VIBEKE QUORNING

CANTOR-BENDIXSON TYPE RANKS & CO-INDUCTION AND INVARIANT RANDOM SUBGROUPS

PHD THESIS

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Abstract

The present thesis consists of two unrelated research projects and is therefore divided into two parts. The first part is based on the paper [27]. The second part is based on the paper [22], which is joint with Alexander S. Kechris.

Part I. For a Polish space X it is well-known that the Cantor-Bendixson rank provides a co-analytic rank on $F_{\aleph_0}(X)$ if and only if X is σ-compact. We construct a family of co-analytic ranks on $F_{\aleph_0}(X)$ for any Polish space X. We study the behaviour of this family and compare the ranks to the Cantor-Bendixson rank. The main results are characterizations of the compact and σ-compact Polish spaces in terms of this behaviour.

Part II. We develop a co-induction operation for invariant random subgroups. We use this operation to construct new examples of continuum size families of non-atomic, weakly mixing invariant random subgroups of certain kinds of wreath products, HNN extensions and free products with normal amalgamation. Moreover, by use of small cancellation theory together with our operation, we construct a new continuum size family of non-atomic invariant random subgroups of $\mathbb{F}_2$ which are all invariant and weakly mixing with respect to the action of $\mathrm{Aut}(\mathbb{F}_2)$. Finally, by studying continuity properties of our operation, we obtain results concerning the continuity of the co-induction operation for weak equivalence classes of measure preserving group actions.

Resumé

Denne afhandling består af to urelaterede forskningsprojekter og er derfor opdelt i to dele. Den første del er baseret på artiklen [27]. Den anden del er baseret på artiklen [22], som er lavet i samarbejde med Alexander S. Kechris.

Del I. For et Polsk rum X er det velkendt at Cantor-Bendixson-rangen udgør en ko-analytisk rang på $F_{\aleph_0}(X)$ hvis og kun hvis X er σ-kompakt. Vi konstruerer en familie af ko-analytiske rangfunktioner på $F_{\aleph_0}(X)$ for ethvert Polsk rum X. Vi undersøger hvordan denne familie opfører sig og sammenligner disse rangfunktioner med Cantor-Bendixson-rangen. Hovedresultaterne er karakteriseringer af de kompakte og σ-kompakte Poliske rum ud fra denne opførsel.

Del II. Vi udvikler en ko-induktionsoperation for invariante tilfældige undergrupper. Vi bruger denne operation til at konstruere nye eksempler på familier med kontinuum mange ikke-atomiske, svagt blandede invariante tilfældige undergrupper af nogle typer af kranseprodukter, HNN-udvidelser og frie produkter med normal amalgamation. Endvidere, ved brug af lille annulleringsteori sammen med vores operation, konstruerer vi en ny familie med kontinuum mange ikke-atomiske invariante tilfældige undergrupper af $\mathbb{F}_2$, som alle er invariante og svagt blandede i forhold til virkningen af $\mathrm{Aut}(\mathbb{F}_2)$. Til sidst, ved at studere kontinuitetsegenskaber ved vores operation, opnår vi resultater angående kontinuiteten af ko-induktionsoperationen for svage ækvivalensklasser af målbevarende gruppevirkninger.

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Contents

Introduction

The present thesis consists of two unrelated parts, which are both based on a corresponding research project. The title of the first part is *Cantor-Bendixson type ranks*. It is based on the paper [27] and is concerned with the study of co-analytic sets in classical descriptive set theory. The title of the second part is *Co-induction and invariant random subgroups*. It is based on the paper [22], which is joint with Alexander S. Kechris, and concerns the study of invariant random subgroups and their connection to measure preserving group actions. This subject lies on the borderline of group theory, ergodic theory and descriptive set theory. Below we will briefly introduce each of these subjects and present the results of this thesis.

Descriptive set theory and co-analytic sets

Descriptive set theory is the study of definable subsets of separable completely metrizable topological spaces. Such topological spaces are said to be Polish and they include $\mathbb{R}$, all separable Banach spaces and ω^ω, where the latter is equipped with the product topology induced by the discrete topology on $\omega = \{0, 1, \ldots\}$. By definable subsets we mean subsets that can be described in terms of the topology. We stratify these sets into hierarchies according to the complexity of their description and study the properties of the sets in each level.

The systematic study of sets in descriptive set theory goes back to the work of Borel, Baire and Lebesgue at the beginning of the 20th century. Since then it has developed extensively and many regularity properties such as Lebesgue measurability, the Baire property and the perfect set property have been proven to hold for subsets of sufficiently low complexity. For example, all of these three properties hold for the Borel sets.

For each Polish space we can use transfinite recursion to construct the Borel sets by starting with the open sets and then recursively closing under the operations of complements and countable unions. We can stratify the Borel sets into the Borel hierarchy, where the complexity of a Borel set is determined by its first appearance in this construction. Thus the open and the closed sets are the simplest.

In the 1910s Suslin discovered that the projection of a Borel set in the plane is not necessarily Borel, thereby pointing out a mistake of Lebesgue in the paper [23] from 1905. This gave rise to a class of sets more complicated than the Borel sets, namely the projective sets. This is the class of sets which is obtained from the Borel sets by recursively closing under the operations of continuous images and complements. Once again, these sets can be stratified into a hierarchy. At the lowest level of the projective hierarchy we have the analytic and the co-analytic sets, which can be defined as the continuous images of Borel sets and their complements, respectively.

We will be focusing on the class of co-analytic sets. A classical example of a co-analytic set is the following: Let $\mathrm{Tree}(\omega)$ denote the set of trees on ω. The subset $\mathrm{WF} \subseteq \mathrm{Tree}(\omega)$ consisting of all well-founded trees is co-analytic. In fact, if $A \subseteq X$ is a co-analytic subset of a Polish space X, then there is a Borel map $f\colon X \to \mathrm{Tree}(\omega)$ such that $x \in A$ if and only if $f(x) \in \mathrm{WF}$. So in many cases we can pass from general co-analytic sets to this specific set in order to obtain results.

A key property of co-analytic sets is that they admit a co-analytic rank into ω_1. Given a set A, a rank into ω_1 is a map $\varphi\colon A \to \omega_1$. If A is a co-analytic subset of a Polish space X, we say that φ is a co-analytic rank if the initial segment

$$A_\alpha^\varphi = \{x \in A \mid \varphi(x) \leq \alpha\}$$

is Borel in X in a uniform manner for all $\alpha < \omega_1$. This ensures that any co-analytic set A is an increasing union of ω_1 Borel sets.

The main result concerning co-analytic ranks is the Boundedness Theorem. It states that if $\varphi\colon A \to \omega_1$ is a co-analytic rank on a co-analytic subset A of a Polish space X, then A is Borel in X if and only if

$$\sup\{\varphi(x) \mid x \in A\} < \omega_1.$$

Moreover, if $B \subseteq A$ is analytic in X, then $\sup\{\varphi(x) \mid x \in B\} < \omega_1$.

The first part of the theorem highlights how these ranks provide a powerful tool for proving that certain subsets are not Borel. For example, in [34] co-analytic ranks are used to prove that, in a certain parametrization of countable groups, the subset of elementary amenable groups is not Borel while the subset of amenable groups is. This result thereby gives a non-constructive existence proof of an amenable group that is not elementary amenable.

The second part of the theorem ensures a uniformity of the co-analytic ranks that a fixed co-analytic set A admits. Indeed, it implies that if $\varphi, \psi \colon A \to \omega_1$ are both co-analytic ranks, then there exists a function $f \colon \omega_1 \to \omega_1$ such that $\varphi(x) \leq f(\psi(x))$ for all $x \in A$. Therefore all co-analytic ranks on A agree on which subsets are bounded and which subsets are not, and so each co-analytic subset of a Polish space admits a natural σ-ideal of bounded sets.

Even though it is known that any co-analytic set admits a co-analytic rank and therefore also the aforementioned σ-ideal, the proof does not provide a concrete rank for a given co-analytic set. In fact, to obtain this result it suffices to construct a co-analytic rank on WF. For a given co-analytic set it is therefore of interest to find explicit co-analytic ranks in order to determine the structure of the co-analytic set in terms of the σ-ideal of bounded sets. Many examples of co-analytic ranks for specific co-analytic sets are collected in [19].

Cantor-Bendixson type ranks

In this part of the thesis we will consider the Effros Borel space $F(X)$ consisting of all closed subsets of a Polish space X, and the co-analytic subset $F_{\aleph_0}(X)$ consisting of the countable closed subsets. It is known that $F_{\aleph_0}(X)$ is co-analytic and not Borel when X is uncountable.

A natural rank on this co-analytic set is the Cantor-Bendixson rank, which assigns to each $F \in F_{\aleph_0}(X)$ the length of the transfinite process of removing isolated points. However, the Cantor-Bendixson rank is only co-analytic when the underlying Polish space is σ-compact. In fact, as mentioned in [19], no explicit co-analytic rank on $F_{\aleph_0}(X)$ for a general Polish space X seems to be known.

In the specific case of ω^ω, there is a natural correspondence between $F(\omega^\omega)$ and $\mathrm{Tree}(\omega)$. Recall that a tree $T \in \mathrm{Tree}(\omega)$ is a set of finite sequences of numbers in ω which is closed under initial segments. One then associates to each $F \in F(\omega^\omega)$ the tree T_F consisting of all finite initial segments of the elements of F. Moreover, on $\mathrm{Tree}(\omega)$ there is a Cantor-Bendixson-like rank, which assigns to each tree the length of the transfinite process of removing isolated branches of the tree. We obtain a co-analytic rank on $F_{\aleph_0}(\omega^\omega)$ by assigning to each $F \in F_{\aleph_0}(\omega^\omega)$ the rank of T_F.

Our first goal is to generalize the construction used in the case of ω^ω to a general Polish space X. The key step in this generalization is to define a correspondence between $F(X)$ and certain subsets of ω^2. This is done by fixing a complete compatible metric d on X and a countable dense sequence $(x_i)_i$ in

X. We then identify each $F \in F(X)$ with the set

$$A_F = \left\{(i,j) \in \omega^2 \mid F \cap \mathrm{B}_d(x_i, 2^{-j-1})\right\}.$$

It turns out that this correspondence shares many of the properties of the correspondence between $F(\omega^\omega)$ and $\mathrm{Tree}(\omega)$. So, since we can define a Cantor-Bendixson-like rank on the subsets of ω^2, we obtain a co-analytic rank on $F_{\aleph_0}(X)$ by assigning to each $F \in F(X)$ the rank of A_F.

A presentation $\mathcal{P} = (X, d, (x_i)_i)$ of a Polish space X is a Polish space X equipped with a fixed choice of a complete compatible metric d and a countable dense sequence $(x_i)_i$. Our construction provides a co-analytic rank $\varphi_{\mathcal{P}}\colon F_{\aleph_0}(X) \to \omega_1$ for each presentation $\mathcal{P}$ of a Polish space X. Hence for a fixed Polish space X we obtain a potentially huge family

$$\{\varphi_{\mathcal{P}} \mid \mathcal{P} \text{ is a presentation of } X\}$$

of co-analytic ranks on $F_{\aleph_0}(X)$.

The second objective is to investigate how this family of co-analytic ranks behaves. We prove results stating how the chosen presentation affects the ranks one obtains, and how the ranks relate to the Cantor-Bendixson rank on $F_{\aleph_0}(X)$. The main results characterizes the compact and σ-compact Polish spaces in terms of the behaviour of the ranks.

We will prove that a Polish space X is compact if and only if the family of ranks is uniformly bounded by the Cantor-Bendixson rank, in the sense that there is a function $f\colon \omega_1 \to \omega_1$ such that

$$\varphi_{\mathcal{P}}(F) \leq f(|F|_{\mathrm{CB}})$$

for all $F \in F_{\aleph_0}(X)$ and all presentations $\mathcal{P}$ of X. Moreover, we will compute a single function $f\colon \omega_1 \to \omega_1$ satisfying the above inequality for any compact Polish space X.

We obtain that a Polish space X is σ-compact if and only if some (equivalently every) rank in the family $(\varphi_{\mathcal{P}})_{\mathcal{P}}$, where $\mathcal{P}$ varies over all presentations of X, is bounded by the Cantor-Bendixson rank. This means that for some (equivalently every) presentation $\mathcal{P}$ of X there is a function $f_{\mathcal{P}}\colon \omega_1 \to \omega_1$ such that

$$\varphi_{\mathcal{P}}(F) \leq f_{\mathcal{P}}(|F|_{\mathrm{CB}})$$

for all $F \in F_{\aleph_0}(X)$. Also in this case, we will for a presentation $\mathcal{P}$ of a σ-compact Polish space X compute a specific function $f_{\mathcal{P}}\colon \omega_1 \to \omega_1$ satisfying the above inequality.

Let us briefly give an overview of the content contained in this part of the thesis.

Chapter 1. This preliminary chapter contains the various notions and basic results needed for the rest of this part of the thesis. In the first section we briefly review some basic results and notions related to Polish spaces, Polish metric spaces and standard Borel spaces, which we assume the reader to be familiar with throughout the rest of this thesis. This includes an introduction to the Effros Borel space $F(X)$ of a Polish space X. In the second section we will introduce the descriptive set-theoretic trees. We will see how these trees are closely related to the closed subsets of certain product spaces. In particular, we will establish the correspondence between $F(\omega^\omega)$ and $\mathrm{Tree}(\omega)$. In the third section we define the analytic and co-analytic sets. We will see that $F_{\aleph_0}(X)$ is in fact a co-analytic subset of $F(X)$ and that is is not Borel when X is uncountable. The fourth section concerns co-analytic ranks. We will among other things discuss the Boundedness Theorem and give a proof of the fact that any co-analytic subset admits a co-analytic rank. In the fifth and final section we will discuss the Cantor-Bendixson rank on $F(X)$ for a Polish space X. We will argue that it is co-analytic on $F_{\aleph_0}(X)$ if and only if X is σ-compact. Moreover, we will show how to use the correspondence between $F(\omega^\omega)$ and $\mathrm{Tree}(\omega)$ to construct a co-analytic rank on $F_{\aleph_0}(\omega^\omega)$.

Chapter 2. The main goal of this chapter is to obtain the aforementioned rank $\varphi_{\mathcal{P}}\colon F_{\aleph_0}(X) \to \omega_1$ for each presentation $\mathcal{P}$ of a Polish space X. In the first section we will give the precise construction of these ranks. In the second section we will provide various properties of this construction and we will see that it does in fact generalize the well-known construction for ω^ω. In the third and final section we investigate the extent to which the rank depends on the chosen metric and on the dense sequence, respectively. We will first isolate classes of Polish metric spaces for which the construction is completely independent of the dense sequence. In general, changes can occur when varying the dense sequence, but we will recover a bound on how much. Afterwards we will see that there is no bound on the changes that may occur when varying the complete metric.

Chapter 3. In this chapter we will compare the ranks that we have constructed to the Cantor-Bendixson rank. In particular, we will prove the characterizations of the compact and σ-compact Polish spaces that we described above.

Chapter 4. This chapter serves as a discussion of some questions related to the subject of this part of the thesis. In the first section we will discuss certain invariance properties one can hope for in a co-analytic rank $\varphi\colon F_{\aleph_0}(X) \to \omega_1$ for a Polish space X. We propose a problem of finding a co-analytic rank

$\varphi\colon F_{\aleph_0}(X) \to \omega_1$ with nicer invariance properties than the ones we construct here. The second section concerns the phenomenon of uniform boundedness that we show for the family $(\varphi_p)_{\mathcal{P}}$, where $\mathcal{P}$ varies over all presentations of a compact Polish space X. We will ask several questions towards understanding if this behaviour holds more generally and, in particular, if it occurs in other cases.

Invariant random subgroups

The study of invariant random subgroups has been an active area of research in recent years.

For a countable group Γ we consider the compact Polish space $\mathrm{Sub}(\Gamma) \subseteq \{0,1\}^\Gamma$ consisting of all subgroups of Γ. A Borel probability measure on $\mathrm{Sub}(\Gamma)$ which is invariant under the natural conjugacy action of Γ on $\mathrm{Sub}(\Gamma)$ is called an invariant random subgroup. The space of invariant random subgroups of Γ is denoted $\mathrm{IRS}(\Gamma)$ and is a closed convex subspace of the space of all Borel probability measures on $\mathrm{Sub}(\Gamma)$. In particular, it is a compact Polish space in the weak*-topology. The study of invariant random subgroups usually concentrates on the ergodic invariant random subgroups, as these constitute the extreme points in $\mathrm{IRS}(\Gamma)$. A notion of invariant random subgroups has also been examined for locally compact groups, where one considers the conjugation action on the space of closed subgroups, but here we will limit our attention to countable groups.

It easy to construct simple examples of invariant random subgroups. For each finite index subgroup the uniform measure on the conjugacy class is an invariant random subgroup. Also, the Dirac measure concentrated at a normal subgroup is an example of an invariant random subgroup. It is a general phenomenon that the invariant random subgroups tend to behave like normal subgroups, rather than arbitrary subgroups. For example, Kersten's theorem and Margulis' normal subgroup theorem have been extended to invariant random subgroups in [2] and [28], respectively. The latter result implies that any ergodic invariant random subgroup of a lattice in a higher rank simple real Lie group, such as $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, is either induced by the trivial subgroup or by a finite index subgroup, as above.

Another more interesting way to obtain invariant random subgroups of a countable group Γ is by use of measure preserving actions. Let (X,μ) be a non-atomic standard probability space, i.e., a probability space isomorphic to $([0,1],\lambda)$, where λ denotes the Lebesgue measure. For each measure preserving action $\Gamma \curvearrowright^a (X,\mu)$ we have an equivariant Borel map $\mathrm{stab}_a\colon X \to \mathrm{Sub}(\Gamma)$ given by $\mathrm{stab}_a(x) = \{\gamma \in \Gamma \mid \gamma \cdot^a x = x\}$. Hence if we let $\mathrm{type}(a)$ denote the pushforward of μ through stab_a, then $\mathrm{type}(a) \in \mathrm{IRS}(\Gamma)$. In fact, it is shown in [2] that all invariant random subgroups of Γ arise in this way. Therefore the study of invariant random subgroups is naturally connected to the study of measure preserving actions.

The term "invariant random subgroup" was introduced in [2]. However, the objects have been studied earlier and the paper [28] is usually considered to be the first on the subject. Throughout the past decade many applications and connections of invariant random subgroups to different areas have been established.

There has also been an interest in studying the structure of the ergodic invariant random subgroups of different classes of groups. As the atomic ergodic invariant random subgroups are exactly the ones induced by a subgroup with only finitely many conjugates, the focus has been to determine the structure of the non-atomic, ergodic invariant random subgroups. It turns out that some groups have only a few, while others admit lots of them. Examples of the first kind include $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, as discussed above. These groups have no non-atomic, ergodic invariant random subgroups. By the results in [30], [13] and [26], the same holds for certain inductive limits of finite alternating groups, the simple Higman-Thompson groups and the groups $\mathrm{PSL}_m(k)$, where k is an infinite field and $m \geq 2$, respectively. In the opposite direction, many classes of groups have been proven to admit continuum many non-atomic, ergodic invariant random subgroups. By the results in [17], [33], [3] and [8], these classes include certain wreath products, the group of finitely supported permutations of ω, every weakly branch group and every group containing a non-abelian free group as a normal subgroup, respectively.

Co-induction and invariant random subgroups

In this part of the thesis we develop a co-induction operation for invariant random subgroups.

Classically, co-induction is an operation which transforms a measure preserving action of a countable group into a measure preserving action of a bigger countable group. To be more precise, let $A(\Gamma, X, \mu)$ denote the Polish space of all measure preserving actions of a countable group Γ on a non-atomic standard probability space (X, μ). Then co-induction is an operation

$$\mathrm{cind}_\Gamma^\Delta \colon A(\Gamma, X, \mu) \to A(\Delta, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma})$$

for each pair of countable groups $\Gamma \leq \Delta$. Co-induction is a quite useful tool. For example, the operation was one of the key ingredients in extending the result that $\mathbb{F}_2$ has continuum many free, ergodic, measure preserving and pairwise orbit inequivalent actions to the result that this holds for any countable group containing a copy of $\mathbb{F}_2$ (see [18]). A few years later Epstein generalized the co-induction construction to pairs of countable groups Γ and Δ that have free measure preserving actions $\Gamma \curvearrowright^a (X, \mu)$ and $\Delta \curvearrowright^b (X, \mu)$ such that the induced orbit equivalence relations satisfy $E_a \subseteq E_b$. This construction together with the result of [16] allowed Epstein to prove that any countable non-amenable group induces continuum many free, ergodic, measure preserving and pairwise orbit inequivalent actions (see [15]).

The objective here is to define an operation $\mathrm{CIND}_\Gamma^\Delta \colon \mathrm{IRS}(\Gamma) \to \mathrm{IRS}(\Delta)$ such that the diagram

$$\begin{tikzcd} A(\Delta, X, \mu) \arrow[r, "\text{type}"] & \mathrm{IRS}(\Delta) \\ A(\Gamma, X, \mu) \arrow[u, "\mathrm{cind}_\Gamma^\Delta"] \arrow[r, "\text{type}"'] & \mathrm{IRS}(\Gamma) \arrow[u, "\mathrm{CIND}_\Gamma^\Delta"'] \end{tikzcd}$$

commutes, for every pair of countable groups $\Gamma \leq \Delta$.

It turns out that the co-induction operation $\mathrm{CIND}_\Gamma^\Delta \colon \mathrm{IRS}(\Gamma) \to \mathrm{IRS}(\Delta)$ provides an elemental method for constructing continuum many non-atomic, weakly mixing invariant random subgroups of certain classes of countable groups. For example, $\mathrm{CIND}_\Gamma^\Delta(\theta)$ is weakly mixing for any $\theta \in \mathrm{IRS}(\Gamma)$ whenever $[\Delta : \Gamma] = \infty$. Note that weakly mixing is the strongest mixing property one can hope to achieve for a non-atomic invariant random subgroup. Indeed, by the result in [31], if $\theta \in \mathrm{IRS}(\Gamma)$ satisfies that the restriction of the conjugacy action $\Gamma \curvearrowright (\mathrm{Sub}(\Gamma), \theta)$ to any infinite subgroup of Γ is ergodic, then there is a finite normal subgroup $\Lambda \leq \Gamma$ such that $\theta(\mathrm{Sub}(\Lambda)) = 1$.

We apply our co-induction operation to construct new examples of continuum size families of non-atomic, weakly mixing invariant random subgroups for the following classes of groups:

(1) All wreath products $H \wr G$, where G and H are countable groups with G infinite and H non-trivial.

(2) All HNN extensions $G = \langle H, t \mid (\forall \in A)\ t^{-1}at = \varphi(a) \rangle$, where H is a countable group, $A \leq H$ and $\varphi \colon A \to H$ is an embedding such that $\langle\langle A \cup \varphi(A) \rangle\rangle \neq H$.

(3) All free products with amalgamation $G *_A H$, where G and H are countable groups satisfying that $A \leq G, H$ is a shared normal subgroup with G/A non-trivial and H/A infinite.

We point out that for some of the aforementioned classes of groups, other examples of such families are already known. In [17] they use a different technique, involving what they call *intersectional invariant random subgroups*, to construct such families for the groups in class (1) and for the non-abelian free groups. In [8] they obtain such families for the groups contained in class (3) by use of completely different techniques, including Pontryagin duality and a deep result of Adian in combinatorial group theory. In contrast, our approach is quite elemental.

In fact, in [8] it is shown that the non-abelian free groups admit continuum many non-atomic, weakly mixing invariant random subgroups that are moreover invariant under the action of the full automorphism group. Invariant random subgroups that are invariant under the action of the full automorphism group are called characteristic random subgroups. We will show how to use our method, together with small cancellation theory, to obtain continuum many non-atomic, characteristic random subgroups of $\mathbb{F}_2$ that are weakly mixing with respect to the action of the full automorphism group.

The original motivation for this project came from a different problem. Inspired by the classical notion for unitary representations, the notions of weak containment and weak equivalence for measure preserving actions were introduced in [20]. Roughly speaking, for $a, b \in A(\Gamma, X, \mu)$ we say that a is weakly contained in b, and write $a \preceq b$, if the action a can be approximated by the action b. If $a \preceq b$ and $b \preceq a$, we say that a and b are weakly equivalent. We denote by $\underset{\sim}{A}(\Gamma, X, \mu)$ the set of weak equivalence classes and equip it with the compact Polish topology introduced in [1].

It is known that the classical co-induction operation descends to a well-defined operation

$$\underset{\sim}{\mathrm{cind}}_{\Gamma}^{\Delta} \colon \underset{\sim}{A}(\Gamma, X, \mu) \to \underset{\sim}{A}(\Gamma, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma})$$

for countable groups $\Gamma \leq \Delta$. In [10] it was asked whether this operation is continuous. We will answer this question negatively. More precisely, we show that if $\Gamma \leq \Delta$ are both amenable and the normal core of Γ in Δ is non-trivial, then the operation is continuous if and only if $[\Delta : \Gamma] < \infty$. Moreover, if we remove the amenability assumption, we obtain that the operation is not continuous if $[\Delta : \Gamma] = \infty$. A negative answer to the continuity question was simultaneously obtained via completely different methods in [4], where examples of two non-amenable groups $\Gamma \leq \Delta$, for which the operation is not continuous and $[\Delta : \Gamma] = 2$, are given.

Our method for obtaining these results relies heavily on the correspondence between $A(\Gamma, X, \mu)$ and $\mathrm{IRS}(\Gamma)$ for a countable group Γ. The surjective map $\mathrm{type}\colon A(\Gamma, X, \mu) \to \mathrm{IRS}(\Gamma)$ descends to a well-defined continuous map $\underset{\sim}{\mathrm{type}}\colon \underset{\sim}{A}(\Gamma, X, \mu) \to \mathrm{IRS}(\Gamma)$, which is moreover a homeomorphism when Γ is amenable (see [32]). Using this, we can for pairs of countable groups $\Gamma \leq \Delta$ settle some questions concerning the continuity of $\underset{\sim}{\mathrm{cind}}{}_{\Gamma}^{\Delta}$ by considering the continuity properties of $\mathrm{CIND}_{\Gamma}^{\Delta}$ instead.

Let us end this introduction by giving a brief overview of the content contained in this part of the thesis.

Chapter 5. This preliminary chapter serves as an introduction to the various notions and results that we need in the following chapters. In the first section we introduce several notions related to measure preserving group actions. Moreover, we will define the Polish space $A(\Gamma, X, \mu)$ of all measure preserving actions of a fixed countable group Γ on a non-atomic standard probability space (X, μ). In the second section we will formally define the relations of weak containment and weak equivalence of measure preserving actions with the goal of obtaining the compact Polish space $\underset{\sim}{A}(\Gamma, X, \mu)$. In the third and final section we will introduce the notion of an invariant random subgroup. We will examine the natural compact Polish topology on $\mathrm{IRS}(\Gamma)$ and establish the connection between $\underset{\sim}{A}(\Gamma, X, \mu)$ and $\mathrm{IRS}(\Gamma)$. Finally, we will discuss the notion of a characteristic random subgroup.

Chapter 6. The main goal of this chapter is to develop the co-induction operation $\mathrm{CIND}_{\Gamma}^{\Delta}\colon \mathrm{IRS}(\Gamma) \to \mathrm{IRS}(\Delta)$ for pairs of countable groups $\Gamma \leq \Delta$. In the first section we will consider the classical co-induction operation for measure preserving actions. We will give the definition and present various properties of this operation that will become useful in the following. In the second section we will obtain the co-induction operation for invariant random subgroups. The third section is devoted to the study of continuity properties of both $\underset{\sim}{\mathrm{cind}}{}_{\Gamma}^{\Delta}$ and $\mathrm{CIND}_{\Gamma}^{\Delta}$. In the fourth and final section we investigate various properties of the co-induced invariant random subgroups. We will focus on the case where $[\Delta : \Gamma] = \infty$. In this case, the co-induced invariant random subgroups will always be weakly mixing and we can characterize when they will be non-atomic. As a by-product, we also obtain a complete characterization of when a co-induced measure preserving action is free.

Chapter 7. In this chapter we will apply the co-induction operation on invariant random subgroups to construct continuum size families consisting of non-atomic, weakly mixing invariant random subgroups for the three classes of groups described above. In the first section we isolate a sufficient criterion for a pair of countable groups $\Gamma \leq \Delta$ that allows us to use our co-induction operation to construct such families for Δ. In the second section we apply this criterion to construct examples of these families for the types of wreath products and HNN extensions described in (1) and (2), respectively. In the third and fourth section we will apply the co-induction operation to construct such families for the non-abelian free groups and more generally for the class of free products with normal amalgamation described in (3) above.

Chapter 8. In this chapter we will use our co-induction operation together with small cancellation theory to construct a continuum size family of non-atomic characteristic random subgroups of $\mathbb{F}_2$ that are weakly mixing with respect to the action of $\mathrm{Aut}(\mathbb{F}_2)$. In the first section we will briefly introduce the small cancellation theory needed for our purposes. We also prove the main result of this chapter, which will be the main ingredient in the construction of the aforementioned family. The construction is done in the second section.

Chapter 9. This chapter contains a discussion of some questions related to the subject of this part of the thesis. In the first section we will examine the question of which groups do admit a continuum sized family of non-atomic, ergodic invariant random subgroups. We will also discuss another operation from $\mathrm{IRS}(\Gamma)$ to $\mathrm{IRS}(\Delta)$ that one can define for pairs of countable groups $\Gamma \leq \Delta$, and how it might be used to come up with new examples of groups with these continuum size families. In the second section we will consider the multiplication operation on $\utilde{A}(\Gamma, X, \mu)$ and discuss its continuity properties for different countable groups Γ. Continuity of this operation is closely related to continuity of the co-induction operation $\utilde{\mathrm{cind}}_\Gamma^\Delta \colon \utilde{A}(\Gamma, X, \mu) \to \utilde{A}(\Gamma, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma})$ in the case where $\Gamma \leq \Delta$ with $[\Delta : \Gamma] < \infty$, and in this case we have not yet completely settled when the co-induction operation is continuous.

Part I

Cantor-Bendixson type ranks

This part constitutes an amended version of the paper

> Vibeke Quorning. Cantor-Bendixson type ranks. *Preprint*, 2018.
> `arXiv:1806.03206`

Parts of the article have been altered and rewritten to fit the format of the thesis. In particular, more preliminary theory has been added, some comments and explanations have been expanded and Chapter 4 is new.

Chapter 1

Studying co-analytic sets

In this part of the thesis we will study the Effros Borel space $F(X)$ consisting of all closed subsets of a Polish space X, and the co-analytic subset $F_{\aleph_0}(X)$ consisting of the countable closed subsets. We will in this preliminary chapter introduce the various notions and basic results needed for the rest of this part of the thesis.

In the first section we will briefly summarize the standard results and notions related to Polish spaces, Polish metric spaces and standard Borel spaces, which we assume the reader to be familiar with throughout the rest of this thesis. This includes a short introduction to the Effros Borel space $F(X)$ of a Polish space X. In the second section we will introduce a very important combinatorial tool in descriptive set theory, namely the descriptive set-theoretic trees. These trees are closely related to the closed subsets of certain product spaces. We will in the next chapter generalize the concept of trees and use the generalization in our study of $F(X)$ for a general Polish space X. In the third section we define the analytic and co-analytic sets. We will provide examples and state various properties of these classes of sets. In particular, we will see that $F_{\aleph_0}(X)$ is a co-analytic subset of $F(X)$ for any Polish space X. We will also see that it is not Borel when X is uncountable. The fourth section concerns ranks and in particular the co-analytic ranks. As we will see, these are important tools in the study of co-analytic sets. In the fifth and final section we will discuss a particular rank, namely the Cantor-Bendixson rank, which plays a huge role in this part of the thesis. In particular, we will show that the Cantor-Bendixson rank is a co-analytic rank on $F_{\aleph_0}(X)$ if and only if X is σ-compact.

A more thorough introduction to these subjects, and proofs of most results contained in this chapter, can be found in [19].

1.1 Polish spaces

In this section we will review the basic notions and properties concerning Polish spaces, Polish metric spaces and standard Borel spaces that we will need later on.

Definition 1.1.1. A *Polish space* is a completely metrizable separable topological space.

Clearly, all countable discrete topological spaces are Polish. Moreover, any countable product of Polish spaces is Polish in the product topology. In particular, the space ω^ω, where $\omega = \{0, 1, 2, \ldots\}$ is considered as a discrete Polish space, is Polish in the product topology. We also have that a subspace of a Polish space is Polish if and only if it is a countable intersection of open sets (see [19, Theorem 3.11]). Such a subset is said to be G_δ.

A subspace of a Polish space X is called *perfect* in X if it is closed and does not contain any isolated points. Note that if such a set is non-empty, then it is uncountable.

Theorem 1.1.2 (Cantor-Bendixson). *Let X be a Polish space. Then there is a unique perfect subset $P \subseteq X$ such that $X \setminus P$ is countable.*

For a proof of this theorem, the reader is referred to [19, Theorem 6.4]. The proof shows that P consists of the points in X for which every neighbourhood is uncountable. In particular, $X \setminus P$ is the union of all countable open sets. Thus we obtain that a Polish space X decomposes uniquely as $X = P \sqcup C$, where P is perfect in X and C is countable and open. Moreover, any perfect subset P_0 in X will satisfy $P_0 \subseteq P$.

Definition 1.1.3. Let X be Polish. The perfect subset $P \subseteq X$ such that $X \setminus P$ is countable is called the *perfect kernel* of X.

Note that a Polish space is countable if and only if the perfect kernel is empty.

A Polish space is called *σ-compact* if it is a countable union of compact subsets. All discrete Polish spaces are σ-compact. A key example of a Polish space that is not σ-compact is ω^ω. In fact, it turns out that having ω^ω as a closed subset is the only obstruction for a Polish space to be σ-compact. A proof of the following theorem can be found in [19, Theorem 7.10].

Theorem 1.1.4 (Hurewicz)**.** *Let X be a Polish space. Then X is not σ-compact if and only if X contains a closed subspace homeomorphic to ω^ω.*

In many cases, the theorem above allows us to pass to ω^ω when working with Polish spaces that are not σ-compact. We will see several examples of this.

Sometimes we are interested in fixing a specific complete compatible metric on a Polish space.

Definition 1.1.5. A *Polish metric space* is a Polish space equipped with a complete compatible metric.

It is easily seen that if (X, d) is a Polish metric space and $Y \subseteq X$, then $(Y, d_{|Y})$ is a Polish metric space if and only if Y is closed.

Next let (X, d) and (Y, δ) be Polish metric spaces. A map $f\colon X \to Y$ is called *isometric* if $\delta(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. If f is also bijective, we say that f is an *isometry* and call (X, d) and (Y, d) *isomorphic*. Clearly, any isometric map is injective and we will therefore also call such a map an *isometric embedding*.

We have the following example of a Polish metric space which is universal in the sense that any Polish metric space is isomorphic to a closed subspace.

Example 1.1.6 (Urysohn's universal metric space)**.** An *Urysohn space* is a Polish metric space (X, d) that satisfies the following extension property: If (A, δ) is a finite metric space, $A_0 \subseteq A$ and $f\colon A_0 \to X$ is an isometric embedding, then there is an isometric embedding $\tilde{f}\colon A \to X$ such that $\tilde{f}(a) = f(a)$ for all $a \in A_0$. A construction of an Urysohn space, as well as proofs of the next couple of statements, can be found in [25]. By using a back-and-forth argument to construct an isometry between countable dense subsets, we obtain that any two Urysohn spaces are isomorphic. Moreover, we can use the extension property of an Urysohn space to construct an isometric embedding of any Polish metric space. We will denote by $(\mathbb{U}, d_{\mathbb{U}})$ the unique (up to isometry) Urysohn space.

From now on we will for a Polish metric space (X, d), $x \in X$ and $r > 0$ denote by $\mathrm{B}_d(x, r)$ and $\overline{\mathrm{B}}_d(x, r)$ the open and closed ball around x with radius r with respect to d.

Instead of fixing a metric on a Polish space and thereby ensuring more structure, we can also go in the other direction and ignore some information.

Definition 1.1.7. A *standard Borel space* is a measurable space $(X, \mathcal{S})$ for which there is some Polish topology on X such that $\mathcal{S}$ is the corresponding Borel σ-algebra. The sets in $\mathcal{S}$ are called *Borel*.

In other words, a standard Borel space is a Polish space equipped with its Borel σ-algebra, where one ignores the underlying topology. We will usually suppress the σ-algebra and denote a standard Borel space by X.

We are often only interested in the properties of subsets of Polish spaces that can be completely determined by the Borel structure. In such cases, and especially if there is a nice and simple description of the Borel structure, it is useful to consider a Polish space as a standard Borel space instead.

Given a standard Borel space X, there is no way to recover a specific underlying Polish topology. In fact, if $B \subseteq X$ is Borel, then there is a Polish topology on X that induces the Borel structure and in which B is clopen, i.e., closed and open (see [19, Theorem 13.1]). Note that this also implies that if $(X, \mathcal{S})$ is a standard Borel space and $B \in \mathcal{S}$, then $(B, \mathcal{S}_{|B})$ is a standard Borel space.

Let X, Y be standard Borel spaces. A map $f \colon X \to Y$ is said to be *Borel* if $f^{-1}(B)$ is Borel in X for any $B \subseteq Y$ Borel. We say that f is a *Borel isomorphism* if f is bijective and both f and f^{-1} are Borel. In general, a map $f \colon X \to Y$ is Borel if and only if $\operatorname{graph}(f) \subseteq X \times Y$ is Borel (see [19, Theorem 12.14]). Thus any bijective Borel map is a Borel isomorphism. It turns out, as the following theorem states, that there is only one uncountable standard Borel space up to Borel isomorphism. A proof can be found in [19, Theorem 15.6].

Theorem 1.1.8. *Let X, Y be uncountable standard Borel spaces. Then there exists a Borel isomorphism $f \colon X \to Y$.*

Even though all uncountable standard Borel spaces are isomorphic, they appear naturally in many situations. One of the most important examples is the following.

Example 1.1.9 (The Effros Borel space)**.** Let X be a Polish space and let $F(X)$ denote the set of all closed subsets of X. We equip $F(X)$ with the Borel structure generated by the sets

$$\{F \in F(X) \mid F \cap U \neq \emptyset\},$$

where $U \subseteq X$ varies over all open subsets. This turns $F(X)$ into a standard Borel space (see [19, Theorem 12.6]). The space $F(X)$ is called the *Effros Borel space* of X.

Note that the Effros Borel space $F(\mathbb{U})$, where $\mathbb{U}$ is the Urysohn space, is a neat parametrization of all Polish (metric) spaces. It provides a useful framework for studying definable subclasses of these spaces.

Remark 1.1.10. In the case where X is compact, there is a natural Polish topology on $F(X)$ which induces the Borel structure, namely the *Vietoris topology.* A subbasis for this topology is given by the collection of sets

$$\{F \in F(X) \mid F \subseteq U\} \quad \text{and} \quad \{F \in F(X) \mid F \cap U \neq \emptyset\},$$

where U varies over all open subsets of X (see [19, Theorem 4.26]).

The Borel structure on $F(X)$ is not always intuitive, as even simple operations need not be Borel. For example, the map $(F_0, F_1) \mapsto F_0 \cap F_1$ from $F(X) \times F(X)$ to $F(X)$ is not in general Borel (see [19, Theorem 27.6]). However, a handy property of the Borel structure is the following selection theorem, for which a proof can be found in [19, Theorem 12.13].

Theorem 1.1.11 (Kuratowski-Ryll-Nardzewski)**.** *Let X be Polish. There is a sequence of Borel functions $d_n \colon F(X) \to X$ for $n \in \mathbb{N}$ such that for each $F \in F(X) \setminus \{\emptyset\}$ the set*

$$\{d_n(F) \mid n \in \mathbb{N}\} \subseteq F$$

is dense.

This means that we can choose a dense subset of every non-empty $F \in F(X)$ in a Borel manner.

1.2 Trees and closed sets

In this section we will introduce a very important tool in descriptive set theory, namely descriptive set-theoretic trees. As we will see, these trees are closely related to the closed subsets of certain product spaces and they allow us to study the closed subsets of such spaces in a combinatorial manner.

To define these set-theoretic trees, we need to fix some terminology. Let A be a non-empty and countable set. We will mainly be interested in the case where $A = \omega$ or $A = \{0, 1\}$, and we will use the notation $2 = \{0, 1\}$. For all $n \in \omega$ and any sequence $s \in A^n$, we use the enumeration $s = (s_0, s_1 \ldots, s_{n-1})$. Moreover, we use A^0 to denote the set $\{\emptyset\}$ and call $\emptyset$ the *empty sequence.* We also let a^n denote the constant sequence $(a, a, \dots, a) \in A^n$ for any $a \in A$ and $n \in \omega$. Now consider the set

$$A^{<\omega} = \bigcup_{n \in \omega} A^n$$

consisting of all finite sequences with elements in A. For any $s \in A^{<\omega}$, we let $\ln(s)$ denote the *length* of s, hence $\ln(s) = n$ if and only if $s \in A^n$. If $s \in A^{<\omega}$ and $m \leq \ln(s)$, the *restriction* $s|m$ is given by

$$s|m = (s_0, s_1, \dots, s_{m-1}).$$

For sequences $s, t \in A^{<\omega}$ we say that s is an *initial segment* of t, and write $s \subseteq t$, if there is $m \leq \ln(t)$ such that $t|m = s$. If $s, t \in A^{<\omega}$ and $s \subseteq t$ or $t \subseteq s$, we say that s and t are *compatible*. Otherwise we say that s and t are *incompatible* and write $s \perp t$. We can also consider the *concatenation* of $s, t \in A^{<\omega}$, which is defined to be the sequence

$$s^\frown t = (s_0, s_1, \dots, s_{n-1}, t_0, t_1, \dots, t_{m-1}),$$

where $s = (s_0, s_1, \dots, s_{n-1})$ and $t = (t_0, t_1, \dots, t_{m-1})$. We write $s^\frown a$ instead of $s^\frown(a)$ if $s \in A^{<\omega}$ and $a \in A$.

Definition 1.2.1. A *tree* on a non-empty countable set A is a subset $T \subseteq A^{<\omega}$ such that if $t \in A^{<\omega}$, $s \in T$ and $t \subseteq s$, then $t \in T$. A tree T on A is said to be *pruned* if for any $s \in T$ there is $a \in A$ such that $s^\frown a \in T$.

We denote by $\mathrm{Tree}(A)$ and $\mathrm{PTree}(A)$ the set of trees on A and the set of pruned trees on A, respectively.

Next let A be a non-empty countable set and consider the product space A^ω, which consist of all the infinite sequences with elements in A. In particular, for each $a \in A$ we have $a^\omega = (a, a, a, \dots) \in A^\omega$. If $x \in A^\omega$ and $m \in \omega$ we define the *restriction* $x|m$ to be

$$x|m = (x_0, x_1, \dots, x_{m-1}).$$

For $x \in A^\omega$ and $s \in A^{<\omega}$ we say that s is an *initial segment* of x, and write $s \subseteq x$, if there is $m \in \omega$ such that $s = x|m$. We also define the *concatenation* of s with x to be the element $s^\frown x \in A^\omega$ given by $s^\frown x(i) = s(i)$ if $i < \ln(s)$ and $s^\frown x(i) = x(i - \ln(s))$ if $i \geq \ln(s)$.

Now equip A with the discrete topology and A^ω with the induced product topology. Then A^ω is Polish and the collection of sets

$$N_s = \{x \in A^\omega \mid s \subseteq x\},$$

where $s \in A^{<\omega}$, constitutes a basis of clopen sets for the topology. Note that $s \subseteq t$ if and only if $N_t \subseteq N_s$, and $s \perp t$ if and only if $N_s \cap N_t = \emptyset$ for any $s, t \in A^{<\omega}$. There is also a natural complete compatible ultra-metric d_{A^ω} on A^ω given by

$$d_{A^\omega}(x,y) = \begin{cases} 3^{-1} \cdot 2^{-\min\{n \in \omega \mid x(n) \neq y(n)\}} & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}.$$

Recall that if d is a metric on a set X, then d is said to be an *ultra-metric* if $d(x,y) \leq \max\{d(x,z), d(y,z)\}$ for all $x, y, z \in X$. In this case, if $y \in \mathrm{B}_d(x,r)$ for some $x, y \in X$ and $r > 0$, then $\mathrm{B}_d(x,r) = \mathrm{B}_d(y,r)$. In other words, every point in a ball is a center of the ball. For the metric d_{A^ω} defined above, note that we have

$$\mathrm{B}_{d_{A^\omega}}(x, 2^{-n-1}) = \overline{\mathrm{B}}_{d_{A^\omega}}(x, 2^{-n-1}) = N_{x|n}$$

for any $x \in A^\omega$ and $n \in \omega$.

We have the following correspondence between the closed subsets of A^ω and the pruned trees on A.

Proposition 1.2.2. *Let A be a non-empty countable set. The map $T \mapsto [T]$ from* $\mathrm{PTree}(A)$ *to* $F(A^\omega)$*, where*

$$[T] = \{x \in A^\omega \mid (\forall n \in \omega)\ x|n \in T\}$$

for all $T \in \mathrm{PTree}(A)$, is a bijection. The inverse is the map $F \mapsto T_F$, where

$$T_F = \left\{s \in A^{<\omega} \mid N_s \cap F \neq \emptyset\right\}$$

for any closed $F \in F(A^\omega)$.

Proof. For $T \in \mathrm{PTree}(A)$ and $s \in A^{<\omega}$ we have

$$s \in T_{[T]} \iff N_s \cap [T] \neq \emptyset \iff s \in T.$$

Moreover, for $F \in F(A^\omega)$ and $x \in A^\omega$ it holds that

$$x \in [T_F] \iff (\forall n \in \omega)\ N_{x|n} \cap F \neq \emptyset \iff x \in F.$$

We can therefore conclude that the map $T \mapsto [T]$ from $\mathrm{PTree}(A)$ to $F(A^\omega)$ is bijective with inverse $F \mapsto T_F$, as wanted. $\square$

We may view each $T \in \mathrm{Tree}(A)$ as an element in the Polish space $2^{(A^{<\omega})}$ via its characteristic function. It is straightforward to check that with this identification $\mathrm{Tree}(A)$ is a closed subset and $\mathrm{PTree}(A)$ is G_δ. Thus both spaces are Polish in the subspace topology. From now on we will always view these sets as Polish spaces equipped with this topology.

Remark 1.2.3. The map $T \mapsto [T]$ from PTree(A) to $F(A)$ given in Proposition 1.2.2 is a Borel isomorphism. Indeed, we have $[T] \cap N_s \neq \emptyset$ if and only if $s \in T$ for any $T \in \mathrm{PTree}(A)$ and $s \in A^{<\omega}$.

Many properties of the closed sets can be translated to properties of the corresponding pruned trees.

Definition 1.2.4. A tree T on a non-empty countable set A is called *perfect* if for any $s \in T$ there are $u, v \in T$ such that $s \subseteq u, v$ and $u \perp v$.

Any perfect tree is pruned. Moreover, the following result characterizes the perfect (closed) subsets via their induced trees.

