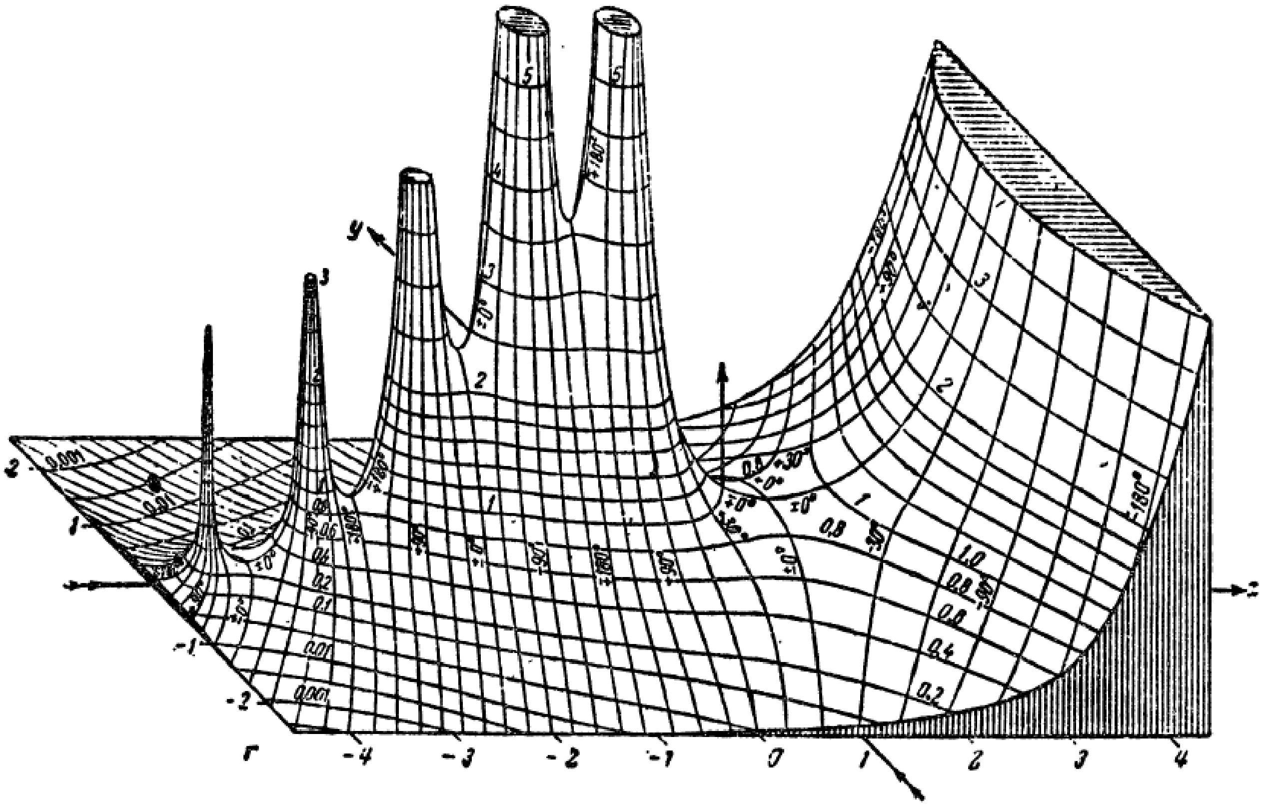
UNIVERSITY OF COPENHAGEN
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICAL SCIENCES



# Higher order monotonicity in the context of beta and gamma functions

PhD Thesis

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Submitted on: 18 May 2018

ISBN: 978-87-7078-911-0

## Abstract

The present thesis investigates higher monotonicity properties in function theory. It consists of three manuscripts. The first one focuses in the beta distribution and its quantiles. Logarithmic concavity of the quantiles with respect to the first parameter is proved. The second manuscript computes asymptotic expansions of the quantiles for the first parameter going to zero or to infinity. The third manuscript is a generalisation of a complete monotonicity result on ratios of gamma functions to entire functions.

## Abstrakt

Den foreliggende afhandling undersøger højere monotonicitetsegenskaber i funktionsteori. Den består af tre manuskripter. Det første fokuserer på beta fordelingen og dens fraktiler. In dette bevises, at fraktilerne er logaritmisk konkave i forhold til den første parameter. Det andet manuskript beregner asymptotiske udviklinger af fraktilerne for den første parameter gående mod nul eller mod uendelig. Det tredje manuscript generaliserer resultater om fuldstaendig
monotonicitet af kvotienter mellem gammafunktioner til hele funktioner.

# Contents

## Acknowledgements

In this section I would like to acknowledge all the people and factors that lead to the actualisation of the present work.

First of all I would like to thank my master and phd thesis supervisor, Henrik Pedersen, for his support, the fruitful discussions we have had and his advice throughtout my phd, and foremost, for his generosity in sharing his lot of ideas. Also, it is worth noting that the balance between similarities and variance among our styles of thinking and approaches regarding some of the questions lead to the development of ideas that resulted in the completion of this thesis, but also made this experience very joyful.

The actualisation of this thesis has strong roots, beginning chronologically with, in my received parental socialisation which included a heavy motivation for intellectual development and academic accomplishment, as well as the personal examples of my parents themselves. Moreover, it was certainly only possible due to their absolute and unconditional support throughout my studies.

My academic journey would probably be much shorter if not for the encouragement of my professor as an undergraduate student in Thessaloniki, Dimitrios Betsakos, and his trust in my potential. Moreover, as long as my undergraduate studies were concerned, it was Lorena who essentially put a structure in my mess in order to complete them, and certainly the advice of Effie and Akis on how to maximise efficiency.

In the whole tormenting journey of my PhD (which is something I do not recommend to anybody choosing, if they do not take some strange pleasure out of it), it was my official and unofficial office mates Francesco, Roberto and Erica with whom together we shared our suffering and were getting short briscola escapes from it, that helped me survive. In the same way, having caring friends and activities both in and out of the mathematical community had a heavy contribution towards that.

Finally, it should be trivial to note that the present work does not exist in a mathematical vacuum, but is inspired and related to work of several mathematicians, whose contributions are discussed in the main body in more detail. I am very happy for having the chance to contribute in a small way in this web that advances mathematical knowledge, which was a dream I had since adolescence.

The image in the title page is the graph of the modulus of gamma function, from Funktionentafeln mit Formeln und Kurven (1909) by Jahnke and Emde.

# Preface

Higher monotonicity is an essential tool in approximation theory and special functions. Beta and gamma functions are two of the most important special functions, with a wide range of applications.

This thesis investigates instances of higher monotonicity in functions that are related to the beta and gamma functions. We have two main results. One is about the logarithmic concavity of the inverse incomplete beta function, as well as asymptotic expansions. The second is the logarithmic complete monotonicity of ratios of entire functions, generalising results on ratios of gamma functions and applying it to multiple gamma functions. Finally, we provide asymptotic expansions for the inverse incomplete beta function wrt its first parameter.

We shall start presenting some general introductory material. In the first chapter, we review two of the most important notions of higher monotonicity: convexity/concavity, and complete monotonicity, as well as their logarithmic analogues. In the second chapter, we review the basic facts on beta and gamma functions and functions related to these. These include the incomplete beta function and its inverse, and multiple gamma functions. In the third chapter, we present some basic facts from the theory of entire functions.

In the appendix, we attach three manuscripts that constitute the main body of the present thesis. The first one focuses in the beta distribution and its quantiles. It proves logarithmic concavity of the quantiles with respect to the first parameters. The second manuscript computes asymptotic expansions for the quantiles for the first parameter going to $0$ or to infinity. The third manuscript is a generalisation of a complete monotonicity result on ratios of gamma functions to entire functions. It also gives a different point of view to previously known results, which were shown only using dedicated properties of the gamma and digamma functions.

CHAPTER 1

# Higher order monotonicity

## 1. Logarithmic convexity and concavity

The usual notion of monotonicity refers to the property of a real valued function on $\mathbb{R}$ being increasing or decreasing. Extending this, we have higher monotonicity properties referring to how fast or slow a function grows or decreases. Such properties are important in function theory and applications. A common example of a higher monotonicity property is concavity/convexity. A convex function is one for which, whenever we chose two points in the graph of a function $f$, the line segment connecting these points lies above or on the graph of the function. A concave function is the one for which the line segment lies below or on the graph. In case of twice differentiable functions, this is equivalent to $f" \geq 0$ for a function $f$ to be convex ($f" \leq 0$ to be concave).

One of the reasons that convexity is important is the following its role in optimisation. Assume a continuously differentiable function $f$ on an interval $(a, b)$ is convex. The definition of convexity for the case of $f$ implies that the graph of the function must lie above its tangent at any point. This is clear by considering two points, as in the definition, and taking the limit as the second approaches the first one. This fact can be expressed analytically as

$$f(x) \geq f(y) + f'(y)(x - y)$$

whenever $x, y \in (a, b)$. Assume, now, that $f$ obtains a local minimum at $y \in (a, b)$, hence $f'(y) = 0$. The previous inequality then implies that for all $x \in (a, b)$, $f(x) \geq f(y)$, hence $f$ obtains a global minimum at $x$. This means that if we know that a function is convex, computing a local minimum with a numerical method (for example with Newton's method) finds a minimum that is guaranteed to also be global. In a similar way, concavity guarantees the globality of local maxima.

Convex and concave functions are not the only ones which guarantee the globality of local minima/maxima. As composition with increasing functions preserve global extrema, the following classes have importance in analysis.

DEFINITION 1.1. A function $f : [0, \infty) \to (0, \infty)$ is called logarithmically convex (concave) if $\log f$ is convex (concave).

By the previous observation we easily deduce that if a logarithmically convex/concave function has a local minimum/maximum, then this is a global one. The further advantage that we get in the logarithmic case is that, because of the fact that convexity/concavity is closed under addition and the properties of the logarithm, logarithmic convexity/concavity is preserved when we consider products of such functions. This has important applications for examples in optimisation problems where such products may occur. For example in [**5**, Proposition 4] a problem of finding the probability of lottery players legitimately claim winning a number of prizes resulted in an optimisation problem involving a product of incomplete beta functions (which we will define later). For the solution of this problem, logarithmic concavity with respect to parameters was crucial.

