work by Kemeny and Snell (1976), the notion of lumping and lumpability has been studied extensively in the statistical literature. However, the questions of statistical interest are not of interest in regards to projections of insurance portfolios. Instead of asking the statistically relevant question; how can we create a partition of the state space that preserves Markovianity? We ask the question; which Markov process on a given partition of the state space behaves most like the original process? Once the Markov process has been found, we also have the further objective of defining a payment stream on the lumped states that imitates the original payment stream. Due to the difference in interest, our definition of lumping is inherently embedded in the insurance domain and vastly different to the one found in the statistical literature.

For convenience, we construct the smaller state space via a so-called reduction function.

Definition 3.3.1. *A function $R : \mathcal{J} \to \mathcal{J}'$, is a Reduction Function if R is a surjection of $\mathcal{J}$ on $\mathcal{J}'$ and*

$$\#\mathcal{J}' < \#\mathcal{J},$$

i.e. there are fewer states in $\mathcal{J}'$ than in $\mathcal{J}$.

A reduction function is basically a look-up table, specifying which states of the non-reduced state space $\mathcal{J}$ that should be grouped. To increase readability, we generally use capital letters for the states of $\mathcal{J}'$ and lower-case letters for the states of $\mathcal{J}$. For some $G \in \mathcal{J}'$ define the inverse of R as

$$R^{-1}(G) := \{i \in \mathcal{J} \mid R(i) = G\},$$

with the informal notation $i \in A \Leftrightarrow i \in R^{-1}(A)$.

We define lumping as the operation of translating an insurance policy on the state space $\mathcal{J}$ to the state space $\mathcal{J}'$. There are many ways to perform the operation of lumping, but for our purposes, a specific way of translating the elements of the insurance policy is particularly useful. Given a group of states A, we propose to construct lumped intensities and payments by creating a probability-weighted average of the intensities and payments within the group.

Definition 3.3.2 (Cash Flow Preserving Lumping). *Given a reduction function $R : \mathcal{J} \to \mathcal{J}'$, a set of transition intensities $\tilde{\mu}_{ij}$, initial probabilities $p_i(0)$, intensities $\mu_{ij}(t)$, sojourn payments $h^i(t)$ and payments on transitions $h^{ij}(t)$, we define the lumped initial probabilities, intensities, sojourn payments and payments on transition by*

$$\begin{aligned}
p_I(0) =& \sum_{i\in I} p_i(0), \\
\mu_{IJ}(t) =& \frac{1}{\sum_{k\in I} \tilde{p}_k(s)} \sum_{i\in I}\sum_{j\in J} \tilde{p}_i(t)\mu_{ij}(t), \\
h^I(s) =& \frac{1}{\sum_{k\in I} \tilde{p}_k(s)} \sum_{i\in I} \tilde{p}_i(s) \left(h^i(s) + \sum_{\substack{j\in I \\ j\neq i}} h^{ij}(s)\tilde{\mu}_{ij}(s) \right), \\
h^{IJ}(s) =& \frac{\displaystyle\sum_{i\in I} \tilde{p}_i(s) \sum_{j\in J} \mu_{ij}(s) h^{ij}(s)}{\displaystyle\sum_{i\in I} \tilde{p}_i(s) \sum_{j\in J} \mu_{ij}(s)},
\end{aligned}$$

respectively. The probability-weights $\tilde{p}_i$ solve the Kolmogorov differential equations

$$\begin{aligned}\tilde{p}_i(0) =& p_i(0)\\ \frac{d}{dt}\tilde{p}_i(t) =& \sum_{k\neq i}\tilde{p}_k(t)\tilde{\mu}_{ki}(t) - \tilde{p}_i(t)\tilde{\mu}_{ik}(t).\end{aligned}$$

We say that the lumping is constructed with respect to $\tilde{\mu}_{ij}$.

The Cash Flow Preserving Lumping (CFPL) is basically a way to probability weigh the benefits and transitions of the policy, where $\tilde{\mu}_{ij}$ determine the weights. The initial distribution and transition intensities between states of $\mathcal{J}'$, can be thought of as characterizing a state-process $\tilde{Z}(t)$ that we assume to exist. If the transition intensities of the non-lumped policy are Borel functions on bounded intervals, then the lumped transition intensities will be Borel functions on bounded intervals, therefore making $\tilde{Z}(t)$ a Markov process. Heuristically, the elements of the lumped policy represents the expected transition intensities and expected payments, given only the information $R(Z(t))$. Note that the payment processes B_1, B_2 and D all can be lumped using Definition 3.3.2, which is the intended use of the definition. Importantly, we can therefore construct lumped reserves, lumped savings accounts and lumped surpluses simply by replacing the original insurance elements with their lumped counterparts.

Definition 3.3.2 may seem somewhat arbitrary, but there are certain appealing properties satisfied by this way of lumping the insurance policy. The relationship between probabilities of the lumped policy and the non-lumped policy is simple and straightforward to interpret.

Lemma 3.3.3 (State-wise Probabilities)**.** *For the CFPL w.r.t μ_{ij}, the state-wise probabilities of the lumped policy, are related to the state-wise probabilities of the non-lumped policy in the following way*

$$p_I(t) = \sum_{i\in I} p_i(t). \tag{3.3.1}$$

See Appendix 3.A for the proof. This is a reasonable property to demand from a lumping. Furthermore, the probability of transition between states of the lumped and non-lumped model have a similar relation.

Lemma 3.3.4 (Transition Probabilities)**.** *For the CFPL w.r.t μ_{ij}, the transition probabilities of the lumped policy, are related to the transition probabilities of the non-lumped policy in the following way*

$$p_{IJ}(t,s) = \sum_{i\in I}\sum_{j\in J}\frac{p_i(t)}{p_I(t)}p_{ij}(t,s). \tag{3.3.2}$$

See Appendix 3.B for the proof. Apart from relations between probabilities of the lumped and non-lumped policy, the expected payments for all states are also preserved.

Theorem 3.3.5 (Preservation of State-wise Cash Flow Contributions)**.** *For any $G \subseteq \mathcal{J}'$, the CFPL w.r.t μ_{ij} preserves all state-wise contributions to the expected accumulated payments*

$$E\left[\int_0^t \mathbb{1}_{\{R(Z(s))\in G\}} dB(s) \middle| \; p_i(0), i \in \mathcal{J}\right] = E\left[\int_0^t \mathbb{1}_{\{\tilde{Z}(s)\in G\}} dB^R(s) \middle| \; p_I(0), I \in \mathcal{J}'\right],$$

where $\tilde{Z}$ is a Markov process with intensities μ_{IJ} and initial distribution given by $p_I(0)$ and B^R is the $\tilde{Z}$-dependent payment stream given by lumping the payments in the B payment stream with the reduction function R.

See Appendix 3.C for the proof.

Remark 3.3.6. Applying Theorem 3.3.5 to the total reduction $R : \mathcal{J} \to 1$, implies that the expected total accumulated payments are preserved. This also means that the cash flow is equal to the sojourn payment of the total lumping

$$\sum_{j\in\mathcal{J}} p_j(t) \left(b^j(t) + \sum_{k:k\neq j} \mu_{jk}(t) b^{jk}(t) \right) = b^T(t).$$

Theorem 3.3.5 states that no matter how states are lumped, the group-wise contribution to the aggregated payments are preserved. Apart from conservation of state-wise accumulated payments it holds for the total lumping that the prospective reserve is equal to the probability-weighted state-wise prospective reserves.

Lemma 3.3.7 (Total Reduction Reserve)**.** *For the CFPL w.r.t μ_{ij}, and the total reduction function, $R : \mathcal{J} \to 1$,*

$$V^T(t) = \sum_{i\in\mathcal{J}} p_i(t) V^i(t).$$

Which is shown simply by noting that the initial conditions and differential equations for the right- and left-hand side are identical – see Appendix 3.D for the derivation. However, it does not in general hold that

$$V^I(t) = \sum_{i\in I} \frac{p_i(t)}{p_I(t)} V^i(t),$$

for any reduction function other than the total reduction, even though it may be a close approximation. By using the cash flow preserving lumping, we get everything we need to produce a with-profit insurance policy on the reduced state space, simply by replacing the payments, intensities and reserves of Theorem 3.2.1, with their lumped counterpart. By replacing the insurance elements of Theorem 3.2.1 with their lumped counterpart we get a system of differential equations for the lumped savings account $\tilde{X}^I$ that we believe approximates the group-wise sum of the non-lumped savings accounts. By projecting the lumped savings account we are losing accuracy, but we are also reducing the computation time.

3.4 Lumping With-Profit Insurance

There is no way to exactly project the with-profit insurance portfolio of state-wise savings accounts and surplus with market-dependent transition intensities, without calculating the $2 \times N \times \#\mathcal{J} + 1$ differential equations specified in Theorem 3.2.1. If there are any market dependent intensities, all probabilities are market-dependent. Our objective is to devise a smaller system of differential equations that describe projections of quantities that carry much of the same information as the projections of the full model. We believe that the approach suggested in this paper is a fairly accurate approximation, but also note that there is no trade-off between accuracy and speed that is optimal for all purposes. Faster, less accurate approaches than the one proposed here definitely exist. Fundamentally we need to discard information in order to reduce the system of differential equations, but there is some information that we certainly do not want to discard.

The main contribution of this paper, is to provide the means to efficiently project a with-profit insurance policy in a setting where the intensities between behavioural groups of states are stochastic. Therefore, the advantage of lumping states, relies on a partition of the state space satisfying the property that two states connected via a stochastic intensity are in different groups. We call this a stochastically-separating partition. With such a partition, we can reduce the number of differential equations from Theorem 3.2.1 while maintaining a model with stochastic intensities. In order to separate states that are connected via a stochastic intensity, the partition may need to be the trivial partition, and even if there exists a non-trivial partition that separates stochastically connected states, it may be hard to find. We do not dive into the rabbit-hole of graph partitioning, but instead refer to the vast literature on the subject, and assume that the stochastically-separating partition is given. See Buluc et al. (2015) for a review of the recent literature on graph partitioning.

As stated in the last section, nothing stops us from using Theorem 3.2.1 with lumped insurance elements to project a lumped savings account to produce an approximation of the aggregated non-lumped savings accounts. In order to understand the gap between the systems of differential equations generated by the original insurance elements, and the ones generated by the lumped insurance elements, we provide an alternative motivation in this section by presenting three assumptions for a with-profit insurance policy that results in a smaller system of differential equations.

3.4.1 Three Assumptions

Assumption One

The first assumption is the essential assumption, in the sense that it allows us to translate individual state-wise savings accounts, to aggregated state-wise savings accounts. We make this translation by equating the distribution of the value of the savings accounts within groups of states, by their conditional probabilities,

$$\frac{p_i(t)}{\sum_{j\in I} p_j(t)} \sum_{j\in I} \tilde{X}^j = \tilde{X}^i, \quad \text{for } i \in I.$$

If it does not hold, this assumption gives an approximate answer to the question: What are the state-wise savings accounts of a policy for which we only know the state-wise probabilities and aggregated savings account?
The information we are discarding is information that distinguishes savings accounts for states in between groups from one another. We are however not discarding information that in a probabilistic sense distinguishes states in between groups.

Assumption Two

The second assumption deals with the influence stochastic intensities have on all state-wise probabilities, and the means to reduce the influence to groups of policies. Instead of using the true market-dependent intensities μ_{ij}, we use some market-independent intensities $\tilde{\mu}_{ij}$ to calculate probabilities $\tilde{p}_i$, and then project the reduced state space insurance policy based on the assumption

$$\frac{\tilde{p}_i(t)}{\tilde{p}_I(t)} = \frac{p_i(t)}{p_I(t)} \Leftrightarrow p_I(t)\frac{\tilde{p}_i(t)}{\tilde{p}_I(t)} = p_i(t).$$

This idea is equivalent to the wrongful assumption that the state-wise group conditional probabilities are invariant to the financial market. Only in the special case of stochastic intensities that influences all states in the group equally, the assumption is actually true. There are several nuances to the choice of $\tilde{\mu}_{ij}$, and what properties it should have, but in this paper we do not digress into the subject.

Assumption Three

Strictly speaking, Assumption Three is not necessary to reduce the number of differential equations, but it allows us to use lumping to formulate a with-profit insurance policy that we can project. Assumption Three is that

$$\sum_{i\in I} \frac{p_i(t)}{p_I(t)} \frac{h^i(t)}{V_2^{i*}(t)} = \frac{1}{V_2^{I*}(t)} \sum_{i\in I} \frac{p_i(t)}{p_I(t)} h^i(t),$$

for any set of deterministic functions h^i. Here, $V_2^{I*}(t)$ is the prospective first-order reserve of the lumped insurance policy. Assumption One and Three are similar in the sense that they both provide relationships between reserves for groups of states and individual states. Using Assumption Three, we do not need to know the value of the state-wise prospective reserves to project the lumped savings account, but can suffice with the lumped prospective reserves.

3.4.2 The Reduced Differential Equations

We can now apply Assumption One, Two and Three to the differential equations from Theorem 3.2.1, to produce a smaller system of differential equations. To exhibit how the assumptions alter the differential equations, consider the retrospective probability weighted third-order reserve

$$\tilde{U}^i(t) = \mathrm{E}_0\left[\mathbb{1}_{\{Z(t)=i\}}\int_0^t e^{\int_s^t r^*(v)dv} d(-B(s)) \mid Z(0)=0\right].$$

This reserve is of interest because it is equal to the state-wise savings account of a policy that does not receive dividends. The differential equation for $\tilde{U}^i(t)$ reads

$$\begin{aligned}\frac{d}{dt}\tilde{U}^i(t) =& r^*(t)\tilde{U}^i(t) - p_i(t)b^i(t) - \sum_{j\neq i} p_j(t)\mu_{ji}(t)b^{ji}(t)\\ &+ \sum_{g\neq i}\mu_{gi}(t)\tilde{U}^g(t) - \mu_{ig}(t)\tilde{U}^i(t),\end{aligned}$$

involving the stochastic intensities μ_{ji}. Let a stochastically separating reduction function $R : \mathcal{J} \to \mathcal{J}'$ be given, and consider the differential equation for the group-wise sum of probability weighted retrospective third-order reserves in $\mathcal{J}$ determined by R. Since the partition of the state space is stochastically separating, all transition intensities within groups of states are deterministic. We write the differential equation, change the order of summation and apply Assumption Two to get

$$\begin{aligned}&\sum_{i\in I}\frac{d}{dt}\tilde{U}^i(t)\\ =& r^*(t)\sum_{i\in I}\tilde{U}^i(t) - p_I(t)\sum_{i\in I}\frac{\tilde{p}_i(t)}{\tilde{p}_I(t)}b^i(t) - \sum_{i\in I}\sum_{j\neq i}p_j(t)\mu_{ji}(t)b^{ji}(t)\\ &+ \sum_{i\in I}\sum_{g\neq i}\mu_{gi}(t)\tilde{U}^g(t) - \mu_{ig}(t)\tilde{U}^i(t)\end{aligned}$$

$$
\begin{aligned}
=& r^*(t)\sum_{i\in I}\tilde{U}^i(t) - p_I(t)\overbrace{\sum_{i\in I}\frac{\tilde{p}_i(t)}{\tilde{p}_I(t)}\left(b^i(t)+\sum_{\substack{j\in I\\ j\neq i}}\mu_{ij}(t)b^{ij}(t)\right)}^{=b^I(t)} \\
&-\sum_{J\neq I}p_J(t)\underbrace{\frac{1}{\tilde{p}_J(t)}\left(\sum_{j\in J}\tilde{p}_j(t)\sum_{i\in I}\mu_{ji}(t)\right)}_{=\mu_{JI}(t)}\underbrace{\frac{\sum_{j\in J}\tilde{p}_j(t)\sum_{i\in I}\mu_{ji}(t)b^{ji}(t)}{\left(\sum_{j\in J}\tilde{p}_j(t)\sum_{i\in I}\mu_{ji}(t)\right)}}_{b^{JI}(t)} \\
&+\sum_{i\in I}\sum_{g\neq i}\mu_{gi}(t)\tilde{U}^g(t)-\mu_{ig}(t)\tilde{U}^i(t) \\
=& r^*(t)\sum_{i\in I}\tilde{U}^i(t) - p_I(t)b^I(t) - \sum_{J\neq I}p_J(t)\mu_{JI}(t)b^{JI}(t) \\
&+\sum_{i\in I}\sum_{G\neq I}\sum_{g\in G}\mu_{gi}(t)\tilde{U}^g(t)-\mu_{ig}(t)\tilde{U}^i(t),
\end{aligned}
$$

recognizing the lumped intensities and payments of the CFPL with respect to intensities $\tilde{\mu}_{ij}(t)$. This differential equation only differs in the Thiele term from the differential equation for the corresponding lumped retrospective probability weighted third-order reserve,

$$
\begin{aligned}
\frac{d}{dt}\tilde{U}^I(t) =& r^*(t)\tilde{U}^I(t) - p_I(t)b^I(t) - \sum_{J\neq I}p_J(t)\mu_{JI}(t)b^{JI}(t) \\
&+\sum_{G\neq I}\mu_{GI}(t)\tilde{U}^G(t)-\mu_{IG}(t)\tilde{U}^I(t).
\end{aligned}
$$

Applying Assumption One and Two we get

$$
\begin{aligned}
&\sum_{i\in I}\frac{d}{dt}\tilde{U}^i(t) \\
=& r^*(t)\sum_{i\in I}\tilde{U}^i(t) - p_I(t)b^I(t) - \sum_{J\neq I}p_J(t)\mu_{JI}(t)b^{JI}(t) \\
&+\sum_{i\in I}\sum_{G\neq I}\sum_{g\in G}\mu_{gi}(t)\tilde{U}^g(t)\frac{\sum_{k\in G}\tilde{U}^k(t)}{\sum_{k\in G}\tilde{U}^k(t)}-\mu_{ig}(t)\frac{\sum_{j\in I}\tilde{U}^j(t)}{\sum_{j\in I}\tilde{U}^j(t)}\tilde{U}^i(t)
\end{aligned}
$$

$$
\begin{aligned}
=& r^*(t)\sum_{i\in I}\tilde{U}^i(t) - p_I(t)b^I(t) - \sum_{J\neq I} p_J(t)\mu_{JI}(t)b^{JI}(t) \\
&+ \sum_{G\neq I}\sum_{k\in G}\tilde{U}^k(t)\overbrace{\sum_{g\in G}\sum_{i\in I}\mu_{gi}(t)\frac{\tilde{p}_g(t)}{\tilde{p}_G(t)}}^{\mu_{GI}(t)} \\
&- \sum_{j\in I}\tilde{U}^j(t)\sum_{G\neq I}\underbrace{\sum_{i\in I}\sum_{g\in G}\mu_{ig}(t)\frac{\tilde{p}_i(t)}{\tilde{p}_I(t)}}_{\mu_{IG}(t)},
\end{aligned}
$$

revealing that if the initial conditions are the same, then

$$
\tilde{U}^I(t) = \sum_{i\in I}\tilde{U}^i(t).
$$

In other words, calculating $\sum_{i\in I}\tilde{U}^i$ by means of Assumption One and Two is equivalent to directly calculating the lumped counterpart $\tilde{U}^I$. Including all three assumptions gives us a lumped version of Theorem 3.2.1.

Theorem 3.4.1 (Lumped projection)**.** *Let a portfolio of with-profit insurance policies indexed by l, and a reduction function R be given. Under Assumption One, Two and Three, the aggregated state-wise savings accounts*

$$
{}_l\widehat{X}^I(t) := \sum_{i\in I}{}_l\tilde{X}^i(t),
$$

and the risk-diversified surplus $\tilde{Y}$, satisfy the system of differential equations

$$
\begin{aligned}
\frac{d}{dt}\,{}_l\widehat{X}^J(t) =& \sum_{I:I\neq J}\Big({}_l\mu_{IJ}(t)\,{}_l\widehat{X}^I(t) - {}_l\mu_{JI}(t)\,{}_l\widehat{X}^J(t)\Big) \\
&+ {}_lg_1^J(t)\,{}_l\widehat{X}^J(t) + {}_lp_J(t)\,{}_lg_0^J(t) + {}_lp_J(t)\,{}_l\delta_2^J(t)\tilde{Y}(t) \\
&+ \sum_{I:I\neq J}{}_l\mu_{IJ}(t)\Big({}_lh_1^{IJ}(t)\,{}_l\widehat{X}^I(t) + {}_lp_I(t)\,{}_lh_0^{IJ}(t)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\tilde{Y}(t) =& r(t)\tilde{Y}(t) - \sum_{l=1}^{N}\sum_{J\in\mathcal{J}'} {}_{l}p_J(t)\, {}_{l}\delta_0^J(t) \\
& - \tilde{Y}(t)\sum_{l=1}^{N}\sum_{J\in\mathcal{J}'} {}_{l}p_J(t)\, {}_{l}\delta_2^J(t) \\
& + \sum_{l=1}^{N}\sum_{J\in\mathcal{J}'} {}_{l}p_J(t)\, {}_{l}c_0^J(t) + {}_{l}\widehat{X}^J(t)\, {}_{l}c_1^J(t), \\
\frac{d}{dt}\, {}_{l}p_J(t) =& \sum_{I\neq J} {}_{l}\mu_{IJ}(t)\, {}_{l}p_I(t) - {}_{l}\mu_{JI}(t)\, {}_{l}p_J(t), \\
{}_{l}\widehat{X}^I(0) =& \mathbb{1}_{\{l_0\in I\}}\, {}_{l}x_0, \\
\tilde{Y}(0) =& y_0, \\
{}_{l}p_J(0) =& \mathbb{1}_{\{l_0\in J\}},
\end{aligned}
$$

where

$$
\begin{aligned}
{}_{l}g_0^J(t) =& {}_{l}\delta_0^J(t) - {}_{l}b_1^J(t) - \sum_{K:K\neq J} {}_{l}R_1^{Jk}(t)\, {}_{l}\mu_{JK}^*(t) \\
& + \frac{{}_{l}V_1^{J*}(t)}{{}_{l}V_2^{J*}(t)}\left({}_{l}b_2^J(t) + \sum_{K:K\neq J} {}_{l}R_2^{JK}(t)\, {}_{l}\mu_{JK}^*(t)\right), \\
{}_{l}g_1^J(t) =& r^*(t) + {}_{l}\delta_1^J(t) - \frac{1}{{}_{l}V_2^{J*}(t)}\left({}_{l}b_2^J(t) + \sum_{K:K\neq J} {}_{l}R_2^{JK}(t)\, {}_{l}\mu_{JK}^*(t)\right), \\
{}_{l}h_0^{JK}(t) =& {}_{l}V_1^{K*}(t) - \frac{{}_{l}V_1^{J*}(t)\, {}_{l}V_2^{K*}(t)}{{}_{l}V_2^{J*}(t)}, \\
{}_{l}h_1^{JK}(t) =& \frac{{}_{l}V_2^{K*}(t)}{{}_{l}V_2^{J*}(t)} - 1, \\
{}_{l}c_0^J(t) =& \sum_{K:K\neq J} {}_{l}\rho_1^{JK}(t) - \frac{{}_{l}V_1^{J*}(t)}{{}_{l}V_2^{J*}(t)}\sum_{K:K\neq J} {}_{l}\rho_2^{JK}(t), \\
{}_{l}c_1^J(t) =& (r(t) - r^*(t)) - {}_{l}\delta_1^J(t) + \frac{1}{{}_{l}V_2^{J*}(t)}\sum_{K:K\neq J} {}_{l}\rho_2^{JK}(t),
\end{aligned}
$$

are given by the lumped intensities and payments of Definition 3.3.2.

See Appendix 3.E for the proof. Denote by $\tilde{X}^I(t)$ the state-wise savings account of the lumped insurance policy. Theorem 3.4.1 states that if Assumptions One, Two and Three are satisfied, the lumped state-wise savings account is equal to the sum over the non-lumped state-wise savings account i.e.

$$\tilde{X}^I(t) = \widehat{X}^I(t).$$

Assumption One, Two and Three are probably not satisfied for most practically relevant insurance policies, but regardless of the assumptions being true or not, we can use the differential equations for the lumped savings account to produce an approximation of $\widehat{X}^I(t)$.

Remark 3.4.2. Assumption One, Two and Three are sufficient conditions to ensure

$$\tilde{X}^I(t) = \sum_{i \in I} \tilde{X}^i(t) \tag{3.4.1}$$

but they are not necessary. For the total lumping, with deterministic intensities and $D(t) = 0$, (3.4.1) is also satisfied. There are also other special cases where (3.4.1) holds, as we show in Section 3.4.3.

Apart from the fulfillment of the three assumptions, there are other conditions under which (3.4.1) is satisfied when $\tilde{X}^I(t)$ is calculated using Theorem 3.4.1, and in the next section we examine these conditions.

3.4.3 Deterministic Intensities and The Single-State Projection

In the special case of deterministic intensities, all probabilities appearing in Theorem 3.2.1 can be calculated once and reused for each financial scenario. In that case, the projection is only influenced by the financial market through the dividends and rate of return on the surplus. Based on the suggestion to partition the state space such that states that are connected via a stochastic intensity are in different groups, we examine the lumping to one state, since a stochastically separating partition in this case is given by $R : \mathcal{J} \to 1$.

As shown by Ahmad et al. (2021), a certain class of dividend strategies imply $\mathcal{F}^S$-adapted Q processes, which leads to significant computational simplifications. We show that the same class of dividend strategies implies that there is no error introduced by lumping to one state. For the particular class of dividend strategies in the form

$$dD(t) = \underbrace{\tilde{\delta}_0(t)V_2^{Z(t)*}(t) - \tilde{\delta}_1(t)V_1^{Z(t)*}(t)}_{\delta_0^{Z(t)}(t)} + \underbrace{\tilde{\delta}_1(t)}_{\delta_1^{Z(t)}(t)} X(t), \tag{3.4.2}$$

we see that

$$dQ(t) = \tilde{\delta}_0(t) + \tilde{\delta}_1(t)Q(t),$$

in which case $\tilde{X}^i$ can be calculated directly by the relation

$$\tilde{X}^i(t) = p_i(t)(Q(t)V_2^{i*}(t) + V_1^{i*}(t)).$$

We now pose the question; given a dividend strategy in the form (3.4.2), how big an error is introduced by lumping to one state? By lumping to one state, the dividend according to Definition 3.3.2 becomes

$$\begin{aligned}
\delta_0^L(t) &= \sum_{i \in \mathcal{J}} \tilde{\delta}_0(t)V_2^{i*}(t)p_i(t) - \tilde{\delta}_1(t)p_i(t)V_1^{i*}(t) \\
\delta_1^L(t) &= \tilde{\delta}_1(t).
\end{aligned}$$

For

$$V_j^*(t) = \sum_{i\in\mathcal{J}} p_i(t)V_j^{i*}(t),$$

the lumped Q process, which by Lemma 3.3.7 is given by $Q^L(t) = \frac{\tilde{X}(t)-V_1^*(t)}{V_2^*(t)}$, has dynamics

$$\begin{aligned} dQ^L(t) =& \frac{\sum_{i\in\mathcal{J}} \tilde{\delta}_0(t)V_2^{i*}(t)p_i(t) - \tilde{\delta}_1(t)p_i(t)V_1^{i*}(t) + \tilde{X}(t)\tilde{\delta}_1(t)}{\sum_{i\in\mathcal{J}} V_2^{i*}(t)p_i(t)} \\ =& \tilde{\delta}_0(t) + Q^L(t)\tilde{\delta}_1(t), \end{aligned}$$

implying that $Q^L(t) = Q(t)$. What we have shown, is that for dividends in the form (3.4.2), the lumped savings account is equal to the sum of the individual savings accounts,

$$\sum_{i\in\mathcal{J}} \tilde{X}^j = Q(t)\sum_{i\in\mathcal{J}} p_i(t)V_2^{i*}(t) + \sum_{i\in\mathcal{J}} p_i(t)V_1^{i*}(t) = \tilde{X}(t).$$

Futhermore, lumping does not introduce any error in the calculation of FDB

$$\begin{aligned} &FDB \\ =& \int_0^n e^{-\int_0^t r(\nu)d\nu} \sum_{i\in\mathcal{J}} \frac{\tilde{X}^i(t)/p_i(t) - V_1^{i*}(t)}{V_2^{i*}(t)} p_i(t) \left(b^i(t) + \sum_{j:j\neq i} \mu_{ij}(t)b^{ij}(t) \right) dt \\ =& \int_0^n e^{-\int_0^t r(\nu)d\nu} Q(t) \sum_{i\in\mathcal{J}} p_i(t) \left(b^i(t) + \sum_{j:j\neq i} \mu_{ij}(t)b^{ij}(t) \right) dt \\ =& \int_0^n e^{-\int_0^t r(\nu)d\nu} \frac{\tilde{X}(t) - V_1^*(t)}{V_2^*(t)} \sum_{i\in\mathcal{J}} p_i(t) \left(b^i(t) + \sum_{j:j\neq i} \mu_{ij}(t)b^{ij}(t) \right) dt. \end{aligned}$$

The natural presumption to make, is that dividends that approximately are in the form (3.4.2), lead to small errors on the calculation of FDB and the lumped savings account. Given dividend functions δ_0^i and δ_1^i, they are approximately in the form (3.4.2) if there exists state-independent functions $\tilde{\delta}_0$ and $\tilde{\delta}_1$ such that

$$\tilde{\delta}_0(t) \approx \frac{\delta_0^{Z(t)}(t) - V_1^{Z(t)*}(t)\delta_1^{Z(t)}(t)}{V_2^{Z(t)*}(t)}, \qquad \tilde{\delta}_1(t) \approx \delta_1^{Z(t)}(t).$$

Conversely, given dividend functions δ_0^i and δ_1^i, we can construct $\tilde{\delta}_0$ and $\tilde{\delta}_1$ to satisfy this approximation via e.g.

$$\tilde{\delta}_0(t) = \frac{\sum_{i\in\mathcal{J}} p_i(t)\delta_0^i(t) - V_1^{T*}(t)\tilde{\delta}_1(t)}{V_2^{T*}(t)}, \qquad \tilde{\delta}_1(t) = \sum_{i\in\mathcal{J}} p_i(t)\delta_1^i(t),$$

and thereby create a strategy satisfying (3.4.2) that approximates the strategy given by δ_0^i and δ_1^i.

3.5 Model Points

The ultimate goal of lumping the with-profit life insurance policy, is to reduce the computation time for a projection of a portfolio of policies. So, in the system of $2 \times N \times \#\mathcal{J} + 1$ differential equations from Theorem 3.2.1, we have focused on decreasing $\#\mathcal{J}$. The approach of this section is to decrease N by combining policies into so-called model points. We present a method to reduce the number of calculations for a particular class of insurance policies where dividends proportional to the savings account are invariant to the policy. The central idea is to calculate the proportion between savings accounts for different policies and realize that it is invariant to the dividend strategy, implying that it can be reused for all financial scenarios.

Given ${}_l\tilde{X}_i(0)$ for $i \in \mathcal{J}$, representing a group of policies $l = 1...N$ experiencing the same intensities and X-proportional dividends, δ_1^i, consider the system of linear differential equations

$$\begin{aligned}\frac{d}{dt}\,{}_k\tilde{X}_i(t) = {}&{}_k a_0^i(t) + {}_k a_1^i(t)\,{}_k\tilde{X}_i(t) + \delta_1^i(t,\omega_t)\,{}_k\tilde{X}_i(t)\\ &+ \sum_{j\neq i}\mu_{ji}(t,\omega_t)\,{}_k\tilde{X}_j(t) - \mu_{ij}(t,\omega_t)\,{}_k\tilde{X}_i(t),\end{aligned}$$

representing the state-wise expected savings accounts for policy k, ignoring payments on transition to simplify notation. We have used the parameter ω_t to highlight which elements of the differential equation that depend on the financial market at time t. The parameter can be thought of as a vector of prices of financial assets at time t which influence the insurance managements decisions in regards to allocation of dividends, as well as the financial assets that influence the behaviour of the policyholders and thereby also the transition intensities between behavioural groups of states. Define the ratio

$$R_i^k(t) := \frac{{}_k\tilde{X}_i(t)}{\sum_{l=1}^N {}_l\tilde{X}_i(t)},$$

representing the policy-k-proportion of the total savings accounts for state i. Note that

$$\frac{d}{dt}R_i^k(t) = \frac{\sum_{l=1}^N {}_l\tilde{X}_i(t)\frac{d}{dt}\,{}_k\tilde{X}_i(t)}{\left(\sum_{l=1}^N {}_l\tilde{X}_i(t)\right)^2} - \frac{{}_k\tilde{X}_i(t)\sum_{l=1}^N \frac{d}{dt}\,{}_l\tilde{X}_i(t)}{\left(\sum_{l=1}^N {}_l\tilde{X}_i(t)\right)^2}$$

$$
\begin{aligned}
=&\frac{\sum_{l=1}^{N} {}_l\tilde{X}_i(t)\left({}_k a_0^i(t) + {}_k a_1^i(t)\,{}_k\tilde{X}_i(t)\right)}{\left(\sum_{l=1}^{N} {}_l\tilde{X}_i(t)\right)^2} \\
&- \frac{\sum_{l=1}^{N} {}_k\tilde{X}_i(t)\left({}_l a_0^i(t) + {}_l a_1^i(t)\,{}_l\tilde{X}_i(t)\right)}{\left(\sum_{l=1}^{N} {}_l\tilde{X}_i(t)\right)^2} \\
&+ \frac{\sum_{l=1}^{N} \sum_{j\neq i} \mu_{ji}(t,\omega_t)\,{}_k\tilde{X}_j(t)\,{}_l\tilde{X}_i(t) - \mu_{ij}(t,\omega_t)\,{}_k\tilde{X}_i(t)\,{}_l\tilde{X}_i(t)}{\left(\sum_{l=1}^{N} {}_l\tilde{X}_i(t)\right)^2} \\
&- \frac{\sum_{l=1}^{N} \sum_{j\neq i} \mu_{ji}(t,\omega_t)\,{}_l\tilde{X}_j(t)\,{}_k\tilde{X}_i(t) - \mu_{ij}(t,\omega_t)\,{}_l\tilde{X}_i(t)\,{}_k\tilde{X}_i(t)}{\left(\sum_{l=1}^{N} {}_l\tilde{X}_i(t)\right)^2} \\
&+ \frac{\overbrace{\sum_{l=1}^{N} {}_l\tilde{X}_i(t)\delta_1^i(t,\omega_t)\,{}_k\tilde{X}_i(t) - \sum_{l=1}^{N} {}_k\tilde{X}_i(t)\delta_1^i(t,\omega_t)\,{}_l\tilde{X}_i(t)}^{=0}}{\left(\sum_{l=1}^{N} {}_l\tilde{X}_i(t)\right)^2},
\end{aligned}
$$

and that this does not depend on δ_1^i. However, the ratios depend on the financial market through the stochastic intensities. By substituting the stochastic intensities μ_{ij} with deterministic intensities μ_{ij}^* from the first-order basis, we can approximate R_i^k with the deterministic R_i^{k*}.

To put these ratios to use, we can calculate $R_i^{k*}(t)$ once, and then calculate the helpful quantities

$$
\begin{aligned}
b_0^i(t) &= \sum_{l=1}^{N} {}_l a_0^i(t), \\
b_1^i(t) &= \sum_{l=1}^{N} {}_l a_1^i(t) R_i^{l*}(t).
\end{aligned}
$$

With b_0^i and b_1^i in hand, we can approximate $\sum_{l=1}^{N} {}_l\tilde{X}_i(t)$ for different realisations of the financial market using the system of differential equations

$$
\begin{aligned}
\frac{d}{dt}\tilde{X}_i^S(t) =& b_0^i(t) + \tilde{X}_i^S(t) b_1^i(t) + \delta_1(t,\omega_t)\tilde{X}_i^S(t) \\
&+ \sum_{j\neq i} \mu_{ji}(t,\omega_t)\tilde{X}_j^S(t) - \mu_{ij}(t,\omega_t)\tilde{X}_i^S(t), \qquad (3.5.1)
\end{aligned}
$$

$$
\tilde{X}_i^S(0) = \sum_{l=1}^{N} {}_l\tilde{X}_i(0). \qquad (3.5.2)
$$

Note that for $dD(t) = 0$, the first-order state-wise savings accounts are equal to the probability weighted retrospective reserves

$$\mathrm{E}_0^*[{}_lX(t)\mathbb{1}_{\{Z(t)=i\}}] = {}_lp_i^*(t)\,{}_lU_1^{i*}(t),$$

which are equal to the probability weighted prospective reserves under the principle of equivalence. This implies that

$$R_i^{k*}(t) = \frac{{}_kp_i^*(t)\,{}_kV_1^{i*}(t)}{\sum_{l=1}^N {}_lp_i^*(t)\,{}_lV_1^{i*}(t)},$$

providing the means to calculate $R_i^{k*}(t)$ from the probability weighted prospective reserves which commonly are available to the insurance company. If the transition intensities μ_{ij} are deterministic, we do not have to rely on an approximation of the ratios, and can calculate them as

$$R_i^k(t) = \frac{{}_kp_i(t)\,{}_kV_1^i(t)}{\sum_{l=1}^N {}_lp_i(t)\,{}_lV_1^i(t)},$$

where the probabilities and prospective reserves are determined on the third-order basis.

The ratios are invariant to dividends linear in the savings account, they are however not invariant to dividends that are not linear in the savings account. Recall the form of the dividends from (3.2.2), where δ_0^i accounts for the dividends that are not linear in the savings account. If δ_0^i does not depend on the financial market, which for instance is the case for $dD(t) = q(t)dC(t)$ where q is some deterministic function, then it can be included as an additional premium to form the payment stream $d\widehat{B}_1(t) = dB_1(t) + \delta_0^{Z(t)}(t)dt$ with corresponding reserves $\widehat{V}_1^{Z(t)*}(t)$. The ratios between savings accounts of different policies in the same state can then be calculated as

$$R_i^{k*}(t) = \frac{{}_kp_i^*(t)\,{}_k\widehat{V}_1^{i*}(t)}{\sum_{l=1}^N {}_lp_i^*(t)\,{}_l\widehat{V}_1^{i*}(t)}.$$

In this section we have presented a way to reduce the number of differential equations from Theorem 3.2.1 and 3.4.1 by combining policies into so-called modelpoints. In a setting where only the dividends that are proportional to the savings account depend on the financial market, and the third-order transition intensities are deterministic, the modelpoint differential equations (3.5.1)-(3.5.2) are equal to the accumulated state-wise savings account for a group of policies.

3.6 Numerical Example

In this section we examine the loss in precision when projecting a lumped with-profit insurance policy, in a setting where all transition intensities are deterministic. The calculations are performed on the industry implemented software product PROBABLI provided by Edlund A/S. How well the approximation performs for stochastic intensities, depends on the specific way in which the intensities depend on the financial market, and the accuracy of the surrogate intensities $\tilde{\mu}_{ij}$, which is a study we do not pursue here.

We showcase characteristics for four different dividends given by the combinations of

$$dD(t) = \pm 0.5 dC_r(t) \pm 0.5 dC_\mu(t),$$

where

$$dC_r(t) = (r(t) - r^*(t))X(t)dt,$$
$$dC_\mu(t) = \sum_{j \neq Z(t)} R^{Z(t)k}(t, X(t))(\mu^*_{Z(t)k}(t) - \mu_{Z(t)k}(t))dt.$$

These dividends have an aspect for both insurance risk and interest risk. Realistically, dividends cannot be taken from the policyholder, but the purpose of the different dividends is not to mimic the real world, but rather to demonstrate how the approximation performs for different kinds of dividends.

We consider a single with-profit insurance policy with a disability annuity of 80.000 per year while disabled until retirement at age 65, as well as a life annuity of 100.000 per year while active or disabled, commencing at retirement. The state space of the policy is depicted in Figure 3.1.

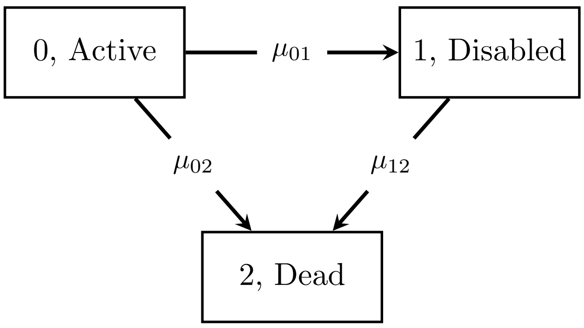

Figure 3.1: *State space of insurance policy*

Both benefits are scaled with bonus. The policyholder is 50 years old and active at time $t = 0$, and all premiums have been paid by $t = 0$. The technical basis consists of the elements

$$r^*(t) = 0.02,$$
$$\mu^*_{01}(t) = \frac{0.85}{0.7} \cdot (0.0006 + 10^{4.71609 - 10 + 0.06 \cdot (50+t)}),$$

as well as a force of mortality, $\mu^*_{02} = \mu^*_{12}$, equal to the 2016 mortality benchmark published by the Danish FSA – see Finanstilsynet.dk, 2019. The third-order basis is determined by

$$
\begin{aligned}
r(t) =& 0.03, \\
\mu_{01}(t) =& \mu^*_{01}(t) \cdot 0.7,
\end{aligned}
$$

and a force of mortality equal to the first-order mortality. We let $Q(0) = 0$ and let all benefits be scaled by bonus, implying that $V_2^{i*}(t) = V_1^{i*}(t)$. The four different dividends in the three state model are given in Table 3.1

$dD(t)$	Positive in transition risk	Negative in transition risk
Positive in interest risk	$0.5dC_r(t)dt + 0.5dC_\mu(t)$	$0.5dC_r(t)dt - 0.5dC_\mu(t)$
Negative in interest risk	$-0.5dC_r(t)dt + 0.5dC_\mu(t)$	$-0.5dC_r(t)dt - 0.5dC_\mu(t)$

Table 3.1: *Overview over the four different dividend strategies*

The dividends correspond to the δ-functions

$$
\begin{gathered}
\delta_0^{Z(t)}(t) = 0, \qquad \delta_2^{Z(t)}(t) = 0 \\
\delta_1^{Z(t)}(t) = \pm(r(t) - r^*(t)) \pm \mathbb{1}_{\{Z(t)=0\}} \left(\frac{V_2^{1*}(t)}{V_2^{0*}(t)} - 1 \right) (\mu^*_{01}(t) - \mu_{01}(t)).
\end{gathered}
$$

A major motivator for projecting with-profit insurance policies, is the calculation of future discretionary benefits, which is why we measure the accuracy of the approximation in terms of differences in the FDB cash flow, defined by

$$
\begin{aligned}
C^3_{FDB}(t) =& \sum_{i \in \mathcal{J}} \frac{\tilde{X}^i(t)/p_i(t) - V_1^{i*}(t)}{V_2^{i*}} p_i(t) \left(b_2^i(t) + \sum_{j:j \neq i} \mu_{ij}(t) b_2^{ij}(t) \right), \\
C^1_{FDB}(t) =& \frac{\tilde{X}(t) - V_1^*(t)}{V_2^*} \sum_{i \in \mathcal{J}} p_i(t) \left(b_2^i(t) + \sum_{j:j \neq i} \mu_{ij}(t) b_2^{ij}(t) \right),
\end{aligned}
$$

representing the expected amount of additional benefits received by the policyholder at time t, for the three- and one state models respectively. Since we are considering a single policy only, we do not pay attention to the diversified surplus, as the surplus provides insignificant additional information. In Figure 3.2 we have plotted C^1_{FDB} and C^3_{FDB}.