Proposition 1.2.5. *Let A be a non-empty countable set and $F \in F(A^\omega)$. Then F is perfect if and only if T_F is perfect.*

Proof. Assume first that F is perfect and let $s \in T_F$. Then there are $x, y \in N_s \cap F$ with $x \neq y$ and hence there must be $u, v \in T_F$ such that $s \subseteq u, v$ and $u \perp v$. Conversely, assume that T_F is perfect and let $s \in T_F$. It suffices to prove that there are $x, y \in N_s \cap F$ with $x \neq y$. Since T_F is perfect, there are $u, v \in T_F$ such that $s \subseteq u, v$ and $u \perp v$. Now, since T_F is pruned, this implies that there must be $x, y \in N_s \cap F$ with $x \neq y$, as wanted. □

We also want to mention the following two easily obtained results concerning compactness. Below we say that a tree T on a non-empty countable set A is *finitely splitting* if for every $s \in T$ there are at most finitely many $a \in A$ such that $s^\frown a \in T$.

Proposition 1.2.6. *Let A be a non-empty countable set and $F \in F(A^\omega)$. Then F is compact if and only if T_F is finitely splitting.*

Proposition 1.2.7 (König's Lemma)**.** *Let T be a finitely splitting tree on a non-empty countable set A. Then $[T] \neq \emptyset$ if and only if T is infinite.*

In the next section, we will be interested in closed subsets of $A^\omega \times B^\omega$ for some non-empty countable sets A and B. In this case, we still have a convenient correspondence between these closed subsets and the pruned trees on $A \times B$. Indeed, we can identify $A^\omega \times B^\omega$ with $(A \times B)^\omega$ and hence, by Proposition 1.2.2, the closed subsets of $A^\omega \times B^\omega$ correspond to the pruned trees on $A \times B$. In the following we will consider any tree T on $A \times B$ as a subset of $A^{<\omega} \times B^{<\omega}$, where $(s,t) \in T$ implies $\mathrm{ln}(s) = \mathrm{ln}(t)$, instead of as a subset of $(A \times B)^{<\omega}$.

Later on, we will also use subsets of the finite sequences as index sets in our constructions and proofs. Let A be a non-empty countable set. For $k \in \omega$ we let

$$A^{\leq k} = \bigcup_{n \leq k} A^n \qquad \text{and} \qquad A^{<k} = \bigcup_{n<k} A^n,$$

i.e., the sets of finite sequences of length at most k and of length less than k, respectively.

1.3 Analytic and co-analytic sets

In this section we introduce the classes of analytic and co-analytic sets. We will see different examples and various properties of these classes of sets.

Definition 1.3.1. Let X be Polish. A subset $A \subseteq X$ is called *analytic* if there is a Polish space Y and a continuous map $f : Y \to X$ such that $f(Y) = A$. We denote by $\boldsymbol{\Sigma}^1_1$ the class of all analytic sets.

The theorem below ensures that any analytic set is a continuous image of ω^ω. A proof can be found in [19, Theorem 7.9].

Theorem 1.3.2. *Let X be a Polish space. There is a continuous surjection $f : \omega^\omega \to X$.*

Clearly, the class of analytic sets is closed under continuous images. It is also closed under countable unions and countable intersections (see [19, Proposition 14.4]). This in particular implies that all Borel sets are analytic. The next theorem states that, for an uncountable Polish space, the family of analytic subsets is strictly larger than the family of Borel subsets. For a proof consult [19, Theorem 14.2].

Theorem 1.3.3 (Souslin)**.** *Let X be an uncountable Polish space. There exists an analytic subset $A \subseteq X$ that is not Borel.*

We also have the following result stating that Borel images and pre-images of analytic sets are analytic. A proof can be found in [19, Proposition 14.4].

Proposition 1.3.4. *Let X, Y be Polish and $f : X \to Y$ Borel. If $A \subseteq X$ and $B \subseteq Y$ are both analytic, then $f(A)$ and $f^{-1}(B)$ are both analytic.*

The proposition above ensures that the analytic sets can be characterized as Borel images of Borel sets. It is therefore natural to say that a subset of a standard Borel space is analytic if it is analytic with respect to some (or equivalently any) Polish topology on X that induces the Borel structure. Note that if X is a standard Borel space, then $A \subseteq X$ is analytic if and only if there is a standard Borel space Y and a Borel map $f \colon Y \to X$ such that $f(Y) = A$.

Definition 1.3.5. Let X be a Polish space. A subset $A \subseteq X$ is called *co-analytic* if $X \setminus A$ is analytic. We denote by $\mathbf{\Pi}^1_1$ the class of all co-analytic sets.

We will also call a subset of a standard Borel space co-analytic if the complement is analytic. Note that it follows directly from the analogous results for analytic sets that the class of co-analytic sets is closed under countable unions, countable intersections and Borel pre-images. Moreover, the class of co-analytic sets includes the Borel sets as well. In fact, by the following theorem, the Borel sets are exactly the sets that are both analytic and co-analytic. A proof can be found in [19, Theorem 14.11].

Theorem 1.3.6 (Souslin's Theorem)**.** *Let X be a Polish space and $B \subseteq X$. Then B is Borel if and only if B is both analytic and co-analytic.*

It is the following example of a co-analytic set that we are going to study in this part of the thesis.

Example 1.3.7. Let X be a Polish space and fix a countable basis $(U_n)_n$ for the topology on X. Consider the Effros Borel space $F(X)$ and the subset

$$F_{\aleph_0}(X) = \{F \in F(X) \mid F \text{ is countable}\}.$$

We will argue that $F_{\aleph_0}(X)$ is co-analytic by arguing that the complement

$$F(X) \setminus F_{\aleph_0}(X) = \{F \in F(X) \mid F \text{ is uncountable}\}$$

is analytic. Note that, by Theorem 1.1.2, we have $F \in F(X)$ is uncountable if and only if there is $P \in F(X)$ such that $P \subseteq F$ and P is non-empty perfect. Moreover, the sets

$$\{(F, P) \in F(X) \times F(X) \mid P \subseteq F\}$$

and

$$\{(F, P) \in F(X) \times F(X) \mid P \text{ is non-empty and perfect}\}$$

are both Borel. Indeed, the first set is Borel since $P \subseteq F$ if and only if for all $n \in \omega$ we have $P \cap U_n = \emptyset$ or $F \cap U_n \neq \emptyset$. The second set is Borel since $P \in F(X)$ is perfect if and only if for all $n \in \omega$ either $U_n \cap F = \emptyset$ or there is $i, j \in \omega$ such that $U_i, U_j \subseteq U_n, U_i \cap U_j = \emptyset$, $U_i \cap P \neq \emptyset$ and $U_j \cap P \neq \emptyset$. So $F(X) \setminus F_{\aleph_0}(X)$ is analytic, being the projection of the intersection of the sets above. Hence we conclude that $F_{\aleph_0}(X)$ is co-analytic, as wanted.

Next we will give another example of an analytic and a co-analytic set. These examples are very important in the general study of these classes of sets, as we will see afterwards.

Example 1.3.8. Recall that $\mathrm{Tree}(\omega)$ denotes the Polish space of all trees on ω. Consider the subsets WF and IF of $\mathrm{Tree}(\omega)$ given by

$$\mathrm{WF} = \{T \in \mathrm{Tree}(\omega) \mid [T] = \emptyset\} \quad \text{and} \quad \mathrm{IF} = \{T \in \mathrm{Tree}(\omega) \mid [T] \neq \emptyset\}\,.$$

A tree $T \in \mathrm{WF}$ is called *well-founded* and a tree $T \in \mathrm{IF}$ is called *ill-founded*. Note that

$$\mathrm{IF} = \mathrm{Proj}_{\mathrm{Tree}(\omega)}\left(\bigcap_{n\in\omega} \{(T,x) \in \mathrm{Tree}(\omega) \times \omega^\omega \mid x|n \in T\}\right)$$

and

$$\{(T,x) \in \mathrm{Tree}(\omega) \times \omega^\omega \mid x|n \in T\} = \bigcup_{s\in\omega^n} (\{T \in \mathrm{Tree}(\omega) \mid s \in T\} \times N_s)$$

is open for all $n \in \omega$. Hence we conclude that IF is analytic and therefore also that WF is co-analytic.

Definition 1.3.9. Let X be a standard Borel space and $A \subseteq X$. We say that A is $\mathbf{\Sigma}^1_1$-complete (resp. $\mathbf{\Pi}^1_1$-complete) if A is analytic (resp. co-analytic) and for any standard Borel space Y and analytic (resp. co-analytic) $B \subseteq Y$ we have a Borel map $f\colon Y \to X$ such that $f^{-1}(A) = B$.

Note that any $\mathbf{\Sigma}^1_1$-complete or $\mathbf{\Pi}^1_1$-complete set can be thought of as a maximally complicated set within its class. Indeed, if we are in the setting of Definition 1.3.9, then

$$x \in B \iff f(x) \in A.$$

Thus the problem of deciding whether $x \in B$ reduces to deciding whether $f(x) \in A$.

We should also point out that, by Theoren 1.3.3, a $\mathbf{\Sigma}^1_1$-complete or $\mathbf{\Pi}^1_1$-complete set cannot be Borel.

The goal for the rest of this section is to prove that IF is $\mathbf{\Sigma^1_1}$-complete and hence also that WF is $\mathbf{\Pi^1_1}$-complete. Note that, by Theorem 1.1.8, it is enough to ensure that Definition 1.3.9 holds for $Y = \omega^\omega$, and so the analytic subsets of ω^ω play a crucial role. We are therefore very interested in the following characterization of analytic subsets and in particular the remark following it.

Proposition 1.3.10. *Let X be Polish and $A \subseteq X$. Then A is analytic if and only if there is closed $F \subseteq X \times \omega^\omega$ such that $A = \mathrm{Proj}_X(F)$.*

Proof. The left implication is clearly true. To prove the right implication, assume that A is analytic. Fix a continuous map $f \colon \omega^\omega \to X$ such that $f(\omega^\omega) = A$. Then the closed set

$$F = \{(x, y) \in X \times \omega^\omega \mid x = f(y)\}$$

satisfies $\mathrm{Proj}_X(F) = A$. □

Remark 1.3.11. If $A \subseteq \omega^\omega$ is analytic, then the previous proposition and Proposition 1.2.2 ensure that there exists a pruned tree T on $\omega \times \omega$ such that $A = \mathrm{Proj}_{\omega^\omega}([T])$. Thus, when working with analytic subsets of ω^ω, we can apply the combinatorial tools of trees.

Using this remark, we can prove the following theorem.

Theorem 1.3.12. *The set* IF *is* $\mathbf{\Sigma}^1_1$*-complete and the set* WF *is* $\mathbf{\Pi}^1_1$*-complete.*

Proof. It suffices to prove that IF is $\mathbf{\Sigma}^1_1$-complete. Let $A \subseteq \omega^\omega$ be analytic. Then there is a pruned tree T on $\omega \times \omega$ such that

$$A = \mathrm{Proj}_{\omega^\omega}([T]).$$

Now for each $x \in \omega^\omega$ let

$$T(x) = \left\{s \in \omega^{<\omega} \mid (x|\mathrm{ln}(s), s) \in T\right\}.$$

Note that $T(x) \in \mathrm{Tree}(\omega)$ for all $x \in \omega^\omega$ and let $f \colon \omega^\omega \to \mathrm{Tree}(\omega)$ be given by $f(x) = T(x)$. It is easily seen that f is continuous, as

$$\begin{aligned} f^{-1}\left(\{S \in \mathrm{Tree}(\omega) \mid s \in S\}\right) &= \{x \in \omega^\omega \mid s \in T(x)\} \\ &= \bigcup_{t \in \omega^{\mathrm{ln}(s)}, (t,s) \in T} N_t. \end{aligned}$$

Hence, since $x \in A$ if and only if $T(x) \in \mathrm{IF}$, we conclude that IF is $\mathbf{\Sigma}^1_1$-complete. □

1.4 Co-analytic ranks

We will here introduce the notion of a co-analytic rank, which is a very important tool in the study of co-analytic sets. They can, for example, be used to settle whether a given co-analytic set is Borel.

Let ORD denote the class of ordinals and let $\omega_1 \in \mathrm{ORD}$ denote the first uncountable ordinal. A *rank* on a set S is a map $\varphi\colon S \to \mathrm{ORD}$. We let $\alpha_\varphi = \sup\{\varphi(x) \mid x \in S\}$ and if $\varphi(S) = \alpha_\varphi$ the rank is called *regular*. Each rank $\varphi\colon S \to \mathrm{ORD}$ induces a prewellordering $\leq_\varphi$ on S given by

$$x \leq_\varphi y \iff \varphi(x) \leq \varphi(y).$$

For the class of co-analytic sets certain ranks are of special interest, as these ensure that this prewellordering has nice definability properties.

Definition 1.4.1. Let X be a Polish space and $A \subseteq X$ a co-analytic set. A rank $\varphi\colon A \to \mathrm{ORD}$ is said to be *co-analytic* if there exist binary relations $R_\varphi^{\mathbf{\Pi}_1^1}, R_\varphi^{\mathbf{\Sigma}_1^1} \subseteq X \times X$ such that $R_\varphi^{\mathbf{\Pi}_1^1}$ is co-analytic, $R_\varphi^{\mathbf{\Sigma}_1^1}$ is analytic and for any $y \in A$ we have

$$xR_\varphi^{\mathbf{\Pi}_1^1}y \iff xR_\varphi^{\mathbf{\Sigma}_1^1}y \iff (x \in A) \wedge (x \leq_\varphi y)$$

for all $x \in X$.

In other words, a rank is co-analytic if the induced prewellordering on A extends to both an analytic and a co-analytic relation on X, which preserve the initial segments of A. This implies in particular that for each $\alpha < \alpha_\varphi$ we have

$$A_\alpha^\varphi = \{x \in A \mid \varphi(x) \leq \alpha\}$$

is Borel in X. Indeed, since φ is a co-analytic rank, we have that A_α^φ is both analytic and co-analytic. Hence, by Theorem 1.3.6, we obtain that A_α^φ is Borel.

Remark 1.4.2. Let A be co-analytic subset of a Polish space X and let $\varphi\colon A \to \mathrm{ORD}$ be a rank. If there exist analytic relations $R_\varphi^{\mathbf{\Sigma}_1^1}, Q_\varphi^{\mathbf{\Sigma}_1^1} \subseteq X \times X$ such that for any $y \in A$ we have

$$xR_\varphi^{\mathbf{\Sigma}_1^1}y \iff (x \in A) \wedge (x \leq_\varphi y) \quad \text{and} \quad xQ_\varphi^{\mathbf{\Sigma}_1^1}y \iff (x \in A) \wedge (x <_\varphi y)$$

for all $x \in X$, then φ is co-analytic. Indeed, define $R_\varphi^{\mathbf{\Pi}_1^1} \subseteq X \times X$ by

$$xR_\varphi^{\mathbf{\Pi}_1^1}y \iff x \in A \wedge \neg(yQ_\varphi^{\mathbf{\Sigma}_1^1}x)$$

for all $x, y \in X$. Then $R_\varphi^{\mathbf{\Pi}_1^1}$ is co-analytic and for $y \in A$ we have

$$xR_\varphi^{\mathbf{\Pi}_1^1}y \iff (x \in A) \wedge (x \leq_\varphi y)$$

for any $x \in X$.

The main interest in these co-analytic ranks comes from the Boundedness Theorem for co-analytic ranks. A proof can be found in [19, Theorem 35.23].

Theorem 1.4.3 (The Boundedness Theorem). *Let X be Polish, $A \subseteq X$ a co-analytic set and $\varphi\colon A \to \omega_1$ a co-analytic rank. Then the following hold:*

(1) A is Borel if and only if $\alpha_\varphi < \omega_1$.

(2) If $B \subseteq A$ is analytic, then $\sup\{\varphi(x) \mid x \in B\} < \omega_1$.

Remark 1.4.4. It follows directly from this theorem that if $\varphi, \psi\colon A \to \omega_1$ are both co-analytic ranks, then there exists $f\colon \omega_1 \to \omega_1$ such that $\varphi(x) \leq f(\psi(x))$ for all $x \in A$. Indeed, put

$$f(\alpha) = \sup\{\varphi(x) \mid \psi(x) \leq \alpha\}$$

for all $\alpha < \omega_1$.

Note that the remark above ensures that if $\varphi, \psi\colon A \to \omega_1$ are both co-analytic ranks on a co-analytic set A, then

$$\sup\{\varphi(x) \mid x \in C\} < \omega_1 \iff \sup\{\psi(x) \mid x \in C\} < \omega_1$$

for any subset $C \subseteq A$. We call a subset $B \subseteq A$ *bounded* if

$$\sup\{\varphi(x) \mid x \in B\} < \omega_1$$

for some (or equivalently any) co-analytic rank $\varphi\colon A \to \omega_1$. It is straightforward to check that the collection of bounded subsets of A forms a σ-ideal.

Definition 1.4.5. Let A be a co-analytic set and I some index set. Moreover, let $\psi, \varphi, \varphi_i\colon A \to \omega_1$ be co-analytic ranks for each $i \in I$.

(1) If $f\colon \omega_1 \to \omega_1$ satisfies $\varphi(x) \leq f(\psi(x))$ for all $x \in A$, we say that φ is *bounded* by ψ via f.

(2) If there is a function $f\colon \omega_1 \to \omega_1$ such that $\varphi_i(x) \leq f(\psi(x))$ for all $x \in A$ and all $i \in I$, we say that the family $(\varphi_i)_{i \in I}$ is *uniformly bounded* by ψ via f.

Next we will show that any co-analytic set admits a co-analytic rank into ω_1. To do so, we will first argue that it is enough to show that a $\mathbf{\Pi}^1_1$-complete set admits such a rank.

Proposition 1.4.6. *Let X and Y be Polish spaces. Assume that $A \subseteq X$ and $B \subseteq Y$ are co-analytic and that $f\colon X \to Y$ is Borel with $f^{-1}(B) = A$. If $\varphi\colon B \to \omega_1$ is a co-analytic rank, then $\psi\colon A \to \omega_1$ given by $\psi(x) = \varphi(f(x))$ is a co-analytic rank.*

Proof. Assume that $R_\varphi^{\mathbf{\Pi}_1^1}, R_\varphi^{\mathbf{\Sigma}_1^1} \subseteq Y \times Y$ are such that $R_\varphi^{\mathbf{\Pi}_1^1}$ is co-analytic, $R_\varphi^{\mathbf{\Sigma}_1^1}$ is analytic and for any $y \in B$ we have

$$xR_\varphi^{\mathbf{\Pi}_1^1}y \iff xR_\varphi^{\mathbf{\Sigma}_1^1}y \iff (x \in B) \wedge (x \leq_\varphi y)$$

for all $x \in Y$. Let $R_\psi^{\mathbf{\Pi}_1^1}, R_\psi^{\mathbf{\Sigma}_1^1} \subseteq X \times X$ be given by

$$xR_\psi^{\mathbf{\Pi}_1^1}y \iff f(x)R_\varphi^{\mathbf{\Pi}_1^1}f(y) \quad \text{and} \quad xR_\psi^{\mathbf{\Sigma}_1^1}y \iff f(x)R_\varphi^{\mathbf{\Sigma}_1^1}f(y)$$

for all $x, y \in X$. Then $R_\psi^{\mathbf{\Pi}_1^1}$ is co-analytic and $R_\psi^{\mathbf{\Sigma}_1^1}$ is analytic. Moreover, if $y \in A$, then

$$xR_\psi^{\Pi_1^1}y \iff xR_\psi^{\Sigma_1^1}y \iff (x \in A) \wedge (x \leq_\varphi y)$$

for all $x \in X$. $\square$

We are now ready to prove that any co-analytic set admits a co-analytic rank into ω_1.

Theorem 1.4.7. *Let X be Polish and $A \subseteq X$ a co-analytic set. There exists a co-analytic rank $\varphi\colon A \to \omega_1$.*

Proof. It suffices to prove that some $\mathbf{\Pi}_1^1$-complete set admits a co-analytic rank. We will here use the set of well-founded trees WF considered in Example 1.3.8.

Define the rank $\rho\colon \mathrm{WF} \to \omega_1$ by

$$\rho(T) = \sup\left\{\rho_T(s) + 1 \mid s \in T\right\},$$

where $\rho_T\colon T \to \omega_1$ is defined recursively such that $\rho_T(s) = 0$ if $s \in T$ is terminal, i.e., has no extension within T, and

$$\rho_T(s) = \sup\left\{\rho_T(s^\frown a) + 1 \;\middle|\; a \in \omega \text{ and } s^\frown a \in T\right\}$$

if $s \in T$ is not terminal. To prove that ρ is co-analytic, we will define analytic relations $R_\rho^{\mathbf{\Sigma}_1^1}, Q_\rho^{\mathbf{\Sigma}_1^1} \subseteq \mathrm{Tree}(\omega) \times \mathrm{Tree}(\omega)$ as in Remark 1.4.2.

Let $S, T \in \mathrm{Tree}(\omega)$. An element $f \in (\omega^{<\omega})^{\omega^{<\omega}}$ is called a *strictly monotone* map from S to T if for all $s, t \in S$ we have $f(s) \in T$ and $s \subsetneq t \implies f(s) \subsetneq f(t)$.

Claim: Let $S, T \in \mathrm{Tree}(\omega)$ be such that $T \in \mathrm{WF}$. Then there is a strictly monotone map from S to T if and only if $S \in \mathrm{WF}$ and $\rho(S) \leq \rho(T)$.

Proof of Claim: For the right implication, assume that $f \in (\omega^{<\omega})^{\omega^{<\omega}}$ is a strictly monotone map from S to T. Then it is easy to see that $S \in \mathrm{WF}$. Moreover, an easy induction argument shows that $\rho_S(s) \leq \rho_T(f(s))$ for any $s \in S$, and hence we obtain that $\rho(S) \leq \rho(T)$.

For the left implication, assume that $S \in \mathrm{WF}$ and that $\rho(S) \leq \rho(T)$. We will now recursively construct a strictly monotone map $f \in (\omega^{<\omega})^{\omega^{<\omega}}$ from S to T such that for any $s \in S$ we have $\rho_S(s) \leq \rho_T(f(s))$. First, let $f(\emptyset) = \emptyset$ and $f(u) = \emptyset$ for any $u \notin S$. Now assume that we have defined $f(s) \in T$ such that $\rho_S(s) \leq \rho_T(f(s))$ for some $s \in S$. Then if $a \in \omega$ such that $s^\frown a \in S$, we let $f(s^\frown a) = f(s)^\frown b$ for some $b \in \omega$ such that $f(s)^\frown b \in T$ and $\rho_S(s^\frown a) \leq \rho_T(f(s)^\frown b)$. Note that such b exists since $\rho_S(s) \leq \rho_T(f(s))$. $\diamond$

Now define $R_\rho^{\boldsymbol{\Sigma}^1_1} \subseteq \mathrm{Tree}(\omega) \times \mathrm{Tree}(\omega)$ by $S R_\rho^{\boldsymbol{\Sigma}^1_1} T$ if and only if there is a strictly monotone map from S to T. It follows directly from the claim that if $T \in \mathrm{WF}$, then

$$S R_\rho^{\boldsymbol{\Sigma}^1_1} T \iff S \in \mathrm{WF} \wedge S \leq_\rho T$$

for any $S \in \mathrm{Tree}(\omega)$. Moreover, the sets

$$\bigcap_{s \in \omega^{<\omega}} \left\{ (f, S, T) \in (\omega^{<\omega})^{\omega^{<\omega}} \times \mathrm{Tree}(\omega)^2 \mid s \notin S \vee f(s) \in T \right\}$$

and

$$\bigcap_{(s,t) \in C} \left\{ (f, S, T) \in (\omega^{<\omega})^{\omega^{<\omega}} \times \mathrm{Tree}(\omega)^2 \mid s \notin S \vee t \notin S \vee f(s) \subsetneq f(t) \right\},$$

where $C = \{(s,t) \in (\omega^{<\omega})^2 \mid s \subsetneq t\}$, are both Borel. Thus, as $R_\rho^{\boldsymbol{\Sigma}^1_1}$ is the projection of the intersection of these sets, we conclude that $R_\rho^{\boldsymbol{\Sigma}^1_1}$ is analytic.

Next note that if $S, T \in \mathrm{Tree}(\omega)$ and $T \in \mathrm{WF}$, then we have $S \in \mathrm{WF}$ and $\rho(S) < \rho(T)$ if and only if there is a strictly monotone map $f \in (\omega^{<\omega})^{\omega^{<\omega}}$ from S to T such that $f(\emptyset) \neq \emptyset$. Indeed, if $S \in \mathrm{WF}$ and $\rho(S) < \rho(T)$ there is $t \in T \setminus \{\emptyset\}$ such that $\rho_S(\emptyset) \leq \rho_T(t)$. So, in the recursive definition in the proof of the claim, we simply let $f(\emptyset) = t$ and then continue as before. Conversely, if there is such $f \in (\omega^{<\omega})^{\omega^{<\omega}}$, then we have $\rho_S(\emptyset) \leq \rho_T(f(\emptyset)) < \rho_T(\emptyset)$. Thus $\rho(S) < \rho(T)$.

Finally, define $Q_\rho^{\boldsymbol{\Sigma}^1_1} \subseteq \mathrm{Tree}(\omega) \times \mathrm{Tree}(\omega)$ by $S Q_\rho^{\boldsymbol{\Sigma}^1_1} T$ if and only if there is a strictly monotone map $f \in (\omega^{<\omega})^{\omega^{<\omega}}$ from S to T such that $f(\emptyset) \neq \emptyset$.

We have, by similar arguments as before, that $Q_\rho^{\mathbf{\Sigma}_1^1}$ is analytic. Moreover if $T \in \mathrm{WF}$, then

$$SQ_\rho^{\mathbf{\Sigma}_1^1}T \iff S \in \mathrm{WF} \wedge S <_\rho T$$

for any $S \in \mathrm{Tree}(\omega)$. $\square$

Note that the proof of Theorem 1.4.7 does not provide a concrete co-analytic rank $\varphi\colon A \to \omega_1$ for a given co-analytic subset A of a Polish space X. However, one way to obtain an explicit co-analytic rank from the proof above is to determine a specific Borel map $f\colon X \to \mathrm{Tree}(\omega)$ such that $f^{-1}(\mathrm{WF}) = A$.

1.5 The Cantor-Bendixson rank

In this section we will examine the Cantor-Bendixson rank, which is a natural rank to consider on $F(X)$ for a Polish space X. We will see that it is a co-analytic rank on $F_{\aleph_0}(X)$ if and only if X is σ-compact. Finally, we will consider a related rank that one can associate to the trees on ω and then show how one uses the correspondence between $F(\omega^\omega)$ and $\mathrm{Tree}(\omega)$ to obtain a co-analytic rank on $F_{\aleph_0}(\omega^\omega)$.

First we will discuss a general way to obtain explicit ranks on $F(X)$ for a Polish space X that are co-analytic when restricted to certain co-analytic subsets of $F(X)$.

Let X be Polish. A derivative on $F(X)$ is a map $D\colon F(X) \to F(X)$ satisfying $D(F) \subseteq F$ and $F_0 \subseteq F_1 \implies D(F_0) \subseteq D(F_1)$ for all $F, F_0, F_1 \in F(X)$. Whenever we have such a derivative, we define the iterated derivatives of $F \in F(X)$ as follows:

$$D^0(F) = F, \qquad D^{\alpha+1}(F) = D(D^\alpha(F)) \quad \text{and} \quad D^\lambda(F) = \bigcap_{\beta<\lambda} D^\beta(F),$$

where $\alpha, \lambda \in \mathrm{ORD}$ and λ is a limit ordinal. The least $\alpha \in \mathrm{ORD}$ satisfying $D^\alpha(F) = D^{\alpha+1}(F)$ is denoted by $|F|_D$. Moreover, we let $D^\infty(F) = D^{|F|_D}(F)$. Note that we obtain an induced rank $F \mapsto |F|_D$ from $F(X)$ to ORD.

Next we will see an example of a rank that is obtained this way and will become useful in Section 3.2.

Example 1.5.1. Let X be a Polish space and let $K(X)$ denote the set of all compact subsets of X. Fix a countable basis $(U_n)_n$ for X and consider the map $D_K\colon F(X) \to F(X)$ given by

$$D_K(F) = \left\{x \in F \mid (\forall n \in \omega)\; x \notin U_n \vee \overline{U_n \cap F} \in K(X)\right\}.$$

Note that D_K does not depend on the chosen basis. Furthermore, D_K removes the points in F that have a pre-compact neighbourhood in F. So we must have $|F|_{D_K} < \omega_1$ for all $F \in F(X)$. Moreover, $D_K^\infty(F) = \emptyset$ if and only if F is σ-compact. Hence the rank $F \mapsto |F|_{D_K}$ from $F(X)$ to ω_1 measures how far from locally compact a σ-compact closed subset is.

Later, we will use the following notation. For any Polish space X and $\alpha < \omega_1$, we let $X_\alpha = D_K^\alpha(X)$ and $|X|_K = |X|_{D_K}$. Then X is σ-compact if and only if $X_{|X|_K} = \emptyset$.

The theorem below provides a sufficient criterion for when a rank induced by a Borel derivative is co-analytic on a certain co-analytic subset. For a proof see [19, Theorem 34.10].

Theorem 1.5.2. *Let X be a σ-compact Polish space. Assume $D\colon F(X) \to F(X)$ is a Borel derivative. Then*

$$\Omega_D = \{F \in F(X) \mid D^\infty(F) = \emptyset\}$$

is a co-analytic set and the rank $\varphi_D\colon \Omega_D \to \mathrm{ORD}$ given by $\varphi_D(F) = |F|_D$ is co-analytic.

Let us now consider a very important application of Theorem 1.5.2. In the example below, we will for a Polish space X introduce the Cantor-Bendixson rank on $F(X)$ and use the aforementioned theorem to prove that it is co-analytic on $F_{\aleph_0}(X)$ when X is σ-compact.

Example 1.5.3 (The Cantor-Bendixson rank)**.** Let X be a Polish space and consider the derivative $D_{\mathrm{CB}}\colon F(X) \to F(X)$ given by

$$D_{\mathrm{CB}}(F) = \{x \in F \mid x \text{ is not isolated in } F\}.$$

Note that $|F|_{D_{\mathrm{CB}}} < \omega_1$ for any $F \in F(X)$, since X has a countable basis. Moreover, $D_{\mathrm{CB}}^\infty(F)$ must be the perfect kernel of F and hence

$$D_{\mathrm{CB}}^\infty(F) = \emptyset \iff F \in F_{\aleph_0}(X)$$

for any $F \in F(X)$. To see that D_{CB} is Borel, fix by Theorem 1.1.11 a Borel map $d_n\colon F(X) \to X$ for each $n \in \omega$ such that

$$\{d_n(F) \mid n \in \omega\} \subseteq F$$

is dense for all $F \in F(X) \setminus \{\emptyset\}$. Furthermore, let $(U_n)_n$ be a countable basis for X. Then for any open $U \subseteq X$, we have $D_{\mathrm{CB}}(F) \cap U = \emptyset$ if and only if either $F = \emptyset$ or for all $n \in \omega$ there exists $m \in \omega$ such that for all $k \in \omega$ we have

$$d_n(F) \notin U \vee (d_n(F) \in U_m \wedge (d_n(F) = d_k(F) \vee d_k(F) \notin U_m)).$$

Therefore we conclude that D_{CB} is Borel and hence it follows by Theorem 1.5.2 that $F \mapsto |F|_{D_{\text{CB}}}$ from $F_{\aleph_0}(X)$ to ω_1 is co-analytic when X is σ-compact.

To ease the notation, we will from now on for any $F \in F(X)$ and $\alpha < \omega_1$ let $F^\alpha = D^\alpha_{\text{CB}}(F)$ and $|F|_{\text{CB}} = |F|_{D_{\text{CB}}}$.

Remark 1.5.4. Let X and Y be Polish spaces. If $F_0 \in F(X)$ and $F_1 \in F(Y)$ are homeomorphic, then $|F_0|_{\text{CB}} = |F_1|_{\text{CB}}$.

We saw in Example 1.5.3 that the Cantor-Bendixson rank $F \mapsto |F|_{\text{CB}}$ from $F_{\aleph_0}(X)$ to ω_1 is co-analytic for any σ-compact Polish space X. Next we will argue that this is not the case when X is not σ-compact. To do so, we prove the following proposition.

Proposition 1.5.5. *Let X be a Polish space. If X is not σ-compact, then the set*

$$\{F \in F_{\aleph_0}(X) \mid |F|_{CB} \leq 1\}$$

is not Borel.

Proof. First note that it suffices to prove that the set

$$\{F \in F_{\aleph_0}(\omega^\omega) \mid |F|_{\text{CB}} \leq 1\}$$

is not Borel. Indeed, if X is not σ-compact, then by Theorem 1.1.4 there exists a closed subset $N \subseteq X$ and a homeomorphism $f \colon \omega^\omega \to N$. Now the map $\overline{f} \colon F(\omega^\omega) \to F(X)$ given by $\overline{f}(F) = f(F)$ is Borel and

$$\overline{f}^{-1}(\{F \in F_{\aleph_0}(X) \mid |F|_{\text{CB}} \leq 1\}) = \{F \in F_{\aleph_0}(\omega^\omega) \mid |F|_{\text{CB}} \leq 1\}.$$

Thus if $\{F \in F_{\aleph_0}(X) \mid |F|_{\text{CB}} \leq 1\}$ is Borel, so is $\{F \in F_{\aleph_0}(\omega^\omega) \mid |F|_{\text{CB}} \leq 1\}$.

To prove that $\{F \in F_{\aleph_0}(\omega^\omega) \mid |F|_{\text{CB}} \leq 1\}$ is not Borel, we will define a Borel map $g \colon \text{Tree}(\omega) \to F(\omega^\omega)$ such that

$$g^{-1}(\{F \in F_{\aleph_0}(\omega^\omega) \mid |F|_{\text{CB}} \leq 1\}) = \text{WF}.$$

For each $T \in \text{Tree}(\omega)$ let $\tilde{T} \in \text{PTree}(\omega)$ be given by

$$\tilde{T} = \left\{ s^\frown 0^n, s^\frown 1^n \mid n \in \omega, s \in T \right\}.$$

It is easy to check that the map $T \mapsto \tilde{T}$ is Borel. So, by Remark 1.2.3, we can conclude that $g\colon \mathrm{Tree}(\omega) \to F(\omega^\omega)$ given by $g(T) = [\tilde{T}]$ is Borel as well. Furthermore, it is straightforward to check that $g(T) \in \{F \in F_{\aleph_0}(X) \mid |F|_{\mathrm{CB}} \leq 1\}$ if and only if $T \in \mathrm{WF}$. Hence we conclude that $\{F \in F_{\aleph_0}(\omega^\omega) \mid |F|_{\mathrm{CB}} \leq 1\}$ is not Borel. $\square$

Remark 1.5.6. The beginning of the proof of Proposition 1.5.5 shows the following fact: If a Polish space X is not σ-compact, then there is an injective Borel map $h\colon F(\omega^\omega) \to F(X)$ such that $h(F)$ is homeomorphic to F for any $F \in F(\omega^\omega)$. In particular, by Proposition 1.4.6, if $\varphi\colon F_{\aleph_0}(X) \to \omega_1$ is a co-analytic rank, then so is $\psi\colon F_{\aleph_0}(\omega^\omega) \to \omega_1$ given by $\psi(F) = \varphi(h(F))$.

Since all initial segments of co-analytic ranks are Borel, we can now conclude the following.

Corollary 1.5.7. *Let X be Polish. Then X is σ-compact if and only if the Cantor-Bendixson rank $F \mapsto |F|_{CB}$ from $F_{\aleph_0}(X)$ to ω_1 is co-analytic.*

We end this section by giving an example of how one may obtain a co-analytic rank on $F_{\aleph_0}(\omega^\omega)$. The construction heavily relies on the correspondence between $F(\omega^\omega)$ and $\mathrm{PTree}(\omega)$ described in Proposition 1.2.2.

Example 1.5.8. We will here construct a co-analytic rank on $F_{\aleph_0}(\omega^\omega)$. First we are going to construct a Borel derivative on $F(\omega^{<\omega})$. Note that $\omega^{<\omega}$ is a σ-compact Polish space when equipped with the discrete topology. Moreover, $F(\omega^{<\omega})$ is just the powerset of $\omega^{<\omega}$. Let $D_{\omega^\omega}\colon F(\omega^{<\omega}) \to F(\omega^{<\omega})$ be given by

$$D_{\omega^\omega}(A) = \{s \in A \mid (\exists u, v \in A)\ s \subseteq u, v \wedge u \perp v\}$$

for any $A \subseteq \omega^{<\omega}$. Then D_{ω^ω} is clearly a Borel derivative. Indeed, for $s \in \omega^{<\omega}$ we have

$$\left\{A \in F(\omega^{<\omega}) \mid s \in A\right\} = \bigcup_{(u,v) \in S_s} \left\{A \in F(\omega^{<\omega}) \mid u, v, s \in A\right\},$$

where $S_s = \{(u, v) \in \omega^{<\omega} \times \omega^{<\omega} \mid s \subseteq u, v \wedge u \perp v\}$. Thus, by Theorem 1.5.2, we obtain a co-analytic rank $A \mapsto |A|_{D_{\omega^\omega}}$ from $\Omega_{D_{\omega^\omega}}$ to ω_1, where

$$\Omega_{D_{\omega^\omega}} = \left\{A \in F(\omega^{<\omega}) \mid D^\infty_{\omega^\omega}(A) = \emptyset\right\}.$$

Next note that for any $F \in F(\omega^\omega)$ we have $D^\infty_{\omega^\omega}(T_F) \neq \emptyset$ if and only if T_F contains a perfect subtree. Furthermore, by Proposition 1.2.5, T_F contains a perfect subtree if and only if F contains a perfect subset. The latter happens if and only if $F \notin F_{\aleph_0}(\omega^\omega)$. Hence, by Proposition 1.4.6 and Remark 1.2.3, we conclude that $F \mapsto |T_F|_{D_{\omega^\omega}}$ from $F_{\aleph_0}(\omega^\omega)$ to ω_1 is a co-analytic rank, as wanted.

Chapter 2

Constructing co-analytic ranks on $F_{\aleph_0}(X)$

In the previous chapter we discussed a natural rank on the co-analytic set $F_{\aleph_0}(X)$ for a Polish space X, namely the Cantor-Bendixson rank. We saw that this rank is co-analytic if and only if X is σ-compact. It remains to find an explicit co-analytic rank on $F_{\aleph_0}(X)$ in the case where X is not σ-compact.

The main goal of this chapter is to obtain a family of concrete co-analytic ranks on $F_{\aleph_0}(X)$ for a general Polish space X by generalizing the idea of Example 1.5.8. The key step is to generalize the well-known correspondence between $F(\omega^\omega)$ and $\mathrm{PTree}(\omega)$. For each fixed complete compatible metric d on X and dense sequence $(x_i)_i$ in X we construct a definable correspondence between $F(X)$ and certain subsets of ω^2, where each $F \in F(X)$ is encoded by the set

$$\left\{(i,j) \in \omega^2 \mid F \cap \mathrm{B}_d(x_i, 2^{-j-1}) \neq \emptyset\right\}.$$

Then, as in Example 1.5.8, we can obtain a co-analytic rank on $F_{\aleph_0}(X)$ by defining an appropriate Borel derivative on $F(\omega^2)$. Doing this for each complete compatible metric and dense sequence, we obtain a potentially huge family of co-analytic ranks.

In the first section we will give the precise construction of the ranks described above. In the second section we will provide various properties of the construction. In particular, we will see that the results concerning the correspondence between $F(\omega^\omega)$ and $\mathrm{PTree}(\omega)$ from Section 1.2 have natural generalizations to this setting. In the third and final section we investigate the extent to which the rank depends on the chosen metric and on the dense sequence. We will first isolate classes of Polish metric spaces for which the construction is completely independent of the dense sequence. In general, changes can occur when varying the dense sequence, but we obtain a bound on how much. Afterwards we will see that there is no bound on the changes that may occur when varying the complete metric.

The results of this chapter have all been obtained by the author in [27] except Proposition 2.2.2 and Proposition 2.2.5, which have been added for the purpose of this exposition.

2.1 The construction

In this section we provide the main construction of this part of the thesis, namely a construction of a family of concrete co-analytic ranks on $F_{\aleph_0}(X)$ for any Polish space X.

Definition 2.1.1. A *presentation* of a Polish space X is a triple $(X, d, \overline{x})$, where d is a complete compatible metric on X and $\overline{x} = (x_n)_n$ is a dense sequence in X.

The following presentation of ω^ω is going to be useful for us.

Example 2.1.2. Consider the natural complete compatible ultra-metric d_{ω^ω} on ω^ω given by

$$d_{\omega^\omega}(x, y) = \begin{cases} 3^{-1} \cdot 2^{-\min\{n\in\omega \mid x(n)\neq y(n)\}} & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

for all $x, y \in \omega^\omega$. Let $\overline{z}$ denote an enumeration of the countable dense subset $\{s^\frown 0^\omega \mid s \in \omega^{<\omega}\} \subseteq \omega^\omega$ and put $\mathcal{P}_{\omega^\omega} = (\omega^\omega, d_{\omega^\omega}, \overline{z})$. We will refer to $\mathcal{P}_{\omega^\omega}$ as the *standard presentation* of ω^ω.

Next let X be a Polish space. We will for each presentation of X construct a co-analytic rank on $F_{\aleph_0}(X)$. Fix towards this end a presentation $\mathcal{P} = (X, d, \overline{x})$ of X. For each $n = (n(0), n(1)) \in \omega^2$ let

$$\mathrm{B}_{\mathcal{P}}(n) = \mathrm{B}_d(x_{n(0)}, 2^{-n(1)-1}) \quad \text{and} \quad \overline{\mathrm{B}}_{\mathcal{P}}(n) = \overline{\mathrm{B}}_d(x_{n(0)}, 2^{-n(1)-1}).$$

Moreover, define the binary relations $\prec_{\mathcal{P}}$ and $\curlywedge_{\mathcal{P}}$ on ω^2 by

$$n \prec_{\mathcal{P}} m \iff n(1) < m(1) \text{ and } \overline{\mathrm{B}}_{\mathcal{P}}(m) \subseteq \mathrm{B}_{\mathcal{P}}(n)$$

$$n \curlywedge_{\mathcal{P}} m \iff \overline{\mathrm{B}}_{\mathcal{P}}(n) \cap \overline{\mathrm{B}}_{\mathcal{P}}(m) = \emptyset$$

for all $n, m \in \omega^2$.

Definition 2.1.3. A subset $A \subseteq \omega^2$ is called $\mathcal{P}$-*closed* if for any $n \in A$ and $m \in \omega^2$ with $m \prec_{\mathcal{P}} n$ we have that $m \in A$. We say that $A \subseteq \omega^2$ is $\mathcal{P}$-*pruned* if for all $n \in A$ there is $m \in A$ satisfying that $n \prec_{\mathcal{P}} m$. Finally, $A \subseteq \omega^2$ is said to be $\mathcal{P}$-*perfect* if for all $n \in A$ there are $u, v \in A$ such that $n \prec_{\mathcal{P}} u, v$ and $u \curlywedge_{\mathcal{P}} v$.

Let $\mathrm{C}_{\mathcal{P}}(\omega^2)$ and $\mathrm{PC}_{\mathcal{P}}(\omega^2)$ denote the set of all $\mathcal{P}$-closed subsets of ω^2 and the set of all $\mathcal{P}$-closed and $\mathcal{P}$-pruned subsets ω^2, respectively. When viewed as subsets of 2^{ω^2} it is straightforward to check that $\mathrm{C}_{\mathcal{P}}(\omega^2)$ is closed and that $\mathrm{PC}_{\mathcal{P}}(\omega^2)$ is G_δ. Hence they are both Polish in the induced subspace topology.

Now we will see how to establish a definable correspondence between $F(X)$ and $\mathrm{PC}_{\mathcal{P}}(\omega^2)$. For $F \in F(X)$ we let

$$A_F^{\mathcal{P}} = \left\{ n \in \omega^2 \mid \mathrm{B}_{\mathcal{P}}(n) \cap F \neq \emptyset \right\}.$$

It is easily seen that $A_F^{\mathcal{P}}$ is $\mathcal{P}$-closed and $\mathcal{P}$-pruned for all $F \in F(X)$ and that the map $F \mapsto A_F^{\mathcal{P}}$ from $F(X)$ to $\mathrm{PC}_{\mathcal{P}}(\omega^2)$ is Borel. For the other direction, consider the set

$$[\omega^2]_{\mathcal{P}} = \left\{ (n_i)_i \in (\omega^2)^\omega \mid (\forall i \in \omega)\ n_i \prec_{\mathcal{P}} n_{i+1} \right\}.$$

A sequence $(n_i)_i \in [\omega^2]_{\mathcal{P}}$ is called $\mathcal{P}$-*compatible*. Note that $[\omega^2]_{\mathcal{P}} \subseteq (\omega^2)^\omega$ is closed and hence Polish. Since d is a complete metric, we obtain a surjective continuous map $\pi_{\mathcal{P}} \colon [\omega^2]_{\mathcal{P}} \to X$ given by

$$\pi_{\mathcal{P}}((n_i)_i) = x \iff \bigcap_{i \in \omega} \mathrm{B}_{\mathcal{P}}(n_i) = \{x\}.$$

For any $A \subseteq \omega^2$ we let

$$[A]_{\mathcal{P}} = \left\{ (n_i)_i \in [\omega^2]_{\mathcal{P}} \mid (\forall i \in \omega)\ n_i \in A \right\}$$

and put $F_A^{\mathcal{P}} = \pi_{\mathcal{P}}([A]_{\mathcal{P}})$. If A is $\mathcal{P}$-closed, then $F_A^{\mathcal{P}}$ is closed in X. Indeed, if $(x_i)_i \in F_A^{\mathcal{P}}$ and $x \in X$ such that $x_i \to x$ as $i \to \infty$, then any $n \in \omega^2$ satisfying $x \in \mathrm{B}_{\mathcal{P}}(n)$ will also satisfy $n \in A$. Hence $x \in \pi_{\mathcal{P}}([A]_{\mathcal{P}})$.

The following proposition is now straightforward to prove by use of the same arguments as in the proofs of Proposition 1.2.2 and Proposition 1.2.5.

Proposition 2.1.4. *Let $\mathcal{P} = (X, d, \overline{x})$ be a presentation of a Polish space X. Then the following hold:*

(1) The map $F \mapsto A_F^{\mathcal{P}}$ is a Borel isomorphism from $F(X)$ to $\mathrm{PC}_{\mathcal{P}}(\omega^2)$ with inverse $A \mapsto F_A^{\mathcal{P}}$.

(2) For any $F \in F(X)$ it holds that F is perfect if and only if $A_F^{\mathcal{P}}$ is $\mathcal{P}$-perfect.

To finish the construction we will define a derivative on $F(\omega^2)$ and use the correspondence given in the previous proposition to obtain a rank on $F(X)$ which is co-analytic when restricted to $F_{\aleph_0}(X)$. Note that $F(\omega^2)$ is just the powerset of ω^2.