It is clear, as the exponential function is convex, that a logarithmically convex function is also convex, i.e. logarithmic convexity is a stronger property than convexity. In the same way, a concave function is also logarithmically concave, hence logarithmic concavity is a weaker property than concavity.

In this thesis, all the functions studied are going to be infinitely many times differentiable, hence these properties are going to be investigated through investigating signs of second logarithmic derivatives.

## 2. Complete monotonicity

Of course, one may continue investigating higher order derivatives to reveal even higher monotonicity properties of functions. An important category of higher order monotonicity is *complete monotonicity*, where all orders of the derivatives of a function are involved. For an account of completely monotonic and related functions (Bernstein, Stjeltjes etc) we refer to [**11**] and [**23**].

DEFINITION 1.2. A function $f : (0, \infty) \to (0, \infty)$ is called completely monotonic ($f \in \mathcal{CM}$) if $(-1)^n f^{(n)}(x) \geq 0 \; \forall n \in \mathbb{N}, x \in (0, \infty)$ .

Completely monotonic functions have a famous characterisation as Laplace transforms of Borel measures on the half-line, what is known as Bernstein-Widder theorem.

THEOREM 1.3. *A function* $f \in C^\infty(0, \infty)$ *is completely monotonic if and only if it is the Laplace transform of a non-negative Borel measure on* $[0, \infty]$*, i.e. if and only if*

$$f(x) = \int_0^\infty e^{-xt} \mathrm{d}\mu t$$

*for a non-negative measure* $\mu$ *on* $[0, \infty]$ *such that*

$$\int_0^\infty e^{-t} \mathrm{d}\mu t < \infty \,.$$

REMARK 1.4. Another way to see completely monotonic functions is in terms of positive definiteness. If we see the closed half-line $[0, \infty)$ as an additive semigroup, the (bounded) completely monotonic functions are exactly the ones that are bounded positive definite. By using shift operators, we may characterise all (also the possibly unbounded, as the shifted functions will necessarily be bounded) completely monotonic functions on $(0, \infty)$ in terms of these operators and positive definiteness. See [**8**, Theorem 6.13].

## 3. Logarithmic complete monotonicity

A stronger form of complete monotonicity involves composing with the logarithm.

DEFINITION 1.5. A function $f : (0, \infty) \to (0, \infty)$ is called logarithmically completely monotonic ($f \in \mathcal{LCM}$) if $-(\log f)'$ is completely monotone.

Though, in first glance, the two notions do not seem to be directly related, it turns out that logarithmically completely monotonic functions form a subclass of $CM$. In particular, we have the following classical result.

THEOREM 1.6. *A function* $f : (0, \infty) \to (0, \infty)$ *is logarithmically completely monotonic if and only if* $f^t$ *is completely monotone for all* $t > 0$.

The property that $f^t$ is completely monotonic for all $t > 0$ is called infinite divisibility. It is an important notion in probability because they represent Laplace transforms of infinitely divisible probability distributions, and arises in relation to variations of the central limit theorem and Lévy processes. See for instance [**22**].

CHAPTER 2

# Entire functions of finite genus

## 1. Hadamard representation

An entire function is a complex-valued function analytic in the whole complex plane. Entire functions are classically studied according to their order of growth. In particular, we categorise entire functions according to whether their growth is comparable to $e^{|z|^a}$ for some $a > 0$. Entire functions that grow faster than that tend to contain pathogenic cases, so the growth restriction provides a nice structure and theory of entire functions to work with. A very nice exposition of the classical theory of entire functions is contained in [**18**].

We shall make the above concept of ”growth” more rigorous.

Let $f$ be an entire function. We define

$$M_f(r) := \max\{|f(z)| : |z| = r\}$$

DEFINITION 2.1. The order $\rho$ of an entire function $f$ is defined by

$$\rho = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\ln r}$$

We say that $f$ is of finite order if $\rho < \infty$. Else, $f$ is of infinite order.

DEFINITION 2.2. The type of an entire function $f$ of finite order $\rho$ is defined by

$$\sigma := \limsup_{r\to\infty} \frac{\ln M_f(r)}{r^\rho}$$

It is a simple application of Cauchy's and Liouville's theorems that if an entire function of finite order has no roots, then it must be in the form $e^{P(z)}$ where $P$ is a polynomial. On the other hand, an entire function $f$ may only have at most countably many roots, whose only possible accumulation point is at infinity, by the identity principle. Hence, dividing $f$ by a converging Weirstrass product whose factors contain exactly these roots, counting multiplicities, we get an entire function with no roots on the complex plane. This shows that an entire function may be uniquely, up to multiplication by some exponential factor, determined by its roots. This sketches the proof of the following formula, called Hadamard's representation theorem.

THEOREM 2.3. *An entire function $f$ of finite order $\rho$ has a Hadamard representation in terms of a canonical product*

$$f(z) = z^m e^{P(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n} + \cdots + \frac{1}{p}\left(\frac{z}{z_n}\right)^p\right)$$

*where $z_{n n\in\mathbb{N}}$ is the sequence of non-zero roots of $f$, typically ordered by increasing order of magnitude, $m$ is the order of a possible root at $0$, $P$ is a polynomial of degree at most $\rho$, and $p \leq \rho$.*

This representation is not unique, in fact one may choose any $p$ smaller or equal to $\rho$ to get a different representation every time. It is however standard in literature in complex analysis to choose $p = \lfloor \rho \rfloor$. This number is called the genus of the entire function $f$.

There is a strong connection between the order of growth of an entire function and the growth of the sequence of its roots. For this we should distinguish between cases where the order is integer and non-integer.

THEOREM 2.4. *Let $f$ be an entire function of order $\rho$. Let $(z_n)_{n\in\mathbf{N}}$ be the sequence of its zeros, having order $\rho_0$. Then, we have that $\rho_0 \leq \rho$. In particular, if $\rho$ is non-integer, then $\rho = \rho_0$.*

One could be let to think that the difference between the integer and non-integer cases is due to the exponential factor. Indeed, if we multiply an entire function of order $\rho = \rho_0$, the same order as its zeros, with an exponential factor $e^{z^n}$, where $n > \rho$, we get an entire function of order $n$ with zeros of order $\rho < n$.

## 2. Entire functions and complete monotonicity

A class of entire functions with a distinct interest is the one consisting of the entire functions with negative zeros. In [**20**], Pedersen studied logarithmic derivatives of entire functions with respect to complete monotonicity. One of the main results in this paper is the following proposition:

PROPOSITION 2.5. *Let $f$ be an entire function of finite genus $p$ having only real, non-positive zeros. Assume that $f$ has a root of $r$th order at $0$ and its negative roots are $\{-\lambda_k\}_{k\in\mathbb{N}}$ where $\lambda_k > 0$. Then, $(-1)^p(x^m(\log f(x))')^{(m+p)}$ is a completely monotonic function for all $m \geq 0$ and it has the representation*

$$(-1)^p(x^m(\log f(x))')^{(m+p)} = \int_0^\infty e^{-sx}s^{m+p}\sum_{k=1}^\infty \lambda_k^m e^{-\lambda_k s}\mathrm{d}s, \quad m \geq 1,$$

$$(-1)^p(\log f(x))^{(p+1)} = \int_0^\infty e^{-sx}s^p\left(r + \sum_{k=1}^\infty e^{-\lambda_k s}\right)\mathrm{d}s\,.$$

We are mostly interested in the case where $m = 0$ above.$(-1)^p(\log f(x))^{(p+1)}$ is completely monotonic, as it is the Laplace transform of a positive measure on the positive half-line. We denote

$$h(s) = \sum_{k=1}^\infty e^{-s\lambda_k}\,.$$

Sumability of this quantity for a given sequence $\{-\lambda_k\}_{k\in\mathbb{N}}$ implies a concrete form of the representing measure in the above Laplace representation. An example is given if we consider $f = 1/\Gamma$, the reciprocal of the gamma function (see Chapter 3). It is an entire function with simple zeros on exactly all the non-positive integers, and hence in this case

$$h_{1/\Gamma}(s) = \sum_{k=1}^\infty e^{-sk} = \frac{e^{-s}}{1-e^{-s}}$$

hence

$$-\psi(x)' = \int_0^\infty e^{-sx}\frac{s}{1-e^{-s}}\mathrm{d}s\,.$$

CHAPTER 3

# Beta and Gamma functions

## 1. The gamma and beta functions

The history of the gamma function runs back to Euler, and to the problem of interpolating the factorial with a continuous function, i.e. finding a function of a continuous variable $x$ that equals $n!$ when $x$ is equal to an integer $n$. This problem was allegedly suggested by Daniel Bernoulli and Goldbach. To illustrate Euler's construction, assume $x, n \geq 0$ are integers. We follow the exposition from [**4**], which is also the standard reference in gamma and beta functions. We may write

$$x! = \frac{(x+n)!}{(x+1)_n} = \frac{n!(n+1)_x!}{(x+1)_n} = \frac{n!n^x}{(x+1)_n}\frac{(n+1)_x}{n^x}$$

We notice that

$$\lim_{n\to\infty}\frac{(n+1)_x}{n^x} = 1$$

hence

$$\begin{aligned} x! &= \lim_{n\to\infty}\frac{n!n^x}{(x+1)_n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^x\prod_{j=1}^{n}\left(1+\frac{x}{j}\right)^{-1}\left(1+\frac{1}{j}\right)^x \\ &= \left(\frac{n}{n+1}\right)^x\prod_{j=1}^{\infty}\left(1+\frac{x}{j}\right)^{-1}\left(1+\frac{1}{j}\right)^x \end{aligned}$$

where the latter infinite product converges for all complex numbers $x$ that are not negative integers because

$$\left(1+\frac{x}{j}\right)^{-1}\left(1+\frac{1}{j}\right)^x = 1 + \frac{x(x-1)}{2j^2} + O\left(\frac{1}{j^3}\right).$$

DEFINITION 3.1. For $x \in \mathbb{C}\setminus\{0,-1,-2,\dots\}$ the gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \lim_{k\to\infty}\frac{k!k^{x-1}}{(x)_k} \tag{1}$$

By the definition we immediately derive the most basic property of the gamma function,

$$\Gamma(x+1) = x\Gamma(x).$$

Moreover, $\Gamma(1) = 1$, hence these two properties give us that

$$\Gamma(n+1) = n!$$

Due to the convergence of the infinite product above, we see that $\Gamma$ is a meromorphic function with simple poles on the non-positive integers. The function $1/\Gamma$ is entire, hence it has the following Hadamard product representation.