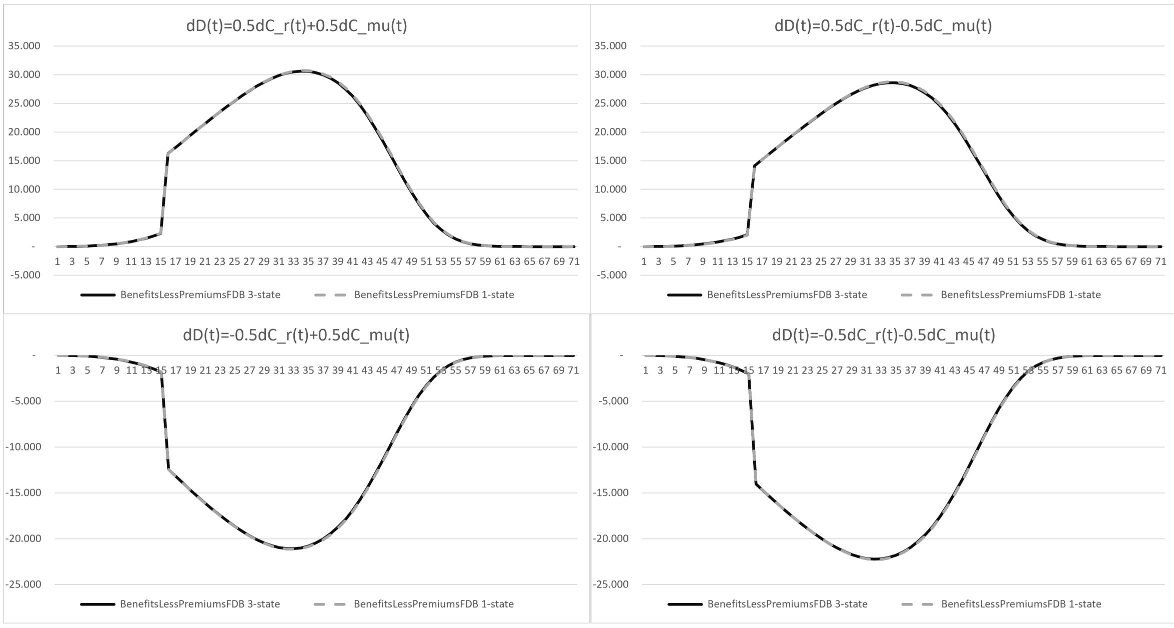

Figure 3.2: *$C_{FDB}(t)$ calculated in the three- and one state models.*

The first thing to notice is that there is almost no visual difference between calculating in the full three state model, and the lumped one state model. Furthermore, the interest risk constitutes a much larger part of the surplus contribution than the transition risk, this might be different for a different policy with e.g. a large life assurance. To highlight the dissimilarity, we plot $C^1_{FDB}(t) - C^3_{FDB}(t)$ in Figure 3.3, showing how much the one state model overestimates the future discretionary benefits.

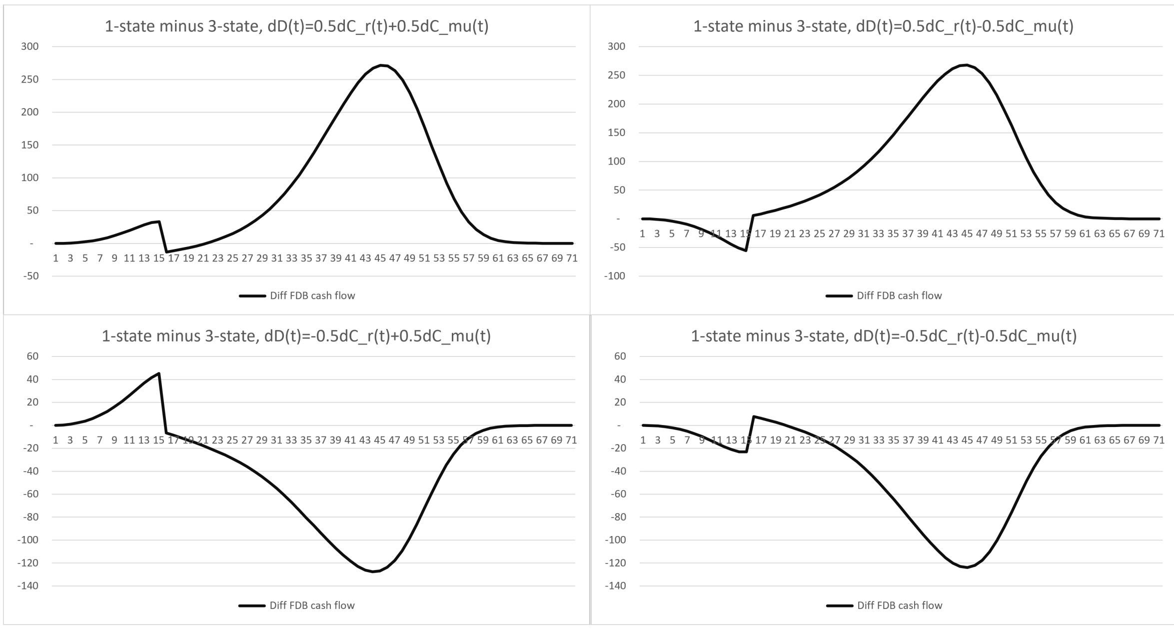

Figure 3.3: *Difference in $C_{FDB}(t)$ between the three- and one state models.*

During the first 15 years there can only be benefits from the disabled state, and for the two plots to the left C_{FDB} is overestimated in the one-state model which implies that

$$\frac{\tilde{X}^1(t)/p_1(t)}{V_2^{1*}(t)} < \frac{\tilde{X}(t)}{p_0(t)V_2^{0*}(t) + p_1(t)V_2^{1*}(t)}.$$

It is not obvious why this is the case – the denominator is largest on the left hand side, but so is the numerator. In order for the policy to receive benefits for $t < 15$ in the three state model, it has to transition to the disabled state, whereas no transition is required (or possible) in the one state model. Furthermore, once the transition to the disabled state has been made in the three state model, $dC_\mu(t) = 0$ and therefore fewer dividends are allotted in absolute value.

After 15 years, we see a jump in the difference because the life annuity benefit commences and the disability pension terminates. The sign of the difference at $t = 15$ is dictated by which of $\tilde{X}^0(t)/p_0(t) + \tilde{X}^1(t)/p_1(t)$ and $\tilde{X}(t)$ is larger, since the benefits and the prospective reserves in both the non-dead states are identical. This mechanism compensates for the approximation error during the first 15 years, since larger benefits imply smaller savings and vice versa. In all four cases, the error at $t = 15$ is small but the absolute error increases, mostly due to the part of the dividend relating to the interest risk.

For a simple policy, we have demonstrated that the approximation error introduced by lumping a three state model to a one state model is small. The performance of the approximation is highly dependent on the specific policy, and our numerical results should be seen in that light. For a large group of policies, some approximation errors will accumulate, while others will cancel each other out – again, this is highly dependent on the specific portfolio and dividend strategy.

3.7 Acknowledgements

We are grateful to Kenneth Bruhn for general thoughts and discussions as well as his assistance in the numerical example.

3.A Proof of Lemma 3.3.3

Proof. By definition,

$$p_I(0) = \sum_{i\in I} p_i(0),$$

and so it only remains to prove that the derivatives coincide. By the Kolmogorov forward equations

$$\frac{d}{dt} p_I(t) = -p_I(t)\mu_{I\cdot}(t) + \sum_{K\neq I} p_K(t)\mu_{KI}(t), \tag{3.A.1}$$

and

$$\frac{d}{dt}\sum_{i\in I} p_i(t) = \sum_{i\in I}\sum_{k\neq i}\left(-p_i(t)\mu_{ik}(t) + p_k(t)\mu_{ki}(t)\right).$$

Note that for $k, k' \in I$ and $k \neq k'$ the contribution to the total sum only relates transitions out of I, as

$$p_k(t)\mu_{kk'}(t) - p_{k'}(t)\mu_{k'k}(t) + p_{k'}(t)\mu_{k'k}(t) - p_k(t)\mu_{kk'}(t) = 0,$$

and so

$$\frac{d}{dt}\sum_{i\in I} p_i(t) = \sum_{i\in I}\sum_{K\neq I}\sum_{k\in K} p_k(t)\mu_{ki}(t) - p_i(t)\mu_{ik}(t).$$

Dividing and multiplying by $\sum_{i\in I} p_i(t)$ and $\sum_{k\in K} p_k(t)$ yields

$$\begin{aligned}\frac{d}{dt}\sum_{i\in I} p_i(t) = &-\sum_{i\in I} p_i(t)\sum_{K\neq I}\sum_{i\in I}\sum_{k\in K}\frac{p_i(t)}{\sum_{i\in I} p_i(t)}\mu_{ik}(t)\\ &+\sum_{K\neq I}\sum_{k\in k} p_k(t)\sum_{k\in K}\sum_{i\in I}\frac{p_k(t)}{\sum_{k\in K} p_k(t)}\mu_{ki}(t).\end{aligned}$$

This differential equation is identical to (3.A.1). □

3.B Proof of Lemma 3.3.4

Proof. Trivially,

$$p_{IJ}(t,t) = \sum_{i\in I}\sum_{j\in J}\frac{p_i(t)}{p_I(t)} p_{ij}(t,t).$$

The Kolmogorov-Chapman equations state that

$$p_J(s) = \sum_{M\in\mathcal{J}'} p_M(t) p_{MJ}(t,s), \tag{3.B.1}$$

which comprises a system of $\#\mathcal{J}'$ equations with $\#\mathcal{J}'$ unknowns, for any given $t < s$. By the Kolmogorv forward equations, and the Kolmogorov-Chapman equations

$$\begin{aligned}\frac{d}{ds}p_J(s) =& \frac{d}{ds}\sum_{j\in j} p_j(s)\\ =&\sum_{j\in J}\sum_{k\neq j}\Big(-p_j(s)\mu_{jk}(s)+p_k(s)\mu_{kj}(s)\Big)\\ =&\sum_{l\in\mathcal{J}}\sum_{j\in J}\sum_{k\neq j}\Big(-p_l(0)p_{lj}(0,s)\mu_{jk}(s)+p_l(0)p_{lk}(0,s)\mu_{kj}(s)\Big)\\ =&\sum_{m\in\mathcal{J}}\sum_{l\in\mathcal{J}}\sum_{j\in J}\sum_{k\neq j}\Big(p_l(0)p_{lm}(0,t)p_{mk}(t,s)\mu_{kj}(s)\Big)\\ &-\sum_{m\in\mathcal{J}}\sum_{l\in\mathcal{J}}\sum_{j\in J}\sum_{k\neq j}\Big(p_l(0)p_{lm}(0,t)p_{mj}(t,s)\mu_{jk}(s)\Big),\end{aligned} \tag{3.B.2}$$

providing us with an expression for the LHS of (3.B.1). We now guess that

$$p_{IJ}(t,s)=\sum_{i\in I}\sum_{j\in J}\frac{p_i(t)}{p_I(t)}p_{ij}(t,s),$$

and set out to verify that the resulting RHS of (3.B.1), equals (3.B.2). Note that, using our guess,

$$\begin{aligned}\frac{d}{ds}p_J(s) =&\frac{d}{ds}\sum_{M\in\mathcal{J}'}p_M(t)\sum_{m\in M}\sum_{j\in J}\frac{p_m(t)}{p_M(t)}p_{mj}(t,s)\\ =&\frac{d}{ds}\sum_{M\in\mathcal{J}'}\sum_{m\in M}\sum_{j\in J}p_m(t)p_{mj}(t,s)\\ =&\frac{d}{ds}\sum_{m\in\mathcal{J}}\sum_{j\in J}p_m(t)p_{mj}(t,s).\end{aligned}$$

We see that indeed

$$\begin{aligned}&\sum_{m\in\mathcal{J}'}\sum_{j\in J}p_m(t)\frac{d}{ds}p_{mj}(t,s)\\ =&\sum_{m\in\mathcal{J}}\sum_{j\in J}p_m(t)\left(\sum_{k\neq j}-p_{mj}(t,s)\mu_{jk}(s)+p_{mk}(t,s)\mu_{kj}(s)\right)\\ =&\sum_{m\in\mathcal{J}}\sum_{j\in J}\sum_{k\neq j}\Big(-p_m(t)p_{mj}(t,s)\mu_{jk}(s)+p_m(t)p_{mk}(t,s)\mu_{kj}(s)\Big)\\ =&\sum_{m\in\mathcal{J}}\sum_{l\in\mathcal{J}}\sum_{j\in J}\sum_{k\neq j}\Big(-p_l(0)p_{lm}(0,t)p_{mj}(t,s)\mu_{jk}(s)+p_l(0)p_{lm}(0,t)p_{mk}(t,s)\mu_{kj}(s)\Big),\end{aligned}$$

thus confirming that

$$p_{IJ}(t,s)=\sum_{i\in I}\sum_{j\in J}\frac{p_i(t)}{p_I(t)}p_{ij}(t,s).$$

□

3.C Proof of Theorem 3.3.5

Proof. Since

$$\begin{aligned}\mathrm{A}_G(0) :=& \mathrm{E}\left[\int_0^0 \mathbb{1}_{\{R(Z(s))\in G\}} dB(s) \middle|\ p_i(0), i\in\mathcal{J}\right]\\ =& \mathrm{E}\left[\int_0^0 \mathbb{1}_{\{\tilde{Z}(s)\in G\}} dB^R(s) \middle|\ p_I(0), I\in\mathcal{J}'\right] =: \mathrm{A}_G^R(0),\end{aligned}$$

we simply have to prove that the derivatives coincide. Note that

$$\frac{d}{dt}\mathrm{A}_G(t) = \sum_{i\in\mathcal{J}} \mathbb{1}_{\{R(i)\in G\}} p_i(t)\left(b^i(t) + \sum_{\substack{j\in\mathcal{J}\\ j\neq i}} b^{ij}(t)\mu_{ij}(t)\right),$$

and

$$\frac{d}{dt}\mathrm{A}_G^R(t) = \sum_{I\in\mathcal{J}'} \mathbb{1}_{\{I\in G\}} p_I(t)\left(b^I(t) + \sum_{\substack{J\in\mathcal{J}'\\ J\neq I}} b^{IJ}(t)\mu_{IJ}(t)\right).$$

By definition of b^I, b^{JI}, μ_{JI} and the fact that $p_I(t) = \sum_{i\in I} p_i(t)$ we get

$$\begin{aligned}\frac{d}{dt}\mathrm{A}_G^R(t) =& \sum_{I\in G} \cancel{p_I(t)} \frac{\sum_{i\in I} p_i(t)\left(b^i(t) + \sum_{\substack{j\in I\\ j\neq i}} b^{ij}(t)\mu_{ij}(t)\right)}{\cancel{p_I(t)}}\\ &+ \sum_{I\in G}\sum_{J\neq I} \cancel{p_I(t)} \sum_{i\in I}\sum_{j\in J} \frac{p_i(t)}{\cancel{p_I(t)}}\mu_{ij}(t) \frac{\sum_{i\in I} p_i(t)\sum_{j\in J}\mu_{ij}(t)b^{ij}(t)}{\sum_{i\in I} p_i(t)\sum_{j\in J}\mu_{ij}(t)}\\ =& \sum_{I\in G}\sum_{i\in I} p_i(t)\left(b^i(t) + \sum_{\substack{j\in I\\ j\neq i}} b^{ij}(t)\mu_{ij}(t)\right)\\ &+ \sum_{I\in G}\sum_{J\neq I}\cancel{\sum_{i\in I}\sum_{j\in J} p_i(t)\mu_{ij}(t)} \frac{\sum_{i\in I} p_i(t)\sum_{j\in J}\mu_{ij}(t)b^{ij}(t)}{\cancel{\sum_{i\in I} p_i(t)\sum_{j\in J}\mu_{ij}(t)}}\\ =& \sum_{I\in G}\sum_{i\in I} p_i(t)b^i(t)\end{aligned}$$

$$+\sum_{I\in G}\underbrace{\left\{\sum_{i\in I}\sum_{\substack{j\in I\\ j\neq i}}p_i(t)b^{ij}(t)\mu_{ij}(t)+\sum_{J\neq I}\sum_{i\in I}p_i(t)\sum_{j\in J}\mu_{ij}(t)b^{ij}(t)\right\}}_{=:\,\clubsuit}.$$

Note that

$$\begin{aligned}\clubsuit&=\underbrace{\sum_{i\in I}\sum_{\substack{j\in I\\ j\neq i}}p_i(t)b^{ij}(t)\mu_{ij}(t)}_{\substack{\text{Expected transition}\\ \text{payments within } I}}+\underbrace{\sum_{J\neq I}\sum_{i\in I}\sum_{j\in J}p_i(t)b^{ij}(t)\mu_{ij}(t)}_{\substack{\text{Expected transition}\\ \text{payments out of } I}}\\&=\sum_{i\in I}\sum_{\substack{j\in\mathcal{J}\\ j\neq i}}p_i(t)b^{ij}(t)\mu_{ji}(t).\end{aligned}$$

Thus

$$\begin{aligned}\frac{d}{dt}\mathrm{A}^R_G(t)&=\sum_{I\in G}\sum_{i\in I}p_i(t)\left(b^i(t)+\sum_{\substack{j\in\mathcal{J}\\ j\neq i}}b^{ij}(t)\mu_{ji}(t)\right)\\&=\sum_{i\in\mathcal{J}}\mathbb{1}_{\{R(i)\in G\}}p_i(t)\left(b^i(t)+\sum_{\substack{j\in\mathcal{J}\\ j\neq i}}b^{ij}(t)\mu_{ij}(t)\right)=\frac{d}{dt}\mathrm{A}_G(t)\end{aligned}$$

$\square$

3.D Proof of Lemma 3.3.7

By Thieles differential equation

$$\begin{aligned}\sum_{i\in\mathcal{J}}\frac{d}{dt}p_i(t)V^i(t)&=r\sum_{i\in\mathcal{J}}p_i(t)V^i(t)-\sum_{i\in\mathcal{J}}p_i(t)b^i(t)\\&-\sum_{i\in\mathcal{J}}p_i(t)\mu_{ik}(t)b^{ik}(t)\\&-\sum_{i\in\mathcal{J}}p_i(t)\sum_{k\neq i}\mu_{ik}(t)(V^k(t)-V^i(t)),\\&+\sum_{i\in\mathcal{J}}V^i\left(\sum_{k\neq i}p_k(t)\mu_{ki}(t)-p_i(t)\sum_{k\neq i}\mu_{ik}(t)\right)\\&=r\sum_{i\in\mathcal{J}}p_i(t)V^i(t)-\sum_{i\in\mathcal{J}}p_i(t)b^i(t)\\&-\sum_{i\in\mathcal{J}}p_i(t)\mu_{ik}(t)b^{ik}(t)\end{aligned}$$

$$\underbrace{-\sum_{i\in\mathcal{J}}\sum_{k\neq i}p_i(t)\mu_{ik}(t)V^k(t)+\sum_{i\in\mathcal{J}}\sum_{k\neq i}p_k(t)\mu_{ki}(t)V^i(t)}_{=0}$$

$$=r\sum_{i\in\mathcal{J}}p_i(t)V^i(t)-\underbrace{\left(\sum_{i\in\mathcal{J}}p_i(t)b^i(t)+\sum_{i\in\mathcal{J}}p_i(t)\mu_{ik}(t)b^{ik}(t)\right)}_{=b^T(t)}$$

$$=r\sum_{i\in\mathcal{J}}p_i(t)V^i(t)-b^T(t).$$

The differential equation for the single-state reserve V^T reads

$$\frac{d}{dt}V^T(t)=rV^T(t)-b^T(s).$$

As the terminal conditions are identical $V^T(n)=0=\sum_{i\in\mathcal{J}}p_i(n)V^i(n)$, we conclude that

$$V^T(t)=\sum_{i\in\mathcal{J}}p_i(t)V^i(t).$$

3.E Proof of Theorem 3.4.1

Disregarding policy indices, the differential equation $\frac{d}{dt}\tilde{X}^j(t)$ from Theorem 3.2.1 can be composed into 11 terms:

$$p_j(t)g_0^j(t),\qquad \tilde{X}^j(t)g_1^j(t),\qquad \frac{p_j(t)}{V_2^{j*}(t)}g_2^j(t),\qquad \frac{\tilde{X}^j(t)}{V_2^{j*}(t)}g_3^j(t),\tag{3.E.1}$$

$$\mu_{ij}(t)p_i(t)h_0^{ij}(t),\qquad \mu_{ij}(t)\tilde{X}^i(t)h_1^{ij}(t),\qquad \mu_{ij}(t)\frac{p_i(t)}{V_2^{i*}(t)}h_2^{ij}(t),\tag{3.E.2}$$

$$\mu_{ij}(t)\frac{\tilde{X}^i(t)}{V_2^{i*}(t)}h_3^{ij}(t),\qquad \mu_{ij}(t)\tilde{X}^i(t),\qquad -\mu_{ji}(t)\tilde{X}^j(t),\qquad p_j(t)\delta_2^j(t)\tilde{Y}(t)\tag{3.E.3}$$

for $i\in\mathcal{J}$, $i\neq j$. We simply have to show that aggregating the terms (3.E.1)-(3.E.3) amounts to the terms from Theorem 3.4.1. Writing $\sum_{j\in J}\frac{d}{dt}\tilde{X}^j(t)$ using the terms (3.E.1)-(3.E.3) we have

$$\begin{aligned}
&\sum_{j\in J}\frac{d}{dt}\tilde{X}^j(t)\\
=&\sum_{j\in J}\Bigg\{\ p_j(t)g_0^j(t)+\tilde{X}^j(t)g_1^j(t)+\frac{p_j(t)}{V_2^{j*}(t)}g_2^j(t)+\frac{\tilde{X}^j(t)}{V_2^{j*}(t)}g_3^j(t)\\
&\qquad+\sum_{i:i\neq j}\mu_{ij}(t)p_i(t)h_0^{ij}(t)+\mu_{ij}(t)\tilde{X}^i(t)h_1^{ij}(t)\\
&\qquad+\sum_{i:i\neq j}\mu_{ij}(t)\frac{p_i(t)}{V_2^{i*}(t)}h_2^{ij}(t)+\mu_{ij}(t)\frac{\tilde{X}^i(t)}{V_2^{i*}(t)}h_3^{ij}(t)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i:i\neq j} \mu_{ij}(t)\tilde{X}^i(t) - \mu_{ji}(t)\tilde{X}^j(t) \\
&\quad + p_j(t)\delta_2^j(t)\tilde{Y}(t) \Bigg\} \\
&= \sum_{j\in J} \Bigg\{ p_j(t)g_0^j(t) + p_j(t)\delta_2^j(t)\tilde{Y}(t) + \tilde{X}^j(t)g_1^j(t) \\
&\quad + \frac{p_j(t)}{V_2^{j*}(t)}g_2^j(t) + \frac{\tilde{X}^j(t)}{V_2^{j*}(t)}g_3^j(t) \Bigg\} \\
&\quad + \sum_{j\in J}\sum_{i:i\neq j} \mu_{ij}(t)p_i(t)h_0^{ij}(t) \\
&\quad + \sum_{j\in J}\sum_{i:i\neq j} \mu_{ij}(t)\tilde{X}^i(t)h_1^{ij}(t) \\
&\quad + \sum_{j\in J}\sum_{i:i\neq j} \mu_{ij}(t)\tilde{X}^i(t) - \mu_{ji}(t)\tilde{X}^j(t) \\
&\quad + \sum_{j\in J}\sum_{i:i\neq j} \mu_{ij}(t)\frac{p_i(t)}{V_2^{i*}(t)}h_2^{ij}(t) \\
&\quad + \sum_{j\in J}\sum_{i:i\neq j} \mu_{ij}(t)\frac{\tilde{X}^i(t)}{V_2^{i*}(t)}h_3^{ij}(t)
\end{aligned}
$$

Noting that payments on transition between states can be split into two types; transitions within J and transitions out of J we rewrite the sums

$$
\begin{aligned}
&= \sum_{j\in J} \Bigg\{ p_j(t)g_0^j(t) + p_j(t)\delta_2^j(t)\tilde{Y}(t) + \tilde{X}^j(t)g_1^j(t) \\
&\quad + \frac{p_j(t)}{V_2^{j*}(t)}g_2^j(t) + \frac{\tilde{X}^j(t)}{V_2^{j*}(t)}g_3^j(t) \Bigg\} \\
&\quad + \sum_{j\in J}\sum_{\substack{i\in J\\ i\neq j}} \mu_{ij}(t)p_i(t)h_0^{ij}(t) + \sum_{I\neq J}\sum_{j\in J}\sum_{i\in I} \mu_{ij}(t)p_i(t)h_0^{ij}(t) \\
&\quad + \sum_{j\in J}\sum_{\substack{i\in J\\ i\neq j}} \mu_{ij}(t)\tilde{X}^i(t)h_1^{ij}(t) + \sum_{I\neq J}\sum_{j\in J}\sum_{i\in I} \mu_{ij}(t)\tilde{X}^i(t)h_1^{ij}(t) \\
&\quad + \sum_{j\in J}\sum_{\substack{i\in J\\ i\neq j}} \mu_{ij}(t)\tilde{X}^i(t) - \mu_{ji}(t)\tilde{X}^j(t) \\
&\quad + \sum_{I\neq J}\sum_{j\in J}\sum_{i\in I} \mu_{ij}(t)\tilde{X}^i(t) - \mu_{ji}(t)\tilde{X}^j(t) \\
&\quad + \sum_{j\in J}\sum_{\substack{i\in J\\ i\neq j}} \mu_{ij}(t)\frac{p_i(t)}{V_2^{i*}(t)}h_2^{ij}(t) + \sum_{I\neq J}\sum_{j\in J}\sum_{i\in I} \mu_{ij}(t)\frac{p_i(t)}{V_2^{i*}(t)}h_2^{ij}(t)
\end{aligned}
$$

$$+\sum_{j\in J}\sum_{\substack{i\in J\\ i\neq j}}\mu_{ij}(t)\frac{\tilde{X}^i(t)}{V_2^{i*}(t)}h_3^{ij}(t)+\sum_{I\neq J}\sum_{j\in J}\sum_{i\in I}\mu_{ij}(t)\frac{\tilde{X}^i(t)}{V_2^{i*}(t)}h_3^{ij}(t)$$

Applying Lemma 3.3.3 and Assumption Two gives

$$\begin{aligned}
&\sum_{j\in J}p_j(t)(g_0^j(t)+\delta_2^j(t)\tilde{Y}(t))+\sum_{\substack{i\in J\\ i\neq j}}\mu_{ij}(t)p_i(t)h_0^{ij}(t)\\
=&p_J(t)\sum_{j\in J}\frac{\tilde{p}_j(t)}{\sum_{k\in J}\tilde{p}_k(t)}\left(g_0^j(t)+\sum_{\substack{i\in J\\ i\neq j}}\mu_{ij}(t)h_0^{ij}(t)\right)\\
&+p_J(t)\tilde{Y}(t)\sum_{j\in J}\frac{\tilde{p}_j(t)}{\sum_{k\in J}\tilde{p}_k(t)}\delta_2^j(t)\\
=&p_J(t)g^J(t)+p_J(t)\tilde{Y}(t)\delta_2^J(t),
\end{aligned}$$

where Definition 3.3.2 has been applied for the sojourn payments g_0^j and the payments on transition h_0^{ij} as well as the dividends δ_2^j. Similarly, by changing the order of summation, and using Assumption One and Two we get

$$\begin{aligned}
&\sum_{j\in J}\tilde{X}^j(t)g_1^j(t)+\sum_{j\in J}\sum_{\substack{i\in J\\ i\neq j}}\mu_{ij}(t)\tilde{X}^i(t)h_1^{ij}(t)\\
=&\sum_{j\in J}\tilde{X}^j(t)g_1^j(t)+\sum_{j\in J}\sum_{\substack{i\in J\\ i\neq j}}\mu_{ji}(t)\tilde{X}^j(t)h_1^{ji}(t)\\
=&\sum_{k\in J}\tilde{X}^k(t)\sum_{j\in J}\frac{\tilde{p}_j(t)}{\sum_{k\in J}\tilde{p}_k(t)}\left(g_1^j(t)+\sum_{\substack{i\in J\\ i\neq j}}\mu_{ji}(t)h_1^{ji}(t)\right),\\
=&g_1^J(t)\sum_{k\in J}\tilde{X}^k(t)
\end{aligned}$$

where Definition 3.3.2 has been applied for the sojourn payments g_1^j and the payments on transition h_1^{ij}. The same principles are applied to the terms that are proportional in $\frac{1}{V_2^i*(t)}$ to achieve

$$\begin{aligned}
&\sum_{j\in J}p_j(t)\frac{g_2^j(t)}{V_2^{j*}(t)}+\sum_{\substack{i\in J\\ i\neq j}}\mu_{ij}(t)p_i(t)\frac{h_2^{ij}(t)}{V_2^{j*}(t)}\\
=&p_J(t)\sum_{j\in J}\frac{\tilde{p}_j(t)}{\sum_{k\in J}\tilde{p}_k(t)}\left(\frac{g_2^j(t)}{V_2^{j*}(t)}+\sum_{\substack{i\in J\\ i\neq j}}\mu_{ij}(t)\frac{h_2^{ij}(t)}{V_2^{j*}(t)}\right)
\end{aligned}$$

and

$$
\begin{aligned}
&\sum_{j\in J}\tilde{X}^j(t)\frac{g_3^j(t)}{V_2^{j*}(t)}+\sum_{\substack{i\in J\\ i\neq j}}\mu_{ij}(t)\tilde{X}^i(t)\frac{h_3^{ij}(t)}{V_2^{j*}(t)}\\
=&\sum_{k\in J}\tilde{X}^k(t)\sum_{j\in J}\frac{\tilde{p}_j(t)}{\sum_{k\in J}\tilde{p}_k(t)}\left(\frac{g_3^j(t)}{V_2^{j*}(t)}+\sum_{\substack{i\in J\\ i\neq j}}\mu_{ji}(t)\frac{h_3^{ji}(t)}{V_2^{j*}(t)}\right).
\end{aligned}
$$

Applying Assumption Three gives

$$
\begin{aligned}
&p_J(t)\sum_{j\in J}\frac{\tilde{p}_j(t)}{\sum_{k\in J}\tilde{p}_k(t)}\left(\frac{g_2^j(t)}{V_2^{j*}(t)}+\sum_{\substack{i\in J\\ i\neq j}}\mu_{ij}(t)\frac{h_2^{ij}(t)}{V_2^{j*}(t)}\right)\\
=&\frac{p_J(t)}{V_2^{J*}(t)}\sum_{j\in J}\frac{\tilde{p}_j(t)}{\sum_{k\in J}\tilde{p}_k(t)}\left(g_2^j(t)+\sum_{\substack{i\in J\\ i\neq j}}\mu_{ij}(t)h_2^{ij}(t)\right)\\
=&\frac{p_J(t)}{V_2^{J*}(t)}g_2^J(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k\in J}\tilde{X}^k(t)\sum_{j\in J}\frac{\tilde{p}_j(t)}{\sum_{k\in J}\tilde{p}_k(t)}\left(\frac{g_3^j(t)}{V_2^{j*}(t)}+\sum_{\substack{i\in J\\ i\neq j}}\mu_{ji}(t)\frac{h_3^{ji}(t)}{V_2^{j*}(t)}\right)\\
=&\frac{\sum_{k\in J}\tilde{X}^k(t)}{V_2^{J*}(t)}\sum_{j\in J}\frac{\tilde{p}_j(t)}{\sum_{k\in J}\tilde{p}_k(t)}\left(g_3^j(t)+\sum_{\substack{i\in J\\ i\neq j}}\mu_{ji}(t)h_3^{ji}(t)\right)\\
=&\frac{p_J(t)}{V_2^{J*}(t)}g_3^J(t)\sum_{k\in J}\tilde{X}^k(t).
\end{aligned}
$$

Note that

$$
\sum_{j\in J}\sum_{\substack{i\in J\\ i\neq j}}\mu_{ij}(t)\tilde{X}^i(t)-\sum_{j\in J}\sum_{\substack{i\in J\\ i\neq j}}\mu_{ji}(t)\tilde{X}^j(t)=0.
$$

We now consider the aggregation of payments between groups of states that are not proportional to $\tilde{X}^i$ or $\frac{1}{V_2^{i*}(t)}$, multiply with $\frac{\sum_{j\in J}\sum_{i\in I}\tilde{p}_i(t)\mu_{ij}(t)}{\sum_{j\in J}\sum_{i\in I}\tilde{p}_i(t)\mu_{ij}(t)}$ and apply Assumption two,

$$
\begin{aligned}
&\sum_{I\neq J}\sum_{j\in J}\sum_{i\in I}\mu_{ij}(t)p_i(t)h_0^{ij}(t)\\
=&\sum_{I\neq J}\frac{p_I(t)}{\sum_{k\in I}\tilde{p}_k(t)}\sum_{j\in J}\sum_{i\in I}\mu_{ij}(t)\tilde{p}_i(t)\frac{\sum_{j\in J}\sum_{i\in I}\tilde{p}_i(t)\mu_{ij}(t)h_0^{ij}(t)}{\sum_{j\in J}\sum_{i\in I}\tilde{p}_i(t)\mu_{ij}(t)}
\end{aligned}
$$

$$= \sum_{I \neq J} p_I(t)\mu_{IJ}(t)h_0^{IJ}(t).$$

We carry on with the remaining transitions between groups of states, and apply Assumption one when the terms are proportional in $\tilde{X}^i$ and Assumption Three when the terms are proportional in $\frac{1}{V_2^{i*}(t)}$. We get

$$\sum_{I \neq J}\sum_{j \in J}\sum_{i \in I} \mu_{ij}(t)\tilde{X}^i(t)h_1^{ij}(t)$$
$$= \sum_{I \neq J}\sum_{k \in I} \tilde{X}^k(t)\frac{1}{\sum_{k \in I}\tilde{p}_k(t)}\sum_{j \in J}\sum_{i \in I}\mu_{ij}(t)\tilde{p}_i(t)\frac{\sum_{j \in J}\sum_{i \in I}\tilde{p}_i(t)\mu_{ij}(t)h_1^{ij}(t)}{\sum_{j \in J}\sum_{i \in I}\tilde{p}_i(t)\mu_{ij}(t)}$$
$$= \sum_{I \neq J}\mu_{IJ}(t)h_1^{IJ}(t)\sum_{k \in I}\tilde{X}^k(t),$$

and

$$\sum_{I \neq J}\sum_{j \in J}\sum_{i \in I}\mu_{ij}(t)p_i(t)\frac{h_2^{ij}(t)}{V_2^{i*}(t)}$$
$$= \sum_{I \neq J}\frac{p_I(t)}{\sum_{k \in I}\tilde{p}_k(t)}\sum_{j \in J}\sum_{i \in I}\mu_{ij}(t)\tilde{p}_i(t)\frac{\sum_{j \in J}\sum_{i \in I}\tilde{p}_i(t)\mu_{ij}(t)\frac{h_2^{ij}(t)}{V_2^{i*}(t)}}{\sum_{j \in J}\sum_{i \in I}\tilde{p}_i(t)\mu_{ij}(t)}$$
$$= \sum_{I \neq J}p_I(t)\mu_{IJ}(t)\frac{h_2^{IJ}(t)}{V_2^{I*}(t)},$$

and

$$\sum_{I \neq J}\sum_{j \in J}\sum_{i \in I}\mu_{ij}(t)\tilde{X}^i(t)\frac{h_3^{ij}(t)}{V_2^{i*}(t)}$$
$$= \sum_{I \neq J}\sum_{k \in I}\tilde{X}^k(t)\frac{1}{\sum_{k \in I}\tilde{p}_k(t)}\sum_{j \in J}\sum_{i \in I}\mu_{ij}(t)\tilde{p}_i(t)\frac{\sum_{j \in J}\sum_{i \in I}\tilde{p}_i(t)\mu_{ij}(t)\frac{h_3^{ij}(t)}{V_2^{i*}(t)}}{\sum_{j \in J}\sum_{i \in I}\tilde{p}_i(t)\mu_{ij}(t)}$$
$$= \sum_{I \neq J}\mu_{IJ}(t)\frac{h_3^{IJ}(t)}{V_2^{I*}(t)}\sum_{k \in I}\tilde{X}^k(t),$$

and

$$\sum_{I \neq J}\sum_{j \in J}\sum_{i \in I}\mu_{ij}(t)\tilde{X}^i(t) - \mu_{ji}(t)\tilde{X}^j(t)$$
$$= \sum_{I \neq J}\sum_{j \in J}\sum_{i \in I}\sum_{k \in I}\tilde{X}^k(t)\mu_{ij}(t)\frac{\tilde{p}_i(t)}{\sum_{k \in I}\tilde{p}_k(t)}\tilde{X}^i(t) - \sum_{k \in J}\tilde{X}^k(t)\mu_{ji}(t)\frac{\tilde{p}_j(t)}{\sum_{k \in J}\tilde{p}_k(t)}$$

$$
\begin{aligned}
&= \sum_{I \neq J} \sum_{k \in I} \tilde{X}^k(t) \sum_{j \in J} \sum_{i \in I} \mu_{ij}(t) \frac{\tilde{p}_i(t)}{\sum_{k \in I} \tilde{p}_k(t)} \\
&\quad - \sum_{k \in J} \tilde{X}^k(t) \sum_{I \neq J} \sum_{j \in J} \sum_{i \in I} \mu_{ji}(t) \frac{\tilde{p}_j(t)}{\sum_{k \in J} \tilde{p}_k(t)} \\
&= \sum_{I \neq J} \left(\mu_{IJ}(t) \sum_{k \in I} \tilde{X}^k(t) - \sum_{k \in I} \tilde{X}^k(t) \mu_{JI}(t) \right).
\end{aligned}
$$

Using analogous calculations for $\frac{d}{dt}\tilde{Y}(t)$ and applying Lemma 3.3.3 shows that the differential equations in Theorem 3.4.1 indeed are the differential equations for the aggregated state-wise savings account.

Chapter 4

Moment Closure for FV-processes using Moment-generating Functions

Abstract

Understanding stochastic processes by calculating their moments, is a relevant task for an unfathomable number of real-world applications. For finite variation processes with non-linear dynamics, the calculation of moments is notoriously difficult due to the infinite system of moment equations that has to be solved approximately. Moment closure techniques that truncate the infinite system of moment equations has been studied since the sixties, and is to this day an active area of research. Instead of approximating the moments directly through the moment equations, we propose to approximate the moment-generating function, based on a derivation of the PDE for the moment-generating function involving infinite partial derivatives. We construct a finite difference scheme that approximates the moment-generating function, and conduct a numerical study to verify its use.

Keywords: Projection, Infinite PDEs, Polynomial dynamics, Moment truncation.

2020 Mathematics Subject Classification: Primary: 60J27; 60-80

Secondary: 60E10

4.1 Introduction

Calculating the future expected value of a stochastic process is a fundamental discipline in probability theory. A standard tool for this task is to employ a system of differential equations that describes the development in the expected value as a function of the expected value itself. For processes with polynomial dynamics, the differential equation for the expected value is going to depend on higher order moments, and the differential equation for these higher order moments depend in turn on even higher order moments. This vicious spiral goes on forever, resulting in an infinite system of ordinary differential equations, called moment equations, that can not be solved numerically. For a class of marked point processes, we use moment-generating functions to cast this infinite system of ordinary differential equations as a partial differential equation involving infinite partial derivatives, and then we truncate the infinite partial derivatives to give PDEs that are numerically viable.

The problem of turning a countable infinite system of moment equations into a finite system of moment equations is referred to as moment closure. Bellman and Richardson (1968) propose a moment closure technique that preserves certain properties of moments. Wilcox and Bellman (1970), Sancho (1970) and Bover (1978) study different ways of truncating the infinite hierarchy of moment equations, for a class of stochastic systems with Gaussian noise by approximating the Gaussian probability density function with Hermite polynomials. Singh and Hespanha (2006) give a summary of the existing moment closure techniques. In contrast to the Fokker-Plank type stochastic systems, we consider processes that are governed by a discrete Markov process. To our knowledge, the use of moment-generating functions for moment closure is new.

Hespanha (2005) shows that for so-called polynomial stochastic hybrid systems, the infinite system of moment equations can be approximated arbitrarily close by a finite system of non-linear ODEs. Continuing this work, Hespanha and Singh (2005) present a certain type of moment closure for polynomial stochastic hybrid systems where the effects of truncated moments are approximated using the non-truncated moments, they also express chemically reacting systems using polynomial stochastic hybrid systems. By assuming that the stochastic process resembles a binomial process, Barzel and Biham (2012) suggest a moment closure technique, and Lakatos et al. (2015) use the same approach for Gaussian, gamma, and lognormal distributions.