Define $D_{\mathcal{P}} \colon F(\omega^2) \to F(\omega^2)$ by

$$D_{\mathcal{P}}(A) = \{n \in A \mid (\exists u, v \in A)\ n \prec_{\mathcal{P}} u, v \text{ and } u \curlywedge_{\mathcal{P}} v\}\,.$$

It is clear that $D_{\mathcal{P}}(A) \subseteq A$ and that $B \subseteq A$ implies $D_{\mathcal{P}}(B) \subseteq D_{\mathcal{P}}(A)$ for all $A, B \in F(\omega^2)$, hence $D_{\mathcal{P}}$ is a derivative on $F(\omega^2)$. Thus for each $A \in P(\omega^2)$ there is a least ordinal $\alpha < \omega_1$ such that $D^{\alpha}_{\mathcal{P}}(A) = D^{\alpha+1}_{\mathcal{P}}(A)$. We let $|A|_{\mathcal{P}}$ denote this least ordinal.

It remains to prove the following theorem.

Theorem 2.1.5. *Let $\mathcal{P} = (X, d, \overline{x})$ be a presentation of a Polish space X. The map $\varphi_{\mathcal{P}} \colon F_{\aleph_0}(X) \to \omega_1$ given by $F \mapsto |A^{\mathcal{P}}_F|_{\mathcal{P}}$ is a co-analytic rank.*

Proof. First, since $D_{\mathcal{P}}$ is continuous, it follows by Theorem 1.5.2 that the set

$$\Omega_{\mathcal{P}} = \left\{A \in F(\omega^2) \mid D^{|A|_{\mathcal{P}}}_{\mathcal{P}}(A) = \emptyset\right\}$$

is co-analytic and that the map $A \mapsto |A|_{\mathcal{P}}$ from $\Omega_{\mathcal{P}}$ to ω_1 is a co-analytic rank. By the definition of $D_{\mathcal{P}}$, we have that $D^{\alpha}_{\mathcal{P}}(A)$ is $\mathcal{P}$-closed for all $\alpha < \omega_1$. Moreover, $D^{|A|_{\mathcal{P}}}_{\mathcal{P}}(A)$ is the largest $\mathcal{P}$-perfect subset of A, in the sense that it contains any $\mathcal{P}$-perfect subset of A. Therefore, by Proposition 2.1.4, we must have that $\pi_{\mathcal{P}}\left([D^{|A|_{\mathcal{P}}}_{\mathcal{P}}(A^{\mathcal{P}}_F)]_{\mathcal{P}}\right)$ is the perfect kernel of F for any $F \in F(X)$. So for any $F \in F(X)$ it holds that

$$F \in F_{\aleph_0}(X) \iff A^{\mathcal{P}}_F \in \Omega_{\mathcal{P}}.$$

Hence, since the map $F \mapsto A^{\mathcal{P}}_F$ from $F(X)$ to $P(\omega^2)$ is Borel, we may conclude that $\varphi_{\mathcal{P}}$ is a co-analytic rank on $F_{\aleph_0}(X)$. $\square$

Since every Polish space X admits many presentations, we obtain a potentially huge family

$$\{\varphi_{\mathcal{P}} \mid \mathcal{P} \text{ is a presentation of } X\}$$

of co-analytic ranks on $F_{\aleph_0}(X)$. In the rest of this part of the thesis we will study the behaviour of this family for different classes of Polish spaces.

2.2 Properties of the construction

In this section we will for a presentation $\mathcal{P}$ of a Polish space X investigate the basic properties of the correspondence between $F(X)$ and $\mathrm{PC}_{\mathcal{P}}(\omega^2)$ described in Proposition 2.1.4. Our main focus will be to recover the results from Section 1.2 concerning the correspondence between $F(\omega^\omega)$ and $\mathrm{PTree}(\omega)$. We end the section by proving that there is a presentation $\mathcal{P}$ of ω^ω such that $\varphi_{\mathcal{P}}\colon F_{\aleph_0}(\omega^\omega) \to \omega_1$ is the rank obtained in Example 1.5.8.

One of the basic properties of the standard correspondence between $F(\omega^\omega)$ and $\mathrm{PTree}(\omega)$ is that every point $x \in \omega^\omega$ is encoded by a unique branch in $\omega^{<\omega}$. Indeed, the branch $\{x|n \mid n \in \omega\}$ satisfies that for every $s \in \omega^{<\omega}$ such that $x \in N_s$ there is $n \in \omega$ with $x|n = s$. In the more general setting this is no longer the case. In particular, for a presentation $\mathcal{P}$ of a Polish space, we do not necessarily have that $\neg(n \prec_{\mathcal{P}} m)$ and $\neg(m \prec_{\mathcal{P}} n)$ imply $n \curlywedge_{\mathcal{P}} m$ for all $n, m \in \omega^2$. The next result states that even though we can have multiple $\mathcal{P}$-compatible sequences that encode the same element, they will be "removed" at the same stage when applying the derivative $D_{\mathcal{P}}$.

Proposition 2.2.1. *Let $\mathcal{P}$ be a presentation of a Polish space X and let $F \in F(X)$. If $(m_i)_i, (n_i)_i \in [A_F^{\mathcal{P}}]_{\mathcal{P}}$ satisfy $\pi_{\mathcal{P}}((m_i)_i) = \pi_{\mathcal{P}}((n_i)_i)$, then*

$$(n_i)_i \in [D^\alpha_{\mathcal{P}}(A_F^{\mathcal{P}})]_{\mathcal{P}} \iff (m_i)_i \in [D^\alpha_{\mathcal{P}}(A_F^{\mathcal{P}})]_{\mathcal{P}}$$

for all $\alpha < \omega_1$.

Proof. For any $\mathcal{P}$-closed $A \subseteq \omega^2$, we have $x \in \pi_{\mathcal{P}}([A]_{\mathcal{P}})$ if and only if for any $m \in \omega^2$ with $x \in \mathrm{B}_{\mathcal{P}}(m)$ we have $m \in A$. $\square$

Next we will prove an analogue of Proposition 1.2.6. The result relates compactness of a Polish space X to a finitely splitting behaviour of ω^2 with respect to any presentation $\mathcal{P}$ of X.

Proposition 2.2.2. *Let $\mathcal{P} = (X, d, \overline{x})$ be a presentation of a Polish space X. Then X is compact if and only if there is a sequence $(N_i)_i \in \omega$ such that for all $i \in \omega$ we have*

(1) $N_i \leq N_{i+1}$.

(2) *If $S \subseteq \omega \times \{i\}$ satisfies that $n \neq m$ implies $n \curlywedge_{\mathcal{P}} m$ for all $n, m \in S$, then S contains at most N_i elements.*

Proof. First assume that X is compact. For each $i \in \omega$ we let $N_i \in \omega$ be least such that there is $x_0, \ldots, x_{N_i-1} \in X$ with

$$\bigcup_{j<N_i} \mathrm{B}_d(x_j, 2^{-i-1}) = X.$$

Then it is straightforward to check that the sequence $(N_i)_i \in \omega$ satisfies (1) and (2).

Conversely, assume that the sequence $(N_i)_i \in \omega$ satisfies (1) and (2). To prove that X is compact, let $(y_j)_j \in X$ be a sequence. We will now recursively construct a sequence $(C_i)_i$ of infinite subsets of ω such that $C_{i+1} \subseteq C_i$ and if $j, l \in C_i$, then $d(y_j, y_l) < 3 \cdot 2^{-i}$ for all $i \in \omega$.

First let $S_0 \subseteq \omega \times \{0\}$ be maximal such that for all $n, m \in S_0$ we have $n \neq m$ implies $n \curlywedge_{\mathcal{P}} m$. Then S_0 contains at most N_0 elements. For each $j \in \omega$, choose $k_j^0 \in \omega$ such that $y_j \in \mathrm{B}_{\mathcal{P}}((k_j^0, 0))$. Then there must be $n_0 \in S_0$ and an infinite subset $C_0 \subseteq \omega$ such that

$$\overline{\mathrm{B}}_{\mathcal{P}}((k_j^0, 0)) \cap \overline{\mathrm{B}}_{\mathcal{P}}(n_0) \neq \emptyset$$

for any $j \in C_0$. So for any $j, l \in C_0$ we have $d(y_j, y_l) \leq 3$.

Next assume that we have constructed C_i for some $i \in \omega$. Let $S_{i+1} \subseteq \omega \times \{i+1\}$ be maximal such that $n \neq m$ implies $n \curlywedge_{\mathcal{P}} m$ for all $n, m \in S_{i+1}$. Note that S_{i+1} contains at most N_{i+1} elements, and for each $j \in C_i$ choose $k_j^{i+1} \in \omega$ such that $y_j \in \mathrm{B}_{\mathcal{P}}((k_j^{i+1}, i+1))$. Once again there must be $n_{i+1} \in S_{i+1}$ and infinite $C_{i+1} \subseteq C_i$ such that

$$\overline{\mathrm{B}}_{\mathcal{P}}((k_j^{i+1}, i+1)) \cap \overline{\mathrm{B}}_{\mathcal{P}}(n_{i+1}) \neq \emptyset$$

for any $j \in C_{i+1}$. So for all $j, l \in C_{i+1}$ we must have $d(y_j, y_l) \leq 3 \cdot 2^{-i-1}$, as wanted. It is now easily seen that the completeness of d implies that there are $y \in X$ and a subsequence $(y_{j_k})_k \subseteq (y_j)_j$ such that $y_{j_k} \to y$ as $k \to \infty$. We may therefore conclude that X is compact. □

We also have the following analogue of Proposition 1.2.7 which will be useful in Chapter 3.

Lemma 2.2.3. *Let X be a compact Polish space, $\mathcal{P} = (X, d, \overline{x})$ a presentation of X and let $A \subseteq \omega^2$ be $\mathcal{P}$-closed. Then $[A]_{\mathcal{P}} \neq \emptyset$ if and only if there is $(n_k)_k \in A$ such that $n_k(1) \to \infty$ as $k \to \infty$.*

Proof. The right implication is clear. To prove the left implication, assume that there is $(n_k)_k \in A$ such that $n_k(1) \to \infty$ as $k \to \infty$ and let $z_k \in \mathrm{B}_{\mathcal{P}}(n_k)$ for all $k \in \omega$. Then, by compactness of X, there is $z \in X$ and a subsequence

$(z_{k_i})_i \subseteq (z_k)_k$ such that $z_{k_i} \to z$ as $i \to \infty$. To prove $z \in [A]_{\mathcal{P}}$, assume that $z \in \mathrm{B}_{\mathcal{P}}(m)$ for some $m \in \omega^2$. Then there is $\varepsilon > 0$ and $N \in \omega$ such that $B_d(z, \varepsilon) \subseteq \mathrm{B}_{\mathcal{P}}(m)$ and $z_{k_i} \in B_d(z, \varepsilon/3)$ for all $i \geq N$. Moreover, as $\mathrm{diam}_d(\mathrm{B}_{\mathcal{P}}(n_k)) \to 0$ when $k \to \infty$, we may choose $M \geq N$ such that $\mathrm{diam}_d(\mathrm{B}_{\mathcal{P}}(n_{k_j})) < \varepsilon/3$ for all $j \geq M$. Then $m \prec n_{k_j}$ for any $j \geq M$ and hence, since A is $\mathcal{P}$-closed, we conclude that $m \in A$. $\square$

Remark 2.2.4. The argument in the proof of Lemma 2.2.3 can also be used to prove the following statement: Let $\mathcal{P}$ be a presentation of any Polish space X, $F \in F_{\aleph_0}(X)$ and $n \in A_F^{\mathcal{P}}$ be such that $\overline{\mathrm{B}_{\mathcal{P}}(n) \cap F}$ is compact. Assume $(n_k)_k \in A_F^{\mathcal{P}}$ satisfies that $n \prec_{\mathcal{P}} n_k$ for each $k \in \omega$ and that $n_k(1) \to \infty$ as $k \to \infty$. Then there exists $x \in \overline{\mathrm{B}_{\mathcal{P}}(n) \cap F}$ such that for any $U \subseteq X$ open with $x \in U$ there is $N \in \omega$ such that $\overline{\mathrm{B}}_{\mathcal{P}}(n_k) \subseteq U$ for all $k \geq N$.

The last result of this section is that the rank we obtain using the standard presentation $\mathcal{P}_{\omega^\omega}$ of ω^ω is the same as the rank constructed in Example 1.5.8.

Proposition 2.2.5. *We have that* $\varphi_{\mathcal{P}_{\omega^\omega}}(F) = |T_F|_{D_{\omega^\omega}}$ *for any* $F \in F_{\aleph_0}(\omega^\omega)$.

Proof. Let $F \in F_{\aleph_0}(\omega^\omega)$. We will show by induction on $\alpha < \omega_1$ that for all $s \in \omega^{<\omega}$ and $n \in \omega^2$ with $\mathrm{B}_{\mathcal{P}_{\omega^\omega}}(n) = N_s$ we have

$$s \in D^\alpha_{\omega^\omega}(T_F) \iff n \in D^\alpha_{\mathcal{P}_{\omega^\omega}}(A_F^{\mathcal{P}_{\omega^\omega}}).$$

Both the induction start and the limit case are straightforward to handle. We will briefly argue for the successor case.

Assume that the statement holds for some $\alpha < \omega_1$. Let $s \in D^{\alpha+1}_{\omega^\omega}(T_F)$ and $n \in \omega^2$ be such that $N_s = \mathrm{B}_{\mathcal{P}_{\omega^\omega}}(n)$. There must be $u, v \in D^\alpha_{\omega^\omega}(T_F)$ such that $s \subseteq u, v$ and $u \perp v$. If we let $m, k \in \omega^2$ satisfy that $\mathrm{B}_{\mathcal{P}_{\omega^\omega}}(m) = N_u$ and $\mathrm{B}_{\mathcal{P}_{\omega^\omega}}(k) = N_v$, then $k, m \in D^\alpha_{\mathcal{P}_{\omega^\omega}}(A_F^{\mathcal{P}_{\omega^\omega}})$, $n \prec_{\mathcal{P}_{\omega^\omega}} m, k$ and $m \curlywedge_{\mathcal{P}_{\omega^\omega}} k$. Hence, since $D^\alpha_{\mathcal{P}_{\omega^\omega}}(A_F^{\mathcal{P}_{\omega^\omega}})$ is $\mathcal{P}$-closed, we obtain that $n \in D^{\alpha+1}_{p_{\omega^\omega}}(A_F^{\mathcal{P}_{\omega^\omega}})$, as wanted. The converse direction is proven similarly. $\square$

2.3 Dependence on presentation

In Section 2.1 we constructed a co-analytic rank $\varphi_{\mathcal{P}} \colon F_{\aleph_0}(X) \to \omega_1$ for each presentation $\mathcal{P} = (X, d, \overline{x})$ of a Polish space X. In this section we look into the extent to which the rank depends on the chosen presentation. We divide this investigation in two parts.

First we consider the variations that occur when varying the dense sequence while holding the metric fixed. We isolate classes of Polish metric spaces for which the construction is completely independent of the dense sequence, and a class for which the ranks agree up to one step. In general, the ranks can change more significantly, but we will recover a bound on this change.

Afterwards we consider the variations that occur when varying the metric while fixing the dense sequence. It is clear that even a small change in the chosen metric can affect the induced rank. We will see that even for the discrete countable Polish space there exist presentations for which the rank of the whole space varies arbitrarily. We will also find a bound for the variation in the case where the two metrics are equivalent (in the strong sense).

The following simple lemma offers a useful tool to compare the ranks induced by these presentations.

Lemma 2.3.1. *Let $\mathcal{P} = (X, d, \overline{x})$ and $\mathcal{S} = (Y, \delta, \overline{y})$ be presentations of the Polish spaces X and Y, respectively. If $A \subseteq \omega^2$ and $\psi\colon A \to \omega^2$ satisfies*

$$n \prec_{\mathcal{P}} m \implies \psi(n) \prec_{\mathcal{S}} \psi(m) \quad \text{and} \quad n \curlywedge_{\mathcal{P}} m \implies \psi(n) \curlywedge_{\mathcal{S}} \psi(m)$$

for all $n, m \in A$, then $|A|_{\mathcal{P}} \leq |\psi(A)|_{\mathcal{S}}$.

Proof. Let $A \subseteq \omega^2$ and $\psi\colon A \to \omega^2$ be as above. We will show by induction on $\alpha < \omega_1$ that $\psi\left(D^{\alpha}_{\mathcal{P}}(A)\right) \subseteq D^{\alpha}_{\mathcal{S}}(\psi(A))$.

The induction start is clear. For the successor case, assume that the inclusion holds for some $\alpha < \omega_1$ and let $n \in D^{\alpha+1}_{\mathcal{P}}(A)$. Then there are $u, v \in D^{\alpha}_{\mathcal{P}}(A)$ such that $n \prec_{\mathcal{P}} u, v$ and $u \curlywedge_{\mathcal{P}} v$. So we obtain $\psi(u), \psi(v) \in D^{\alpha}_{\mathcal{S}}(\psi(A))$, $\psi(n) \prec_{\mathcal{S}} \psi(u), \psi(v)$ and $\psi(u) \curlywedge_{\mathcal{S}} \psi(v)$, hence $n \in D^{\alpha+1}_{\mathcal{S}}(\psi(A))$. Finally, if the inclusion holds for all $\beta < \lambda$ for some limit ordinal $\lambda < \omega_1$, then we have

$$\psi(D^{\lambda}_{\mathcal{P}}(A)) \subseteq \bigcap_{\beta<\lambda} \psi\left(D^{\beta}_{\mathcal{P}}(A)\right) \subseteq D^{\lambda}_{\mathcal{S}}(\psi(A)).$$

Therefore we conclude that the inclusion holds for all $\alpha < \omega_1$ and hence that $|A|_{\mathcal{P}} \leq |\psi(A)|_{\mathcal{S}}$, as wanted. ☐

Note that $|A|_{\mathcal{P}} = |\psi(A)|_{\mathcal{S}}$ if we have bi-implications instead of implications in the lemma above.

2.3.1 Change of dense sequence

We will now investigate what happens when we change the dense sequence and keep the metric fixed. More precisely, we will for a Polish metric space (X, d) consider the variation among the ranks of the form $\varphi_{\mathcal{P}}$, where $\mathcal{P} = (X, d, \overline{x})$ for some dense sequence $\overline{x}$ in X. We will refer to this family of ranks as the family of induced ranks of the Polish metric space (X, d). Our first objective is to isolate two classes of Polish metric spaces for which the induced ranks does not depend on the chosen sequence. Thus for these spaces the family of induced ranks is a singleton. The first result states that this holds for all Polish ultra-metric spaces, that is, all Polish metric spaces (X, d), where d is an ultra-metric.

Proposition 2.3.2. *Let (X, d) be a Polish ultra-metric space. Then any pair of presentations $\mathcal{P} = (X, d, \overline{x})$ and $\mathcal{S} = (X, d, \overline{y})$ satisfies $\varphi_{\mathcal{P}} = \varphi_{\mathcal{S}}$.*

Proof. Fix presentations $\mathcal{P} = (X, d, \overline{x})$ and $\mathcal{S} = (X, d, \overline{y})$ of X. Since d is an ultrametric, we have

$$\{\mathrm{B}_{\mathcal{P}}((i, j)) \mid i \in \omega\} = \{\mathrm{B}_{\mathcal{S}}((i, j)) \mid i \in \omega\}$$

for each $j \in \omega$. Therefore we can choose $f\colon \omega \to \omega$ such that $\mathrm{B}_{\mathcal{P}}((i, j)) = \mathrm{B}_{\mathcal{S}}((f(i), j))$ for all $i \in \omega$, and define $\psi\colon \omega^2 \to \omega^2$ by $\psi(n) = (f(n(0)), n(1))$. Note that $n \prec_{\mathcal{P}} m \iff \psi(n) \prec_{\mathcal{S}} \psi(m)$ and $n \curlywedge_{\mathcal{P}} m \iff \psi(n) \curlywedge_{\mathcal{S}} \psi(m)$ for all $n, m \in \omega^2$. Now let $F \in F_{\aleph_0}(X)$ and note that $\psi(A_F^{\mathcal{P}}) \subseteq A_F^{\mathcal{S}}$. Hence it follows by Lemma 2.3.1 that $\varphi_{\mathcal{P}}(F) \leq \varphi_{\mathcal{S}}(F)$. By symmetry we conclude $\varphi_{\mathcal{P}} = \varphi_{\mathcal{S}}$, as wanted. $\square$

The next result implies that for the class of compact Polish spaces the induced rank only depends on the chosen metric. Below we say that a metric is *proper* if all the closed balls are compact. Moreover, a Polish metric space (X, d) is called *proper* if d is a proper metric. We will now show that for all proper Polish metric spaces, the rank is independent of the choice of dense sequence.

Theorem 2.3.3. *Let (X, d) be a proper Polish metric space. Then any pair of presentations $\mathcal{P} = (X, d, \overline{x})$ and $\mathcal{S} = (X, d, \overline{y})$ satisfies $\varphi_{\mathcal{P}} = \varphi_{\mathcal{S}}$.*

Proof. Let $F \in F_{\aleph_0}(X)$. First we prove by induction on $\alpha < \omega_1$ that if $n \in D^{\alpha}_{\mathcal{P}}(A_F^{\mathcal{P}})$ and $\varepsilon > 0$, then there is $m \in D^{\alpha}_{\mathcal{S}}(A_F^{\mathcal{S}})$ such that

$$\overline{\mathrm{B}}_{\mathcal{S}}(m) \subseteq \mathrm{B}_d(x_{n(0)}, 2^{-n(1)-1} + \varepsilon)$$

and $m(1) \geq n(1)$.

For $\alpha = 0$ the statement is easily seen to be true. For the successor case, assume the statement holds for some $\alpha < \omega_1$. Moreover, let $n \in D^{\alpha+1}_{\mathcal{P}}(A_F^{\mathcal{P}})$ and $\varepsilon > 0$ be given. Then there are $u, v \in D^{\alpha}_{\mathcal{P}}(A_F^{\mathcal{P}})$ such that $n \prec_{\mathcal{P}} u, v$ and $u \curlywedge_{\mathcal{P}} v$. The compactness of $\overline{\mathrm{B}}_{\mathcal{P}}(n)$ implies that there exists $\rho > 0$ such that

$$\mathrm{B}_d(x_{u(0)}, 2^{-u(1)-1} + \rho), \mathrm{B}_d(x_{v(0)}, 2^{-v(1)-1} + \rho) \subseteq \mathrm{B}_d(x_{n(0)}, 2^{-n(1)-1} - \rho)$$

and
$$\mathrm{B}_d(x_{u(0)}, 2^{-u(1)-1} + \rho) \cap \mathrm{B}_d(x_{v(0)}, 2^{-v(1)-1} + \rho) = \emptyset.$$
By the induction hypothesis there are $\tilde{v}, \tilde{u} \in D^{\alpha}_{\mathcal{S}}(A^{\mathcal{S}}_F)$ such that
$$\overline{\mathrm{B}}_{\mathcal{S}}(\tilde{u}) \subseteq \mathrm{B}_d(x_{u(0)}, 2^{-u(1)-1} + \rho), \qquad \overline{\mathrm{B}}_{\mathcal{S}}(\tilde{v}) \subseteq \mathrm{B}_d(x_{v(0)}, 2^{-v(1)-1} + \rho),$$
$\tilde{u}(1) \geq u(1)$ and $\tilde{v}(1) \geq v(1)$. Now choose $m \in \omega^2$ such that $d(x_{n(0)}, y_{m(0)}) < \min\{\varepsilon, \rho\}$ and $m(1) = n(1)$. Then $m \prec_{\mathcal{S}} \tilde{u}, \tilde{v}$ and $\tilde{u} \curlywedge_{\mathcal{S}} \tilde{v}$. Therefore $m \in D^{\alpha+1}_{\mathcal{S}}(A^{\mathcal{S}}_F)$, as wanted.

Finally, assume that the statement holds for all $\beta < \lambda$ for some limit ordinal $\lambda < \omega_1$. Let $n \in D^{\lambda}_{\mathcal{P}}(A^{\mathcal{P}}_F)$ and let $\varepsilon > 0$ be given. Now choose $(\beta_k)_k \in \lambda$ such that $\bigcup_{k\in\omega} \beta_k = \lambda$. Then there exists $(n_k)_k \in A^{\mathcal{P}}_F$ such that $n \prec_{\mathcal{P}} n_k$, $n_k \in D^{\beta_k}_{\mathcal{P}}(A^{\mathcal{P}}_F)$ and $n_k(1) \to \infty$ as $k \to \infty$. Therefore we may let $z \in \overline{\mathrm{B}_{\mathcal{P}}(n) \cap F}$ satisfy the statement of Remark 2.2.4. Now choose $m \in \omega^2$ such that $m(1) \geq n(1)$, $2^{-m(1)-1} < \varepsilon/2$ and $z \in \mathrm{B}_{\mathcal{S}}(m)$. Then there is $N \in \omega$ such that $\overline{\mathrm{B}}_{\mathcal{P}}(n_k) \subseteq \mathrm{B}_{\mathcal{S}}(m)$ for all $k \geq N$. By use of compactness of $\overline{\mathrm{B}}_{\mathcal{S}}(m)$ and the induction hypothesis as above, we deduce that there exists $m_k \in D^{\beta_k}_{\mathcal{S}}(A^{\mathcal{S}}_F)$ with $m \prec_{\mathcal{S}} m_k$ for all $k \geq N$. This implies that $m \in D^{\lambda}(A^{\mathcal{S}}_F)$, as wanted.

We have now established that if for some $\alpha < \omega_1$ we have $n \in D^{\alpha}_{\mathcal{P}}(A^{\mathcal{P}}_F)$, then there is $m \in D^{\alpha}_{\mathcal{S}}(A^{\mathcal{S}}_F)$. Therefore $D^{\alpha}_{\mathcal{S}}(A^{\mathcal{S}}_F) = \emptyset$ implies $D^{\alpha}_{\mathcal{P}}(A^{\mathcal{P}}_F) = \emptyset$ for all $\alpha < \omega_1$. By symmetry we conclude that $\varphi_{\mathcal{P}} = \varphi_{\mathcal{S}}$. □

We will now isolate the two most important properties of the Polish metric space that we used in the proof above. We will prove that when these properties hold for a Polish metric space, then the induced ranks are almost independent of the dense sequence.

Proposition 2.3.4. *Let (X, d) be a Polish metric space. Suppose that for all $x, y \in X$ and $\varepsilon, \xi > 0$ we have*

(1) $\overline{\mathrm{B}}_d(x, \varepsilon) \subseteq \mathrm{B}_d(y, \xi)$ implies that there exists $\rho > 0$ such that
$$\mathrm{B}_d(x, \varepsilon + \rho) \subseteq \mathrm{B}_d(y, \xi - \rho).$$

(2) $\overline{\mathrm{B}}_d(x, \varepsilon) \cap \overline{\mathrm{B}}_d(y, \xi) = \emptyset$ implies that there exists $\rho > 0$ such that
$$\mathrm{B}_d(x, \varepsilon + \rho) \cap \mathrm{B}_d(y, \xi + \rho) = \emptyset.$$

Then for any pair of presentations $\mathcal{P} = (X, d, \overline{x})$ and $\mathcal{S} = (X, d, \overline{y})$ we have $\varphi_{\mathcal{P}}(F) \leq \varphi_{\mathcal{S}}(F) + 1$ for all $F \in F_{\aleph_0}(X)$.

Proof. Let $F \in F_{\aleph_0}(X)$. We prove by induction on $\alpha < \omega_1$ that for all $n \in D^{\alpha+1}_{\mathcal{P}}(A^{\mathcal{P}}_F)$ and all $\varepsilon > 0$ there is $m \in D^{\alpha}_{\mathcal{S}}(A^{\mathcal{S}}_F)$ such that $\overline{\mathrm{B}}_{\mathcal{S}}(m) \subseteq \mathrm{B}_d(x_{n(0)}, 2^{-n(1)-1} + \varepsilon)$ and $m(1) \geq n(1)$.

The induction start and the successor case are done exactly as in the proof of Theorem 2.3.3 and are therefore left to the leader. Assume now that the statement holds for all $\beta < \lambda$ for some limit $\lambda < \omega_1$. Let $n \in D^{\lambda+1}_{\mathcal{P}}(A^{\mathcal{P}}_F)$ and $\varepsilon > 0$ be given. Then there is $\tilde{n} \in D^{\lambda}_{\mathcal{P}}(A^{\mathcal{P}}_F)$ with $n \prec_{\mathcal{P}} \tilde{n}$, and therefore also $0 < \rho < \varepsilon$ such that

$$\mathrm{B}_d(x_{\tilde{n}(0)}, 2^{-\tilde{n}(1)-1} + \rho) \subseteq \mathrm{B}_d(x_{n(0)}, 2^{-n(1)-1} - \rho).$$

Then, since $\tilde{n} \in D^{\beta+1}_{\mathcal{P}}(A^{\mathcal{P}}_F)$ for all $\beta < \lambda$, it follows by the induction hypothesis that there is $m_\beta \in D^{\beta}_{\mathcal{S}}(A^{\mathcal{S}}_F)$ with $m_\beta(1) \geq \tilde{n}(1)$ and

$$\overline{\mathrm{B}}_{\mathcal{S}}(m_\beta) \subseteq \mathrm{B}_d(x_{\tilde{n}(0)}, 2^{-\tilde{n}(1)-1} + \rho) \subseteq \mathrm{B}_d(x_{n(0)}, 2^{-n(1)-1} - \rho)$$

for all $\beta < \omega_1$. Now choose $m \in \omega^2$ such that $d(x_{n(0)}, y_{m(0)}) < \min\{\varepsilon, \rho\}$ and $m(1) = n(1)$. Then one easily checks that $m \prec_{\mathcal{S}} m_\beta$ for all $\beta < \omega_1$. Hence we must have $m \in D^{\lambda}_{\mathcal{S}}(A^{\mathcal{S}}_F)$, as wanted.

Now we may conclude that the statement holds for all $\alpha < \omega_1$. In particular, we obtain that $\varphi_{\mathcal{P}}(F) \leq \varphi_{\mathcal{S}}(F) + 1$. $\square$

Remark 2.3.5. The conditions of the previous proposition hold for all separable Banach spaces and the Urysohn space $(\mathbb{U}, d_{\mathbb{U}})$, which we discussed in Example 1.1.6.

The example below shows that the conclusion of Proposition 2.3.4 is optimal.

Example 2.3.6. Let H be the real infinite dimensional separable Hilbert space. Denote by $(e_i)_i$ an orthonormal basis and by d the metric induced by the inner product. Consider the presentations $\mathcal{P} = (H, d, \overline{x})$ and $\mathcal{S} = (H, d, \overline{y})$, where $\overline{x}$ is an enumeration of the dense subset

$$D = \left\{ \sum_{i \leq n} \lambda_i e_i \mid n \in \omega, \lambda_i \in \mathbb{Q} \right\}$$

and $\overline{y}$ is an enumeration of $D \setminus \{0\}$. For each $i \in \omega$ let

$$F_i \subseteq \left\{ \lambda e_i \mid \lambda \in \left[\frac{1}{2} - \frac{1}{2^{i+1}}, \frac{1}{2} \right) \right\}$$

be finite with $\varphi_{\mathcal{S}}(F_i), \varphi_{\mathcal{P}}(F_i) \geq i$. Then put $F = \bigcup_{i \in \omega} F_i$. We must have $\varphi_{\mathcal{S}}(F) \leq \varphi_{\mathcal{P}}(F)$ and $D^{\omega+1}_{\mathcal{P}}(A^{\mathcal{P}}_F) = \emptyset$. Moreover, $n \in D^{\omega}_{\mathcal{P}}(A^{\mathcal{P}}_F)$ if and only if $x_{n(0)} = 0$ and $n(1) = 0$. Therefore we conclude that $\varphi_{\mathcal{S}}(F) = \omega$ and $\varphi_{\mathcal{P}}(F) = \omega + 1$.

Now we will consider how much the induced ranks can vary for general Polish metric spaces.

Theorem 2.3.7. *There exist a Polish metric space (X,d) together with presentations $\mathcal{P} = (X,d,\overline{x})$ and $\mathcal{S} = (X,d,\overline{y})$ such that*

$$\varphi_{\mathcal{P}}(F) = \omega + 1 \qquad \textit{and} \qquad \varphi_{\mathcal{S}}(F) = 2$$

for some $F \in F_{\aleph_0}(X)$.

Proof. First we construct a Polish space X_k, presentations $\mathcal{P}_k = (X_k, d_k, \overline{x}_k)$, $\mathcal{S}_k = (X_k, d_k, \overline{y}_k)$ of X_k and $F_k \in F_{\aleph_0}(X_k)$ such that $\varphi_{\mathcal{P}_k}(F) = 2 + k$ and $\varphi_{\mathcal{S}_k}(F) = 2$ for each $2 \leq k < \omega$.

Fix $2 \leq k < \omega$. Put

$$Y_k = \{w_i \mid i \in [0, 2^{-3}]\} \cup \{z_l \mid l \geq k\}$$

and define for each $m \leq k$ a complete metric δ^k_m on Y_k by

$$\delta^k_m(w_i, w_j) = 2^{-m-1}|i-j|, \qquad \delta^k_m(z_l, w_i) = 2^{-m-1}(1 - 2^{-l} - i)$$

and

$$\delta^k_m(z_l, z_n) = \begin{cases} 0 & \text{if} \quad l = n \\ 2^{-k-1} & \text{if} \quad l \neq n \end{cases}$$

for all $i, j \in [0, 2^{-3}]$ and $l, n \geq k$. Note that δ^k_m induce the same Polish topology on Y_k for all $m \leq k$.

Next for $s \in 2^{\leq k}$ let $Y^s_k = \{y^s \mid y \in Y_k\}$ denote a copy of Y_k and let $F_k = \{f_t \mid t \in 2^{k+1}\}$. Put

$$X_k = \left(\bigsqcup_{s \in 2^{\leq k}} Y^s_k \right) \sqcup F_k.$$

Define a complete metric d_k on X_k by letting

(1) $d_{k|Y^s_k} = \delta^k_{\ln(s)}$ for all $s \in 2^{\leq k}$.

(2) $d_k(w^s_0, w^{s^\frown j}_0) = 2^{-\ln(s)-2} + \frac{1}{k+1}2^{-k-3}$ for all $s \in 2^{<k}$ and $j \in 2$.

(3) $d_k(z^s_l, z^{s^\frown j}_l) = 2^{-\ln(s)-l-2} + \frac{1}{k+1}2^{-k-l-1}$ for all $s \in 2^{<k}$, $j \in 2$ and $l \geq k$.

(4) $d_k(w^s_0, f_{s^\frown j}) = 2^{-k-1} - 2^{-k-3}$ for all $s \in 2^k$ and $j \in 2$.

That d_k is indeed a metric on X_k follows from the fact that for each $s \in 2^{<k}$, $j \in 2$, $n \in \omega$ and $l \geq k$ both of the squares

$$\begin{tikzcd}
w_0^s \arrow[r, dash, "2^{-\ln(s)-1}(1-2^{-l})"] \arrow[d, dash, "2^{-\ln(s)-2}+\frac{1}{k+1}2^{-k-3}"'] & z_l^s \arrow[d, dash, "2^{-\ln(s)-l-2}+\frac{1}{k+1}2^{-k-l-1}"] \\
w_0^{s^\frown j} \arrow[r, dash, "2^{-\ln(s)-2}(1-2^{-l})"'] & z_l^{s^\frown j}
\end{tikzcd}$$

and

$$\begin{tikzcd}
z_l^s \arrow[r, dash, "2^{-k-1}"] \arrow[d, dash, "2^{-\ln(s)-l-2}+\frac{1}{k+1}2^{-k-l-1}"'] & z_{l+n}^s \arrow[d, dash, "2^{-\ln(s)-l-n-2}+\frac{1}{k+1}2^{-k-l-n-1}"] \\
z_l^{s^\frown j} \arrow[r, dash, "2^{-k-1}"'] & z_{l+n}^{s^\frown j}
\end{tikzcd}$$

satisfy the triangle inequality.

Now let

$$\overline{v}_k = \left\{ w_i^s, z_l^s \mid i \in [0, 2^{-3}] \cap \mathbb{Q},\ l \geq k,\ s \in 2^{\leq k} \right\} \cup F_k$$

and

$$\overline{u}_k = \left\{ w_i^s, z_l^s \mid i \in (0, 2^{-3}] \cap \mathbb{Q},\ l \geq k,\ s \in 2^{\leq k} \right\} \cup F_k,$$

and note that both $\overline{v}_k$ and $\overline{u}_k$ are countable dense in X_k. Put

$$\mathcal{P}_k = (X_k, d_k, \overline{v}_k) \qquad \text{and} \qquad \mathcal{S}_k = (X_k, d_k, \overline{u}_k).$$

We will first argue that $\varphi_{\mathcal{P}_k}(F_k) = 2 + k$. It is clear that $\varphi_{\mathcal{P}_k}(F_k) \leq 2 + k$, as F_k contains 2^{k+1} elements. Moreover, it is straightforward to check that

$$\mathrm{B}_{d_k}(w_0^s, 2^{\ln(s)-1}) = \overline{\mathrm{B}}_{d_k}(w_0^s, 2^{\ln(s)-1}) = \left(\bigsqcup_{t \in 2^{\leq k},\ s \subseteq t} Y_k^t \right) \sqcup \left\{ f_t \mid t \in 2^{k+1},\ s \subseteq t \right\}$$

for all $s \in 2^{\leq k}$. Therefore we have

$$\overline{\mathrm{B}}_{d_k}(w_0^{s^\frown 0}, 2^{\ln(s)-2}), \overline{\mathrm{B}}_{d_k}(w_0^{s^\frown 1}, 2^{\ln(s)-2}) \subseteq \mathrm{B}_{d_k}(w_0^s, 2^{\ln(s)-1})$$

and

$$\overline{\mathrm{B}}_{d_k}(w_0^{s^\frown 0}, 2^{\ln(s)-2}) \cap \overline{\mathrm{B}}_{d_k}(w_0^{s^\frown 1}, 2^{\ln(s)-2}) = \emptyset$$

for all $s \in 2^{<k}$. Since it also holds that

$$f_{s^\frown 0}, f_{s^\frown 1} \in \mathrm{B}_{d_k}(w_0^s, 2^{\ln(s)-1})$$

for all $s \in 2^k$, we obtain $\varphi_{\mathcal{P}_k}(F_k) = k + 2$.

We will now argue that $\varphi_{\mathcal{S}_k}(F_k) = 2$. In this case it is clear that $\varphi_{\mathcal{S}_k}(F_k) \geq 2$, as $f_{s^\frown 0}, f_{s^\frown 1} \in F_k$ with $d_k(f_{s^\frown 0}, f_{s^\frown 1}) < 2^{-1}$ for all $s \in 2^k$. Moreover, it is straightforward to check that if $x \in X_k \setminus \{w_0^s \mid s \in 2^{\leq k}\}$, $t \in 2^k$ and $m \in \omega$ such that

$$f_{t^\frown 0}, f_{t^\frown 1} \in \mathrm{B}_{d_k}(x, 2^{-m-1}),$$

then for all $s \in 2^{\leq k}$ there is $N_s \in \omega$ such that

$$z_l^s \in \mathrm{B}_{d_k}(x, 2^{-m-1})$$

for all $l \geq N_s$. Hence if $x, y \in \overline{u}_k$ and $n, m \in \omega$ satisfy that there are $s, t \in 2^k$ such that

$$f_{s^\frown 0}, f_{s^\frown 1} \in \mathrm{B}_{d_k}(x, 2^{-n-1}) \quad \text{and} \quad f_{t^\frown 0}, f_{t^\frown 1} \in \mathrm{B}_{d_k}(y, 2^{-m-1}),$$

then $\mathrm{B}_{d_k}(x, 2^{-n-1}) \cap \mathrm{B}_{d_k}(y, 2^{-m-1}) \neq \emptyset$. We can therefore conclude that $\varphi_{\mathcal{S}_k}(F_k) = 2$.

Finally, let

$$X = [0,1] \sqcup \left(\bigsqcup_{2 \leq k < \omega} X_k \right), \qquad F = \bigsqcup_{2 \leq k < \omega} F_k, \qquad \overline{x} = ([0,1] \cap \mathbb{Q}) \sqcup \left(\bigsqcup_{2 \leq k < \omega} \overline{v}_k \right)$$

and

$$\overline{y} = ((0,1] \cap \mathbb{Q}) \sqcup \left(\bigsqcup_{2 \leq k < \omega} \overline{u}_k \right).$$

Moreover, define a complete metric d on X by letting

(1) $d_{|X_k} = 2^{-1} d_k$ for all $2 \leq k < \omega$.

(2) $d(i, j) = |i - j|$ for all $i, j \in [0, 1]$.

(3) $d(0, w_0^{\emptyset,k}) = 2^{-2}$ for all $2 \leq k < \omega$. Here $w_0^{\emptyset,k}$ denotes the element $w_0^{\emptyset} \in X_k$.

Put

$$\mathcal{S} = (X, d, \overline{x}) \qquad \text{and} \qquad \mathcal{P} = (X, d, \overline{y}).$$

Since

$$\overline{\mathrm{B}}_d(w_0^{\emptyset,k}, 2^{-2}) \subseteq \mathrm{B}_d(0, 2^{-1})$$

and $\overline{\mathrm{B}}_{d_k}(w_0^{\emptyset,k}, 2^{-1}) = \overline{\mathrm{B}}_d(w_0^{\emptyset,k}, 2^{-2}) \cap X_k$ for all $2 \leq k < \omega$, we must have $\varphi_{\mathcal{P}}(F) = \omega + 1$. Moreover, if $2 \leq k < \omega$, $m \in \omega$ and $y \in X_k$ satisfy that there is $t \in 2^k$ such that

$$f_{t^\frown 0}, f_{t^\frown 1} \in B_d(y, 2^{-m-1}),$$

then

$$\mathrm{B}_d(y, 2^{-m-1}) \setminus \mathrm{B}_d(r, 2^{-1}) \neq \emptyset$$

for any $r > 0$. Therefore we can conclude that $\varphi_{\mathcal{S}}(F) = 2$. □

Our last result gives a bound on the variation one can obtain within the family of induced ranks of a Polish metric space.

Theorem 2.3.8. *Let $\mathcal{P} = (X, d, \overline{x})$ and $\mathcal{S} = (X, d, \overline{y})$ be presentations of X. Then*

$$\varphi_{\mathcal{P}}(F) \leq \omega\varphi_{\mathcal{S}}(F) + 2$$

for all $F \in F_{\aleph_0}(X)$.

Proof. We will prove by induction on $\alpha < \omega_1$ that if $n \in D_{\mathcal{P}}^{\omega\alpha+1}(A_F^{\mathcal{P}})$ with $n(1) > 0$ and $\varepsilon > 0$, then there is $m \in D_{\mathcal{S}}^{1+\alpha}(A_F^{\mathcal{S}})$ such that $d(x_{n(0)}, y_{m(0)}) < \varepsilon$ and $m(1) = n(1) - 1$. Note that this is sufficient, since if $\varphi_{\mathcal{S}}(F) = \alpha_0$ for some $\alpha_0 < \omega_1$, then $D_{\mathcal{S}}^{\alpha_0}(A_F^{\mathcal{S}}) = \emptyset$ and hence $D_{\mathcal{P}}^{\omega\alpha_0+1}(A_F^{\mathcal{P}}) \subseteq \omega \times \{0\}$. So $\varphi_{\mathcal{P}}(F) \leq \omega\alpha_0 + 2$, as desired.

For $\alpha = 0$ the statement above is easily seen to be true. Assume that the statement holds for some $\alpha < \omega_1$. Let $\varepsilon > 0$ and $n \in D_{\mathcal{P}}^{\omega(\alpha+1)+1}(A_F^{\mathcal{P}})$ with $n(1) > 0$. Then there are $u, v \in D_{\mathcal{P}}^{\omega(\alpha+1)}(A_F^{\mathcal{P}})$ with $n \prec_{\mathcal{P}} u, v$ and $u \curlywedge_{\mathcal{P}} v$. Therefore for each $k \in \omega$ there exist $n_k^u, n_k^v \in D_{\mathcal{P}}^{\omega\alpha+1}(A_F^{\mathcal{P}})$ such that $u \prec_{\mathcal{P}} n_k^u$, $v \prec_{\mathcal{P}} n_k^v$ and $n_k^u(1), n_k^v(1) > n(1) + k + 1$. Next, as d is complete and $u \curlywedge_{\mathcal{P}} v$, one of the following three cases holds:

(1) There is $k \in \omega$ such that $d(x_{n_k^u(0)}, x_{n_k^v(0)}) > 2^{-k}$.

(2) There are $k_0, k_1 \in \omega$ such that $d(x_{n_{k_0}^u(0)}, x_{n_{k_1}^u(0)}) > 2^{-\min\{k_0,k_1\}}$.

(3) There are $k_0, k_1 \in \omega$ such that $d(x_{n_{k_0}^v(0)}, x_{n_{k_1}^v(0)}) > 2^{-\min\{k_0,k_1\}}$.

If we are in the first case, it follows by the induction hypothesis that there exist $m_0, m_1 \in D_{\mathcal{S}}^{1+\alpha}(A_F^{\mathcal{S}})$ with $m_0(1), m_1(1) \geq n(1) + k + 1$, $d(x_{n_k^u(0)}, y_{m_0(0)}) < 2^{-n_k^u(1)-1}$ and $d(x_{n_k^v(0)}, y_{m_1(0)}) < 2^{-n_k^v(1)-1}$. It is easy to check that $m_0 \curlywedge_{\mathcal{S}} m_1$. Moreover, if we choose $m \in \omega^2$ with $d(x_{n(0)}, y_{m(0)}) < 2^{-n(1)-2}$ and $m(1) = n(1) - 1$, we get $m \prec_{\mathcal{S}} m_0, m_1$. Therefore $m \in D_{\mathcal{S}}^{1+\alpha+1}(A_F^{\mathcal{S}})$, as wanted. The second and third case are handled analogously.

Finally, assume that the statement holds for all $\beta < \lambda$ for some limit ordinal $\lambda < \omega_1$. Let $\varepsilon > 0$ and $n \in D_{\mathcal{P}}^{\omega\lambda+1}(A_F^{\mathcal{P}})$ be given. Then there is $\tilde{n}_\beta \in D_{\mathcal{P}}^{\omega\beta+1}(A_F^{\mathcal{P}})$ for all $\beta < \lambda$ such that $n \prec_{\mathcal{P}} \tilde{n}_\beta$ and $\tilde{n}_\beta(1) > n(1) + 1$. Thus for each $\beta < \lambda$ there is $m_\beta \in D_{\mathcal{S}}^{1+\beta}(A_F^{\mathcal{S}})$ with

$$d(x_{\tilde{n}_\beta(0)}, y_{m_\beta(0)}) < 2^{-\tilde{n}_\beta(1)-1}$$

and $m_\beta(1) = \tilde{n}_\beta(1)-1$. Now choose any $m \in \omega^2$ with $d(x_{n(0)}, y_{m(0)}) < 2^{-n(1)-2}$ and $m(1) = n(1) - 1$. Then $m \prec_{\mathcal{S}} m_\beta$ for all $\beta < \lambda$ and hence $m \in D^{1+\lambda}_{\mathcal{S}}(A^{\mathcal{S}}_F)$, as wanted. $\square$

2.3.2 Change of metric

Now we will investigate what happens when we change the complete metric while keeping the dense sequence fixed. It is clear, as we will also see, that the rank depends heavily on the chosen metric. Let us begin with two easy observations.