THEOREM 3.2. *We have*

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n} \tag{2}$$

*where* $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right)$ *is Euler's constant.*

Probably its most standard representation is the Euler integral for the gamma function.

THEOREM 3.3. *For $\Re z > 0$, we have*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t \tag{3}$$

All these representations form alternative, standard ways to define the gamma function. Another equivalent, standard definition of the gamma function is following characterisation.

THEOREM 3.4 (Bohr–Mollerup theorem). *The gamma function is the only function $f$ on $(0, \infty)$ for which the three following properties simultaneously hold:*

*i.* $f(1) = 1$
*ii.* $f(x+1) = xf(x)$, $x > 0$
*iii.* *$f$ is logarithmically convex*

One important property of the gamma function is the Euler's reflection formula.

THEOREM 3.5.

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \tag{4}$$

We denote the logarithmic derivative of $\Gamma'/\Gamma$ of the gamma function by $\psi$. Sometimes this is called the digamma function, and it has the following representations.

$$\psi(z) = \gamma + \sum_{k=1}^{\infty} \left( \frac{1}{z+k-1} - \frac{1}{k} \right) \tag{5}$$

$$\psi(z) = \int_0^\infty \left( \frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right) dt = -\gamma + \int_0^1 \left( \frac{1-s^{z-1}}{1-s} \right) ds \tag{6}$$

A function closely related to the gamma function is the beta function. The beta function is defined for $\Re x, \Re y > 0$ by the beta integral of Euler,

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t \tag{7}$$

It can be extended (meromorphically) to the whole complex plane as

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{8}$$

A generalisation of the gamma function is the multiple gamma function of order $N$. There are many related kinds of multiple gamma functions in the literature. In this thesis, we specifically deal with the $N$-multiple gamma function with parameters $(1, 1)$. See **[21]** and **[7]**. The definition of multiple gamma function that we use here is the following:

DEFINITION 3.6. The function $\Gamma_N(z)$ is called multiple gamma function and is defined by the following recurrence relations:

(1) $\Gamma_N(1) = 1$
(2) $\Gamma_1(z) = \Gamma(z)$
(3) $\Gamma_{N+1}(z+1) = \dfrac{\Gamma_{N+1}(z)}{\Gamma_N(z)}$

## 2. Ratios of gamma functions and complete monotonicity

The study of ratios of gamma functions with respect to complete monotonicity stems back at Bustoz and Ismail [**12**] who showed that the ratio

$$\frac{\Gamma(x)\Gamma(x+a+b)}{\Gamma(x+a)\Gamma(x+b)}$$

is logarithmically completely monotonic. Several extensions of this result for ratios of the form

$$\frac{\prod_{j=1}^{p}\Gamma(x+a_j)}{\prod_{j=1}^{p}\Gamma(x+b_j)}$$

were found by Ismail and Muldoon [**16**], Alzer [**3**], and Grinshpan and Ismail [**15**].

Karp and Prilepkina in [KarpPril] extended the previous results by including weights, i.e. studying the following ratio:

$$W(x)=\frac{\prod_{k=1}^{p}\Gamma(A_k x+a_k)}{\prod_{k=1}^{q}\Gamma(B_k x+b_k)} \tag{9}$$

This ratio appears in the context of probability, as it is the representing measure in the Laplace transform integral representation of some special cases of the Mejer's G function, a special function that generalises hypergeometric and several other hypergeometric functions. Their results rely on the integral representations of gamma and digamma functions.

Their main results are:

LEMMA 3.7. *The function* $(\log W)''$ *is completely monotonic if and only if*

$$P(u)=\sum_{i=1}^{p}\frac{e^{-a_i u/A_i}}{1-e^{-u/A_i}}-\sum_{i=1}^{q}\frac{e^{-b_j u/B_j}}{1-e^{-u/B_j}}\geq 0 \textit{ for all } u>0. \tag{10}$$

*In the affirmative case,*

$$(\log W)''=\int_0^\infty e^{-xu}uP(u)\mathrm{d}u \tag{11}$$

THEOREM 3.8. *The function* $W$ *is logarithmically completely monotonic if*

$$\sum_{j=1}^{q}B_j=\sum_{i=1}^{q}A_i,\quad \rho=\prod_{i=1}^{p}A_i^{A_i}\prod_{i=1}^{p}B_j^{B_j}\leq 1 \tag{12}$$

*and condition* (10) *holds. In the affirmative case,*

$$-(\log W)'=\int_0^\infty e^{-xu}P(u)\mathrm{d}u+\log(1/\rho) \tag{13}$$

## 3. The median of the gamma distribution

In probability theory, there is an important probability distribution that is closely related to the gamma function, the gamma distribution. It is defined by considering the incomplete integral in (3):

DEFINITION 3.9. The gamma distribution with parameter $x>0$ is the probability distribution on $[0,\infty)$ having cumulative distribution function defined by

$$s\mapsto\frac{1}{\Gamma(x)}\int_0^s e^{-t}t^{x-1}\mathrm{d}t \tag{14}$$

Its median $m$ is defined implicitly through the equation

$$\frac{1}{\Gamma(x)}\int_0^{m(x)} e^{-t}t^{x-1}\mathrm{d}t = \frac{1}{2} \tag{15}$$

or equivalently

$$\int_0^{m(x)} e^{-t}t^{x-1}\mathrm{d}t = \frac{1}{2}\int_0^{\infty} e^{-t}t^{x-1}\mathrm{d}t \tag{16}$$

So, one can consider the median as a function of the parameter $x$ and study its analytic properties. The median of the gamma distribution has been studied in several occasions. In [**13**], Chen and Rubin proved that

$$x - 1/3 < m(x) < x \tag{17}$$

and further conjectured that $m(n) - n$ is decreasing. This conjecture was proved by Alm in [**2**]. In [**14**], Alzer further proved that $m(n+1) - an$ decreases for all $n \geq 0$ if and only if $a \geq 1$ and increases if and only if $a \leq m(2) - \log 2$. In [**1**], Adell and Jodrá explore a very interesting connection with a sequence due to Ramanujan.

In [**9**], Berg and Pedersen prove the following result

THEOREM 3.10. *Let* $x > 0$. *Then,* $0 < m'(x) < 1$.

which is in fact the continuous version of the Chen-Rubin conjecture.

First, they prove that $m$ is increasing, by showing a more general result on convolution semigroups of measure on the positive half-line.

DEFINITION 3.11. A family $\{\mu_x\}_{x>0}$ of probabilities on $[0,\infty)$ is called a convolution semigroup if it has the properties

i. $\mu_x([0,\infty)) = 1$ for all $x > 0$
ii. $\mu_x * \mu_y = \mu_{x+y}$ for all $x, y > 0$
iii. $\mu_x \to \delta_0$ for $x \to 0$ in the vague topology

where $\delta_0$ is the Dirac mass at zero.

A probability measure $\mu$ on $[0,\infty)$ has median $m$ if $\mu([0,m]) = 1/2$. A probability distribution may not have a median, or even if it has, it does not have to be unique. To guarantee existence it is enough to require that its density is absolutely continuous with respect to the Lebesgue measure. To further guarantee uniqueness, the density is enough to be a.e. strictly positive. The following proposition gives a monotonicity result for probability semigroups with strictly positive density with respect to Lebesgue measure:

PROPOSITION 3.12. *Let* $\{\mu_x\}_{x>0}$ *be a conolution semigroup of probabilities on* $[0,\infty)$ *having a.e. strictly positive densities with respect to Lebesgue measure. Then, the median* $m(x)$ *of* $\mu_x$ *is a continuous and strictly increasing function on* $(0,\infty)$. *Furthermore,* $\lim_{x\to 0} m(x) = 0$ *and* $\lim_{x\to\infty} m(x) = \infty$

The gamma distribution indeed forms a convolution semigroup. Property i. holds trivially. To show that ii. holds, denote the density of the gamma distribution with parameter $x$ by $f_x$. Let $x, y > 0$. Then, the density of the convolution of the gamma measures with parameters $x$ and $y$ is

$$
\begin{aligned}
\int_0^s f_x(t) f_y(s-t)\,\mathrm{d}t &= \int_0^s \frac{t^{x-1}e^{-t}}{\Gamma(x)} \frac{(s-t)^{y-1}e^{-(s-t)}}{\Gamma(y)}\,\mathrm{d}t \\
&= e^{-s} \int_0^s \frac{t^{x-1}(s-t)^{y-1}}{\Gamma(x)\Gamma(y)}\,\mathrm{d}t \\
&= e^{-s} s^{x+y-1} \int_0^1 \frac{u^{x-1}(1-u)^{y-1}}{\Gamma(x)\Gamma(y)}\,\mathrm{d}u \\
&= e^{-s} s^{x+y-1} \frac{B(x,y)}{\Gamma(x)\Gamma(y)} \\
&= \frac{e^{-s} s^{x+y-1}}{\Gamma(x+y)}
\end{aligned}
$$

by substituting $u = t/s$ in the prelast equality. The third property is also easy to check, by considering a compactly supported function $g$ and taking limits to zero. Hence, Proposition 3.12 can be applied to show that $m$ is increasing and continuous.

Further, they show that (15) implies that $m$ is a real analytic function of $x$, as differentiating it (using the implicit function theorem) gives a differential equation of the form $m'(x) = G(x, m(x))$, where $G$ is real analytic in both variables.