Apart from using differential equations to calculate expected values, Monte Carlo simulations is another tool at our disposal for this task. By simulating a process sufficiently many times and taking the average, we produce an unbiased estimator of the expected value of the process. This method of calculating expected values is versatile, but computationally heavy. For calculating high orders of moments, the Monte Carlo approach is even less efficient. The Monte Carlo approach does not utilise what we know about the dynamics of the processes, and using it is like using a sledgehammer to crack a nut. Moment-generating functions carry information about the distribution of the process, and by tapping into this information we can determine the expected evolution of the process.

Instead of producing infinitely many equations for the moments of the process, we produce a single equation involving infinitely many moments. Classical moment closure solves M equations to produce an approximation of the first M moments. In contrast, we only solve a single equation for the moment-generating function. However, this single equation involves infinite partial derivatives, and the finite difference method for calculating the truncated partial derivatives may involve more than M equations. It is definitely of interest to study the difference in efficiency between solving moment equations, solving moment-generating functions and a Monte-Carlo method, as well as finding use cases for each, but it is outside the scope of this paper.

The main contribution of this paper is to provide a novel system of ODEs that acts as a tool to project processes with polynomial dynamics. Overall, the paper deals with two topics

1) Constructing a system of PDEs that describe the state-wise moment-generating functions.

2) Turning this system of PDEs into a system of ODEs that can be fed to a computer.

Topic 1) is purely theoretical, as the resulting system of PDEs cannot directly be approximated by a computer, but it forms the foundation for producing a system of differential equations that can. The problems of topic 2) concern the error introduced by truncating the system of PDEs from topic 1), as well as the computational considerations in constructing a suitable finite difference scheme. Using moment-generating functions, we can easily put a bound on the error between the truncated and non-truncated PDEs. Putting a bound on the difference between the truncated and non-truncated moment equations is much more cumbersome, due to the interconnected relation between the moment equations.

The paper is structured as follows: in Section 4.2 we present the family of stochastic processes that we study. In Section 4.3 we derive the system of infinite partial derivatives that are the foundation of the paper. In Section 4.4 we present the measures we take to turn the system of infinite PDEs into a system of finite PDEs, and in Section 4.5 we define the approximating state-wise moment-generating function. Finally, in Section 4.4, we conduct a numerical study.

4.2 Setup

In this section we present the family of processes that we study. Most of the literature on moment closure deals with diffusion processes, as opposed to the finite variation processes that we consider, but the concept of using moment-generating functions also applies to diffusion processes.

Let $Z(t)$ be a state process on a state space $\mathcal{J} = \{0, ..., J\}$. We assume that Z is

Markov with transition intensities $\mu_{ij}(t)$ $i, j \in \mathcal{J}$. By a permutation argument, we assume $Z(0) = 0$ without loss of generality. We associate with Z the counting processes

$$N^k(t) = \#\{s \in (0,t] : Z(s-) \neq k, Z(s) = k\},$$

counting the number of jumps into state k. Analogously $N^{jk}(t)$ counts the number of jumps into state k from state j.

We restrict ourselves to consider finite variation marked point processes that can be represented as functionals of the state process Z in the form

$$X(t) = x_0 + \int_0^t g^{Z(s)}(s, X(s))ds + \int_0^t \sum_{k:k\neq Z(s-)} h^{Z(s-)k}(s, X(s-))dN^k(s),$$

where $g^i(s,x)$ represents the continuous development of X while Z is in state i and $h^{jk}(s,x)$ represents the jump in X on transition of Z from state j to state k. We consider processes with polynomial dynamics, defined by the form of g^i and h^{jk} as

$$g^i(t,x) = \sum_{l=0}^{p} g_l^i(t)x^l, \tag{4.2.1}$$

$$h^{jk}(t,x) = \sum_{l=0}^{p} h_l^{jk}(t)x^l. \tag{4.2.2}$$

Polynomial dynamics do not encompass all dynamics of practical interest, but they can form approximations of non-polynomial dynamics. Indeed, a strand of literature on stochastic chemical kinetics deals with the problem of formulating moment equations for processes with non-polynomial dynamics – see e.g. Sotiropoulos and Kaznessis (2011) and Ale et al. (2013). The simplest non-trivial case of concern, is a process X determined by

$$dX(t) = \sum_{l=0}^{p} g_l^{Z(t)}(t)X(t)^l dt, \qquad X(0) = x_0.$$

Our ambition in this paper is to calculate $\mathrm{E}[\mathbb{1}_{\{Z(t)=i\}}X(t)]$, being the probability weighted expected value of $X(t)$, as well as higher-order moments $\mathrm{E}[\mathbb{1}_{\{Z(t)=i\}}X(t)^j]$. Note that the 0^th moment is the state-wise probabilities

$$p_{0i}(0,t) := P(Z(t) = i | Z(0) = 0) = \mathrm{E}[\mathbb{1}_{\{Z(t)=i\}}X(t)^0],$$

which can be computed using the Kolmogorov forward differential equations. For affine dynamics, i.e. for $p = 1$ in (4.2.1)-(4.2.2), a hierarchy of $J \times N$ ordinary linear differential equations describe moments of the first N moments, which can be derived using linearity of expectations.

When the dynamics of X are polynomial, the p^{th} moment in general depends on the $(p+1)^{th}$ moment. Consider as an example the process with polynomial dynamics

$$dX(t) = (g_0^{Z(t)}(t) + g_1^{Z(t)}(t)X(t) + g_2^{Z(t)}(t)X(t)^2)dt,$$

and note that

$$\begin{aligned}d(X(t)^p) =& pX(t)^{(p-1)}\left(g_0^{Z(t)}(t)+g_1^{Z(t)}(t)X(t)+g_2^{Z(t)}(t)X(t)^2\right)dt\\ =& p\left(g_0^{Z(t)}(t)X(t)^{(p-1)}+g_1^{Z(t)}(t)X(t)^p+g_2^{Z(t)}(t)X(t)^{(p+1)}\right)dt,\end{aligned}$$

such that by induction, the differential equation for $\mathrm{E}[\mathbb{1}_{\{Z(t)=i\}}X(t)^p]$ involves all lower order moments as well as all higher order moments ad infinitum.

This dependence on all higher-order moments suggests the question; How much does the high-order moments of X influence the low-order moments? If the answer is "not very much", we can perhaps neglect the influence of all moments higher than some order, which facilitates a numerical solution. This is the standard approach in moment closure, but to further develop this idea, we present a way to handle and quantify the influence that moments have on each other through the moment-generating function.

4.3 A PDE for the State-wise Moment-generating Function

In this section we present a system of PDEs that describes the state-wise moment-generating functions of state-wise processes with polynomial dynamics. With access to the state-wise moment-generating functions, we can evaluate the expected state-wise processes, as well as higher order moments by differentiation.

We first consider a simple one-dimensional case without discrete dynamics. We later deal with a two-dimensional process with discrete dynamics. The proof is inspired by Bruhn and Lollike (2021), who derive a system of ODEs for a class of affine insurance related processes, but do so without the moment-generating functions. Assume that the dynamics of X are given by

$$dX(t)=\sum_{l=0}^{p}g_l^{Z(t)}(t)X(t)^l dt, \qquad X(0)=x_0,$$

where $g_l^i(t)$ are bounded functions. Then, by Itô's Lemma for finite variation processes,

$$\begin{aligned}\mathbb{1}_{\{Z(t)=i\}}d\left(e^{\lambda X(s)}\right) =& \mathbb{1}_{\{Z(t)=i\}}\sum_{l=0}^{p}\lambda e^{\lambda X(s)}g_l^{Z(s)}(s)X(s)^l\\ =& \mathbb{1}_{\{Z(t)=i\}}\sum_{l=0}^{p}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{k!}g_l^{Z(s)}(s)X(s)^{l+k}.\end{aligned}$$

Since $g_l^i(t)$ are bounded functions, $X(t)$ is bounded, and the moment-generating function of $X(t)$ exists. We define the state-wise moment-generating function as

$$m_i(\lambda,t):=\mathrm{E}[\mathbb{1}_{\{Z(t)=i\}}e^{\lambda X(t)}].$$

Applying Itô's lemma, we can write $e^{\lambda X(t)}$ as an integral of the dynamics of $e^{\lambda X(s)}$ over $(0,t]$,

$$\begin{aligned}
m_i(\lambda,t) =&\mathrm{E}[\mathbb{1}_{\{Z(t)=i\}}e^{\lambda X(t)}] \\
=&\mathrm{E}[\mathbb{1}_{\{Z(t)=i\}}e^{\lambda X(0)}] + \int_0^t \mathrm{E}\left[\mathbb{1}_{\{Z(t)=i\}}\sum_{l=0}^{p}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{k!}g_l^{Z(s)}(s)X(s)^{l+k}\right]ds \\
=&p_{0i}(0,t)e^{\lambda x_0} \\
&+\sum_{l=0}^{p}\int_0^t\sum_{j\in\mathcal{J}}p_{0j}(0,s)g_l^j(s)\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{k!}\mathrm{E}\left[\mathbb{1}_{\{Z(t)=i\}}X(s)^{l+k}\middle|\, Z(s)=j\right]ds.
\end{aligned}$$

We have applied Fubini's theorem and interchanged expectation and infinite sum, which is permissible because $X(t)$ is bounded, and because the convergence of the infinite sum towards the exponential function is uniform on any bounded interval. By the Markov property

$$\mathrm{E}\left[\mathbb{1}_{\{Z(t)=i\}}X(s)^{l+k}\middle|\, Z(s)=j\right] = \frac{\mathrm{E}[\mathbb{1}_{\{Z(s)=j\}}X(s)^{l+k}]}{p_{0j}(0,s)}p_{ji}(s,t). \tag{4.3.1}$$

It is due to (4.3.1) that we can separate the present and the future to get a forward differential equation for $m_i(\lambda,t)$. We now use (4.3.1) to write

$$m_i(\lambda,t) =p_{0i}(0,t)e^{\lambda x_0} + \sum_{l=0}^{p}\int_0^t\sum_{j\in\mathcal{J}}g_l^j(s)\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{k!}\mathrm{E}[\mathbb{1}_{\{Z(s)=j\}}X(s)^{l+k}]p_{ji}(s,t)ds.$$

Futhermore, since $X(s)$ is bounded, the dominated convergence theorem implies that

$$\mathrm{E}[\mathbb{1}_{\{Z(s)=j\}}X(s)^{l+k}] = \frac{\partial^{l+k}}{\partial\lambda^{l+k}}m_j(\lambda,s)\big|_{\lambda=0}.$$

This is another crucial step, and a novel one compared to Bruhn and Lollike (2021). It allows us to deal with higher-order moments of X, and the relation is a consequence of the unique property of the exponential function

$$\frac{d^i}{d\lambda^i}e^{\lambda x}\big|_{\lambda=0} = x^i.$$

To ease readability we introduce the notation

$$\frac{\partial^{l+k}}{\partial\lambda^{l+k}}m_j(0,s) = \frac{\partial^{l+k}}{\partial\lambda^{l+k}}m_j(\lambda,s)\big|_{\lambda=0}.$$

Differentiating m_i with respect to t yields the following differential equation,

$$
\begin{aligned}
&\frac{\partial}{\partial t} m_i(\lambda, t) \\
=& e^{\lambda x_0} \frac{\partial}{\partial t} p_{0i}(0,t) \\
&+ \sum_{l=0}^{p} \sum_{j \in \mathcal{J}} g_l^j(t) \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{k!} \frac{\partial^{l+k}}{\partial \lambda^{l+k}} m_j(0,t) p_{ji}(t,t) \\
&+ \sum_{l=0}^{p} \int_0^t \sum_{j \in \mathcal{J}} g_l^j(s) \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{k!} \frac{\partial^{l+k}}{\partial \lambda^{l+k}} m_j(0,s) \frac{\partial}{\partial t} p_{ji}(s,t) ds \\
=& \sum_{l=0}^{p} \sum_{j \in \mathcal{J}} g_l^j(t) \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{k!} \frac{\partial^{l+k}}{\partial \lambda^{l+k}} m_j(0,t) p_{ji}(t,t) \\
&+ \sum_{g \neq i} \mu_{gi}(t) \underbrace{\left(p_{0g}(0,t) e^{\lambda x_0} + \sum_{l=0}^{p} \int_0^t \sum_{j \in \mathcal{J}} g_l^j(s) \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{k!} \frac{\partial^{l+k}}{\partial \lambda^{l+k}} m_j(0,s) p_{jg}(s,t) ds \right)}_{m_g(\lambda,t)} \\
&- \sum_{g \neq i} \mu_{ig}(t) \underbrace{\left(p_{0i}(0,t) e^{\lambda x_0} + \sum_{l=0}^{p} \int_0^t \sum_{j \in \mathcal{J}} g_l^j(s) \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{k!} \frac{\partial^{l+k}}{\partial \lambda^{l+k}} m_j(0,s) p_{ji}(s,t) ds \right)}_{m_i(\lambda,t)} \\
=& \sum_{l=0}^{p} g_l^i(t) \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{k!} \frac{\partial^{l+k}}{\partial \lambda^{l+k}} m_i(0,t) + \sum_{g \neq i} \left(\mu_{gi}(t) m_g(\lambda,t) - \mu_{ig}(t) m_i(\lambda,t) \right),
\end{aligned}
$$

where $\frac{d}{dt} p_{ji}(s,t)$ is given by the Kolmogorov forward differential equations,

$$
\begin{aligned}
\frac{\partial}{\partial t} p_{ji}(s,t) =& \sum_{k: k \neq i} p_{jk}(s,t) \mu_{ki}(t) - \mu_{ik}(t) p_{ji}(s,t), \\
p_{ji}(s,s) =& \mathbb{1}_{\{j=i\}}.
\end{aligned}
$$

The result is restated in Theorem 4.3.1.

Theorem 4.3.1 (PDE for state-wise moment-generating function, one dimension, no jumps)**.** *Assume $X(t)$ has dynamics*

$$
dX(t) = \sum_{l=0}^{p} g_l^{Z(t)}(t) X(t)^l, \qquad X(0) = x_0,
$$

for bounded functions $g_l^i(t)$. Then $m_i(\lambda,t) = E[\mathbb{1}_{\{Z(t)=i\}}e^{\lambda X(t)}]$ satisfies the PDE

$$\frac{\partial}{\partial t}m_i(\lambda,t) = \sum_{g\neq i}(\mu_{gi}(t)m_g(\lambda,t) - \mu_{ig}(t)m_g(\lambda,t)) \tag{4.3.2}$$

$$+\sum_{l=0}^{p} g_l^i(t)\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{k!}\frac{\partial^{l+k}}{\partial\lambda^{l+k}}m_i(0,t) \tag{4.3.3}$$

$$m_i(\lambda,0) = \mathbb{1}_{\{i=Z(0)\}}e^{\lambda x_0}. \tag{4.3.4}$$

Solving the PDE of Theorem 4.3.1, yields an expression for $m_i(\lambda,t)$. By differentiating $m_i(\lambda,t)$ with respect to λ and evaluating in $\lambda = 0$, we are given the quantity of interest; the state-wise expected value

$$\frac{\partial}{\partial\lambda}m_i(0,t) = \mathrm{E}[\mathbb{1}_{\{Z(t)=i\}}X(t)].$$

Due to the limit $k\to\infty$, Theorem 4.3.1 is of little practical use, unless a closed form solution to the PDE can be derived. In the PDE (4.3.2)-(4.3.4) we have quantified the influence of all moments on the time-derivative of the moment-generating function, which helps to give us an idea about how many moments we need to accurately solve the moment-generating function. This influence decays like $\frac{1}{k!}$, meaning that very high moments have a very small influence on the changes over time in the state-wise moment-generating function.

Before we discuss how Theorem 4.3.1 can be turned into a practically useful system of differential equations, we extend the result to two-dimensional processes $(X(t), Y(t))$ with affine jump dynamics. The proof follows the steps already seen, with the addition of step two, in order to account for the discrete dynamics.

1) Use the boundedness of (X, Y) and uniform convergence of the exponential function over an arbitrary bounded interval to interchange expectation and infinite sum.

2) Use predictability of the integrand to exchange integrator $\mathbb{1}_{\{Z(t)=i\}}N^{rj}(s)$ with its predictable compensator.

3) Use the tower property to condition on $Z(s)$.

4) Use the Markov property to separate the future ($\mathbb{1}_{\{Z(t)=i\}}$) from the present (the condition on $Z(s)$).

5) Use the dominated convergence theorem to replace the expectation of the moments of X and Y with derivatives of the moment-generating function.

6) Differentiate the integral equation for the state-wise moment-generating functions to achieve a system of differential equations.

Following the steps yields the following theorem.

Theorem 4.3.2 (PDE for state-wise moment-generating function, two dimensions, with jumps)**.** *Assume*

$$dW(s) = \sum_{l=0}^{p} g_l^{Z(s)}(s)W(s)^l ds + \sum_{l=0}^{1} \sum_{j\neq Z(s-)} h_l^{Z(s-)k}(s)W(s)^l dN^j(s), \qquad W(0) = w_0$$

where $W(s)^l$ *are the element-wise powers of* W*, and*

$$g_l^r(s) = \begin{pmatrix} {}_{xx}g_l^r(s) & {}_{xy}g_l^r(s) \\ {}_{yx}g_l^r(s) & {}_{yy}g_l^r(s) \end{pmatrix}$$

$$h_l^{rj}(s) = \begin{pmatrix} {}_{xx}h_l^{rj}(s) & {}_{xy}h_l^{rj}(s) \\ {}_{yx}h_l^{rj}(s) & {}_{yy}h_l^{rj}(s) \end{pmatrix}.$$

then $m_i(\lambda, t) = E[\mathbb{1}_{\{Z(t)=i\}} e^{\lambda^T W(t)}]$ *solves the PDE*

$$\begin{aligned}
\frac{\partial}{\partial t} m_i(\lambda, t) =& \sum_{r\neq i} \mu_{ri}(t) m_r(\lambda, t) - \mu_{ir}(t) m_i(\lambda, t) \\
&+ \sum_{k=0}^{\infty} \sum_{q=0}^{k} \sum_{l=0}^{p} \frac{1}{(k-q)!q!} \lambda_1^{k-q+1} \lambda_2^q \, {}_{xx}g_l^i(t) \frac{\partial^{k+l}}{\partial \lambda_1^{k-q+l} \partial \lambda_2^q} m_i(0, t) \\
&+ \sum_{k=0}^{\infty} \sum_{q=0}^{k} \sum_{l=0}^{p} \frac{1}{(k-q)!q!} \lambda_1^{k-q+1} \lambda_2^q \, {}_{xy}g_l^i(t) \frac{\partial^{k+l}}{\partial \lambda_1^{k-q} \partial \lambda_2^{q+l}} m_i(0, t) \\
&+ \sum_{k=0}^{\infty} \sum_{q=0}^{k} \sum_{l=0}^{p} \frac{1}{(k-q)!q!} \lambda_1^{k-q} \lambda_2^{q+1} \, {}_{yx}g_l^i(t) \frac{\partial^{k+l}}{\partial \lambda_1^{k-q+l} \partial \lambda_2^q} m_i(0, t) \\
&+ \sum_{k=0}^{\infty} \sum_{q=0}^{k} \sum_{l=0}^{p} \frac{1}{(k-q)!q!} \lambda_1^{k-q} \lambda_2^{q+1} \, {}_{yy}g_l^i(t) \frac{\partial^{k+l}}{\partial \lambda_1^{k-q} \partial \lambda_2^{q+l}} m_i(0, t) \\
&+ \sum_{k=0}^{\infty} \sum_{q_1+q_2+q_3=k} \frac{1}{q_1!q_2!q_3!} \sum_{r\neq i} \frac{\partial^{q_1+q_2}}{\partial \lambda_1^{q_1} \partial \lambda_2^{q_2}} m_r(0, s) \\
&\times (\lambda_1(1 + {}_{xx}h_1^{ri}(t)) + \lambda_2 \, {}_{yx}h_1^{ri}(t))^{q_1} \\
&\times (\lambda_1 \, {}_{xy}h_1^{ri}(t) + \lambda_2(1 + {}_{yy}h_1^{ri}(t)))^{q_2} \\
&\times (\lambda_1({}_{xx}h_0^{ri}(t) + {}_{xy}h_0^{ri}(t)) + \lambda_2({}_{yx}h_0^{ri}(t) + {}_{yy}h_0^{ri}(t)))^{q_3} \\
&\times \mu_{ri}(t) \\
&- \sum_{k=0}^{\infty} \sum_{q=0}^{k} \frac{1}{(k-q)!q!} \lambda_1^{k-q} \lambda_2^q \sum_{r\neq i} \frac{\partial^k}{\partial \lambda_1^{k-q} \partial \lambda_2^q} m_r(0, s) \mu_{ri}(t) \\
m_i(\lambda, 0) =& \mathbb{1}_{\{i=0\}} e^{\lambda^T w_0}
\end{aligned}$$

See Appendix 4.C for the proof. The extension to non-affine jump dynamics and higher dimensions is straightforward and follows an analogous proof, but for notational convinience we only consider affine jump dynamics and two dimensions. Also for notational convenience, we do not consider dynamics that involve the product of powers of X and Y such as e.g. $\sum_{l=0}^{p}\sum_{j=0}^{p} X(t)^l Y(t)^j g_{l,j}^{Z(t)}(t)$, even though such terms in the dynamics of W could be included directly, noting that

$$\mathrm{E}[\mathbb{1}_{\{Z(t)=i\}} X(t)^l Y(t)^j] = \frac{\partial^{j+l}}{\partial\lambda_1^l \partial\lambda_2^j} m_i(0,t).$$

Theorem 4.3.1 is a special case of Theorem 4.3.2, for

$$h_l^{rj}(s) = 0,$$
$$Y(0) = {}_{xy}g_l^r(s) = {}_{yx}g_l^r(s) = {}_{yy}g_l^r(s) = 0.$$

Theorem 4.3.2 is a purely theoretical result and not applicable in practice due to the infinite series. However, using the methods presented in the next section to translate the infinitely large system of partial differential equations to a finite system of linear ordinary differential equations, Theorem 4.3.2 allows for a wide range of processes with non-linear dynamics to be projected.

4.4 Truncation and Discretisation

Ultimately we want a system of differential equations that is viable for numerical procedures, and to that end we face two obstacles;

- **Discretisation**. Computers only do discrete calculations, and we want to calculate the state-wise moment-generating function for all time-points within some interval. To do this we need an algorithm that approximately solves the PDE of Theorem 4.3.2.

- **Infinite Series**. The PDE of Theorem 4.3.2 involves the infinite series

$$\sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{k!} \frac{\partial^{l+k}}{\partial\lambda^{l+k}} m_i(0,t).$$

  If we can not find an analytical solution to infinite series, we need to approximate the solution with a finite number of calculations.

To keep notation simple we consider the PDE from Theorem 4.3.1 to highlight central mathematical concepts, but these concepts also apply to the more involved PDE from Theorem 4.3.2. The results in this section naturally depend on the specific dynamics of the process in question, but we refrain from discussing special cases that lead to particularly simple or particularly complex relations.

4.4.1 Infinite Series

A blunt idea to deal with the infinite series is to truncate the series and simply ignore all terms where k is larger than some value M. This idea is in line with moment closure as seen in e.g. Bover (1978) and Bellman and Richardson (1968), with the notable difference that we truncate the infinite partial derivatives instead of the infinite system of ODEs. We split the sum over k in the differential equation of Theorem 4.3.1 into terms smaller than M and terms larger than M,

$$\begin{aligned}\frac{\partial}{\partial t}m_i(\lambda,t) =& \sum_{l=0}^{p} g_l^i(t)\sum_{k=0}^{M}\frac{\lambda^{k+1}}{k!}\frac{\partial^{l+k}}{\partial\lambda^{l+k}}m_i(0,t)\\ &+\sum_{g\neq i}\left(\mu_{gi}(t)m_g(\lambda,t)-\mu_{ig}(t)m_i(\lambda,t)\right)\\ &+\varepsilon_i(\lambda,t),\end{aligned} \tag{4.4.1}$$

where

$$\varepsilon_i(\lambda,t)=\sum_{l=0}^{p} g_l^i(t)\sum_{k=M+1}^{\infty}\frac{\lambda^{k+1}}{k!}\frac{\partial^{l+k}}{\partial\lambda^{l+k}}m_i(0,t),$$

and we call this value the cut-off. Even though the assumption $\varepsilon_i(\lambda,t)=0$ seems crude, it is not entirely unreasonable since $k!$ rapidly diverges to infinity. As $X(t)$ is bounded, we have for $k\to\infty$,

$$\begin{gathered}\frac{\lambda^{k+1}}{k!}X(t)^{p+k}\to 0\\ \Rightarrow\\ \frac{\lambda^{k+1}}{k!}\frac{\partial^{p+k}}{\partial\lambda^{p+k}}m_i(0,t)\to 0,\end{gathered}$$

supporting the claim that the influence of high-order moments on the time-derivative of the state-wise moment-generating function is minuscule.

It is well known that any moment of a process with polynomial dynamics of degree larger than one, depend on infinitely many moments. We have also just argued that the dependence on the high orders of moments is small for the partial derivative in time of the state-wise moment-generating function, and that we therefore can ignore the corresponding terms in the infinite PDE. This argument is however based only on the direct influence of the higher order moments on the time derivative. Ultimately we are interested in the state-wise moments of processes, corresponding to the partial derivatives with respect to λ in the state-wise moment-generating function evaluated in $\lambda=0$. Note that

$$\frac{\partial}{\partial\lambda}\varepsilon_i(\lambda,t)=\sum_{l=0}^{p} g_l^i(t)\sum_{k=M+1}^{\infty}(k+1)\frac{\lambda^{k}}{k!}\frac{\partial^{l+k}}{\partial\lambda^{l+k}}m_i(0,t),$$

which means that the influence of the $(l+k)^{th}$ moment on $\frac{\partial}{\partial\lambda}\varepsilon_i(\lambda,t)$, is a factor $(k+1)$ larger than the influence of the same moment on $\varepsilon_i(\lambda,t)$. Simply put, the cut-off is sensitive to changes in λ. This means that the state-wise moment-generating function is sensitive to changes in λ, and we should therefore be careful about ignoring the cut-off, as it is precisely the sensitivity in λ for the state-wise moment-generating function we are interested in. This does not nullify our argument to assume $\varepsilon_i(\lambda,t)=0$, but we realize that the error introduced by ignoring the cut-off, generally speaking is larger for the partial derivative with respect to λ, and that this should be taken into account when choosing a value for M.

A significant advantage of the use of state-wise moment-generating functions for moment closure, is that we can put a bound on the size of the cut-off. For bounded g functions, define

$$G_l(t) = \sup_{i\in\mathcal{J}}\{|g_l^i(t)|\},$$

and note that $|X(t)| \le \tilde{B}(t)$ for

$$\tilde{B}(t) = x_0 + \int_0^t \sum_{l=0}^p G_l(s)\tilde{B}(s)^l ds.$$

Define $B(t) := \max(1,\tilde{B}(t))$, then

$$\begin{aligned}
|\varepsilon_i(\lambda,t)| &= \left|\sum_{l=0}^p g_l^i(t) \sum_{k=M+1}^{\infty} \frac{\lambda^{k+1}}{k!}\frac{\partial^{l+k}}{\partial\lambda^{l+k}} m_i(0,t)\right| \\
&\le \sum_{l=0}^p G_l(t) \sum_{k=M+1}^{\infty} \frac{\lambda^{k+1}}{k!} B(t)^{l+k} \\
&= \sum_{l=0}^p G_l(t)\lambda B(t)^l \left(e^{B(t)\lambda} - \sum_{k=0}^{M} \frac{(B(t)\lambda)^k}{k!}\right).
\end{aligned}$$

For $M\to\infty$ the sum in the parenthesis tends to $e^{B(t)\lambda}$ notoriously fast. For given $\delta > 0$ we can choose M such that

$$\sum_{l=0}^p G_l(t)\lambda B(t)^l \left(e^{B(t)\lambda} - \sum_{k=0}^{M} \frac{(B(t)\lambda)^k}{k!}\right) \le \delta,$$

and thus ensure

$$|\varepsilon_i(\lambda,t)| \le \delta,$$

making the cut-off arbitrarily small, thereby restricting the local truncation error. Analogous calculations lead to bounds on the partial derivatives of λ in the cut-off. A similar result is presented in Hespanha (2005) for the difference between the truncated and non-truncated system of moment equations for polynomial stochastic hybrid systems, but due to the interconnected moment equations, the bounds are complicated to construct.

Compared to classical moment closure where a system of M equations provide approximations of M moments, it may seem unnatural to use moments up to order $M+p$ to propagate the differential equation for the state-wise moment-generating function where moments of order higher than M are truncated. When we use state-wise moment-generating functions for moment closure, there is indeed a blurry line between the moments we use in the differential equations, and the moments we are approximating. This is a consequence of the less restrictive form of the PDEs from Theorem 4.3.2.

Alternatively, we could truncate the infinite PDEs of Thoerem 4.3.2 to more closely resemble classical moment closure, where the moments of order higher than M have no direct influence on moments of order lower than M, using

$$\begin{aligned}\frac{\partial}{\partial t}m_i(\lambda,t) =& \sum_{l=0}^{p} g_l^i(t) \sum_{k=0}^{M-l} \frac{\lambda^{k+1}}{k!}\frac{\partial^{l+k}}{\partial \lambda^{l+k}} m_i(0,t) \\ &+ \sum_{g\neq i}\left(\mu_{gi}(t)m_g(\lambda,t) - \mu_{ig}(t)m_i(\lambda,t)\right) \\ &+ \widehat{\varepsilon}_i(\lambda,t), \end{aligned} \tag{4.4.2}$$

where

$$\widehat{\varepsilon}_i(\lambda,t) = \sum_{l=0}^{p} g_l^i(t) \sum_{k=M-l+1}^{\infty} \frac{\lambda^{k+1}}{k!}\frac{\partial^{l+k}}{\partial \lambda^{l+k}} m_i(0,t).$$

This would closer resemble classical moment closure techniques, but there is no reason to conform to this practice when we have the ability to improve it.

4.4.2 Discretisation

In order to numerically solve the PDEs of Theorem 4.3.1 and 4.3.2, we need a finite difference method that approximates the partial derivatives with respect to both λ and t. There is a wide range of tools available for the numerical solution of systems of one-dimensional ordinary differential equations such as the Runge-Kutta methods. Our focus in regards to discretisation therefore lies on the translation of PDEs to ODEs, leaving the additional discretisation of these ODEs to the preferences of the reader. This method of solving PDEs numerically is often referred to as the method of lines.

The call for discretisation in the λ-dimension is a consequence of solving the moment-generating function. No discretisation is required for the infinite system of moment equations. One of the disadvantages of using moment-generating functions is that this extra discretisation step introduces another layer of implementation considerations, such as the discretisation mesh size, for which there is no a priori good choice.

The unusual property of the PDEs for $m_i(\lambda, t)$ is that the derivative in t depends on the derivative in λ only for $\lambda = 0$. This can be exploited to produce an estimate of $\mathrm{E}[\mathbb{1}_{\{Z(t)=i\}}X(t)]$ and any higher order moment. We can approximate

$$E[\mathbb{1}_{\{Z(t)=0\}}X(t)] = \frac{\partial}{\partial\lambda} m_i(0, t),$$

by a finite difference method if we have access to a set of values, $m_i(\vec{\lambda}, t)$ for some vector $\vec{\lambda}$, with indices or nodes $\vec{\lambda}[q]$ for $q = 1, ..., K$, that lie in the vicinity of zero. In principle, the state-wise moment-generating functions could have state dependent nodes, but for the sake of simplicity we assume that all states use the same vector of nodes. Under the assumption that $\varepsilon_M(\vec{\lambda}[q], t) = 0$, we can approximate $m_i(\vec{\lambda}[q], t)$ by $\hat{m}_i(\vec{\lambda}[q], t)$ using the ODEs given by

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{m}_i(\vec{\lambda}[q], t) =& \sum_{l=0}^{p} g_l^i(t) \sum_{k=0}^{M} \frac{\lambda^{k+1}}{k!} \tilde{m}_i^{(l+k)}(0, t) \\
&+ \sum_{g \neq i} \left(\mu_{gi}(t) m_g(\vec{\lambda}[q], t) - \mu_{ig}(t) m_i(\vec{\lambda}[q], t)\right), \\
\hat{m}_i(\vec{\lambda}[q], 0) =& \mathbb{1}_{\{i=0\}} e^{\vec{\lambda}[q]x_0}
\end{aligned}$$

where $\tilde{m}_i^{(k)}(0, t) = B_k^T \cdot \hat{m}_i(\vec{\lambda}, t)$ for some vector B_k satisfying

$$\tilde{m}_i^{(k)}(0, t) \approx \frac{\partial^k}{\partial\lambda^k} m_i(0, t). \tag{4.4.3}$$

By replacing the partial derivatives in λ with a finite difference approximation, we are turning PDEs into ODEs.

We return to the matter of constructing $\tilde{m}_i^{(n)}(0, t)$ that satisfies (4.4.3) in Section 4.5. Note that a by-product of this calculation method for $\tilde{m}_i^{(1)}(0, t) \approx \mathrm{E}[\mathbb{1}_{\{Z(t)=i\}}X(t)]$, are the approximated higher order moments up to degree $M + p$.

4.4.3 Infinite Series and Discretisation

The two obstacles also have a combined effect that deserves attention. When we choose the cut-off point M, we have to keep the discretisation error in mind and vice verca. We do not have access to the exact partial derivatives in λ, so they have to be estimated by finite difference methods, but the accuracy of these methods depend on the cut-off error.

Say that we choose $M = 10$. The direct influence of the higher-than-ten order moments on the numerically approximated first moment is probably small since $\frac{1}{k!}$, for $k > 10$, is small, but the influence of the 11^{th} moment on the numerically approximated 10^{th}

moment is larger since the 10^{th} derivative in λ for the cut-off is

$$\frac{\partial^{10}}{\partial \lambda^{10}} \varepsilon_i(\lambda, t) = \sum_{l=0}^{p} g_l^i(t) \sum_{k=10+1}^{\infty} \frac{(k+1)\lambda^{k-9}}{(k-9)!} \frac{\partial^{l+k}}{\partial \lambda^{l+k}} m_i(0, t),$$

and $\frac{1}{(k-9)!} > \frac{1}{k!}$. This cut-off is completely ignored in the approximating ODEs for the state-wise moment-generating functions. Even if the finite-difference method is accurate, the numerically approximated 10^{th} moment is probably a poor approximation, due to the cut-off. This poorly approximated 10^{th} moment directly influences all the lower order moments, but the higher the moment the higher the influence. In a sense, the cut-off error creates a rippling effect downward through the moments, where the error in the approximated 10^{th} moment carries over to the error in the approximated 9^{th} moment and so on. Our confidence in the approach to ignore the cut-off is based on the fact that the ripples are dampened each time the cut-off error carries over to a lower moment.

4.5 Approximating a PDE with an ODE

In the last section we presented two steps to transform the PDEs of Theorem 4.3.1 and 4.3.2 into ODEs,

1) Ignore cut-off to make a system of PDEs that require only finitely many moments.

2) Use finite difference methods to approximate partial derivatives in λ to turn system of PDEs into system of ODEs.

There are several considerations that come into play when constructing the finite difference scheme of step 2). These considerations are important, but mathematically uninteresting – see Appendix 4.B for the details.

In the derivations of Appendix 4.B, we use standard methods from Numerical Analysis to construct an approximation of $\frac{\partial^n}{\partial \lambda^n} m_i(0, t)$ with a small local truncation error. The result is highlighted in Lemma 4.5.1.

Lemma 4.5.1. *Let $\vec{\lambda}$ be a vector given by*

$$\vec{\lambda} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{K-1} \\ \alpha_K \end{pmatrix} \cdot h.$$

We call h the step size and α_i the step repetitions. Assume that K is odd, and that the step repetitions are symmetric around zero, i.e $\alpha_{\lfloor K/2 \rfloor + 1} = 0$ and $\alpha_i = -\alpha_{K-i+1}$.

For the diagonal matrix

$$C = Diag\left(\left\{\frac{0!}{h^0}, \frac{1!}{h^1}, \frac{2!}{h^2}, ..., \frac{(K-1)!}{h^{K-1}}\right\}\right),$$

and the Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{K-1} & \alpha_K \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_{K-1}^2 & \alpha_K^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \alpha_1^{K-1} & \alpha_2^{K-1} & \cdots & \alpha_{K-1}^{K-1} & \alpha_K^{K-1} \end{pmatrix},$$

the partial derivatives in λ are given by

$$\frac{\partial^n}{\partial \lambda^n} m_i(0,t) = (V^{-1}C_n)^T m_i(\vec{\lambda}, t) + O(h^{K-n-1}),$$

where C_n is the n^{th} column of C.

Using Lemma 4.5.1, we can calculate high-order partial derivatives in λ with a arbitrarily small local truncation error. The computationally demanding part of calculating $\tilde{m_i}^{(n)}(0,t)$ is the inversion of the Vandermonde matrix. Fortunately, this inversion can be done with an explicit formula

$$V^{-1}_{[i,j]} = \frac{(-1)^{K-j} \mathrm{e}_{K-j}(\{\alpha_1, ..., \alpha_K\} \setminus \{\alpha_i\})}{\prod\limits_{\substack{m=1 \\ m \neq i}}^{n} (\alpha_i - \alpha_m)},$$

where

$$\mathrm{e}_m(\{\alpha_1, ..., \alpha_K\}) = \sum_{1 \le j_1 < j_2 <, ..., < j_m \le K} \alpha_{j_1} \alpha_{j_2} \cdots \alpha_{j_m}$$

are the symmetric elementary polynomials – see Turner (1966) for a derivation.

Remark 4.5.2 (Cumulant Function). Depending on the specific dynamics of the process, the numerical values of the state-wise moment-generating functions can be so large or small that it causes computational problems due to insufficient machine precision. A way to mitigate this problem is to log-scale the moment-generating functions and instead consider the state-wise cumulant functions defined as

$$\kappa_i(\lambda, t) = \log\left(m_i(\lambda, t)\right),$$

with differential equation

$$\frac{\partial}{\partial t}\kappa_i(\lambda, t) = \frac{\partial}{\partial t}m_i(\lambda, t)\frac{1}{m_i(\lambda, t)}.$$

Expressing the partial derivatives of $m_i(\lambda, t)$ in λ in terms of partial derivatives of $\kappa_i(\lambda, t)$ in λ is done via the relation

$$\begin{aligned}\frac{\partial^n}{\partial \lambda^n}m_i(0,t) =& m_i(0,t)\sum_{k=0}^{n} B_{n,k}\left(\frac{\partial}{\partial \lambda}\kappa_i(0,t), \frac{\partial^2}{\partial \lambda^2}\kappa_i(0,t), ..., \frac{\partial^{n-k+1}}{\partial \lambda^{n-k+1}}\kappa_i(0,t)\right)\\ =& p_i(t)\sum_{k=0}^{n} B_{n,k}\left(\frac{\partial}{\partial \lambda}\kappa_i(0,t), \frac{\partial^2}{\partial \lambda^2}\kappa_i(0,t), ..., \frac{\partial^{n-k+1}}{\partial \lambda^{n-k+1}}\kappa_i(0,t)\right)\end{aligned}$$

where $B_{n,k}(x_1, x_2, ..., x_{n-k+1})$ are the incomplete Bell Polynomials. △

An important remark is that the cut-off is not invariant to transformations of the moment-generating functions. The cut-off for the state-wise cumulant functions may therefore be closer to zero than the corresponding cut-off for the moment-generating functions. Wilcox and Bellman (1970) examine different ways of transforming the system of moment equations, and the influence on the truncation error. They find that for a stochastic nonlinear oscillator, using cumulants resulted in the smallest truncation error. This speaks in favour of using the cumulant functions instead of moment-generating functions. It is outside the scope of this paper to study the performance of different transformations of the moment-generating functions, but it is an obvious area of future research.

Based on the approximation of the partial derivatives in λ from Lemma 4.5.1, we can approximate the differential equations of Theorem 4.3.2 by the system of ODEs given in Definition 4.5.3.