First, if $\mathcal{P} = (X, d, \overline{x})$ is a presentation of a Polish space X, then the metric d_b given by

$$d_b(x, y) = \min\{d(x, y), 1\}$$

for all $x, y \in X$ satisfies $\operatorname{diam}_{d_b}(X) < \infty$. Moreover, the presentation $\mathcal{S} = (X, d_b, \overline{x})$ will satisfy $\varphi_{\mathcal{P}} = \varphi_{\mathcal{S}}$. Thus all the ranks we obtain using our construction are induced by presentations with bounded metrics.

Secondly, if for some $k \in \omega$ we have presentations $\mathcal{P} = (X, d, \overline{x})$ and $\mathcal{S} = (X, 2^{-k}d, \overline{x})$ of a Polish space X, then

$$\varphi_{\mathcal{P}}(F) \leq \varphi_{\mathcal{S}}(F) \leq \varphi_{\mathcal{P}}(F) + k$$

for all $F \in F_{\aleph_0}(X)$.

The first result of this section generalizes the second observation. It provides a bound on the change that may occur when passing to an equivalent metric. Here we say that two compatible metrics d, δ on a Polish space X are *equivalent* if there exists $N > 0$ such that

$$\frac{1}{N} d(x, y) \leq \delta(x, y) \leq N d(x, y)$$

for all $x, y \in X$.

Proposition 2.3.9. *Let $\mathcal{P} = (X, d, \overline{x})$ and $\mathcal{S} = (X, \delta, \overline{x})$ be presentations of a Polish space X such that d and δ are equivalent. Then*

$$\varphi_{\mathcal{P}}(F) \leq \omega(\varphi_{\mathcal{S}}(F) + 1)$$

for all $F \in F_{\aleph_0}(X)$.

Proof. First, since d and δ are equivalent, we may fix $l \in \omega$ such that

$$2^{-l} d(x, y) \leq \delta(x, y) \leq 2^l d(x, y)$$

for all $x, y \in X$.

By a similar argument as the one in the proof of Theorem 2.3.8, one can prove the following: For all $\alpha < \omega_1$ it holds that if $n \in D_{\mathcal{P}}^{\omega\alpha+1}(A_F^{\mathcal{P}})$ with $n(1) > l$, then $\tilde{n} = (n(0), n(1) - l - 1) \in D_{\mathcal{S}}^{1+\alpha}(A_F^{\mathcal{S}})$.

This completes the proof, since if $\varphi_{\mathcal{S}}(F) = \alpha_0$ for some $\alpha_0 < \omega_1$, then $D_{\mathcal{S}}^{\alpha_0}(A_F^{\mathcal{S}}) = \emptyset$ and hence $D_{\mathcal{P}}^{\omega\alpha_0+1}(A_F^{\mathcal{P}}) \subseteq \omega \times \{0, \ldots, l\}$. Therefore we obtain $D_{\mathcal{P}}^{\omega(\alpha_0+1)}(A_F^{\mathcal{P}}) = \emptyset$, as wanted. $\square$

Our goal for the rest of this section is to prove that there is no bound on the change in general. The idea is to begin with the standard presentation $\mathcal{P}_{\omega^\omega}$ of ω^ω and a certain closed subset $F \in F_{\aleph_0}(\omega^\omega)$ such that $\mathcal{P}_{\omega^\omega}(F) = 2$. Then for each $\alpha < \omega_1$ we will construct a new presentation $\mathcal{P}_\alpha$ of ω^ω such that $\varphi_{\mathcal{P}_\alpha}(F) = \alpha$. To obtain these $\mathcal{P}_\alpha$ for $\alpha < \omega_1$, we will make use of the construction in the following definition.

Definition 2.3.10. Let $\mathcal{P} = (X, d, \overline{x})$ be a presentation of a Polish space X and let $f\colon X \to X$ be a homeomorphism. The *induced presentation* of X is the presentation

$$\mathcal{P}_f = (X, d_f, \overline{y}),$$

where $d_f(x, y) = d(f(x), f(y))$ for all $x, y \in X$ and $y_i = f^{-1}(x_i)$ for all $i \in \omega$.

It is clear that the ranks $\varphi_{\mathcal{P}}$ and $\varphi_{\mathcal{P}_f}$ must be closely related. The precise connection is described in the proposition below.

Proposition 2.3.11. *Let $\mathcal{P}$ be a presentation of a Polish space X and $f\colon X \to X$ a homeomorphism. Then $\varphi_{\mathcal{P}_f}(F) = \varphi_{\mathcal{P}}(f(F))$ for all $F \in F_{\aleph_0}(X)$.*

Proof. This follows directly from Lemma 2.3.1 and the fact that

$$x \in \mathrm{B}_{\mathcal{P}_f}(n) \iff f(x) \in \mathrm{B}_{\mathcal{P}}(n)$$

for all $x \in X$ and $n \in \omega^2$. $\square$

We have the following natural way to obtain homeomorphisms of ω^ω. Consider two subsets $A = \{s_i \in \omega^{<\omega} \mid i \in \omega\}$ and $B = \{t_i \in \omega^{<\omega} \mid i \in \omega\}$. If A and B satisfy that $\ln(s_i) = \ln(t_i)$, $s_i \perp s_j$, $t_i \perp t_j$ and $s_i \perp t_j$ for all $i, j \in \omega$ with $i \neq j$, we say that they are *compatible sets of initial segments*. In this case, we define the map $f_{A,B}\colon \omega^\omega \to \omega^\omega$ given by

$$f_{A,B}(x) = \begin{cases} t_i^\frown z & \text{if} \quad (\exists i \in \omega)(\exists z \in \omega^\omega)\ x = s_i^\frown z \\ s_i^\frown z & \text{if} \quad (\exists i \in \omega)(\exists z \in \omega^\omega)\ x = t_i^\frown z \\ x & \text{otherwise} \end{cases}.$$

It is straightforward to check that $f_{A,B}$ is a homeomorphism. We call $f_{A,B}$ the *induced switch map*.

Theorem 2.3.12. *There is $F \in F_{\aleph_0}(\omega^\omega)$ and for each $2 \leq \alpha < \omega_1$ a presentation $\mathcal{P}_\alpha$ of ω^ω such that*

$$\varphi_{\mathcal{P}_{\omega^\omega}}(F) = 2 \qquad \textit{and} \qquad \varphi_{\mathcal{P}_\alpha}(F) = \alpha$$

for all $2 \leq \alpha < \omega_1$.

Proof. First let $F = \{n^\omega \mid n \in \omega\}$ and note that $\varphi_{\mathcal{P}_{\omega^\omega}}(F) = 2$. We will now recursively construct compatible sets of initial segments A_α and B_α such that the induced switch map f_α satisfies $\varphi_{\mathcal{P}_{\omega^\omega}}(f_\alpha(F)) = \alpha$ for all $2 \leq \alpha < \omega_1$.

For $\alpha = 2$, we let $A_2 = \{s_i \in \omega^{<\omega} \mid i \in \omega\}$ and $B_2 = \{t_i \in \omega^{<\omega} \mid i \in \omega\}$, where $s_i = t_i = (i)$ for all $i \in \omega$. Clearly, A_2 and B_2 are compatible sets of initial segments.

Now assume that we have built the compatible sets of initial segments $A_\alpha = \{s_i \in \omega^{<\omega} \mid i \in \omega\}$ and $B_\alpha = \{t_i \in \omega^{<\omega} \mid i \in \omega\}$ for some $\alpha < \omega_1$. Then put

$$A_{\alpha+1} = \{\tilde{s}_i \in \omega^{<\omega} \mid i \in \omega\},$$

where $\tilde{s}_{2i}$ and $\tilde{s}_{2i+1}$ are obtained from s_i by replacing any occurrence of n in $s_i(0)^\frown s_i$ with $2n$ and $2n+1$, respectively, for all $n \in \omega$. Also put

$$B_{\alpha+1} = \{\tilde{t}_i \in \omega^{<\omega} \mid i \in \omega\},$$

where $\tilde{t}_{2i} = (0)^\frown \tilde{t}^0_{2i}$ and $\tilde{t}_{2i+1} = (1)^\frown \tilde{t}^0_{2i+1}$, and $\tilde{t}^0_{2i}$ and $\tilde{t}^0_{2i+1}$ are obtained from t_i by replacing any occurrence of n in t_i with $2n$ and $2n+1$, respectively, for all $n \in \omega$. It is straightforward to check that if A_α and B_α are compatible sets of initial segments, then $A_{\alpha+1}$ and $B_{\alpha+1}$ are compatible as well.

Next let $\lambda < \omega_1$ be a limit ordinal and assume that we have constructed compatible sets of initial segments

$$A_\beta = \left\{s^\beta_k \in \omega^{<\omega} \mid k \in \omega\right\} \quad \text{and} \quad B_\beta = \left\{t^\beta_k \in \omega^{<\omega} \mid k \in \omega\right\}$$

for all $\beta < \lambda$. Then let $(\beta_i)_i < \lambda$ be an increasing sequence such that $\bigcup_{i\in\omega} \beta_i = \lambda$ and fix an enumeration $(p_i)_i$ of the prime numbers. Then put

$$A_\lambda = \{\tilde{s}_{i,k} \in \omega^{<\omega} \mid i, k \in \omega\},$$

where $\tilde{s}_{i,k}$ is obtained from $s^{\beta_i}_k$ by replacing any occurrence of n in $s^{\beta_i}_k(0)^\frown s^{\beta_i}_k$ with p_i^{n+1} for all $n \in \omega$. Also put

$$B_\lambda = \{\tilde{t}_{i,k} \in \omega^{<\omega} \mid i, k \in \omega\},$$

where $\tilde{t}_{i,k} = (p_i)^\frown \tilde{t}^0_{i,k}$ and $\tilde{t}^0_{i,k}$ is obtained from $t^{\beta_i}_k$ by replacing any occurrence of n in $t^{\beta_i}_k$ by p_i^{n+1} for all $n \in \omega$. Again, it is easy to check that A_λ and B_λ are compatible sets of initial segments.

Finally, the construction of A_α and B_α for $2 \leq \alpha < \omega_1$ ensures that a straightforward induction argument shows that $\varphi_{\mathcal{P}_{\omega^\omega}}(f_\alpha(F)) = \alpha$. So if we let $\mathcal{P}_\alpha = \mathcal{P}_{(\mathcal{P}_{\omega^\omega})_{f_\alpha}}$ for each $2 \leq \alpha < \omega_1$, then $\varphi_{\mathcal{P}_\alpha}(F) = \alpha$ for all $2 \leq \alpha < \omega_1$, as wanted. □

Note that if d is an ultra-metric on a Polish space X and $f\colon X \to X$ is a homeomorphism, then d_f is also an ultra-metric. Hence it follows from Proposition 2.3.2 that Theorem 2.3.12 also holds if we moreover want the presentations $\mathcal{P}_{\omega^\omega}$ and $\mathcal{P}_\alpha$ to have the same dense sequence for all $2 \leq \alpha < \omega_1$.

Before we end this section, we will point out two direct consequences of Theorem 2.3.12 that will turn out to be useful in the next chapter.

Corollary 2.3.13. *For each $\alpha < \omega_1$ there is a discrete $F \in F_{\aleph_0}(\omega^\omega)$ with $\varphi_{\mathcal{P}_{\omega^\omega}}(F) = \alpha$.*

Proof. Let F and f_α be as in the proof of Theorem 2.3.12 for each $2 \leq \alpha < \omega_1$. Then $f_\alpha(F) \in F_{\aleph_0}(\omega^\omega)$ is discrete and satisfies $\varphi_{\mathcal{P}}(f_\alpha(F)) = \alpha$. Moreover, $\varphi_{\mathcal{P}}(\emptyset) = 0$ and $\varphi_{\mathcal{P}}(\{x\}) = 1$ for all $x \in \omega^\omega$. □

Corollary 2.3.14. *For each $1 \leq \alpha < \omega_1$ there is a presentation $\mathcal{P}_\alpha = (\omega, d_\alpha, \omega)$ of the discrete Polish space ω such that $\varphi_{\mathcal{P}_\alpha}(\omega) = \alpha$.*

Proof. For each $2 \leq \alpha < \omega_1$ let F and f_α be as in the proof of Theorem 2.3.12 and fix a homeomorphism $g_\alpha\colon \omega \to f_\alpha(F)$. Let d_α denote the metric on ω given by $d_\alpha(i,j) = \rho(g_\alpha(i), g_\alpha(j))$ for all $i,j \in \omega$ and put $\mathcal{P}_\alpha = (\omega, d_\alpha, \omega)$ for all $2 \leq \alpha < \omega_1$. Then we must have $\varphi_{\mathcal{P}_\alpha}(\omega) = \alpha$. It is clear that the presentation $\mathcal{P}_1 = (\omega, d_1, \omega)$, where $d_1(i,j) = 1$ for all $i,j \in \omega$ with $i \neq j$, satisfies $\varphi_{\mathcal{P}}(\omega) = 1$. □

Chapter 3

The relation to the Cantor-Bendixson rank

In this chapter we will compare the Cantor-Bendixson rank to the ranks that we constructed in Chapter 2. The main results of this chapter are characterizations of the compact Polish spaces and the σ-compact Polish spaces in terms of how the family of ranks

$$\{\varphi_{\mathcal{P}} \mid \mathcal{P} \text{ is a presentation of } X\}$$

behaves in relation to the Cantor-Bendixson rank for a given Polish space X.

In the first section we will prove that a Polish space is compact if and only if the family of ranks is uniformly bounded by the Cantor-Bendixson rank. Moreover, if a Polish space X is compact, we will compute a specific function $f\colon \omega_1 \to \omega_1$ such that

$$\varphi_{\mathcal{P}}(F) \leq f(|F|_{\mathrm{CB}})$$

for all $F \in F_{\aleph_0}(X)$ and all presentations $\mathcal{P}$ of X. In fact, we obtain one function that works for all compact Polish spaces. In the second section we will prove that a Polish space is σ-compact if and only if some (equivalently every) rank in the family is bounded by the Cantor-Bendixson rank. Also in this case we will for a presentation $\mathcal{P}$ of a σ-compact Polish space X compute a specific function $f_{\mathcal{P}}\colon \omega_1 \to \omega_1$ such that

$$\varphi_{\mathcal{P}}(F) \leq f_{\mathcal{P}}(|F|_{\mathrm{CB}})$$

for all $F \in F_{\aleph_0}(X)$. Here the functions depend on the chosen presentation and hence also on the σ-compact Polish space in question.

Unless otherwise specifically stated, all results in this chapter have been obtained by the author in [27].

First we will argue that the ranks constructed in Chapter 2 refine the Cantor-Bendixson rank. Recall from Example 1.5.3 that for any Polish space X and $F \in F(X)$ we let $|F|_{\mathrm{CB}}$ denote the Cantor-Bendixson rank of F, and for all $\alpha < \omega_1$ we let F^α denote the iterated Cantor-Bendixson derivative of F.

Proposition 3.0.1. *Let $\mathcal{P}$ be a presentation of a Polish space X and let $F \in F(X)$. Then $A^{\mathcal{P}}_{F^\alpha} \subseteq D^\alpha_{\mathcal{P}}(A^{\mathcal{P}}_F)$ for all $\alpha < \omega_1$. In particular, $F^\alpha \subseteq F^{\mathcal{P}}_{D^\alpha_{\mathcal{P}}(A^{\mathcal{P}}_F)}$ for all $\alpha < \omega_1$.*

Proof. The result is proven by induction on $\alpha < \omega_1$. The induction start is trivial. Assume therefore that $A^{\mathcal{P}}_{F^\alpha} \subseteq D^\alpha_{\mathcal{P}}(A^{\mathcal{P}}_F)$ for some $\alpha < \omega_1$ and let $n \in A^{\mathcal{P}}_{F^{\alpha+1}}$. Then $\mathrm{B}_{\mathcal{P}}(n) \cap F^{\alpha+1} \neq \emptyset$ and hence there must be $x, y \in \mathrm{B}_{\mathcal{P}}(n) \cap F^\alpha$ with $x \neq y$. This implies that we can find $k, l \in A^{\mathcal{P}}_{F^\alpha} \subseteq D^\alpha_{\mathcal{P}}(A^{\mathcal{P}}_F)$ such that $n \prec_{\mathcal{P}} k, l$ and $k \curlywedge_{\mathcal{P}} l$. Therefore we conclude that $n \in D^{\alpha+1}_{\mathcal{P}}(A^{\mathcal{P}}_F)$.

Next assume $A^{\mathcal{P}}_{F^\beta} \subseteq D^\beta_{\mathcal{P}}(A^{\mathcal{P}}_F)$ for all $\beta < \lambda$ for some limit ordinal $\lambda < \omega_1$ and let $n \in A^{\mathcal{P}}_{F^\lambda}$. Then $A^{\mathcal{P}}_{F^\lambda} \subseteq A^{\mathcal{P}}_{F^\beta} \subseteq D^\beta_{\mathcal{P}}(A^{\mathcal{P}}_F)$ for all $\beta < \lambda$ and hence we obtain that $n \in D^\lambda_{\mathcal{P}}(A^{\mathcal{P}}_F)$, as desired. □

From this proposition we easily get the following corollary.

Corollary 3.0.2. *Let $\mathcal{P}$ be any presentation of a Polish space X. For all $F \in F_{\aleph_0}(X)$ we have $|F|_{CB} \leq \varphi_{\mathcal{P}}(F)$.*

3.1 A characterization of compact spaces

In this section we will characterize the compact Polish spaces in terms of how the ranks constructed in Chapter 2 relate to the Cantor-Bendixson rank. More precisely, we will prove the following theorem.

Theorem 3.1.1. *Let X be a Polish space. The following are equivalent:*

(1) X is compact.

(2) The family $(\varphi_{\mathcal{P}})_{\mathcal{P}}$, where $\mathcal{P}$ varies over all presentations of X, is uniformly bounded by the Cantor-Bendixson rank.

(3) For any presentation $\mathcal{P}$ of X we have $\varphi_{\mathcal{P}}(F) < \omega|F|_{CB}$ for all $F \in F_{\aleph_0}(X)$.

The theorem above implies that for a compact Polish space there is a uniformity of the family of induced ranks. We will discuss this phenomenon further in Section 4.2.

It is clear that (3) $\implies$ (2) in Theorem 3.1.1. Below we will show the implications (2) $\implies$ (1) and (1) $\implies$ (3).

We will begin with the latter. The goal is to prove that if X is compact and $\mathcal{P}$ is a presentation of X, then $\varphi_{\mathcal{P}}(F) < \omega|F|_{\mathrm{CB}}$ for all $F \in F_{\aleph_0}(X)$. In order to obtain strict inequality we will prove that $\varphi_{\mathcal{P}}(F)$ is a successor for any $F \in F_{\aleph_0}(X)$. Note that this is also the case for the Cantor-Bendixson rank.

Proposition 3.1.2. *Let $\mathcal{P}$ be a presentation of a compact Polish space X. If $F \in F_{\aleph_0}(X)$ is non-empty and $\alpha < \omega_1$ is least such that $[D^{\alpha}_{\mathcal{P}}(A^{\mathcal{P}}_F)]_{\mathcal{P}} = \emptyset$, then α is a successor.*

Proof. First, since F is non-empty, we have $\alpha > 0$. Now assume for a contradiction that $[D^{\beta}_{\mathcal{P}}(A^{\mathcal{P}}_F)]_{\mathcal{P}} \neq \emptyset$ for all $\beta < \lambda$ and $[D^{\lambda}_{\mathcal{P}}(A^{\mathcal{P}}_F)]_{\mathcal{P}} = \emptyset$ for some limit ordinal $\lambda < \omega_1$. Fix $(\beta_i)_i < \lambda$ such that $\beta_i \leq \beta_{i+1}$ for all $i \in \omega$ and $\bigcup_{i\in\omega} \beta_i = \lambda$. For each $i \in \omega$ choose $x_i \in \pi_{\mathcal{P}}([D^{\beta_i}_{\mathcal{P}}(A^{\mathcal{P}}_F)]_{\mathcal{P}})$. By compactness of X, there is $x \in X$ and a subsequence $(x_{i_k})_k \subseteq (x_i)_i$ such that $x_{i_k} \to x$ as $k \to \infty$. So, since $\pi_p([D^{\beta_i}_{\mathcal{P}}(A^{\mathcal{P}}_F)]_{\mathcal{P}})$ is closed and

$$\pi_{\mathcal{P}}([D^{\beta_{i+1}}_{\mathcal{P}}(A^{\mathcal{P}}_F)]_{\mathcal{P}}) \subseteq \pi_{\mathcal{P}}([D^{\beta_i}_{\mathcal{P}}(A^{\mathcal{P}}_F)]_{\mathcal{P}})$$

for all $i \in \omega$, we get that $x \in \pi_{\mathcal{P}}([D^{\beta_i}_{\mathcal{P}}(A^{\mathcal{P}}_F)]_{\mathcal{P}})$ for all $i \in \omega$. Thus, by Proposition 2.2.1, we obtain $x \in \pi_{\mathcal{P}}([D^{\lambda}_{\mathcal{P}}(A^{\mathcal{P}}_F)]_{\mathcal{P}})$, which contradicts that $[D^{\lambda}_{\mathcal{P}}(A^{\mathcal{P}}_F)]_{\mathcal{P}} = \emptyset$. □

Proposition 3.1.3. *Let $\mathcal{P}$ be a presentation of a compact Polish space X and assume that $[D^{\beta}_{\mathcal{P}}(A^{\mathcal{P}}_F)]_{\mathcal{P}} = \emptyset$ for some $\beta < \omega_1$. Then there is $k \in \omega$ such that $D^{\beta+k}_{\mathcal{P}}(A^{\mathcal{P}}_F) = \emptyset$.*

Proof. Suppose for a contradiction that this is not the case. Then for each $k \in \omega$ there is $n_k \in D^{\beta}_{\mathcal{P}}(A^{\mathcal{P}}_F)$ with $n_k(1) \geq k$. Thus, by Lemma 2.2.3, we obtain $[D^{\beta}_{\mathcal{P}}(A^{\mathcal{P}}_F)]_{\mathcal{P}} \neq \emptyset$, which is a contradiction. □

Combining Proposition 3.1.2 and Proposition 3.1.3 we get that $\varphi_{\mathcal{P}}(F)$ is a successor for any $F \in F_{\aleph_0}(X)$ whenever $\mathcal{P}$ is a presentation of a compact Polish space X.

Corollary 3.1.4. *Let $\mathcal{P}$ be a presentation of a compact Polish space X. For all $F \in F_{\aleph_0}(X)$ there is $\beta < \omega_1$ such that $\varphi_{\mathcal{P}}(F) = \beta + 1$.*

The following lemma will be crucial in the proof of (1) $\implies$ (3) in Theorem 3.1.1.

Lemma 3.1.5. *Let $\mathcal{P}$ a presentation of a Polish space X. If we have $F \in F_{\aleph_0}(X)$, $\alpha < \omega_1$ and $n \in D_{\mathcal{P}}^{\omega\alpha}(A_F^{\mathcal{P}})$ such that $\overline{B_{\mathcal{P}}(n) \cap F}$ is compact, then there is $x \in \overline{B_{\mathcal{P}}(n) \cap F}$ such that $x \in F^{\alpha}$.*

Proof. The statement is trivial for $\alpha = 0$. Assume that the statement is true for some $\alpha < \omega_1$ and that $n \in D_{\mathcal{P}}^{\omega(\alpha+1)}(A_F^{\mathcal{P}})$. For each $k \in \omega$ there must be $m_k \in D_{\mathcal{P}}^{\omega\alpha+k}(A_F^{\mathcal{P}})$ such that $n \prec_{\mathcal{P}} m_k$. We will now recursively construct a sequence $(x_k)_k \in F \cap \mathrm{B}_{\mathcal{P}}(n)$ such that $x_k \in F^{\alpha}$ for all $k \in \omega$ and $x_i \neq x_j$ whenever $i \neq j$. First, since $m_0 \in D^{\omega\alpha}(A_F)$, it follows by the induction hypothesis that there is

$$x_0 \in \overline{\mathrm{B}_{\mathcal{P}}(m_0) \cap F} \subseteq \mathrm{B}_{\mathcal{P}}(n) \cap F$$

with $x_0 \in F^{\alpha}$. Now assume we have constructed $x_0, \dots, x_{k-1}$ for some $k > 0$ satisfying the above. Then, as $m_k \in D^{\omega\alpha+k}(A_F)$, there is $(l_s)_{s \in 2^{\leq k}} \in D_{\mathcal{P}}^{\omega\alpha}(A_F^{\mathcal{P}})$ such that $l_{\emptyset} = m_k$, $l_s \prec_{\mathcal{P}} l_{s^\frown 0}, l_{s^\frown 1}$ and $l_{s^\frown 0} \not\curlywedge_{\mathcal{P}} l_{s^\frown 1}$ for all $s \in 2^{<k}$. Next, since $k < 2^k$, there must be $s \in 2^k$ such that $x_i \notin \overline{\mathrm{B}}_{\mathcal{P}}(l_s)$ for all $i < k$. Moreover, as $l_s \in D_{\mathcal{P}}^{\omega\alpha}(A_F^{\mathcal{P}})$, it follows by the induction hypothesis that there is

$$x_k \in \overline{\mathrm{B}_{\mathcal{P}}(l_s) \cap F} \subseteq \mathrm{B}_{\mathcal{P}}(n) \cap F$$

with $x_k \in F^{\alpha}$. By the choice of l_s, we ensure that $x_k \neq x_i$ for all $i < k$. Continuing this way we obtain a sequence $(x_k)_k \in \mathrm{B}_{\mathcal{P}}(n) \cap F$ satisfying the above. By compactness of X, it follows that there is $x \in \overline{\mathrm{B}_{\mathcal{P}}(n) \cap F}$ with $x \in F^{\alpha+1}$.

To finish the proof, let $\lambda < \omega_1$ be a limit ordinal and assume that the statement is true for all $\beta < \lambda$. Moreover, let $n \in D_{\mathcal{P}}^{\omega\lambda}(A_F^{P})$ and fix $(\beta_i)_i < \lambda$ with $\beta_i \leq \beta_{i+1}$ for all $i \in \omega$ and $\bigcup_{i \in \omega} \beta_i = \lambda$. Then, by the induction hypothesis, we may for each $i \in \omega$ choose $x_i \in \overline{\mathrm{B}_{\mathcal{P}}(n) \cap F}$ with $x_i \in F^{\beta_i}$. Since X is compact, there are $x \in \overline{\mathrm{B}_{\mathcal{P}}(n) \cap F}$ and a subsequence $(x_{i_k})_k \subseteq (x_i)_i$ such that $x_{k_i} \to x$ as $k \to \infty$. Thus $x \in F^{\beta_{i_k}}$ for all $k \in \omega$ and hence $x \in F^{\lambda}$. $\square$

We will next, by use of Corollary 3.1.4 and Lemma 3.1.5, conclude the proof of (1) $\implies$ (3) in Theorem 3.1.1.

Theorem 3.1.6. *Let $\mathcal{P}$ be a presentation of a compact Polish space X. Then $\varphi_{\mathcal{P}}(F) < \omega|F|_{CB}$ for all $F \in F_{\aleph_0}(X)$.*

Proof. Let $F \in F_{\aleph_0}(X)$. If $|F|_{\mathrm{CB}} = \alpha$, then $F^{\alpha} = \emptyset$ and hence, by Lemma 3.1.5, we must have $D_{\mathcal{P}}^{\omega\alpha}(A_F^{\mathcal{P}}) = \emptyset$. Therefore, since $\varphi_{\mathcal{P}}(F)$ is a successor by Corollary 3.1.4, we obtain $\varphi_{\mathcal{P}}(F) < \omega\alpha$. $\square$

Now we turn to the proof of (2) $\implies$ (1) in Theorem 3.1.1. Our strategy is to fix a non-compact Polish space X and a discrete closed infinite subset $F \in F_{\aleph_0}(X)$. Then given $\alpha < \omega_1$ we will construct a presentation $\mathcal{P}_\alpha$ of X such that $\varphi_{\mathcal{P}_\alpha}(F) \geq \alpha$. To construct this presentation we will need the following extension theorem for complete metrics due to Hausdorff and Bacon (see [5, Theorem 3.2]).

Theorem 3.1.7 (Hausdorff-Bacon)**.** *Let X be a completely metrizable space, $K \subseteq X$ a closed subset and d_k a complete metric on K. Then there exists a complete compatible metric d on X such that $d_{|K} = d_K$.*

Theorem 3.1.8. *Let X be a non-compact Polish space and $F \in F_{\aleph_0}(X)$ an infinite discrete subset. For each $\alpha < \omega_1$ there is a presentation $\mathcal{P}_\alpha$ of X such that $\varphi_{\mathcal{P}_\alpha}(F) \geq \alpha$.*

Proof. Let $\alpha < \omega_1$ be given and fix an enumeration $F = \{y_i \mid i \in \omega\}$. Moreover, by Corollary 2.3.14, we can fix an ultra-metric d_F on F that induces the discrete topology and such that the presentation $\mathcal{S} = (F, d_F, F)$ satisfies $\varphi_{\mathcal{S}}(F) > 3\alpha$. Using the presentation $\mathcal{S}$ of F, we will construct a presentation $\mathcal{P}$ of X that satisfies $\varphi_{\mathcal{P}}(F) \geq \alpha$. Let $\overline{x}$ be a countable dense sequence such that $x_{2k} = y_k$ for all $k \in \omega$. By applying Theorem 3.1.7, let d be a complete metric on X that extends d_F. Then put $\mathcal{P} = (X, d, \overline{x})$. It now suffices to prove that $\varphi_{\mathcal{S}}(F) \leq 3\varphi_{\mathcal{P}}(F) + 1$.

For all $l, k, i, j \in \omega$ the triangle inequality implies that

$$\mathrm{B}_{\mathcal{S}}(l, k+1) \prec_{\mathcal{S}} \mathrm{B}_{\mathcal{S}}(i, j) \implies \mathrm{B}_{\mathcal{P}}(2l, k) \prec_{\mathcal{P}} \mathrm{B}_{\mathcal{P}}(2i, j)$$

and

$$\overline{\mathrm{B}}_{\mathcal{S}}(l, k) \curlywedge_{\mathcal{S}} \overline{\mathrm{B}}_{\mathcal{S}}(i, j) \implies \overline{\mathrm{B}}_{\mathcal{P}}(2l, k+1) \curlywedge_{\mathcal{P}} \overline{\mathrm{B}}_{\mathcal{P}}(2i, j+1).$$

In the following we will for any $n \in \omega^2$ with $n(1) > 0$ let $\tilde{n} = (2n(0), n(1) - 1)$. We will prove by induction on $\beta < \omega_1$ that if $n \in D^{3\beta}_{\mathcal{S}}(A^{\mathcal{S}}_F)$ and $n(1) > 0$, then $\tilde{n} \in D^{\beta}_{\mathcal{P}}(A^{\mathcal{P}}_F)$. This is immediate for $\beta = 0$ and, by the induction hypothesis, when β is a limit ordinal. We will therefore concentrate on the successor case. Assume that the statement holds for some $\beta < \omega_1$ and that $n \in D^{3\beta+3}_{\mathcal{S}}(A^{\mathcal{S}}_F)$ with $n(1) > 0$. Then there exists $(n_s)_{s \in 2^{\leq 3}} \in D^{3\beta}_{\mathcal{S}}(A^{\mathcal{S}}_F)$ satisfying $n = n_\emptyset$, $n_s \prec_{\mathcal{S}} n_{s^\frown 0}, n_{s^\frown 1}$ and $n_{s^\frown 0} \curlywedge_{\mathcal{S}} n_{s^\frown 1}$ for all $s \in 2^{<3}$. Since d_F is an ultra-metric, it follows from the above implications that there are $s, t \in 2^3$ such that $\tilde{n}_s \curlywedge_{\mathcal{P}} \tilde{n}_t$ and $\tilde{n} \prec_{\mathcal{P}} \tilde{n}_s, \tilde{n}_t$. Therefore, by the induction hypothesis, we must have $\tilde{n} \in D^{\beta+1}_{\mathcal{P}}(A^{\mathcal{P}}_F)$, as wanted. $\square$

Putting together Theorem 3.1.6 and Theorem 3.1.8, we have finalized the proof of Theorem 3.1.1, which was the goal of this section.

3.2 A characterization of σ-compact spaces

In this section we characterize the σ-compact Polish spaces in terms of how the family of ranks constructed in Chapter 2 relates to the Cantor-Bendixson rank. More precisely, we will prove the following theorem, where $|\cdot|_K$ denotes the rank from Example 1.5.1.

Theorem 3.2.1. *Let X be a Polish space. The following are equivalent:*

(1) X is σ-compact.

(2) For some presentation $\mathcal{P}$ of X there exists $f\colon \omega_1 \to \omega_1$ such that $\varphi_{\mathcal{P}}(F) \leq f(|F|_{CB})$ for all $F \in F_{\aleph_0}(X)$.

(3) For each presentation $\mathcal{P}$ of X there exist ordinals $\alpha_{\mathcal{P}}, \beta_{\mathcal{P}} < \omega_1$ such that

$$\varphi_{\mathcal{P}}(F) \leq (\omega|F|_{CB} + \alpha_{\mathcal{P}})|X|_K + \beta_{\mathcal{P}}$$

for all $F \in F_{\aleph_0}(X)$.

It is clear that (3) $\implies$ (2), hence it suffices to prove (2) $\implies$ (1) and (1) $\implies$ (3). First we will see that (2) $\implies$ (1) can be obtained as a consequence of Theorem 1.4.3 and Corollary 2.3.13.

Proposition 3.2.2. *Let $\mathcal{P}$ be a presentation of some Polish space X. If there is $f\colon \omega_1 \to \omega_1$ such that $\varphi_{\mathcal{P}}(F) \leq f(|F|_{CB})$ for all $F \in F_{\aleph_0}(X)$, then X is σ-compact.*

Proof. Assume that there is $f\colon \omega_1 \to \omega_1$ such that $\varphi_{\mathcal{P}}(F) \leq f(|F|_{CB})$ for all $F \in F_{\aleph_0}(X)$. Then there is $\alpha_0 \in \omega_1$ such that $\varphi_{\mathcal{P}}(F) \leq \alpha_0$ whenever $F \in F_{\aleph_0}(X)$ is discrete. Now assume towards a contradiction that X is not σ-compact. Then, by Remark 1.5.6, we obtain a co-analytic rank $\psi\colon F_{\aleph_0}(\omega^\omega) \to \omega_1$ for which $\psi(F) \leq \alpha_0$ for all discrete $F \in F_{\aleph_0}(\omega^\omega)$. Therefore, by Theorem 1.4.3, all co-analytic ranks on $F_{\aleph_0}(\omega^\omega)$ are bounded on the discrete subsets. This contradicts Corollary 2.3.13. □

Next we will prove the implication (1) $\implies$ (3) of Theorem 3.2.1. The proof uses that for each σ-compact Polish space X we obtain the iterated derivatives X_α for $\alpha < |X|_K$. Moreover, if we let $O_\alpha = X_\alpha \setminus X_{\alpha+1}$, then O_α is open and locally compact in X_α for each $\alpha < |X|_K$ and

$$X = \bigsqcup_{\alpha < |X|_K} O_\alpha.$$

The idea is then to deal with each of these pieces one at a time, as we know that any point $x \in O_\alpha$ will have a pre-compact neighbourhood in X_α and, by Lemma 3.1.5, we know how to deal with balls contained in such neighbourhoods.

Theorem 3.2.3. *Let $\mathcal{P}$ be a presentation of a σ-compact Polish space X. Then there are $\alpha_\mathcal{P}, \beta_\mathcal{P} < \omega_1$ such that*

$$\varphi_\mathcal{P}(F) \leq (\omega|F|_{CB} + \alpha_\mathcal{P})|X|_K + \beta_\mathcal{P}$$

for all $F \in F_{\aleph_0}(X)$.

Proof. Let $\lambda = |X|_K$. For each $\beta < \lambda$ define the subsets

$$A^0_\beta = \left\{n \in \omega^2 \mid \mathrm{B}_\mathcal{P}(n) \cap X_\beta \neq \emptyset, \mathrm{B}_\mathcal{P}(n) \cap X_{\beta+1} = \emptyset, \overline{\mathrm{B}_\mathcal{P}(n) \cap X_\beta} \in K(X)\right\}$$
$$A^1_\beta = \left\{n \in \omega^2 \mid \mathrm{B}_\mathcal{P}(n) \cap X_\beta \neq \emptyset, \mathrm{B}_\mathcal{P}(n) \cap X_{\beta+1} = \emptyset, \overline{\mathrm{B}_\mathcal{P}(n) \cap X_\beta} \notin K(X)\right\}$$

and put $C = \omega^2 \setminus (\bigcup_{\beta<\lambda}(A^0_\beta \cup A^1_\beta))$. For $i, j \in \{0,1\}$ and $\beta, \beta' < \lambda$ the following observations hold:

(1) If $i \neq j$ or $\beta \neq \beta'$, then $A^i_\beta \cap A^j_{\beta'} = \emptyset$.

(2) If $n \in A^0_\beta$ and $m \in \omega^2$ satisfy $n \prec_\mathcal{P} m$, then $m \in A^0_\gamma$ or $m \in A^1_{\gamma'}$ for some $\gamma \leq \beta$ or $\gamma' < \beta$.

(3) If $n \in A^1_\beta$ and $m \in \omega^2$ satisfy $n \prec_\mathcal{P} m$, then $m \in A^0_\gamma$ or $m \in A^1_\gamma$ for some $\gamma \leq \beta$.

(4) We have $[A^1_\beta]_\mathcal{P} = [C]_\mathcal{P} = \emptyset$.

It is straightforward to check that observation (1), (2) and (3) hold. Observation (4) holds since if $[A^1_\beta]_\mathcal{P} \neq \emptyset$ for some $\beta < \lambda$, then there would be $x \in X_\beta \setminus X_{\beta+1}$ without a pre-compact neighbourhood in X_β. Moreover, if $[C]_\mathcal{P} \neq \emptyset$, then there would be a point $x \in X$ such that $x \notin X_\beta$ for any $\beta < \lambda$.

From observation (4) it follows that we may choose $\alpha_\mathcal{P}, \beta_\mathcal{P} < \omega_1$ such that $|C|_\mathcal{P} \leq \beta_\mathcal{P}$ and $|A^1_\beta|_\mathcal{P} \leq \alpha_\mathcal{P}$ for all $\beta < \lambda$. Note that if λ is a successor, then $C = \emptyset$ and hence we may choose $\beta_\mathcal{P} = 0$.

Now fix $F \in F_{\aleph_0}(X)$. We will argue that

$$\varphi_\mathcal{P}(F) \leq (\omega|F|_{\mathrm{CB}} + \alpha_\mathcal{P})\lambda + \beta_\mathcal{P}$$

by proving the following two claims.

Claim 1: Let $\beta < \lambda$. If $n \in D^{\omega\alpha}_\mathcal{P}(A^0_\beta \cap A^\mathcal{P}_F)$ for some $1 \leq \alpha < \omega_1$, then there is $x \in \overline{\mathrm{B}_\mathcal{P}(n) \cap X_\beta} \cap F^\alpha$.

Proof of Claim 1: We prove this claim by induction on $1 \leq \alpha < \omega_1$. First assume that $n \in D^{\omega}_{\mathcal{P}}(A^0_\beta \cap A^{\mathcal{P}}_F)$. Then there are $(m_i)_i \in A^0_\beta \cap A_F$ and $(x_i)_i, (y_i)_i \in X$ such that for all $i, j \in \omega$ we have

(a) $n \prec_{\mathcal{P}} m_i$ and $m_i(1) \geq n(1) + i$.

(b) $x_i \in \mathrm{B}_{\mathcal{P}}(m_i) \cap X_\beta$ and $y_i \in \mathrm{B}_{\mathcal{P}}(m_i) \cap F$.

(c) $x_i \neq x_j$ and $y_i \neq y_j$ whenever $i \neq j$.

Since $\overline{\mathrm{B}_{\mathcal{P}}(n) \cap X_\beta}$ is compact by definition of A^0_β, there must exist some $x \in \overline{\mathrm{B}_{\mathcal{P}}(n) \cap X_\beta}$ and a subsequence $(x_{i_j})_j \subseteq (x_i)_i$ such that $x_{i_j} \to x$ as $j \to \infty$. Now, since $x_i, y_i \in \mathrm{B}_{\mathcal{P}}(m_i)$ for all $i \in \omega$ and $\mathrm{diam}(\mathrm{B}_{\mathcal{P}}(m_i)) \to 0$ as $i \to \infty$, we must have $y_{i_j} \to x$ as $j \to \infty$, as well. Therefore, as $(y_{i_j})_j \in F$ and F is closed, we obtain that $x \in F^1$, as desired.

The proof of the successor and the limit case can now be done as in the proof of Lemma 3.1.5. $\diamond$

Let $\beta < \lambda$. It is a consequence of observation (2) and (3) that an easy induction argument on $\eta < \omega_1$ shows that

$$n \in A^0_\beta \cap D^{\eta}_{\mathcal{P}}\left(A^{\mathcal{P}}_F \setminus (\bigcup_{\gamma<\beta}(A^0_\gamma \cup A^1_\gamma))\right) \implies n \in D^{\eta}_{\mathcal{P}}(A^{\mathcal{P}}_F \cap A^0_\beta)$$

and

$$n \in A^1_\beta \cap D^{\eta}_{\mathcal{P}}\left(A^{\mathcal{P}}_F \setminus (A^0_\beta \cup \bigcup_{\gamma<\beta}(A^0_\gamma \cup A^1_\gamma))\right) \implies n \in D^{\eta}_{\mathcal{P}}(A^{\mathcal{P}}_F \cap A^1_\beta)$$

for all $\eta < \omega_1$. Using these implications we will obtain the next claim.

Claim 2: We have

$$D^{(\omega|F|_{\mathrm{CB}}+\alpha_{\mathcal{P}})\beta}_{\mathcal{P}}(A^{\mathcal{P}}_F) \subseteq A^{\mathcal{P}}_F \setminus \bigcup_{\gamma<\beta}(A^0_\gamma \cup A^1_\gamma)$$

and

$$D^{(\omega|F|_{\mathrm{CB}}+\alpha_{\mathcal{P}})\beta+\omega|F|_{\mathrm{CB}}}_{\mathcal{P}}(A^{\mathcal{P}}_F) \subseteq A^{\mathcal{P}}_F \setminus (A^0_\beta \cup \bigcup_{\gamma<\beta}(A^0_\gamma \cup A^1_\gamma))$$

for all $\beta < \lambda$.

Proof of Claim 2: First we consider the case $\beta = 0$. The first inclusion is trivial. For the second inclusion, note that if $n \in D_{\mathcal{P}}^{\omega|F|_{\mathrm{CB}}}(A_F^{\mathcal{P}})$, then it follows by Claim 1 that $n \notin A_0^0$.

Next assume that the inclusions hold for some $\beta < \lambda$. To prove that the first inclusion holds for $\beta + 1$, note that

$$\begin{aligned} D_{\mathcal{P}}^{(\omega|F|_{\mathrm{CB}}+\alpha_{\mathcal{P}})(\beta+1)}(A_F^{\mathcal{P}}) &= D_{\mathcal{P}}^{\alpha_{\mathcal{P}}}\left(D_{\mathcal{P}}^{(\omega|F|_{\mathrm{CB}}+\alpha_{\mathcal{P}})\beta+\omega|F|_{\mathrm{CB}}}(A_F^{\mathcal{P}})\right) \\ &\subseteq D_{\mathcal{P}}^{\alpha_{\mathcal{P}}}\left(A_F^{\mathcal{P}} \setminus (A_\beta^0 \cup \bigcup_{\gamma<\beta}(A_\gamma^0 \cup A_\gamma^1))\right). \end{aligned}$$

By the implications above and the fact that $D_{\mathcal{P}}^{\alpha_{\mathcal{P}}}(A_F^{\mathcal{P}} \cap A_\beta^1) = \emptyset$, we conclude

$$D_{\mathcal{P}}^{(\omega|F|_{\mathrm{CB}}+\alpha_{\mathcal{P}})(\beta+1)}(A_F^{\mathcal{P}}) \subseteq A_F^{\mathcal{P}} \setminus \bigcup_{\gamma\leq\beta}(A_\gamma^0 \cup A_\gamma^1),$$

as wanted.

For the second inclusion, we then obtain

$$D_{\mathcal{P}}^{(\omega|F|_{\mathrm{CB}}+\alpha_{\mathcal{P}})(\beta+1)+\omega|F|_{\mathrm{CB}}}(A_F^{\mathcal{P}}) \subseteq D_{\mathcal{P}}^{\omega|F|_{\mathrm{CB}}}\left(A_F^{\mathcal{P}} \setminus \bigcup_{\gamma\leq\beta}(A_\gamma^0 \cup A_\gamma^1)\right).$$

By use of the implications above and the fact that $D_{\mathcal{P}}^{\omega|F|_{\mathrm{CB}}}(A_F^{\mathcal{P}} \cap A_{\beta+1}^0) = \emptyset$, we conclude

$$D_{\mathcal{P}}^{(\omega|F|_{\mathrm{CB}}+\alpha_{\mathcal{P}})(\beta+1)+\omega|F|_{\mathrm{CB}}}(A_F^{\mathcal{P}}) \subseteq A_F^{\mathcal{P}} \setminus (A_{\beta+1}^0 \cup \bigcup_{\gamma\leq\beta}(A_\gamma^0 \cup A_\gamma^1)),$$

as wanted.

Finally, assume that the claim holds for all $\beta < \xi$ for some limit $\xi < \omega_1$. Then we have

$$\begin{aligned} D_{\mathcal{P}}^{(\omega|F|_{\mathrm{CB}}+\alpha_{\mathcal{P}})\xi}(A_F^{\mathcal{P}}) &= \bigcap_{\beta<\xi} D_{\mathcal{P}}^{(\omega|F|_{\mathrm{CB}}+\alpha_{\mathcal{P}})\beta}(A_F^{\mathcal{P}}) \\ &\subseteq \bigcap_{\beta<\xi}\left(A_F^{\mathcal{P}} \setminus (\bigcup_{\gamma<\beta}(A_\gamma^0 \cup A_\gamma^1)\right) \\ &= A_F^{\mathcal{P}} \setminus \bigcup_{\gamma<\xi}(A_\gamma^0 \cup A_\gamma^1), \end{aligned}$$

as wanted.

Now for the second inclusion, we have

$$D_{\mathcal{P}}^{(\omega|F|_{\mathrm{CB}}+\alpha_{\mathcal{P}})\xi+\omega|F|_{\mathrm{CB}}}(A_F^{\mathcal{P}}) \subseteq D_{\mathcal{P}}^{\omega|F|_{\mathrm{CB}}}\left(A_F^{\mathcal{P}} \setminus (\bigcup_{\gamma<\xi}(A_\gamma^0 \cup A_\gamma^1)\right).$$

As before, we use the implications above and the fact that $D_{\mathcal{P}}^{\omega|F|_{\mathrm{CB}}}(A_F^{\mathcal{P}} \cap A_\xi^0) = \emptyset$, to obtain that

$$D_{\mathcal{P}}^{(\omega|F|_{\mathrm{CB}}+\alpha_{\mathcal{P}})\xi+\omega|F|_{\mathrm{CB}}}(A_F^{\mathcal{P}}) \subseteq A_F \setminus (A_\xi^0 \cup \bigcup_{\gamma<\xi}(A_\gamma^0 \cup A_\gamma^1).$$

Thus we conclude that the inclusions hold for all $\beta < \lambda$. $\diamond$

To finish the proof, note that Claim 2 implies $D_{\mathcal{P}}^{(\omega|F|_{\mathrm{CB}}+\alpha_{\mathcal{P}})\lambda}(A_F^{\mathcal{P}}) \subseteq C$ and therefore that $D_{\mathcal{P}}^{(\omega|F|_{\mathrm{CB}}+\alpha_{\mathcal{P}})\lambda+\beta_{\mathcal{P}}}(A_F^{\mathcal{P}}) = \emptyset$, as wanted. $\square$

Putting together Proposition 3.2.2 and Theorem 3.2.3, we have finalized the proof of Theorem 3.2.1, which was the goal of this section.