In particular, they study the median $m$ through the function

$$\phi(x) := \log \frac{x}{m(x)}$$

and prove that for $x > 0$,

$$1 - x\phi'(x) < e^{\phi(x)},$$

from which the above theorem follows. They also show that

$$\frac{1}{3} < x\phi(x) < \log 2$$

from which it follows that

$$xe^{-\log 2/x} < m(x) < xe^{-1/3x}$$

improving the inequality (17). For example, using $e^{-a} < 1 - a + a^2/2$ for $a > 0$, it gives

$$m(x) < x - \frac{1}{3} + \frac{1}{18x}$$

Moreover, asymptotically at $0$

$$\frac{m(x)}{e^{-\gamma} 2^{-1/x}} \to_{x\to 0} 1$$

while at infinity, the authors give the fallowing asymptotic expansion

$$m(x) = x - \frac{1}{3} + \frac{8}{405x} + \frac{184}{25515x^2} + o(x^{-2})$$

Both expressions can be differentiated to give asymptotic expressions for $m'$.

Subsequently in [**10**], the authors prove that $m$ is a convex function. These analytic results on the median of the gamma distribution were the main motivation for the first two articles attached, investigating the analytic properties of the quantiles of the beta distribution.

## 4. Beta distribution and inverse incomplete beta function

The beta distribution is another important parametrised family of probability distributions. It is defined similarly to the gamma distribution, by ”cutting out” the Euler’s beta integral.

DEFINITION 3.13. The beta distribution with parameters $a, b > 0$ is the probability distribution on $[0, 1]$ having cumulative distribution function

$$I(p; a, b) := \frac{1}{B(a, b)} \int_0^p t^{a-1}(1-t)^{b-1}\mathrm{d}t \tag{18}$$

The function $I$ is called regularised incomplete beta function. In [**17**], Karp and Prilepkina show logarithmic convexity/concavity with respect to parameters by analytic methods.

Its inverse with respect to the $p$ variable is the inverse (regularised) incomplete beta function and is usually denoted by $I^{-1}(p; a, b)$. This quantity is exactly the $p$-quantile of the beta distribution with parameters $a$ and $b$. Quantiles have great importance in statistics and probability, in particular in computing confindence intervals. For a standard reference on the beta distribution see [**6**, Chapter 2].

In [**24**], Temme gives asymptotic expansions of $p$-quantiles when the parameters $a$ and $b$ are large. In particular, these approximations hold with high enough accuracy when $a + b > 5$ and they are implemented in several programming language packages that compute the inverse incomplete beta function. For lower values, though, the method commonly used is a variation of Newton’s method, which is slow for many practical needs like optimisation tasks where several instances of this function have to be repeatedly computed. See also [**19**] for some interesting inequalities for the median.

Our purpose in the first two manuscripts is to study the behaviour the inverse incomplete beta function when we fix $p$ and the parameter $b$, as a function of $a$.

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# LOGARITHMIC CONCAVITY OF THE INVERSE INCOMPLETE BETA FUNCTION WITH RESPECT TO THE FIRST PARAMETER

DIMITRIS ASKITIS

Abstract. The beta distribution is a two-parameter family of probability distributions whose distribution function is the (regularised) incomplete beta function. In this paper, the inverse incomplete beta function is studied analytically as univariate function of the first parameter. Monotonicity, limit results and convexity properties are provided. In particular, logarithmic concavity of the inverse incomplete beta function is established. In addition, we provide monotonicity results on inverses of a larger class of parametrised distributions that may be of independent interest.

## 1. Introduction

Let a probability distribution on $I \subset \mathbb{R}$ have a cumulative distribution function (CDF) $F$. Its median is defined as a point on $I$ that leaves half of the "mass" on the left and half on the right, i.e. a value $m \in I$ such that $F(m) = 1/2$. In a similar way, we consider the more general notion of a $p$-quantile:

Definition 1.1. Let a probability distribution on $I \subset \mathbb{R}$ have cumulative distribution function $F$, and let $p \in (0, 1)$. A value $q_p \in I$ is a $p$-quantile of it if $F(q) = p$.

In this notation, the 1/2-quantile is exactly the median. It is not always the case that a $p$-quantile exists for a probability distribution, or that it is unique. However, existence and uniqueness are guaranteed in the case of a.e. positive density wrt Lebesgue measure. Then, we may consider the inverse distribution function of $F$. The median and $p$-quantiles have importance in statistics as measures of position less affected by extreme values than e.g. the mean, and they have further uses considering levels of significance.

We are interested in parametrised families of probability distributions and the behaviour of the $p$-quantile with respect to the parameter, with $p$ being fixed. In case we have a family of cumulative distribution functions $F_a$, $a$ being the parameter of the family, such that for each $a$ the corresponding $p$-quantile exists and is unique, we may define it as a function of $a$ implicitly through the functional equation $F_a(q_p(a)) = p$.

On the case of the median of the gamma distribution, such studies have been done in several occasions, e.g. in [2], [7] and [8]. In [1], Adell and Jodrá explore a very interesting connection with a sequence due to Ramanujan. In [5] and [6], Berg and Pedersen give a proof of the continuous version of the Chen-Rubin conjecture, originally stated in [7], and they moreover prove convexity and find asymptotic expansions.

In the present article, the main focus is on the $p$-quantile of the beta distribution, or equivalently the inverse of the (regularised) incomplete beta function (3), as a function of the parameter $a$. For a standard reference on the beta distribution see [4, Chapter 2]. This inverse has also been considered by Temme [15] who studied its uniform asymptotic behaviour. In particular, his results give a very accurate approximation for the inverse for $a + b > 5$. This is used in computer algorithms approximating the inverse incomplete beta function. Also, see [14] for some interesting inequalities for the median. In [10], logarithmic convexity/concavity results are proved for the regularised incomplete beta function wrt to parameters, though the methods employed there are quite different, and there does not seem to be any direct connection with the results in the present article. In applications, (strict) logarithmic concavity is an important property, as it ensures the uniqueness of minimum and it is invariant under taking products.

The beta function is defined for $\Re a, \Re b > 0$ as the integral

$$\mathrm{B}(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t\,. \tag{1}$$

It also has the following representation as ratio of gamma functions

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{2}$$

which gives a meromorphic continuation of the beta function in $\mathbb{C}^2$. More information on the beta function can be found on [3]. The beta distribution is the 2-parameter family of probability distributions, whose cumulative distribution function is the regularised incomplete beta function

$$I(x;a,b) := \frac{\int_0^x t^{a-1}(1-t)^{b-1}\mathrm{d}t}{\mathrm{B}(a,b)}\,. \tag{3}$$

We fix $p \in (0,1)$ and $b > 0$, and we consider the first parameter $a$ as a variable. We shall see in the Appendix that, due to a reflection formula for the regularised incomplete beta function, we can translate the results to the case when we fix the other parameter instead. We consider the $p$-quantile of the beta distribution, which in the literature is often also called the inverse incomplete beta function, as a function of $a$. We denote it by $q : (0,\infty) \to (0,1)$ and define it implicitly by the equation $I(q(a);a,b) = p$, or equivalently by

$$\int_0^{q(a)} t^{a-1}(1-t)^{b-1}\mathrm{d}t = p\int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t\,. \tag{4}$$

In the literature this value is often denoted by $I_p^{-1}(a,b)$, and in our case $q$ is the function $a \mapsto I_p^{-1}(a,b)$. Moreover, we consider the function

$$\phi(a) := -a\log q(a)\,, \tag{5}$$

which turns out to be containing further information on $q$. In the following plots we can get an idea on how the median of the beta distribution behaves wrt $a$.

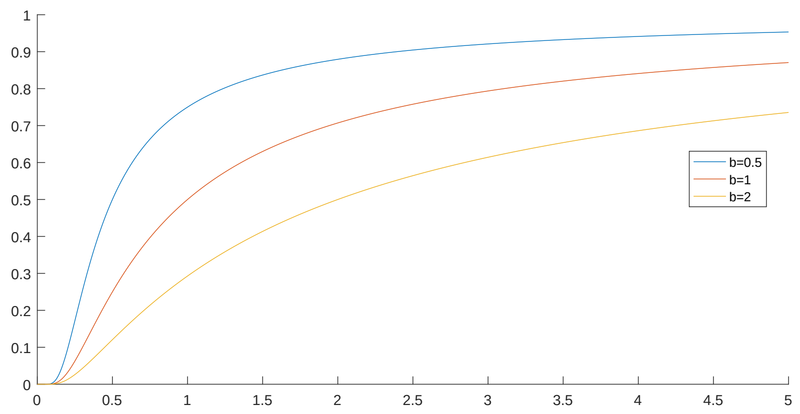


FIGURE 1. Plot of $q$ for $p = 1/2$

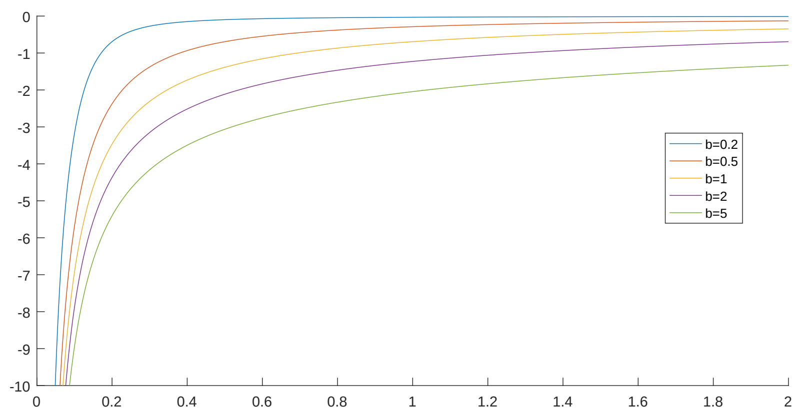


FIGURE 2. Plot of $\log q$ for $p = 1/2$

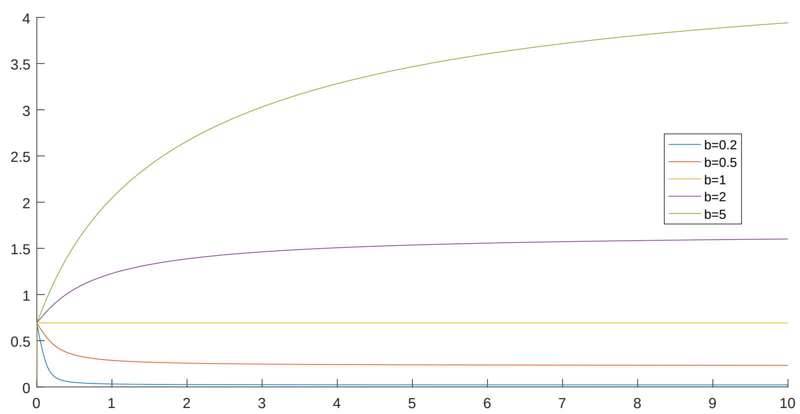


FIGURE 3. Plot of $\phi$ for $p = 1/2$

In the rest of the paper we fix $p \in (0,1)$. We first get the following two propositions, regarding monotonicity and first order asymptotics:

Proposition 1.2. *The function $q$ in (4) is a real analytic and increasing function on $(0,\infty)$. It has limits*
$$\lim_{a\to 0} q(a) = 0$$
*and*
$$\lim_{a\to\infty} q(a) = 1\,.$$

Proposition 1.3. *The function $\phi$ in (5) is real analytic on $(0,\infty)$. It is decreasing if $b<1$, constant if $b=1$ and increasing if $b>1$. It has limits*
$$\lim_{a\to 0} \phi(a) = -\log p$$
*and*
$$\lim_{a\to\infty} \phi(a) = \gamma_b\,,$$
*where $\gamma_b$ is the $(1-p)$-quantile of the gamma distribution with parameter $b$.*

Then, we investigate the analytic properties of the inverse incomplete beta function deeper. In particular, investigating its logarithm, we obtain the following two results, which constitute the main contribution of this paper:

Theorem 1.4. *For fixed $b \in (0,1)$, $\phi$ in (5) is (strictly) convex.*

Theorem 1.5. *For fixed $b \in (0,\infty)$, $q$ in (4) is (strictly) log-concave.*

Remark 1.6. One can infer from Figure 1 that $q$ is neither concave nor convex; its reciprocal $1/q$, though, is logarithmically convex by Theorem 1.5, hence also convex. Moreover, based on Figure 3, as well as numerical results, for $b>1$ we conjecture that $\phi$ is concave.

The article is organised in the following way. In section 2 we present some general results regarding $p$-quantiles of more general probability distributions, that may be of independent interest. For instance, Lemma 2.2 is a generalisation of results concerning monotonicity properties of ratios of power series and polynomials to ratios of integrals. In section 3 we study the monotonicity and limit properties of $q$ and $\phi$ and prove Propositions 1.2 and 1.3. In section 4 we prove convexity of $\phi$ for $b<1$, while in section 5 we prove logarithmic concavity of $q$. In the Appendix, we look into the dependence on the parameter $b$ with $a$ being fixed and translate some of the results to this case.

## 2. General results on $p$-quantiles of probability distributions

The following lemma is a standard result in measure theory, that lets us interchange integration and differentiation [11, Theorem 6.28]. In the rest of the paper, $\partial_x$ denotes differentiation with respect to the variable $x$.

Lemma 2.1. *Let $(\Omega,\mathcal{B},\mu)$ be a measure space, $I\subset\mathbb{R}$ an open interval and $f: I\times\Omega\to\mathbb{R}$ a function such that:*

*i.* $a\mapsto f(a,t)$ *is differentiable for $\mu$-a.e.* $t\in\Omega$
*ii.* $t\mapsto f(a,t)$ *is $\mu$-integrable for all* $a\in I$
*iii.* $\exists g\in L^1(\Omega,\mathrm{d}\mu)$ *such that* $|\partial_a f(a,t)|\leqslant g(t)$ *for all $a\in I$ and $\mu$-a.e.* $t\in\Omega$

*Then, the function $a\mapsto\int_\Omega f(a,t)\mathrm{d}\mu(t)$ is differentiable and*
$$\partial_a\int_\Omega f(a,t)\mathrm{d}\mu(t) = \int_\Omega \partial_a f(a,t)\mathrm{d}\mu(t)\,.$$

LEMMA 2.2. *Let $I \subset \mathbb{R}$ be an open interval, $A \subset \mathbb{R}$ a non-empty Borel set, $\mu$ a $\sigma$-finite Borel measure on $A$ and $u, v : A \to [0, +\infty)$ measurable functions, not simultaneously* 0. *Let $f : I \times A \to (0, +\infty)$ be such that*

*i.* *$a \mapsto f(a,t)$ is differentiable for $\mu$-a.e. $t \in A$*

*ii.* *$t \mapsto u(t)f(a,t)$ and $t \mapsto v(t)f(a,t)$ are $\mu$-integrable for all $a \in I$.*

*iii.* *For each compact subset $K \subset I$, there exists a function $g_K : A \to [0, +\infty)$ such that $ug_K, vg_K$ are $\mu$-integrable and $|\partial_a f(a,t)| \leqslant g_K(t)$ for all $a \in K$ and $\mu$-a.e. $t \in A$.*

*Let $F : I \to \mathbb{R}$ be defined by:*

$$F(a) := \frac{\int_A f(a,t)u(t)\mathrm{d}\mu(t)}{\int_A f(a,t)v(t)\mathrm{d}\mu(t)}\,.$$

*Then, the following hold:*

*I.* *If for all $a \in I$ and for $\mu$-a.e. $t \in A$, $\partial_a f(a,t)/f(a,t)$ and $u(t)/v(t)$ both increase or both decrease wrt $t$, then $F$ is increasing.*

*II.* *If for all $a \in I$ and for $\mu$-a.e. $t \in A$, $\partial_a f(a,t)/f(a,t)$ increases (decreases) wrt $t$ and $u(t)/v(t)$ decreases (increases), then $F$ is decreasing.*

*Proof.* Let $U(a) = \int_A f(a,t)u(t)\mathrm{d}\mu(t)$, $V(a) = \int_A f(a,t)v(t)\mathrm{d}\mu(t)$. By the fact that $u(t)\partial_a f(a,t)$ and $v(t)\partial_a f(a,t)$ are dominated on compact subsets of $I$ by a $\mu$-integrable function of $t$, Lemma 2.1 gives that both $U$ and $V$ are differentiable, and the derivatives can be given by differentiating the integrands. Then, $F'$ also exists and hence we need to investigate the derivative

$$F'(a) = \frac{U'(a)V(a) - U(a)V'(a)}{V^2(a)}\,.$$

We find

$$\begin{aligned}
&U'(a)V(a) - U(a)V'(a) = \\
&= \int_A \int_A u(s)v(t)(\partial_a f(a,s)f(a,t) - \partial_a f(a,t)f(a,s))\mathrm{d}\mu(s)\mathrm{d}\mu(t) \\
&= \int_A \int_{A\cap\{s<t\}} u(s)v(t)(\partial_a f(a,s)f(a,t) - \partial_a f(a,t)f(a,s))\mathrm{d}\mu(s)\mathrm{d}\mu(t) \\
&\qquad + \int_A \int_{A\cap\{s>t\}} u(s)v(t)(\partial_a f(a,s)f(a,t) - \partial_a f(a,t)f(a,s))\mathrm{d}\mu(s)\mathrm{d}\mu(t) \\
&= \int_A \int_{A\cap\{s<t\}} u(s)v(t)(\partial_a f(a,s)f(a,t) - \partial_a f(a,t)f(a,s))\mathrm{d}\mu(s)\mathrm{d}\mu(t) \\
&\qquad + \int_A \int_{A\cap\{s<t\}} u(t)v(s)(\partial_a f(a,t)f(a,s) - \partial_a f(a,s)f(a,t))\mathrm{d}\mu(s)\mathrm{d}\mu(t) \\
&= \int_A \int_{A\cap\{s<t\}} (u(s)v(t) - u(t)v(s))(\partial_a f(a,s)f(a,t) - \partial_a f(a,t)f(a,s))\mathrm{d}\mu(s)\mathrm{d}\mu(t)
\end{aligned}$$

where in the pre-last equality we have made use of Fubini's theorem. The last integrand, as $s < t$, is non-negative (non-positive) if $\partial_a f/f$ and $u/v$ have the same (opposite) monotonicity properties, which proves the lemma. □

REMARK 2.3. In the proceeding Lemma, the same conclusion holds we allow $u$, $v$ to assume the value zero at the same time, as then, without loss of generality, we can just integrate over the set $A' = A\backslash(\{u(t) = 0\} \cap \{v(t) = 0\})$, which is again a Borel set, and we consider the condition $u/v$ being increasing (or decreasing) in $A'$.

REMARK 2.4. Lemma 2.2 is a general case of results concerning monotonicity properties of ratios of power series and polynomials. For instance, it gives [12, Lemma 2.2], if we set $\mu$ to be the counting measure on $\mathbb{N}$.