Definition 4.5.3 (ODE Approximation of infinite PDE)**.** *Let M be given, and let the dynamics of*

$$W(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$$

be polynomial of degree p. For odd values of $K > M + p$, let $\vec{\lambda}$ be a vector of length K given by $\vec{\lambda} = (\alpha_1, \alpha_2, ..., \alpha_{K-1}, \alpha_K)^T \cdot h$, for $\alpha_{\lfloor K/2\rfloor+1=0}$ and $\alpha_i = -\alpha_{K-i+1} = i$. We define the approximating state-wise moment-generating function $\hat{m}_i$ as the function *that solves the ordinary differential equation*

$$
\begin{aligned}
&\frac{d}{dt}\hat{m}_i(\vec{\lambda}_{[q_1]}, \vec{\lambda}_{[q_2]}, t) \\
&= \sum_{r \neq i} \mu_{ri}(t)\hat{m}_r(\vec{\lambda}_{[q_1]}, \vec{\lambda}_{[q_2]}, t) - \mu_{ir}(t)\hat{m}_i(\vec{\lambda}_{[q_1]}, \vec{\lambda}_{[q_2]}, t) \\
&+ \sum_{k=0}^{M}\sum_{j=0}^{k}\sum_{l=0}^{p} \frac{1}{(k-j)!j!}\vec{\lambda}_{[q_1]}^{k-j+1}\vec{\lambda}_{[q_2]}^{j}\,{}_{xx}g_l^i(t)D^i_{[k-j+l+1,j+1]}(t) \\
&+ \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{p} \frac{1}{(k-j)!j!}\vec{\lambda}_{[q_1]}^{k-j+1}\vec{\lambda}_{[q_2]}^{j}\,{}_{xy}g_l^i(t)D^i_{[k-j+1,j+l+1]}(t) \\
&+ \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{p} \frac{1}{(k-j)!j!}\vec{\lambda}_{[q_1]}^{k-j}\vec{\lambda}_{[q_2]}^{j+1}\,{}_{yx}g_l^i(t)D^i_{[k-j+l+1,j+1]}(t) \\
&+ \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{p} \frac{1}{(k-j)!j!}\vec{\lambda}_{[q_1]}^{k-j}\vec{\lambda}_{[q_2]}^{j+1}\,{}_{yy}g_l^i(t)D^i_{[k-j+1,j+l+1]}(t) \\
&+ \sum_{k=0}^{\infty}\sum_{j_1+j_2+j_3=k} \frac{1}{j_1!j_2!j_3!}\sum_{r \neq i} D^r_{[j_1+1,j_2+1]}(t) \\
&\times (\vec{\lambda}_{[q_1]}(1 + {}_{xx}h_1^{ri}(t)) + \vec{\lambda}_{[q_2]}\,{}_{yx}h_1^{ri}(t))^{j_1} \\
&\times (\vec{\lambda}_{[q_1]}\,{}_{xy}h_1^{ri}(t) + \vec{\lambda}_{[q_2]}(1 + {}_{yy}h_1^{ri}(t)))^{j_2} \\
&\times (\vec{\lambda}_{[q_1]}({}_{xx}h_0^{ri}(t) + {}_{xy}h_0^{ri}(t)) + \vec{\lambda}_{[q_2]}({}_{yx}h_0^{ri}(t) + {}_{yy}h_0^{ri}(t)))^{j_3} \\
&\times \mu_{ri}(t) \\
&- \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{1}{(k-j)!j!}\vec{\lambda}_{[q_1]}^{k-j}\vec{\lambda}_{[q_2]}^{j}\sum_{r \neq i} D^r_{[k-j+1,j+1]}(t)\mu_{ri}(t) \\
\hat{m}_i(\vec{\lambda}_{[q_1]}, \vec{\lambda}_{[q_2]}, 0) &= \mathbb{1}_{\{i=0\}} e^{\vec{\lambda}_{[q_1]}X(0) + \vec{\lambda}_{[q_2]}Y(0)},
\end{aligned}
$$

where the i^{th} column of D^r is given by

$$D^r_{[,i]}(t) = (V^{-1}C)^T \cdot B^r_{[i,]}(t),$$

and the k'^{th} column of B^r is given by

$$B^r_{[,k]}(t) = (V^{-1}C)^T \cdot \hat{m}_r(\vec{\lambda}, \vec{\lambda}_{[k]}, t),$$

with matrices V and C given in Lemma 4.5.1. This implies that the k^{th} column of B is the vector of approximate partial derivatives in λ_1 for $\lambda_2 = \vec{\lambda}_{[k]}$,

$$B^r_{[,k]}(t) \approx \left(m_r((0, \vec{\lambda}[k])^T, t), \frac{\partial}{\partial \lambda_1} m_r\left((0, \vec{\lambda}[k])^T, t\right), ..., \frac{\partial^K}{\partial \lambda_1^K} m_r\left((0, \vec{\lambda}[k])^T, t\right)\right)^T,$$

and the indices of D are the approximate partial derivatives in λ_1 and λ_2 evaluated in zero,

$$D^r_{[i,k]}(t) \approx \frac{\partial^{i+k-2}}{\partial \lambda_1^{i-1} \partial \lambda_2^{k-1}} m_r(0, t).$$

Definition 4.5.3 is our main contribution as it combines our derivation of the infinite PDEs of Theorem 4.3.2, and the implementation-specific considerations of Section 4.4. After discretisation over the time parameter, it provides us with a finite difference scheme that approximates the moment-generating function in a three-dimensional grid of λ_1, λ_2 and t values. Using the calculations of Appendix 4.B, one can easily show that the scheme is consistent, assuming that the cut-off is zero. Showing that the scheme is stable is a subject of further research.

The ODE of Definition 4.5.3 does not utilize the standard tricks that serve to improve the performance of finite difference schemes, and there are several of these tricks that readily could be applied. To list a few ideas;

- A Crank-Nicolson type of implementation could be used, where the calculation of the forward differential equation relies on a solution that is consistent with a backwards differential equation for each step in time.
- We know that the $0^t h$ moment corresponds to the state-wise probabilities, that can be calculated using Kolmogorovs differential equations. For each calculation of the forward differential equation of $\widehat{m}_i$, a constraint could be enforced to ensure that
$$D^i_{[0,0]}(t) \approx m_i(0,t) = p_{0i}(0,t).$$
- Following the ideas of Bellman and Richardson (1968), there are also relations between different moments that should be maintained, e.g.
$$\mathrm{E}[\mathbb{1}_{\{Z(t)=i\}} X(t)^2] \geq \mathrm{E}[\mathbb{1}_{\{Z(t)=i\}} X(t)]^2.$$
- In general, there is a list of properties of the moment-generating function that we should try to make sure $\widehat{m}_i$ preserves. To list a few;
  - Chernoff bound
$$\mathrm{P}\left(X(t) \geq a(t)\right) \leq e^{-\lambda a(t)} \sum_{i \in \mathcal{J}} m_i(\lambda, t).$$
  - Jensens inequality
$$m_i(\lambda, t) \geq e^{\mathrm{E}[X(t)]\lambda}.$$
  - For non-negative processes we have that
$$\mathbb{1}_{\{Z(t)=i\}} X(t)^k \leq \left(\frac{k}{\lambda e}\right)^k \mathbb{1}_{\{Z(t)=i\}} e^{\lambda X(t)},$$
    and therefore
$$\frac{\partial^k}{\partial \lambda^k} m_i(0,t) \leq \left(\frac{k}{\lambda e}\right)^k m_i(\lambda, t).$$

4.6 Numerical Example

As discussed in Section 4.4, the system of differential equations where the cut-off is omitted is probably a good approximation of the true system of differential equations for the state-wise moment-generating functions, but it is still only an approximation of the true system of infinite partial differential equations. We test the performance of the approximation for two cases.

4.6.1 Set-up for Numerical Study

We use Definition 4.5.3 on a process that is relevant in the context of with-profit life insurance. For a state-proces Z representing the state of life of an insured, we consider a with-profit insurance contract with a savings account X and a surplus Y. We investigate a life-death model of a with-profit insurance contract, with state 0 representing 'alive' and state 1 representing 'dead' depicted in Figure 4.1.

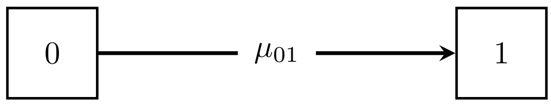

Figure 4.1: *Life-death model*

The policyholder is assumed to be alive at time zero. At the death of the policyholder, the remaining savings are transferred to the surplus, and no further payments are made to the savings account, i.e for $Z(s) = 1$, $\{X(t)\}_{s\leq t} = 0$. The insurance related functions and parameters that determine the dynamics of X and Y are not essential to the numerical results, but serve an interesting example. It is only essential that the dynamics of X and Y are polynomial. The parameters and functions that determine the dynamics of X and Y are given in Appendix 4.A.

In order to determine the true probability weighted state-wise expected values in the simple two state model, we condition on the time of the first jump and calculate the deterministic state wise values of $X(t)$ and $Y(t)$ for all $t \in (0, T]$. For jump-times that occur after t, we can calculate

$$\mathrm{E}[W(t)|Z(t) = 0], \tag{4.6.1}$$

and then multiply by $p_{00}(0, t)$ to get the probability weighted expected value of W in state 0. By aggregating and weighting the jump-time dependent values of X and Y with the jump-time density given jump before time t, we can form the expected values

$$\mathrm{E}[W(t)|Z(t) = 1], \tag{4.6.2}$$

and then multiply by $p_{01}(0,t)$ to get the probability weighted expected value of W in state 1. To ease notation, we define

$$\tilde{X}^0(t) := \mathrm{E}[\mathbb{1}_{\{Z(t)=0\}}X(t)], \tag{4.6.3}$$
$$\tilde{Y}^0(t) := \mathrm{E}[\mathbb{1}_{\{Z(t)=0\}}Y(t)], \tag{4.6.4}$$
$$\tilde{Y}^1(t) := \mathrm{E}[\mathbb{1}_{\{Z(t)=1\}}Y(t)]. \tag{4.6.5}$$

We do not include $\mathrm{E}[\mathbb{1}_{\{Z(t)=1\}}X(t)] = 0$ in our figures.

4.6.2 Affine Dynamics

To validate the use of state-wise moment-generating functions, we calculate the state-wise expected retrospective reserves in a setting where the dynamics of X and Y are affine. In Figure 4.2 we have plotted $\tilde{X}^0, \tilde{Y}^0$ and $\tilde{Y}^1$, calculated using the approximating state-wise moment-generating function of Definition 4.5.3.

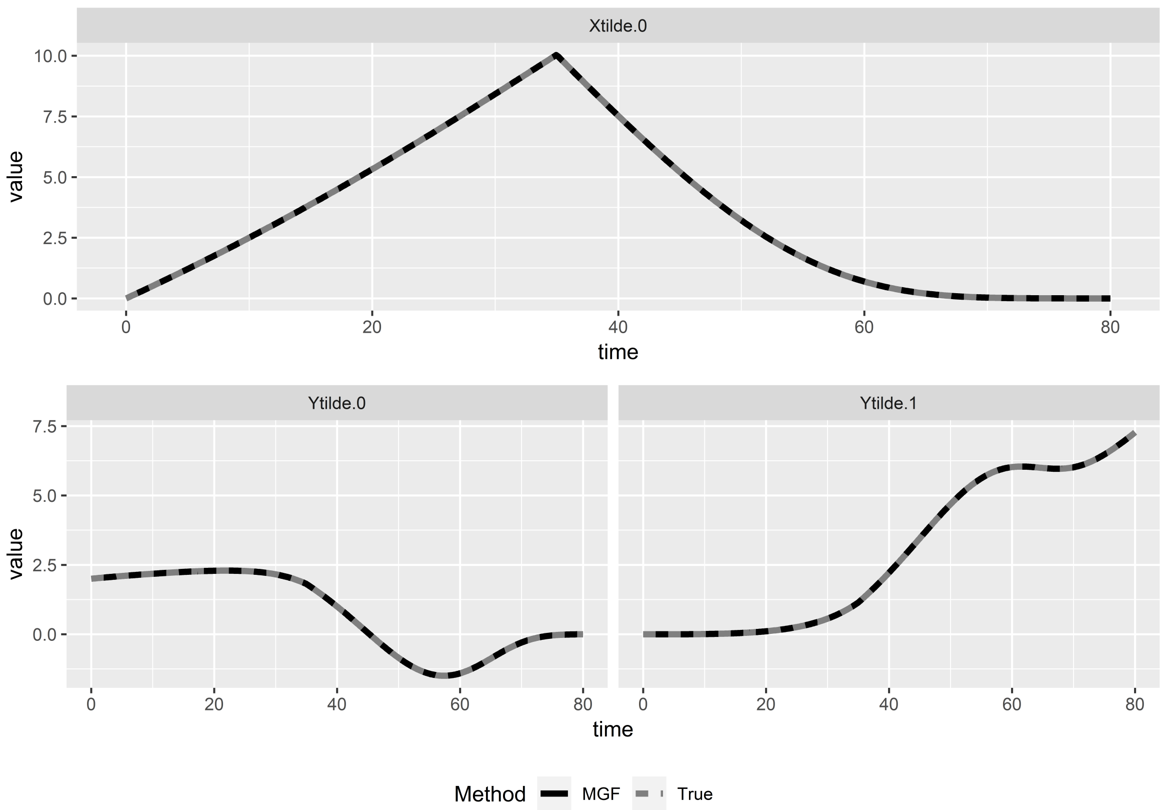

Figure 4.2: *Calculation of $\tilde{X}^0(t), \tilde{Y}^0(t)$ and $\tilde{Y}^1(t)$ using approximation of state-wise moment-generating function and true solution. The results were produced using a cut-off at $M = 8$, and $\vec{\lambda} = (\alpha_1, ..., \alpha_K)^t \cdot h$ where $\alpha_i = -\alpha_{k-i+1} = i$ for $K = 15$ and $h = 0.05$.*

The biggest absolute relative difference between the two calculation methods is $1.2 \cdot 10^{-12}$ and within the margin of numerical error.

As the dynamics of X and Y are affine, there is no need to resort to the state-wise moment-generating functions, but the example shows us that the the error introduced by ignoring the cut-off is small.

4.6.3 Polynomial Dynamics

To create non-affine polynomial dynamics, we introduce a payment from the surplus to the savings account which are the to so-called dividends. The dividend payment is determined by a polynomial approximation of the dividend function

$$\max(0, Y(t)) \cdot 0.1,$$

corresponding to a continuous payout of 10% of the surplus, if the surplus is positive. We perform a fifth-degree polynomial approximation of the $\max(0, x) \cdot 0.1$ function on the interval $(-14.85, 18.15)$, where the coefficients are determined using Remez algorithm, resulting in the $|| \cdot ||_\infty$ norm closest polynomial approximation. See Figure 4.5 for a plot of the dividend function and the corresponding polynomial approximation. The state-wise retrospective reserves calculated using the state-wise moment-generating functions are seen in Figure 4.3.

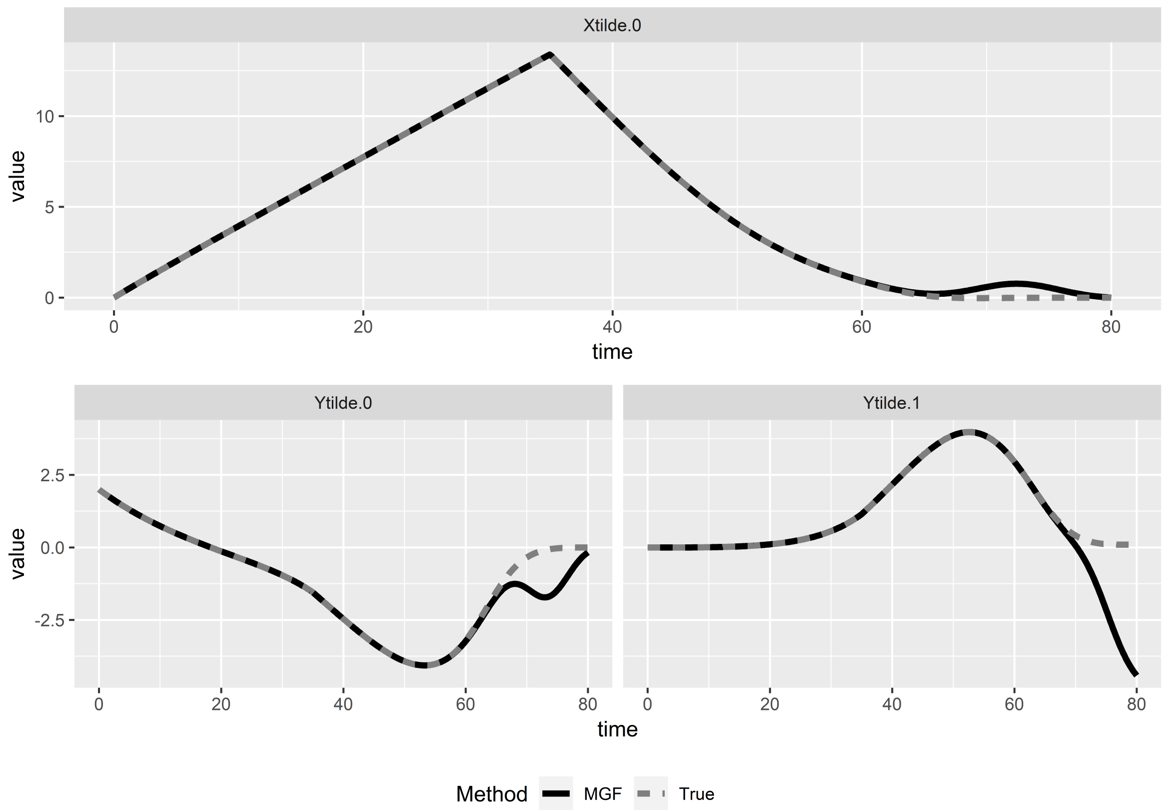

Figure 4.3: *Calculation of $\tilde{X}^0(t), \tilde{Y}^0(t)$ and $\tilde{Y}^1(t)$ using approximation of state-wise moment-generating function and true solution. The results were produced using a cut-off at $M = 8$, and $\vec{\lambda} = (\alpha_1, ..., \alpha_K)^t \cdot h$ where $\alpha_i = -\alpha_{k-i+1} = i$ for $K = 15$ and $h = 0.05$.*

Unlike the case with linear dynamics, there is a visible difference in all three state-wise reserves after $t = 64$. This error can come from three sources; ignoring the cut-off, discretisation and computing precision. Concerning computing precision, the absolute value of the inverse Vandermonde matrix ranges from $1.5 \cdot 10^{-6}$ to 1.5, and the C matrix ranges from 1 to $1.7 \cdot 10^{39}$, pushing the machine precision of standard software and hardware to its limits. The same cut-off value, Vandermonde matrix and C matrix were used in the previous example without any approximation error. This shows that the approximation error is highly sensitive to the degree of the polynomial dynamics, and that one should be careful about blindly accepting the ODE of Definition 4.5.3 as an accurate approximation of the true moment-generating function.

Examining the marginal moment-generating functions for X and Y for $t \in (0, 64]$ seen in Figure 4.4, we see that there are values of (λ, t) for which $\hat{m}_i$ is negative highlighting a flaw in the approximation, as the true state-wise moment-generating function is strictly positive.

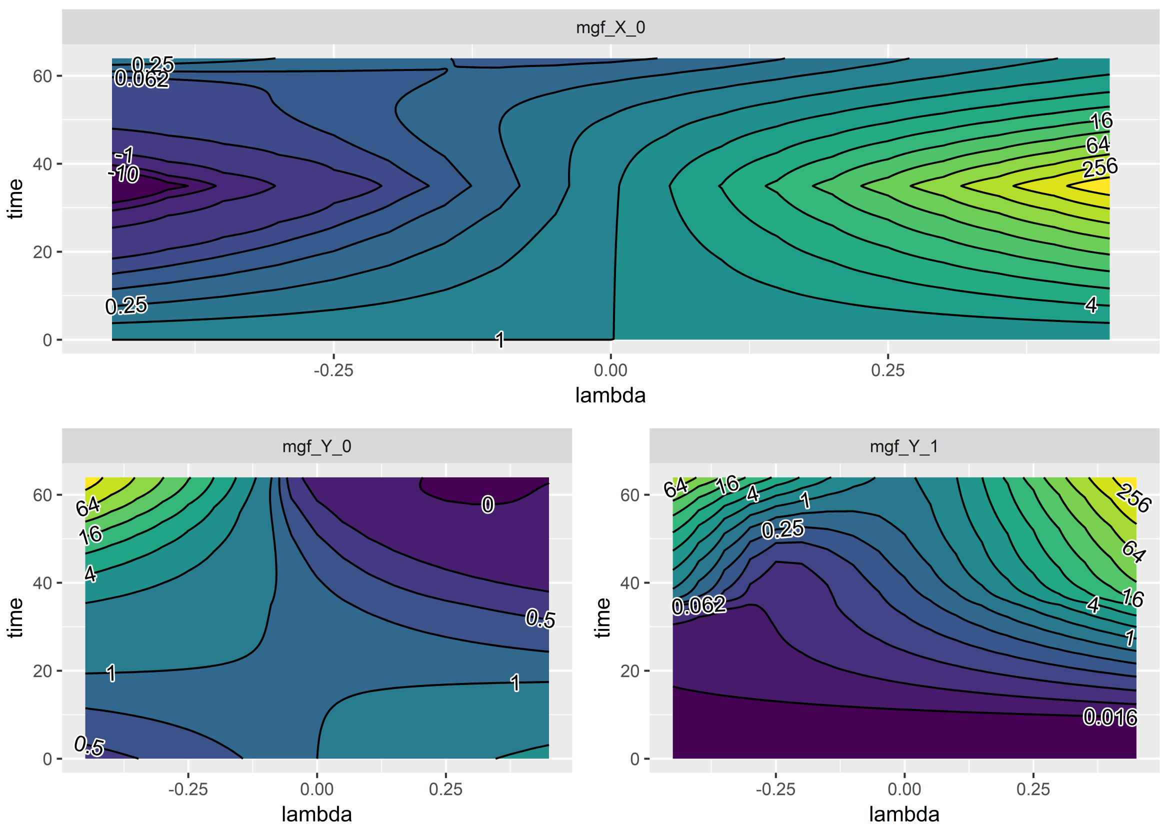

Figure 4.4: *Top: contour plot of $\widehat{m}_0(\lambda_1, 0, t)$, bottom left: contour plot of $\widehat{m}_0(0, \lambda_2, t)$, bottom right: contour plot of $\widehat{m}_1(0, \lambda_2, t)$*

We encourage the reader to replicate the example and to experiment with the parameters of the approximating PDE to get rid of the large approximation error for $t > 64$.

4.A Dynamics of Insurance Contract

The dynamics of the with-profit insurance contract savings account is

$$dX(t) = g_X^{Z(t)}(X(t), Y(t), t)dt + \sum_{k \neq Z(t-)} h_X^{Z(t-)k}(X(t-), Y(t-), t)dN^k,$$

where

$$\begin{aligned} g_X^j(x, y, t) =& r^*(t)x + \delta^j(x, y, t) - b_1^j(t) - \frac{x - V_1^{j*}(t)}{V_2^{j*}(t)} b_2^j(t) - \sum_{k \neq j} R^{jk}(x, t)\mu_{jk}^*(t), \\ h_X^{jk}(x, y, t) =& V_1^{*k}(t) - x + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)}, V_2^{*k}(t) + \delta^{jk}(x, y, t) \\ R^{jk}(x, t) =& b_1^{jk}(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)}(b_2^{jk}(t) + V_2^{*k}(t)) + V_1^{*k}(t) - x. \end{aligned}$$

The functions b_i^j and b_i^{jk} respectively represent payments while Z is in state j and on transition from state j to state k. The dividends that flow from the surplus to the savings account are determined by the dividend functions δ^j. The reserves V_i^{*j} are given by

$$V_i^{*j}(t) = \int_t^n e^{-\int_t^s r^*(\nu)d\nu} \sum_{k \in \mathcal{J}} p_{jk}(t, s) \left(b_i^k(s) + \sum_{l: l \neq k} \mu_{kl}(s) b_i^{kl}(s) \right) ds.$$

The dynamics of the surplus are

$$dY(t) = g_Y^{Z(t)}(X(t), Y(t), t)dt + \sum_{k \neq Z(t-)} h_Y^{Z(t-)k}(X(t-), Y(t-), t)dN^k,$$

where

$$\begin{aligned} g_Y^j(x, y, t) =& r(t)y + (r(t) - r^*(t))x - \delta^j(x, y, t) + \sum_{k \neq Z(t-)} R^{Z(t-)k}(X(t), t)\mu_{jk}^*(t), \\ h_Y^{jk}(x, y, t) =& -R^{jk}(x, t) - \delta^{jk}(x, y, t). \end{aligned}$$

Notice that the only terms that are not affine in x and y are the δ-functions which represent the dividend. It is therefore solely through the specification of the dividend that the dynamics of X and Y may become polynomial.

We construct a policy with a life assurance on death before retirement at age 65 which is not scaled with bonus, and a life annuity commencing at retirement that is scaled with bonus. Before retirement, an equivalence premium is paid by the policyholder. All elements of the insurance contract are presented in Table 4.1.

Quantity	Value	Description
T	110	Termination of contract
a_0	30	Age at time zero
T_{ret}	65	Age of retirement
$b_1^j(t)$	$\mathbb{1}_{\{j=0\}}(\mathbb{1}_{\{t\geq T_{ret}\}}\cdot 0.8-\pi\mathbb{1}_{\{t<T_{ret}\}})$	Unscaled sojourn payment in state j
$b_2^j(t)$	$\mathbb{1}_{\{j=0\}}(\mathbb{1}_{\{t\geq T_{ret}\}}\cdot 0.8)$	Scaled sojourn payment in state j
$b_1^{01}(t)$	$\mathbb{1}_{\{t\leq 35\}}\cdot 3$	Unscaled payment on transition from 0 to 1
$b_2^{01}(t)$	0	Scaled payment on transition from 0 to 1
π	0.2379553	Equivalence premium
$r^*(t)$	0.015	Technical basis interest
$r(t)$	$0.01+\frac{t\cdot 0.015}{110-30}$	Market basis interest
$\mu_{01}^*(t)$	$0.0005+10^{5.6+0.04\cdot(s+a_0)-10}$	Technical basis force of mortality
$\mu_{01}(t)$	$0.9\cdot(0.0005+10^{5.6+0.04\cdot(s+a_0)-10})$	Market basis force of mortality

Table 4.1: *Parameters used in numerical section.*

4.B Small Local Truncation Error Finite Difference Scheme

In this section we construct a finite difference scheme that approximates the partial derivatives $\frac{\partial^n}{\partial\lambda^n}m_i(0,t)$ via a linear combinations of $m_i(\vec{\lambda}[q],t)$ for some vector of nodes $\vec{\lambda}$. The results in this section are not new – see *Introduction to Numerical Methods in Differential Equations* (2007) for an introduction to numerical methods in differential equations – but they serve an important role as they make our contributions relevant for the practitioner. The basic principle behind the finite difference scheme is based on the expression of a function through its Taylor expansion. Let

$$\vec{\lambda}=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_{K-1}\\ \alpha_K\end{pmatrix}h,$$

where we call h the step size and α_i the step repetitions. For $\vec{\lambda} \in \mathbb{R}^K$, were K is an odd number, we wish to approximate the n^{th} order derivative using a weighted sum of known values of $m_i(\vec{\lambda}, t)$

$$\begin{aligned}\tilde{m}_i^{(n)}(0,t) =& \frac{a_1 m_i(\alpha_1 h, t) + a_2 m(\alpha_2 h, t) + \cdots + a_K m_i(\alpha_K h, t)}{h^n} \\ =& (\vec{a})^T \frac{m_i(\vec{\lambda}, t)}{h^n} \\ =& \frac{\partial^n}{\partial \lambda^n} m_i(0,t) + O(h^?)\end{aligned}$$

for some vector of weights $\vec{a}$ yet to be determined, and some order of convergence rate in h that should be as high as possible. We use the set of nodes defined by $\alpha_i = -\alpha_{K-i+1} = i$, and $\alpha_{\lfloor K/2 \rfloor + 1} = 0$, but the results of this section apply to any set of α_i that satisfy $\alpha_i = -\alpha_{K-i+1}$. By choosing the nodes most efficiently, we can decrease the local truncation error by one order, making the finite difference approximation better. However, the most efficient values of α_i depend on the order of the derivative that we wish to approximate, and as we need to calculate many orders of derivatives, it is convenient to consider fixed values of α_i. By a Taylor expansion we see that

$$\begin{aligned}m_i(\alpha_i h, t) =& m_i(0,t) + m_i'(0,t)\alpha_i h + \frac{m_i''(0,t)}{2!}(\alpha_i h)^2 + \frac{m_i^{(3)}(0,t)}{3!}(\alpha_i h)^3 + \cdots \\ &+ \frac{m_i^{(K)}(0,t)}{K!}(\alpha_i h)^K + O(h^K).\end{aligned}$$

Therefore we may write

$$\begin{aligned}\tilde{m}_i^{(n)}(0,t) =& \frac{a_1 \left(m_i(0,t) + m_i'(0,t)\alpha_1 h + \cdots + \frac{m_i^{(K)}(0,t)}{K!}(\alpha_1 h)^K + O(h^K) \right)}{h^n} \\ &+ \frac{a_2 \left(m_i(0,t) + m_i'(0,t)\alpha_2 h + \cdots + \frac{m_i^{(K)}(0,t)}{K!}(\alpha_2 h)^K + O(h^K) \right)}{h^n} \\ &\vdots \\ &+ \frac{a_K \left(m_i(0,t) + m_i'(0,t)\alpha_K h + \cdots + \frac{m_i^{(K)}(0,t)}{K!}(\alpha_K h)^K + O(h^K) \right)}{h^n} \\ =& \frac{c_1 m(0,t) + c_2 m'(0,t) h + \cdots + c_{K+1} \frac{m_i^{(K)}(0,t)}{K!} h^K}{h^n} + O(h^{K-n}),\end{aligned}$$

for c_i and a_i that solve

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{K-1} & \alpha_K \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_{K-1}^2 & \alpha_K^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \alpha_1^{K-1} & \alpha_2^{K-1} & \cdots & \alpha_{K-1}^{K-1} & \alpha_K^{K-1} \\ \alpha_1^K & \alpha_2^K & \cdots & \alpha_{K-1}^K & \alpha_K^K \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{K-1} \\ a_K \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{K-1} \\ c_K \\ c_{K+1} \end{pmatrix}. \tag{4.B.1}$$

Choosing

$$c_i = \begin{cases} n!, & \text{for } i = n+1 \\ 0, & \text{otherwise} \end{cases}, \tag{4.B.2}$$

we see that

$$\begin{aligned} \tilde{m}_i^{(n)}(0,t) =& \frac{n! \dfrac{m_i^{(n)}(0,t)h^n}{n!}}{h^n} + O(h^{K-n}) \\ =& m_i^{(n)}(0,t) + O(h^{K-n}). \end{aligned}$$

Thus we have to find a_i that solve (4.B.1) for c_i given by (4.B.2) in order to get the smallest possible error. However, (4.B.1) is an overdetermined system of $K+1$ equations with K unknowns, which is not guaranteed to have a solution. At the cost of accuracy, we discard the requirement for c_{K+1} and reduce the system to

$$\tilde{m_i}^{(n)}(0,t) = \frac{c_1 m(0,t) + c_2 m'(0,t)h + \cdots + c_K \dfrac{m_i^{(K-1)}(0,t)}{(K-1)!}(h)^{K-1}}{h^n} + O(h^{K-n-1}),$$

such that finding a_i amounts to inverting the Vandermonde matrix

$$V := \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{K-1} & \alpha_K \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_{K-1}^2 & \alpha_K^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \alpha_1^{K-1} & \alpha_2^{K-1} & \cdots & \alpha_{K-1}^{K-1} & \alpha_K^{K-1} \end{pmatrix},$$

which can be done using explicit formulas. Now $\tilde{m_i}^{(n)}(0,t)$ can be determined by

$$\tilde{m_i}^{(n)}(0,t) = (V^{-1}C_n)^T m_i(\vec{\lambda}, t)$$

where C_n is the n^{th} column of the diagonal matrix

$$C = \text{Diag}\left(\left\{\frac{0!}{h^0}, \frac{1!}{h^1}, \frac{2!}{h^2}, ..., \frac{(K-1)!}{h^{K-1}}\right\}\right).$$

Note that for $A = V^{-1}C$,

$$\left(A^T m_i(\vec{\lambda}, t)\right)^T = (\tilde{m_i}^{(0)}(0,t), \tilde{m_i}^{(1)}(0,t), ..., \tilde{m_i}^{(K-2)}(0,t), \tilde{m_i}^{(K-1)}(0,t))$$

providing us with an approximation for all derivatives in a single matrix multiplication. An attractive feature in terms of practical implementations is that A only has to be calculated once for the given choice of step repetitions α_i.

4.C Proof of Theorem 4.3.2

Proof. Consider a two-dimensional process W, with dynamics

$$dW(s) = \sum_{l=0}^{p} g_l^{Z(s)}(s)W(s)^l + \sum_{l=0}^{p_j} \sum_{j\neq Z(s-)} h_l^{Z(s-)k}(s)W(s)^l dN^j(s),$$

where $W(s)^l$ are the element-wise powers of W, and

$$g_l^r(s) = \begin{pmatrix} {}_{xx}g_l^r(s) & {}_{xy}g_l^r(s) \\ {}_{yx}g_l^r(s) & {}_{yy}g_l^r(s) \end{pmatrix}$$

$$h_l^{rj}(s) = \begin{pmatrix} {}_{xx}h_l^{rj}(s) & {}_{xy}h_l^{rj}(s) \\ {}_{yx}h_l^{rj}(s) & {}_{yy}h_l^{rj}(s) \end{pmatrix}.$$

The dynamics of the moment-generating function are by Itô's Lemma for FV-functions

$$\begin{aligned}
d(\exp(\lambda^T W(s))) =& \lambda_1 \sum_{l=0}^{p}\sum_{k=0}^{\infty}\sum_{q=0}^{k} \frac{1}{(k-q)!q!}\lambda_1^{k-q}X(s)^{k-q+l}\lambda_2^q Y(s)^q \; {}_{xx}g_l^{Z(s)}(s) \\
&+ \lambda_1 \sum_{l=0}^{p}\sum_{k=0}^{\infty}\sum_{q=0}^{k} \frac{1}{(k-q)!q!}\lambda_1^{k-q}X(s)^{k-q}\lambda_2^q Y(s)^{q+l} \; {}_{xy}g_l^{Z(s)}(s) \\
&+ \lambda_2 \sum_{l=0}^{p}\sum_{k=0}^{\infty}\sum_{q=0}^{k} \frac{1}{(k-q)!q!}\lambda_1^{k-q}X(s)^{k-q+l}\lambda_2^q Y(s)^q \; {}_{yx}g_l^{Z(s)}(s) \\
&+ \lambda_2 \sum_{l=0}^{p}\sum_{k=0}^{\infty}\sum_{q=0}^{k} \frac{1}{(k-q)!q!}\lambda_1^{k-q}X(s)^{k-q}\lambda_2^q Y(s)^{q+l} \; {}_{yy}g_l^{Z(s)}(s) \\
&+ \sum_{j\neq Z(s-)} \left(e^{\lambda^T W(s)} - e^{\lambda^T W(s-)}\right) dN^j(s),
\end{aligned}$$

The last line accounts for the jump-dynamics, which we deal with separately;

$$
\begin{aligned}
&\sum_{j\neq Z(s-)} \left(e^{\lambda^T W(s)} - e^{\lambda^T W(s-)}\right) dN^j(s)\\
&= \sum_{j\neq Z(s-)} \left(\sum_{k=0}^{\infty} \frac{(\lambda_1 X(s) + \lambda_2 Y(s))^k}{k!} - \sum_{k=0}^{\infty} \frac{(\lambda_1 X(s-) + \lambda_2 Y(s-))^k}{k!}\right) dN^j(s).
\end{aligned}
$$

We now insert

$$
\begin{aligned}
\sum_{j\neq Z(s-)} X(s) dN^j(s) = \sum_{j\neq Z(s-)} X(s-) + \sum_{l=0}^{p_j} \Big({}_{xx}h_l^{Z(s-)j}(s) X(s-)^l &\\
+ {}_{xy}h_l^{Z(s-)j}(s) Y(s-)^l \Big) dN^j(s)&,\\
\sum_{j\neq Z(s-)} Y(s) dN^j(s) = \sum_{j\neq Z(s-)} Y(s-) + \sum_{l=0}^{p_j} \Big({}_{yx}h_l^{Z(s-)j}(s) X(s-)^l &\\
+ {}_{yy}h_l^{Z(s-)j}(s) Y(s-)^l \Big) dN^j(s)&,
\end{aligned}
$$

to get

$$
\begin{aligned}
&\sum_{j\neq Z(s-)} \left(\sum_{k=0}^{\infty} \frac{(\lambda_1 X(s) + \lambda_2 Y(s))^k}{k!} - \sum_{k=0}^{\infty} \frac{(\lambda_1 X(s-) + \lambda_2 Y(s-))^k}{k!}\right) dN^j(s)\\
&= \sum_{j\neq Z(s-)} \sum_{k=0}^{\infty} \frac{1}{k!} \Bigg[\lambda_1 \left(X(s-) + \sum_{l=0}^{p_j} {}_{xx}h_l^{Z(s-)j}(s) X(s-)^l + {}_{xy}h_l^{Z(s-)j}(s) Y(s-)^l \right)\\
&+ \lambda_2 \left(Y(s-) + \sum_{l=0}^{p_j} {}_{yx}h_l^{Z(s-)j}(s) X(s-)^l + {}_{yy}h_l^{Z(s-)j}(s) Y(s-)^l \right) \Bigg]^k dN^j(s)\\
&- \sum_{j\neq Z(s-)} \sum_{k=0}^{\infty} \left(\frac{(\lambda_1 X(s-) + \lambda_2 Y(s-))^k}{k!} \right) dN^j(s).
\end{aligned}
$$

Raising the sum of the $4(p_j+1)+2$ terms within the square brackets to the power of k, can be done with the multinomial theorem,

$$
\left(\sum_{q=0}^{4(p_j+1)+2} x_q \right)^k = \sum_{q_1+q_2+\ldots+q_{4(p_j+1)+2}=k} \binom{k}{q_0, q_1, \ldots, q_{4(p_j+1)+2}} \prod_{l=0}^{4(p_j+1)+2} x_l^{q_l}.
$$

For $p_j = 1$, corresponding to affine dynamics on transition between states we get

$$
\begin{aligned}
&\sum_{j\neq Z(s-)}\sum_{k=0}^{\infty}\frac{1}{k!}\Bigg[\lambda_1\left(X(s-)+\sum_{l=0}^{p_j}{}_{xx}h_l^{Z(s-)j}(s)X(s-)^l+{}_{xy}h_l^{Z(s-)j}(s)Y(s-)^l\right)\\
&+\lambda_2\left(Y(s-)+\sum_{l=0}^{p_j}{}_{yx}h_l^{Z(s-)j}(s)X(s-)^l+{}_{yy}h_l^{Z(s-)j}(s)Y(s-)^l\right)\Bigg]^k dN^j(s)\\
&-\sum_{j\neq Z(s-)}\sum_{k=0}^{\infty}\frac{(\lambda_1X(s-)+\lambda_2Y(s-))^k}{k!}dN^j(s)\\
=&\sum_{j\neq Z(s-)}\sum_{k=0}^{\infty}\frac{1}{k!}\Bigg[\lambda_1\,{}_{xx}h_0^{Z(s-)j}(s)+\lambda_1\,{}_{xy}h_0^{Z(s-)j}(s)\\
&+\lambda_2\,{}_{yx}h_0^{Z(s-)j}(s)+\lambda_2\,{}_{yy}h_0^{Z(s-)j}(s)\\
&\qquad+X(s-)\left(\lambda_1(1+{}_{xx}h_1^{Z(s-)j}(s))+\lambda_2\,{}_{yx}h_1^{Z(s-)j}(s)\right)\\
&\qquad+Y(s-)\left(\lambda_1({}_{xy}h_1^{Z(s-)j}(s))+\lambda_2(1+{}_{yy}h_1^{Z(s-)j}(s)\right)\Bigg]^k dN^j(s)\\
&-\sum_{j\neq Z(s-)}\sum_{k=0}^{\infty}\frac{(\lambda_1X(s-)+\lambda_2Y(s-))^k}{k!}dN^j(s),
\end{aligned}
$$

we now use the trinomial and binomial theorems to write out the powers of k

$$
\begin{aligned}
\frac{(\lambda_1X(s-)+\lambda_2Y(s-))^k}{k!}&=\sum_{q=0}^{k}\frac{1}{(k-q)!q!}\lambda_1^{k-q}\lambda_2^qX(s-)^{k-q}Y(s-)^q\\
(A(s)+B(s)+C(s))^k&=\sum_{q_1+q_2+q_3=k}\frac{k!}{q_1!q_2!q_3!}A(s)^{q_1}B(s)^{q_2}C(s)^{q_3}
\end{aligned}
$$

where

$$
\begin{aligned}
A(s)&=X(s-)\left(\lambda_1(1+{}_{xx}h_1^{Z(s-)j}(s))+\lambda_2\,{}_{yx}h_1^{Z(s-)j}(s)\right),\\
B(s)&=Y(s-)\left(\lambda_1\,{}_{xy}h_1^{Z(s-)j}(s)+\lambda_2(1+{}_{yy}h_1^{Z(s-)j}(s))\right),\\
C(s)&=\lambda_1\,{}_{xx}h_0^{Z(s-)j}(s)+\lambda_1\,{}_{xy}h_0^{Z(s-)j}(s)+\lambda_2\,{}_{yx}h_0^{Z(s-)j}(s)+\lambda_2\,{}_{yy}h_0^{Z(s-)j}(s).
\end{aligned}
$$

We now write out the dynamics of $e^{\lambda^T W(s)}$

$$
\begin{aligned}
d(\exp(\lambda^T W(s))) =& \lambda_1 \sum_{l=0}^{p}\sum_{k=0}^{\infty}\sum_{q=0}^{k} \frac{1}{(k-q)!q!} \lambda_1^{k-q} X(s)^{k-q+l} \lambda_2^q Y(s)^q \, {}_{xx}g_l^{Z(s)}(s) \\
&+ \lambda_1 \sum_{l=0}^{p}\sum_{k=0}^{\infty}\sum_{q=0}^{k} \frac{1}{(k-q)!q!} \lambda_1^{k-q} X(s)^{k-q} \lambda_2^q Y(s)^{q+l} \, {}_{xy}g_l^{Z(s)}(s) \\
&+ \lambda_2 \sum_{l=0}^{p}\sum_{k=0}^{\infty}\sum_{q=0}^{k} \frac{1}{(k-q)!q!} \lambda_1^{k-q} X(s)^{k-q+l} \lambda_2^q Y(s)^q \, {}_{yx}g_l^{Z(s)}(s) \\
&+ \lambda_2 \sum_{l=0}^{p}\sum_{k=0}^{\infty}\sum_{q=0}^{k} \frac{1}{(k-q)!q!} \lambda_1^{k-q} X(s)^{k-q} \lambda_2^q Y(s)^{q+l} \, {}_{yy}g_l^{Z(s)}(s) \\
&\sum_{j\neq Z(s-)} \sum_{k=0}^{\infty} \sum_{q_1+q_2+q_3=k} \frac{1}{q_1!q_2!q_3!} \\
&\times X(s-)^{q_1} \left(\lambda_1(1 + {}_{xx}h_1^{Z(s-)j}(s)) + \lambda_2 \, {}_{yx}h_1^{Z(s-)j}(s)\right)^{q_1} \\
&\times Y(s-)^{q_2} \left(\lambda_1 \, {}_{xy}h_1^{Z(s-)j}(s) + \lambda_2(1 + {}_{yy}h_1^{Z(s-)j}(s))\right)^{q_2} \\
&\times \big(\lambda_1 \, {}_{xx}h_0^{Z(s-)j}(s) + \lambda_1 \, {}_{xy}h_0^{Z(s-)j}(s) \\
&+ \lambda_2 \, {}_{yx}h_0^{Z(s-)j}(s) + \lambda_2 \, {}_{yy}h_0^{Z(s-)j}(s)\big)^{q_3} dN^j(s) \\
&- \sum_{j\neq Z(s-)} \sum_{k=0}^{\infty}\sum_{q=0}^{k} \frac{1}{(k-q)!q!} \lambda_1^{k-q}\lambda_2^q X(s-)^{k-q} Y(s-)^q dN^j(s).
\end{aligned}
$$