Chapter 4

Related questions

This chapter contains a discussion of some questions related to the subject of this part of the thesis.

In the first section we will discuss certain invariance properties one can hope for in a co-analytic rank $\varphi\colon F_{\aleph_0}(X) \to \omega_1$. We will argue that a Polish space X is σ-compact if and only if there is a co-analytic rank $\varphi\colon F_{\aleph_0}(X) \to \omega_1$ such that $\varphi(F_0) = \varphi(F_1)$ whenever F_0 and F_1 are homeomorphic. Afterwards we will see that for each Polish metric space (X, d) there is a co-analytic rank $\varphi\colon F_{\aleph_0}(X) \to \omega_1$ such that $\varphi(F_0) = \varphi(F_1)$ whenever $(F_0, d_{|F_0})$ and $(F_1, d_{|F_1})$ are isomorphic. The proof is not constructive, so it leaves open the problem of finding a concrete co-analytic rank with this property. The second section concerns the phenomenon of uniformly bounded families of ranks. We have seen that the family $\{\varphi_{\mathcal{P}} \mid \mathcal{P} \text{ is a presentation of } X\}$ is uniformly bounded by the Cantor-Bendixson rank if X is a compact Polish space. We will ask several questions towards understanding if this behaviour holds more generally and, in particular, if it occurs in other cases.

The results in Section 4.1 are all well-known or direct consequences of well-known results.

4.1 Invariant ranks on $F_{\aleph_0}(X)$

In this section we will discuss which invariance properties one can obtain for a co-analytic rank $\varphi\colon F_{\aleph_0}(X) \to \omega_1$ for a general Polish space X. Let us begin by making this notion of invariance more precise.

Let X be a set, $A \subseteq X$ and E an equivalence relation on X. Then A is said to be *E-invariant* if $x \in A$ and xEy imply $y \in A$ for all $x, y \in X$. For co-analytic ranks we have the following definition of invariance.

Definition 4.1.1. Let X be a Polish space and E an equivalence relation on X. Moreover, let $A \subseteq X$ be co-analytic and E-invariant. A co-analytic rank $\varphi\colon A \to \omega_1$ is called *E-invariant* if

$$xEy \implies \varphi(x)E\varphi(y)$$

for all $x, y \in A$.

The next result, due to Solovay, states that if the equivalence relation E in the definition above is analytic, then there exists an E-invariant co-analytic rank $\varphi\colon A \to \omega_1$. For a sketch of the proof, the reader is referred to the hint of [19, Exercise 34.6].

Theorem 4.1.2 (Solovay). *Let X be a Polish space and E an analytic equivalence relation on X. If $A \subseteq X$ is co-analytic and E-invariant, then there exists an E-invariant co-analytic rank $\varphi\colon A \to \omega_1$.*

First let X be a Polish space and consider the equivalence relation $\sim_h$ on $F(X)$ given by $F_0 \sim_h F_1$ if and only if there is homeomorphism $f\colon F_0 \to F_1$. It is clear that $F_{\aleph_0}(X)$ is a $\sim_h$-invariant subset. We will now briefly argue that for any Polish space X there exists a $\sim_h$-invariant co-analytic rank $\varphi\colon F_{\aleph_0}(X) \to \omega_1$ if and only if X is σ-compact. By Corollary 1.5.7 and Remark 1.5.4, the Cantor-Bendixson rank is a $\sim_h$-invariant co-analytic rank on $F_{\aleph_0}(X)$ when X is σ-compact. Below we prove the opposite implication.

Proposition 4.1.3. *Let X be a Polish space. If there exists a $\sim_h$-invariant co-analytic rank $\varphi\colon F_{\aleph_0}(X) \to \omega_1$, then X is σ-compact.*

Proof. Assume for contradiction that X is not σ-compact and that such a rank exists. Then, by Remark 1.5.6, we obtain a $\sim_h$-invariant co-analytic rank $\varphi\colon F_{\aleph_0}(\omega^\omega) \to \omega_1$. In particular, there is $\alpha_0 < \omega_1$ such that $\varphi(F) \leq \alpha_0$ for all discrete $F \in F_{\aleph_0}(X)$. However, this contradicts Corollary 2.3.13 and Remark 1.4.4. $\square$

Combining Proposition 4.1.3 and Theorem 4.1.2, we may conclude that $\sim_h$ is not an analytic equivalence relation on $F(X)$ when X is not σ-compact.

Next let X be a Polish space and consider the equivalence relation $\sim_i$ on $F(X)$ given by $F_0 \sim_i F_1$ if and only if $(F_0, d_{|F_0})$ and $(F_1, d_{|F_1})$ are isomorphic. Once again, it is clear that $F_{\aleph_0}(X)$ is $\sim_i$-invariant. We will now, by applying Theorem 4.1.2, argue that for any Polish metric space (X, d), there exists a $\sim_i$-invariant co-analytic rank $\varphi\colon F_{\aleph_0}(X) \to \omega_1$.

Proposition 4.1.4. *For any Polish metric space (X, d) there exists a $\sim_i$-invariant co-analytic rank $\varphi\colon F_{\aleph_0}(X) \to \omega_1$.*

Proof. It suffices to prove that $\sim_i$ is analytic. First, by Theorem 1.1.11, fix for each $n \in \omega$ a Borel map $\rho_n\colon F(X) \to X$ such that

$$\{\rho_n(F) \mid n \in \omega\} \subseteq F$$

is dense for all non-empty $F \in F(X)$. Now define the Borel map $\psi\colon F(X) \setminus \{\emptyset\} \to \mathbb{R}^{\omega \times \omega}$ given by

$$\psi(F)(i, j) = d(\rho_i(F), \rho_j(F))$$

for all $i, j \in \omega$ and $F \in F(X)$. Then for each $F \in F(X) \setminus \{\emptyset\}$ the element $\psi(F)$ satisfies that the completion of ω equipped with the pseudo metric $\delta(i, j) = \psi(F)(i, j)$ is isomorphic to $(F, d_{|F})$. An element in $\mathbb{R}^{\omega \times \omega}$ which represents a pseudo metric on ω is called a *code* for a Polish metric space. It is shown in [11, Lemma 4] that the equivalence relation of coding the same Polish metric space (up to isometry) is an analytic equivalence relation. Thus $\sim_i$ must be analytic as well. □

Now we know that for any Polish metric space (X, d) there is a $\sim_i$-invariant co-analytic rank $\varphi\colon F_{\aleph_0}(X) \to \omega_1$. However, this result relies entirely on Theorem 4.1.2, and the proof of this theorem is not constructive. Indeed, as in the proof of Theorem 1.4.7, it is enough to prove the result for some $\mathbf{\Pi}^1_1$-complete set such as WF. So this leaves us with the following problem.

Problem 4.1.5. *For a general Polish metric space (X, d) find a concrete $\sim_i$-invariant co-analytic rank $\varphi\colon F_{\aleph_0}(X) \to \omega_1$.*

If X is σ-compact, then the Cantor-Bendixson rank has this property. It is also easy to deduce that if (X, d) is an ultra-metric Polish space, then for any dense sequence $\overline{x}$ in X, the presentation $\mathcal{P} = (X, d, \overline{x})$ satisfies that $\varphi_{\mathcal{P}}\colon F_{\aleph_0}(X) \to \omega_1$ is co-analytic and $\sim_i$-invariant.

In general, we can for a Polish metric space (X, d) use the ranks constructed in Chapter 2 to obtain a rank $\varphi\colon F_{\aleph_0}(X) \to \omega_1$ such that

$$F_0 \sim_i F_1 \implies \varphi(F_0) = \varphi(F_1)$$

for all $F_0, F_1 \in F_{\aleph_0}(X)$. Indeed, fix a Polish metric space (X, d) and let $\mathcal{P}_F = (F, d_{|F}, F)$ for each $F \in F_{\aleph_0}(X)$. Then put $\varphi(F) = \varphi_{\mathcal{P}_F}(F)$ for all $F \in F_{\aleph_0}(X)$. However, it seems unlikely that this rank is co-analytic in general.

4.2 Uniformly bounded families of ranks

In Chapter 4 we saw that the family of induced ranks $\varphi_{\mathcal{P}}\colon F_{\aleph_0}(X) \to \omega_1$, where $\mathcal{P}$ varies over all presentations of some Polish space X, is uniformly bounded by the Cantor-Bendixson rank if and only if X is compact. In this section we will discuss questions related to this behaviour. A natural question to ask is if this phenomenon holds more generally.

Question 4.2.1. *Let X be a compact Polish space. Is the family of all regular co-analytic ranks on $F_{\aleph_0}(X)$ uniformly bounded by the Cantor-Bendixson rank?*

It is clear that the assumption of regularity of the ranks is necessary. Indeed, we can easily construct a counterexample if the assumption is removed. For each $\alpha < \omega_1$ consider the rank $\psi_\alpha\colon F_{\aleph_0}(X) \to \omega_1$ given by $\psi_\alpha(F) = \alpha + |F|_{\mathrm{CB}}$. It will be co-analytic, as it induces the same prewellordering as the Cantor-Bendixson rank. It is also clear that the family $\{\psi_\alpha \mid \alpha < \omega_1\}$ is not uniformly bounded by the Cantor-Bendixson rank.

Unless one has an idea to construct a counterexample to Question 4.2.1, the question seems hard to tackle. The following question might be easier to begin with.

Question 4.2.2. *Let X be a compact Polish space and for each $i \in I$ let $D_i\colon F(X) \to F(X)$ be a Borel derivative such that*

$$\{F \in F(X) \mid D_i^\infty(F) = \emptyset\} = F_{\aleph_0}(X).$$

Under what circumstances is the family of induced co-analytic ranks $(\varphi_{D_i})_{i\in I}$ uniformly bounded by the Cantor-Bendixson rank?

If the answer to Question 4.2.1 is positive, it will suggest that for a compact Polish space X, the co-analytic subset $F_{\aleph_0}(X)$ has a very rigid structure of the co-analytic ranks it admits.

Generally, it would be interesting to investigate when this phenomenon occurs and to find other examples of such families.

Definition 4.2.3. Let A be a co-analytic subset of a Polish space X and let I be some index set. A family of co-analytic ranks $\varphi_i\colon A \to \omega_1$ for $i \in I$ is said to be *uniformly bounded* if there is a co-analytic rank $\psi\colon A \to \omega_1$ and a function $f\colon \omega_1 \to \omega_1$ such that $\varphi_i(x) \leq f(\psi(x))$ for all $x \in A$ and $i \in I$.

Clearly, all countable families are uniformly bounded. Indeed, if $\varphi_n\colon A \to \omega_1$ is a co-analytic rank for each $n \in \omega$, then it follows by Remark 1.4.4 that for every $n \in \omega$ there is $f_n \colon \omega_1 \to \omega_1$ such that $\varphi_n(x) \leq f_n(\varphi_0(x))$ for all $x \in A$. Hence $f \colon \omega_1 \to \omega_1$ defined by $f(\alpha) = \sup\{f_n(\alpha) \mid n \in \omega\}$ will satisfy that $\varphi_n(x) \leq f(\varphi_0(x))$ for all $x \in A$ and $n \in \omega$. We are therefore only interested in examples of uncountable uniformly bounded families of ranks.

Problem 4.2.4. *Find examples of uncountable families of co-analytic ranks that are uniformly bounded.*

Finally, we should point out that it is not the case that the family $(\varphi_{\mathcal{P}})_{\mathcal{P}}$, where $\mathcal{P}$ varies over all presentations of a Polish space X, is countable whenever X is compact.

Proposition 4.2.5. *The family $\{\varphi_{\mathcal{P}} \mid \mathcal{P}$ is a presentation of $2^\omega\}$ is uncountable.*

Before we begin the proof, let us fix some notation. Let $\mathcal{P}_{2^\omega} = (2^\omega, d_{2^\omega}, \overline{z})$, where $\overline{z}$ is some countable dense sequence in 2^ω and

$$d_{2^\omega}(x,y) = \begin{cases} 3^{-1} \cdot 2^{-\min\{n \in \omega \mid x(n) \neq y(n)\}} & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}.$$

Moreover, if $A = \{s_i \in 2^{<\omega} \mid i \in I\}$ and $B = \{t_i \in 2^{<\omega} \mid i \in I\}$ satisfy $\ln(s_i) = \ln(t_i)$, $s_i \perp s_j$, $t_i \perp t_j$ and $s_i \perp t_j$ for all $i, j \in I$ with $i \neq j$, we say that A and B are *compatible sets of initial segments*. In this case, we define a homeomorphism $f_{A,B} \colon 2^\omega \to 2^\omega$ given by

$$f_{A,B}(x) = \begin{cases} t_i^\frown z & \text{if} \quad (\exists i \in I)(\exists z \in \omega^\omega)\ x = s_i^\frown z \\ s_i^\frown z & \text{if} \quad (\exists i \in I)(\exists z \in \omega^\omega)\ x = t_i^\frown z \\ x & \text{otherwise} \end{cases}$$

and call $f_{A,B}$ the *induced switch map*.

Proof of Proposition 4.2.5. First we will for each $3 \leq n < \omega$ recursively construct $F_n^0 \in F_{\aleph_0}(2^\omega)$ with $\varphi_{\mathcal{P}_{2^\omega}}(F_n^0) = n$ and elements $s_n^u, t_n^u \in 2^{<\omega}$ for each $u \in 2^{n-3}$ such that

$$A_n^0 = \left\{s_n^u \in 2^{<\omega} \mid u \in 2^{n-3}\right\} \qquad \text{and} \qquad B_n^0 = \left\{t_n^u \in 2^{<\omega} \mid u \in 2^{n-3}\right\}$$

are compatible sets of initial seqments and such that the induced switch map $f_n \colon 2^\omega \to 2^\omega$ satisfies $\varphi_{\mathcal{P}_{2^\omega}}(f_n(F_n^0)) = n - 1$.

Let $F_3^0 = \left\{0^\omega, 0^\frown 1^\omega, 1^\omega, (1,1)^\frown 0^\omega\right\}$. Moreover, put $s_3^\emptyset = (0,1)$ and $t_3^\emptyset = (1,0)$. Then it is immediate that $\varphi_{\mathcal{P}_{2^\omega}}(F_3^0) = 3$ and $\varphi_{\mathcal{P}_{2^\omega}}(f_3(F_3^0)) = 2$.

Now assume that we have carried out the construction for some $3 \leq n < \omega$. Then let

$$F^0_{n+1} = \left\{0^\frown x \mid x \in F^0_n\right\} \cup \left\{1^\frown x \mid x \in F^0_n\right\}$$

and for each $u \in 2^{n-3}$ put

$$s^{u^\frown 0}_{n+1} = 0^\frown s^u_n, \qquad s^{u^\frown 1}_{n+1} = 1^\frown s^u_n, \qquad t^{u^\frown 0}_{n+1} = 0^\frown t^u_n \quad \text{and} \quad t^{u^\frown 1}_{n+1} = 1^\frown t^u_n.$$

It is straightforward to check that $\varphi_{\mathcal{P}_{2^\omega}}(F^0_n) = n$ and that $\varphi_{\mathcal{P}_{2^\omega}}(f_n(F^0_n)) = n-1$. This finishes our recursive construction.

Next for each $3 \leq n < \omega$ and $u \in 2^{n-3}$, we put $F_n = \left\{1^{n\frown}0^\frown x \mid x \in F^0_n\right\}$, $\tilde{s}^u_n = 1^{n\frown}0^\frown s^u_n$ and $\tilde{t}^u_n = 1^{n\frown}0^\frown t^u_n$. Note that

$$A_n = \left\{\tilde{s}^u_n \mid u \in 2^{n-3}\right\} \qquad \text{and} \qquad B_n = \left\{\tilde{u}^u_n \mid u \in 2^{n-3}\right\}$$

are still compatible sets of initial segments and that their induced switch map $\tilde{f}_n \colon 2^\omega \to 2^\omega$ still satisfies $\varphi_{\mathcal{P}_{2^\omega}}(F_n) = n$ and $\varphi_{\mathcal{P}_{2^\omega}}(\tilde{f}_n(F_n)) = n-1$ for all $3 \leq n < \omega$. Hence for each $y \in 2^\omega$ we have that

$$A_y = \bigcup_{n \in \omega, y(n)=1} \left\{\tilde{s}^u_n \mid u \in 2^{n-3}\right\} \quad \text{and} \quad B_y = \bigcup_{n \in \omega, y(n)=1} \left\{\tilde{t}^u_n \mid u \in 2^{n-3}\right\}$$

are compatible sets of initial segments. Furthermore, if we let $f_y \colon 2^\omega \to 2^\omega$ be the induced switch map, then

$$\varphi_{\mathcal{P}_{2^\omega}}(f_y(F_n)) = n \iff y(n) = 0.$$

So if we let $\mathcal{P}_y = (\mathcal{P}_{2^\omega})_{f_y}$ for all $y \in 2^\omega$, it follows by Proposition 2.3.11 that $\varphi_{\mathcal{P}_y} \neq \varphi_{\mathcal{P}_x}$ for all $x, y \in 2^\omega$ with $x \neq y$. $\square$

Part II

Co-induction and invariant random subgroups

This part constitutes an amended version of the paper

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Parts of the article have been altered and rewritten to fit the format of the thesis. In particular, more preliminary theory has been added, some comments and explanations have been expanded and Chapter 9 is new.

Chapter 5

Actions and invariant random subgroups

In this part of the thesis we will be concerned with invariant random subgroups and their connection to the measure preserving group actions. We will in this preliminary chapter introduce the various notions and results that we need for this part of the thesis. We assume the reader to be familiar with the basic notions discussed in Section 1.1.

In the first section we introduce notions related to measure preserving groups actions. In particular, we will discuss various properties that such actions can have and define the Polish space of all measure preserving actions of a fixed countable group. In the second section we will define the relations of weak containment and weak equivalence of measure preserving group actions, with the goal of obtaining a compact Polish space of weak equivalence classes. In the third and final section we will introduce the main notion of this part of the thesis, namely the notion of an invariant random subgroup. We will see that the set of invariant random subgroups of a fixed countable group admits a natural compact Polish topology. Furthermore, we will establish the connection between the weak equivalence classes of the measure preserving actions and the invariant random subgroups of a fixed countable group. Finally, we will discuss the notion of a characteristic random subgroup, which is a special kind of well-behaved invariant random subgroup.

Most of the results in this chapter are standard. Proposition 5.1.6 and Lemma 5.3.5 can also be found in [22]. For a more thorough introduction to the subjects of this chapter, the reader is referred to [20] and [10].

5.1 Measure preserving group actions

In this section we will focus on measure preserving group actions. We will briefly discuss properties of such actions and define the Polish space of measure preserving actions of a fixed countable group.

First let us specify the underlying measure spaces that we are interested in.

Definition 5.1.1. Let X be a standard Borel space. A *Borel measure* μ on X is a measure on the Borel sets of X. A Borel measure is called *atomic* if $\mu(\{x\}) > 0$ for some $x \in X$.

If X is a standard Borel space and $x \in X$, then the *Dirac measure* concentrated at x is the measure given by

$$\delta_x(B) = \begin{cases} 1 & \text{if} \quad x \in B \\ 0 & \text{if} \quad x \notin B \end{cases}$$

for any $B \subseteq X$ Borel. Clearly, every Dirac measure is atomic.

Definition 5.1.2. A *standard probability space* is a probability space (X, μ), where X is a standard Borel space and μ is a Borel probability measure. If μ is non-atomic, we say that (X, μ) is a non-atomic standard probability space.

If (X, μ) is a standard probability space, Y is a standard Borel space and $f \colon X \to Y$ is Borel, then the *pushforward* of μ through f is the Borel probability measure ν on Y given by

$$\nu(B) = \mu(f^{-1}(B))$$

for all $B \subseteq Y$ Borel. It is easily seen that (Y, ν) is non-atomic if (X, μ) is non-atomic.

Let (X, μ) and (Y, ν) be standard probability spaces. A Borel map $f \colon X \to Y$ is called *measure preserving* if $\mu(f^{-1}(B)) = \nu(B)$ for all Borel $B \subseteq Y$. Note that if f is a measure preserving Borel isomorphism, then f^{-1} is measure preserving. In this case, we say that f is a *measure isomorphism* and call (X, μ) and (Y, ν) *isomorphic*.

Recall that any two uncountable standard Borel spaces are Borel isomorphic. The next theorem states that there is only one (up to isomorphism) non-atomic Borel probability measure to put on such spaces. A proof can be found in [19, Theorem 7.41].

Theorem 5.1.3. *Any non-atomic standard probability space is isomorphic to* $([0,1],\lambda)$, *where* λ *is the Lebesgue measure.*

Note that if $(X_n,\mu_n)_n$ is a sequence of standard probability spaces, then $X = \Pi_{n\in\omega}X_n$ is a standard Borel space when equipped with the product Borel structure. Moreover, there exists a unique Borel probability measure $\mu = \Pi_{n\in\omega}\mu_n$ on X satisfying that $\mu\left(\Pi_{n\in\omega}B_n\right) = \Pi_{n\in\omega}\mu_n(B_n)$ for all sequences $(B_n)_n$ of Borel subsets of X with $B_n \neq X$ for only finitely many $n \in \omega$. Thus we obtain that (X,μ) is a standard probability space. The same holds for finite products of standard probability spaces. If (Y,ν) is a standard probability space and $n \in \omega \cup \{\omega\}$, we will use (Y^n,ν^n) to denote the standard probability space $(\Pi_{i<n}Y, \Pi_{i<n}\nu)$. If (Y,ν) is non-atomic, so is (Y^n,ν^n).

Definition 5.1.4. Let $\Gamma \curvearrowright^a X$ be a Borel action of a countable group Γ on a standard Borel space X. A Borel probability measure μ on X is said to be *invariant* if $\mu(\gamma \cdot^a B) = \mu(B)$ for all $\gamma \in \Gamma$ and $B \subseteq X$ Borel. If μ is invariant, we say that $\Gamma \curvearrowright^a (X,\mu)$ is *measure preserving.*

In other words, an action $\Gamma \curvearrowright^a (X,\mu)$ is measure preserving if the map $x \mapsto \gamma \cdot^a x$ from X to X is measure preserving for all $\gamma \in \Gamma$.

A measure preserving action $\Gamma \curvearrowright^a (X,\mu)$ is called *free* if $\mu(\mathrm{Fix}_a(\gamma)) = 0$ for all $\gamma \in \Gamma \setminus \{e\}$, where

$$\mathrm{Fix}_a(\gamma) = \{x \in X \mid \gamma \cdot^a x = x\}.$$

As we will see in Section 5.3, the non-free measure preserving actions will be of great importance to us.

Two key properties of measure preserving actions are of special interest to us.

Definition 5.1.5. Let $\Gamma \curvearrowright^a X$ be a Borel action of a countable group Γ on a standard probability space X.

An invariant Borel probability measure μ on X is said to be *ergodic* if any Borel set $A \subseteq X$ which satisfies $\mu(A \triangle (\gamma \cdot^a A)) = 0$ for all $\gamma \in \Gamma$ also must satisfy $\mu(A) \in \{0,1\}$. If μ is ergodic, we say that the action $\Gamma \curvearrowright^a (X,\mu)$ is *ergodic.*

An invariant Borel probability measure μ on X is said to be *weakly mixing* if the action $\Gamma \curvearrowright^{a^2} (X^2,\mu^2)$ given by $\gamma \cdot^{a^2} (x,y) = (\gamma \cdot^a x, \gamma \cdot^a y)$ is ergodic. If μ is weakly mixing, we say that the action $\Gamma \curvearrowright^a (X,\mu)$ is *weakly mixing.*

Note that every weakly mixing action $\Gamma \curvearrowright^a (X, \mu)$ must be ergodic. Indeed, if $A \subseteq X$ is Borel and $\mu(A \triangle (\gamma \cdot^a A)) = 0$ for all $\gamma \in \Gamma$, then

$$\mu^2\left((A \times A) \triangle ((\gamma \cdot^a A) \times (\gamma \cdot^a A))\right) = 0$$

for all $\gamma \in \Gamma$. Hence, by ergodicity of a^2, we obtain $\mu(A)^2 = \mu^2(A \times A) \in \{0, 1\}$.

Proposition 5.1.6. *Let $\Gamma \curvearrowright^a (X, \mu)$ be weakly mixing and assume that μ is atomic. Then there is $x \in X$ such that $\mu = \delta_x$.*

Proof. Let $x \in X$ satisfy $\mu(\{x\}) > 0$. Then the orbit of x satisfies

$$\operatorname{card}(\{\gamma \cdot^a x \mid \gamma \in \Gamma\}) = n$$

for some $n \geq 1$ and, by ergodicity of a, we must have $\mu(\{\gamma \cdot^a x\}) = \frac{1}{n}$ for all $\gamma \in \Gamma$. Moreover, as the set $\left\{(\gamma \cdot^a x, \gamma \cdot^a x) \in X^2 \mid \gamma \in \Gamma\right\}$ is invariant under the diagonal action a^2 and a is weakly mixing, we obtain that

$$\mu^2(\{(\gamma \cdot^a x, \gamma \cdot^a x)\}) = \frac{1}{n}$$

for all $\gamma \in \Gamma$. Since we also have

$$\mu^2(\{(\gamma \cdot^a x, \gamma \cdot^a x)\}) = \frac{1}{n^2}$$

for all $\gamma \in \Gamma$, we conclude that $n = 1$ and therefore that $\mu = \delta_x$, as desired. $\square$

In other words, if a measure is weakly mixing with respect to some Borel action of a countable group, then it is either a Dirac measure or non-atomic. This simplifies the process of checking if a weakly mixing measure is non-atomic.

We will now construct the Polish space of measure preserving actions of a fixed countable group. For the remainder of this part, we will assume that (X, μ) is a non-atomic standard probability space.

The *measure algebra* of μ, denoted by MALG_μ, is the algebra consisting of the Borel subsets of X considered modulo μ-null sets. This algebra can be equipped with a Polish topology induced by the complete metric d_μ given by

$$d_\mu(A, B) = \mu(A \triangle B)$$

for all $A, B \in \mathrm{MALG}_\mu$.

Let $\mathrm{Aut}(X, \mu)$ denote the group of measure isomorphisms of (X, μ), where we identify two isomorphisms if they agree almost everywhere. There are two natural topologies on $\mathrm{Aut}(X, \mu)$ which turn it into a topological group.

The *weak topology* on $\mathrm{Aut}(X,\mu)$ is the topology generated by the maps $\varphi_A\colon \mathrm{Aut}(X,\mu) \to \mathrm{MALG}_\mu$ given by $\varphi_A(T) = T(A)$, where A varies over all elements in MALG_μ. A left invariant metric d_w on $\mathrm{Aut}(X,\mu)$ inducing this topology is given by

$$d_w(T,S) = \sum_{n\in\omega} 2^{-n-1}\mu(T(A_n)\triangle S(A_n)),$$

where $(A_n)_n \in \mathrm{MALG}_\mu$ is a dense sequence.

The *uniform topology* on $\mathrm{Aut}(X,\mu)$ is defined by the two-sided invariant complete metric d_u on $\mathrm{Aut}(X,\mu)$ given by

$$d_u(S,T) = \mu\left(\{x \in X \mid S(x) \neq T(x)\}\right).$$

It is clear that the uniform topology is finer than the weak topology. Moreover, $\mathrm{Aut}(X,\mu)$ is a Polish group when equipped with the weak topology, while the uniform topology is not separable (see [20, Section 1]). Recall that a *Polish group* is a topological group whose topology is Polish.

Now fix a countable group Γ. Each measure preserving action $\Gamma \curvearrowright^a (X,\mu)$ can be represented by a group homomorphism $h_a\colon \Gamma \to \mathrm{Aut}(X,\mu)$ given by $h_a(\gamma)(x) = \gamma \cdot^a x$. We can therefore identify the space of measure preserving actions $A(\Gamma,X,\mu)$ with the subset of $\mathrm{Aut}(X,\mu)^\Gamma$ consisting of all group homomorphisms. Both the uniform and the weak topology on $\mathrm{Aut}(X,\mu)$ satisfies that $A(\Gamma,X,\mu)$ is closed in the induced product topology on $\mathrm{Aut}(X,\mu)^\Gamma$. Hence $A(\Gamma,X,\mu)$ is Polish in the topology induced by the weak topology on $\mathrm{Aut}(X,\mu)$ and completely metrizable in the topology induced by the uniform topology on $\mathrm{Aut}(X,\mu)$. If nothing is specified, we will assume that $A(\Gamma,X,\mu)$ is equipped with the weak topology. Later we will use the notation γ^a instead of $h_a(\gamma)$ for each $a \in A(\Gamma,X,\mu)$ and $\gamma\in\Gamma$.

In the next section we will consider a binary relation on $A(\Gamma,X,\mu)$ for a countable group Γ, namely the relation of weak containment, which is going to play an important role for us. In the end of this section we will briefly discuss a more classical relation, of which weak containment is a generalization.

Definition 5.1.7. Let Γ be a countable group and $a,b \in A(\Gamma,X,\mu)$. We say that a is a *factor* of b, and write $a \sqsubseteq b$, if there is a measure preserving map $f\colon X \to X$ such that $f(\gamma \cdot^b x) = \gamma \cdot^a f(x)$ for all $\gamma \in \Gamma$ and almost all $x \in X$. The map f is called a *factor map* from b to a. If f is moreover a measure isomorphism, we say that a and b are *isomorphic* and call f an *isomorphism* of a and b.

We have that both ergodicity and weak mixing are passed on to factors.

Proposition 5.1.8. *Let Γ be a countable group and $a, b \in A(\Gamma, X, \mu)$. If $a \sqsubseteq b$ and b is ergodic (resp. weakly mixing), then a is ergodic (resp. weakly mixing).*

Proof. First assume b is ergodic and that $a \sqsubseteq b$. Let $f\colon X \to X$ be a factor map from b to a and let $A \subseteq X$ be Borel such that $\mu\left((\gamma \cdot^a A) \triangle A\right) = 0$ for all $\gamma \in \Gamma$. Then put $B = f^{-1}(A)$ and note that we must have $\mu((\gamma \cdot^b B) \triangle B) = 0$ for all $\gamma \in \Gamma$. Hence, by ergodicity of b, we obtain $\mu(A) = \mu(B) \in \{0, 1\}$, as wanted.

Now assume that b is weakly mixing and that $a \sqsubseteq b$. Then it is easy to check that also $a^2 \sqsubseteq b^2$. Hence it follows from the first part that a^2 is ergodic and therefore that a is weakly mixing. □

5.2 The space of weak equivalence classes

We will here introduce the notions of weak containment and weak equivalence of measure preserving group actions. Moreover, we will see how to equip the set of weak equivalence classes with a compact Polish topology.

The notion of weak containment of actions is motivated by the analogous notion for unitary representations and is defined as follows: Let Γ be a fixed countable group and let $a, b \in A(\Gamma, X, \mu)$. We say that a is *weakly contained* in b if for all $A_0, \dots, A_{n-1} \in \mathrm{MALG}_\mu$, $F \subseteq \Gamma$ finite and $\varepsilon > 0$ there are $B_0, \dots, B_{n-1} \in \mathrm{MALG}_\mu$ such that

$$|\mu(\gamma^a(A_i) \cap A_j) - \mu(\gamma^b(B_i) \cap B_j)| < \varepsilon$$

for all $i, j < n$ and $\gamma \in F$. If a is weakly contained in b, we write $a \preceq b$. If $a \preceq b$ and $b \preceq a$, we say that a and b are *weakly equivalent* and write $a \simeq b$. It is easily seen that if $a, b \in A(\Gamma, X, \mu)$ and $a \sqsubseteq b$, then $a \preceq b$. The converse does not hold in general (see [10, Section 6]).

Another way to characterize weak containment is as follows: Fix an enumeration $\Gamma = \{\gamma_i \mid i \in \omega\}$ and let $\mathcal{P}_k$ denote the set of all Borel partitions of X into k pieces for each $k > 1$. For each $a \in A(\Gamma, X, \mu)$, $n, k > 1$ and $P = (A_0, \dots, A_{k-1}) \in \mathcal{P}_k$ we let $M^P_{n,k}(a) \in [0,1]^{n \times k \times k}$ be given by

$$M^P_{n,k}(a)(m, i, j) = \mu(\gamma^a_m(A_i) \cap A_j)$$

for $m < n$ and $i, j < k$. Put

$$C_{n,k}(a) = \overline{\left\{M^P_{n,k}(a) \mid P \in \mathcal{P}_k\right\}},$$

that is, the closure of the set $\{M^P_{n,k}(a) \mid P \in \mathcal{P}_k\}$ in $[0,1]^{n\times k\times k}$. Then it is straightforward to check that we have

$$\begin{aligned} a \preceq b &\iff (\forall n, k > 1)(C_{n,k}(a) \subseteq C_{n,k}(b)) \\ a \simeq b &\iff (\forall n, k > 1)(C_{n,k}(a) = C_{n,k}(b)) \end{aligned}$$

for all $a, b \in A(\Gamma, X, \mu)$.

Now consider the set of weak equivalence classes $\underaccent{\sim}{A}(\Gamma, X, \mu)$ for a fixed countable group Γ. For each action $a \in A(\Gamma, X, \mu)$ we let $\underaccent{\sim}{a} \in \underaccent{\sim}{A}(\Gamma, X, \mu)$ denote its weak equivalence class. By the above, the map $\iota\colon \underaccent{\sim}{A}(\Gamma, X, \mu) \to \prod_{n,k>1} F([0,1]^{n\times k\times k})$ given by

$$\iota(\underaccent{\sim}{a}) = (C_{n,k}(a))_{n,k>1}$$

is an injection.

Recall that $F([0,1]^{n\times k\times k})$ denotes the Effros Borel space of $[0,1]^{n\times k\times k}$. It follows by Remark 1.1.10 that $F([0,1]^{n\times k\times k})$ is a compact Polish space when equipped with the Vietoris topology. We obtain a complete metric that induces this topology as follows: Fix a complete metric d on $[0,1]^{n\times k\times k}$ with $\operatorname{diam}_d([0,1]^{n\times k\times k}) = 1$ and let

$$\delta_{n,k}(K, L) = \max_{x\in K} \inf_{y\in L} d(x, y)$$

for all $K, L \in F([0,1]^{n\times k\times k})$. Then

$$d_{n,k}(K, L) = \begin{cases} 0 & \text{if } L = K = \emptyset \\ 1 & \text{if } (L = \emptyset \vee K = \emptyset) \wedge K \neq L \\ \max\{\delta_{n,k}(K, L), \delta_{n,k}(L, K)\} & \text{if } K, L \neq \emptyset \end{cases}$$

is a complete metric on $F([0,1]^{n\times k\times k})$ that induces the Vietoris topology.

It is proven in [1, Theorem 4] that the image $\iota(\underaccent{\sim}{A}(\Gamma, X, \mu))$ is a closed subset of $\prod_{n,k>1} F([0,1]^{n\times k\times k})$. Therefore, by transferring back the subspace topology, we obtain a compact Polish topology on $\underaccent{\sim}{A}(\Gamma, X, \mu)$. Moreover, the metric $\underaccent{\sim}{d}$ on $\underaccent{\sim}{A}(\Gamma, X, \mu)$ given by

$$\underaccent{\sim}{d}(\underaccent{\sim}{a}, \underaccent{\sim}{b}) = \sum_{n,k>1} 2^{-n-k} d_{n,k}(C_{n,k}(a), C_{n,k}(b))$$

is complete and induces the topology on $\underaccent{\sim}{A}(\Gamma, X, \mu)$. We will for the remainder of this part assume that $\underaccent{\sim}{A}(\Gamma, X, \mu)$ is equipped with this topology.

5.3 Invariant random subgroups

We will now introduce the notion of an invariant random subgroup and establish a connection between the space of these and the space of weak equivalence classes of measure preserving actions of the group. Afterwards we will consider a special kind of invariant random subgroups, namely the characteristic random subgroups.

Fix a countable group Γ and let $\mathrm{Sub}(\Gamma)$ denote the set of all subgroups of Γ. We can identify each $\Lambda \in \mathrm{Sub}(\Gamma)$ with its characteristic function in $\{0,1\}^{\Gamma}$. It is easily checked that with this identification $\mathrm{Sub}(\Gamma)$ is a closed subset and hence a compact Polish space in the subspace topology. Moreover, if we for each finite $F \subseteq \Gamma$ let

$$N_F^{\Gamma} = \{\Lambda \in \Gamma \mid F \subseteq \Lambda\} \quad \text{and} \quad M_F^{\Gamma} = \{\Lambda \in \Gamma \mid F \cap \Lambda = \emptyset\},$$

then the collection of sets

$$V_{F,K} = N_F^{\Gamma} \cap M_K^{\Gamma},$$

where $F, K \subseteq \Gamma$ are finite, constitutes a basis of clopen sets for this topology.

Now consider the conjugation action $\Gamma \curvearrowright^c \mathrm{Sub}(\Gamma)$ given by $\gamma \cdot^c \Lambda = \gamma\Lambda\gamma^{-1}$. It is easily seen to be continuous and the fixed points are exactly the normal subgroups of Γ.

Definition 5.3.1. Let Γ be a countable group. An *invariant random subgroup* of Γ is a conjugation invariant Borel probability measure on $\mathrm{Sub}(\Gamma)$. We denote by $\mathrm{IRS}(\Gamma)$ the set of all invariant random subgroups of Γ.

For any normal subgroup $\Lambda \leq \Gamma$ the Dirac measure δ_Λ is an invariant random subgroup. An invariant random subgroup can therefore be seen as a random version of a normal subgroup. We also have the following more interesting source of examples.

Example 5.3.2. Let Γ be a countable group and $a \in A(\Gamma, X, \mu)$. Consider the map $\mathrm{stab}_a \colon X \to \mathrm{Sub}(\Gamma)$ given by

$$\mathrm{stab}_a(x) = \{\gamma \in \Gamma \mid \gamma \cdot^a x = x\}.$$

It is straightforward to verify that this map is Borel and that it satisfies that

$$\mathrm{stab}_a(\gamma \cdot^a x) = \gamma\,(\mathrm{stab}_a(x))\,\gamma^{-1} = \gamma \cdot^c \mathrm{stab}_a(x)$$

for all $x \in X$ and $\gamma \in \Gamma$. Therefore we obtain that the pushforward of μ via this map, which we denote by $\mathrm{type}(a)$, is an invariant random subgroup of Γ.

The example above allows us to construct invariant random subgroups by use of measure preserving actions. In fact, by the result below, every invariant random subgroup arises this way. A proof can be found in [2, Proposition 13].

Proposition 5.3.3. *Let Γ be a countable group and $\theta \in \mathrm{IRS}(\Gamma)$. There exists $a \in A(\Gamma, X, \mu)$ such that $\mathrm{type}(a) = \theta$.*

Let Γ be a countable group. We say that an invariant random subgroup $\theta \in \mathrm{IRS}(\Gamma)$ is *ergodic* (resp. *weakly mixing*) if the action $\Gamma \curvearrowright^c (\mathrm{Sub}(\Gamma), \theta)$ is ergodic (resp. weakly mixing). Note that the action $\Gamma \curvearrowright^c (\mathrm{Sub}(\Gamma), \mathrm{type}(a))$ is a factor of $\Gamma \curvearrowright^a (X, \mu)$ for any $a \in A(\Gamma, X, \mu)$. Hence if a is ergodic (resp. weakly mixing), so is $\mathrm{type}(a)$. The converse does not hold in general. For example, $a \in A(\Gamma, X, \mu)$ can be a free non-ergodic action, but $\mathrm{type}(a) = \delta_{\{e_\Gamma\}}$ is weakly mixing.

We will now discuss the surjective map $\mathrm{type}\colon A(\Gamma, X, \mu) \to \mathrm{IRS}(\Gamma)$ given by $a \mapsto \mathrm{type}(a)$. Note that all free measure preserving actions $a \in A(\Gamma, X, \mu)$ satisfy that $\mathrm{type}(a) = \delta_{\{e\}}$. We are therefore mainly interested in the non-free actions.

The next result describes how the map type interacts with the relation of weak equivalence. The first part of the result below is proved in [1, Section 4]. A proof of the second part is found in [9, Proposition 5.1].

Theorem 5.3.4. *Let Γ be a countable group and $a, b \in A(\Gamma, X, \mu)$.*

(1) If $a \simeq b$, then $\mathrm{type}(a) = \mathrm{type}(b)$.

(2) If Γ is amenable and $\mathrm{type}(a) = \mathrm{type}(b)$, then $a \simeq b$.

The previous theorem ensures that the map $\underset{\sim}{\mathrm{type}}\colon \underset{\sim}{A}(\Gamma, X, \mu) \to \mathrm{IRS}(\Gamma)$ given by $\underset{\sim}{\mathrm{type}}(\underset{\sim}{a}) = \mathrm{type}(a)$ is well-defined and, moreover, that $\underset{\sim}{\mathrm{type}}$ is a bijection when Γ is amenable. This clearly fails in the case of non-amenable groups, as these have several weakly inequivalent free actions (see [32, Remark 4.3]).

Now we move on to consider the natural topology on $\mathrm{IRS}(\Gamma)$ for a fixed countable group Γ. First consider the space $P(\mathrm{Sub}(\Gamma))$ consisting of all Borel probability measures on $\mathrm{Sub}(\Gamma)$. We have that $P(\mathrm{Sub}(\Gamma))$ is a compact Polish space in the topology generated by the maps $\mu \mapsto \int f d\mu$, where $f\colon \mathrm{Sub}(\Gamma) \to \mathbb{R}$ is a continuous function (see [19, Theorem 17.22]).

We will next obtain a useful description of this topology. The collection of positive basic sets

$$N_F^\Gamma = \{\Lambda \in \mathrm{Sub}(\Gamma) \mid F \subseteq \Lambda\},$$

where $F \subseteq \Gamma$ finite, constitutes a family of clopen subsets which generates the Borel structure of $\mathrm{Sub}(\Gamma)$ and is closed under finite intersections. Therefore it follows from the π-λ Theorem (see [19, Theorem 10.1]) that if $\mu, \nu \in P(\mathrm{Sub}(\Gamma))$ satisfy $\mu(N_F^\Gamma) = \nu(N_F^\Gamma)$ for all finite $F \subseteq \Gamma$, then $\mu = \nu$. This fact together with the compactness of $P(\mathrm{Sub}(\Gamma)$ yields the following useful lemma.

Lemma 5.3.5. *Let Γ be a countable group and $(\mu_n)_n, \mu \in P(\mathrm{Sub}(\Gamma))$. Then $\mu_n \to \mu$ as $n \to \infty$ if and only if $\mu_n(N_F^\Gamma) \to \mu(N_F^\Gamma)$ as $n \to \infty$ for all finite $F \subseteq \Gamma$.*

Proof. The right implication follows directly from The Portmanteau Theorem (see [19, Theorem 17.20]). For the left implication, assume that $\mu_n \nrightarrow \mu$ as $n \to \infty$. By compactness, there is a subsequence $(\mu_{n_i})_i$ and $\nu \in P(\mathrm{Sub}(\Gamma))$ such that $\nu \neq \mu$ and $\mu_{n_i} \to \nu$ as $i \to \infty$. Since $\nu \neq \mu$, there exists finite $F \subseteq \Gamma$ such that $\nu(N_F^\Gamma) \neq \mu(N_F^\Gamma)$. Hence $\mu_{n_i}(N_F^\Gamma) \nrightarrow \mu(N_F^\Gamma)$ when $n \to \infty$, as wanted. ☐

Next let Γ be a countable group and consider the action $\Gamma \curvearrowright P(\mathrm{Sub}(\Gamma))$ given by $(\gamma \cdot \mu)(B) = \mu(\gamma^{-1} \cdot^c B)$ for all Borel $B \subseteq \mathrm{Sub}(\Gamma)$. This action is clearly continuous and hence we obtain that $\mathrm{IRS}(\Gamma)$ is closed in $P(\mathrm{Sub}(\Gamma))$. Therefore $\mathrm{IRS}(\Gamma)$ is a compact Polish space in the subspace topology.

It turns out that the topology on $\mathrm{IRS}(\Gamma)$ behaves nicely with respect to the map $\underset{\sim}{\mathrm{type}}\colon \underset{\sim}{A}(\Gamma, X, \mu) \to \mathrm{IRS}(\Gamma)$. A proof of the following theorem can be found in [32, Theorem 5.2].

Theorem 5.3.6. *Let Γ be a countable group. The map $\underset{\sim}{\mathrm{type}}\colon \underset{\sim}{A}(\Gamma, X, \mu) \to \mathrm{IRS}(\Gamma)$ is continuous. In particular, it is a homeomorphism if Γ is amenable.*

If we let $\underset{\sim}{\mathrm{FR}}(\Gamma, X, \mu) = \underset{\sim}{\mathrm{type}}^{-1}(\delta_{\{e\}})$, we obtain that $\underset{\sim}{\mathrm{FR}}(\Gamma, X, \mu)$ is a closed subspace of $\underset{\sim}{A}(\Gamma, X, \mu)$. Note that $\underset{\sim}{\mathrm{FR}}(\Gamma, X, \mu)$ is the space consisting of all the weak equivalence classes of the free actions of Γ.

5.3.1 Characteristic random subgroups

In the last part of this section we will introduce the notion of a characteristic random subgroup, which is a particularly neat invariant random subgroup.

Let Γ be a countable group and let $\mathrm{Aut}(\Gamma)$ denote the group of automorphisms of Γ. We have a natural action $\mathrm{Aut}(\Gamma) \curvearrowright \mathrm{Sub}(\Gamma)$ given by $\varphi \cdot \Lambda = \varphi(\Lambda)$. Next let $\mathrm{Inn}(\Gamma) \subseteq \mathrm{Aut}(\Gamma)$ denote the subgroup of inner automorphisms, i.e., the automorphisms given by $\gamma_0 \mapsto \gamma\gamma_0\gamma^{-1}$ for some $\gamma \in \Gamma$. Then an invariant random subgroup is a Borel probability measure on $\mathrm{Sub}(\Gamma)$ which is invariant under the action of $\mathrm{Inn}(\Gamma) \curvearrowright \mathrm{Sub}(\Gamma)$. We will now consider the Borel probability measures that are invariant under the action of the full automorphism group.

Definition 5.3.7. Let Γ be a countable group. A *characteristic random subgroup* of Γ is an automorphism invariant Borel probability measure on $\mathrm{Sub}(\Gamma)$. We denote by $\mathrm{CRS}(\Gamma)$ the set of all characteristic random subgroups of Γ.