Lemma 2.5. *Let $I, J$ be two open intervals. Let $f : I \times J \to (0, \infty)$ be such that:*

i. *$a \mapsto f(a, x)$ is differentiable for a.e. $x \in J$*
ii. *$x \mapsto f(a, x)$ is integrable for all $a \in I$*
iii. *For each compact subset $K \subset I$, there exists an integrable function $g_K : J \to [0, +\infty)$ such that $|\partial_a f(a, t)| \leqslant g_K(t)$ for all $a \in K$ and $\mu$-a.e. $t \in A$.*
iv. *The logarithmic derivative of $f$ wrt $a$ is increasing (decreasing) wrt $x$ for a.e. $x$, i.e.*

$$\frac{\partial_a f(a, x)}{f(a, x)} \uparrow_x (\downarrow_x) .$$

*Then, the $p$-quantile $q(a)$ of the probability distribution with density $f(a, x)/\int_J f(a, t)dt$ is increasing (decreasing) wrt $a$.*

*Proof.* We will deal with the case that the logarithmic derivative of $f$ is increasing, and the other case, that it is decreasing, is analogous. Let $x \in J = (c, d)$, where $-\infty \leqslant c < d \leqslant +\infty$. Then the cumulative distribution function is

$$F(a; x) = \frac{\int_c^x f(a, t)\mathrm{d}t}{\int_c^d f(a, t)\mathrm{d}t} = \frac{\int_c^d f(a, t)1_{[c,x]}(t)\mathrm{d}t}{\int_c^d f(a, t)\mathrm{d}t} .$$

We set $u(t) = 1_{[c,x]}(t)$ and $v(t) = 1$. As $u/v = u$ decreases and $\partial_a f/f$ increases wrt $t$, by Lemma 2.2 we get that $F$ decreases pointwise wrt $a$. This means

$$\frac{\int_c^{q(a+h)} f(a, t)\mathrm{d}t}{\int_c^d f(a, t)\mathrm{d}t} \geqslant \frac{\int_c^{q(a+h)} f(a + h, t)\mathrm{d}t}{\int_c^d f(a + h, t)\mathrm{d}t} = p = \frac{\int_c^{q(a)} f(a, t)\mathrm{d}t}{\int_c^d f(a, t)\mathrm{d}t}$$

so that $q(a + h) \geqslant q(a)$ and hence that the $p$-quantile is increasing. □

Remark 2.6. In Lemma 2.2, if the logarithmic derivative $\partial_a f/f$ is strictly monotone (and $u \not\equiv v$), it is easy to see from the proof that the ratio of the integrals in the conclusion should also be strictly monotone. Hence, also in Lemma 2.5, if the logarithmic derivative is strictly increasing (decreasing), then the $p$-quantile is also strictly increasing (decreasing).

The following Lemma deals with the question of convergence of $p$-quantiles of a convergent sequence of probability distributions. We denote the extended real line $\mathbb{R} \cup \{\pm\infty\}$ by $\hat{\mathbb{R}}$, with its usual topology.

Lemma 2.7. *Let $F_n : \hat{\mathbb{R}} \to [0, 1]$ be a sequence of cumulative distribution functions on $\mathbb{R}$, extended by $F_n(-\infty) := 0$ and $F_n(+\infty) := 1$. Let $q_n$ be a $p$-quantile of $F_n$, i.e. $F_n(q_n) = p \in (0, 1)$, $\forall n \in \mathbb{N}$. Assume the following conditions:*

i. *The sequence $(F_n(x))_{n\in\mathbb{N}}$ converges pointwise to a limit $F_\infty(x) := \lim_{n\to\infty} F_n(x)$*
ii. *The sequence of $p$-quantiles converges to a limit $q_\infty := \lim_{n\to\infty} q_n \in \hat{\mathbb{R}}$*

*Then,*

$$q_\infty \in [\sup\{x \in \hat{\mathbb{R}} | F_\infty(x) < p\}, \inf\{x \in \hat{\mathbb{R}} | F_\infty(x) > p\}] . \tag{6}$$

*Thus, if $F_\infty$ is continuous, $q_\infty$ is a $p$-quantile of $F_\infty$.*

*Proof.* Let some $w \in \hat{\mathbb{R}}$ be such that $F_\infty(w) < p$. By condition (i) we have that there is some $n_0 \in \mathbb{N}$ such that $\forall n > n_0 : F_n(w) < p = F_n(q_n)$. As each $F_n$ is non-decreasing, we have that $\forall n > n_0 : w < q_n$ and hence $q_\infty \geqslant w$. As this holds $\forall w \in \{x \in \hat{\mathbb{R}} | F_\infty(x) < p\}$, we get that $q_\infty \geqslant \sup\{x \in \hat{\mathbb{R}} | F_\infty(x) < p\}$. In a similar way we may prove that $q_\infty \leqslant \inf\{x \in \hat{\mathbb{R}} | F_\infty(x) > p\}$. In case $F_\infty$ is continuous, we have $[\sup\{x \in \hat{\mathbb{R}} | F_\infty(x) < p\}, \inf\{x \in \hat{\mathbb{R}} | F_\infty(x) > p\}] = \{x \in \mathbb{R} | F_\infty(x) = p\}$, hence then $q_\infty$ is a $p$-quantile of $F_\infty$. □

Remark 2.8. As $\hat{\mathbb{R}}$ is compact, $p$-quantiles always have limit points, and the above Lemma shows that convergence of distribution functions for which $p$-quantiles exist implies that all their limit points lie in the interval in (6). This interval either consists of the closure of $F_\infty^{-1}(\{p\})$, or, if this set is empty, it degenerates to a point, which is a point of discontinuity of $F_\infty$.

Lemma 2.9. *Let $I, J \subset \mathbb{R}$ be open intervals, and $(F(a;x))_{a\in I}$ be a family of cumulative probability distribution functions of $x$ on $J$, having positive densities $f(a;t)$ with respect to Lebesgue measure. Moreover assume that the corresponding densities are real analytic in both variables. Denote the respective $p$-quantiles by $q(a)$. Then, $q$ is a real analytic function of $a$.*

*Proof.* As the densities are positive functions, the $p$-quantile exists and is unique for each $a$. Hence, the function $q(a)$ is well defined implicitly as the solution $y = q(a)$ to the equation $F(a;y)-p = 0$. Let some $y_0 \in J$ and $a_0 \in I$ such that $F(a_0;y_0)-p = 0$. As $F$ is real analytic and $\partial_y F(a;y) = f(a;y) \neq 0$, by [13, Theorem 6.1.2] the equation $F(a;y) - p = 0$ has a real analytic solution $y = y(a)$ in a neighbourhood of $a_0$ such that $F(a_0;y(a_0)) - p = 0$. By uniqueness of the $p$-quantile this solution must be exactly $q(a)$, and hence $q$ is real analytic. ☐

## 3. Monotonicity and limits

*Proof of Proposition 1.2* Fix $b > 0$. As the regularised incomplete beta function $I(x;a,b)$ is real analytic in $x$ and $a$, Lemma 2.9 gives real analyticity of $q$. Let $\beta(a;x) := x^{a-1}(1-x)^{b-1}$. Its logarithmic derivative wrt $a$ is

$$\frac{\partial_a \beta(a,b;x)}{\beta(a,b;x)} = \frac{x^{a-1}(1-x)^{b-1}\log x}{x^{a-1}(1-x)^{b-1}} = \log x\,,$$

which is an increasing function of $x$, and Lemma 2.5 gives us that $q$ is also increasing. Its limits at 0 and $\infty$ are classical results. They can also be obtained by considering limits of the incomplete beta function and using Lemma 2.7. Let, for instance, some limit point $\lim_{n\to\infty} q(a_n) = q_\infty \in [0,1]$ for a sequence $a_n \to \infty$. Then, the fact that $\lim_{a\to\infty} I(x;a,b)$ vanishes for $x \in [0,1)$ and is a unit at $x = 1$ gives $q_\infty = 1$, hence $\lim_{a\to\infty} q(a) = 1$. A similar argument shows $\lim_{a\to 0} q(a) = 0$. ☐

*Proof of Proposition 1.3* By Proposition 1.2 already, $\phi$ can be seen to be a real analytic function. Regarding monotonicity, if $b = 1$ then $\phi(a) \equiv -\log p$. Assume $b > 1$. By using a change of variables in (4) we get

$$\int_{\phi(a)}^{\infty} e^{-s}(1-e^{-s/a})^{b-1}\mathrm{d}s = p\int_0^{\infty} e^{-s}(1-e^{-s/a})^{b-1}\mathrm{d}s \tag{7}$$

and hence the function $\phi$ is the $(1-p)$-quantile of the distribution with density function

$$x \mapsto \frac{e^{-x}(1-e^{-x/a})^{b-1}}{\int_0^{+\infty} e^{-s}(1-e^{-s/a})^{b-1}\mathrm{d}s}\,.$$

We set $f(a;x) := e^{-x}(1-e^{-x/a})^{b-1}$. The logarithmic derivative of $f$ wrt $a$ is

$$\frac{\partial_a f(a;x)}{f(a;x)} = -\frac{(b-1)xe^{-x/a}}{a^2(1-e^{-x/a})}\,.$$

The derivative of this wrt $x$ is

$$\partial_x\left(\frac{\partial_a f(a;x)}{f(a;x)}\right) = \frac{b-1}{a^3}\mathrm{e}^{-\frac{x}{a}}\left(a\mathrm{e}^{-\frac{x}{a}} - a + x\right)\left(-1+\mathrm{e}^{-\frac{x}{a}}\right)^{-2} \geqslant 0$$

as the function $x \mapsto a\mathrm{e}^{-\frac{x}{a}} - a + x$ has positive derivative for $x > 0$ and vanishes at 0. Thus, by Lemma 2.5 we have that $\phi$ is increasing. The case $b < 1$ is similar.