Which we can use to get an integral expression for $m_i(\lambda, t)$

$$
m_i(\lambda, t) = \mathrm{E}[\mathbb{1}_{\{Z(t)=i\}} e^{\lambda^T W(0)}] + \mathrm{E}\left[\mathbb{1}_{\{Z(t)=i\}} \int_0^t d(\exp(\lambda^T W(s)))\right].
$$

As the integrand in the integrals accounting for the jump dynamics is predictable, we may integrate wrt. the $\sigma\{\{Z(\tau)\}_{\tau\leq s}, Z(t) = i\}$-predictable compensator for the FV-process $\mathbb{1}_{\{Z(t)=i\}} N^{rj}(s)$ given by

$$
\mathbb{1}_{\{Z(t)=i\}} \mathbb{1}_{\{Z(s-)=r\}} \mu_{rj}(s) \frac{p_{ji}(s,t)}{p_{ri}(s,t)},
$$

as derived in Appendix A of Bruhn and Lollike (2021). Using the Markov property, and the dominated convergence theorem to replace the expectation of the moments of X and Y with derivatives of the moment-generating function, we are left with the expression

$$
\begin{aligned}
&m_i(\lambda,t)\\
=&p_{0i}(0,t)e^{\lambda^T w_0}\\
&+\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{\lambda_1^{k-q+1}\lambda_2^q}{(k-q)!q!}\int_0^t\sum_{r\in\mathcal{J}}p_{ri}(s,t)\,{}_{xx}g_l^r(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q+l}\partial\lambda_2^q}m_r(0,s)ds\\
&+\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{\lambda_1^{k-q+1}\lambda_2^q}{(k-q)!q!}\int_0^t\sum_{r\in\mathcal{J}}p_{ri}(s,t)\,{}_{xy}g_l^r(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q}\partial\lambda_2^{q+l}}m_r(0,s)ds\\
&+\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{\lambda_1^{k-q}\lambda_2^{q+1}}{(k-q)!q!}\int_0^t\sum_{r\in\mathcal{J}}p_{ri}(s,t)\,{}_{yx}g_l^r(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q+l}\partial\lambda_2^q}m_r(0,s)ds\\
&+\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{\lambda_1^{k-q}\lambda_2^{q+1}}{(k-q)!q!}\int_0^t\sum_{r\in\mathcal{J}}p_{ri}(s,t)\,{}_{yy}g_l^r(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q}\partial\lambda_2^{q+l}}m_r(0,s)ds\\
&+\sum_{k=0}^{\infty}\sum_{q_1+q_2+q_3=k}\frac{1}{q_1!q_2!q_3!}\int_0^t\sum_{r\in\mathcal{J}}\sum_{j\neq r}\frac{\partial^{q_1+q_2}}{\partial\lambda_1^{q_1}\partial\lambda_2^{q_2}}m_r(0,s)\\
&\times(\lambda_1(1+{}_{xx}h_1^{rj}(s))+\lambda_2\,{}_{yx}h_1^{rj}(s))^{q_1}\\
&\times(\lambda_1\,{}_{xy}h_1^{rj}(s)+\lambda_2(1+{}_{yy}h_l^{rj}(s)))^{q_2}\\
&\times(\lambda_1({}_{xx}h_0^{rj}(s)+{}_{xy}h_0^{rj}(s))+\lambda_2({}_{yx}h_0^{rj}(s)+{}_{yy}h_0^{rj}(s)))^{q_3}\\
&\times\mu_{rj}(s)p_{ji}(s,t)ds\\
&-\sum_{k=0}^{\infty}\sum_{q=0}^{k}\frac{\lambda_1^{k-q}\lambda_2^q}{(k-q)!q!}\int_0^t\sum_{r\in\mathcal{J}}\sum_{j\neq r}\frac{\partial^k}{\partial\lambda_1^{k-q}\partial\lambda_2^q}m_r(0,s)\mu_{rj}(s)p_{ji}(s,t)ds.
\end{aligned}
$$

Which we can differentiate with respect to t. We split the differentiation into three parts. First we consider only the probability weighted initial value

$$
\frac{\partial}{\partial t}p_{0i}(0,t)e^{\lambda^T w_0}=\sum_{\substack{q\in\mathcal{J}\\ q\neq i}}\mu_{qi}(t)p_{0q}(0,t)e^{\lambda^T w_0}-\mu_{iq}(t)p_{0i}(0,t)e^{\lambda^T w_0}.
$$

Then we consider the derivative of the terms accounting for the continuous dynamics

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Bigg\{\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q+1}\lambda_2^{q}\int_0^t\sum_{r\in\mathcal{J}}p_{ri}(s,t)\,{}_{xx}g_l^r(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q+l}\partial\lambda_2^{q}}m_r(0,s)ds\\
&\quad+\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q+1}\lambda_2^{q}\int_0^t\sum_{r\in\mathcal{J}}p_{ri}(s,t)\,{}_{xy}g_l^r(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q}\partial\lambda_2^{q+l}}m_r(0,s)ds\\
&\quad+\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q}\lambda_2^{q+1}\int_0^t\sum_{r\in\mathcal{J}}p_{ri}(s,t)\,{}_{yx}g_l^r(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q+l}\partial\lambda_2^{q}}m_r(0,s)ds\\
&\quad+\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q}\lambda_2^{q+1}\int_0^t\sum_{r\in\mathcal{J}}p_{ri}(s,t)\,{}_{yy}g_l^r(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q}\partial\lambda_2^{q+l}}m_r(0,s)ds\Bigg\}\\
&=\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q+1}\lambda_2^{q}\,{}_{xx}g_l^i(t)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q+l}\partial\lambda_2^{q}}m_i(0,t)\\
&\quad+\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q+1}\lambda_2^{q}\,{}_{xy}g_l^i(t)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q}\partial\lambda_2^{q+l}}m_i(0,t)\\
&\quad+\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q}\lambda_2^{q+1}\,{}_{yx}g_l^i(t)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q+l}\partial\lambda_2^{q}}m_i(0,t)\\
&\quad+\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q}\lambda_2^{q+1}\,{}_{yy}g_l^i(t)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q}\partial\lambda_2^{q+l}}m_i(0,t)\\
&+\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q+1}\lambda_2^{q}\int_0^t\sum_{r\in\mathcal{J}}\frac{\partial}{\partial t}p_{ri}(s,t)\,{}_{xx}g_l^r(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q+l}\partial\lambda_2^{q}}m_r(0,s)ds\\
&\quad+\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q+1}\lambda_2^{q}\int_0^t\sum_{r\in\mathcal{J}}\frac{\partial}{\partial t}p_{ri}(s,t)\,{}_{xy}g_l^r(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q}\partial\lambda_2^{q+l}}m_r(0,s)ds\\
&\quad+\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q}\lambda_2^{q+1}\int_0^t\sum_{r\in\mathcal{J}}\frac{\partial}{\partial t}p_{ri}(s,t)\,{}_{yx}g_l^r(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q+l}\partial\lambda_2^{q}}m_r(0,s)ds\\
&\quad+\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q}\lambda_2^{q+1}\int_0^t\sum_{r\in\mathcal{J}}\frac{\partial}{\partial t}p_{ri}(s,t)\,{}_{yy}g_l^r(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q}\partial\lambda_2^{q+l}}m_r(0,s)ds.
\end{aligned}
$$

For notational ease, consider a single pair of lines in the equation that both contain ${}_{\diamondsuit\diamondsuit}g_l^m(s)$ for $\diamondsuit\in\{x,y\}$, for instance $\diamondsuit\diamondsuit=xx$. By Kolmogorov's forward differential equation,

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q+1}\lambda_2^{q}\,{}_{xx}g_l^{i}(t)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q+l}\partial\lambda_2^{q}}m_i(0,t)\\
&\int_0^t\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q+1}\lambda_2^{q}\sum_{r\in\mathcal{J}}\frac{\partial}{\partial t}p_{ri}(s,t)\,{}_{xx}g_l^{m}(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q+l}\partial\lambda_2^{q}}m_r(0,s)ds\\
=&\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{1}{(k-q)!q!}\lambda_1^{k-q+1}\lambda_2^{q}\,{}_{xx}g_l^{i}(t)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q+l}\partial\lambda_2^{q}}m_i(0,t)\\
&-\sum_{\substack{q\in\mathcal{J}\\ q\neq i}}\mu_{iq}(t)\int_0^t\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{\lambda_1^{k-q+1}\lambda_2^{q}}{(k-q)!q!}\sum_{r\in\mathcal{J}}p_{ri}(s,t)\,{}_{xx}g_l^{m}(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q+l}\partial\lambda_2^{q}}m_r(0,s)ds\\
&+\sum_{\substack{q\in\mathcal{J}\\ q\neq i}}\mu_{qi}(t)\int_0^t\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{l=0}^{p}\frac{\lambda_1^{k-q+1}\lambda_2^{q}}{(k-q)!q!}\sum_{r\in\mathcal{J}}p_{rq}(s,t)\,{}_{xx}g_l^{m}(s)\frac{\partial^{k+l}}{\partial\lambda_1^{k-q+l}\partial\lambda_2^{q}}m_r(0,s)ds.
\end{aligned}
$$

The calculations for $\Diamond\Diamond \in \{xy, yy, yx\}$ are analogous. Differentiating the terms accounting for the jump-dynamics we get

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Bigg\{\sum_{k=0}^{\infty}\sum_{q_1+q_2+q_3=k}\frac{1}{q_1!q_2!q_3!}\int_0^t\sum_{r\in\mathcal{J}}\sum_{j\neq r}\frac{\partial^{q_1+q_2}}{\partial\lambda_1^{q_1}\partial\lambda_2^{q_2}}m_r(0,s)\\
&\times(\lambda_1(1+{}_{xx}h_1^{rj}(s))+\lambda_2\,{}_{yx}h_1^{rj}(s))^{q_1}\\
&\times(\lambda_1\,{}_{xy}h_1^{rj}(s)+\lambda_2(1+{}_{yy}h_l^{rj}(s)))^{q_2}\\
&\times(\lambda_1({}_{xx}h_0^{rj}(s)+{}_{xy}h_0^{rj}(s))+\lambda_2({}_{yx}h_0^{rj}(s)+{}_{yy}h_0^{rj}(s)))^{q_3}\\
&\times\mu_{rj}(s)p_{ji}(s,t)ds\\
&-\sum_{k=0}^{\infty}\sum_{q=0}^{k}\frac{1}{(k-q)!q!}\lambda_1^{k-q}\lambda_2^{q}\int_0^t\sum_{r\in\mathcal{J}}\sum_{j\neq r}\frac{\partial^{k}}{\partial\lambda_1^{k-q}\partial\lambda_2^{q}}m_r(0,s)\mu_{rj}(s)p_{ji}(s,t)ds.\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
=&\sum_{k=0}^{\infty}\sum_{q_1+q_2+q_3=k}\frac{1}{q_1!q_2!q_3!}\sum_{r\neq i}\frac{\partial^{q_1+q_2}}{\partial\lambda_1^{q_1}\partial\lambda_2^{q_2}}m_i(0,s)\\
&\times(\lambda_1(1+{}_{xx}h_1^{ri}(t))+\lambda_2\,{}_{yx}h_1^{ri}(t))^{q_1}\\
&\times(\lambda_1\,{}_{xy}h_1^{ri}(t)+\lambda_2(1+{}_{yy}h_l^{ri}(t)))^{q_2}\\
&\times(\lambda_1({}_{xx}h_0^{ri}(t)+{}_{xy}h_0^{ri}(t))+\lambda_2({}_{yx}h_0^{ri}(t)+{}_{yy}h_0^{ri}(t)))^{q_3}\\
&\times\mu_{ri}(s)\\
+&\sum_{n:n\neq i}\mu_{ni}(t)\sum_{k=0}^{\infty}\sum_{q_1+q_2+q_3=k}\frac{1}{q_1!q_2!q_3!}\int_0^t\sum_{r\in\mathcal{J}}\sum_{j\neq r}\frac{\partial^{q_1+q_2}}{\partial\lambda_1^{q_1}\partial\lambda_2^{q_2}}m_r(0,s)\\
&\times(\lambda_1(1+{}_{xx}h_1^{rj}(s))+\lambda_2\,{}_{yx}h_1^{rj}(s))^{q_1}\\
&\times(\lambda_1\,{}_{xy}h_1^{rj}(s)+\lambda_2(1+{}_{yy}h_l^{rj}(s)))^{q_2}\\
&\times(\lambda_1({}_{xx}h_0^{rj}(s)+{}_{xy}h_0^{rj}(s))+\lambda_2({}_{yx}h_0^{rj}(s)+{}_{yy}h_0^{rj}(s)))^{q_3}\\
&\times\mu_{rj}(s)\frac{\partial}{\partial t}p_{jn}(s,t)ds\\
-&\sum_{n:n\neq i}\mu_{in}(t)\sum_{k=0}^{\infty}\sum_{q_1+q_2+q_3=k}\frac{1}{q_1!q_2!q_3!}\int_0^t\sum_{r\in\mathcal{J}}\sum_{j\neq r}\frac{\partial^{q_1+q_2}}{\partial\lambda_1^{q_1}\partial\lambda_2^{q_2}}m_r(0,s)\\
&\times(\lambda_1(1+{}_{xx}h_1^{rj}(s))+\lambda_2\,{}_{yx}h_1^{rj}(s))^{q_1}\\
&\times(\lambda_1\,{}_{xy}h_1^{rj}(s)+\lambda_2(1+{}_{yy}h_l^{rj}(s)))^{q_2}\\
&\times(\lambda_1({}_{xx}h_0^{rj}(s)+{}_{xy}h_0^{rj}(s))+\lambda_2({}_{yx}h_0^{rj}(s)+{}_{yy}h_0^{rj}(s)))^{q_3}\\
&\times\mu_{rj}(s)\frac{\partial}{\partial t}p_{ji}(s,t)ds\\
&-\sum_{n:n\neq i}\mu_{ni}(t)\sum_{k=0}^{\infty}\sum_{q=0}^{k}\frac{\lambda_1^{k-q}\lambda_2^{q}}{(k-q)!q!}\int_0^t\sum_{r\in\mathcal{J}}\sum_{j\neq r}\frac{\partial^{k}}{\partial\lambda_1^{k-q}\partial\lambda_2^{q}}m_r(0,s)\mu_{rj}(s)p_{jn}(s,t)ds\\
&+\sum_{n:n\neq i}\mu_{in}(t)\sum_{k=0}^{\infty}\sum_{q=0}^{k}\frac{\lambda_1^{k-q}\lambda_2^{q}}{(k-q)!q!}\int_0^t\sum_{r\in\mathcal{J}}\sum_{j\neq r}\frac{\partial^{k}}{\partial\lambda_1^{k-q}\partial\lambda_2^{q}}m_r(0,s)\mu_{rj}(s)p_{ni}(s,t)ds.
\end{aligned}
$$

Combining all three differentiated parts of the state-wise moment-generating function, we recognise $m_i(\lambda, t)$ and $m_r(\lambda, t)$ to arrive at the differential equation

$$
\begin{aligned}
\frac{\partial}{\partial t} m_i(\lambda, t) = &\sum_{r \neq i} \mu_{ri}(t) m_r(\lambda, t) - \mu_{ir}(t) m_i(\lambda, t) \\
&+ \sum_{k=0}^{\infty} \sum_{q=0}^{k} \sum_{l=0}^{p} \frac{1}{(k-q)!q!} \lambda_1^{k-q+1} \lambda_2^{q} {}_{xx}g_l^i(t) \frac{\partial^{k+l}}{\partial \lambda_1^{k-q+l} \partial \lambda_2^{q}} m_i(0, t) \\
&+ \sum_{k=0}^{\infty} \sum_{q=0}^{k} \sum_{l=0}^{p} \frac{1}{(k-q)!q!} \lambda_1^{k-q+1} \lambda_2^{q} {}_{xy}g_l^i(t) \frac{\partial^{k+l}}{\partial \lambda_1^{k-q} \partial \lambda_2^{q+l}} m_i(0, t) \\
&+ \sum_{k=0}^{\infty} \sum_{q=0}^{k} \sum_{l=0}^{p} \frac{1}{(k-q)!q!} \lambda_1^{k-q} \lambda_2^{q+1} {}_{yx}g_l^i(t) \frac{\partial^{k+l}}{\partial \lambda_1^{k-q+l} \partial \lambda_2^{q}} m_i(0, t) \\
&+ \sum_{k=0}^{\infty} \sum_{q=0}^{k} \sum_{l=0}^{p} \frac{1}{(k-q)!q!} \lambda_1^{k-q} \lambda_2^{q+1} {}_{yy}g_l^i(t) \frac{\partial^{k+l}}{\partial \lambda_1^{k-q} \partial \lambda_2^{q+l}} m_i(0, t) \\
&+ \sum_{k=0}^{\infty} \sum_{q_1+q_2+q_3=k} \frac{1}{q_1! q_2! q_3!} \sum_{r \neq i} \frac{\partial^{q_1+q_2}}{\partial \lambda_1^{q_1} \partial \lambda_2^{q_2}} m_r(0, s) \\
&\times (\lambda_1(1 + {}_{xx}h_1^{ri}(t)) + \lambda_2 \, {}_{yx}h_1^{ri}(t))^{q_1} \\
&\times (\lambda_1 \, {}_{xy}h_1^{ri}(t) + \lambda_2(1 + {}_{yy}h_l^{ri}(t)))^{q_2} \\
&\times (\lambda_1({}_{xx}h_0^{ri}(t) + {}_{xy}h_0^{ri}(t)) + \lambda_2({}_{yx}h_0^{ri}(t) + {}_{yy}h_0^{ri}(t)))^{q_3} \\
&\times \mu_{ri}(t) \\
&- \sum_{k=0}^{\infty} \sum_{q=0}^{k} \frac{1}{(k-q)!q!} \lambda_1^{k-q} \lambda_2^{q} \sum_{r \neq i} \frac{\partial^{k}}{\partial \lambda_1^{k-q} \partial \lambda_2^{q}} m_r(0, s) \mu_{ri}(t).
\end{aligned}
$$

$\square$

4.D Figures

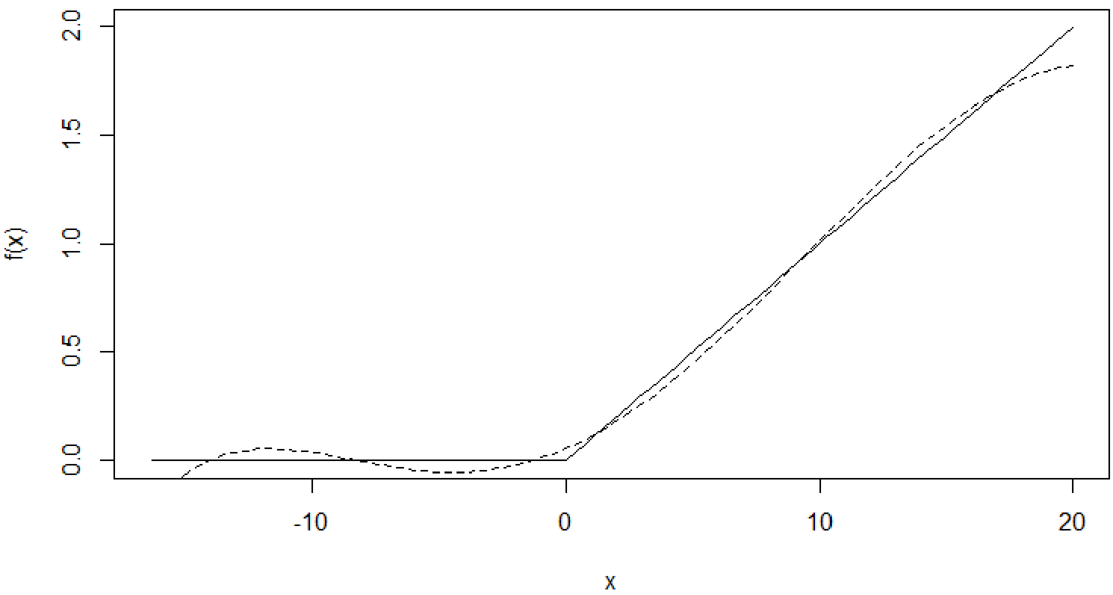

Figure 4.5: *Polynomial approximation of* $\max(0, x) \cdot 0.1$, *degree=5. Outside the interval of convergence* $(-14.85, 18.15)$ *the polynomial diverges.*

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Competing Interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

Chapter 5

Introduction to Utility Optimization

In this chapter we introduce utility optimization and equilibrium theory to provide the background for Chapter 6 and Chapter 7. We also highlight the most important concepts from the last two chapters in a non-technical fashion.

5.1 Background

On the topic of personal finance, the question of how wealth should be consumed and invested has been of interest since the inception of modern finance. The consumption-investment strategy is inevitably dependent on the preferences of the investor. Understanding in which way the preferences of the investor influence the optimal consumption-investment strategy by studying utility theory is a practical problem in theoretical disguise. The real-world portfolio managers are not investing according to some theoretically optimal strategy, but understanding how the features of the theoretically optimal control are connected to the preferences of the investor, is a relevant real-world problem.

A utility function is a mathematical object that represents the preferences of an investor. Utility optimization is all about studying models of preferences and the financial market. By finding the optimal strategy for a given utility function in a given financial market, we get a better understanding of the factors that influence the objectives of real-world investors, if the model is sensible. Utility functions cannot be deduced from a brain scan or a questionnaire, and the real-world financial market is too complex to model accurately.

A common phrase in statistic is,

> "All models are wrong, but some are useful."

The same can be said about utility theory. In his 100-page paper with 650 references Schultz (2015), the Cambridge professor and Brain Prize winner Wolfram Schultz describes how neuron signals induce learning and thereby ultimately influence behaviour. On the matter of utility he writes,

> "Although all reward, reinforcement, and decision variables are theoretical constructs, their neuronal signals constitute measurable physical implementations and as such confirm the validity of these concepts."

5.1.1 Dynamic Programming

The Danish former elite soldier B.S. Christiansen advocates for what he calls a 'backwards schedule'. If you have to be at work by 9:00 it means that you have to leave at 8:37, which means that you have to get dressed by 8:26, which means that you have to start breakfast at 7:56, which means that you have to get up at 7:43 if you want a shower. B.S. Christiansen has simply applied what Richard E. Bellman called the dynamic programming principle.

The dynamic programming principle is to take a complicated problem (as for instance 'what time should I get up?'), and break it into sub-problems that can be solved recursively in time. This idea can be formulated mathematically and applied rigorously via the so-called Hamilton-Jacobi-Bellman (HJB) equation, which is an indispensable tool for the derivation of optimal consumption-investment strategies.

A value function represents the value of a certain control for a certain utility function. Value functions allow us to compare and rank different controls, and define the optimal control as the one that achieves the highest value function. The HJB equation describes how a value function has to unfold backwards in time in order for the control to be optimal.

5.1.2 Equilibrium Theory

Have you ever planned to go for a run, but end up eating popcorn and watching a movie instead? or postponed the completion of your tax returns, even though you had designated time for it? If you have ever changed your mind, you have experienced time-inconsistent preferences. Having different objectives at different points in time is completely natural, but one has to be careful about defining optimality when the objective is not fixed.

A naïve treatment of time-inconsistent preferences is to solve the problem as if the preferences were time-consistent. The naïve investor constantly re-evaluates his control, and does not take into account that his preferences are changing over time. The value function that emerges from adopting this strategy, is not going to correspond to the value function that the naïve investor optimises over, which is why the label as 'naïve' is fitting.

A more sophisticated way to deal with time-inconsistent preferences, is to acknowledge that the preferences are going to change, and incorporate this knowledge into the definition of optimality. An optimal control defined as a Nash subgame equilibrium does exactly this.

To explain the idea behind a Nash subgame equilibrium, imagine a group of people who are playing a game of knocking over cans at a carnival together. They all share the prize, and each player gets to throw one ball. The first player wants a stuffed unicorn, and accordingly throws his ball at the pyramid of cans that will give him the unicorn. The second player wants a candy cane, but since the first player already knocked over one can in the unicorn pyramid, the second player also goes for the unicorn pyramid, as it is an acceptable trade-off for him. The third player wants neither a unicorn or a candy cane, and instead aims at the chocolate-bar pyramid. In this way, each player is playing a subgame, and the optimal action for the n'th player is determined by the actions of the first $n-1$ players. An equilibrium is reached when every single player is satisfied with their own subgame strategy given the strategy of the players that come before.

The value function for the equilibrium control satisfies an extended HJB equation, where suitable adjustments to the original HJB equations account for the change in the objective over time.

Prospect theory is an experimentally attested descriptive theory of behavioural finance that explains several anomalies stemming from the classical expected utility hypothesis, where agents are assumed to be rational. An essential feature of prospect theory, is the notion that utility is relative to a status quo, and not absolute. Furthermore, the status quo is dynamic. Incorporating these properties in a utility function result in time-inconsistent preferences.

5.2 Overview of Chapters 6 and 7

This thesis deals with time-inconsistent finite horizon investment problems, where optimality is defined in terms of equilibria. In the following two subsections we outline the central ideas of Chapters 6 and 7. Chapter 6 deals with an application of the extended HJB equation, on polynomial approximations of utility functions. Chapter 7 deals with a single utility function of particular interest with ties to prospect theory.

5.2.1 Polynomial Approximations

Polynomials are relatively simple, and can be used to form approximations of non-polynomial functions. It is natural to assume that if a polynomial closely approximates a utility function, then the optimal control for the polynomial closely approximates the optimal control for the utility function. It is exactly this idea we investigate in Chapter 6.

The trouble with polynomial utility functions is that they diverge, which can induce diverging optimal controls. To mitigate this problem, the polynomial utility function has to be constructed, so that the approximation is accurate for the most likely values of terminal wealth. Increasing the degree of the polynomial provides a closer approximation over a fixed interval, but higher degrees of polynomials diverge faster, which increases the influence on the optimal control. The polynomial utility function has to satisfy two interconnected properties;

- A close approximation of the original utility function over the likely values of terminal wealth.
- A degree of polynomial so low that the divergence outside the likely values of terminal wealth does not influence the optimal control.

We produce two numerical examples to study the accuracy of the control induced by polynomial utility functions. One of the examples has a known solution and serves to prove that the method is sound. The other example provides an approximation of an optimal control, for a utility function with an unknown solution.

5.2.2 Avoiding Dynamic Programming

One of the utility functions studied in the examples of Chapter 6, is the focal point of Chapter 7. Motivated by (among other things) the numerical results of Chapter 6, we assume that the equilibrium control is independent of wealth. This paves the way for finding the equilibrium control through a direct use of the definition of the equilibrium control, instead of solving the extended HJB equation.

An Authors Note on Chapter 7

Less than two months before submission of this thesis, Chapters 2, 3, 4 and 6 were finished, and I had to choice between squeezing in an extra chapter and having a long time to write an introduction. Apparently i chose the latter. Unfortunately I did not have the time to bring Chapter 7 to a point where I would deem it ready for submission. Please forgive the crudeness of Chapter 7.

Chapter 6

Polynomial Utility

Abstract

We approximate the utility function by polynomial series and solve the related dynamic portfolio optimization problems. We study the quality of the approximation for the Taylor and Bernstein series in response to the center and the degree of the expansion. The issue of time-inconsistency, arising from a dynamically adapted center of the expansion, is approached by equilibrium theory. In the numerical study we focus on two specific utility functions: For power utility, access to the optimal portfolio allows for a more complete illustration of the approximations; for the S-shaped utility function of prospect theory, the use of equilibrium theory allows for approximating the solution to the (obviously interesting but yet unsolved) case of current wealth as a dynamic reference point.

Keywords: Dynamic programming, Optimal Asset Allocation, Expected Utility Theory, Polynomial Expansions.

MSC Classification: 91B51, 91-08, 91-10, 37N40

Statements and Declarations

The research leading to these results received funding from Innovation Fund Denmark under grant number 7076-00029, project title "ProBaBLI - Projection of Balances and Benefits in Life Insurance". The authors have no competing interests to declare that are relevant to the content of this article.

6.1 Introduction

We approximate the portfolio utility maximization problem by series expansions of the utility function of which the second order Taylor expansion is a special case leading to mean-variance utility. Conforming with the modern approach to dynamic mean-variance optimization we approach the problem by equilibrium theory. We study the convergence of two expansions via Taylor and Bernstein polynomials, respectively, discuss their weaknesses, and propose ways to circumvent them. We illustrate the insight obtained in numerical studies covering both power utility and S-shaped utility with an adapted reference point.

The starting point for this work is the mean-variance portfolio optimization problem introduced by Markowitz (1952) in a one-period framework. Here, we think of the mean-variance objective as an appreciation of expected wealth minus variance multiplied by the absolute risk aversion. Levy and Markowitz (1979) formalize the objective as an approximation based on a second order Taylor expansion around the expected return. Thus, the mean-variance objective can be interpreted as an approximation to an underlying regular utility maximization problem from which the absolute risk aversion is derived and plugged in as weight on the variance. Levy and Markowitz (1979) conclude that the approximation is good for a logarithmic utility, but others find the approximation to be poor in general/other settings, see e.g. Loistl (1976) and Jondeau and Rockinger (2006). From a mathematical point, there are other series expansions available than the Taylor expansion and one can also vary over the point of expansion. The discussion about the quality should be seen in that light.

The issue about the quality of the methods becomes even more delicate when formalizing a dynamic problem. In the variance appears the point of expansion, expected wealth, as the point to which the variance measures the expected quadratic distance. For a one-period version of the problem this is unambiguously meaningful. When forming a dynamic problem the question arises: What should be the point of expansion at a given point in time? If the point is the initially expected wealth, then a time-consistent dynamic problem arises, and it can be solved by standard dynamic programming as a quadratic utility problem with a Lagrangian constraint, see e.g. Xia (2005). However, the objective is only of mean-variance type seen from the initial time point and not from later time points. If the point of expansion is conditionally expected wealth, the objective is always a (conditional) mean- (conditional) variance problem. However, since the objective now contains the conditional expectation squared, the problem becomes time-inconsistent and other methods than standard dynamic programming are needed. So, the point of expansion is crucial for both the interpretation and the solvability of the problem.

The dynamic mean-variance problem has gained a lot attention since the solution proposed by Basak and Chabakauri (2010). They essentially formulate and solve the problem by an equilibrium approach. That approach was formalized for time-inconsistent problems in continuous time by Ekeland and Lazrak (2010) and Ekeland and Pirvu (2008), mainly to cope with the source of time-inconsistency known as non-exponential discounting. But the approach works generally for time-inconsistent problems. This is studied intensively in Björk and Murgoci (2014) and Björk et al. (2014) and followed up by many others in more recent works, see e.g. Kryger et al. (2020) and the references therein. Here, we also take the equilibrium approach but, in contrast to the extensive work on the mean-variance problem, we go beyond both the second moment as the object of interest and the conditional expectation as the point to which distance is measured.

The Taylor expansion around the conditional expectation for a general utility function was first studied by Nordfang and Steffensen (2017), and they identify clear weaknesses. We have three main objectives in continuation of their work:

a) We identify and discuss further the reason for these weaknesses. The key issue is that, independently of the order, the Taylor expansion provides convergence within a limited interval only, specified indirectly by the point of expansion. Since the approximately optimal control takes the wealth process outside this interval with a sufficiently high probability, it has the harmful impact on the optimal control that it does not converge in general. We propose here to use a point of expansion that depends on the order to cope with the problem. The weakness of the approximation is also what drives Fahrenwaldt and Sun (2020) to study intensively of the remainder term of the expansion.

b) We propose and study the Bernstein polynomium as an alternative to the Taylor polynomium. As for the Taylor polynomium, the Bernstein polynomium converges over an interval. But since this interval is specified directly, this gives a more direct control over the area of convergence. However, again the relation between the interval and the risk of the wealth ending there prevents convergence of the strategy. And again, we propose to set the interval as a function of the order of expansion. We discuss advantages and disadvantages among the two expansions and present supporting numerical results for the power utility case where the solution to the underlying problem is explicitly known and the quality of the approximation, therefore, can be illustrated directly.

c) The equilibrium approach is mainly introduced to deal with the time-consistency arising from higher order moments in the expansion. However, its presence allows us to study series expansion methods even when even the original utility function contains time-consistency issues, without really adding much difficulty. We therefore study the feature of Prospect Theory known as S-shaped utility around a reference point (and disregard probability distortion). Similarly to the point from which the quadratic distance is measured in the mean-variance case, also the reference point for S-shaped utility can be elaborated on in a dynamic context. Almost everywhere in the literature, this point is taken to be initial wealth. We study the version of the problem where the reference point at every point in time and space is taken to be current wealth. Then the solution to the underlying problem is not explicitly known but we present numerical solutions based on the Taylor and Bernstein polynomials.

The outline of the paper is as follows: In Section 2, we provide the setup and present results from the equilibrium theory. Sections 3 and 4 go through the solution, ideas, advantages, and disadvantages of the Taylor and Bernstein polynomials, respectively. Numerical studies for both power utility and S-shaped utility with adapted reference point are presented in Section 5.

6.2 Setup and Useful Results

In this section we present the problem and present results about equilibrium strategies. We assume that the investor trades at a Black-Scholes market equipped with a single stock, S, and a bank account, B, with dynamics

$$\begin{aligned} dB(t) &= rB(t)dt, \qquad B(0) = 1 \\ dS(t) &= \alpha S(t)dt + \sigma S(t)dW(t), \qquad S(0) = s_0, \end{aligned}$$

where W is a standard Brownian motion. The wealth of the investor is invested in the two assets according to the control π which denotes the proportion of wealth invested in the stock. The wealth is assumed to be self-financing such that the dynamics are given by

$$dX^\pi(t) = \left(r + \pi(t, X^\pi(t))(\alpha - r)\right) X^\pi(t)dt + \pi(t, X^\pi(t))\sigma X^\pi(t)dW(t), \tag{6.2.1}$$

$$X^\pi(0) = x_0 > 0. \tag{6.2.2}$$

Controls that are independent of wealth, i.e. deterministic, are particularly appealing, both from a computational and practical point of view.

Lemma 6.2.1. *For wealth independent controls, $\pi(t,x) = \pi(t)$, $X^\pi(t)$ is log-normally distributed with*

$$X^\pi(t) \sim LN\left(\log(x_0) + \int_0^t \left(r + \pi(s)(\alpha - r) - \frac{(\sigma\pi(s))^2}{2}\right) ds, \int_0^t (\sigma\pi(s))^2 ds\right),$$

and the expectation of the i^{th} power of $X^\pi(T)$, conditional on $X^\pi(t) = x$, is given by

$$E_{t,x}[(X^\pi(T))^i] = x^i \exp\left(i \int_t^T \left(r + \pi(s)(\alpha - r) + (i-1)\frac{(\sigma\pi(s))^2}{2}\right) ds\right). \tag{6.2.3}$$

Proof. Proof of Lemma 6.2.1. Since $\pi(t)$ is deterministic, $\int_0^t \sigma\pi(s)dW(s) \sim \mathcal{N}(0, \int_0^t (\sigma\pi(s))^2 ds)$, applying Ito's lemma gives

$$\log(X^\pi(T)) \sim \mathcal{N}\left(\log(x_0) + \int_0^t \left(r + \pi(s)(\alpha - r) - \frac{(\sigma\pi(s))^2}{2}\right) ds, \int_0^t (\sigma\pi(s))^2 ds\right).$$

Conditioning on $X^\pi(t) = x$ and integrating to T gives (6.2.3), which is the expression for the i^{th} moment of the log-normal random variable $X^\pi(T) \mid X^\pi(t) = x$. □

The goal of the investor is to optimize expected terminal utility. But unlike the standard formulation, we allow terminal utility to not only depend on terminal wealth but also current wealth. Thus, We introduce the function

$$J^\pi(t, x) = E_{t,x}[u(t, X^\pi(t), X^\pi(T))]. \tag{6.2.4}$$

We are going to approximate the true problem above by an approximated problem by simply by replacing u with a polynomial utility function in the form

$$\tilde{u}_n(t, x, z) := \sum_{i=0}^{n} a_{i,n}(t, x) z^i,$$

for a set of coefficients $a_{i,n}$ that are determined in such a way that $\tilde{u}_n$ is an nth degree polynomial approximation of u. The standard approach to optimization is to set a supremum in front of (6.2.4). However, no matter whether the expectation be over u or $\tilde{u}_n$, the presence of current wealth violates Bellman's Principle of Optimality, as formulated by Bellman:

> An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

In other words, no matter what happens, we should not deviate from our initial objective. If the property cited above is fulfilled, the policy is said to be time-consistent, and an optimization problem that has a time-consistent policy as optimal solution is said to be a time-consistent problem. But clearly, the introduction of current wealth in the objective violates exactly that property.

It may seem odd to have preferences that change according to the current wealth and to have time-inconsistent policies, but this is by no means unrealistic, in fact quite the opposite - see Tversky and Kahneman (1986). Daniel Kahneman was awarded the Nobel Memorial Prize in Economics in 2002 "for having integrated insights from psychological research into economic science, especially concerning human judgment and decision-making under uncertainty"[1]. The work by Kahneman shows that utility

[1] Nobel Media AB, 2021.

should be measured relative to a reference wealth level which could naturally be current wealth. Thus, according to Kahneman, it is basic human psychology to have time-inconsistent preferences. Interestingly, time-inconsistent preferences were introduced in economic decision making long before Kahneman, see Strotz (1955) and Samuelson (1937).

Note that even if the original function u is not wealth-dependent and time-inconsistent, it may well be that we want to approximate it by a polynomial utility function $\tilde{u}$ which is. Actually, this is a core idea of the present paper to actually do that. As discussed above we cannot put a supremum in front of (6.2.4) and solve by standard Bellman theory. Instead we approach the problem by so-called equilibrium theory, introducing an equilibrium value function (instead of an optimal value function) and an equilibrium strategy (instead of an optimal strategy). This route starts by defining the set of admissible strategies in the following way.

Definition 6.2.2 (Admissibility)**.** *For n real functions $g_1(x), ..., g_n(x)$, a control π is admissible w.r.t $g_1, ..., g_n$ (or simply admissible) if for each g_i, there exists a function $G^{i,\pi}(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}$ from $C^{1,2}$ such that*

1) $G_t^{i,\pi} = -(r + \pi(\alpha - r))xG_x^{i,\pi} - \frac{1}{2}\sigma^2\pi^2x^2G_{xx}^{i,\pi}, \quad G^{i,\pi}(T,x) = g_i(x).$

2) $\sigma\pi(t, X^\pi(t))X^\pi(t)G_x^{i,\pi}(t, X^\pi(t)) \in \mathcal{L}^2$*, for X^π following the SDE given by* (6.2.1)-(6.2.2).

The set of admissible strategies is denoted by $\mathcal{U}$.

The next step is to define an equilibrium strategy. This definition follows the definition by Björk et al. (2014).

Definition 6.2.3 (Equilibrium control)**.** *Consider a control $\hat{\pi}$. Choose a real number $h \in (0,T)$ and a fixed initial point (t,x), such that $t \in [0, T-h)$. Define for a control π the control π^h by*

$$\pi^h_{(t,x)}(s,y) = \begin{cases} \pi(s,y), & \text{for } (s,y) \in [t, t+h) \times \mathbb{R}, \\ \hat{\pi}(s,y), & \text{for } (s,y) \in [t+h, T] \times \mathbb{R}. \end{cases}$$

If

$$\liminf_{h \to 0} \frac{J^{\hat{\pi}}(t,x) - J^{\pi^h}(t,x)}{h} \geq 0,$$

for all controls π for which π^h is admissible, then $\hat{\pi}$ is an equilibrium control.

In other words, for all points in time there is no other strategy that yields a smaller marginal change in the value function than the equilibrium control. Equivalently, the equilibrium control yields the largest marginal change to the value function with going backwards in time. This makes it optimal in the equilibrium sense but not (always) the control that mazimizes (6.2.4).

Having defined the objective of the investor, we are ready to address the problem for the case of polynomial utility functions. We are interested in the equilibrium optimal control $\hat{\pi}_n$ for the value function $J_n^{\hat{\pi}_n}$ determined by

$$J_n^{\hat{\pi}_n}(t,x) = \sum_{i=0}^{n} a_{i,n}(t,x) E_{t,x}[(X^{\hat{\pi}_n}(T))^i]. \tag{6.2.5}$$

The fact that we can find the equilibrium control in Equation (6.2.5), and deal with polynomial utility functions in a general framework, is made possible by the extended HJB equation, first conceived by Kryger and Steffensen (2010), with a revised version of the proof provided by Kryger et al. (2020).