For any countable group Γ we have $\mathrm{CRS}(\Gamma) \subseteq \mathrm{IRS}(\Gamma)$. Since $\mathrm{Aut}(\Gamma)$ acts on $\mathrm{Sub}(\Gamma)$ by homeomorphisms, we also obtain that $\mathrm{CRS}(\Gamma)$ is closed in $P(\mathrm{Sub}(\Gamma))$ and hence compact Polish in the subspace topology.

We may naturally view $\mathrm{Aut}(\Gamma)$ as a subspace of Γ^Γ. It is straightforward to check that $\mathrm{Aut}(\Gamma) \subseteq \Gamma^\Gamma$ is G_δ and that $\mathrm{Aut}(\Gamma)$ is a topological group when equipped with the subspace topology. Hence $\mathrm{Aut}(\Gamma)$ is a Polish group and it is easily seen that the action $\mathrm{Aut}(\Gamma) \curvearrowright \mathrm{Sub}(\Gamma)$ is continuous. By use of Lemma 5.3.5, we also obtain the following result.

Proposition 5.3.8. *Let Γ be a countable group. Then $\mathrm{Aut}(\Gamma) \curvearrowright^\beta P(\mathrm{Sub}(\Gamma))$ given by*

$$(\varphi \cdot^\beta \mu)(B) = \mu(\varphi^{-1}(B))$$

for all $B \subseteq X$ Borel is continuous.

Proof. Assume $(\varphi_i)_i, \varphi \in \mathrm{Aut}(\Gamma)$ and $(\mu_i)_i, \mu \in P(\mathrm{Sub}(\Gamma))$ satisfy that $\varphi_i \to \varphi$ and $\mu_i \to \mu$ as $i \to \infty$. Let $F \subseteq \Gamma$ be finite. Since $\varphi_i \to \varphi$ as $i \to \infty$, there is $N \in \omega$ such that $\varphi_i^{-1}(F) = \varphi^{-1}(F)$ for all $i \geq N$. Therefore, as

$$(\varphi_i \cdot^\beta \mu_i)(N_F^\Gamma) = \mu_i(N^\Gamma_{\varphi_i^{-1}(F)}) \qquad \text{and} \qquad (\varphi \cdot^\beta \mu)(N_F^\Gamma) = \mu(N^\Gamma_{\varphi^{-1}(F)}),$$

we must have

$$(\varphi_i \cdot^\beta \mu_i)(N_F^\Gamma) \to (\varphi \cdot^\beta \mu)(N_F^\Gamma)$$

when $i \to \infty$. Hence, by Lemma 5.3.5, we conclude that $\mathrm{Aut}(\Gamma) \curvearrowright^\beta P(\mathrm{Sub}(\Gamma))$ is continuous. $\square$

The following corollary is a direct consequence of the previous proposition.

Corollary 5.3.9. *Let Γ be a countable group and let $\Delta \leq \mathrm{Aut}(\Gamma)$ be a dense subgroup. If $\theta \in P(\mathrm{Sub}(\Gamma))$ is invariant under the action of Δ, then $\theta \in \mathrm{CRS}(\Gamma)$.*

Proof. For each $\theta \in P(\mathrm{Sub}(\Gamma))$ it follows by Proposition 5.3.8 that the set

$$\left\{ \varphi \in \mathrm{Aut}(\Gamma) \mid \varphi \cdot^\beta \theta = \theta \right\}$$

is closed in $\mathrm{Aut}(\Gamma)$. $\square$

In other words, in order to obtain characteristic random subgroups, it is enough to ensure that a Borel probability measure on $\mathrm{Sub}(\Gamma)$ is invariant under the action of some countable dense subgroup of $\mathrm{Aut}(\Gamma)$.

Chapter 6

Co-induction of invariant random subgroups

The main goal of this chapter is to develop a co-induction operation for invariant random subgroups. To be more specific, if $\Gamma \leq \Delta$ are countable groups, then there is a well-known co-induction operation $\mathrm{cind}_\Gamma^\Delta \colon A(\Gamma, X, \mu) \to A(\Delta, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma})$. It turns out that this operation descends to a well-defined operation

$$\underset{\sim}{\mathrm{cind}}_\Gamma^\Delta \colon \underset{\sim}{A}(\Gamma, X, \mu) \to \underset{\sim}{A}(\Delta, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma}).$$

We will then construct a co-induction operation $\mathrm{CIND}_\Gamma^\Delta \colon \mathrm{IRS}(\Gamma) \to \mathrm{IRS}(\Delta)$ such that the diagram

$$\begin{tikzcd}
\underset{\sim}{A}(\Delta, X, \mu) \arrow[r, "\underset{\sim}{\mathrm{type}}"] & \mathrm{IRS}(\Delta) \\
\underset{\sim}{A}(\Gamma, X, \mu) \arrow[r, "\underset{\sim}{\mathrm{type}}"'] \arrow[u, "\underset{\sim}{\mathrm{cind}}_\Gamma^\Delta"] & \mathrm{IRS}(\Gamma) \arrow[u, "\mathrm{CIND}_\Gamma^\Delta"']
\end{tikzcd}$$

commutes. This operation is going to be the foundation of the rest of this part of the thesis, where we will apply it to obtain various families of well-behaved invariant random subgroups.

In the first section we will examine the classical co-induction operation for measure preserving group actions. We will give the definition, and present various properties of this operation that will become useful in the following. In the second section we will obtain the co-induction operation for invariant random subgroups that we described above. The third section is devoted to the study of continuity properties of both $\underset{\sim}{\mathrm{cind}}_\Gamma^\Delta$ and $\mathrm{CIND}_\Gamma^\Delta$ for pairs of countable groups $\Gamma \leq \Delta$. We will obtain a complete characterization of when $\mathrm{CIND}_\Gamma^\Delta$ is continuous. Moreover, we will see that neither $\underset{\sim}{\mathrm{cind}}_\Gamma^\Delta$ nor $\mathrm{CIND}_\Gamma^\Delta$ is continuous when $[\Delta : \Gamma] = \infty$. In the fourth and final section we investigate properties of the co-induced invariant random subgroups. We will focus on the case where $[\Delta : \Gamma] = \infty$. In this case, the co-induced invariant random subgroups will always be weakly mixing and we can characterize when they will be non-atomic. As a by-product we also obtain a complete characterization of when a co-induced action is free.

Unless specifically stated otherwise, all results in this chapter have been obtained in joint work with Alexander S. Kechris and can also be found in [22].

6.1 Co-induction of actions

We will here review the co-induction operation for actions, which is an operation that transforms an action of a subgroup into an action of the bigger group. We will consider various properties of this operation. In particular, we will establish that it descends to an operation on the weak equivalence classes of measure preserving group actions.

Let $\Gamma \leq \Delta$ be countable groups. Fix a transversal $T \subseteq \Delta$ for the left cosets in Δ/Γ, that is, a set of representatives for the left cosets. We have a natural action $\sigma_T \colon \Delta \times T \to T$ given by

$$\sigma_T(\delta, t) = \tilde{t} \iff \tilde{t}\Gamma = \delta t\Gamma,$$

and a cocycle $\rho_T \colon \Delta \times T \to \Gamma$ for this action given by

$$\rho_T(\delta, t) = \sigma_T(\delta, t)^{-1}\delta t.$$

Now for $a \in A(\Gamma, X, \mu)$ we obtain the co-induced action $a_T \in A(\Delta, X^T, \mu^T)$ given by

$$(\delta \cdot^{a_T} f)(t) = \rho_T(\delta^{-1}, t)^{-1} \cdot^{a} f(\sigma_T(\delta^{-1}, t))$$

for all $t \in T$. By considering the natural bijection $\iota_S \colon \Delta/\Gamma \to T$, we may view the co-induced action $a_T \in A(\Delta, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma})$ by letting

$$(\delta \cdot^{a_T} f)(\delta_0\Gamma) = \rho(\delta^{-1}, \iota_T(\delta_0\Gamma))^{-1} \cdot^{a} f(\delta^{-1}\delta_0\Gamma)$$

for all $\delta_0 \in \Delta$.

We will now prove that the co-induced action is independent (up to isometry) of the choice of transversal.

Proposition 6.1.1. *Let $\Gamma \leq \Delta$ be countable groups. If $T, S \subseteq \Delta$ are transversals for the left cosets in Δ/Γ and $a \in A(\Gamma, X, \mu)$, then the actions $a_T, a_S \in A(\Delta, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma})$ are isomorphic.*

Proof. First consider the map $\iota\colon \Delta/\Gamma \to \Gamma$ given by

$$\iota(\delta_0\Gamma) = \iota_S(\delta_0\Gamma)^{-1}\iota_T(\delta_0\Gamma)$$

and note that

$$\rho_S(\delta, \iota_S(\delta_0\Gamma)) = \iota(\delta\delta_0\Gamma)\rho_T(\delta, \iota_T(\delta_0\Gamma))\iota(\delta_0\Gamma)^{-1}$$

for all $\delta, \delta_0 \in \Delta$. So $\varphi\colon X^{\Delta/\Gamma} \to X^{\Delta/\Gamma}$ given by $\varphi(f)(\delta_0\Gamma) = \iota(\delta_0\Gamma) \cdot^a f(\delta_0\Gamma)$ will satisfy

$$\varphi(\delta \cdot^{a_T} f)(\delta_0\Gamma) = \delta \cdot^{a_S} \varphi(f)(\delta_0\Gamma)$$

for all $f \in X^{\Delta/\Gamma}$ and $\delta, \delta_0 \in \Delta$. It is therefore easily seen that φ is an isomorphism of a_T and a_S. $\square$

In light of Proposition 6.1.1, we will for each $a \in A(\Gamma, X, \mu)$ let $\operatorname{cind}_\Gamma^\Delta(a) \in A(\Delta, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma})$ denote the co-induced action with respect to some transversal. Note that a is a factor of $\operatorname{cind}_\Gamma^\Delta(a)_{|\Gamma}$ via the map $f \mapsto f(\Gamma)$ from $X^{\Delta/\Gamma}$ to X, hence $a \preceq \operatorname{cind}_\Gamma^\Delta(a)_{|\Gamma}$. We also have the following "chain rule".

Proposition 6.1.2. *Let $\Lambda \leq \Gamma \leq \Delta$ be countable groups and $a \in A(\Lambda, X, \mu)$. The actions $\operatorname{cind}_\Gamma^\Delta\left(\operatorname{cind}_\Lambda^\Gamma(a)\right)$ and $\operatorname{cind}_\Lambda^\Delta(a)$ are isomorphic.*

Proof. Let $T \subseteq \Delta$ and $S \subseteq \Gamma$ be transversals for the left cosets in Δ/Γ and Γ/Λ, respectively. Then it is easily seen that $TS = \{ts \mid t \in T, s \in S\}$ is a transversal for the left cosets in Δ/Λ. One may check that

$$\sigma_{TS}(\delta, ts) = \sigma_T(\delta, t)\sigma_S(\rho_T(\delta, t), s)$$

and hence that

$$\rho_{TS}(\delta, ts) = \rho_S(\rho_T(\delta, t), s)$$

for all $s \in S, t \in T$ and $\delta \in \Delta$.

Next consider the map $\varphi\colon (X^S)^T \to X^{TS}$ given by $\varphi(f)(ts) = (f(t))(s)$ for all $t \in T$ and $s \in S$. Since

$$\begin{aligned}
\varphi(\delta \cdot^{(a_S)_T} f)(ts) &= \Big((\delta \cdot^{(a_S)_T} f)(t)\Big)(s) \\
&= \left(\rho_T(\delta^{-1}, t)^{-1} \cdot^{a_S} f(\sigma_T(\delta^{-1}, t))\right)(s) \\
&= \rho_S(\rho_T(\delta^{-1}, t), s)^{-1} \cdot^a \left(f(\sigma_T(\delta^{-1}, t))\right)(\sigma_S(\rho_T(\delta^{-1}, t), s)) \\
&= \rho_{TS}(\delta^{-1}, ts)^{-1} \cdot^a \varphi(f)(\sigma_{TS}(\delta^{-1}, ts)) \\
&= (\delta \cdot^{a_{TS}} \varphi(f))(ts)
\end{aligned}$$

for all $f \in (X^S)^T$ and $\delta \in \Delta$, we conclude that $\mathrm{cind}_\Gamma^\Delta\left(\mathrm{cind}_\Lambda^\Gamma(a)\right)$ and $\mathrm{cind}_\Lambda^\Delta(a)$ are isomorphic, as wanted. $\square$

The next result, due to Ioana, characterizes when the co-induced action is weakly mixing. For a proof see [18, Lemma 2.2.].

Proposition 6.1.3 (Ioana). *Let $\Gamma \leq \Delta$ be countable groups and $a \in A(\Gamma, X, \mu)$.*

(1) If $[\Delta : \Gamma] = \infty$, then $\mathrm{cind}_\Gamma^\Delta(a)$ is weakly mixing.

(2) If $[\Delta : \Gamma] < \infty$, then $\mathrm{cind}_\Gamma^\Delta(a)$ is weakly mixing if and only if a is weakly mixing.

(3) The action $\mathrm{cind}_\Gamma^\Delta(a)_{|\Gamma}$ is weakly mixing if and only if a is weakly mixing.

Finally, we have the following result, due to Kechris, which ensures that the co-induction operation is invariant under weak equivalence. The proof can be found in [21, Proposition A.1].

Proposition 6.1.4 (Kechris). *Let $\Gamma \leq \Delta$ be countable groups and $a, b \in A(\Gamma, X, \mu)$. If $a \preceq b$, then we have $\mathrm{cind}_\Gamma^\Delta(a) \preceq \mathrm{cind}_\Gamma^\Delta(b)$.*

Since any pair of isomorphic actions are weakly equivalent, it follows by Proposition 6.1.1 and Proposition 6.1.4 that the map

$$\underset{\sim}{\mathrm{cind}}_\Gamma^\Delta \colon \underset{\sim}{A}(\Gamma, X, \mu) \to \underset{\sim}{A}(\Delta, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma})$$

given by $\underset{\sim}{\mathrm{cind}}_\Gamma^\Delta(\underset{\sim}{a}) = \underset{\sim}{\mathrm{cind}_\Gamma^\Delta(a)}$ is well-defined. In Section 6.3 we will address the question of continuity of this map.

6.2 A co-induction operation on invariant random subgroups

In this section we will use the connection between measure preserving actions and invariant random subgroups to obtain a co-induction operation for invariant random subgroups.

In the following, whenever we have countable groups $\Gamma \leq \Delta$, the *normal core* of Γ in Δ is the subgroup $\mathrm{core}_\Delta(\Gamma) \leq \Gamma$ given by

$$\mathrm{core}_\Delta(\Gamma) = \bigcap_{\delta \in \Delta} \delta \Gamma \delta^{-1}.$$

Clearly, $\mathrm{core}_\Delta(\Gamma)$ is normal in Δ and it is straightforward to prove that $\mathrm{core}_\Delta(\Gamma) = \bigcap_{t \in T} t \Gamma t^{-1}$ for any transversal $T \subseteq \Delta$ for the left cosets in Δ/Γ.

Theorem 6.2.1. *Let $\Gamma \leq \Delta$ be countable groups and $T \subseteq \Delta$ a transversal for the left cosets in Δ/Γ. There exists a co-induction operation*

$$\mathrm{CIND}_\Gamma^\Delta \colon \mathrm{IRS}(\Gamma) \to \mathrm{IRS}(\Delta)$$

such that

$$\mathrm{CIND}_\Gamma^\Delta(\theta)(N_F^\Delta) = \begin{cases} 0 & \text{if} \quad F \not\subseteq \mathrm{core}_\Delta(\Gamma) \\ \prod_{t \in T} \theta(N_{t^{-1}Ft}^\Gamma) & \text{if} \quad F \subseteq \mathrm{core}_\Delta(\Gamma) \end{cases}$$

and $\mathrm{CIND}_\Gamma^\Delta(\mathrm{type}(a)) = \mathrm{type}(\mathrm{cind}_\Gamma^\Delta(a))$ *for all* $a \in A(\Gamma, X, \mu)$.

Proof. Let $\theta \in \mathrm{IRS}(\Gamma)$ and fix $a \in A(\Gamma, X, \mu)$ such that $\mathrm{type}(a) = \theta$. We will show that $\mathrm{type}(\mathrm{cind}_\Gamma^\Delta(a)) \in \mathrm{IRS}(\Delta)$ satisfies

$$\mathrm{type}(\mathrm{cind}_\Gamma^\Delta(a))(N_F^\Delta) = \begin{cases} 0 & \text{if} \quad F \not\subseteq \mathrm{core}_\Delta(\Gamma) \\ \prod_{t \in T} \theta(N_{t^{-1}Ft}^\Gamma) & \text{if} \quad F \subseteq \mathrm{core}_\Delta(\Gamma) \end{cases}.$$

First we will prove the following claim.

Claim: We have that $\mathrm{type}(\mathrm{cind}_\Gamma^\Delta(a))(\mathrm{Sub}\,(\mathrm{core}_\Delta(\Gamma))) = 1$ and

$$\mathrm{type}(\mathrm{cind}_\Gamma^\Delta(a))_{|\,\mathrm{Sub}(\mathrm{core}_\Delta(\Gamma))} = \mathrm{type}(\mathrm{cind}_\Gamma^\Delta(a)_{|\,\mathrm{core}_\Delta(\Gamma)}).$$

Proof of Claim: Let $\sigma \colon \Delta \times T \to T$ and $\rho \colon \Delta \times T \to \Gamma$ be given by

$$\sigma(\delta, t) = \tilde{t} \iff \delta t \Gamma = \tilde{t}\Gamma$$

and

$$\rho(\delta, t) = \sigma(\delta, t)^{-1}\delta t.$$

Note that $\sigma(\delta, \cdot) \colon T \to T$ is the identity if and only if $\delta \in \mathrm{core}_\Delta(\Gamma)$ for all $\delta \in \Delta$. Now recall that for each $a \in A(\Gamma, X, \mu)$ the co-induced action $a_T = \mathrm{cind}_\Gamma^\Delta(a)$ is given by

$$(\delta \cdot^{a_T} f)(t) = \rho(\delta^{-1}, t)^{-1} \cdot^a f(\sigma(\delta^{-1}, t))$$

for all $t \in T$, $\delta \in \Delta$ and $f \in X^T$. Hence if $\delta \cdot^{a_T} f = f$, then

$$\big(f(t), f(\sigma(\delta^{-1}, t))\big) \in E_a$$

for all $t \in T$. Here E_a denotes the orbit equivalence relation induced by the action a. This implies that

$$\mu^T(\mathrm{Fix}_{a_T}(\delta)) = \mu^T\left(\{f \in X^T \mid \delta \cdot^{a_T} f = f\}\right) = 0$$

for all $\delta \in \Delta \setminus \mathrm{core}_\Delta(\Gamma)$. Therefore, since

$$X^T \setminus \mathrm{stab}_{a_T}^{-1}(\mathrm{Sub}(\mathrm{core}_\Delta(\Gamma))) = \bigcup_{\delta \in \Delta \setminus \mathrm{core}_\Delta(\Gamma)} \mathrm{Fix}_{a_T}(\delta),$$

we obtain $\mathrm{type}(\mathrm{cind}_\Gamma^\Delta(a))(\mathrm{Sub}(\mathrm{core}_\Delta(\Gamma))) = 1$. Moreover,

$$\mathrm{stab}_{a_T}(f) \in \mathrm{Sub}(\mathrm{core}_\Delta(\Gamma)) \iff \mathrm{stab}_{a_T}(f) = \mathrm{stab}_{a_T|\,\mathrm{core}_\Delta(\Gamma)}(f)$$

for all $f \in X^T$. Hence $\mathrm{type}(\mathrm{cind}_\Gamma^\Delta(a))_{|\,\mathrm{Sub}(\mathrm{core}_\Delta(\Gamma))} = \mathrm{type}(\mathrm{cind}_\Gamma^\Delta(a)_{|\,\mathrm{core}_\Delta(\Gamma)})$, as wanted. $\diamond$

Next note that $\mathrm{cind}_\Gamma^\Delta(a)_{|\,\mathrm{core}_\Delta(\Gamma)} = \Pi_{t\in T} a_t$, where $a_t \in A(\mathrm{core}_\Delta(\Gamma), X, \mu)$ is given by $\gamma \cdot^{a_t} x = t^{-1}\gamma t \cdot^a x$ for each $t \in T$. Hence it follows from the claim that

$$\mathrm{type}(\mathrm{cind}_\Gamma^\Delta(a))(N_F^\Delta) = \begin{cases} 0 & \text{if} \quad F \nsubseteq \mathrm{core}_\Delta(\Gamma) \\ \mathrm{type}(\prod_{t\in T} a_t)(N_F^{\mathrm{core}_\Delta(\Gamma)}) & \text{if} \quad F \subseteq \mathrm{core}_\Delta(\Gamma) \end{cases}.$$

Therefore, as

$$\mathrm{type}(\prod_{t\in T} a_t)(N_F^{\mathrm{core}_\Delta(\Gamma)}) = \prod_{t\in T} \mathrm{type}(a)(N_{t^{-1}Ft}^\Gamma) = \prod_{t\in T} \theta(N_{t^{-1}Ft}^\Gamma),$$

the conclusion follows. $\square$

Note that it follows by the invariance of θ that if $T, S \subseteq \Delta$ are both transversals for the left cosets Δ/Γ, then

$$\prod_{t\in T} \theta(N_{t^{-1}Ft}) = \prod_{s\in S} \theta(N_{s^{-1}Fs}).$$

So $\mathrm{CIND}_\Gamma^\Delta(\theta)$ does not depend on the chosen transversal.

An easy consequence of the previous theorem is that the type of the co-induced action only depends on the type of the action.

Corollary 6.2.2. *Let $\Gamma \leq \Delta$ be countable groups and $a, b \in \mathrm{A}(\Gamma, X, \mu)$. If $\mathrm{type}(a) = \mathrm{type}(b)$, then $\mathrm{type}(\mathrm{cind}_\Gamma^\Delta(a)) = \mathrm{type}(\mathrm{cind}_\Gamma^\Delta(b))$.*

Remark 6.2.3. If $\Gamma \leq \Delta$ are countable groups and $\theta \in \mathrm{IRS}(\Gamma)$, we may also view $\mathrm{CIND}_\Gamma^\Delta(\theta)$ as an element of $\mathrm{IRS}(\Gamma)$, since $\mathrm{CIND}_\Gamma^\Delta(\theta)$ is supported on $\mathrm{core}_\Delta(\Gamma) \leq \Gamma$. Moreover, any group Γ is contained in a countable group Δ in such a way that for densely many $\varphi \in \mathrm{Aut}(\Gamma)$ there is $\delta \in \Delta$ such that $\varphi(\gamma) = \delta\gamma\delta^{-1}$ for all $\gamma \in \Gamma$. So we can use the co-induction operation

$\mathrm{CIND}_\Gamma^\Delta$ to transform invariant random subgroups of Γ into characteristic random subgroups of Γ. In the case where Γ is centerless, we may simply identify Γ with the subgroup of inner automorphisms in $\mathrm{Aut}(\Gamma)$. We will show how this strategy can be applied to construct characteristic random subgroups of $\mathbb{F}_2$ in Chapter 8.

Let $\Gamma \leq \Delta$ be countable groups. For each $\theta \in \mathrm{IRS}(\Gamma)$ we can also obtain $\mathrm{CIND}_\Gamma^\Delta(\theta) \in \mathrm{IRS}(\Delta)$ in the following alternative way: Fix a transversal T for the left cosets in Δ/Γ and $\theta \in \mathrm{IRS}(\Gamma)$. View θ as a probability Borel measure on $\mathrm{Sub}(\Delta)$ and for each $t \in T$ define the Borel probability measure θ_t on $\mathrm{Sub}(\Delta)$ to be the pushforward of θ through the map $\Lambda \mapsto t\Lambda t^{-1}$ from $\mathrm{Sub}(\Delta)$ to $\mathrm{Sub}(\Delta)$. Then

$$\theta_t(N_F^\Delta) = \theta(N_{t^{-1}Ft}^\Delta)$$

for all finite $F \subseteq \Delta$ and

$$\theta_\infty = \prod_{t\in T} \theta_t$$

is a Borel probability measure on $\mathrm{Sub}(\Delta)^T$. Moreover, we have an action $\Delta \curvearrowright^a \mathrm{Sub}(\Delta)^T$ given by

$$\delta \cdot^a (\Lambda_t)_{t\in T} = (\delta \Lambda_{\sigma_T(\delta^{-1},t)} \delta^{-1})_{t\in T},$$

where $\sigma_T \colon \Delta \times T \to T$ is given by $\sigma_T(\delta, t) = \tilde{t}$ if and only if $\delta t \Gamma = \tilde{t}\Gamma$. Next note that θ_∞ is invariant under a and that $I \colon \mathrm{Sub}(\Delta)^T \to \mathrm{Sub}(\Delta)$ given by

$$I((\Lambda_t)_{t\in T}) = \bigcap_{t\in T} \Lambda_t$$

is a Borel map. In fact, we have

$$I\left(\delta \cdot^a (\Lambda_t)_{t\in T}\right) = \bigcap_{t\in T} \delta \Lambda_t \delta^{-1} = \delta I\left((\Lambda_t)_{t\in T}\right) \delta^{-1}$$

for all $(\Lambda_t)_{t\in T} \in \mathrm{Sub}(\Delta)^T$ and $\delta \in \Delta$. So if we let θ^* denote the pushforward of θ_∞ through I, we obtain that $\theta^* \in \mathrm{IRS}(\Delta)$. To see that $\theta^* = \mathrm{CIND}_\Gamma^\Delta(\theta)$, note that

$$\theta^*(N_F^\Delta) = \prod_{t\in T} \theta_t(N_F^\Delta) = \begin{cases} 0 & \text{if} \quad F \not\subseteq \mathrm{core}_\Delta(\Gamma) \\ \prod_{t\in T} \theta(N_{t^{-1}Ft}^\Gamma) & \text{if} \quad F \subseteq \mathrm{core}_\Delta(\Gamma) \end{cases}$$

for all $F \subseteq \Delta$ finite.

Remark 6.2.4. The action $\Delta \curvearrowright^a (\mathrm{Sub}(\Delta)^T, \theta_\infty)$ is weakly mixing when $[\Delta : \Gamma] = \infty$. Indeed, let $b \in A(\Gamma, X, \mu)$ satisfy that $\mathrm{type}(b) = \theta$ and consider the map $F \colon X^{\Delta/\Gamma} \to \mathrm{Sub}(\Delta)^T$ given by

$$F((x_{t\Gamma})_{t\in T}) = (t\ \mathrm{stab}_b(x_{t\Gamma})\ t^{-1})_{t\in T}.$$

It is straightforward to check that F is a factor map from $\mathrm{cind}_{\Gamma}^{\Delta}(b)$ to a. Therefore, since $\mathrm{cind}_{\Gamma}^{\Delta}(b)$ is weakly mixing when $[\Delta : \Gamma] = \infty$, we obtain that the same holds for a. From this we may also conclude that $\mathrm{CIND}_{\Gamma}^{\Delta}(\theta)$ is weakly mixing when $[\Delta : \Gamma] = \infty$. We will also provide a short proof of this fact in Section 6.4 without appealing to this alternative construction.

Remark 6.2.5. One can also define $J\colon \mathrm{Sub}(\Delta)^T \to \mathrm{Sub}(\Delta)$ by

$$J((\Lambda_t)_{t\in T}) = \langle \Lambda_t \mid t \in T\rangle,$$

where $\langle \Lambda_t \mid t \in T\rangle$ denotes the subgroup generated by $\bigcup_{t\in T} \Lambda_t$. Then

$$J\left(\delta \cdot^a (\Lambda_t)_{t\in T}\right) = \delta J\left((\Lambda_t)_{t\in T}\right)\delta^{-1}$$

for all $(\Lambda_t)_{t\in T} \in \mathrm{Sub}(\Delta)^T$ and $\delta \in \Delta$. So if we let θ^{**} denote the pushforward of θ_∞ by J, then $\theta^{**} \in \mathrm{IRS}(\Delta)$ and θ^{**} is weakly mixing when $[\Delta\colon \Gamma] = \infty$. We will discuss this operation a bit further in Section 9.1.

6.3 Continuity of co-induction

In this section we will consider continuity properties of the co-induction operations $\underset{\sim}{\mathrm{cind}}_{\Gamma}^{\Delta}\colon \underset{\sim}{A}(\Gamma, X, \mu) \to \underset{\sim}{A}(\Delta, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma})$ and $\mathrm{CIND}_{\Gamma}^{\Delta}\colon \mathrm{IRS}(\Gamma) \to \mathrm{IRS}(\Delta)$ for pairs of countable groups $\Gamma \leq \Delta$.

On the level of invariant random subgroups, we have the following complete characterization of when the co-induction operation is continuous.

Proposition 6.3.1. *Let* $\Gamma \leq \Delta$ *be countable groups. The co-induction operation* $\mathrm{CIND}_{\Gamma}^{\Delta}\colon \mathrm{IRS}(\Gamma) \to \mathrm{IRS}(\Delta)$ *is continuous if and only if either* $[\Delta\colon \Gamma] < \infty$ *or* $\mathrm{core}_{\Delta}(\Gamma) = \{e_\Gamma\}$.

Proof. It is easily seen that if $\mathrm{core}_{\Delta}(\Gamma) = \{e_\Gamma\}$, then $\mathrm{CIND}_{\Gamma}^{\Delta}(\theta) = \delta_{\{e_\Gamma\}}$ for any $\theta \in \mathrm{IRS}(\Gamma)$. Therefore in this case, the co-induction operation is continuous. If $[\Delta : \Gamma] < \infty$, then the operation is continuous because the product in the description of $\mathrm{CIND}_{\Gamma}^{\Delta}$ given in Theorem 6.2.1 is finite.

Conversely, assume $[\Delta : \Gamma] = \infty$ and $\mathrm{core}_{\Delta}(\Gamma) \neq \{e_\Gamma\}$. For each $n \in \omega$ let $\theta_n = 2^{-n}\delta_{\{e\}} + (1 - 2^{-n})\delta_\Gamma$ and note that $\theta_n \to \delta_\Gamma$ as $n \to \infty$ in $\mathrm{IRS}(\Gamma)$. However,

$$\mathrm{CIND}_{\Gamma}^{\Delta}(\theta_n)(N_F^{\Delta}) = \prod_{t\in T} \theta_n(N_{t^{-1}Ft}^{\Gamma}) = 0$$

for any $\{e_\Gamma\} \subsetneq F \subseteq \mathrm{core}_{\Delta}(\Gamma)$ finite, while

$$\mathrm{CIND}_{\Gamma}^{\Delta}(\theta)(N_F^{\Delta}) = \prod_{t\in T} \delta_\Gamma(N_{t^{-1}Ft}^{\Gamma}) = 1$$

for all $F \subseteq \mathrm{core}_\Delta(\Gamma)$ finite. Hence in this case, $\mathrm{CIND}_\Gamma^\Delta$ is not continuous. $\square$

Next, since $\underset{\sim}{\mathrm{type}}\colon \underset{\sim}{A}(\Gamma, X, \mu) \to \mathrm{IRS}(\Gamma)$ is a homeomorphism when Γ is a countable amenable group, we obtain the following corollary.

Corollary 6.3.2. *Let $\Gamma \leq \Delta$ be countable groups and assume that Δ is amenable. The co-induction operation $\underset{\sim}{\mathrm{cind}}{}_\Gamma^\Delta\colon \underset{\sim}{A}(\Gamma, X, \mu) \to \underset{\sim}{A}(\Delta, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma})$ is continuous if and only if either $[\Delta : \Gamma] < \infty$ or $\mathrm{core}_\Delta(\Gamma) = \{e\}$.*

Next we will prove that the right implication of Corollary 6.3.2 holds in general. Below we will for a countable group Γ let $i_\Gamma \in A(\Gamma, X, \mu)$ denote the trivial action, that is, the action given by $\gamma \cdot^{i_\Gamma} x = x$.

Proposition 6.3.3. *Let $\Gamma \leq \Delta$ be countable groups. If $[\Delta : \Gamma] = \infty$ and $\mathrm{core}_\Delta(\Gamma) \neq \{e\}$, then $\underset{\sim}{\mathrm{cind}}{}_\Gamma^\Delta\colon \underset{\sim}{A}(\Gamma, X, \mu) \to \underset{\sim}{A}(\Delta, X, \mu)$ is not continuous.*

Proof. Consider a sequence of actions $(a_n)_n \in A(\Gamma, X, \mu)$ for which there exists a sequence of Borel sets $(B_n)_n \subseteq X$ such that $\mu(B_n) = 2^{-n}$, $a_{n|\Gamma \times B_n}$ is free and $a_{n|\Gamma \times (X \setminus B_n)}$ is trivial for all $n \in \omega$. Then, since

$$|\mu(A \cap C) - \mu((\gamma \cdot^{a_n} A) \cap C)| < 2^{-n}$$

for all $\gamma \in \Gamma$ and $A, C \subseteq X$ Borel, we must have $\underset{\sim}{a_n} \to \underset{\sim}{i_\Gamma}$ as $n \to \infty$ in $\underset{\sim}{A}(\Gamma, X, \mu)$. Moreover,

$$\mathrm{type}(a_n) = 2^{-n}\delta_{\{e\}} + (1 - 2^{-n})\delta_\Gamma$$

for all $n \in \omega$ and hence it follows, as in the proof of Proposition 6.3.1, that $\mathrm{type}(\mathrm{cind}_\Gamma^\Delta(a_n)) \not\to \mathrm{type}(\mathrm{cind}_\Gamma^\Delta(i_\Gamma))$ when $n \to \infty$ in $\mathrm{IRS}(\Delta)$. So $\underset{\sim}{\mathrm{cind}}{}_\Gamma^\Delta$ cannot be continuous. $\square$

Let Γ be a countable group. For each $a \in A(\Gamma, X, \mu)$ and $n \in \omega \cup \{\omega\}$ we let $a^n \in A(\Gamma, X^n, \mu^n)$ be given by

$$(\gamma \cdot^{a^n} f)(i) = \gamma \cdot^a f(i)$$

for all $i < n$. Then the operation $\underset{\sim}{a} \mapsto \underset{\sim}{a^n}$ from $\underset{\sim}{A}(\Gamma, X, \mu)$ to $\underset{\sim}{A}(\Gamma, X^n, \mu^n)$ is well-defined by [10, Proposition 3.28]. Moreover, by arguments similar to those above, we obtain the following result.

Proposition 6.3.4. *Let Γ be a countable group. The map $\underset{\sim}{a} \mapsto \underset{\sim}{a^\omega}$ from $\underset{\sim}{A}(\Gamma, X, \mu)$ to $\underset{\sim}{A}(\Gamma, X^\omega, \mu^\omega)$ is not continuous when $\Gamma \neq \{e_\Gamma\}$.*

Proof. Let $(a_n)_n \in A(\Gamma, X, \mu)$ and $(B_n)_n \subseteq X$ be as in the proof of Proposition 6.3.3. Assume towards a contradiction that the map is continuous and that $\Gamma \neq \{e_\Gamma\}$. Then $\mathrm{type}(a_n^\omega) \to \mathrm{type}(i_\Gamma^\omega)$ as $n \to \infty$ in $\mathrm{IRS}(\Gamma)$. However,

$$\mathrm{type}(a_n^\omega)(\{\Gamma\}) = \mu\left(\left\{\prod_{m\in\omega}(X \setminus B_n)\right\}\right) = 0$$

for all $n \in \omega$, while $\mathrm{type}(i_\Gamma^\omega)(\{\Gamma\}) = 1$. □

Remark 6.3.5. For each pair of countable groups $\Gamma \leq \Delta$ with a fixed transversal T for the left cosets in Δ/Γ there is a connection between the continuity of the map $\underset{\sim}{a} \mapsto \underset{\sim}{\mathrm{cind}}_\Gamma^\Delta(\underset{\sim}{a})$ from $\underset{\sim}{A}(\Gamma, X, \mu)$ to $\underset{\sim}{A}(\Delta, X^T, \mu^T)$ and the continuity of the map $\underset{\sim}{a} \mapsto \underset{\sim}{a^T}$ from $\underset{\sim}{A}(\Gamma, X, \mu)$ to $\underset{\sim}{A}(\Gamma, X^T, \mu^T)$. Indeed, it follows by [10, Proposition 3.27 and Proposition 10.10] that the restriction operation $b \mapsto b_{|\Gamma}$ from $A(\Delta, X^T, \mu^T)$ to $A(\Gamma, X^T, \mu^T)$ descends to a well-defined continuous operation $\underset{\sim}{b} \mapsto \underset{\sim}{b_{|\Gamma}}$ from $\underset{\sim}{A}(\Delta, X^T, \mu^T)$ to $\underset{\sim}{A}(\Gamma, X^T, \mu^T)$. Moreover, if $t\gamma t^{-1} = \gamma$ for all $t \in T$ and $\gamma \in \Gamma$, then

$$\mathrm{cind}_\Gamma^\Delta(a)_{|\Gamma} = a^T$$

for all $a \in A(\Gamma, X, \mu)$. Hence in such cases, if $\underset{\sim}{a} \mapsto \underset{\sim}{\mathrm{cind}}_\Gamma^\Delta(\underset{\sim}{a})$ is continuous, then so is $\underset{\sim}{a} \mapsto \underset{\sim}{a^n}$.

Remark 6.3.6. Our methods for proving the results in this section provide no information about the continuity properties of the restricted map

$$\underset{\sim}{\mathrm{cind}}_\Gamma^\Delta \colon \underset{\sim}{\mathrm{FR}}(\Gamma, X, \mu) \to \underset{\sim}{A}(\Delta, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma})$$

for general pairs of countable groups $\Gamma \leq \Delta$. Nor do we know what happens with the operation $\underset{\sim}{\mathrm{cind}}_\Gamma^\Delta \colon \underset{\sim}{A}(\Gamma, X, \mu) \to \underset{\sim}{A}(\Delta, X, \mu)$ in the case where $[\Delta : \Gamma] < \infty$ and the groups are non-amenable.

It is shown in [4, Theorem 1.2], by completely different methods, that the operation $\underset{\sim}{a} \mapsto \underset{\sim}{a^2}$ from $\underset{\sim}{A}(\Gamma, X, \mu)$ to $\underset{\sim}{A}(\Gamma, X^2, \mu^2)$ is not continuous when Γ is a non-abelian free group. As a corollary, the map

$$\underset{\sim}{\mathrm{cind}}_\Gamma^{\Gamma\times(\mathbb{Z}/2\mathbb{Z})} \colon \underset{\sim}{A}(\Gamma, X, \mu) \to \underset{\sim}{A}(\Gamma \times (\mathbb{Z}/2\mathbb{Z}), X^2, \mu^2)$$

is not continuous when Γ is a non-abelian free group. Moreover, both of these continuity results hold if we restrict the maps to the space $\underset{\sim}{\mathrm{FR}}(\Gamma, X, \mu)$. In particular, this shows that the map $\underset{\sim}{\mathrm{cind}}_\Gamma^\Delta \colon \underset{\sim}{A}(\Gamma, X, \mu) \to \underset{\sim}{A}(\Delta, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma})$ is not necessarily continuous when $\Gamma \leq \Delta$ are countable groups with $[\Delta : \Gamma] < \infty$.

6.4 Properties of the co-induced invariant random subgroups

In this section we have collected a series of useful results concerning the properties of the co-induced invariant random subgroups.

The first result ensures that a co-induced invariant random subgroup is weakly mixing when $\Gamma \leq \Delta$ are countable groups with $[\Delta : \Gamma] = \infty$.

Proposition 6.4.1. *Let $\Gamma \leq \Delta$ be countable groups with $[\Delta : \Gamma] = \infty$. Then $\mathrm{cind}_\Gamma^\Delta(\theta) \in \mathrm{IRS}(\Delta)$ is weakly mixing for any $\theta \in \mathrm{IRS}(\Gamma)$.*

Proof. Let $\theta \in \mathrm{IRS}(\Gamma)$ and $a \in A(\Gamma, X, \mu)$ be such that $\mathrm{type}(a) = \theta$. Then $\mathrm{cind}_\Gamma^\Delta(a)$ is weakly mixing by Lemma 6.1.3 and hence, by Theorem 6.2.1, so is $\mathrm{CIND}_\Gamma^\Delta(\theta)$. □

The previous proposition highlights why the co-induction operation is useful if one is interested in obtaining weakly mixing invariant random subgroups.

Next we will provide a characterization of when the co-induced invariant random subgroup is non-atomic in the infinite index case. Below we will for a countable group Γ and $\gamma \in \Gamma$ write N_γ^Γ instead of $N_{\{\gamma\}}^\Gamma$. Moreover, for each $\theta \in \mathrm{IRS}(\Gamma)$ we let

$$\ker(\theta) = \left\{\gamma \in \Gamma \mid \theta(N_\gamma^\Gamma) = 1\right\}.$$

Note that $\ker(\theta)$ is a normal subgroup of Γ. Moreover, for any normal $\Lambda \in \mathrm{Sub}(\Gamma)$ we have $\ker(\delta_\Lambda) = \Lambda$. For the co-induced measure we have the following result concerning the kernel.

Proposition 6.4.2. *Let $\Gamma \leq \Delta$ be countable groups. Then*

$$\ker(\mathrm{CIND}_\Gamma^\Delta(\theta)) = \mathrm{core}_\Delta(\ker(\theta))$$

for any $\theta \in \mathrm{IRS}(\Gamma)$.

Proof. Fix a transversal T for the left cosets in Δ/Γ and let $\theta \in \mathrm{IRS}(\Gamma)$. Then for $\delta \in \Delta$ we have

$$\begin{aligned} \delta \in \ker(\mathrm{CIND}_\Gamma^\Delta(\theta)) &\iff \delta \in \mathrm{core}_\Delta(\Gamma) \wedge (\forall t \in T)\ \theta(N_{t^{-1}\delta t}^\Gamma) = 1 \\ &\iff (\forall t \in T)\ \delta \in t \ker(\theta) t^{-1} \\ &\iff \delta \in \mathrm{core}_\Delta(\ker(\theta)), \end{aligned}$$

as wanted. □

The proposition above ensures that if $\mathrm{CIND}_\Gamma^\Delta(\theta)$ is a Dirac measure, then $\mathrm{CIND}_\Gamma^\Delta(\theta) = \delta_{\mathrm{core}_\Delta(\ker(\theta))}$. By use of this fact we easily obtain the following result.

Proposition 6.4.3. *Let $\Gamma \leq \Delta$ be countable groups with $[\Delta : \Gamma] = \infty$ and $\theta \in \mathrm{IRS}(\Gamma)$. If $T \subseteq \Delta$ is a transversal for the left cosets in Δ/Γ, then $\mathrm{CIND}_\Gamma^\Delta(\theta)$ is non-atomic if and only if there is $\gamma \in \mathrm{core}_\Delta(\Gamma) \setminus \mathrm{core}_\Delta(\ker(\theta))$ such that*

$$\sum_{t \in T} \left(1 - \theta(N^\Gamma_{t^{-1}\gamma t})\right) < \infty$$

and $\theta(N^\Gamma_{t^{-1}\gamma t}) > 0$ for all $t \in T$.

Proof. First note that $\mathrm{CIND}_\Gamma^\Delta(\theta)$ is weakly mixing by Proposition 6.4.1. Hence it follows by Proposition 5.1.6 and Proposition 6.4.2 that $\mathrm{CIND}_\Gamma^\Delta(\theta)$ is non-atomic if and only if $\mathrm{CIND}_\Gamma^\Delta(\theta) \neq \delta_{\mathrm{core}_\Delta(\ker(\theta))}$. The latter is easily seen to be equivalent to the statement in the proposition. □

It follows by Theorem 6.2.1 that if $\Gamma \leq \Delta$ are countable groups and $a \in A(\Gamma, X, \mu)$, then $\mathrm{cind}_\Gamma^\Delta(a)$ is free if and only if $\mathrm{CIND}_\Gamma^\Delta(\mathrm{type}(a)) = \delta_{\{e_\Gamma\}}$. Moreover, note that for any $\gamma \in \Gamma$ we have

$$\mathrm{type}(a)(N^\Gamma_\gamma) = \mu(\mathrm{Fix}_a(\gamma)).$$

Therefore by similar argumentation as in the proof of Proposition 6.4.3, we obtain the following characterization of when the co-induced action is free.

Proposition 6.4.4. *Let $\Gamma \leq \Delta$ be countable groups, $a \in A(\Gamma, X, \mu)$ and let $T \subseteq \Delta$ be a transversal for the left cosets in Δ/Γ. Then $\mathrm{cind}_\Gamma^\Delta(a)$ is not free if and only if for some $\gamma \in \mathrm{core}_\Delta(\Gamma) \setminus \{e_\Gamma\}$ we have*

$$\sum_{t \in T} \left(1 - \mu(\mathrm{Fix}_a(t^{-1}\gamma t))\right) < \infty$$

and $\mu\left(\mathrm{Fix}_a(t^{-1}\gamma t)\right) > 0$ for all $t \in T$.

Note that for any $\gamma \in \Gamma$ we have $1 - \mu(\mathrm{Fix}_a(\gamma)) = d_u(\gamma^a, 1)$, where $1 \in \mathrm{Aut}(X, \mu)$ denotes the identity map and d_u refers to the metric inducing the uniform topology on $\mathrm{Aut}(X, \mu)$. Hence if $[\Delta : \Gamma] = \infty$ and the co-induced action is non-free, then the conjugates of some $\gamma \in \mathrm{core}_\Delta(\Gamma) \setminus \{e_\Gamma\}$ under the transversal T must uniformly converge very fast to the identity in $\mathrm{Aut}(X, \mu)$.

We end this section with a short discussion of cases where the image of the co-induction operation is minimal. It is clear that if $\Gamma \leq \Delta$ are countable groups and $\Lambda \leq \Delta$ is normal with $\Lambda \subseteq \Gamma$, then $\delta_\Lambda \in \mathrm{IRS}(\Gamma)$ will satisfy that $\mathrm{CIND}_\Gamma^\Delta(\delta_\Lambda) = \delta_\Lambda$. Thus all Dirac measures in $\mathrm{IRS}(\Delta)$ with support in $\mathrm{Sub}(\Gamma)$ are contained in the image of the co-induction operation. In some cases these are the only ones.

Proposition 6.4.5. *Let Δ be a countable group. If $\mathbb{Z} \leq \Delta$ with $[\Delta : \mathbb{Z}] = \infty$, then any $\theta \in A(\mathbb{Z}, X, \mu)$ satisfies $\mathrm{CIND}_\Gamma^\Delta(\theta) = \delta_{\mathrm{core}_\Delta(\ker(\theta))}$.*

Proof. Let $n \in \omega$ be such that $n\mathbb{Z} = \mathrm{core}_\Delta(\mathbb{Z})$. For each $t \in T$ and $m \in n\mathbb{Z}$ we have $t^{-1}mt = \pm m$. So, since $\theta(N_m^{\mathbb{Z}}) = \theta(N_{-m}^{\mathbb{Z}})$, we obtain

$$\mathrm{CIND}_\Gamma^\Delta(\theta)\left(N_m^\Delta\right) = 1 \iff m \in \ker(\theta)$$

and

$$\mathrm{CIND}_\Gamma^\Delta(\theta)\left(N_m^\Delta\right) = 0 \iff m \notin \ker(\theta),$$

as wanted. □

Note that if every co-induced invariant random subgroup is a Dirac measure, then every co-induced action will have almost everywhere fixed stabilizers.