For the asymptotic results, we notice that for $a \to 0$, we have that

$$\lim_{a\to 0} \frac{e^{-x}(1-e^{-x/a})^{b-1}}{\int_0^\infty e^{-s}(1-e^{-s/a})^{b-1}\mathrm{d}s} = \frac{e^{-x}}{\int_0^\infty e^{-s}\mathrm{d}s} = e^{-x}\,.$$

The corresponding distributions, whose $p$-quantiles are equal to $\phi(a)$, converge to the gamma distribution with parameter 1, and hence by Lemma 2.7 $\lim_{a\to 0}\phi(a) = -\log p$. Similarly, for $a \to \infty$

$$\lim_{a\to\infty} \frac{e^{-x}(1-e^{-x/a})^{b-1}}{\int_0^\infty e^{-s}(1-e^{-s/a})^{b-1}\mathrm{d}s} = \lim_{a\to\infty} \frac{e^{-x}}{\int_0^\infty e^{-s}\frac{(1-e^{-s/a})^{b-1}}{(1-e^{-x/a})^{b-1}}\mathrm{d}s} = \frac{e^{-x}x^{b-1}}{\int_0^\infty e^{-s}s^{b-1}\mathrm{d}s}$$

hence the distribution converges to the gamma distribution with parameter $b$ and $\lim_{a\to\infty}\phi(a) = \gamma_b$, the $(1-p)$-quantile of the gamma distribution with parameter $b$. □

## 4. Convexity of $\phi$ for $b < 1$

We rewrite (7) as

$$\int_0^{\phi(a)} e^{-s}(1-e^{-s/a})^{b-1}\mathrm{d}s = (1-p)\int_0^\infty e^{-s}(1-e^{-s/a})^{b-1}\mathrm{d}s\,. \tag{8}$$

We denote $f(a;s) = e^{-s}(1-e^{-s/a})^{b-1}$ and differentiating the above equation we have

$$\phi'(a)f(a;\phi(a)) + \int_0^{\phi(a)} \partial_1 f(a;t)\mathrm{d}t = (1-p)\int_0^\infty \partial_1 f(a;t)\mathrm{d}t \tag{9}$$

Differentiating again,

$$\begin{aligned}\phi''(a)f(a;\phi(a)) =&(1-p)\int_{\phi(a)}^\infty \partial_1^2 f(a;t)\mathrm{d}t - p\int_0^{\phi(a)} \partial_1^2 f(a;t)\mathrm{d}t \\ &- (\phi'(a))^2\partial_2 f(a;\phi(a)) - 2\phi'(a)\partial_1 f(a;\phi(a))\end{aligned} \tag{10}$$

where $\partial_j, j \in \mathbb{N}$, denotes differentiation wrt to the $j$th variable.

*Proof of Theorem 1.4* Let $b \in (0,1)$. By Proposition 1.3 $\phi' < 0$, and as

$$\partial_s f(a;s) = -e^{-s}(1-e^{-s/a})^{b-1} + \frac{b-1}{a}e^{-s}(1-e^{-s/a})^{b-2}e^{-s/a} < 0$$

and

$$\partial_a f(a;s) = -s\frac{b-1}{a^2}e^{-s}(1-e^{-s/a})^{b-2} > 0$$

we see that $\phi'(a)^2\partial_2 f(a;\phi(a)) < 0$ and $\phi'(a)\partial_1 f(a;\phi(a)) < 0$. In order to show that $\phi'' > 0$, using (10) what is left is to show that

$$(1-p)\int_{\phi(a)}^\infty \partial_1^2 f(a;t)\mathrm{d}t - p\int_0^{\phi(a)} \partial_1^2 f(a;t)\mathrm{d}t \geqslant 0 \tag{11}$$

We shall rewrite the above integrals in another way. We have

$$
\begin{aligned}
&\int_{\phi(a)}^{\infty} \partial_1^2 f(a;t)\mathrm{d}t = \\
&\quad= \frac{b-1}{a^4}\int_{\phi(a)}^{\infty} 2te^{-\frac{2t}{a}}e^{-t}(1-e^{-\frac{t}{a}})^{b-3}\left(\left(a-\frac{t}{2}\right)e^{\frac{t}{a}}+\frac{(b-1)t}{2}-a\right)\mathrm{d}t \\
&\quad= \frac{2(b-1)}{a}\int_{\frac{\phi(a)}{a}}^{\infty} se^{-as}e^{-2s}(1-e^{-s})^{b-3}\left(\left(1-\frac{s}{2}\right)e^{s}+\frac{b-1}{2}s-1\right)\mathrm{d}s \\
&\quad= \frac{2(b-1)}{a}\int_{\frac{\phi(a)}{a}}^{\infty} e^{-at}\eta(t)\mathrm{d}t
\end{aligned}
$$

where

$$
\eta(x) := xe^{-2x}(1-e^{-x})^{b-3}\left(e^{x}-1-\frac{x}{2}e^{x}+\frac{b-1}{2}x\right) \tag{12}
$$

and similarly

$$
\int_0^{\phi(a)} \partial_1^2 f(a;t)\mathrm{d}t = \frac{2(b-1)}{a}\int_0^{\frac{\phi(a)}{a}} e^{-at}\eta(t)\mathrm{d}t
$$

Hence we can rewrite

$$
\begin{aligned}
&(1-p)\int_{\phi(a)}^{\infty} \partial_1^2 f(a;t)\mathrm{d}t - p\int_0^{\phi(a)} \partial_1^2 f(a;t)\mathrm{d}t = \\
&\qquad \frac{2(b-1)}{a}\left((1-p)\int_{\phi(a)/a}^{\infty} e^{-at}\eta(t)\mathrm{d}t - p\int_0^{\phi(a)/a} e^{-at}\eta(t)\mathrm{d}t\right)
\end{aligned} \tag{13}
$$

We now proceed to show (11). We see in Lemma 4.1 below that the function

$$
w(x) := \left(1-\frac{x}{2}\right)e^{x}+\frac{b-1}{2}x-1 \tag{14}
$$

has a unique root $\rho$ on $(0,+\infty)$, and it is positive on $(0,\rho)$ and negative on $(\rho,\infty)$. Assume that $\phi(a)\geqslant \rho a$. As $w$ and $\eta$ have the same sign, we have that $\int_{\phi(a)/a}^{\infty} e^{-at}\eta(t)\mathrm{d}t<0$. For the other integral, we have

$$
\begin{aligned}
\int_0^{\phi(a)/a} e^{-at}\eta(t)\mathrm{d}t &= \int_0^{\rho} e^{-at}\eta(t)\mathrm{d}t + \int_{\rho}^{\phi(a)/a} e^{-at}\eta(t)\mathrm{d}t \\
&\geqslant e^{-a\rho}\left(\int_0^{\rho}\eta(t)\mathrm{d}t + \int_{\rho}^{\phi(a)/a}\eta(t)\mathrm{d}t\right) \\
&\geqslant e^{-a\rho}\left(\int_0^{\rho}\eta(t)\mathrm{d}t + \int_{\rho}^{\infty}\eta(t)\mathrm{d}t\right) = e^{-a\rho}\int_0^{\infty}\eta(t)\mathrm{d}t = 0
\end{aligned}
$$

by Lemma 4.2 below. Hence

$$
\frac{2(b-1)}{a}\left((1-p)\int_{\phi(a)/a}^{\infty} e^{-at}\eta(t)\mathrm{d}t - p\int_0^{\phi(a)/a} e^{-at}\eta(t)\mathrm{d}t\right)\geqslant 0
$$

and by (13), (11) is proved for $\phi(a)\geqslant \rho a$.

Now, assume that $\phi(a)<\rho a$. We define

$$
h(a;t) := \frac{\partial_1^2 f(a;t)}{(b-1)f(a;t)} = \frac{2t((a-t/2)e^{t/a}+(b-1)t/2-a)}{a^4(e^{t/a}-1)^2} \tag{15}
$$

We further denote

$$
h_0(s) := \frac{a^2}{2}h(a;as) = \frac{s((1-s/2)e^{s}+(b-1)s/2-1)}{(e^{s}-1)^2} = \frac{sw(s)}{(e^{s}-1)^2} \tag{16}
$$

By Lemma 4.3, $h_0$ is decreasing on $(0, \rho)$, hence $h(a; s)$ is also decreasing wrt $s$ on $(0, \rho a)$. Hence, for $t \in (0, \phi(a)) \subset (0, \rho a)$ we have $h(a; t) > h(a; \phi(a))$. For $t \in (\phi(a), \rho a)$, we analogously have $h(a; \phi(a)) > h(a; t)$, and if $t \in (\rho a, \infty)$, then $h(a; \phi(a)) > 0 > h(a; t)$. Hence,

$$
\begin{aligned}
&(1-p)\int_{\phi(a)}^{\infty} \partial_1^2 f(a;t)\mathrm{d}t - p\int_0^{\phi(a)} \partial_1^2 f(a;t)\mathrm{d}t = \\
&\quad =(b-1)\left((1-p)\int_{\phi(a)}^{\infty} h(a;t)f(a;t)\mathrm{d}t - p\int_0^{\phi(a)} h(a;t)f(a;t)\mathrm{d}t\right) \\
&\quad \geqslant (b-1)h(a;\phi(a))\left((1-p)\int_{\phi(a)}^{\infty} f(a;t)\mathrm{d}t - p\int_0^{\phi(a)} f(a;t)\mathrm{d}t\right) = 0
\end{aligned}
$$

by (8). Thus (11) is proved. As the RHS of (10) is positive, then $\phi'' > 0$. □

LEMMA 4.1. *Fix $b > 0$. The function $w$ in (14) has a unique root $\rho$ on $(0, \infty)$. We have that $w(x) > 0$ for $x < \rho$ and $w(x) < 0$ for $x > \rho$.*

*Proof.* We have

$$w'(x) = \frac{1-x}{2}e^x + \frac{b-1}{2}$$

and

$$w''(x) = -\frac{x}{2}e^x < 0 \quad \text{for } x > 0$$

Hence $w'$ is strictly decreasing, and as $w'(0) = b/2$ and $\lim_{x\to+\infty} w'(x) = -\infty$, it changes its sign exactly once and we get that $w$ is initially increasing and then decreasing, concave function. As $w(0) = 0$ and $\lim_{x\to+\infty} w(x) = -\infty$, we get that $w$ has a unique root $\rho \in (0, \infty)$, and $w(x) > 0$ for $x < \rho$ and $w(x) < 0$ for $x > \rho$. □

LEMMA 4.2. *For $b > 0$, it holds that*

$$\int_0^{\infty} se^{-2s}(1-e^{-s})^{b-3}\left(e^s - 1 - \frac{s}{2}e^s + \frac{b-1}{2}s\right)\mathrm{d}s = 0$$