Theorem 6.2.4 (Extended HJB equation)**.** *Let $f : [0,T] \times \mathbb{R}^{n+2} \to \mathbb{R}$ be a function that is once differentiable in the first argument, and twice differentiable in all other arguments. Let $g_0, ..., g_n$ be real functions. Consider the investor with value function*

$$J^{\pi}(t,x) = f(t, x, E_{t,x}[g_0(X^{\pi}(T))], ..., E_{t,x}[g_n(X^{\pi}(T))]),$$

with wealth dynamics described by the SDE

$$\begin{aligned} dX^{\pi}(t) &= (r + \pi(t, X^{\pi}(t))(\alpha - r))\, X^{\pi}(t)dt + \pi(t, X^{\pi}(t))\sigma X^{\pi}(t)dW(t), \\ X^{\pi}(0) &= x_0 > 0, \end{aligned}$$

and equilibrium optimal control π^*. If

$$\hat{\pi} = \arg\inf_{\pi} \left\{ -(r + \pi(\alpha - r))x \sum_{i=0}^{n} f_{G^i} G_x^{i,\pi^*} - \frac{1}{2}\sigma^2 \pi^2 x^2 \sum_{i=0}^{n} f_{G^i} G_{xx}^{i,\pi^*} \right\} \tag{6.2.6}$$

is an admissible strategy with respect to the functions $g_0, ..., g_n$, then $\hat{\pi}(t,x) = \pi^*(t,x)$, and the optimal value function is determined by

$$J^{\hat{\pi}}(t,x) = f(t, x, G^{0,\pi^*}(t,x), ..., G^{n,\pi^*}(t,x)).$$

For $\sum_{i=0}^{n} f_{G^i} G_{xx}^{i,\pi^*} < 0$, the control $\hat{\pi}$ in Equation (6.2.6) can be expressed as

$$\hat{\pi}(t,x) = -\frac{\alpha - r}{\sigma^2 x} \frac{\sum_{i=0}^{n} f_{G^i}(t, x, G^{0,\pi^*}(t,x), ..., G^{n,\pi^*}(t,x)) G_x^{i,\pi^*}(t,x)}{\sum_{i=0}^{n} f_{G^i}(t, x, G^{0,\pi^*}(t,x), ..., G^{n,\pi^*}(t,x)) G_{xx}^{i,\pi^*}(t,x)},$$

which for polynomial utility functions with coefficients $a_{i,n}$ equates to

$$\hat{\pi}_n(t,x) = -\frac{\alpha - r}{\sigma^2 x} \cdot \frac{\sum_{i=0}^{n} a_{i,n}(t,x) m_x^{\hat{\pi}_n}(i,t,x)}{\sum_{i=0}^{n} a_{i,n}(t,x) m_{xx}^{\hat{\pi}_n}(i,t,x)}, \tag{6.2.7}$$

where $m^{\hat{\pi}_n}(i,t,x)$ denotes the i^{th} moment of the stochastic variable describing the wealth at termination using $\hat{\pi}_n$ as a control, i.e. $m^{\hat{\pi}_n}(i,t,x) = \mathrm{E}_{t,x}[(X^{\hat{\pi}_n}(T))^i]$.

By Theorem 6.2.4 we can find the optimal control for polynomial utility functions. In terms of interpretation and implementation, simple optimal controls are preferred. It turns out that a class of polynomial coefficients lead to optimal controls that are independent of wealth, making the controls deterministic. This class is immediately detected by looking for wealth-independent solutions to (6.2.7), recalling Lemma 6.2.1.

Lemma 6.2.5 (Deterministic Controls)**.** *For any function $h(x)$, coefficients of the form*

$$a_{i,n}(t,x) = h(x)x^{-i}c_{i,n}(t)$$

lead to wealth independent optimal controls, and $\pi(t)$ is the solution to

$$\hat{\pi}(t) = -\frac{\alpha - r}{\sigma^2} \cdot \frac{\sum_{i=0}^n c_{i,n}(t)i\left(i\int_t^T \left(r + \hat{\pi}(s)(\alpha - r) + (i-1)\frac{(\sigma\hat{\pi}(s))^2}{2}\right)ds\right)}{\sum_{i=0}^n c_{i,n}(t)i(i-1)\left(i\int_t^T \left(r + \hat{\pi}(s)(\alpha - r) + (i-1)\frac{(\sigma\hat{\pi}(s))^2}{2}\right)ds\right)}.$$

Proof. Proof of Lemma 6.2.5. Assume we have coefficients of the form $a_{i,n}(t,x) = h(x)x^{-i}c_{i,n}(t)$. We now guess that the solution to (6.2.7) using these coefficients, is independent of wealth. By Lemma 6.2.1,

$$m^{\hat{\pi}_n}(i,t,x) = x^i \exp\left(i\int_t^T \left(r + \hat{\pi}(s)(\alpha - r) + (i-1)\frac{(\sigma\hat{\pi}(s))^2}{2}\right)ds\right),$$

and so

$$\begin{aligned} &-\frac{\alpha - r}{\sigma^2 x} \cdot \frac{\sum_{i=0}^n a_{i,n}(t,x)m_x^{\hat{\pi}_n}(i,t,x)}{\sum_{i=0}^n a_{i,n}(t,x)m_{xx}^{\hat{\pi}_n}(i,t,x)} \\ &= -\frac{\alpha - r}{\sigma^2} \cdot \frac{\sum_{i=0}^n c_{i,n}(t)i\left(i\int_t^T \left(r + \hat{\pi}(s)(\alpha - r) + (i-1)\frac{(\sigma\hat{\pi}(s))^2}{2}\right)ds\right)}{\sum_{i=0}^n c_{i,n}(t)i(i-1)\left(i\int_t^T \left(r + \hat{\pi}(s)(\alpha - r) + (i-1)\frac{(\sigma\hat{\pi}(s))^2}{2}\right)ds\right)} \end{aligned} \tag{6.2.8}$$

confirming that, for $a_{i,n}(t,x) = h(x)x^{-i}c_{i,n}(t)$, $\pi(t,x) = \pi(t)$ is a solution to equation (6.2.7). □

In the numerical section, all coefficients of the polynomial utility functions are on the form of Lemma 6.2.5, and the calculations of optimal controls where performed by finding a solution to (6.2.8).

6.3 The Taylor Optimal Control

The main focus of the paper is the way in which the polynomial utility function is constructed from the original utility function. Note that different polynomial approximations lead to different controls. This section presents some results, that are also available in Nordfang and Steffensen (2017) and Fahrenwaldt and Sun (2020), but at the same time we highlight some of the general problems with polynomial utility functions.

An nth order Taylor expansion in z around the point $d(t,x)$ of the utility function $u(t,x,z)$ is given by

$$\begin{aligned}\tilde{u}_n^T(t,x,z) =& \sum_{k=0}^{n} \frac{1}{k!} u^{(k)}(t,x,d(t,x))(z-d(t,x))^k \\ =& \sum_{k=0}^{n} \frac{1}{k!} u^{(k)}(t,x,d(t,x)) \sum_{i=0}^{k} \binom{k}{i} z^i d(t,x)^{k-i}(-1)^{k-i} \\ =& \sum_{i=0}^{n} a_{i,n}^T(t,x) z^i,\end{aligned}$$

where

$$a_{i,n}^T(t,x) = \frac{1}{i!} \sum_{k=0}^{n-i} \frac{(-1)^k d(t,x)^k}{k!} u^{(k+i)}(t,x,d(t,x)), \tag{6.3.1}$$

where $u^{(k)}(t,x,z)$ denotes $\frac{\partial^k}{\partial z^k} u(t,x,z)$, and where we have introduced $a_{i,n}^T$ as the coefficients of the polynomial. The topscript T refers to Taylor as we later introduce a different expansion and need to disunguish between the two. The choice of $d(t,x)$ may have a large influence on the equilibrium control. If $d(t,x)$ is too large or too small, the Taylor approximation around the likely values of $X^\pi(T)$ may be poor. Furthermore, specific choices of $d(t,x)$ may lead to particularly simple equilibrium controls which we analyse in Example 6.3.1.

Example 6.3.1 (Power Utility Coefficients)**.** The power utility function u_P defined by

$$u_P(t,x,z) = \frac{1}{1-\gamma} z^{1-\gamma},$$

where $\gamma > 0$, $\gamma \neq 1$ is the risk aversion parameter, has kth derivative given by

$$u_P^{(k)}(z) = \frac{1}{1-\gamma} z^{1-\gamma-k} \prod_{l=0}^{k-1} (1-\gamma-l).$$

Expanding the Taylor polynomial around $d(t,x)$ gives the coefficients

$$\begin{aligned}a_{i,n}^T(t,x) =& \frac{1}{i!} \sum_{k=0}^{n-i} \frac{(-1)^k d(t,x)^k}{k!} \frac{1}{1-\gamma} d(t,x)^{1-\gamma-k-i} \prod_{l=0}^{k+i-1} (1-\gamma-l) \\ =& \frac{d(t,x)^{1-\gamma-i}(-1)^{n+i} \prod_{l=1}^{n}(1-\gamma-l)}{i!(n-i)!(1-\gamma-i)}.\end{aligned}$$

For any function f we can consider the point of expansion given by $d(t,x) = xf(t)$. Then the coefficients can be expressed as $x^{1-\gamma-i}c_{i,n}(t)$, and the Taylor optimal control is independent of wealth by Lemma 6.2.5.

Nordfang and Steffensen (2017) choose $d(t,x) = E_{t,x}[X^{\hat{\pi}_n}(T)]$ as their point of expansion. Making the ansatz that $\hat{\pi}_n$ is independent of wealth for this point of expansion, Lemma 6.2.1 states that

$$d(t,x) = x \exp\left(\int_t^T r + \hat{\pi}_n(s)(\alpha - r)ds\right).$$

By the observation above this fits with $\hat{\pi}_n$ becoming independent of wealth such that we have detected such a case. △

Taylor polynomials are widely used to approximate differentiable functions, and it is therefore meaningful to derive the Taylor optimal control. In the following section we shed light on a fundamental problem with Taylor polynomials.

6.3.1 Problems with the Taylor Optimal Control

Fahrenwaldt and Sun (2020) perform a thorough analysis of the remainder term that accounts for the difference between the original value function and the approximate polynomial value function, for $d(t,x) = E_{t,x}[X^{\hat{\pi}_n}(T)]$ and the case of a power utility function. A particularly interesting result of Fahrenwaldt and Sun (2020) is that the remainder term *increases* in n for n sufficiently large. In other words, increasing the degree of the Taylor polynomial produces at some point a worse approximation in that particular sense. In this section we provide some intuition behind this behavior.

To an understanding of the trouble with the Taylor optimal control, we consider the interval over which the Taylor expansion of the power utility function converges to the power utility function. Performing a ratio test shows that

$$\lim_{n\to\infty} u_{2n}^T(t,x,z) = \begin{cases} -\infty, & \text{for } z > 2d(t,x), \\ u(t,x,z), & \text{for } z < 2d(t,x), \end{cases}$$

and

$$\lim_{n\to\infty} u_{2n+1}^T(t,x,z) = \begin{cases} \infty, & \text{for } z > 2d(t,x), \\ u(t,x,z), & \text{for } z < 2d(t,x). \end{cases}$$

Thus, if terminal wealth exceeds $2d(t,x)$, then for large n, the utility function actually diverges in such a way that it goes to infinity for large even n and to minus infinity for large odd n. The event that $X^{\hat{\pi}}(T) > 2d(t,x)$ (called the divergence event below) may be unlikely, but even a small probability can in this case have a large influence due to the special divergence of the utility function. In Figure 6.1 we have plotted two Taylor polynomial approximations of the power utility function, along with the density of the terminal wealth achieved for the true optimal constant control $\pi^* = 0.25$.

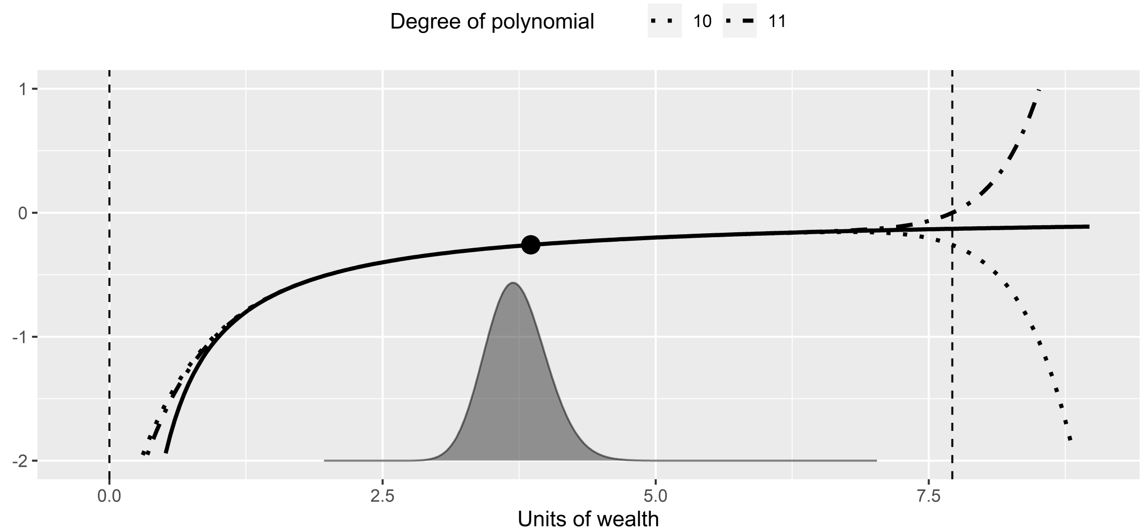

Figure 6.1: *Point at $z = 3.86$ signifies the point of expansion, which is the expected value of terminal wealth. Vertical lines indicate convergence radius from ratio test*

While the divergence event seems extremely unlikely, its influence cannot be understated. For fixed (t, x) and n, the probability that the terminal wealth is going to exceed $2d(t, x)$ is fixed. But increasing the degree of the Taylor polynomial, increases (or decreases depending on whether n is odd or even) the marginal utility of this extreme event, as the Taylor polynomial diverges faster. Increasing the odd degree of the polynomial eventually increases the incentive to reach the divergence event becomes dominating such that the optimal control is $\hat{\pi}(t, x) = \infty$. Conversely, increasing the even degree, the strategy $\hat{\pi}(t, x) = 0$ becomes optimal as the incentive to avoid the divergence event becomes dominating.

As explained, the problem is that for increasing n, the probability of the divergence event is unchanged, but the influence of it is increasing. For the case of power utility this problem can perhaps be remedied by considering instead a point of evaluation that increases as n increases, e.g. $d_n(t, x) = 0.5 \cdot \inf\{q \in \mathbb{R} \mid P_{t,x}(X^{\hat{\pi}}(T) \leq q) = 1 - n^{-\eta}\}$ for some constant $\eta > 0$. There is no way, however, to generally control the interval over which the Taylor polynomial converges to the true utility function. A ratio test can identify how far away from $d(t, x)$ the Taylor series converges to the true utility function, but for a given utility function, it may not be possible to chose a point $d(t, x)$ that leads to a satisfactory interval of convergence. Furthermore, the Taylor polynomial requires a smooth utility function such that e.g. a piecewise utility function is not allowed. Another example is the S-shaped utility function, where either the utility in gains or utility in losses can be approximated but not both.

One may try to ignore the behaviour of the polynomial utility function for values larger than some extremely large B. Why should we care about unrealistically high values of wealth? However, unrealistically high values of wealth do not exist as the following argument shows. If we consider the constant proportion $\pi(s,x) = C$ for $s \in [t, t+\Delta t]$, then

$$\begin{aligned} &P(X^\pi(t+\Delta t) > B \mid X^\pi(t) = x) \\ =&P\left(x \exp\left(\Delta t\left(r + C(\alpha - r) - \frac{\sigma^2 C^2}{2}\right) + \int_t^{t+\Delta t} \sigma C dW(s)\right) > B\right) \\ =&P\left(\int_t^{t+\Delta t} \sigma C dW(s) > \log(B) - \log(x) - \Delta t\left(r + C(\alpha - r) - \frac{\sigma^2 C^2}{2}\right)\right). \end{aligned} \tag{6.3.2}$$

With

$$\int_t^{t+\Delta t} \sigma C dW(s) \sim \mathcal{N}\left(0, \Delta t \sigma^2 C^2\right),$$

we get, for $Y \sim \mathcal{N}(0,1)$,

$$\begin{aligned} &P(X^\pi(t+\Delta t) > B \mid X^\pi(t) = x) \\ =&1 - P\left(Y \leq \frac{\log(B) - \log(x) - \Delta t\left(r + C(\alpha - r) - \frac{\sigma^2 C^2}{2}\right)}{\sigma C \sqrt{\Delta t}}\right). \end{aligned} \tag{6.3.3}$$

Thus, if the investor wants to reach B over the next small interval with probability $\lambda \in (0,1)$, he can do so by solving (6.3.3) equal to λ with respect to C. This amounts to solving the quadratic equation,

$$C^2 \frac{\Delta t \sigma}{2\sqrt{\Delta t}} - C\left(\Phi^{-1}(1-\lambda) + \frac{\Delta t(\alpha - r)}{\sigma\sqrt{\Delta t}}\right) - \frac{\log(B) - \log(x) - \Delta t r}{\sigma\sqrt{\Delta t}} = 0,$$

where Φ^{-1} is the quantile function of the standard normal distribution. If B or λ is sufficiently large the investor must borrow to invest more than his wealth, though. What is important to realize here is that the investor can achieve any fixed wealth with any fixed probability over any fixed time interval. Divergence of the utility function could give him the incentive to do so. Therefore divergence even for extreme values of wealth cannot be ignored.

It poses a question – if all polynomial diverges, how come numerical studies such as the ones provided in Nordfang and Steffensen (2017) show that we can get reasonable polynomial controls at all? We don't know, but apparently we can. We have merely shown that the investor basically can achieve any wealth he desires over an arbitrarily small time interval, but we do not know the exact conditions under which this is optimal. Part of the answer probably lies in the fact that the point at which the polynomial starts to diverge, changes in time and wealth as well as in π^*. The fact that we can get any reasonable results, indicates that if the required control to achieve divergent utility is sufficiently extreme, the investor won't do it. Perhaps there is some combination of probability of the event, and degree of polynomial that results in a divergent control. This hypothesis is corroborated by the very same numerical studies in Nordfang and Steffensen (2017) where the optimal control is seen to diverge as $(T-t)$ increases, resulting in a wider distribution of terminal wealth. The exact relation between probability and increase in utility that results in this dangerous cocktail, is a matter for further research.

That problem exposed in the previous paragraph does not mean that the approximate policy based on the approximate utility is always extreme. Obviously, we may have meaningful strategies as long as n is not too large. But even for n going to infinity, the policy seems to converge for some parameters and some time horizons. This is well-documented by Nordfang and Steffensen (2017). This indicates that, in spite of the ability to reach any level with any probability over any time interval, there is always also the risk of not getting there perhaps followed by a risk of harmful outcomes. Therefore, for some parameters and time horizons it is more valuable to the investor to choose his policy according to what happens within the interval of convergence of the utility function. So, there seems to be a balance between being dominated by the utility gained inside and outside the interval of convergence. Obviously, the fact that the point of expansion $d(t,x)$ moves with time and wealth makes the balance utterly involved and we do not pursue this question further here. Instead we now turn to different method of expansion that allows is to explicitly control the area of convergence.

6.4 The Bernstein Optimal Control

Bernstein polynomials were first used for a constructive proof for the Weierstrass approximation theorem. While their convergence is slow, their coefficients are explicit and simple. The Bernstein polynomial of degree n on the unit interval of the function f is defined as

$$B_n(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) b_{n,i}(x),$$

where $b_{n,i}$ are the Bernstein basis polynomials

$$b_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}.$$

The Bernstein polynomial is essentially a weighted average of the points $f(i/n)$, where the Bernstein basis polynomials represent how much weight should be given to $f(i/n)$

for a given value of x. In Figure 6.2 we have plotted the Bernstein basis polynomials for $n = 4$.

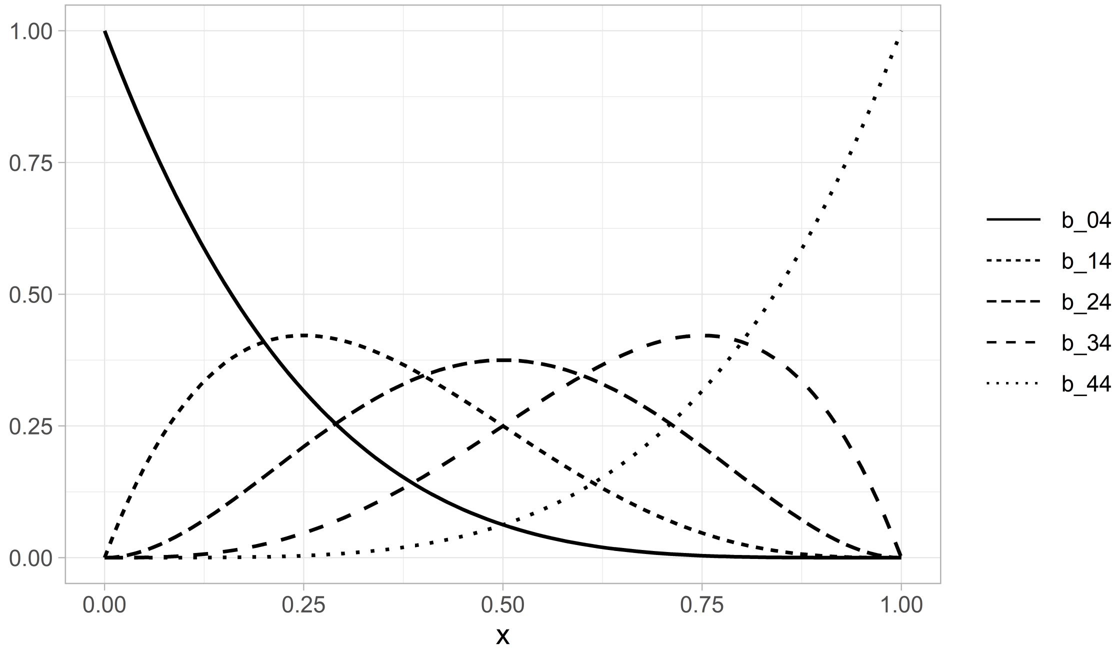

Figure 6.2: *$b_{i,4}(x)$ for $i = 0...4$, and $x \in [0, 1]$*

We do not want to restrict the wealth of the investor to the interval [0,1], so we consider a shifted and scaled version of the Bernstein polynomials.

Definition 6.4.1 (Generalized Bernstein Polynomial)**.** *The nth degree generalized Bernstein polynomial, $\tilde{u}_n^B$, of u for $(t, x, z) \in [0, T] \times \mathbb{R}^2$ is defined as*

$$\tilde{u}_n^B(t, x, z) := \sum_{i=0}^{n} u(t, x, x_{n,i}(t, x))b_{n,i}(t, x, z),$$

where

$$b_{n,i}(t, x, z) = \binom{n}{i} \left(\frac{z - x_{n,0}(t, x)}{x_{n,n}(t, x) - x_{n,0}(t, x)}\right)^i \left(1 - \frac{z - x_{n,0}(t, x)}{x_{n,n}(t, x) - x_{n,0}(t, x)}\right)^{n-i},$$

and

$$x_{n,i}(t, x) = x_{n,0}(t, x) + \frac{(x_{n,n}(t, x) - x_{n,0}(t, x))i}{n},$$

for some upper and lower points $x_{n,n}$ and $x_{n,0}$.

For notational reasons we often omit the dependence in t and x for $x_{n,i}(t, x)$. In order to easily apply Theorem 6.2.4 and Equation (6.2.7) we need to rewrite the form of the extended Bernstein Polynomial from definition 6.4.1, to a polynomial in the form

$$\tilde{u}_n^B(t,x,z) = \sum_{i=0}^{n} a_{i,n}^B(t,x) z^i.$$

From Definition 6.4.1 we can write

$$\begin{aligned}
&\tilde{u}_n^B(t,x,z) \\
&= \sum_{i=0}^{n} u(t,x,x_{n,i}) \binom{n}{i} \left(\frac{z - x_{n,0}}{x_{n,n} - x_{n,0}}\right)^i \left(1 - \frac{z - x_{n,0}}{x_{n,n} - x_{n,0}}\right)^{n-i} \\
&= \sum_{j=0}^{n} z^j \sum_{k=j}^{n} \binom{k}{j} (-1)^{k-j} x_{n,0}^{k-j} (x_{n,n} - x_{n,0})^{-k} \sum_{i=0}^{k} u(t,x,x_{n,i}) \binom{n}{i} \binom{n-i}{k-i} (-1)^{k-i} \\
&= \sum_{j=0}^{n} z^j a_{j,n}^B(t,x),
\end{aligned}$$

where

$$a_{j,n}^B(t,x) = (x_{n,0})^{-j} \sum_{k=j}^{n} \left(\frac{x_{n,0}}{x_{n,n} - x_{n,0}}\right)^k \sum_{i=0}^{k} u(t,x,x_{n,i}) \binom{n}{i} \binom{n-i}{k-i} \binom{k}{j} (-1)^{-i-j}.$$

Unlike the Taylor polynomial that needs a single point of expansion, the Bernstein polynomial needs two, a lower and an upper point.

In between the lower and upper points, Theorem 6.4.2 below states that there is a bound on the difference between u and $\tilde{u}_n$, that depends on the modulus of continuity defined by

$$\omega_n(u(t,x,z),\delta) := \sup_{\substack{z,y \in [x_{n,0}, x_{n,n}] \\ |z-y| < \delta}} \mid u(t,x,z) - u(t,x,y) \mid .$$

The upper bound is proportional to the modulus of continuity, and quadratic in the distance between $x_{n,0}$ and $x_{n,n}$.

Theorem 6.4.2 (Bound on Generalized Bernstein Polynomial)**.** *For any (t,x) and any function $u(t,x,z)$ continuous in z,*

$$\begin{aligned}
&\mid u(t,x,z) - \tilde{u}_n^B(t,x,z) \mid \\
&\leq \omega_n(u(t,x,z), n^{-1/2}) \left(\frac{(x_{n,0} + x_{n,n})((x_{n,0} + x_{n,n}) - 4) + 8}{4}\right),
\end{aligned}$$

for z in $[x_{n,0}, x_{n,n}]$. Furthermore, for $x_{n,0} = a > -\infty$ and $x_{n,n} = b < \infty$ and $z \in [a,b]$,

$$\tilde{u}_n^B(t,x,z) \to u(t,x,z) \quad \textit{uniformly.}$$

Proof. Proof of Theorem 6.4.2. The proof is analogous to the constructive proof for the Weierstrass's Approximation theorem using Bernstein polynomials on $(0,1)$ (see for instance Theorem 36.4 of Estep (2002)), with the obvious modifications required to apply the theorem for Generalized Bernstein Polynomials. □

Using Theorem 6.4.2 we can control the interval over which the generalized Bernstein power series converges uniformly to the true utility function. The last part of the theorem is a version of the Weierstrass Approximation Theorem. Outside the interval of convergence, however, the polynomial is going to diverge, and if the probability of the wealth falling outside that interval of convergence is sufficiently high, it can influence the optimal control. The higher the degree of the polynomial, the faster the divergence, and the larger the influence on the optimal control. We wish to keep the influence of the divergence under control. In order to do that, we need to increase the length of the interval between the lower and upper points of evaluation, such that the probability of ending outside is sufficiently low, without compromising the quality of the polynomial approximation to the original utility function. For this purpose we define a sequence of $x_{n,0}(t,x)$ and $x_{n,n}(t,x)$ that satisfies precisely this requirement.

Definition 6.4.3 (Maximally Increasing Sequence)**.** *For any (t,x), and any uniformly continuous function $u(t,x,z)$ for $z \in (p_1,p_2)$ the Maximally Increasing Sequence (MIS) of lower and upper points is defined as*

$$x_{n,0} = y_0(m(n)), \quad x_{n,n} = y_n(m(n)),$$

where

$$y_0(m) := \begin{cases} -m, & \text{if } -\infty = p_1, \\ p_1 + d/m, & \text{otherwise,} \end{cases}$$

$$y_n(m) := \begin{cases} m, & \text{if } \infty = p_2, \\ p_2 - d/m, & \text{otherwise,} \end{cases},$$

$$d := \begin{cases} (p_2 - p_1)/3, & \text{if } -\infty < p_1 < p_2 < \infty, \\ 1, & \text{otherwise.} \end{cases},$$

The increasing sequence $m(n)$ is defined iteratively as $m(1) := 1$,

$$m(n) := \begin{cases} m(n-1)+1, & \text{if (6.4.1) holds for } (z,y) \in \Omega_n, \\ m(n-1), & \text{otherwise,} \end{cases},$$

where both

$$\begin{aligned} \Omega_n := \{&(z,y) \in [y_0(m(n-1)+1), y_n(m(n-1)+1)]^2 \\ &\text{such that} \quad | z - y |< n^{-1/2}\}, \end{aligned}$$

$$\begin{aligned} | u(t,x,z) - u(t,x,y) |<4\varepsilon\Bigg(&(y_0(m(n-1)+1) + y_n(m(n-1)+1)) \\ &\times (y_0(m(n-1)+1) + y_n(m(n-1)+1)-4) + 8\Bigg)^{-1}. \end{aligned} \tag{6.4.1}$$

The sequence is maximally increasing in the sense that the interval grows whenever it can do so without compromising the absolute difference between $u(t,x,z)$ and $u(t,x,y)$. The MIS is constructed to satisfy two needs. We want the lower and upper points to converge to the support of $X^\pi(T)$, but we also want the generalized Bernstein polynomial to produce a close approximation of the original utility function. In essence the interval $(x_{n,0}, x_{n,n})$ gets larger and larger, but it only gets larger when we can be sure that we maintain the inequality given by (6.4.1). If we cannot increase the interval under that condition, we decrease the maximum distance between z and y, and we do this until we can, again, increase the interval. Using the MIS, we get the following theorem.

Theorem 6.4.4 (Convergence of Generalized Bernstein Polynomials)**.** *For any (t,x) and $p_1, p_2 \in \mathbb{R}^{\pm\infty}$ and a uniformly continuous function $u(t,x,z)$ on (p_1,p_2), the MIS satisfies*

$$x_{n,0} \to p_1^+ \qquad x_{n,n} \to p_2^-,$$

and

$$\sup_{\substack{z,y\in[x_{n,0},x_{n,n}] \\ |z-y|<n^{-1/2}}} |\, u(z) - u(y)\, | \left(\frac{(x_{n,0}+x_{n,n})((x_{n,0}+x_{n,n})-4)+8}{4} \right) \to 0.$$

See Appendix 6.A.1 for the proof. Combining Theorems 6.4.2 and 6.4.4 we get the following lemma.

Lemma 6.4.5. *For arbitrary (t,x) and any uniformly continuous function in z, $u(t,x,z)$, for $z \in (p_1,p_2)$ where $p_1, p_2 \in \mathbb{R}^{\pm\infty}$, the MIS ensures that $\tilde{u}_n^B(t,x,z)$ converges towards $u(t,x,z)$. The convergence is uniform for $z \in [a,b]$ for $p_1 < a < b < p_2$.*

See Appendix 6.A.2 for the proof. This lemma states that for a uniformly continuous utility function, the generalized Bernstein polynomial converges towards u,

$$u(t,x,z) = \lim_{n\to\infty} \sum_{i=0}^{n} a_{i,n}^B(t,x) z^i, \tag{6.4.2}$$

and the convergence is uniform for z in any closed subset contained in (p_1,p_2). This is a vast generalization compared to the Taylor polynomial. However, there is still an issue with the generalized Bernstein Polynomial. The support of $X^\pi(T)$ is a subset of $[0,\infty)$ for any control π, and therefore $p_1 = 0, p_2 = \infty$ is of particular interest. The convergence is uniform for any bounded closed interval only, and the expectation of an infinite sum is equal to the sum of an infinite series of expectations only if the series converges uniformly in the whole range of the support of the random variable. As $b < \infty$, we can never guarantee that the convergence is uniform on the support of $X^\pi(T)$. In fact, it is definitely not uniform due to the divergent nature of polynomials., and the limit of the original and polynomial value functions do not coincide. One may try to ignore the support of $X^\pi(T)$ that falls outside $[a,b]$ but as we discussed towards the end of the last section, this may not be possible. If however, the support of $X^\pi(T)$ is restricted to a subset of $[a,b]$, we have that

$$E_{t,x}[u(t,X^\pi(t),X^\pi(T))] = \lim_{n\to\infty}\sum_{i=0}^{n} a_{i,n}^B(t,x)E_{t,x}[(X^\pi(T))^i].$$

Then Theorem 6.2.4 provides us with the equilibrium control to the original problem via the limit of the equilibrium control for the generalized Bernstein polynomial utility function. Unfortunately, the support of $X^\pi(T)$ is only restricted to a subset of $[a,b]$ for a particular set of controls, and only if the equilibrium control is contained within that set, we get the desired convergence of the control,

$$\hat{\pi} = \lim_{n\to\infty} \hat{\pi}_n^B.$$

A seemingly different issue is that Lemma 6.4.5 applies to uniformly continuous utility functions only. But this is actually related to the issue as above since the generalized Bernstein polynomials converge to any continuous function on any bounded interval $[a,b]$. So regardless of u being uniformly continuous or just continuous, the convergence is uniform for a bounded interval only. For uniformly continuous u we also get

$$\lim_{n\to\infty} \tilde{u}_n^B(t,x,z) = u(t,x,z) \text{ for } z\in(p_1,p_2),$$

but it does not help us produce the original equilibrium control as a limit of the Bernstein optimal controls.

The fact that we can choose the interval of convergence for the generalized Bernstein Polynomials is an improvement compared to Taylor polynomials, but it comes at a cost of slower convergence. Basically, we want two things to be satisfied at all times, namely

$$\tilde{u}_n^B \approx u \text{ for } z\in A, \text{ and } P_{t,x}(X^{\hat{\pi}}(T)\in A)\approx 1.$$

This trade-off between interval of convergence and approximation error is crucial, as numerical problems with high values of n have to be dealt with. In fact, the MIS is practically intractable, as the convergence $(x_{n,0},x_{n,n}) \to (p_1,p_2)$ is slow. The MIS only sees to how close $\tilde{u}_n^B$ approximates u, and adjusts $x_{n,0}$ and $x_{n,n}$ only when it can be done without compromising the closeness of that approximation. The other extreme is to choose $x_{n,0}$ and $x_{n,n}$ as a small and large quantile of $X^{\hat{\pi}}(T)$, and then not worry about how well $\tilde{u}_n^B$ approximates u.

It is reasonable to ask what we have gained by using generalized Bernstein polynomials in contrast to Taylor polynomials, and there is not a definitive answer. In general Taylor polynomials converge faster than Bernstein polynomials, but this is only useful if the interval of convergence captures the required contours of the utility function.

6.5 Numerical Analysis

Polynomial utility functions give rise to controls that are difficult to interpret based on analytical solutions, due to the intuitively challenging ratio of sums in (6.2.7). Plotting the polynomial utility functions and the corresponding optimal controls is perhaps the most accessible way of understanding how the former influence the latter. This section is dedicated to examine the differences between a Taylor optimal control and a Bernstein optimal control, as well as how the divergence and interval of convergence for the polynomial utility function influence the optimal control. The coefficients of all four polynomial utility functions in this section result, by Lemma 6.2.5, in wealth independent optimal controls.

Numerically, the optimal controls are found through an iterative scheme, that locates a fixed point of the mapping C given by

$$C(\{\pi(t)\}_{t\in[0,T]}) = \left\{ -\frac{\alpha - r}{\sigma^2 x} \cdot \frac{\sum_{i=0}^{n} a_i(t,x) m_x^{\pi}(i,t,x)}{\sum_{i=0}^{n} a_i(t,x) m_{xx}^{\pi}(i,t,x)} \right\}_{t\in[0,T]}$$

for $m^{\pi^*}(i,t,x) = E_{t,x}[(X^{\pi}(T))^i]$ and $\sum_{i=0}^{n} a_i(t,x) m_{xx}^{\pi}(i,t,x) < 0$. Equation (6.2.7) and Theorem 6.2.4 state that a fixed point of C is an optimal control for the polynomial with coefficients a_i. In all numerical calculations convergence of $\max_t \mid \pi_i(t) - \pi_{i+1}(t) \mid$ was achieved with a precision of 10^{-5}.

The control problems of the numerical section represent a lifetime investment problem with a 30-year investment horizon, i.e. $T = 30$, and market parameters given by

$$\begin{pmatrix} \alpha \\ r \\ \sigma \end{pmatrix} = \begin{pmatrix} 0.06 \\ 0.04 \\ 0.2 \end{pmatrix}.$$

The first axis of all plots are in units of current wealth, as all controls are independent of wealth.

6.5.1 Numerical Comparison for Power Utility

By examining the polynomial controls that arise by approximating a power utility function, we can benchmark their performance to the true optimal control. The constant relative risk aversion is $\gamma = 2$, leading to the optimal control

$$\pi^* = \frac{\alpha - r}{\sigma^2 \gamma} = 0.25.$$

Taylor Polynomials

In Figure 6.3 we have plotted a 10'th order Taylor approximation of a power utility function evaluated around the conditional mean of terminal wealth, for different time points.

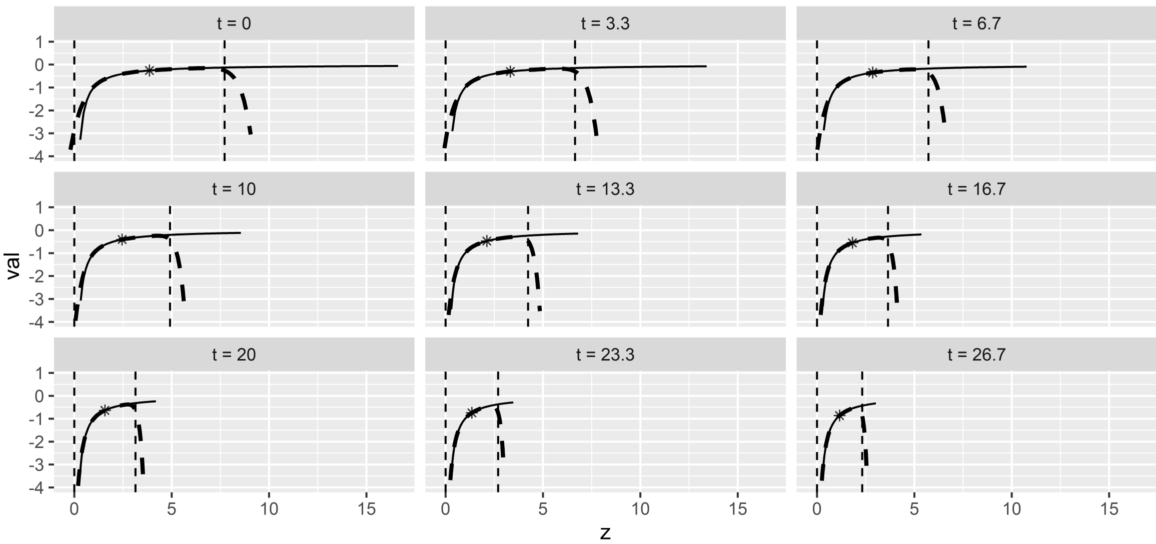

Figure 6.3: *The vertical lines indicate the interval of convergence for the Taylor series, found by performing a ratio test*

Based on these plots alone, it is hard to say whether or not the polynomial control is going to approximate the true optimal control well. The Taylor polynomial is a good approximation within the interval of convergence, but the influence of the divergence outside the interval is unknown. In Figure 6.4 we have plotted 6 polynomial controls for 6 different degrees of Taylor polynomials that are formed by expanding around the conditional mean of terminal wealth.

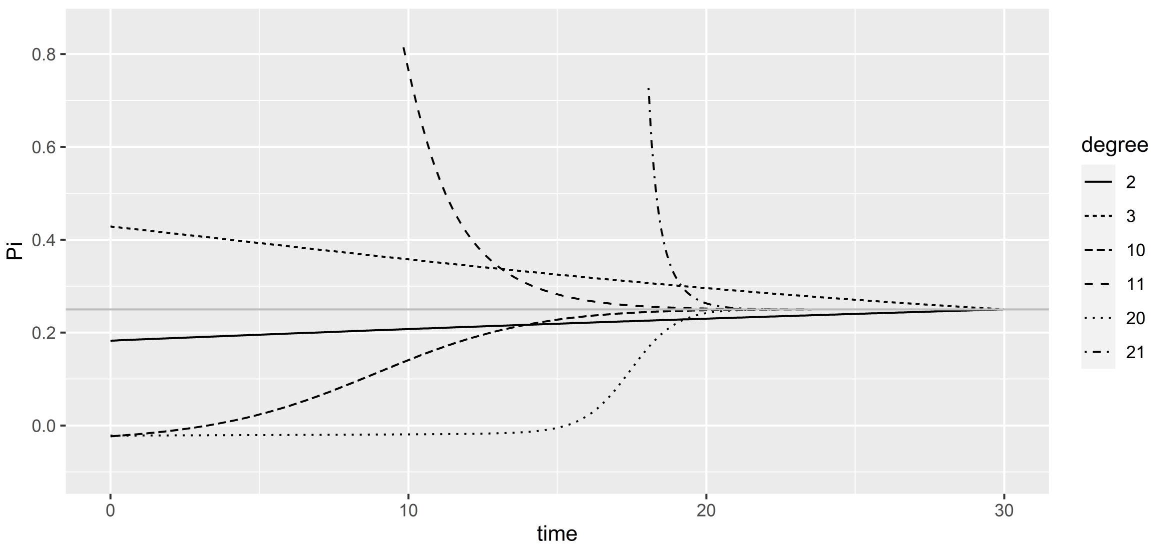

Figure 6.4: *The Taylor optimal control with expansion around* $E[X^{\hat{\pi}_n}(T)]$*. True optimal control is* $\pi^* = 0.25$

This plot demonstrates the problem caused by divergence. When the degree is even and the polynomial diverges towards $-\infty$, the preferences of the investor become wealth-fearing. He seeks to avoid large values of wealth, and his will to avoid this is determined by the probability of the event. For that reason he has an incentive to avoid uncertainty, and therefore decreases his holdings in the stock. As a consequence of Lemma 6.2.1, the probability of the extreme event is increasing in $(T - t)$, which explains why the influence of the divergence is small when close to maturity. Conversely, when the degree is odd, and the polynomial diverges towards ∞, the preferences of the investor become risk-loving, and therefore he increases his holdings in the stock to the point of infinite gearing.

We can remedy the problem of the Taylor optimal control, by evaluating the polynomial around a quantile of $X^{\pi}(T)$, thus ensuring that the probability of the extreme event is kept at bay. In Figure 6.5 we have plotted the polynomial control of a Taylor expansion around the 99.9% quantile of $X_n^{\hat{\pi}}(T)$.