Chapter 7

New constructions of non-atomic, weakly mixing invariant random subgroups

In this chapter we will apply the co-induction operation for invariant random subgroups that we developed in the previous chapter to construct new examples of continuum size families of non-atomic, weakly mixing invariant random subgroups for several classes of groups.

In the first section we isolate a sufficient criterion for a pair of countable groups $\Gamma \leq \Delta$ that allows us to use the co-induction operation to construct such families for Δ. In the second section we apply this criterion to construct examples of these families for some classes of wreath products and HNN extensions. In the third and fourth section, we will apply the co-induction operation to construct such families for the non-abelian free groups and, more generally, for certain free products of groups with normal amalgamation.

We should point out that for many of these classes of groups, other examples of such families have already been constructed. In [8] they use a completely different technique (including Pontryagin duality and a deep result of Adian in combinatorial group theory) to obtain continuum size families of non-atomic, weakly mixing characteristic random subgroups for the non-abelian free groups Thus these provide examples of such families for any group containing a non-abelian free group as a normal subgroup. In [17] they obtain constructions of continuum size families of non-atomic weakly mixing invariant random subgroups for the same class of wreath products as we will consider here and for the non-abelian free groups. Their method is again different and involves what they call *intersectional invariant random subgroups.* Other results concerning invariant random subgroups of the lamplighter groups can be found in [7].

The results and constructions in this chapter have all been obtained in joint work with Alexander S. Kechris and can also be found in [22].

7.1 A sufficient criterion

The goal of this section is to provide a sufficient criterion for an infinite index subgroup to generate continuum many non-atomic, weakly mixing co-induced invariant random subgroups of the bigger group.

For a countable group Γ and a subset $S \subseteq \Gamma$ we let $\langle S \rangle_\Gamma$ denote the subgroup generated by S in Γ and $\langle\langle S \rangle\rangle_\Gamma$ denote the normal subgroup generated by S in Γ.

Proposition 7.1.1. *Let $\Gamma \leq \Delta$ be countable groups with $[\Delta : \Gamma] = \infty$ and consider the statements:*

(1) There exists a transversal $T = \{t_i \mid i \in \omega\}$ for the left cosets in Δ/Γ and $\gamma_0 \in \mathrm{core}_\Delta(\Gamma)$ such that the chain of normal subgroups $(\overline{\Gamma}_{k,T,\gamma_0})_{k\in\omega}$ given by

$$\overline{\Gamma}_{k,T,\gamma_0} = \langle\langle t_i^{-1}\gamma_0 t_i \mid i \geq k \rangle\rangle_\Gamma$$

is not constant.

(2) There exists a continuum size family $(\theta_i)_{i\in I} \in \mathrm{IRS}(\Gamma)$ such that the elements in the family $\left(\mathrm{CIND}_\Gamma^\Delta(\theta_i)\right)_{i\in I} \in \mathrm{IRS}(\Delta)$ are all non-atomic, weakly mixing and satisfy that $\mathrm{CIND}_\Gamma^\Delta(\theta_i) \neq \mathrm{CIND}_\Gamma^\Delta(\theta_j)$ for all $i, j \in I$ with $i \neq j$.

(3) There exists $\theta \in \mathrm{IRS}(\Gamma)$ such that $\mathrm{CIND}_\Gamma^\Delta(\theta) \in \mathrm{IRS}(\Delta)$ is non-atomic.

(4) There exists $\theta \in \mathrm{IRS}(\Gamma)$ such that $\mathrm{CIND}_\Gamma^\Delta(\theta) \in \mathrm{IRS}(\Delta)$ is not a Dirac measure.

(5) For any transversal $T = \{t_i \mid i \in \omega\}$ for the left cosets in Δ/Γ there is $\gamma_0 \in \mathrm{core}_\Delta(\Gamma)$ such that the chain of subgroups $(\Gamma_{k,T,\gamma_0})_{k\in\omega}$ given by

$$\Gamma_{k,T,\gamma_0} = \langle t_i^{-1}\gamma_0 t_i \mid i \geq k \rangle_\Gamma$$

is not constant.

It holds that $(1) \implies (2) \implies (3) \implies (4) \implies (5)$.

Proof. It is clear that (2) $\implies$ (3) $\implies$ (4). Below we will prove (1) $\implies$ (2) and (4) $\implies$ (5).

For the implication (1) $\implies$ (2), assume (1) holds for T and γ_0. We will first construct one $\theta \in \mathrm{IRS}(\Gamma)$ such that $\mathrm{CIND}_\Gamma^\Delta(\theta)$ is non-atomic and weakly mixing. Afterwards we will argue how to obtain continuum many. Let

$$\theta = \sum_{k\in\omega} 2^{-k-1}\delta_{\overline{\Gamma}_{k,T,\gamma_0}}.$$

Then the non-constant assumption on the sequence $(\overline{\Gamma}_{k,T,\gamma_0})_{k\in\omega}$ ensures that $\gamma_0 \notin \mathrm{core}_\Delta(\ker(\theta))$, as for some $j,k \in \omega$ with $j \leq k$ we have $t_j^{-1}\gamma_0 t_j \notin \overline{\Gamma}_{k,T,\gamma_0}$. Moreover, it follows directly by the definition of θ that $\theta(N^\Gamma_{t_i^{-1}\gamma_0 t_i}) > 0$ for all $i \in \omega$ and that

$$\sum_{i\in\omega}\Big(1-\theta(N^\Gamma_{t_i^{-1}\gamma_0 t_i})\Big) < \infty.$$

The assumptions of Proposition 6.4.3 are therefore satisfied and hence we obtain that $\mathrm{CIND}_\Gamma^\Delta(\theta)$ is non-atomic. Since $[\Delta\colon\Gamma] = \infty$, it will also be weakly mixing by Proposition 6.4.1.

To construct continuum many of these, let $N \in \omega$ be least such that $\overline{\Gamma}_{N+1,T,\gamma_0} \subsetneq \overline{\Gamma}_{N,T,\gamma_0}$ and let

$$\lambda = \sum_{k\leq N+1} 2^{-k-1}.$$

Next fix $S \subseteq \{0,\ldots,N\}$ such that for all $k \in \omega$ we have $t_k^{-1}\gamma_0 t_k \notin \overline{\Gamma}_{N+1,T,\gamma_0}$ if and only if $k \in S$. For each $r \in (0,\lambda)$ put

$$\theta_r = r\delta_{\overline{\Gamma}_{0,T,\gamma_0}} + (\lambda - r)\delta_{\overline{\Gamma}_{N+1,T,\gamma_0}} + \sum_{N+1<k} 2^{-k-1}\delta_{\overline{\Gamma}_{k,T,\gamma_0}}$$

and note that

$$\theta_r(N^\Gamma_{t_k^{-1}\gamma_0 t_k}) = r$$

for all $k \in S$, while

$$\theta_r(N^\Gamma_{t_k^{-1}\gamma_0 t_k}) = \theta(N^\Gamma_{t_k^{-1}\gamma_0 t_k})$$

for all $k \notin S$. Hence, by the description of the co-induction operation given in Theorem 6.2.1, we obtain that

$$\mathrm{CIND}_\Gamma^\Delta(\theta_r)\left(N^\Delta_{\gamma_0}\right) = \prod_{k\in\omega}\theta_r(N^\Gamma_{t_k^{-1}\gamma_0 t_k}) = r^{|S|}\prod_{k\in\omega\setminus S}\theta(N^\Gamma_{t_k^{-1}\gamma_0 t_k})$$

for all $r \in (0,\lambda)$. We can therefore conclude that $\left(\mathrm{CIND}_\Gamma^\Delta(\theta_r)\right)_{r\in(0,\lambda)}$ is a continuum size family of non-atomic, weakly mixing invariant random subgroups of Δ, as wanted.

For the implication (4) $\implies$ (5), assume that $\theta \in \mathrm{IRS}(\Gamma)$ satisfies that $\mathrm{CIND}_{\Gamma}^{\Delta}(\theta)$ is not a Dirac measure. Moreover, let $T = \{t_i \mid i \in \omega\}$ be a transversal for Δ/Γ. Then

$$\mathrm{CIND}_{\Gamma}^{\Delta}(\theta)\left(N_F^{\Delta}\right) = \prod_{t \in T} \theta(N_{t^{-1}Ft}^{\Gamma})$$

for all finite $F \subseteq \mathrm{core}_{\Delta}(\Gamma)$. Since $\mathrm{CIND}_{\Gamma}^{\Delta}(\theta)$ is not a Dirac measure, there exists $\gamma_0 \in \mathrm{core}_{\Delta}(\Gamma)$ such that $\mathrm{CIND}_{\Gamma}^{\Delta}(\theta)\left(N_{\gamma_0}^{\Delta}\right) \in (0,1)$. So if we let $b \in \mathrm{A}(\Gamma, X, \mu)$ satisfy that $\mathrm{type}(b) = \theta$, then $\mu(\mathrm{Fix}_b(t_m^{-1}\gamma_0 t_m)) = \lambda_m < 1$ for some $m \in \omega$ and

$$\sum_{i \in \omega} \left(1 - \mu\left(\mathrm{Fix}_b(t_i^{-1}\gamma_0 t_i)\right)\right) < \infty.$$

By convergence of the series, it follows that there is some $N \in \omega$ such that

$$\mu\left(\left\{x \in X \mid (\forall \gamma \in \Gamma_{N,T,\gamma_0})\gamma \cdot^b x = x\right\}\right) > \lambda_m.$$

We can therefore conclude that $t_m^{-1}\gamma_0 t_m \notin \Gamma_{N,T,\gamma_0}$, as wanted. $\square$

Note that if Γ in Proposition 7.1.1 is abelian, then all the statements are equivalent. Moreover, the invariant random subgroups constructed in the proof of (1) $\implies$ (2) are not weakly mixing when restricted to Γ.

Remark 7.1.2. In general, if $\Gamma \leq \Delta$ is a normal subgroup, then T is a transversal for the left cosets in Δ/Γ if and only if $T^{-1} = \left\{t^{-1} \mid t \in T\right\}$ is a transversal, as well. Therefore in this case, the statement

(1') There exists a transversal R for the left cosets in Δ/Γ and $\gamma_0 \in \Gamma$ such that the chain of normal subgroups $(\overline{\Lambda}_{k,R,\gamma_0})_{k \in \omega}$ given by

$$\overline{\Lambda}_{k,R,\gamma_0} = \langle\langle r_i \gamma_0 r_i^{-1} \mid i \geq k \rangle\rangle_{\Gamma}$$

is not constant.

is equivalent to condition (1) in Propostition 7.1.1.

7.2 Wreath products and HNN extensions

We will in this section show how to apply the criterion in Proposition 7.1.1 to obtain continuum size families of non-atomic, weakly mixing invariant random subgroups for certain wreath products and HNN extensions.

First we consider wreath products. Let G and H be countable groups and consider the action $G \curvearrowright^{\alpha} \oplus_G H$ given by $g \cdot^{\alpha} f(g_0) = f(g^{-1}g_0)$. The *wreath product* of G by H is the semidirect product $(\oplus_G H) \rtimes_{\alpha} G$, which is denoted by $H \wr G$.

Construction 7.2.1 (Wreath products)**.** We will here construct continuum many non-atomic, weakly mixing invariant random subgroups for wreath products of the form $H \wr G$, where G and H are countable groups such that G is infinite and H is non-trivial.

Let $\Gamma = \oplus_G H$ and $\Delta = H \wr G$. Then $\Gamma \leq \Delta$ is normal and $G \subseteq \Delta$ is a transversal for the left cosets Δ/Γ. Fix an enumeration $G = \{g_i \mid i \in \omega\}$ and let $h_0 \in H \setminus \{e_H\}$. Define $\gamma_0 \in \Gamma$ by

$$\gamma_0(g_i) = \begin{cases} e_H & \text{if} \quad i \neq 0 \\ h_0 & \text{if} \quad i = 0 \end{cases}.$$

Now since $\left(\overline{\Gamma}_{k,G,\gamma_0}\right)_{k\in\omega}$ is not constant, the proof of (1) $\implies$ (2) in Proposition 7.1.1 provides a construction of continuum many non-atomic, weakly mixing invariant random subgroups of $H \wr G$. •

Let Ω is a countable set and G a countable group with an action $G \curvearrowright^{\alpha} \Omega$. Consider the induced action $G \curvearrowright^{\overline{\alpha}} \oplus_{\Omega} H$ given by $(g \cdot^{\overline{\alpha}} f)(w) = f(g^{-1} \cdot^{\alpha} w)$. The semidirect product $(\oplus_{\Omega} H) \rtimes_{\overline{\alpha}} G$ is called a *generalized wreath product* and we denote such a wreath product by $H \wr_{\Omega} G$. Arguments similar to those in Construction 7.2.1 also work for $H \wr_{\Omega} G$ when the action $G \curvearrowright^{\alpha} \Omega$ has an infinite orbit with finite stabilizers.

Next we will consider HNN extensions over "small" subgroups. Let H be a countable group, $A \leq H$ a subgroup and $\varphi\colon A \to H$ an embedding. The *HNN extension* of H relative to A and φ is the group

$$G = \langle H, t \mid (\forall a \in A)\ t^{-1}at = \varphi(a) \rangle,$$

i.e., the quotient of the free product $H * \langle t \rangle$ by $\langle\langle t^{-1}at\varphi(a)^{-1} \mid a \in A \rangle\rangle_{H*\langle t \rangle}$. We will in our construction use the following theorem, which identifies an HNN extension with a semidirect product. A proof can be found in [6, Theorem 17.1].

Theorem 7.2.2. *Consider the HNN extension*

$$G = \langle H, t \mid (\forall a \in A)\ t^{-1}at = \varphi(a) \rangle,$$

where H is a countable group, $A \leq H$ and $\varphi\colon A \to H$ is an embedding. Let $H_n = \{h_n \mid h \in H\}$ for each $n \in \mathbb{Z}$ be a copy of H and put

$$F = \langle *_{n\in\mathbb{Z}} H_n \mid (\forall j \in \mathbb{Z})(\forall a \in A)\ a_{j+1} = \varphi(a)_j \rangle.$$

Then $G \cong F \rtimes_\psi \mathbb{Z}$, where

$$\psi(h^1_{i_1} h^2_{i_2} \cdots h^k_{i_k}) = h^1_{i_1+1} h^2_{i_2+1} \cdots h^k_{i_k+1}$$

for all $h^1, \ldots, h^k \in H$ and $i_1, \ldots, i_k \in \mathbb{Z}$.

We will now show how to use this decomposition of an HNN extension to obtain a continuum size family of non-atomic, weakly mixing invariant random subgroups of certain HNN extensions.

Construction 7.2.3 (HNN extensions)**.** We will here construct continuum many non-atomic, weakly mixing invariant normal subgroups for HNN extensions of the form $G = \langle H, t \mid (\forall a \in A)\ t^{-1}at = \varphi(a)\rangle$, where H is a countable group, $A \leq H$ and $\varphi\colon A \to H$ is an embedding such that $\langle\langle A \cup \varphi(A)\rangle\rangle_H \neq H$.

Let F and ψ be as in Theorem 7.2.2 so that $G \cong F \rtimes_\psi \mathbb{Z}$. Put $\Gamma = F$ and $\Delta = F \rtimes_\psi \mathbb{Z}$. Then $\mathbb{Z} \subseteq \Delta$ is a transversal for the left cosets in Δ/Γ. Now let $\Lambda = \langle\langle A \cup \varphi(A)\rangle\rangle_H$ and consider for each $i \in \mathbb{Z}$ the homomorphism $f_i\colon H_i \to H/\Lambda$ given by $f_i(h_i) = e\Lambda$ if $i \neq 0$ and $f_0(h_0) = h\Lambda$ for all $h \in H$. Then let $f\colon \Gamma \to H/\Lambda$ be the homomorphism induced by $(f_i)_{i\in\mathbb{Z}}$. For a fixed $x \in H \setminus \Lambda$ we must have $f(x_0) \neq e$, while $f(x_i) = e$ for all $i \neq 0$. This implies that

$$x_0 \notin \langle\langle \psi^i(x_0) \mid i \in \mathbb{Z} \setminus \{0\}\rangle\rangle_\Gamma = \langle\langle x_i \mid i \in \mathbb{Z} \setminus \{0\}\rangle\rangle_\Gamma$$

and hence, using the proof of (1) $\implies$ (2) in Proposition 7.1.1, we construct continuum many non-atomic, weakly mixing invariant random subgroups of G. •

The previous construction covers the case where φ is an automorphism of a non-trivial normal subgroup of H. We also have the following application.

Corollary 7.2.4. *If $n, m \in \mathbb{Z} \setminus \{0\}$ are not relatively prime, then there are continuum many non-atomic, weakly mixing invariant random subgroups of $\mathrm{BS}(n, m) = \langle x, t \mid tx^n t^{-1} = x^m\rangle$.*

Proof. We have that $\mathrm{BS}(n, m)$ is the HNN extension of $\mathbb{Z}$ with respect to the isomorphism $\varphi\colon n\mathbb{Z} \to m\mathbb{Z}$ given by $\varphi(n) = m$. So, since $\langle\langle n\mathbb{Z} \cup m\mathbb{Z}\rangle\rangle_\mathbb{Z} = \gcd(n, m)\mathbb{Z}$, it follows by Construction 7.2.3 that there are continuum many non-atomic, weakly mixing invariant random subgroups of $\mathrm{BS}(n, m)$. □

7.3 Non-abelian free groups

We will in this section turn our attention towards the non-abelian free groups. First we will use the co-induction operation to construct continuum many non-atomic, weakly mixing invariant random subgroups of $\mathbb{F}_\infty$. Afterwards we will present a different construction that works for all non-abelian free groups.

Let F be a free non-abelian group. A set $S \subseteq F$ is called a *basis* of F if F is freely generated by S.

The idea in the first construction is to consider various semidirect products of the form $\mathbb{F}_\infty \rtimes_\varphi \mathbb{Z}$, where φ is induced by a permutation of some basis of $\mathbb{F}_\infty$. For each such semidirect product, we apply the co-induction operation on a certain invariant random subgroup of $\mathbb{F}_\infty$ to obtain a non-atomic, weakly mixing invariant random subgroup of $\mathbb{F}_\infty \rtimes_\varphi \mathbb{Z}$ with support in $\mathbb{F}_\infty$. We will moreover ensure that the restriction to $\mathbb{F}_\infty$ is weakly mixing and thereby obtain a non-atomic, weakly mixing invariant random subgroup of $\mathbb{F}_\infty$.

Since we need the co-induced invariant random subgroups to be weakly mixing with respect to the action of $\mathbb{F}_\infty$, we cannot just apply Proposition 7.1.1 as we did in the previous section.

Construction 7.3.1 ($\mathbb{F}_\infty$)**.** We will here construct continuum many non-atomic, weakly mixing invariant random subgroups of $\mathbb{F}_\infty$.

Fix a basis $\mathbb{F}_\infty = \langle b, a_k \mid k \in \omega\rangle$. Let $(B_k)_k \subseteq X$ be a sequence of Borel sets such that for each $k, i, j \in \omega$ we have $\mu(B_k) = 2^{-k-1}$ and $B_j \cap B_i = \emptyset$ if $i \neq j$. For each $k \in \omega$ put

$$A_k = X \setminus \left(\bigcup_{k<j} B_j \right)$$

and fix a free action $\mathbb{F}_\infty \curvearrowright^{\alpha_k} B_k$. Then define $\alpha \in A(\mathbb{F}_\infty, X, \mu)$ by $a_k \cdot^\alpha x = x$ if $x \in A_k$ and $a_k \cdot^\alpha x = a_k \cdot^{\alpha_j} x$ if $x \in B_j$ for some $j > k$. Finally, let b act as a weakly mixing transformation on X to ensure that α is weakly mixing.

Next let $S \subseteq \omega$ be infinite and let $\pi_S \colon \omega \to \omega$ be a permutation which is transitive on S and fixes every element of $\omega \setminus S$. Define $\varphi_S \colon \mathbb{Z} \to \mathrm{Aut}(\mathbb{F}_\infty)$ by $\varphi_z(b) = b$ and $\varphi_z(a_k) = a_{\pi_S^z(k)}$ for all $z \in \mathbb{Z}$ and $k \in \omega$. Consider $\Delta_S = \mathbb{F}_\infty \rtimes_{\varphi_S} \mathbb{Z}$ and let

$$\theta_S = \mathrm{type}\left(\mathrm{cind}_\Gamma^{\Delta_S}(\alpha)_{|\mathbb{F}_\infty}\right) = \left(\mathrm{CIND}_\Gamma^{\Delta_S}(\mathrm{type}(\alpha))\right)_{|\,\mathrm{Sub}(\mathbb{F}_\infty)} \in \mathrm{IRS}(\mathbb{F}_\infty).$$

Note that θ_S is weakly mixing by Proposition 6.1.3 and that θ_S is non-atomic, since

$$\theta_S(N_{a_k}^{\Delta_S}) = \prod_{k \in S} \mu(\mathrm{Fix}_\alpha(a_k)) \in (0,1) \iff k \in S$$

for all $k \in \omega$. Moreover, this implies that whenever $S, T \subseteq \omega$ are infinite with $S \neq T$ we have $\theta_S \neq \theta_T$. Hence the family $\{\theta_S \mid S \subseteq \omega \text{ infinite}\}$ constitutes a continuum size family of non-atomic, weakly mixing invariant random subgroups of $\mathbb{F}_\infty$. •

In the next construction we consider $\mathbb{F}_n$ for some $n \in \omega \cup \{\omega\}$ with $n \geq 2$ and a certain copy of $\mathbb{F}_\infty$ inside $\mathbb{F}_n$. The idea is to construct a continuum size family of invariant random subgroups of $\mathbb{F}_\infty$ such that the co-induced invariant random subgroups constitute a family of non-atomic, weakly mixing invariant random subgroups of $\mathbb{F}_n$. We will moreover ensure that the obtained invariant random subgroups are weakly mixing when restricted to this copy of $\mathbb{F}_\infty$ as well.

Construction 7.3.2 (Non-abelian free groups)**.** Let $n \in \omega \cup \{\infty\}$ with $n \geq 2$. We will here construct continuum many non-atomic, weakly mixing invariant random subgroups of $\mathbb{F}_n$.

Fix a basis $\mathbb{F}_n = \langle a_i \mid i < n \rangle$ and consider the surjective group homomorphism $\varphi\colon \mathbb{F}_n \to \mathbb{Z}$ given by $\varphi(a_0) = 1$ and $\varphi(a_i) = 0$ for $0 < i < n$. Then let $\Gamma = \ker(\varphi)$ and note that

$$\Gamma = \langle a_0^{-k} a_i a_0^k \mid k \in \mathbb{Z}, 0 < i < n \rangle.$$

The set $\left\{a_0^{-k} a_i a_0^k \mid k \in \mathbb{Z}, 0 < i < n\right\}$ freely generates Γ as a copy of $\mathbb{F}_\infty$ inside $\mathbb{F}_n$. Moreover, the set $T = \left\{a_0^k \mid k \in \mathbb{Z}\right\}$ constitutes a transversal for the left cosets in $\mathbb{F}_n/\Gamma$.

Now for each $\lambda \in (0,1)$ let $(T_k^\lambda)_{k \in \mathbb{Z}} \in \mathrm{Aut}(X, \mu)$ satisfy that the action induced by $\langle T_{-1}^\lambda, T_0^\lambda, T_1^\lambda \rangle$ is weakly mixing and

$$\mu(\mathrm{Fix}(T_k^\lambda)) = \begin{cases} 3^{-1} & \text{if} \quad |k| < 2 \\ \lambda & \text{if} \quad |k| = 2 \\ 1 - 2^{-k-1} & \text{if} \quad |k| > 2 \end{cases}$$

for each $k \in \mathbb{Z}$. One way to choose $T_{-1}^\lambda, T_0^\lambda, T_1^\lambda$ is to decompose X as $X = X_{-1} \sqcup X_0 \sqcup X_1$ such that $\mu(X_{-1}) = \mu(X_0) = \mu(X_1) = 3^{-1}$ and then let T_j be weakly mixing when restricted to $X \setminus X_j$ and trivial on X_j for $j \in \{-1, 0, 1\}$. Next define an action $\alpha_\lambda \in A(\Gamma, X, \mu)$ by letting $a_0^{-k} a_1 a_0^k \cdot^{\alpha_\lambda} x = T_k^\lambda(x)$ and $a_0^{-k} a_i a_0^k \cdot^{\alpha_\lambda} x = x$ for all $1 < i < n$ and $k \in \mathbb{Z}$. Then put $\theta_\lambda = \mathrm{type}(\alpha_\lambda)$

and note that all conditions of Proposition 6.4.3 are satisfied with respect to $a_1 \in \Gamma$. Hence $\mathrm{CIND}_\Gamma^\Delta(\theta_\lambda)$ is non-atomic. Finally, since $[\mathbb{F}_n : \Gamma] = \infty$ and

$$\mathrm{CIND}_\Gamma^{\mathbb{F}_n}(\theta_\lambda)(N_{a_1}^{\mathbb{F}_n}) = \prod_{k\in\mathbb{Z}} \theta_\lambda(N^\Gamma_{a_0^{-k}a_1a_0^k}) = \lambda^2 3^{-3} \prod_{k\in\mathbb{Z}\setminus\{-2,\ldots,2\}} (1-2^{-k-1})$$

for each $\lambda \in (0,1)$, the family $(\mathrm{CIND}_\Gamma^{\mathbb{F}_n}(\theta_\lambda))_{\lambda\in(0,1)}$ constitutes a continuum size family of non-atomic, weakly mixing invariant random subgroups of $\mathbb{F}_n$. In fact, since each $a_\lambda \in A(\Gamma, X, \mu)$ is weakly mixing, it follows by Proposition 6.1.3 that each $\mathrm{CIND}_\Gamma^{\mathbb{F}_n}(\theta_\lambda)_{|\,\mathrm{Sub}(\Gamma)} \in \mathrm{IRS}(\Gamma)$ is weakly mixing as well. •

Note that if in Construction 7.3.2 we did not care that $\mathrm{CIND}_\Gamma^{\mathbb{F}_n}(\theta_\lambda)_{|\,\mathrm{Sub}(\Gamma)} \in \mathrm{IRS}(\Gamma)$ is weakly mixing for each $\lambda \in (0,1)$, one could just omit the requirement that the action induced by $\langle T_{-1}^\lambda, T_0^\lambda, T_1^\lambda\rangle$ is weakly mixing.

Using an action similar to the one in Construction 7.3.2, we obtain the following algebraic fact.

Corollary 7.3.3. *Let $s, w_1, w_2, \ldots \in \mathbb{F}_\infty$ satisfy $w_i^{-1}sw_i \neq w_j^{-1}sw_j$ for all $i, j \in \omega$ with $i \neq j$. Then the set*

$$\{w_n^{-1}sw_n \mid n \in \omega\}$$

does not extend to a basis of $\mathbb{F}_\infty$.

Proof. Assume towards a contradiction that we may extend the set to a basis of $\mathbb{F}_\infty$. Then we would have that $\{w_n^{-1}sw_n \mid n \in \omega\}$ generates a copy F_∞ of $\mathbb{F}_\infty$ as a subgroup of $\mathbb{F}_\infty$. So let $a \in A(F_\infty, X, \mu)$ be an action such that $d_u((w_n^{-1}sw_n)^a, 1) \neq 0$ for all $n \in \omega$ and

$$d_u((w_n^{-1}sw_n)^a, 1) \to 0$$

as $n \to \infty$ in $\mathrm{Aut}(X, \mu)$. Now, since the set extends to a basis, we may extend a to an action $b \in A(\mathbb{F}_\infty, X, \mu)$. Hence we obtain that

$$d_u((w_n^{-1}sw_n)^a, 1) = d_u((w_n^{-1}sw_n)^b, 1) = d_u((w_n^{-1})^b s^b w_n^b, 1) = d_u(s^b, 1)$$

for all $n \in \omega$, which contradicts the convergence above. □

7.4 Free products with normal amalgamation

In this section we will use the co-induction operation to give constructions of non-atomic, weakly mixing invariant random subgroups of certain free products of groups with normal amalgamation.

Recall that if G and H are countable groups and $A \leq G, H$ is a shared subgroup, then we may form the group

$$G *_A H = \langle G * H \mid (\forall a \in A)\ a_G = a_H \rangle,$$

where $a_G \in G$ and $a_H \in H$ denote the copies of an element $a \in A$ in the groups G and H, respectively. Note that $G *_A H$ is the quotient of the free product $G * H$ by $\langle\langle a_G a_H^{-1} \mid a \in A \rangle\rangle_{G*H}$. Groups of this form are called *free products with amalgamation*.

First we will consider free products without amalgamation and afterwards argue how this construction generalizes to some free products of groups with normal amalgamation.

Construction 7.4.1 (Free products)**.** We will here construct continuum many non-atomic, weakly mixing invariant random subgroups of $G * H$, where G and H are non-trivial countable groups with H infinite. The invariant random subgroups will have support in

$$\Gamma = \langle [g, h] \mid g \in G, h \in H \rangle$$

and we can ensure that these invariant random subgroups are weakly mixing when restricted to Γ as well.

Let $\Delta = G * H$. Consider the natural group homomorphism $\varphi\colon \Delta \to G \times H$ and put $\Gamma = \ker \varphi$. Then Γ is freely generated by the commutators

$$\Gamma = \langle [g, h] \mid g \in G \setminus \{e_G\}, h \in H \setminus \{e_H\} \rangle,$$

where $[g, h] = ghg^{-1}h^{-1}$. Moreover, $T = \{gh \mid g \in G, h \in H\}$ is a transversal for the left cosets in Δ/Γ. Now fix $g_0 \in G \setminus \{e_G\}$ and $h_0 \in H \setminus \{e_H\}$. For each $\lambda \in (0,1)$ let $a_\lambda \in A(\Gamma, X, \mu)$ satisfy that $\mu(\mathrm{Fix}_{a_\lambda}([g_0, h_0])) = \lambda$ and $\mu(\mathrm{Fix}_{a_\lambda}([g, h])) = 1$ for all $g \in G \setminus \{e_G, g_0\}$ and $h \in H \setminus \{e_H, h_0\}$. Put $\theta_\lambda = \mathrm{type}(a_\lambda)$ and note that

$$\mathrm{CIND}_\Gamma^\Delta(\theta_\lambda)\left(N_{[g_0,h_0]}^\Delta\right) = \prod_{gh \in T} \theta_\lambda\left(N^\Gamma_{h^{-1}g^{-1}[g_0,h_0]gh}\right).$$

We have

$$h^{-1}g^{-1}[g_0, h_0]gh = [h^{-1}, g^{-1}g_0][g^{-1}g_0, h^{-1}h_0][h^{-1}h_0, g^{-1}][g^{-1}, h^{-1}]$$

for all $g \in G$ and $h \in H$. So $[g_0, h_0]$ or its inverse is in the word of

$$h^{-1}g^{-1}[g_0, h_0]gh$$

if and only if

$$(g,h) \in \left\{(e_G, h_0^{-1}), (e_G, e_H), (g_0^{-1}, e_H), (g_0^{-1}, h_0^{-1})\right\}.$$

Therefore $\mathrm{CIND}_\Gamma^\Delta(\theta_\lambda)\left(N^\Delta_{[g_0,h_0]}\right) = \lambda^4$ and so $\left(\mathrm{CIND}_\Gamma^\Delta(\theta_\lambda)\right)_{\lambda\in(0,1)}$ constitutes a continuum size family of non-atomic, weakly mixing invariant random subgroups of Δ.

To ensure that the co-induced invariant random subgroups are weakly mixing when restricted to Γ, let $h_1, h_2, h_3 \in H \setminus \{e_H\}$ satisfy that h_0, h_1, h_2, h_3 are distinct and that $h_j h_0 \neq h_i$, $h_j h_0^{-1} \neq h_i$ for all $i, j \in \{0,1,2,3\}$. Then modify a_λ such that the action of

$$\langle [g_0, h_1], [g_0, h_2], [g_0, h_3]\rangle$$

is weakly mixing and satisfies $\mu(\mathrm{Fix}_{a_\lambda}([g_0, h_i])) = 1/3$ for $1 \leq i \leq 3$. Note that the constrains on h_0, h_1, h_2, h_3 ensure that at most one of $[g_0, h_i]$ satisfies that it or its inverse is in the word

$$[h^{-1}, g^{-1}g_0][g^{-1}g_0, h^{-1}h_0][h^{-1}h_0, g^{-1}][g^{-1}, h^{-1}]$$

when $g \in G$ and $h \in H$. Moreover, as with $[g_0, h_0]$, each will appear exactly four times. So we have

$$\mathrm{CIND}_\Gamma^\Delta(\theta_\lambda)\left(N^\Delta_{[g_0,h_0]}\right) = \lambda^4 3^{-12}.$$

Once again, $\left(\mathrm{CIND}_\Gamma^\Delta(\theta_\lambda)\right)_{\lambda\in(0,1)}$ constitutes a continuum size family of non-atomic, weakly mixing invariant random subgroups of Δ. These will now also be weakly mixing when restricted to Γ by Proposition 6.1.3. •

In order to generalize Construction 7.4.1 to some free products with normal amalgamation, we note the following well-known simple fact.

Proposition 7.4.2. *Let Δ and Γ be countable groups and $\varphi\colon \Delta \to \Gamma$ a surjective group homomorphism. There is an embedding $\Psi\colon \mathrm{IRS}(\Gamma) \to \mathrm{IRS}(\Delta)$ such that if $\theta \in \mathrm{IRS}(\Gamma)$ is ergodic, weakly mixing or non-atomic, then so is $\Psi(\theta)$.*

Proof. First note that the map $\Phi\colon \mathrm{Sub}(\Gamma) \to \mathrm{Sub}(\Delta)$ given by $\Phi(\Lambda) = \varphi^{-1}(\Lambda)$ is a continuous injection with image

$$\Phi(\mathrm{Sub}(\Gamma)) = \{\Lambda \in \mathrm{Sub}(\Delta) \mid \ker(\varphi) \subseteq \Lambda\}.$$

Therefore $\Phi(\mathrm{Sub}(\Gamma)) \subseteq \mathrm{Sub}(\Delta)$ is closed and $\Phi\colon \mathrm{Sub}(\Gamma) \to \Phi(\mathrm{Sub}(\Gamma))$ is a homeomorphism.

Next let $\theta \in \mathrm{IRS}(\Gamma)$ and let θ^* denote the pushforward of θ through Φ. Then

$$\theta^*(N_F^{\Delta}) = \theta(N_{\varphi(F)}^{\Gamma})$$

for any finite $F \subseteq \Delta$. Hence $\theta^* \in \mathrm{IRS}(\Delta)$ and θ^* is supported on $\Phi(\mathrm{Sub}(\Gamma))$. Moreover, the map $\Psi\colon \mathrm{IRS}(\Gamma) \to \mathrm{IRS}(\Delta)$ given by $\Psi(\theta) = \theta^*$ is injective and continuous by Proposition 5.3.5. Therefore $\Psi(\mathrm{IRS}(\Gamma)) \subseteq \mathrm{IRS}(\Delta)$ is closed and $\Psi\colon \mathrm{IRS}(\Gamma) \to \Psi(\mathrm{IRS}(\Gamma))$ is a homeomorphism. It is now straightforward to check that if θ is ergodic, weakly mixing or non-atomic, then so is $\Psi(\theta)$. □

Note that it follows by Proposition 7.4.2 that for any countable group Γ there is an embedding $\iota\colon \mathrm{IRS}(\Gamma) \to \mathrm{IRS}(\mathbb{F}_\infty)$.

Remark 7.4.3. Assume we are in the setting of Proposition 7.4.2 and its proof. If $\theta \in \mathrm{IRS}(\Gamma)$ and $a \in A(\Gamma, X, \mu)$ satisfy $\mathrm{type}(a) = \theta$, then the action $b \in A(\Delta, X, \mu)$ given by $\delta \cdot^b x = \varphi(\delta) \cdot^a x$ will satisfy $\mathrm{type}(b) = \Psi(\theta)$.

We can now apply Proposition 7.4.2 to obtain the following two constructions.

Construction 7.4.4 (Free products with normal amalgamation)**.** We will here construct continuum many non-atomic, weakly mixing invariant random subgroups of $G *_A H$, where G and H are countable groups, $A \leq G, H$ is a shared normal subgroup such that G/A is non-trivial and H/A is infinite.

First note that we have a natural surjective group homomorphism $\varphi\colon G *_A H \to G/A * H/A$. Hence it follows by Proposition 7.4.2 that there is an embedding $\Psi\colon \mathrm{IRS}(G/A * H/A) \to \mathrm{IRS}(G *_A H)$ such that $\Psi(\theta)$ is non-atomic and weakly mixing if θ is non-atomic and weakly mixing. Now let $(\nu_\lambda)_{\lambda\in(0,1)}$ denote the continuum size family of non-atomic, weakly mixing invariant random subgroups of $G/A*H/A$ obtained by Construction 7.4.1. Then the family $(\Psi(\theta_\lambda))_{\lambda\in(0,1)}$ constitutes a continuum size family of non-atomic, weakly mixing invariant random subgroups of $G *_A H$. •

Construction 7.4.5 (Countable free products)**.** We will here construct continuum many non-atomic, weakly mixing invariant random subgroups of the countable free products of the form $*_{i\in\omega} H_i$, where $(H_i)_i$ is a sequence of countable groups with H_0 infinite and H_1 non-trivial.

This is done exactly as in Construction 7.4.4 by considering the natural surjective group homomorphism $\varphi\colon *_{i\in\omega} H_i \to H_0 * H_1$. •

Chapter 8

Characteristic random subgroups of $\mathbb{F}_2$

In this chapter we will use the co-induction operation to construct continuum many non-atomic characteristic random subgroups of $\mathbb{F}_2$. Moreover, these characteristic random subgroups will be weakly mixing with respect to the action of $\mathrm{Aut}(\mathbb{F}_2)$. Other examples of such families are already known. As already mentioned, in [8] they obtain continuum size families of non-atomic characteristic random subgroups on the non-abelian free groups that are weakly mixing with respect to the usual conjugation action of the group itself. Their construction uses Pontryagin duality and a deep result of Adian in combinatorial group theory, whereas the construction we present here is very elemental.

Our idea is to identify $\mathbb{F}_2$ with the normal subgroup of inner automorphisms $\mathrm{Inn}(\mathbb{F}_2) \leq \mathrm{Aut}(\mathbb{F}_2)$. Note that the action $\mathrm{Aut}(\mathbb{F}_2) \curvearrowright \mathrm{Sub}(\mathbb{F}_2)$ given by $\varphi \cdot \Lambda = \varphi(\Lambda)$ corresponds to the conjugation action $\mathrm{Aut}(\mathbb{F}_2) \curvearrowright \mathrm{Sub}(\mathrm{Inn}(\mathbb{F}_2))$. Therefore in order to obtain a continuum size family of non-atomic characteristic random subgroups of $\mathbb{F}_2$ that are each weakly mixing with respect to the action of $\mathrm{Aut}(\mathbb{F}_2)$, it suffices to ensure that Condition (1') in Remark 7.1.2 is satisfied for the pair $\mathrm{Inn}(\mathbb{F}_2) \leq \mathrm{Aut}(\mathbb{F}_2)$. To do so, we will use small cancellation theory, which is a useful tool to show that a given element of a non-abelian free group does not lie in a specific normal subgroup.

In the first section we will briefly introduce the small cancellation theory needed for our purposes. We will then prove the main result of this chapter, which will allow us to conclude that Condition (1') in Remark 7.1.2 is satisfied. In the second section we will construct a continuum size family of non-atomic characteristic invariant random subgroups of $\mathbb{F}_2$ that are weakly mixing with respect to the action of $\mathrm{Aut}(\mathbb{F}_2)$. This will be an easy consequence of the main result from the first section.

All results and constructions in this chapter, except Theorem 8.1.2, have been obtained in joint work with Alexander S. Kechris and can also be found in [22].

8.1 Small cancellation theory

We will here present a few notions and a result from small cancellation theory. Afterwards we will state and prove the main theorem of this chapter.

We will below consider the non-abelian free groups. So let $n \in \omega \cup \{\infty\}$ with $n \geq 2$ and choose a basis $\mathbb{F}_n = \langle a_i \mid i < n \rangle$. We will now fix some terminology. A *word* is a product $s_0 s_1 \cdots s_{k-1}$, where $s_j \in \left\{a_i, a_i^{-1} \mid i < n\right\}$ for all $j < k$. We say that a word $s_0 s_1 \cdots s_{k-1}$ is *reduced* if $s_j \neq s_{j+1}^{-1}$ for all $j < k-1$, and a reduced word $s_0 s_1 \cdots s_{k-1}$ is called *cyclically reduced* if $s_0 \neq s_{k-1}^{-1}$. We think of an element $x \in \mathbb{F}_n$ as represented by the unique reduced word $s_0 s_1 \cdots s_{k-1}$ such that $x = s_0 s_1 \cdots s_{k-1}$. In particular, when we talk about a word or an element of $\mathbb{F}_n$, we mean the reduced word that represents it. Moreover, we will denote by $|x|$ the length of this word. For a subset $S \subseteq \mathbb{F}_n$ we let $\tilde{S}$ denote the set of all cyclically reduced cyclic conjugates of the words in S and their inverses.

Definition 8.1.1. Let $n \in \omega \cup \{\infty\}$ be such that $n \geq 2$. A subset $S \subseteq \mathbb{F}_n$ satisfies the $C'(1/6)$ *cancellation property* if whenever $u \in \mathbb{F}_n$ is an initial segment of $x, y \in \tilde{S}$ with $x \neq y$, then $|u| < \frac{1}{6} \min\{|x|, |y|\}$.

The next theorem highlights why this property is of interest to us. A proof can be found in [24, Theorem 4.5 in Chapter V].

Theorem 8.1.2. *Let $n \in \omega \cup \{\infty\}$ be such that $n \geq 2$. If $S \subseteq \mathbb{F}_n$ satisfies the $C'(1/6)$ cancellation property and $z \in \langle\langle S \rangle\rangle \setminus \tilde{S}$ is a cyclically reduced word, then there is $x \in \tilde{S}$ such that $|x| < |z|$.*

The previous theorem states that if we consider the normal subgroup $\langle\langle S \rangle\rangle$ of $\mathbb{F}_n$ induced by a set S of words satisfying the $C'(1/6)$ cancellation property, then any cyclically reduced $x \in \langle\langle S \rangle\rangle$, which is not a cyclic conjugate of an element in S, must be longer than the shortest element in $\tilde{S}$.

The goal for the rest of this section is to prove the following theorem.

Theorem 8.1.3. *Fix a basis $\mathbb{F}_2 = \langle a, b \rangle$ and let $w = aba^2b^2 \cdots a^n b^n$ for some $n > 101$. Then there is a transversal T for the left cosets in* $\mathrm{Aut}(\mathbb{F}_2)/\mathrm{Inn}(\mathbb{F}_2)$

such that the set

$$\{\eta(w) \mid \eta \in T\}$$

satisfies the $C'(1/6)$ cancellation property.

Fix a basis $\mathbb{F}_2 = \langle a, b\rangle$ and consider the automorphisms $\chi, \xi, \varphi, \psi, \tau \in \mathrm{Aut}(\mathbb{F}_2)$ given by

$$\chi(a) = a, \qquad \chi(b) = b^{-1}, \qquad \xi(a) = a^{-1}, \qquad \xi(b) = b, \qquad \tau(a) = b,$$

$$\tau(b) = a, \qquad \varphi(a) = ab, \qquad \varphi(b) = b, \qquad \psi(a) = a \quad \text{and} \quad \psi(b) = ba.$$

Let $\mathrm{Fr}_+(\varphi, \psi)$ denote the set of automorphisms generated by using only φ and ψ (and not φ^{-1}, ψ^{-1}). Then it follows by [12, Section 3] that

$$T = \{\rho,\ \rho\tau,\ \xi\sigma, \xi\sigma\tau,\ \rho\xi,\ \rho\tau\xi,\ \xi\sigma\xi,\ \xi\sigma\tau\xi,\ \rho\chi,\ \rho\tau\chi,\ \xi\sigma\chi,$$
$$\xi\sigma\tau\chi,\ \rho\xi\chi,\ \rho\tau\xi\chi,\ \xi\sigma\xi\chi,\ \xi\sigma\tau\xi\chi \mid \sigma, \rho \in \mathrm{Fr}_+(\varphi, \psi),\ \sigma \neq 1\}$$

is a transversal for the left cosets in $\mathrm{Aut}(\mathbb{F}_2)/\mathrm{Inn}(\mathbb{F}_2)$. Note that $1 \in \mathrm{Fr}_+(\varphi, \psi)$ denotes the identity map. Consider also the word

$$w = aba^2b^2a^3b^3 \cdots a^nb^n$$

for some fixed $n > 101$. The rest of this section constitutes a proof of the fact that the family

$$\{\eta(w) \mid \eta \in T\}$$

satisfies the $C'(1/6)$ cancellation property. This is done by a case-by-case analysis.

Put

$$\begin{aligned}
w_0 &= w = aba^2b^2 \cdots a^nb^n \\
w_1 &= \xi\chi(w) = a^{-1}b^{-1}a^{-2}b^{-2} \cdots a^{-n}b^{-n} \\
w_2 &= \xi(w) = a^{-1}ba^{-2}b^2 \cdots a^{-n}b^n \\
w_3 &= \chi(w) = ab^{-1}a^2b^{-2} \cdots a^nb^{-n} \\
w_4 &= \tau w = bab^2a^2 \cdots b^na^n \\
w_5 &= \tau\xi\chi(w) = b^{-1}a^{-1}b^{-2}a^{-2} \cdots b^{-n}a^{-n} \\
w_6 &= \tau\xi(w) = b^{-1}ab^{-2}a^2 \cdots b^{-n}a^n \\
w_7 &= \tau\chi(w) = ba^{-1}b^2a^{-2} \cdots b^na^{-n}.
\end{aligned}$$

and let $v_i = w_i^{-1}$ for all $0 \leq i \leq 7$. Below we will use the following terminology. For two words $x, y \in \mathbb{F}_2$ a *cancellation* of x and y is a string $u \in \mathbb{F}_2$

which appears in the reduced cycles of both x and y. We say that u is a *bad cancellation* of x and y if

$$|u| \geq 1/6 \min\{\|x\|, \|y\|\}.$$

Here $\|\cdot\|$ denotes the length of the induced cyclically reduced word. We call a cancellation *maximal* if it cannot be extended. The goal is then to prove that there is no bad cancellation between any pair of words in the set

$$B = \{\rho(w_i), \xi\sigma(w_i), \rho(v_i), \xi\sigma(v_i) \mid \sigma, \rho \in \mathrm{Fr}_+(\varphi, \psi),\ \sigma \neq 1,\ 0 \leq i \leq 7\}.$$

Let

$$\begin{aligned} B_0 &= \{\rho(w_i), \rho(v_i) \mid \rho \in \mathrm{Fr}_+(\varphi, \psi),\ 0 \leq i \leq 7\} \\ B_1 &= \{\xi\sigma(w_i), \xi\sigma(v_i) \mid \sigma \in \mathrm{Fr}_+(\varphi, \psi) \setminus \{1\},\ 0 \leq i \leq 7\}. \end{aligned}$$

Then it suffices to prove that there is no bad cancellation among the words in B_0, and then prove that there cannot be any bad cancellation between a word from B_0 and a word from B_1.