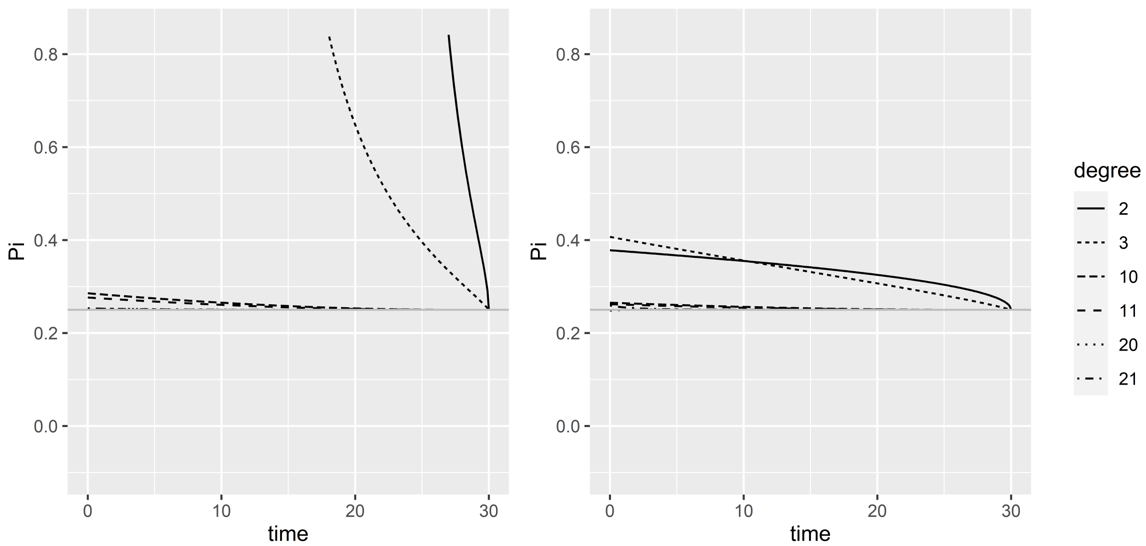

Figure 6.5: *Left: Taylor optimal control with expansion around 99.9% quantile of terminal wealth. Right: Taylor optimal control with expansion around $1-n^{-2}$-quantile of terminal wealth*

For degrees $n = 2, 3$, the Taylor expansion around the 99.9% quantile does a poorer job than if the expansion had been done around the mean value, but for higher degrees of polynomials the polynomial controls do a much better job at approximating the true optimal control. A more refined alternative is to choose an n-dependent quantile such as the $1-n^{-2}$-quantile, which produces the optimal controls seen to the right in Figure 6.5. So far we have achieved Taylor controls that are considerably closer to the true optimal control, than the Taylor controls produced by Nordfang and Steffensen (2017). In the next subsection we find the Bernstein controls for a power utility function.

Bernstein Polynomials

In order to construct the Bernstein polynomials, we need to decide upon a time and wealth dependent lower and upper point of expansion. These lower and upper points could for instance be the points defined by the MIS. The MIS is, however, a poor choice as it favours the approximation of the utility function but completely disregards the distribution of terminal wealth. Instead, we could use an n-dependent lower and upper quantile of $X^{\hat{\pi}}(T)$ given via Lemma 6.2.1 by

$$x_{n,0}(t,x) = xe^{\left(\Phi(n^{-\eta})\sigma\sqrt{\int_t^T \hat{\pi}(s)^2 ds}+(T-t)r+(\alpha-r)\int_t^T \hat{\pi}(s)ds-\frac{\sigma^2}{2}\int_t^T \hat{\pi}(s)^2 ds\right)},$$
$$x_{n,n}(t,x) = xe^{\left(\Phi(1-n^{-\eta})\sigma\sqrt{\int_t^T \hat{\pi}(s)^2 ds}+(T-t)r+(\alpha-r)\int_t^T \hat{\pi}(s)ds-\frac{\sigma^2}{2}\int_t^T \hat{\pi}(s)^2 ds\right)},$$

corresponding to the $n^{-\eta}$ and $1-n^{-\eta}$ quantiles of terminal wealth. We have settled on $\eta = 2$ for the numerical studies. The trouble with this choice of points is that they both converge to x for $t \to T$, which results in polynomials with numerically challenging coefficients. Instead, we subtract and add an amount to the lower and upper point respectively, to ensure that the lower and upper points do not converge.

$$\tilde{x}_{n,0}(t,x) = x_{n,0}(t,x) - x_{n,0}(0,x)\xi_L$$
$$\tilde{x}_{n,n}(t,x) = x_{n,n}(t,x) + x_{n,0}(0,x)\xi_U$$

For approximations of a power utility function, we found that $\xi_U = \xi_L = 0.3$ produces polynomials with numerically practical coefficients. The resulting optimal controls are plotted in Figure 6.6.

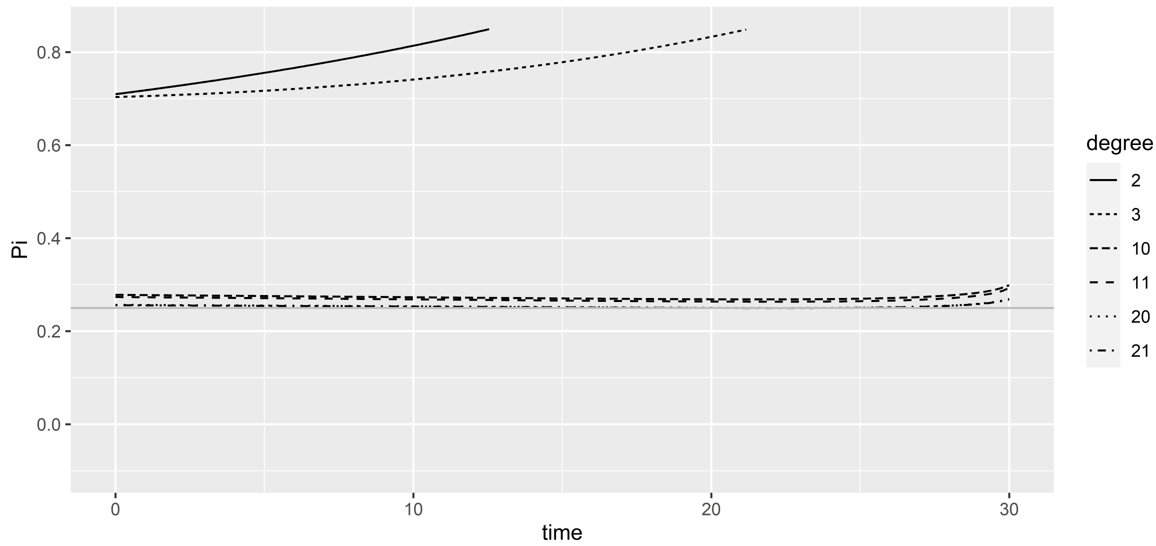

Figure 6.6: *Bernstein induced optimal controls using $\tilde{x}_{n,0}$ and $\tilde{x}_{n,n}$*

Note that the optimal controls all have a higher proportion invested in the stock than the true optimal control. This is because the Bernstein polynomial utility function has a higher marginal utility on the interval $(x_{n,0}, x_{n,n})$. For degrees of polynomials larger than 3, the Bernstein controls are more stable and closer to the true optimal control, than the corresponding Taylor controls. This is in line with how well the Bernstein- and Taylor polynomials approximate the power utility function - the Bernstein polynomials form a good approximation over an interval even larger than $(x_{n,0}, x_{n,n})$, whereas the Taylor polynomials only form a good approximation over the interval $(0, 2d(t,x))$. See Figure 10 and 11 in the Appendix for the Bernstein and Taylor polynomial utility functions.

The comparison of Taylor and Bernstein polynomials for a Power utility function does not provide any new insights concerning the optimal control, as the true optimal solution is known. It does however show that the polynomial controls can be used to approximate a non-polynomial control. In the next section we step into uncharted territory and examine the polynomial control of a prospect theory, S-shaped utility function with wealth dependent reference point, where the true optimal control is unknown.

6.5.2 Numerical Comparison for S-shaped utility

An optimal control for the S-shaped utility function proposed by Tversky and Kahneman (1986) with a wealth dependent reference point has, to our knowledge, not yet been found. Tversky and Kahneman (1986) proposed an asymmetric S-shaped utility function that allowed for different relative risk aversions in losses and gains. We consider the symmetric S-shaped utility function

$$u_s(t,x,z) = \frac{\text{sign}\{z - xh(t)\}}{1-\gamma} \mid z - xh(t) \mid^{1-\gamma} \qquad h(t) = e^{\rho(T-t)},$$

where z should be thought of as the wealth at termination, and ρ is a minimum return target. Note that we require $\gamma < 1$ for the utility function to be continuous. Following Nordfang and Steffensen (2017) we choose $\gamma = 0.5$. Døskeland and Nordahl (2008) find the optimal control of an investor with an S-shaped utility function, pre-committed to a fixed reference point K. Kryger et al. (2020) find the optimal control that minimizes the quadratic distance of a sophisticated investor who continuously reevaluates his return target based on his current wealth, thereby having the value function

$$\frac{1}{2}\mathrm{E}_{t,x}\left[\left(X^\pi(T) - xh(t)\right)^2\right]. \tag{6.5.1}$$

Kryger et al. (2020) conduct a numerical study for $h(t) = e^{\rho(T-t)}$. In this subsection we contribute with an approximate solution, based on Taylor and Bernstein polynomials. As an exact solution has not yet been found, we cannot benchmark the polynomial controls to the true optimal control. However, the value function (6.5.1) in many ways resembles the value function for the investor with an S-shaped utility function given by,

$$J(t,x) = \mathrm{E}_{t,x}\left[\frac{\text{sign}\{z - xh(t)\}}{1-\gamma} \mid z - xh(t) \mid^{1-\gamma}\right] \tag{6.5.2}$$

Unlike (6.5.1), (6.5.2) cannot be partitioned into elements of $E_{t,x}[X^\pi(T)]$ and $E_{t,x}[X^\pi(T)^2]$, which give access to the solution found by Kryger et al. (2020). We believe that the financial incentives for the investors with value functions (6.5.1) and (6.5.2) are similar; both investors compare their wealth at termination with their current wealth compounded with some target rate of return. This leads us to believe that the optimal control for the value function (6.5.2) will resemble that of (6.5.1), in particular we expect the optimal control to be a decreasing function of time. We have chosen $\rho = 0.02 = r/2$, in order to reflect the preferences of an investor who has realistic market expectations.

Taylor Polynomials

A ratio test shows that the Taylor polynomial around $d(t,x) > xh(t)$ converges to u_s for $z \in (xh(t); 2d(t,x) - xh(t))$, and so the risk willingness in losses cannot be captured by a Taylor polynomial. Due to infinite marginal utility in $z = xh(t)$, the investor has a limitless incentive to ensure that the terminal wealth does not fall below the reference point. Hence, risk willingness in losses is an irrelevant trait of the investor with value function (6.5.2). That is not to say that risk willingness in losses never is relevant. In any incomplete market, where such a downside constraint cannot be hedged, risk-willingness in losses cannot be ignored.

In order to avoid an interval of convergence that vanishes as $t \to T$, we expand the Taylor polynomial around the 99% quantile plus a constant. The resulting controls are seen in Figure 6.7

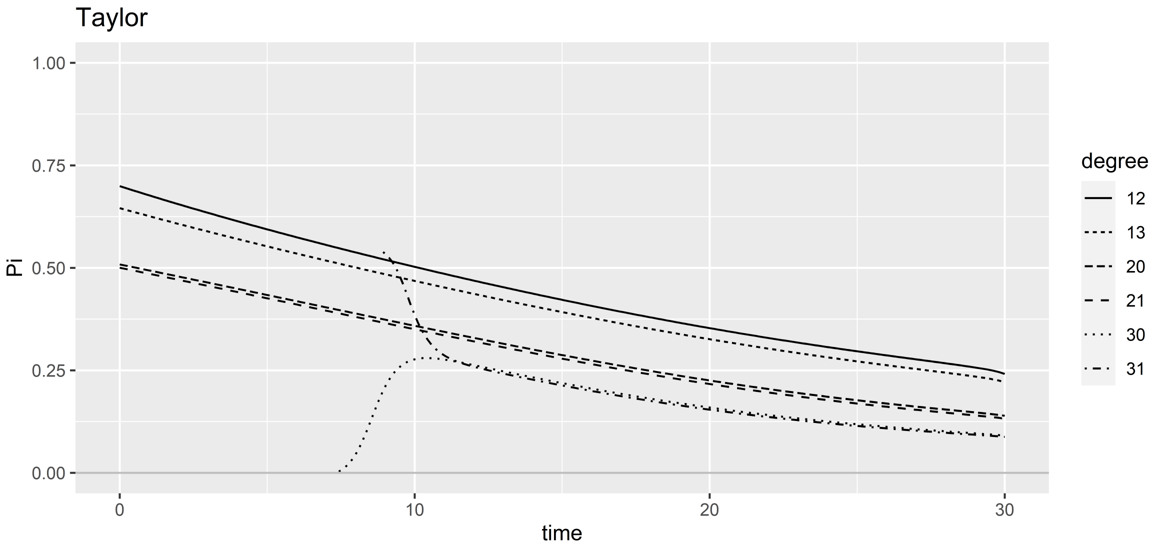

Figure 6.7: *Taylor polynomial controls for S-shaped utilit with* $\rho = 0.02$, $d(t,x) = F^{-1}(0.99) + 1.75$

As we expect, the optimal control is a decreasing function of time, but unlike the optimal control for (6.5.1), the Taylor controls do not result in a risk-free position at termination. This discrepancy is probably caused by the fact that the Taylor polynomials do not have infinite marginal utility in x at termination, and therefore the incentive to avoid wealth below the minimum return target is not uncompromising. We also see that the divergence of the polynomials has an influence on the optimal control for $n = 30, 31$, which can be corrected by enlarging the interval of convergence.

As a sanity check, we also find the optimal control for $\rho = r$, which ought to result in a constant optimal control with all wealth in the risk free asset.

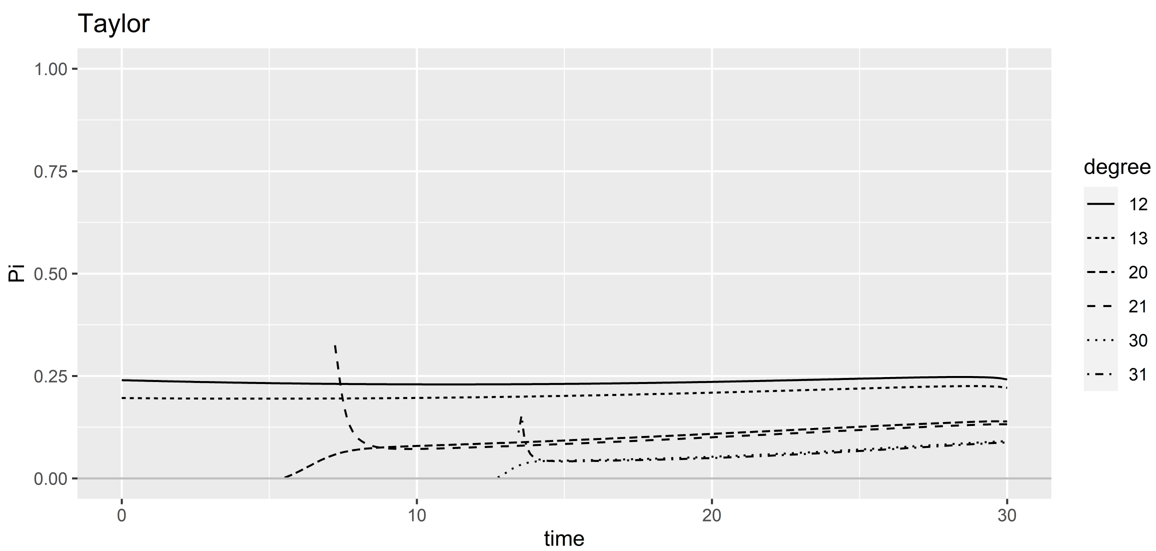

Figure 6.8: *Taylor controls, $\rho = r$*

The Taylor control for $\rho = r$ is close to constant but not equal to zero, probably due to the Taylor polynomial not having infinite marginal utility in the reference point. Interestingly, the optimal allocation in the stock at termination is close to identical for both values of ρ, as seen in Table 6.1 Perhaps these terminal values indicate the size of

Degree	$\rho = 0.02$	$\rho = r$
12	$\hat{\pi}(T)$ =0.2414	$\hat{\pi}(T)$ =0.2415
13	$\hat{\pi}(T)$ =0.2211	$\hat{\pi}(T)$ =0.2211
20	$\hat{\pi}(T)$ =0.1391	$\hat{\pi}(T)$ =0.1391
21	$\hat{\pi}(T)$ =0.1321	$\hat{\pi}(T)$ =0.1321
30	$\hat{\pi}(T)$ =0.0909	$\hat{\pi}(T)$ =0.0909
31	$\hat{\pi}(T)$ =0.0879	$\hat{\pi}(T)$ =0.0879

Table 6.1: *Allocation in risky asset at termination*

the error for the Taylor controls, and a corresponding downwards shift might provide a closer approximation to the true optimal control.

The Taylor polynomial has to be constructed based on a single point of expansion, and it cannot take risk willingness in losses into account. The Taylor approximation of the S-shaped utility function is therefore equivalent to a shifted power utility function, with a wealth- and time-dependent shift $xe^{\rho(T-t)}$. If we want to include the "S-shape" we have to use another type of polynomial.

Bernstein Polynomials

Unlike Taylor polynomials, Bernstein polynomials can incorporate the risk willingness in losses that are characteristic for S-shaped utility functions. Due to the slow convergence of Bernstein polynomials, a high degree of the polynomial is necessary for the approximation to be close - but even so, the lack of infinite marginal utility in the reference point affects the optimal control. The upper and lower points for the Bernstein polynomials are chosen in the same way as in Section 6.5.1 with $\xi_L = 0$ and $\xi_U = 0.6$, resulting in the optimal controls seen in Figure 6.9 for $\rho = 0.02$ and $\rho = r$.

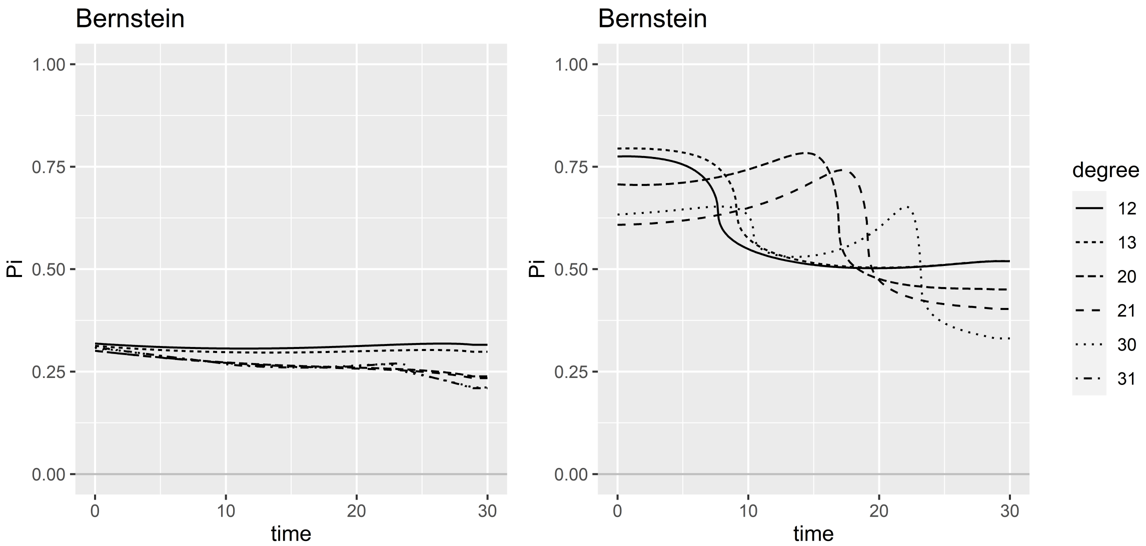

Figure 6.9: *Left: Bernstein controls for $\rho = 0.02$. Right: Bernstein controls for $\rho = r$*

For $\rho = 0.02$ there is a slight decrease in the holdings of the risky asset over time, but it is not as pronounced as in the Taylor case. As can be seen in Figure 13 and 12, the Bernstein polynomials form a good approximation over a larger interval than the corresponding Taylor polynomials. However, the behaviour of the polynomials for large values of wealth, are not as important as the behaviour for values of wealth close to the reference point, where the Taylor approximation is better. For $\rho = r$ the Bernstein controls are far from constant and confusing to interpret, but nonetheless optimal for the corresponding polynomial utility function. Bernstein polynomials of degree n are essentially a weighted average of n points on an interval. The position of these points on the function that we are trying to approximate, has an influence on the accuracy of the approximation. In the case of S-shaped utility functions, the 'bend' around the reference point has a critical influence on the optimal control, which can be seen if the points used by the Bernstein polynomial are placed poorly. When the approximation is poor, the punishment for falling below the reference point is small, resulting in large holdings of the risky asset - these are the humps seen in the optimal controls. As the placement of points depends on the degree of the polynomial as well as the time to maturity, the humps are placed differently depending on time and n.

By their construction, Bernstein polynomials have a smoothing effect, which for the case of an S-shaped utility function implies that the marginal utility in the reference point is nowhere as large as for a corresponding Taylor polynomial. For the S-shaped utility function and the complete Black-Scholes market of consideration in the present paper, Bernstein polynomials are inferior to Taylor polynomials. In an incomplete market where uncertainty is prevalent, the smoothing of the utility function is not destructive. We may even motivate the smoothing of the utility function as the constructive feature in an incomplete market.

6.5.3 Conclusion of Numerical results

There is no way to objectively compare the Bernstein and Taylor controls, as the quality of their approximation depends on the subjectively chosen way of constructing the polynomials. Nevertheless, we believe that Bernstein controls are superior to Taylor controls for a class of utility functions where the marginal utility for all likely values of terminal wealth is modest - power utility for instance. When the marginal utility is high, the Bernstein polynomial will perform a poor approximation, and high marginal utility tends to have a large influence on the optimal control. So in the case of for instance S-shaped utility, a Taylor control is probably better suited, provided that the interval of convergence contains most of the possible values of terminal wealth.

6.A Proofs

6.A.1 Proof of Theorem 6.4.4.

Proof. Let $\varepsilon > 0$ be given. If p_1 and p_2 are finite we wish to show that there exists an N such that for $n > N$,

$$x_{n,0} - p_1 < \varepsilon, \qquad p_2 - x_{n,n} < \varepsilon,$$

and

$$\sup_{\substack{z,y\in[x_{n,0},x_{n,n}]\\ |z-y|<n^{-1/2}}} | u(z) - u(y) | \left(\frac{(x_{n,0} + x_{n,n})((x_{n,0} + x_{n,n}) - 4) + 8}{4} \right) < \varepsilon.$$

If $p_1 = -\infty$ we need to show that $x_{n,0} < -1/\varepsilon$, and similarly that $x_{n,n} > 1/\varepsilon$ for $p_2 = \infty$.

Note by the uniform continuity of u, that for any integer M_1 there exists an integer M_2 such that

$$\begin{aligned}
&\forall z, y \in [y_0(M_1), y_n(M_1)]; \quad | z - y |< M_2^{-1/2}\\
&\Rightarrow\\
&| u(z) - u(y) |< \frac{4\varepsilon}{(y_0(M_1) + y_n(M_1))((y_0(M_1) + y_n(M_1)) - 4) + 8}.
\end{aligned}$$

This implies that no matter the value of $m(n)$, it will increases by one eventually. Thus we can conclude that $m(n) \to \infty$. Let N be the smallest integer for which

$$m(N) > \max\left(\frac{d}{\varepsilon}, 1\right).$$

By construction of the sequence $m(n)$ we see that for all $n > N$

- if $-\infty = p_1$ then $x_{n,0} = y_0(m(n)) = -m(n) \le -m(N) < -1/\varepsilon$,
- if $-\infty < p_1$ then $x_{n,0} - p_1 = y_0(m(n)) - p_1 = d/m(n) \le d/m(N) < \varepsilon$,
- if $\infty = p_2$ then $x_{n,n} = y_n(m(n)) = m(n) \ge m(N) > 1/\varepsilon$,
- if $\infty > p_2$ then $p_2 - x_{n,n} = p_2 - y_n(m(n)) = d/m(n) \le d/m(N) < \varepsilon$.

Furthermore,

$$\begin{aligned}
&\sup_{\substack{z,y\in[x_{n,0},x_{n,n}]\\ |z-y|<n^{-1/2}}} | u(z) - u(y) | \left(\frac{(x_{n,0} + x_{n,n})((x_{n,0} + x_{n,n}) - 4) + 8}{4} \right)\\
&<\varepsilon \left(\frac{4}{(y_0(m(n)) + y_n(m(n)))(y_0(m(n)) + y_n(m(n)) - 4) + 8} \right)\\
&\quad \times \left(\frac{(y_0(m(n)) + y_n(m(n)))(y_0(m(n)) + y_n(m(n)) - 4) + 8}{4} \right)\\
&=\varepsilon.
\end{aligned}$$

□

6.A.2 Proof of Lemma 6.4.5.

Proof. By Theorem 6.4.2

$$\begin{aligned} &\mid u(t,x,z) - \tilde{u}_n^B(t,x,z) \mid \\ \leq &\omega_n(u(t,x,z), n^{-1/2}) \left(\frac{(x_{n,0}+x_{n,n})((x_{n,0}+x_{n,n})-4)+8}{4} \right), \end{aligned}$$

for $z \in [x_{n,0}, x_{n,n}]$. By Theorem 6.4.4 there exists a sequence $\{x_{n,0}, x_{n,n}\}_n$ such that

$$\omega_n(u(t,x,z), n^{-1/2}) \left(\frac{(x_{n,0}+x_{n,n})((x_{n,0}+x_{n,n})-4)+8}{4} \right) \to 0,$$

and $\{x_{n,0}, x_{n,n}\} \to (p_1, p_2)$. Note that

$$\left(\frac{(x_{n,0}+x_{n,n})((x_{n,0}+x_{n,n})-4)+8}{4} \right) \geq 1,$$

and that there exists an M such that for $m > M$, $[a,b] \subset [x_{m,0}, x_{m,m}]$ whereby

$$\mid u(t,x,z) - \tilde{u}_m^B(t,x,z) \mid < \varepsilon, \quad \text{for } z \in [a,b],$$

proving that the convergence is uniform on $[a,b]$. □

6.B Figures

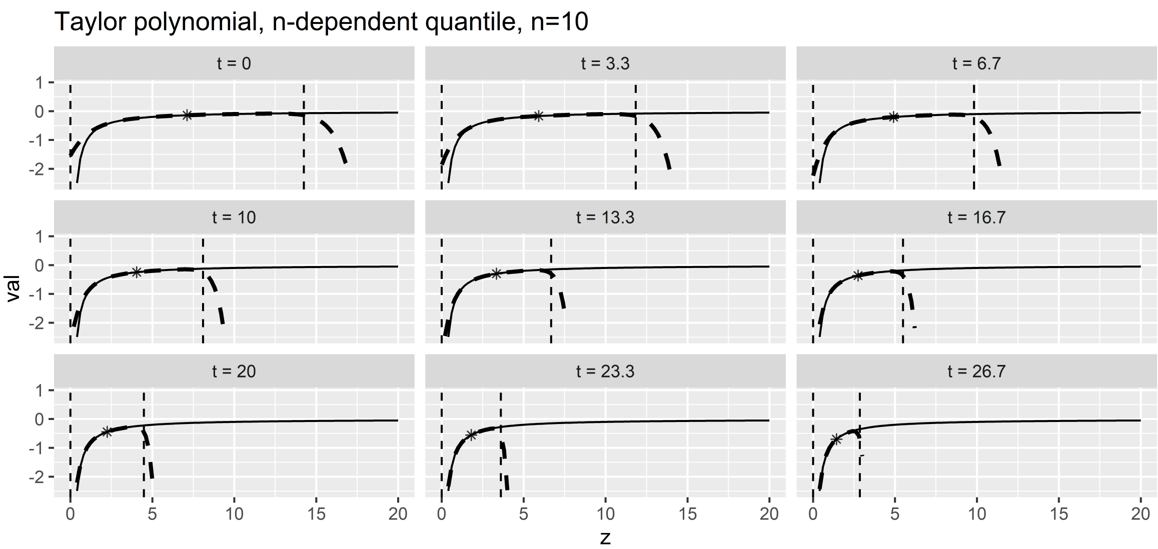

Figure 10: *Taylor power polynomials for $n = 10$, with n-dependent quantiles*

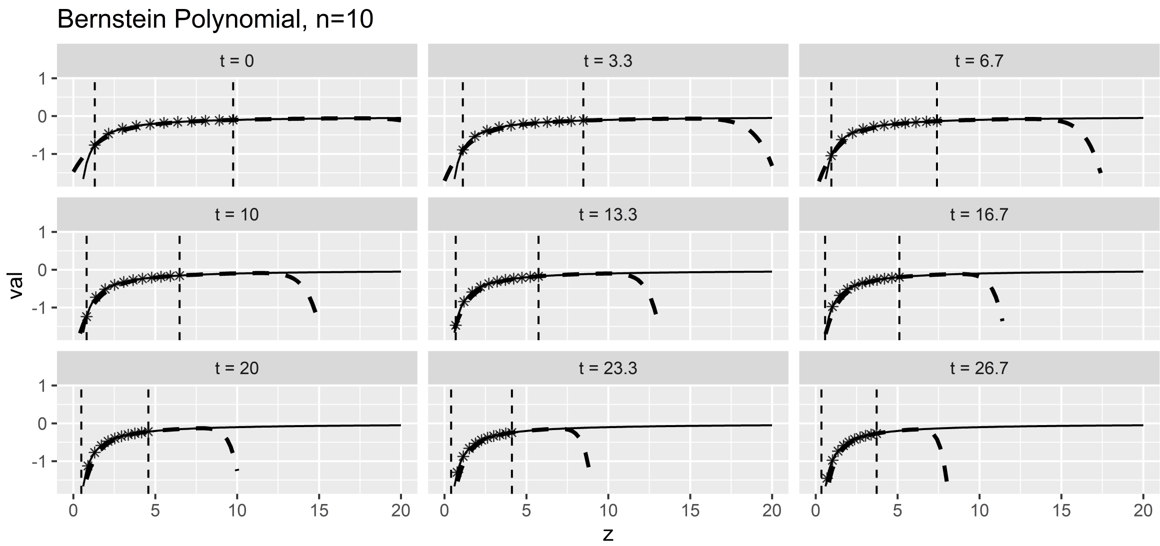

Figure 11: *Bernstein power polynomials for $n = 10$, using $\tilde{x}_{n,0}$ and $\tilde{x}_{n,n}$ as lower and upper points*

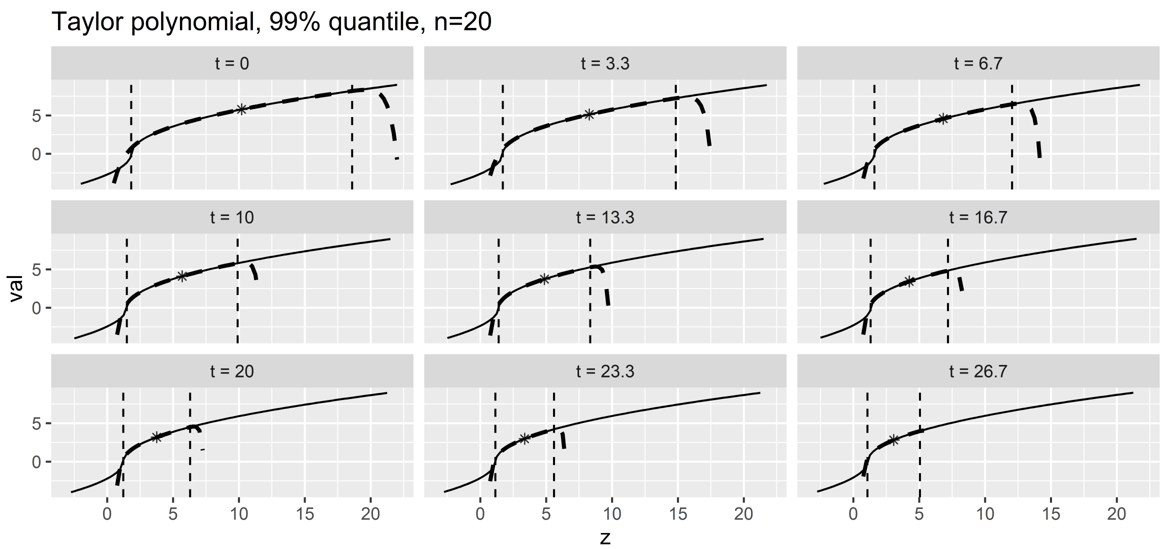

Figure 12: *Taylor power polynomials for $n = 20$, $\rho = 0.02$, expanding around the 99% conditional quantile of terminal wealth*

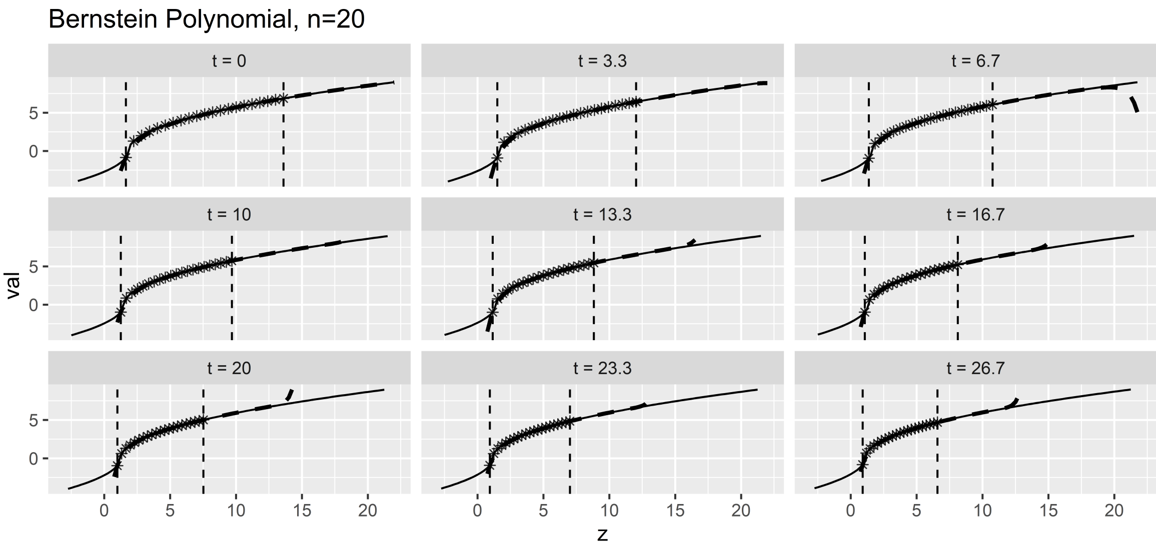

Figure 13: *Bernstein power polynomials for $n = 20$, $\rho = 0.02$, using $\tilde{x}_{n,0}$ and $\tilde{x}_{n,n}$ as lower and upper points. The "bend" is included, even though it is hard to confirm visually*

Chapter 7

Power Utility with Dynamic Reference Point

Abstract

We study a power utility function where utility of terminal wealth is determined by the excess over current discounted wealth, motivated by central concepts from prospect theory. The ever-changing target in the utility function implies time-inconsistent preferences, and accordingly, the control problem is approached by equilibrium theory. Due to the form of the utility function, standard guesses on the solution of the HJB equation of the problem are unfruitful. By assuming a wealth-independent control, the definition of the equilibrium control as the minimizer of the time-derivative in the value function, yields a fixed-point equation for the equilibrium control that we study numerically.

Keywords: Stochastic control, Prospect theory, Time-inconsistent preferences, Equilibrium theory.

7.1 Introduction

We study the finite horizon investment problem of an investor who trades at a Black-Scholes market and has a power utility function with a dynamic wealth- and time-dependent reference point. The utility function is motivated by prospect theory, and the dynamic time-inconsistent objective of the investor calls for a game-theoretic equilibrium solution. We argue for and assume that the equilibrium control is independent of wealth, and derive a fixed-point equation for the equilibrium control, and examine its properties in a numerical study.

Delegating difficult decisions by relying on a mathematical recipe that tells you how to invest your assets, is an appealing concept. Accordingly, portfolio selection problems have been of interest since the inception of modern finance. Markowitz (1952) was the first to formulate the portfolio selection problem as a problem of finding an optimal trade-off between maximizing the expected return and minimizing the variance. Merton (1969) formulated the balance of return and risk in continuous time through utility functions, and derived the so-called Merton's fraction representing the optimal investment proportion in risky assets for a constant relative risk aversion utility function. Utility functions with constant relative risk aversion are appealing from a mathematical standpoint, but difficult to justify for real-world modelling.

Introduced by Tversky and Kahneman (1979), prospect theory presented a descriptive model of decision making under risk, and is now one of the prevalent theories in behavioural economics. Under prospect theory, decisions are made based on gains and losses relative to a target/reference point as opposed to absolute values. The ever-changing reference point makes mathematical treatment of portfolio selection problems under prospect theory difficult. The typical approach to marry prospect theory and portfolio selection, is to fix the reference point (Døskeland and Nordahl (2008), Zhou and He (2011), Dong and Zheng (2020)) or to specify it as a stochastic variable (Berkelaar et al. (2004), Jin and Yu Zhou (2008), Rasonyi and Rodrigues (2012)), thus avoiding the time-inconsistent nature of the problem.

Perhaps because of the very human nature of having objectives that are inconsistent accross time, the study of time-inconsistent problems has a long history dating back to Strotz (1955). More recently, Björk and Murgoci (2010) define the solution to time-inconsistent consumption-investment problems as a Nash equilibrium, where the control problem conceptually is regarded as a game between a continuum of agents. Adopting the equilibrium approach, Kryger et al. (2020) provide a verification theorem for a general class of time-inconsistent problems.

Using the verification theorem of Kryger et al. (2020), Nordfang and Steffensen (2017) and Lollike and Steffensen (2021) find the equilibrium control for a class of polynomial utility functions, including polynomial approximations of a power utility function with dynamic wealth- and time-dependent reference point.

In this paper we directly solve the equilibrium control problem of an investor with a power utility function with dynamic reference point. To our knowledge, this is the first closed-form solution to a power utility function with a dynamic reference point. To facilitate the solution we assume that the equilibrium control is deterministic, and provide arguments to justify this assumption.

The paper is structured as follows. In Section 7.2 we provide the framework for the problem by defining the Black-Scholes market, the equilibrium control, and the utility function. In Section 7.3 we present the verification theorem of Björk et al. (2017), which would have provided us with the equilibrium control, if we had been able to guess a solution to the extended HJB equation. In Section 7.4 we argue that the equilibrium control is independent of wealth, facilitating another way of finding the equilibrium control as the minimizer of the time-derivative in the value function. In Section 7.5 we present the fixed-point equation for the equilibrium control, and in Section 7.6 we conduct a numerical study.

7.2 The Problem

7.2.1 Framework

We consider a Black-Scholes market consisting of a bank account B and a single stock S, expressed by the SDEs

$$\begin{aligned} dB(t) =& rB(t)dt, \quad B(0) = 1 \\ dS(t) =& \alpha S(t)dt + \sigma S(t)dW(t), \quad S(0) = s_0 > 0, \end{aligned}$$

where W is a standard Brownian motion. We assume that $r < \alpha$ and $\sigma > 0$. The investor with initial wealth x_0, who invests in this market and allocates the proportion $\pi(t)$ in the stock at time t, has dynamics of wealth given by

$$X^\pi(t) =(r + \pi(t)(\alpha - r))X^\pi(t)dt + \pi(t)\sigma X^\pi(t)dW(t), \tag{7.2.1}$$

$$X^\pi(0) =x_0. \tag{7.2.2}$$

We assume that the portfolio is self-financing – i.e. there is no withdrawal or deposit of wealth for $t > 0$. The investment horizon is finite and terminates at a deterministic point in time T. The central topic of this paper is to find the (in some sense) optimal proportion of wealth to be invested in the stock, called the control law, given the utility function of the investor. We misuse notation and let π represent both a stochastic optimal control law as well as a feedback control law, which is a deterministic function of the current time and wealth. If not stated otherwise, π is a feedback control law.

Wealth-independent controls are a essential to this paper because they imply that the distribution of terminal wealth follows a log-normal distribution.

Lemma 7.2.1. *For wealth-independent controls $\pi(t,x) = \pi(t)$, the distribution of terminal wealth, $X^\pi(T)$ given the current wealth $X^\pi(t) = x$ for the SDE with dynamics given by* (7.2.1) *follows a log-normal distribution,*

$$X^\pi(T) \sim LN\left(\log(x) + \int_t^T \left(r + \pi(s)(\alpha - r) - \frac{(\sigma\pi(s))^2}{2}\right) ds, \int_t^T (\sigma\pi(s))^2 ds\right). \tag{7.2.3}$$

Proof of Lemma 7.2.1. Define the process

$$dZ^\pi(t) = r + \pi(t)(\alpha - r) - \frac{\sigma^2\pi(t)^2}{2} dt + \pi(t)\sigma dW(t), \tag{7.2.4}$$

$$Z^\pi(t) = \log(x), \tag{7.2.5}$$

and note that the time-T value follows a normal distribution

$$Z^\pi(T) \sim \mathcal{N}\left(\log(x) + \int_t^T \left(r + \pi(s)(\alpha - r) - \frac{(\sigma\pi(s))^2}{2}\right) ds, \int_t^T (\sigma\pi(s))^2 ds\right).$$

Applying Itô's lemma to find the dynamics of $e^{Z^\pi(t)}$ gives the relation

$$e^{Z^\pi(t)} = X^\pi(t).$$

□

Lemma 7.2.1 allows us to formulate the equilibrium problem in probabilistic terms, which is the essential tool that provides us with a solution.

7.2.2 Defining Optimality

For time-consistent problems, the optimal control is elegantly defined as the control that achieves the supremum over the expected utility of terminal wealth,

$$\sup_\pi \mathrm{E}_{t,x}[u(X^\pi(T))].$$

Defining optimality for time-inconsistent problems is not as simple. As a way to quantify the value of a control, we introduce the value function,

$$J^\pi(t,x) = E_{t,x}[u(t, X^\pi(t), X^\pi(T))].$$

For fixed (t_1, x_1) we can find the control $\pi^*_{t_1,x_1}(t,x)$ for $(t,x) \in [t_1, T] \times \mathbb{R}$ that achieves the supremum

$$\pi^*_{t_1,x_1}(t,x) = \arg\sup_\pi J^\pi(t,x).$$

At another point in time, with another value of wealth (t_2, x_2), the control that achieves the supremum is not necessarily going to be the same for all values of (t, x), i.e. $\pi^*_{t_1,x_1}(t,x) \neq \pi^*_{t_2,x_2}(t,x)$. Instead of accepting this inconsistent, naïve control as a solution to the problem, we reformulate the objective of the investor to be more sophisticated.

Following Björk and Murgoci (2010) and Björk et al. (2017) we define the optimal control as a Nash subgame equilibrium control.

Definition 7.2.2 (Equilibrium control)**.** *Consider a control $\hat{\pi}$. Choose a real number $h \in (0, T)$ and a fixed initial point (t, x), such that $t \in [0, T-h)$. Define for a control π the control π^h by*

$$\pi^h_{(t,x)}(s,y) = \begin{cases} \pi(s,y), & \text{for } (s,y) \in [t, t+h) \times \mathbb{R}, \\ \hat{\pi}(s,y), & \text{for } (s,y) \in [t+h, T] \times \mathbb{R}. \end{cases}$$

If

$$\liminf_{h \to 0} \frac{J^{\hat{\pi}}(t,x) - J^{\pi^h}(t,x)}{h} \geq 0,$$

for all controls π for which π^h is admissible, then $\hat{\pi}$ is an equilibrium control.

We return to the definition of admissibility, and for now assume that all controls are admissible. Intuitively, the equilibrium control in the fixed point (t, x) corresponds to the optimal control for the investor with (t, x)-dependent preferences, given that the future control is fixed. Rewriting the differences in value functions

$$\begin{aligned} \frac{J^{\hat{\pi}}(t,x) - J^{\pi^h}(t,x)}{h} &= \frac{J^{\hat{\pi}}(t,x) - J^{\pi^h}(t,x) + \overbrace{J^{\pi^h}(t+h,x) - J^{\hat{\pi}}(t+h,x)}^{=0}}{h} \\ &= \frac{\partial}{\partial t} J^{\pi^h}(t,x) - \frac{\partial}{\partial t} J^{\hat{\pi}}(t,x) + o(h), \end{aligned}$$

and taking the limit as $h \to 0$, we see that the equilibrium control by definition yields the smallest marginal change in the value function. By finding the control that minimizes the derivative of the value function in t, we find the equilibrium control.

The definition of the equilibrium control is essential for time-inconsistent problems, and the one provided above may not be suited for all time-inconsistent problems – see He and Jiang (2021), Hernández and Possamaï (2021) and the references therein for a discussion on the topic. For the relatively well-behaved utility function of consideration in this paper, Definition 7.2.3 will suffice.

7.2.3 Power Utility with Dynamic Reference Point

Following Nordfang and Steffensen (2017) where it was first presented, we define the power utility function with dynamic wealth- and time dependent reference point as

$$u(t, x, z) = \frac{1}{1-\gamma}\left(z - xe^{\rho(T-t)}\right)^{1-\gamma}, \quad \gamma > 0. \tag{7.2.6}$$

The values of t, x and z represent the current time, wealth and terminal wealth respectively. The investor with the dynamic-reference power utility function constantly evaluates his utility of terminal wealth relative to his current wealth compounded with a target rate of return, ρ. For $\rho \to -\infty$ the dynamic reference point disappears, and we are left with the classical power utility function studied since Merton (1969). We note that $x < ye^{\rho(T-t)}$ can lead to complex-valued utility for certain values of γ. However, we also note that there is infinite marginal utility in the reference point, and therefore the investor hedges the risk of terminal wealth below the reference point. In an incomplete market where this risk cannot be hedged, one would have to be more careful about dealing with terminal wealth falling below the reference point.

The utility function is motivated by the S-shaped utility function suggested by prospect theory as presented in Tversky and Kahneman (1979), where utility of an uncertain outcome is relative to the status quo. The experimental evidence provided by Tversky and Kahneman (1979, 1986) shows that the status quo is dynamic. Based on a series of experiments they state;

> "These observations show that the effective carriers of values are gains and losses, or changes in wealth, rather than states of wealth as implied by the rational model."
>
> –from Tversky and Kahneman (1986)

For the power utility function with dynamic reference point, the objective is not to achieve a high value of terminal wealth – indeed any value of wealth has zero utility at termination – but rather to achieve changes in wealth that are satisfactory. A notable difference between the S-shaped utility function from prospect theory and (7.2.6), is that we do not incorporate risk-seeking behaviour in losses. We do not have to incorporate the lower part of the S-shape as the Black-Scholes market is complete and, the risk of wealth below the reference point can be hedged.

In order for a control to be admissible, the value function has to be well-defined. The value function changes as a function of (t, x), not only because of the (t, x)-conditional expectation, but also because the preferences of the investor depend on (t, x). To define admissibility we separate these effects by fixing the preferences of the investor.

Definition 7.2.3 (Admissibility)**.** *A control π is admissible with respect to* (7.2.6) *if there exists a function $G^\pi(t, x, s, y) : [0, T] \times \mathbb{R} \times [0, T] \times \mathbb{R} \to \mathbb{R}$ from $C^{1,2,1,1}$ such that*

1)

$$\begin{aligned}\frac{\partial}{\partial t}G^{\pi}(t,x,s,y) =& -(r+\pi(t,x)(\alpha-r))x\frac{\partial}{\partial x}G^{\pi}(t,x,s,y)\\ &-\frac{1}{2}\sigma^2\pi(t,x)^2x^2\frac{\partial^2}{\partial x^2}G^{\pi}(t,x,s,y),\\ G^{\pi}(T,x,s,y) =& \frac{1}{1-\gamma}\left(x-ye^{\rho(T-s)}\right)^{1-\gamma}.\end{aligned}$$

2)

$$\sigma\pi(t,X^{\pi}(t))X^{\pi}(t)\frac{\partial}{\partial x}G^{\pi}(t,X^{\pi}(t),t,X^{\pi}(t))\in\mathcal{L}^2,$$

for X^{π} following the SDE given by (7.2.1)-(7.2.2).

We denote the set of admissible strategies by $\mathcal{U}$.

For an admissible control, G has the interpretation

$$G^{\pi}(t,x,s,y) = \mathrm{E}_{t,x}[u(s,y,X^{\pi}(T))],$$

by the Feynmann-Kac formula – in fact this ad hoc definition of admissibility is constructed precisely to satisfy the conditions of the Feynman-Kac formula. This means that

$$J^{\pi}(t,x) = G^{\pi}(t,x,t,x),$$

and implies the existence of the value function. An important detail is that the first two arguments of G represent the variable time and wealth in the conditional expectation, whereas the last two arguments of G represent the preferences of the investor. To avoid misunderstandings about which arguments are variable and which are fixed, we introduce the notation

$$G(t,x,\underline{t},\underline{x}) = G(t,x,s,y)\big|_{s=t,y=x},$$

which is helpful in the next section.

7.3 Trying to Guess a Solution

The standard approach to stochastic control problems, is to guess a solution to the HJB-equation that encodes the problem. The same approach is in principle also applicable here for the extended HJB-equations that applies to time-inconsistent problems, but the form of the utility function makes guessing a solution difficult. In this section we present what we characterize as our closest, but still unsuccessful, guess.

Theorem 6.3 of Björk et al. (2017) is a verification theorem for equilibrium control problems in the form

$$J^{\pi}(t,x) = \mathrm{E}_{t,x}[u(t,x,X^{\pi}(T))].$$

We state the Theorem here in a version tailored to the utility function (7.2.6).

Theorem 7.3.1 (Verification of equilibrium control). *Assume that the function $\widehat{G}$ satisfies the following three equations*

$$\frac{\partial}{\partial t}\widehat{G}(t,x,\underline{t},\underline{x}) = \inf_{\pi}\left\{-(r+\pi(\alpha-r))x\frac{\partial}{\partial x}\widehat{G}(t,x,\underline{t},\underline{x}) - \frac{1}{2}\sigma^2\pi^2x^2\frac{\partial^2}{\partial x^2}\widehat{G}(t,x,\underline{t},\underline{x})\right\}, \tag{7.3.1}$$

$$\widehat{G}(T,x,s,y) = \frac{1}{1-\gamma}\left(x - ye^{\rho(T-s)}\right)^{1-\gamma}, \tag{7.3.2}$$

$$\widehat{G}(T,x,T,x) = 0. \tag{7.3.3}$$

Then

$$\widehat{G}(T,x,s,y) = E_{t,x}[u(s,y,X^{\widehat{\pi}}(T))],$$

and $\widehat{G}(t,x,t,x) = V(t,x)$ where V is the value function for the equilibrium control. Furthermore, the equilibrium control is given by

$$\arg\inf_{\pi}\left\{-(r+\pi(\alpha-r))x\frac{\partial}{\partial x}\widehat{G}(t,x,\underline{t},\underline{x}) - \frac{1}{2}\sigma^2\pi^2x^2\frac{\partial^2}{\partial x^2}\widehat{G}(t,x,\underline{t},\underline{x})\right\}. \tag{7.3.4}$$

Note that the equilibrium control achieves the infimum in (7.3.4), which is a quadratic equation in π, and therefore

$$\widehat{\pi}(t,x) = -\frac{\alpha-r}{x\sigma^2}\frac{\widehat{G}_x(t,x,\underline{t},\underline{x})}{\widehat{G}_{xx}(t,x,\underline{t},\underline{x})}, \tag{7.3.5}$$

for $\widehat{G}_{xx}(t,x,\underline{t},\underline{x}) < 0$.

We were unsuccessful in guessing a function that satisfies (7.3.1), (7.3.2) and (7.3.3). We did however manage to produce a guess, $\bar{G}$ presented below, that satisfies (7.3.1) and (7.3.3) but not (7.3.2). The proof of Theorem 6.3 from Björk et al. (2017) consists of two steps;

1) Prove that $\widehat{G}(T,x,s,y) = \mathrm{E}_{t,x}[u(s,y,X^{\widehat{\pi}}(T))]$.

2) Prove that (7.3.4) is indeed an equilibrium control law.

The proof provided by Björk et al. (2017) relies on all three equations, and we cannot use their proof technique to verify that the control given by

$$\bar{\pi}(t,x) = -\frac{\alpha-r}{x\sigma^2}\frac{\bar{G}_x(t,x,\underline{t},\underline{x})}{\bar{G}_{xx}(t,x,\underline{t},\underline{x})},$$

is an equilibrium control. Our guess perhaps forms a stepping-stone for more fruitful guesses, and for that reason we state it here.

Making the ansatz that the equilibrium control is independent of wealth, and rewriting (7.3.5)

$$\widehat{G}_x(t,x,\underline{t},\underline{x}) = \widehat{G}_{xx}(t,x,\underline{t},\underline{x})x\frac{-\widehat{\pi}(t)\sigma^2}{\alpha - r},$$

implying that $\widehat{G}$ is in the form

$$\widehat{G}(t,x,s,y) = {}_1H(t,s,y) + {}_2H(t,s,y)x^{1-\dfrac{\alpha-r}{\sigma^2\widehat{\pi}(t)}}.$$

Based on this, we make the guess

$$\bar{G}(t,x,s,y) = e^{I(t,s,y)}y^{\dfrac{\alpha-r}{\sigma^2\pi(s)}-\gamma}x^{1-\dfrac{\alpha-r}{\sigma^2\pi(t)}}g(s),$$

where

$$I(t,s,y) = \frac{1}{2\sigma^2}\int_t^s \frac{2\log(y)(\alpha-r)\pi'(\tau)}{\pi(\tau)^2} + \frac{((\alpha-r)\pi(\tau)+2r)(\sigma^2\pi(\tau)+r-\alpha)}{\pi(\tau)}d\tau,$$

and $g(t)$ is some unknown function that satisfies $g(T) = 0$. Through straightforward but tedious calculation of partial derivatives one can verify that

$$\begin{aligned}
\frac{\partial}{\partial t}\bar{G}(t,x,\underline{t},\underline{x}) =& \inf_{\pi}\Bigg\{ -(r+\pi(\alpha-r))x\frac{\partial}{\partial x}\bar{G}^{\pi}(t,x,\underline{t},\underline{x}) \\
&\quad -\frac{1}{2}\sigma^2\pi^2x^2\frac{\partial^2}{\partial x^2}\bar{G}^{\pi}(t,x,\underline{t},\underline{x})\Bigg\}, \\
\bar{G}(T,x,T,x) =& 0,
\end{aligned}$$

and

$$\bar{\pi}(t) = -\frac{\alpha-r}{x\sigma^2}\frac{\bar{G}_x(t,x,\underline{t},\underline{x})}{\bar{G}_{xx}(t,x,\underline{t},\underline{x})}.$$

In order for (7.3.2) to hold, g would have to solve

$$e^{I(T,s,y)}y^{\dfrac{\alpha-r}{\sigma^2\pi(s)}-\gamma}x^{1-\dfrac{\alpha-r}{\sigma^2\pi(t)}}g(s) = \frac{1}{1-\gamma}(x-ye^{\rho(T-s)})^{1-\gamma},$$

which is absurd as g does not depend on x or y. Since $\bar{G}$ does not satisfy (7.3.2), it is not the equilibrium value function.

We were not successful in guessing a solution to (7.3.1)-(7.3.3). This might just be due to poor guesses, but in any case, we need another way of finding the equilibrium control. The central idea in this paper, is to impose a structure on the equilibrium control, which leads to a structure on the value function that facilitates a solution. In the next section we motivate the structure we impose on the equilibrium control and provide arguments to justify it.

7.4 Wealth-independent Controls

For the set of controls that are independent of wealth, we know from Lemma 7.2.1 that the terminal wealth is log-normal distributed and therefore

$$
\begin{aligned}
&\frac{1}{1-\gamma}\mathrm{E}_{t,x}\left[\left(X^\pi(T)-X^\pi(t)e^{\rho(T-t)}\right)^{1-\gamma}\right]\\
=&\frac{1}{1-\gamma}x^{1-\gamma}e^{\rho(T-t)(1-\gamma)}\mathrm{E}\left[\left(e^{\int_t^T r-\rho+\pi(s)(\alpha-r)-\frac{\sigma^2\pi(s)^2}{2}ds+Z\sqrt{\int_t^T\sigma^2\pi(s)^2ds}}-1\right)^{1-\gamma}\right],
\end{aligned}
\tag{7.4.1}
$$

for a standard normal random variable Z. This implies that the value function is given by

$$
J^\pi(t,x)=x^{1-\gamma}g(t),
$$

for a π-dependent function g. Recall that the equilibrium control achieves the smallest marginal change in the value function over time. If the equilibrium control is independent of wealth, we can find it as the control that minimizes the derivative of g, and in Section 7.5 we find the control that does exactly this. In this section we argue, but do not prove, that the equilibrium control is independent of wealth.

7.4.1 Special Choice of γ

We are not able to guess the form of G a solution for $\gamma > 0$, but the problem simplifies when $(1-\gamma)=2$. In this case the value function is in the form

$$
J^\pi(t,x)=\frac{1}{2}\mathrm{E}_{t,x}[X^\pi(T)^2]+\frac{1}{2}\left(xe^{\rho(T-t)}\right)^2-xe^{\rho(T-t)}\mathrm{E}_{t,x}[X^\pi(T)],
\tag{7.4.2}
$$

representing the value function of an investor who seeks to minimize the quadratic distance to a target return, studied in Section 4.2.3 of Kryger et al. (2020). They show that the equilibrium control in this case is wealth-independent, and decreasing to zero for $t\to T$. The utility function

$$
u(t,x,z)=\frac{1}{1-\gamma}(z-xe^{\rho(T-t)})^2,
$$

is very similar to (7.2.6), differing only by the risk-aversion parameter. Heuristically they also describe the same incentives; the investor has a target rate of return ρ he seeks to uphold. The difference between $\gamma=-1$ and $\gamma>0$, is that for $\gamma=-1$ there is a disincentive to deviate from the target rate of return ρ, even when the rate of return is higher than ρ. For $\gamma>0$ a rate of return in excess of ρ is welcome, which seems to be the more sensible behaviour. Based on the similarities in the objectives, the special case for $\gamma=-1$ provides the first hint that the equilibrium optimal control for the utility function (7.2.6) is independent of wealth.

7.4.2 Polynomial Approximation

The second hint for a wealth-independent control, comes from Lollike and Steffensen (2021) where approximations of the optimal control through polynomial expansions of the utility function in (7.2.6) are considered. For two different forms of polynomial expansions of the utility function in (7.2.6), the resulting equilibrium control is independent of wealth. Furthermore, the approximating control is decreasing over time, similar to the control for the investor with $\gamma = -1$, corroborating that the solutions are similar.

Polynomial utility functions with (t, x)-dependent coefficients c_i and wealth-independent controls, have a value function in the form

$$\begin{aligned} J_n^\pi(t,x) &= \mathrm{E}_{t,x}\left[\sum_{i=0}^{n} c_i(t, X^\pi(t))X^\pi(T)^i\right] \\ &= \int_{-\infty}^{\infty}\sum_{i=0}^{n} c_i(t,x)x^i e^{(za^\pi(t)+d^\pi(t))i}\varphi(z)dz. \end{aligned}$$

where φ is the density of a standard normal distribution. The functions a^π and d^π are defined as

$$\begin{aligned} a^\pi(t) :=& \sigma\sqrt{\int_t^T \pi^2(s)ds}, \\ d^\pi(t) :=& \int_t^T r + \pi(s)(\alpha - r) - \frac{\sigma^2\pi(s)^2}{2}ds, \end{aligned}$$

which is notationally convenient for the representation of the (t, x)-dependent density of the log-normal distribution of $X^\pi(T)$. If the polynomial value function converges to the true value function, and the polynomial value function has wealth-independent equilibrium controls for all n, then we can conclude that the true value function also has a wealth-independent control.

There are two conditions under which the polynomial value function converges to the true value function,

1) Point-wise convergence of the polynomial to the original utility function

$$\lim_{n\to\infty}\sum_{i=0}^{n} c_i(t,x)x^i e^{(za^\pi(t)+d^\pi(t))i} = u(t,x,z),$$

on the support of $X^\pi(T)$.

2) Existence of an integrable dominating function D,

$$\left|\sum_{i=0}^{n} c_i(t,x)x^i e^{(za^\pi(t)+d^\pi(t))i}\phi(z)\right| \leq D(t,x,z),$$
$$\int_{-\infty}^{\infty} D(t,x,z)dz < \infty.$$

These two conditions allow us to apply the dominated convergence theorem and conclude that the original equilibrium control is equal to the limit of the polynomial equilibrium controls. If the coefficients imply wealth-independent polynomial equilibrium controls, we can therefore conclude that the original equilibrium control is independent of wealth.

The first condition is satisfied for a set of coefficients that are constructed as an approximating polynomial on an n-dependent interval that grows to the support of $X^\pi(T)$. The second condition is much more difficult, and maybe even impossible, to satisfy due to the unboundedness of e^{nz-z^2} – but perhaps the dominated convergence theorem is the wrong tool. There is a balance between the divergence of the polynomial and the density of $X^\pi(T)$ that determines weather or not the polynomial value function forms a good approximation of the original value function. Essentially we are looking for a set of coefficients that lead to ever better approximations of the original value function, and although it is hard to prove, we believe that these coefficients exist.

7.4.3 The Nature of the Utility Function

The perhaps most compelling argument for a wealth-independent equilibrium control, comes from the nature of the utility function itself. The utility of terminal wealth is measured by how much it exceeds the current discounted wealth. This means that for every change in the wealth of the investor, there is a proportional change in the reference point. Increasing the wealth of the investor, and keeping the reference point fixed, creates an incentive to invest less in the stock. Conversely, increasing the reference point of the investor, and keeping the wealth fixed, creates an incentive to invest more in the stock. By simultaneously increasing wealth and reference point by a proportional amount, these opposing incentives cancel each other out.

We have provided three arguments to support the claim that the equilibrium control is independent of wealth, but even if the claim is wrong, the best control in the subclass of wealth-independent controls still an interesting control to study, because it is deterministic.

7.5 Solution to the Problem

7.5.1 Other Solutions

Before we consider the solution to our time-inconsistent problem, we consider the solution to a similar problem. An approximation of the utility function (7.2.6) is studied by Nordfang and Steffensen (2017), using a Taylor polynomial expanded around the dynamic expected terminal wealth. They argue that it makes sense to compare the time-inconsistent problem to the solvable time-consistent problem given by

$$\sup_{\pi} \frac{1}{1-\gamma} \mathrm{E}\left[(X^{\pi}(T) - K))^{1-\gamma} \right].$$

This value function represents the pre-committed investor who has a fixed target return K, which remains the target return until termination. The optimal control for this problem is given by

$$\pi^*(t,x) = \frac{\alpha - r}{\sigma^2 x \gamma} \left(x - K e^{-r(T-t)} \right),$$

as derived by Nordfang and Steffensen (2017). Døskeland and Nordahl (2008) study the pre-committed investor for $K = X(0)e^{\rho T}$, corresponding to the dynamic reference point at time zero. If instead the investor naïvely updates his reference point continuously corresponding to $K = X^{\pi}(t)e^{\rho(T-t)}$, but invests as if it is fixed, the control is given by

$$\pi^N(t) = \frac{\alpha - r}{\sigma^2 \gamma} \left(1 - e^{(\rho - r)(T-t)} \right),$$

notably being independent of wealth. Furthermore the naïve control has an intuitive interpretation; the investor hedges his reference point by allocating a proportion equal to $e^{(\rho-r)(T-t)}$ in the bank, and the rest is invested according to Merton's fraction. The intuitive reason is that once the investor has ensured that his wealth does not fall below the reference point, he can invest his remaining wealth according to the optimal control for classic power utility.

The naïve control can be seen as the 'standard behaviour' of the investor – how he would invest if he did not account for the control at future points in time. For this reason the naïve control acts as a baseline result in the numerical study of Section 7.6.

7.5.2 The Wealth-independent Equilibrium Control

Based on the arguments in Section 7.4, we assume that there exists an equilibrium control that is independent of wealth, and now set out to compute it. The wealth-independent equilibrium control can be found as the control that minimizes the derivative in time of the time and wealth separable value function from (7.4.1), given by

$$J^{\pi}(t,x) = x^{1-\gamma} g(t)$$

where

$$g(t) = \frac{1}{1-\gamma} e^{\rho(T-t)(1-\gamma)} \mathrm{E}\left[\left(e^{Za^\pi(t)+d^\pi(t)-\rho(T-t)} - 1\right)^{1-\gamma}\right], \qquad Z \sim \mathcal{N}(0,1).$$

To shorten notation we define

$$b^\pi(t) := d^\pi(t) - \rho(T-t),$$

such that $X^\pi(t)e^{Za^\pi(t)+b^\pi(t)}$ corresponds to the wealth at termination, discounted with the target rate of return ρ. The wealth-independent equilibrium control at time t is thus given by

$$\begin{aligned} &\arg\inf_{\pi(t)} \frac{d}{dt}\left(\frac{e^{\rho(T-t)(1-\gamma)}\mathrm{E}\left[\left(e^{Za^\pi(t)+b^\pi(t)} - 1\right)^{1-\gamma}\right]}{1-\gamma}\right) \\ =&\arg\inf_{\pi(t)} \frac{d}{dt}\int_{-\infty}^{\infty}\left(e^{za^\pi(t)+b^\pi(t)} - 1\right)^{1-\gamma}\varphi(z)dz, \end{aligned} \tag{7.5.1}$$

where we have used that the control at a single point in time has no influence on $a^\pi(t)$ and $b^\pi(t)$ as it has Lebesgue measure zero. To avoid complex-valued utility, we restrict Z to the values that ensure positive expected utility at termination given the time-t preferences. We find the time-dependent lower limit of the integral,

$$l(t) := \inf\{z : 1 \leq e^{za^\pi(t)+b^\pi(t)}\} \quad \Leftrightarrow \quad l(t) = \frac{-b^\pi(t)}{a^\pi(t)}.$$

Restricting the standard normal random variable is equivalent to redefining the utility function as

$$u(t,x,z) = \frac{1}{1-\gamma}\max\left(z - xe^{\rho(T-t)}, 0\right)^{1-\gamma}.$$

As argued in Section 7.2.3, the max-function is redundant for the equilibrium control as there is infinite marginal utility in the reference point, and the investor therefore hedges the risk of falling below the reference point. The restriction on the distribution of terminal wealth helps us here because we are not only considering the equilibrium control.

For notational convenience we define

$$\lambda^\pi_{i,k}(t,z) = \left(e^{za^\pi(t)+b^\pi(t)} - 1\right)^{1-\gamma}\varphi(z)\left(\frac{e^{za^\pi(t)+b^\pi(t)}}{e^{za^\pi(t)+b^\pi(t)} - 1}\right)^k z^i,$$

where φ is the density of a standard normal distribution. This class of functions appears to be essential for the study of the equilibrium control. We are particularly interested in the integrals

$$\int_{l(t)}^{\infty}\lambda_{i,k}(t,z)dz,$$

which in general do not have closed form solutions, and therefore the equilibrium control is given in terms of these integrals. The main result of the paper is given in Theorem 7.5.1.

Theorem 7.5.1. *An equilibrium control for the set of wealth-independent admissible controls satisfies the following equation*

$$\widehat{\pi}(t) = \frac{(\alpha - r)}{\sigma^2 \left(1 - \dfrac{\displaystyle\int_{l(t)}^{\infty} \lambda_{1,1}^{\widehat{\pi}}(t,z)dz}{a^{\widehat{\pi}}(t) \displaystyle\int_{l(t)}^{\infty} \lambda_{0,1}^{\widehat{\pi}}(t,z)dz} \right)}.$$

See Appendix 7.A for the proof. Due to the complex integrals over z, not much can be said about $\widehat{\pi}$ based solely on Theorem 7.5.1. Unsurprisingly, the proportion in the stock is increasing in $(\alpha - r)$ and decreasing in σ. The form of the equilibrium control does however resemble Merton's fraction $\frac{\alpha - r}{\sigma^2 \gamma}$ where γ has been replaced with

$$1 - \frac{\displaystyle\int_{l(t)}^{\infty} \lambda_{1,1}^{\widehat{\pi}}(t,z)dz}{a^{\widehat{\pi}}(t) \displaystyle\int_{l(t)}^{\infty} \lambda_{0,1}^{\widehat{\pi}}(t,z)dz}.$$

This similarity suggests that the fraction

$$\frac{\displaystyle\int_{l(t)}^{\infty} \lambda_{1,1}^{\widehat{\pi}}(t,z)dz}{a^{\widehat{\pi}}(t) \displaystyle\int_{l(t)}^{\infty} \lambda_{0,1}^{\widehat{\pi}}(t,z)dz},$$

represents the investors risk aversion at time t taking the future control into account, in the same way as $1 - \gamma$ does it in the classical power utility problem.

We note that

$$\begin{aligned}\frac{\partial}{\partial t} \lambda_{i,k}^{\pi}(t,z) =& (1 - \gamma - k) \left(a^{\pi\prime}(t) \lambda_{i+1,k+1}^{\pi}(t) + b^{\pi\prime}(t) \lambda_{i,k+1}^{\pi}(t) \right) \\ &+ k \left(a^{\pi\prime}(t) \lambda_{i+1,k}^{\pi}(t) + b^{\pi\prime}(t) \lambda_{i,k}^{\pi}(t) \right),\end{aligned}$$

which is useful if one wants to calculate derivatives of $\widehat{\pi}$, but we found the derivatives to be even less informative than the fixed-point equation of Theorem 7.5.1.

7.5.3 Infinite Horizon

As $T \to \infty$ the expected terminal wealth $e^{Za^{\widehat{\pi}}(t) + d^{\widehat{\pi}}(t)}$ tends to infinity, and the influence of subtracting one from this value becomes negligible which means

$$J(t,x) \sim \frac{1}{1-\gamma} x^{1-\gamma} \mathrm{E}\left[(e^{Za^{\pi}(t) + d^{\pi}(t)})^{1-\gamma} \right] = \frac{1}{1-\gamma} \mathrm{E}_{t,x}\left[X^{\pi}(T)^{1-\gamma} \right],$$

corresponding to a classical power utility value function. For classical power utility the optimal wealth allocation is determined by

$$\pi^* = \frac{\alpha - r}{\sigma^2 \gamma},$$

implying that $\widehat{\pi}(t) \to \frac{\alpha - r}{\sigma^2\gamma}$ for $T \to \infty$. This result is corroborated in the numerical study.

By decreasing the target return we are making the reference point more easily attainable, thereby removing some of the investors incentive to deviate from Merton's fraction. Indeed

$$\lim_{\rho \to -\infty} \frac{1}{1-\gamma}(z - xe^{\rho(T-t)})^{1-\gamma} = \frac{1}{1-\gamma} z^{1-\gamma},$$

implying that $\widehat{\pi}(t) = \frac{\alpha - r}{\sigma\gamma}$ for $\rho \to -\infty$. The same investment proportion is achieved for $T \to \infty$, and in that sense, decreasing the value of the target return has the same effect on the control as increasing the investment horizon.

7.6 Numerical Study

In this section we conduct a numerical analysis of the equilibrium control from Theorem 7.5.1. To calculate the equilibrium control, we iterate the fixed-point equation for $\widehat{\pi}$ until convergence.

The parameters are chosen to represent a lifetime investment problem with a 30-year horizon. The parameters are given by

$$\begin{pmatrix} T \\ \alpha \\ r \\ \sigma \\ \gamma \\ \rho \end{pmatrix} = \begin{pmatrix} 30 \\ 0.09 \\ 0.04 \\ 0.5 \\ \frac{1}{2.75} \\ 0.02 \end{pmatrix},$$

resulting in a market price of risk of $(\alpha - r)/\sigma = 0.1$ and Mertons fraction of $(\alpha - r)/(\sigma^2\gamma) = 0.55$.

7.6.1 Basecase

In Figure 7.1 we have plotted the equilibrium control, together with the naive control and Mertons fraction.

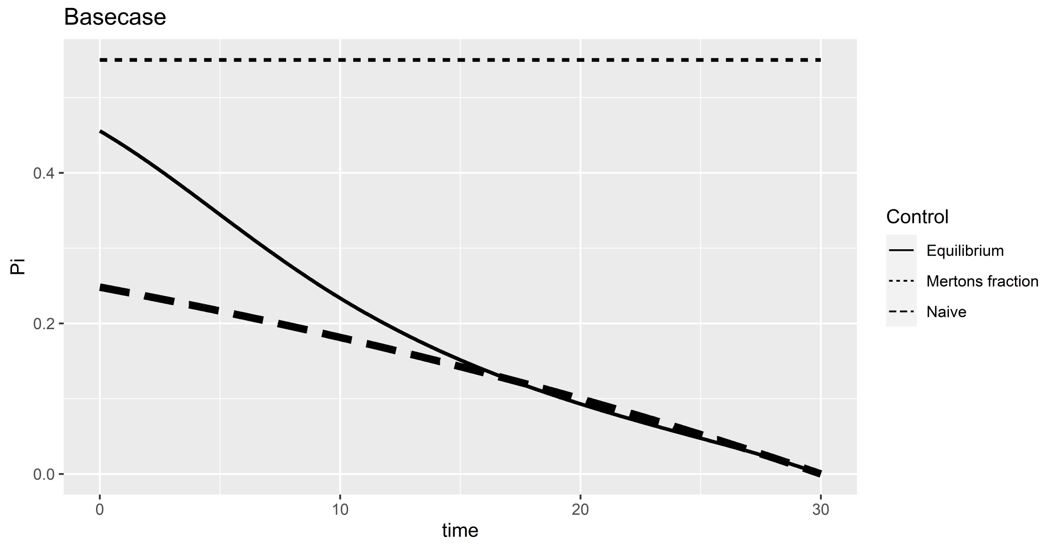

Figure 7.1: *Equilibrium and naive control for time-inconsistent problem as well as Merton's fraction*

There is a significant difference between the naïve and equilibrium control, with a higher investment in the risky asset in the equilibrium control for $t < 15$.

We know that both the naïve and equilibrium control tend to Merton's fraction for $T \to \infty$ (or equivalently $t \to -\infty$), but from the looks of Figure 7.1, they seem to converge at very different rates. By increasing T, we get a better view of the different convergence rates.

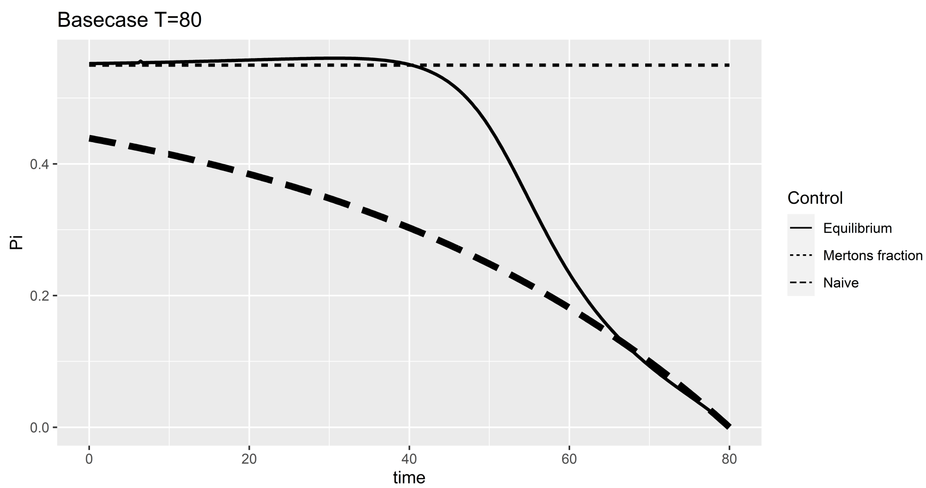

Figure 7.2: *The convergence towards Merton's fraction is more evident here, than in Figure 7.1.*

The equilibrium control is almost constant until the hump at $t = 40$ whereafter the control changes drastically. Apart from a change in the units of time, this control is similar to the implemented investment strategy for one of the largest Danish pension providers seen in Figure 7.3.

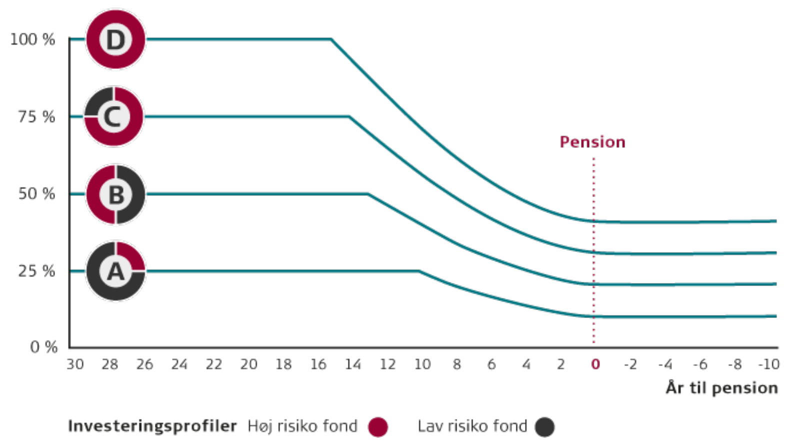

Figure 7.3: *Proportions allocated in high-risk funds versus low-risk funds for different 'risk profiles'. The x-axis indicates years until retirement, and the y-axis indicates percentage in high-risk fund. Picture grabbed from `https://pfa.dk/privat/opsparing/pfa-investerer/` January 6th 2022.*

Even the concave descent of the control to its final resting point looks similar.

7.6.2 Smaller Target Return and Less Risk Averse

The difference between the naïve and equilibrium control, is more pronounced for some parameter-values than others.

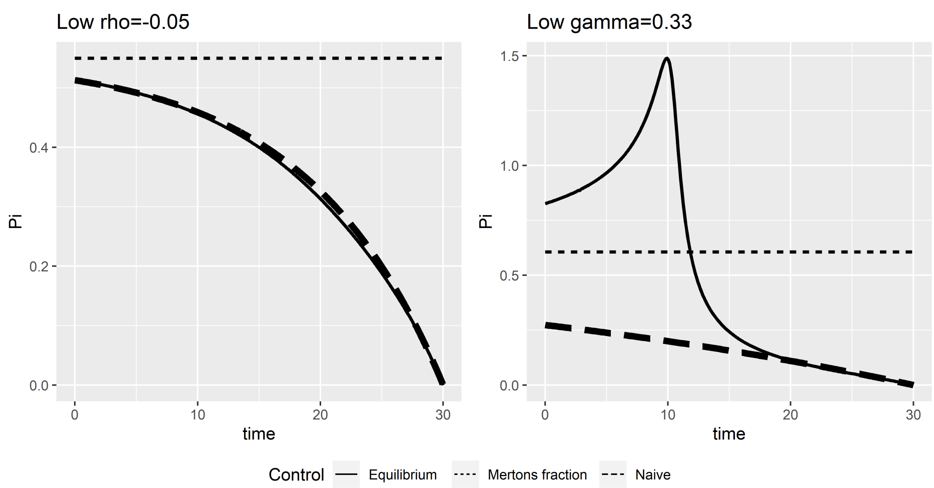

Figure 7.4: *Naive and equilibrium controls that are similar and different.*

For a very low target return, the naïve and equilibrium control are close to each other. The same effect can be achieved by adjusting other parameters e.g. $\gamma = 2$.

By decreasing γ to 0.33, the hump above Merton's fraction has turned into a peak. When γ is small the incentive to surpass the target return is strong, which might be part of the explanation, but to fully understand why this peak appears a better understanding of the $\lambda_{i,k}$ functions is needed. It seems natural to ask why the equilibrium control does not diverge – what makes the investor turn on a dime? We believe that this rapid change in the growth of the equilibrium control is caused by the $a^{\widehat{\pi}}$ and $b^{\widehat{\pi}}$ functions that increase with $\widehat{\pi}$. Increases in $a^{\widehat{\pi}}$ and $b^{\widehat{\pi}}$ in lead to expected returns surpassing the target return, which removes some of the incentive to deviate from Merton's fraction, just as we saw for $T \to \infty$.

7.7 Further Research

While the fixed-point equation of Theorem 7.5.1 paves the way for a calculation of the equilibrium control, it has poor value in terms of economic interpretation. A better understanding of the $\lambda_{i,k}$-functions and their economic interpretation, would improve the value of Theorem 7.5.1.

A guess that solves (7.3.1)-(7.3.3) would not only determine whether or not the equilibrium control is independent of wealth, it would also provide an economic interpretation of the equilibrium control. Somehow applying the fixed-point equation of Theorem 7.5.1 to produce a guess, may aide in forming the guess.

Finally, the same method can be applied on other, similar utility functions with a prospect theoretic origin. A utility function with finite marginal utility in the reference point that also includes risk willingness in losses, is an obvious candidate.

7.A Proof of Theorem 7.5.1

Proof. Realizing that $(e^{za^\pi(t)+b^\pi(t)}-1)^{1-\gamma}\varphi(z) = \lambda_{0,0}(t,z)$, we now differentiate under the integral in (7.5.1),

$$\int_{l(t)}^{\infty} \frac{\partial}{\partial t}\lambda_{0,0}(t,z)dz =(1-\gamma)\left(a^{\pi\prime}(t)\int_{l(t)}^{\infty}\lambda_{1,1}(t,z)dz + b^{\pi\prime}(t)\int_{l(t)}^{\infty}\lambda_{0,1}(t,z)dz\right).$$

Plugging in

$$\begin{aligned} a^{\pi\prime}(t) =&\frac{-\sigma^2\pi(t)^2}{2a^\pi(t)},\\ b^{\pi\prime}(t) =&\rho - r - \pi(t)(\alpha - r) + \frac{\sigma^2\pi(t)^2}{2}, \end{aligned}$$

we get

$$\begin{aligned} \int_{l(t)}^{\infty} \frac{\partial}{\partial t}\lambda_{0,0}(t,z)dz =& -\frac{\sigma\pi(t)^2}{2a^\pi(t)}(1-\gamma)\int_{l(t)}^{\infty}\lambda_{1,1}(t,z)dz\\ &+(1-\gamma)\left(\rho - r - \pi(t)(\alpha-r) + \frac{\sigma^2\pi(t)^2}{2}\right)\int_{l(t)}^{\infty}\lambda_{0,1}(t,z)dz\\ =&\pi(t)^2(1-\gamma)\frac{\sigma^2}{2}\left(\int_{l(t)}^{\infty}\lambda_{0,1}(t,z)dz - \frac{1}{a^\pi(t)}\int_{l(t)}^{\infty}\lambda_{1,1}(t,z)dz\right)\\ &-\pi(t)(1-\gamma)(\alpha-r)\int_{l(t)}^{\infty}\lambda_{0,1}(t,z)dz\\ &+(1-\gamma)\left(\rho-r\right)\int_{l(t)}^{\infty}\lambda_{0,1}(t,z)dz. \end{aligned}$$

This quadratic equation in $\pi(t)$ achieves its infimum in

$$\pi(t) = \frac{(\alpha-r)\displaystyle\int_{l(t)}^{\infty}\lambda_{0,1}(t,z)dz}{\sigma^2\left(\displaystyle\int_{l(t)}^{\infty}\lambda_{0,1}(t,z)dz - \frac{1}{a^\pi(t)}\int_{l(t)}^{\infty}\lambda_{1,1}(t,z)dz\right)}$$

for

$$\sigma^2\left(\int_{l(t)}^{\infty}\lambda_{0,1}(t,z)dz - \frac{1}{a^\pi(t)}\int_{l(t)}^{\infty}\lambda_{1,1}(t,z)dz\right) > 0,$$

and $\pi(t) = \pm\infty$ otherwise implying that it is not admissible. By definition, the equilibrium control achieves this infimum for all values of t. $\square$

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