Before we do this, we will prove two lemmas upon which most of the remaining arguments are based.

To state the first lemma, we will for a word $x \in \mathbb{F}_2$ let $\overline{x} \in \mathbb{F}_2$ denote the word obtained from x by switching every negative power of a and b to be positive.

Lemma 8.1.4. *Let $x, y \in \mathbb{F}_2$, $\rho \in \mathrm{Fr}_+(\varphi, \psi)$ and let q be a cancellation of $\rho(x)$ and $\rho(y)$. Assume $N \in \omega$ satisfies that for any cancellation c of x and y the total number of a's and the total number of b's in $\overline{c}$ are both less than N. Then*

$$|q| \leq (N+2)\left(|\rho(a)| + |\rho(b)|\right).$$

Proof. First let $S \in \{\varphi, \psi\}$ and $u \in \{a, b\}$ be such that $u = a \iff S = \psi$. Assume that q is a maximal cancellation of $S(x)$ and $S(y)$. By checking the preimages of all possible neighbourhoods of q in the reduced cycle induced by $S(x)$ and $S(y)$, one finds that there is a maximal cancellation c of x and y such that q is equal to one of the strings:

$$S(c), \quad uS(c), \quad S(c)u^{-1} \quad \text{or} \quad uS(c)u^{-1}.$$

Next for $\rho \in \mathrm{Fr}_+(\varphi, \psi)$ we let $S_0, \dots, S_N \in \{\varphi, \psi\}$ and $u_0, \dots, u_N \in \{a, b\}$ be such that $\rho = S_N \cdots S_0$ and $u_i = a \iff S_i = \psi$ for all $0 \leq i \leq N$. Observe that

$$\rho(ab) = abu_N S_N(u_{N-1})(S_N S_{N-1})(u_{N-2}) \cdots (S_N \cdots S_1)(u_0)$$

and $|\rho(ab)| = |\rho(a)| + |\rho(b)|$. Now let q be a maximal cancellation of $\rho(x)$ and $\rho(y)$. Then, by repeating the argument above, there is a maximal cancellation c of x and y such that

$$\begin{aligned}|q| &\leq |u_N| + \sum_{j=1}^{N} |(S_N \cdots S_j)(u_{j-1})| + |\rho(c)| + \sum_{j=1}^{N} |(S_N \cdots S_j)(u_{j-1}^{-1})| + |u_N^{-1}| \\ &\leq 2|\rho(ab)| + |\rho(c)| \\ &\leq (N+2)\left(|\rho(a)| + |\rho(b)|\right),\end{aligned}$$

as wanted. □

Note that the proof above also shows that if C is the set of cancellations between x and y, then for any cancellation q of $\rho(x)$ and $\rho(y)$ we have

$$|q| \leq \max\{\ |\rho(c)| \ \mid c \in C\} + 2(|\rho(a)| + |\rho(b)|).$$

The previous lemma provides a tool to bound the length of a cancellation between two words in B_0 from above. The next lemma bounds the length of a word in B_0 from below. We will in the following call $x \in \mathbb{F}_2$ *positive* if x consists only of positive powers of a and b. Similarly, we say $x \in \mathbb{F}_2$ is *negative* if x consists only of negative powers of a and b. It is clear that if x is either positive or negative, then $|x| = \|x\|$.

Lemma 8.1.5. *Let $\rho \in \mathrm{Fr}_+(\varphi, \psi)$ and $z \in \{w_0, v_0, \ldots, w_7, v_7\}$. Then*

$$\|\rho(z)\| \geq \left(\frac{n(n-1)}{2} - 2n\right)\left(|\rho(a)| + |\rho(b)|\right).$$

Proof. It is enough to consider $w_0, \ldots, w_7$. If $z \in \{w_0, w_1, w_4, w_5\}$, then

$$\|\rho(z)\| = \frac{n(n-1)}{2}\left(|\rho(a)| + |\rho(b)|\right),$$

since there is no cancellation.

For the remaining cases, we will begin with some observations. Assume $p, p_+, p_- \in \mathbb{F}_2$ satisfy that $p = p_+p_-$ is reduced, p_+ is positive and p_- is negative. Moreover, let $S \in \{\varphi, \psi\}$ and $u \in \{a, b\}$ be such that $u = a \iff S = \psi$. Then $S(p) = S(p_+)S(p_-)$. If both p_+ and p_- are non-trivial, then $S(p_+)$ will end with u and $S(p_-)$ will begin with u^{-1}. Thus uu^{-1} will be removed in the product. Since p_+p_- is reduced, there will not be any other reduction in $S(p_+)S(p_-)$. Note also that $S(p_+)$ is positive and $S(p_-)$ is negative. Lastly, note that if instead $p = p_-p_+$ is reduced, then $S(p) = S(p_-)S(p_+)$ is also reduced.

Now let $x \in \mathbb{F}_2$ be neither positive nor negative and let $\rho \in \mathrm{Fr}_+(\varphi, \psi)$. Fix $S_0 \ldots, S_N \in \{\varphi, \psi\}$ and $u_0, \ldots, u_N \in \{a, b\}$ such that $\rho = S_N \cdots S_0$ and $u_i = a \iff S_i = \psi$ for all $0 \leq i \leq N$.

First assume that $S_j \cdots S_0(x)$ is neither positive nor negative for all $0 \leq j < N$. Then for each $0 \leq i \leq N$ we fix $k_i \geq 1$ together with positive $p^1_{(i,+)}, \ldots p^{k_i}_{(i,+)} \in \mathbb{F}_2 \setminus \{e_{\mathbb{F}_2}\}$ and negative $p^1_{(i,-)}, \ldots, p^{k_i}_{(i,-)} \in \mathbb{F}_2 \setminus \{e_{\mathbb{F}_2}\}$ such that for each $0 \leq i < N$ there is a cyclically reduced cyclic conjugate of x and of $S_i \cdots S_0(x)$ of the form

$$p^1_{(0,+)} p^1_{(0,-)} p^2_{(0,+)} p^2_{(0,-)} \cdots p^{k_0}_{(0,+)} p^{k_0}_{(0,-)}$$

and

$$p^1_{(i+1,+)} p^1_{(i+1,-)} p^2_{(i+1,+)} p^2_{(i+1,-)} \cdots p^{k_{i+1}}_{(i+1,+)} p^{k_{i+1}}_{(i+1,-)},$$

respectively. Then $k_0 \geq k_1 \geq \cdots \geq k_N$ and hence, by the observations above, we have

$$\begin{aligned}\|\rho(x)\| &= |\rho(\overline{x})| - 2k_N |u_N| - \sum_{i=0}^{N-1} 2k_i |S_N \cdots S_{i+1}(u_i)| \\ &\geq |\rho(\overline{x})| - 2k_0 \left(|\rho(a)| + |\rho(b)|\right).\end{aligned}$$

Next assume that $0 \leq j < N$ is least such that $S_j \cdots S_0(x)$ is either positive or negative. Then, as before, we may for each $0 \leq i \leq j$ chose $l_i \geq 1$ together with positive $q^1_{(i,+)}, \ldots q^{l_i}_{(i,+)} \in \mathbb{F}_2 \setminus \{e_{\mathbb{F}_2}\}$ and negative $q^1_{(i,-)}, \ldots, q^{l_i}_{(i,-)} \in \mathbb{F}_2 \setminus \{e_{\mathbb{F}_2}\}$ such that for each $0 \leq i < j$ there is a cyclically reduced cyclic conjugate of x and of $S_i \cdots S_0(x)$ of the form

$$q^1_{(0,+)} q^1_{(0,-)} q^2_{(0,+)} q^2_{(0,-)} \cdots q^{l_0}_{(0,+)} q^{l_0}_{(0,-)}$$

and

$$q^1_{(i+1,+)} q^1_{(i+1,-)} q^2_{(i+1,+)} q^2_{(i+1,-)} \cdots q^{l_{i+1}}_{(i+1,+)} q^{l_{i+1}}_{(i+1,-)},$$

respectively. Then $l_0 \geq l_1 \geq \cdots \geq l_j$ and hence, by the observations above, we have

$$\begin{aligned}\|\rho(x)\| &= |\rho(\overline{x})| - \sum_{i=0}^{j} 2l_i |S_N \cdots S_{i+1}(u_i)| \\ &\geq |\rho(\overline{x})| - 2l_0 \left(|\rho(a)| + |\rho(b)|\right).\end{aligned}$$

Finally, note that for any $z \in \{w_2, w_3, w_6, w_7\}$ we can chose $k_0, l_0 = n$ in the argument above. Thus, as

$$|\rho(\overline{z})| = \frac{n(n-1)}{2} \left(|\rho(a)| + |\rho(b)|\right),$$

we obtain

$$\|\rho(z)\| \geq \left(\frac{n(n-1)}{2} - 2n\right)(|\rho(a)| + |\rho(b)|),$$

as desired. □

Note that

$$8n < \frac{1}{6}\left(\frac{n(n-1)}{2} - 2n\right),$$

since $n > 101$. We will use this repeatedly below to conclude that there is no bad cancellation in the various cases.

We will now begin to argue that there is no bad cancellation between two words from B_0. The following decomposition will be useful. For $m \in \{1, \dots, n\}$ let

$$w_0^m = a^m b^m \qquad w_1^m = a^{-m} b^{-m} \qquad w_3^m = a^m b^{-m}$$
$$w_4^m = b^m a^m \qquad w_5^m = b^{-m} a^{-m} \qquad w_7^m = b^m a^{-m}.$$

Moreover, for $m \in \{1, \dots, n-1\}$ put

$$w_2^m = b^m a^{-m-1} \qquad w_2^n = b^n a^{-1} \qquad w_6^m = a^m b^{-m-1} \qquad w_6^n = a^n b^{-1}.$$

Then for $i \in \{0, \dots, 7\}$ we have, up to cyclic permutation, that

$$w_i = w_i^1 w_i^2 \cdots w_i^n,$$

and that

$$\rho(w_i^1)\rho(w_i^2)\cdots\rho(w_i^n)$$

is a reduced word whenever each factor is reduced for all $\rho \in \mathrm{Fr}_+(\varphi, \psi)$. However, the latter is not necessarily cyclically reduced. If for some $k \geq 1$ we have $\rho = \varphi^k$ or $\rho = \psi^k$ or $i \in \{0, 1, 3, 4, 5, 7\}$, then

$$\rho(w_i^1)\rho(w_i^2)\cdots\rho(w_i^n)$$

is cyclically reduced. If $i \in \{2, 6\}$ and $\rho \notin \{\varphi^k, \psi^k \mid k \in \omega\}$, then any possible reduction in the induced cycle of $\rho(w_i^1)\rho(w_i^2)\cdots\rho(w_i^n)$ is contained in $\rho(w_i^n)\rho(w_i^1)$.

For the remaining part of this section, let $x, y \in \{w_0, v_0, \dots, w_7, v_7\}$ be fixed.

Claim 1: If $\rho \in \mathrm{Fr}_+(\varphi, \psi)$ and $x \neq y$, then there is no bad cancellation between $\rho(x)$ and $\rho(y)$.

Proof of Claim 1: It is easy to check that $N = 2n - 2$ satisfies the assumption of Lemma 8.1.4, since $x \neq y$. Hence any cancellation q between $\rho(x)$ and $\rho(y)$ satisfies

$$|q| \leq 2n\left(|\rho(a)| + |\rho(b)|\right).$$

Therefore, by Lemma 8.1.5, there cannot be any bad cancellation between $\rho(x)$ and $\rho(y)$. ◇

In the following, we let

$$A_\varphi = \{\varphi\rho \mid \rho \in \mathrm{Fr}_+(\varphi, \psi)\} \text{ and } A_\psi = \{\psi\rho \mid \rho \in \mathrm{Fr}_+(\varphi, \psi)\}.$$

Note that $\mathrm{Fr}_+(\varphi, \psi) \setminus \{1\} = A_\varphi \sqcup A_\psi$.

Claim 2: If $\rho, \sigma, \eta \in \mathrm{Fr}_+(\varphi, \psi)$ with $\rho = \eta\sigma$ and $\sigma \neq 1$, then there is no bad cancellation between $\rho(x)$ and $\eta(y)$.

Proof of Claim 2: Note that either $\sigma \in A_\varphi$ or $\sigma \in A_\psi$. Assume that we are in the first case. Then the only powers of a occurring in $\sigma(x)$ are a and a^{-1}. Hence any cancellation between $\sigma(x)$ and y is a substring of the cycle induced by y that only contains these powers. Therefore it is easily seen that $N = n+1$ satisfies the assumptions of Lemma 8.1.4 and hence any cancellation q between $\rho(x)$ and $\eta(y)$ satisfies

$$|q| \leq (n+3)\left(|\eta(a)| + |\eta(b)|\right).$$

Moreover, by Lemma 8.1.5, it holds that

$$\|\rho(x)\|, \|\eta(y)\| \geq \left(\frac{n(n-1)}{2} - 2n\right)(|\eta(a)| + |\eta(b)|),$$

since $|\rho(a)| + |\rho(b)| \geq |\eta(a)| + |\eta(b)|$. Therefore there is no bad cancellation between $\rho(x)$ and $\eta(y)$. A similar argument applies if $\sigma \in A_\psi$. ◇

Now we will take care of the case where $\rho_1, \rho_2 \in \mathrm{Fr}_+(\varphi, \psi)$ are distinct, but none of them extends the other.

Claim 3: If $\rho_1, \rho_2, \sigma_1, \sigma_2, \eta_1, \eta_2 \in \mathrm{Fr}_+(\varphi, \psi)$ satisfy

$$\sigma_1 \in A_\varphi, \qquad \sigma_2 \in A_\psi, \qquad \rho_1 = \eta_1\sigma_1 \qquad \text{and} \qquad \rho_2 = \eta_2\sigma_2,$$

then there is no bad cancellation between $\rho_1(x)$ and $\rho_2(y)$.

Proof of Claim 3: First note that $\sigma_1(x)$ will only contain a and a^{-1} as powers of a, while $\sigma_2(y)$ will only contain b and b^{-1} as powers of b.

We claim that for each $i \in \{0, \ldots, 7\}$ and $m \in \{3, \ldots, n-1\}$ we have that $\sigma_1(w_i^m)$ contains the string b^l or b^{-l} for some $l \geq 2$. Indeed, σ_1 is of one of the forms φ^k, $\sigma_1^0\psi\varphi^k$, $\varphi\psi^k$ or $\sigma_1^0\varphi\psi^k$ for some $k \geq 1$ and $\sigma_1^0 \in A_\varphi$. By straightforward calculations, the statement is clearly true for $\sigma_1 = \varphi^k$ or $\sigma_1 = \varphi\psi^k$. To see that the statement also holds in the remaining cases, one may consider $\varphi\psi^k(w_i^m)$ and $\psi\varphi^k(w_i^m)$, and then use the fact that $\sigma_1^0(a) = aub$ and $\sigma_1^0(b) = bzb$ or $\sigma_1^0(b) = b$ for some positive $u, z \in \mathbb{F}_2$.

Similarly, for each $i \in \{0, \ldots, 7\}$ and $m \in \{3, \ldots, n-1\}$ we have that $\sigma_2(w_i^m)$ contains the string a^l or a^{-l} for some $l \geq 2$.

Now let $i, j \in \{0, \ldots, 7\}$ satisfy that $x \in \{w_i, v_i\}$ and $y \in \{w_j, v_j\}$. Then any cancellation q between $\sigma_1(x)$ and $\sigma_2(y)$ is contained in either

$$\sigma_1(w_i^{n-1})\sigma_1(w_i^n)\sigma_1(w_i^1)\sigma_1(w_i^2)\sigma_1(w_i^3),$$

$$\sigma_1(w_i^m)\sigma_1(w_i^{m+1})$$

or in one of their inverses for some $m \in \{3, \ldots, n-2\}$. Therefore, by Lemma 8.1.4, we have

$$|q| \leq 3n(|\rho_1(a)| + |\rho_1(b)|)$$

and hence, by Lemma 8.1.5, we obtain $|q| < \frac{1}{6}\|\rho_1(x)\|$.

Similarly, any cancellation q between $\sigma_1(x)$ and $\sigma_2(y)$ is contained in either

$$\sigma_2(w_j^{n-1})\sigma_2(w_j^n)\sigma_2(w_j^1)\sigma_2(w_j^2)\sigma_2(w_j^3),$$

$$\sigma_2(w_j^m)\sigma_2(w_j^{m+1})$$

or in one of their inverses for some $m \in \{3, \ldots, n-2\}$. Therefore, by Lemma 8.1.4 and Lemma 8.1.5, we also have $|q| < \frac{1}{6}\|\rho_2(y)\|$. Hence there cannot be any bad cancellation between $\rho_1(x)$ and $\rho_2(y)$. $\diamond$

Putting together Claim 1, Claim 2 and Claim 3 we may conclude that there is no bad cancellation between two words in B_0, i.e., that B_0 satisfies the $C'(1/6)$ cancellation property.

We will now prove that there is no bad cancellation between a word from B_0 and a word from B_1. To do so, let $A_\varphi^0 = \{\varphi^k \mid k \geq 1\}$, $A_\psi^0 = \{\psi^k \mid k \geq 1\}$ and

$$A^1 = \left\{\eta\varphi\psi^k, \eta\psi\varphi^k \mid k \geq 1, \eta \in \mathrm{Fr}_+(\varphi, \psi)\right\}.$$

Then $\mathrm{Fr}_+(\varphi, \psi) \setminus \{1\} = A_\varphi^0 \sqcup A_\psi^0 \sqcup A^1$. We will still consider fixed $x, y \in \{w_0, v_0, \ldots, w_7, v_7\}$. Let also $i, j \in \{0, \ldots, 7\}$ be fixed such that $x \in \{w_i, v_i\}$ and $y \in \{w_j, v_j\}$.

Claim 4: If $\rho, \sigma \in \mathrm{Fr}_+(\varphi, \psi)$ with $\rho = 1$ and $\sigma \neq 1$, then there is no bad cancellation between $\rho(x)$ and $\xi\sigma(y)$.

Proof of Claim 4: This follows by the same arguments as the ones used in the beginning of the proof of Claim 2. $\diamond$

Claim 5: If $(\rho, \sigma) \in A_\varphi \times A_\psi$ or $(\rho, \sigma) \in A_\psi \times A_\varphi$, then there is no bad cancellation between $\rho(x)$ and $\xi\sigma(y)$.

Proof of Claim 5: This follows by arguments similar to those in the beginning of the proof of Claim 3. $\diamond$

Claim 4 ensures that we may assume that both $\rho, \psi \in A^0_\varphi \sqcup A^0_\psi \sqcup A^1$. Moreover, by Claim 5, there is no bad cancellation between $\rho(x)$ and $\xi\sigma(y)$ in the case where $\rho \in A^0_\varphi$ and $\sigma \in A^0_\psi$ or in the case where $\rho \in A^0_\psi$ and $\sigma \in A^0_\varphi$. Through the next three claims, we prove that there is no bad cancellation within each of these sets.

Claim 6: If $k, l \geq 1$, then there is no bad cancellation between $\varphi^k(x)$ and $\xi\varphi^l(y)$.

Proof of Claim 6: Consider $\varphi^t(u)$ and $\xi\varphi^t(z)$ for $u, z \in \{w_0, v_0, \dots, w_7, v_7\}$ and $t \geq 1$. Within each of these words either all the powers of a are positive or all the powers of a are negative. Below we have put these observations into a table. Here $+$ and $-$ refer to the sign of the occurring powers of a.

	w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7
φ^k	$+$	$-$	$-$	$+$	$+$	$-$	$+$	$-$
$\xi\varphi^l$	$-$	$+$	$+$	$-$	$-$	$+$	$-$	$+$

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7
φ^k	$-$	$+$	$+$	$-$	$-$	$+$	$-$	$+$
$\xi\varphi^l$	$+$	$-$	$-$	$+$	$+$	$-$	$+$	$-$

It is easily seen that if the signs of the powers of a do not match, then there is no bad cancellation between $\varphi^k(z)$ and $\xi\varphi^l(u)$ for the corresponding $u, z \in \{w_0, v_0, \dots, w_7, v_7\}$. Assume therefore that the sign of the powers of a in $\varphi^k(x)$ and the sign of the powers of a in $\xi\varphi^l(y)$ match. Then the signs of the powers of a in x and y do not match. We assume that the powers of a in x are negative and that the powers of a in y are positive. The other case is handled similarly. If $x = w_i$, then the string $(\varphi^k(a^{-1}))^2 = (b^{-k}a^{-1})^2$ is contained in $\varphi^k(w_i^m)$ for all $m \in \{3, \dots, n-1\}$. If $x = v_i$, then the string $(b^{-k}a^{-1})^2$ is contained in $\varphi^k(w_i^m)^{-1}$ for all $m \in \{3, \dots, n-1\}$. Similarly, if $y = w_j$, then the string $(\xi\varphi^l(a))^2 = (a^{-1}b^l)^2$ is contained in $\xi\varphi^l(w_j^m)$ for all $m \in \{3, \ldots, n-1\}$. If $y = v_j$, then the string $(a^{-1}b^l)^2$ is contained in $\xi\varphi^l(w_j^m)^{-1}$ for all $m \in \{3, \ldots, n-1\}$. Moreover, the string $(b^{-k}a^{-1})^2$ does not appear in $\xi\varphi^l(y)$ and the string $(a^{-1}b^l)^2$ does not appear in $\varphi^k(x)$. Any cancellation q between $\varphi^k(x)$ and $\xi\varphi^l(y)$ is therefore contained in either

$$\varphi^k(w_i^{n-1})\varphi^k(w_i^n)\varphi^k(w_i^1)\varphi^k(w_i^2)\varphi^k(w_i^3),$$

$$\varphi^k(w_i^m)\varphi^k(w_i^{m+1})$$

or in one of their inverses for some $m \in \{3, \ldots, n-2\}$. Similarly, q is also contained in either

$$\xi\varphi^l(w_j^{n-1})\xi\varphi^l(w_j^n)\xi\varphi^l(w_j^1)\xi\varphi^l(w_j^2)\xi\varphi^l(w_j^3),$$

$$\xi\varphi^l(w_j^m)\xi\varphi^l(w_j^{m+1})$$

or in one of their inverses for some $m \in \{3, \ldots, n-2\}$. Therefore, by Lemma 8.1.4, we have

$$|q| \leq 3n \min\Big\{|\varphi^k(a)| + |\varphi^k(b)|, |\xi\varphi^l(a)| + |\xi\varphi^l(b)|\Big\}$$

and hence, by Lemma 8.1.5, there is no bad cancellation between $\varphi^k(x)$ and $\xi\varphi^l(y)$. $\diamond$

Claim 7: If $k, l \geq 1$, then there is no bad cancellation between $\psi^k(x)$ and $\xi\psi^l(y)$.

Proof of Claim 7: First consider the sign of the powers of b occurring in $\psi^k(u)$ and $\xi\psi^l(z)$ for $u, z \in \{w_0, v_0, \ldots, w_7, v_7\}$.

	w_0	w_1	w_2	w_3	w_4	w_5	w_6	w_7
ψ^k	+	−	+	−	+	−	−	+
$\xi\psi^l$	+	−	+	−	+	−	−	+

	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7
ψ^k	−	+	−	+	−	+	+	−
$\xi\psi^l$	−	+	−	+	−	+	+	−

It is easily seen that if the sign in two cells does not match, then there is no bad cancellation between $\psi^k(u)$ and $\xi\psi^l(z)$ for the corresponding $u, z \in \{w_0, v_0, \ldots, w_7, v_7\}$. In the case where the signs of the powers of b are the same, one may use a similar argument as the one in Claim 7. Assume first that the signs of b are both positive. If $x = w_i$, then the string $(ba^k)^2$ is contained in $\psi^k(w_i^m)$ for all $m \in \{3, \ldots, n-1\}$. If $x = v_i$, then the string $(ba^k)^2$ is contained in $\psi^k(w_i^m)^{-1}$ for all $m \in \{3, \ldots, n-1\}$. Similarly, if $y = w_j$, then the string $(ba^{-l})^2$ is contained in $\xi\psi^l(w_j^m)$ for all $m \in \{3, \ldots, n-1\}$. If $y = v_j$, then the string $(ba^{-l})^2$ is contained in $\xi\psi^l(w_j^m)^{-1}$ for all $m \in \{3, \ldots, n-1\}$. Moreover, the string $(ba^k)^2$ will not be contained in $\xi\psi^l(y)$, while the string $(ba^{-l})^2$ will not be contained in $\psi^k(x)$. Thus one may deduce, as in the proof of Claim 6, that there cannot be any bad cancellation between $\psi^k(x)$ and $\xi\psi^l(y)$ in this case. In the case where the powers of b are both negative a similar argument will apply. $\diamond$

Claim 8: If $\rho, \sigma \in A^1$, then there is no bad cancellation between $\rho(x)$ and $\xi\sigma(y)$.

Proof of Claim 8: First let $k \geq 1$. By considering the form of $\psi\varphi^k(z)$ and $\varphi\psi^k(z)$ for the words $z \in \{w_0, v_0, \ldots, w_7, v_7\}$, one finds that for every $\eta \in \mathrm{Fr}_+(\varphi, \psi)$ the words $\eta\psi\varphi^k(z)$ and $\eta\varphi\psi^k(z)$ will contain at most one occurrence of one of the strings

$$ab^{-t}a, \qquad a^{-1}b^ta^{-1}, \qquad ba^{-t}b \quad \text{or} \quad b^{-1}a^tb^{-1}$$

for some $t \geq 1$. Moreover, for all $r \in \{0, \ldots, 7\}$ and $m \in \{2, \ldots, n-1\}$ the words $\eta\varphi\psi^k(w_r^m)$ and $\eta\psi\varphi^k(w_r^m)$ contain at least one of the strings

$$ab^ta, \qquad a^{-1}b^{-t}a^{-1}, \qquad ba^tb \quad \text{or} \quad b^{-1}a^{-t}b^{-1}$$

for some $t \geq 1$. This is again straightforward to check by considering the form of $\psi\varphi^k(w_r^m)$ and $\varphi\psi^k(w_r^m)$.

The above implies that $\rho(x)$ contains at most one of the strings

$$ab^{-t}a, \qquad a^{-1}b^ta^{-1}, \qquad ba^{-t}b \quad \text{or} \quad b^{-1}a^tb^{-1}$$

for some $t \geq 1$. Moreover, for all $m \in \{2, \ldots, n-1\}$ the word $\rho(w_i^m)$ contains at least one of the strings

$$ab^ta, \qquad a^{-1}b^{-t}a^{-1}, \qquad ba^tb \quad \text{or} \quad b^{-1}a^{-t}b^{-1}$$

for some $t \geq 1$.

Conversely, the above also implies that $\xi\sigma(y)$ contains at most one of the strings

$$ab^ta, \qquad a^{-1}b^{-t}a^{-1}, \qquad ba^tb \quad \text{or} \quad b^{-1}a^{-t}b^{-1}$$

for some $t \geq 1$. Moreover, for all $m \in \{2, \ldots, n-1\}$ the word $\xi\sigma(w_j^m)$ contains at least one of the strings

$$ab^{-t}a, \qquad a^{-1}b^ta^{-1}, \qquad ba^{-t}b \quad \text{or} \quad b^{-1}a^tb^{-1}$$

for some $t \geq 1$.

Therefore, by making considerations and use of Lemma 8.1.4 and Lemma 8.1.5 as in the proofs of the previous claims, we may conclude that there cannot be any bad cancellation between $\rho(x)$ and $\xi\sigma(y)$. $\diamond$

Finally, we will prove that there is no bad cancellation between $\rho(x)$ and $\xi\sigma(y)$ if $\rho \in A^1$ and $\sigma \in A^0_\varphi \cup A^0_\psi$.

Claim 9: If $\rho \in A^1$ and $\sigma \in A^0_\varphi \cup A^0_\psi$, then there is no bad cancellation between $\rho(x)$ and $\xi\sigma(y)$.

Proof of Claim 9: From earlier results it is enough to consider the case where $\rho, \sigma \in A_\varphi$ or $\rho, \sigma \in A_\psi$. Assume that we are in the first case and let $l, k \geq 1$ be such that $\sigma = \varphi^k$ and $\rho = \varphi^l \eta$ for some $\eta \in A_\psi$. Then the only possible powers of b occurring in $\rho(x)$ are $b, b^{-1}, b^l, b^{-l}, b^{l+1}, b^{-l-1}$. Hence if q is a cancellation between $\rho(x)$ and $\sigma(y)$, then there is $m \in \{1, 2, \ldots, n-7\}$ or $t \in \{0, \ldots, 6\}$ such that q is contained in either

$$\sigma(w_j^m)\sigma(w_j^{m+1}) \cdots \sigma(w_j^{m+7}),$$

$$\sigma(w_j^{n-t})\sigma(w_j^{n-t+1}) \cdots \sigma(w_j^n)\sigma(w_j^1)\sigma(w_j^2) \cdots \sigma(w_j^{8-t-1})$$

or in one of their inverses. Therefore, by Lemma 8.1.4, we obtain

$$|q| \leq 8n(|\sigma(a)| + |\sigma(b)|)$$

and hence, by Lemma 8.1.5, we have $|q| < \frac{1}{6}\|\sigma(y)\|$.

To see that $|q| < \frac{1}{6}\|\rho(x)\|$, assume first that $l \geq k$. Then, by Lemma 8.1.5 and the fact that $|\rho(a)| + |\rho(b)| \geq |\sigma(a)| + |\sigma(b)|$, we have

$$\begin{aligned}
\|\rho(x)\| &\geq \left(\frac{n(n-1)}{2} - 2n\right)(|\rho(a)| + |\rho(b)|) \\
&\geq \left(\frac{n(n-1)}{2} - 2n\right)(|\sigma(a)| + |\sigma(b)|) \\
&> 6|q|,
\end{aligned}$$

as wanted. Next assume that $l < k$. Then each of b^l and b^{-l} occurs at most once in $\sigma(y)$. However, by the arguments in Claim 3, we have that for all $m \in \{3, \ldots, n-1\}$ there is $t \geq 2$ such that a^t or a^{-t} occur in $\eta(w_i^m)$. Hence b^l or b^{-l} occur in $\rho(w_i^m)$ for all $m \in \{3, \ldots, n-1\}$. Therefore q is contained in either

$$\rho(w_i^{n-3})\rho(w_i^{n-2})\rho(w_i^{n-1})\rho(w_i^n)\rho(w_i^1)\rho(w_i^2)\rho(w_i^3)\rho(w_i^4)\rho(w_i^5),$$

$$\rho(w_i^{m-1})\rho(w_i^m)\rho(w_i^{m+1})\rho(w_i^{m+1})$$

or in one of their inverses for some $m \in \{4, \ldots, n-3\}$. Thus we obtain

$$|q| \leq 5n\,(|\rho(a)| + |\rho(b)|)$$

and hence, by Lemma 8.1.5, we get $|q| < \frac{1}{6}\|\rho(x)\|$.

We may therefore conclude that there is no bad cancellation between $\rho(x)$ and $\xi\sigma(y)$. If $\rho, \sigma \in A_\psi$ a similar argument works. $\diamond$

Putting all the claims together we finally have a proof of Theorem 8.1.3, which was the goal of this section.

8.2 Application to characteristic invariant subgroups

In this section we apply Theorem 8.1.3 to ensure that Criterion (1') of Remark 7.1.2 is satisfied for the pair $\mathrm{Inn}(\mathbb{F}_2) \leq \mathrm{Aut}(\mathbb{F}_2)$. We use this to construct a continuum size family of non-atomic characteristic random subgroups of $\mathbb{F}_2$ that are all weakly mixing with respect to the action of $\mathrm{Aut}(\mathbb{F}_2)$.

Construction 8.2.1 (Characteristic random subgroups of $\mathbb{F}_2$.)**.** We will here construct continuum many non-atomic characteristic random subgroups of $\mathbb{F}_2$. These characteristic random subgroups will moreover be weakly mixing with respect to the action of $\mathrm{Aut}(\mathbb{F}_2)$.

Fix a basis $\mathbb{F}_2 = \langle a, b\rangle$ and let $w = aba^2b^2 \cdots a^nb^n$ for some $n > 101$. Moreover, by Theorem 8.1.3, we may choose a transversal T for the left cosets in $\mathrm{Aut}(\mathbb{F}_2)/\mathrm{Inn}(\mathbb{F}_2)$ such that the set

$$\{\eta(w) \mid \eta \in T\}$$

satisfies the $C'(1/6)$ cancellation property. Next fix an enumeration $T = \{\eta_i \mid i \in \omega\}$ such that η_0 is the identity. Then, since it follows by Lemma 8.1.5 that $\|\eta_i(w)\| \geq |\eta_0(w)|$ for all $i \geq 1$, we must have

$$\eta_0(w) \notin \langle\langle \eta_i(w) \mid i \geq 1\rangle\rangle_{\mathbb{F}_2}$$

by Theorem 8.1.2. This ensures that Condition (1') of Remark 7.1.2 is satisfied for w and T. Hence it follows by Proposition 7.1.1 that we may construct a continuum size family of non-atomic invariant random subgroups of $\mathrm{Inn}(\mathbb{F}_2)$ which are moreover invariant and weakly mixing with respect to the conjugation action of $\mathrm{Aut}(\mathbb{F}_2)$. By use of the natural identification of $\mathbb{F}_2$ with $\mathrm{Inn}(\mathbb{F}_2)$, we obtain continuum many characteristic random subgroups of $\mathbb{F}_2$ which are weakly mixing with respect to the natural action of $\mathrm{Aut}(\mathbb{F}_2)$. $\bullet$

The method for constructing characteristic random subgroups of $\mathbb{F}_2$ presented above is rather elemental. We end this section by discussing the possibility of generalizing the idea behind the construction to a general non-abelian free group.

Let $n \in \omega \cup \{\infty\}$ with $n \geq 2$ and fix a basis $\mathbb{F}_n = \langle a_i \mid i < n \rangle$. For all $z \in \mathbb{F}_n$ and $i, j < n$ with $i \neq j$ consider the automorphisms $\rho_z, \varphi_i, \psi_{i,j} \in \mathrm{Aut}(\mathbb{F}_n)$ given by $\rho_z(x) = zxz^{-1}$,

$$\varphi_i(a_k) = \begin{cases} a_k & \text{if} \quad k \neq i \\ a_k^{-1} & \text{if} \quad k = i \end{cases}$$

and

$$\varphi_{i,j}(a_k) = \begin{cases} a_k & \text{if} \quad k \neq i \\ a_k a_j & \text{if} \quad k = i \end{cases}.$$

Moreover, let

$$\mathrm{Aut}_f(\mathbb{F}_n) = \langle \rho_z, \varphi_i, \psi_{i,j} \mid z \in \mathbb{F}_n,\ i, j < n,\ i \neq j \rangle.$$

It follows by [24, Proposition 4.1 in Chapter 1] that $\mathrm{Aut}_f(\mathbb{F}_n) = \mathrm{Aut}(\mathbb{F}_n)$ if $n \in \omega$, and that $\mathrm{Aut}_f(\mathbb{F}_\infty)$ is dense in $\mathrm{Aut}(\mathbb{F}_\infty)$. Hence in order to obtain continuum many non-atomic characteristic random subgroups on $\mathbb{F}_n$ that are weakly mixing with respect to the action of $\mathrm{Aut}_f(\mathbb{F}_n)$, it suffices to find $x \in \mathbb{F}_n$ and a transversal T for the left cosets in $\mathrm{Aut}_f(\mathbb{F}_n)/\operatorname{Inn}(\mathbb{F}_n)$ such that

(1) $\{\eta(x) \mid \eta \in T\}$ satisfies the $C'(1/6)$ cancellation property.

(2) $\|\eta(x)\| \geq |x|$ for all $\eta \neq 1$.

Indeed, if (1) and (2) are satisfied, then the argument in Construction 8.2.1 works.

Chapter 9

Related questions

This chapter contains a discussion of some questions related to the subject of this part of the thesis. We have seen several examples of classes of countable groups that admit a continuum size family of non-atomic, ergodic invariant random subgroups. In the first section we will examine the question of which groups admit such a family. First we briefly review some well-known results concerning this question. Afterwards we discuss the operation mentioned in Remark 6.2.5 and the possibilities for it to be used to come up with new examples of groups with such continuum size families. In the second section we will consider the multiplication operation $\underset{\sim}{\times}\colon \underset{\sim}{A}(\Gamma, X, \mu)^2 \to \underset{\sim}{A}(\Gamma, X^2, \mu^2)$ and discuss its continuity properties for different countable groups Γ. As we have already hinted at in Remark 6.3.5, the continuity of this operation is closely related to the continuity of the co-induction operation $\underset{\sim}{\mathrm{cind}}_\Gamma^\Delta\colon \underset{\sim}{A}(\Gamma, X, \mu) \to \underset{\sim}{A}(\Gamma, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma})$ in the case where $\Gamma \leq \Delta$ are countable groups with $[\Delta : \Gamma] < \infty$, and in this case we have not completely settled when the co-induction operation is continuous.

9.1 Groups with many invariant random subgroups

In the past decade it has been of great interest to study the structure of the ergodic invariant random subgroups of various classes of groups. As every atomic, ergodic invariant random subgroup is induced by a subgroup with only finitely many conjugates, it is natural to focus on the non-atomic, ergodic invariant random subgroups. We will first discuss the question of which groups admit continuum many such invariant random subgroups and afterwards discuss a possible strategy for obtaining new examples of groups with this property.

We have seen several examples of classes of groups that admit continuum many non-atomic, ergodic invariant random subgroups, namely certain classes of wreath products, HNN extensions and free products with normal amalgamation. Other such examples include the group of finitely supported permutations of ω, every weakly branch group and every group containing a non-abelian free group as a normal subgroup (see [33], [3] and [8], respectively).

Conversely, there are also groups with no non-atomic ergodic invariant random subgroups. These include lattices in simple higher rank Lie groups, the simple Higman-Thompson groups, certain inductive limits of finite alternating groups and the groups $\mathrm{PSL}_m(k)$, where k is an infinite field and $m \geq 2$ (see [28], [13], [30] and [26], respectively). Finally, for certain limits of finite symmetric and alternating groups there are only countably many ergodic invariant random subgroups and these have all been classified (see [29], [30] and [14]).

In light of the above, it seems natural to ask the following question.

Question 9.1.1. *Which groups admit a continuum size family of non-atomic, ergodic invariant random subgroups?*

Note that in Proposition 7.1.1 we isolate the following algebraic condition on a countable group, which ensures that the group admits continuum many non-atomic, ergodic invariant random subgroups.

Proposition 9.1.2. *Let Δ be a countable group. If there exists a subgroup $\Gamma \leq \Delta$ with $[\Delta : \Gamma] = \infty$ together with a transversal $T = \{t_i \mid i \in \omega\}$ for the left cosets in Δ/Γ and $\gamma_0 \in \mathrm{core}_\Delta(\Gamma)$ such that the chain of normal subgroups $(\overline{\Gamma}_{k,T,\gamma_0})_{k\in\omega}$ given by*

$$\overline{\Gamma}_{k,T,\gamma_0} = \langle\langle t_i^{-1}\gamma_0 t_i \mid i \geq k\rangle\rangle_\Gamma$$

is not constant, then Δ admits continuum many non-atomic, weakly mixing invariant random subgroups.

In Remark 6.2.5 we point out another operation one can consider in order to construct weakly mixing invariant random subgroups of a countable group. Recall the construction below.

Assume that $\Gamma \leq \Delta$ are countable groups and fix a transversal T for the left cosets in Δ/Γ. Now fix $\theta \in \mathrm{IRS}(\Gamma)$ and view θ as a probability Borel measure on $\mathrm{Sub}(\Delta)$. For each $t \in T$ define the probability Borel measure θ_t on $\mathrm{Sub}(\Delta)$ to be the pushforward of θ through the map $\Lambda \mapsto t\Lambda t^{-1}$ from $\mathrm{Sub}(\Delta)$ to $\mathrm{Sub}(\Delta)$. Then

$$\theta_t(N_F^\Delta) = \theta(N_{t^{-1}Ft}^\Delta),$$

for all finite $F \subseteq \Delta$ and

$$\theta_\infty = \prod_{t \in T} \theta_t$$

is a probability Borel measure on $\mathrm{Sub}(\Delta)^T$. Moreover, we have a measure preserving action $\Delta \curvearrowright^a (\mathrm{Sub}(\Delta)^T, \theta_\infty)$ given by

$$\delta \cdot^a (\Lambda_t)_{t \in T} = (\delta \Lambda_{\delta^{-1}.t} \delta^{-1})_{t \in T}.$$

Now consider $J\colon \mathrm{Sub}(\Delta)^T \to \mathrm{Sub}(\Delta)$ given by

$$J((\Lambda_t)_{t \in T}) = \langle \Lambda_t \colon t \in T \rangle.$$

and let θ^{**} denote the pushforward of θ_∞ through J. Then $\theta^{**} \in \mathrm{IRS}(\Delta)$ and θ^{**} is weakly mixing when $[\Delta \colon \Gamma] = \infty$.

In order to be able to use this operation to construct new families of non-atomic, weakly mixing invariant random subgroups, one has to answer the following question.

Question 9.1.3. *When is θ^{**} non-atomic?*

Assume that $[\Delta : \Gamma] = \infty$. Then θ^{**} is non-atomic if and only if there is $\delta \in \Delta$ such that $\theta^{**}(N_\delta^\Delta) \in (0, 1)$. As it is clear that

$$\theta^{**}(N_\delta^\Delta) \geq \theta_t(N_\delta^\Delta)$$

for any $t \in T$ and $\delta \in \Delta$, it is easy to ensure that $\theta^{**}(N_\delta^\Delta) > 0$. The harder part is to get the other inequality to hold at the same time. This is the opposite to the situation of the co-induction operation, where it is quite easy to ensure $\mathrm{CIND}_\Gamma^\Delta(\theta)(N_\delta^\Delta) < 1$, but one has to cook up special circumstances in order to ensure that we also have $\mathrm{CIND}_\Gamma^\Delta(\theta)(N_\delta^\Delta) > 0$. Therefore it seems that a sufficient criterion for non-atomicity of θ^{**} will be quite different from the characterization for non-atomicity of the co-induced invariant random subgroup given in Proposition 6.4.3.

9.2 Continuity of multiplication of weak equivalence classes

We will here discuss another interesting project related to the content of this part of the thesis, namely the project of deciding for which countable groups Γ the multiplication operation on $\underset{\sim}{A}(\Gamma, X, \mu)$ is continuous.

Fix a countable group Γ. Given actions $a, b \in A(\Gamma, X, \mu)$, we define the product action $a \times b \in A(\Gamma, X^2, \mu^2)$ to be given by

$$\gamma \cdot^{a \times b} (x, y) = (\gamma \cdot^{a} x, \gamma \cdot^{b} y).$$

It is easily seen that if $a_0 \simeq a_1$ and $b_0 \simeq b_1$, then

$$a_0 \times b_0 \simeq a_1 \times b_1$$

for all $a_0, a_1, b_0, b_1 \in \underset{\sim}{A}(\Gamma, X, \mu)$. Therefore multiplication descends to a well-defined operation $\underset{\sim}{\times}\colon \underset{\sim}{A}(\Gamma, X, \mu)^2 \to \underset{\sim}{A}(\Gamma, X^2, \mu^2)$ given by

$$\underset{\sim}{a} \underset{\sim}{\times} \underset{\sim}{b} = \underset{\sim}{a \times b}.$$

We may view $\underset{\sim}{a \times b} \in \underset{\sim}{A}(\Gamma, X, \mu)$ for any $\underset{\sim}{a}, \underset{\sim}{b} \in \underset{\sim}{A}(\Gamma, X, \mu)$, since (X^2, μ^2) is isomorphic to (X, μ). Hence $\underset{\sim}{\times}$ can be seen as an operation on $\underset{\sim}{A}(\Gamma, X, \mu)$ and, with this identification, it is straightforward to check that $(\underset{\sim}{A}(\Gamma, X, \mu), \underset{\sim}{\times})$ is an abelian semigroup. The main question to consider here is when $(\underset{\sim}{A}(\Gamma, X, \mu), \underset{\sim}{\times})$ is in fact a topological semigroup.

Question 9.2.1. *For which countable groups Γ is the multiplication operation on $\underset{\sim}{A}(\Gamma, X, \mu)$ continuous?*

In [10, Problem 10.36] it was first asked if the multiplication operation is continuous in general. The authors, together with Tamuz, proved that if Γ is an amenable group, then the multiplication operation is continuous (see [10, Theorem 10.37]). The proof relies on the correspondence between $\underset{\sim}{A}(\Gamma, X, \mu)$ and $\mathrm{IRS}(\Gamma)$, so it did not give any insight to the general case. Recently, Bernshteyn showed that the operation may be discontinuous (see [4]). More precisely, he proves that if Γ is isomorphic to a Zariski dense subgroup of $\mathrm{SL}_d(\mathbb{Z})$ for some $d \geq 2$, for example if Γ is a non-abelian free group, then the multiplication operation is not continuous, even when restricted to $\underset{\sim}{\mathrm{FR}}(\Gamma, X, \mu)$. As Bernshteyn mentions, it is tempting to conjecture that the operation is continuous if and only if the group is amenable. However, a natural first step is to ask if the operation is discontinuous for any countable group containing a non-abelian free group.

In Section 6.3 we investigated the continuity properties of the co-induction operation $\underset{\sim}{\mathrm{cind}}_{\Gamma}^{\Delta}\colon \underset{\sim}{A}(\Gamma, X, \mu) \to \underset{\sim}{A}(\Delta, X^{\Delta/\Gamma}, \mu^{\Delta/\Gamma})$ for countable groups $\Gamma \leq \Delta$. In the case where $[\Delta : \Gamma] = \infty$, we settled that this operation is never continuous unless $\mathrm{core}_{\Delta}(\Gamma) = \{e_{\Gamma}\}$. In the case where $[\Delta : \Gamma] < \infty$, we found that the operation is continuous when Δ is amenable. If $1 < [\Delta : \Gamma] < \infty$ and Δ is not amenable, then our methods provide no information.

Question 9.2.2. *For which pairs of countable groups $\Gamma \leq \Delta$ with Δ non-amenable,* $\mathrm{core}_\Delta(\Gamma) \neq \{e_\Gamma\}$ *and* $1 < [\Delta : \Gamma] < \infty$ *is the operation*

$$\underset{\sim}{\mathrm{cind}}_\Gamma^\Delta \colon \underset{\sim}{A}(\Gamma, X, \mu) \to \underset{\sim}{A}(\Delta, X, \mu)$$

continuous?

As we discussed in Remark 6.3.5, the previous question is closely related to Question 9.2.1. Moreover, as mentioned in Remark 6.3.6, the result in [4] implies that for any non-abelian free group Γ the map

$$\underset{\sim}{\mathrm{cind}}_\Gamma^{\Gamma\times(\mathbb{Z}/2\mathbb{Z})} \colon \underset{\sim}{A}(\Gamma, X, \mu) \to \underset{\sim}{A}(\Gamma \times (\mathbb{Z}/2\mathbb{Z}), X^2, \mu^2)$$

is not continuous. So, in contrast to the case where Δ is amenable, the operation does not need to be continuous when $[\Delta : \Gamma] < \infty$.

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