*Proof.* In the course of the proof we assume that $b \neq 1, 2$, which may be lifted in the end by taking limits. We split the integral into 3 parts. The first one is

$$
\begin{aligned}
I_1 &= \int_0^{\infty} se^{-2s}(1-e^{-s})^{b-3}(e^s-1)\mathrm{d}s \\
&= \int_0^{\infty} se^{-s}(1-e^{-s})^{b-2}\mathrm{d}s \\
&= -\int_0^{\infty} \log(1-e^{-t})e^{-(b-1)t}\mathrm{d}t \\
&= -\int_0^{\infty} \log(1-e^{-t})\left(\frac{1-e^{-(b-1)t}}{b-1}\right)'\mathrm{d}t \\
&= \frac{1}{b-1}\int_0^{\infty} \frac{e^{-t}-e^{-bt}}{1-e^{-t}}\mathrm{d}t \\
&= \frac{\psi(b)+\gamma}{b-1}
\end{aligned}
$$

where $\psi := \Gamma'/\Gamma$ is the digamma function (see [3, Chapter 1]). For the second part,

$$
\begin{aligned}
I_2 =& \frac{b-1}{2}\int_0^\infty s^2 e^{-2s}(1-e^{-s})^{b-3}\mathrm{d}s \\
=& \frac{b-1}{2(b-2)}\left(\int_0^\infty s^2 e^{-s}(1-e^{-s})^{b-2}\mathrm{d}s - 2\int_0^\infty s e^{-s}(1-e^{-s})^{b-2}\mathrm{d}s\right) \\
=& \frac{b-1}{2(b-2)}\int_0^1 \log^2 t(1-t)^{b-2}\mathrm{d}t - \frac{\psi(b)+\gamma}{b-2} \\
=& \frac{b-1}{2(b-2)}\partial_1^2 \mathrm{B}(1,b-1) - \frac{\psi(b)+\gamma}{b-2}
\end{aligned}
$$

using that $\partial_1^n \mathrm{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\log^n t\mathrm{d}t$ for $b > -n$, which is derived by differentiating the integral representation of the beta function for $b > 0$ and using the identity principle. Finally,

$$
\begin{aligned}
I_3 =& -\frac{1}{2}\int_0^\infty s^2 e^{-s}(1-e^{-s})^{b-3}\mathrm{d}s \\
=& -\frac{1}{2}\int_0^1 (\log t)^2(1-t)^{b-3}\mathrm{d}t \\
=& -\frac{1}{2}\partial_1^2 \mathrm{B}(1,b-2) \\
=& -\frac{1}{2}\partial_a^2\left(\mathrm{B}(a,b-1)\frac{a+b-2}{b-2}\right)\Bigg|_{a=1} \\
=& -\frac{b-1}{2(b-2)}\partial_1^2\mathrm{B}(1,b-1) - \frac{\partial_1 \mathrm{B}(1,b-1)}{b-2} \\
=& -\frac{b-1}{2(b-2)}\partial_1^2\mathrm{B}(1,b-1) + \frac{\gamma+\psi(b)}{(b-2)(b-1)}
\end{aligned}
$$

where we have used that $\partial_1 \mathrm{B}(1,b-1) = \frac{\gamma+\psi(b)}{b-1}$. We see that $I_1+I_2+I_3 = 0$, and the Lemma is proved. □

Lemma 4.3. *Fix $b > 0$. The function $h_0$ in (16) is decreasing between* 0 *and its root $\rho \in (0,\infty)$.*

*Proof.* It is easy to see that $x/(e^x-1)$ is decreasing. The rest is also decreasing as

$$
\frac{\left(1-\frac{x}{2}\right)e^x + \frac{b-1}{2}x - 1}{e^x-1} = \frac{b}{2}\frac{x}{e^x-1} + 1 - \frac{1}{2}\frac{x(e^x+1)}{e^x-1}
$$

and

$$
\left(\frac{x(e^x+1)}{e^x-1}\right)' = \frac{e^{2x} - 2e^x x - 1}{(e^x-1)^2} \geqslant 0
$$

as $\left(e^{2x} - 2e^x x - 1\right)' = 2e^x(e^x - x - 1) \geqslant 0$ and the numerator vanishes at 0. Hence, on $(0,\rho)$, $h_0$ is the product of two decreasing, positive functions, hence decreasing. □

## 5. Logarithmic concavity of $q$

In this section, we shall prove Theorem 1.5. In order to have a more concise notation, we shall often omit the argument $a$ from the notation of the functions of $a$ ($q, \phi$ and $\xi$), without their argument. Using [9, 8.17.7], we can rewrite (4), as

$$
\frac{q^a}{a}\,{}_2F_1(a, 1-b; a+1; q) = p\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{17}
$$

and expanding the hypergeometric sum,

$$q^a \sum_{n=0}^{\infty} \frac{(1-b)_n q^n}{(a+n)n!} = p\Gamma(b)\frac{\Gamma(a)}{\Gamma(a+b)} \tag{18}$$

Of course, if $b \in \mathbb{N}$, the sum above terminates at $b-1$, as $(1-b)_b = 0$. Using that

$$\frac{(1-b)_n}{n!} = \frac{(b-1)(b-1-1)\cdots(b-1-n-1)}{n!}(-1)^n = \binom{b-1}{n}(-1)^n$$

and denoting

$$\xi := -\log q \tag{19}$$

we can rewrite (18) further as

$$e^{-a\xi}\sum_{n=0}^{\infty}\binom{b-1}{n}\frac{(-1)^n e^{-n\xi}}{(a+n)} = p\Gamma(b)\frac{\Gamma(a)}{\Gamma(a+b)}$$

that is

$$\sum_{n=0}^{\infty}\frac{\Gamma(a+b)}{\Gamma(a)(a+n)}\binom{b-1}{n}(-1)^n e^{-(n+a)\xi} = p\Gamma(b) \tag{20}$$

We shall show that $\xi$ is convex, which shall imply the logarithmic concavity. The following lemma will be the key to this proof.

Lemma 5.1. *We have that*

$$\xi' = \sum_{n=0}^{\infty}\frac{1}{a+b+n}Y_{n+b}(\xi) - \sum_{n=0}^{\infty}\frac{1}{a+n}Y_n(\xi) \tag{21}$$

*where*

$$Y_c(\xi) := \frac{\int_0^{\xi} e^{ct}(1-e^{-t})^{b-1}\mathrm{d}t}{e^{c\xi}(1-e^{-\xi})^{b-1}} \tag{22}$$

*Proof.* Differentiating (20) we get

$$\begin{aligned} 0 = &\sum_{n=0}^{\infty}\binom{b-1}{n}(-1)^n e^{-(n+a)\xi}\left[\left(\frac{\Gamma(a+b)}{\Gamma(a)(a+n)}\right)' - (\xi+(n+a)\xi')\frac{\Gamma(a+b)}{\Gamma(a)(a+n)}\right] \\ =&e^{-a\xi}\sum_{n=0}^{\infty}\binom{b-1}{n}(-1)^n e^{-n\xi}\left[\left(\frac{\Gamma(a+b)}{\Gamma(a)(a+n)}\right)' - \xi'\frac{\Gamma(a+b)}{\Gamma(a)} - \xi\frac{\Gamma(a+b)}{\Gamma(a)(a+n)}\right] \end{aligned}$$

Using the fact that $\sum_{n=0}^{\infty}\binom{b-1}{n}(-1)^n e^{-n\xi} = (1-e^{-\xi})^{b-1}$, we get

$$
\begin{aligned}
&\xi'(1-e^{-\xi})^{b-1} = \\
&= \sum_{n=0}^{\infty} \binom{b-1}{n} (-1)^n e^{-n\xi} \left( \left( \frac{\Gamma(a+b)}{\Gamma(a)(a+n)} \right)' \Big/ \left( \frac{\Gamma(a+b)}{\Gamma(a)} \right) - \frac{\xi}{a+n} \right) \\
&= \sum_{n=0}^{\infty} \binom{b-1}{n} (-1)^n e^{-n\xi} \left( \frac{\psi(a+b)-\psi(a)}{a+n} - \frac{1}{(a+n)^2} - \frac{\xi}{a+n} \right) \\
&= \sum_{n=0}^{\infty} \binom{b-1}{n} (-1)^n e^{-n\xi} \left( \sum_{k=0}^{\infty} \left( \frac{1}{k+a} - \frac{1}{k+a+b} \right) \frac{1}{a+n} - \frac{1}{(a+n)^2} - \frac{\xi}{a+n} \right) \\
&= \sum_{n=0}^{\infty} \binom{b-1}{n} (-1)^n e^{-n\xi} \times \\
&\quad \left( \sum_{k\neq n} \left( \frac{1}{(k+a)(a+n)} - \frac{1}{(k+a+b)(a+n)} \right) - \frac{1}{(n+a+b)(a+n)} - \frac{\xi}{a+n} \right) \\
&= \sum_{n=0}^{\infty} \binom{b-1}{n} (-1)^n e^{-n\xi} \times \\
&\quad \sum_{k\neq n} \left( \left( \frac{1}{a+n} - \frac{1}{a+k} \right) \frac{1}{k-n} - \left( \frac{1}{a+n} - \frac{1}{a+k+b} \right) \frac{1}{k+b-n} \right) \\
&\quad - \sum_{n=0}^{\infty} \left( \binom{b-1}{n} (-1)^n e^{-n\xi} \left( \frac{1}{a+n} - \frac{1}{a+n+b} \right) \frac{1}{b} - \frac{\xi}{a+n} \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{a+n} \binom{b-1}{n} (-1)^n e^{-n\xi} \left( \sum_{k\neq n} \left( \frac{1}{k-n} - \frac{1}{k+b-n} \right) \right) \\
&\quad - \sum_{n=0}^{\infty} \frac{1}{a+n} \binom{b-1}{n} (-1)^n e^{-n\xi} \xi + \sum_{n=0}^{\infty} \binom{b-1}{n} (-1)^n e^{-n\xi} \times \\
&\quad \left( \sum_{k\neq n} \left( \frac{1}{a+k+b} \frac{1}{k+b-n} - \frac{1}{a+k} \frac{1}{k-n} \right) + \left( \frac{1}{a+n+b} - \frac{1}{a+n} \right) \frac{1}{b} \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{a+n} \binom{b-1}{n} (-1)^n e^{-n\xi} \left( \sum_{k\neq n} \left( \frac{1}{k-n} - \frac{1}{k+b-n} \right) \right) \\
&\quad + \sum_{n=0}^{\infty} \left( \frac{1}{a+n+b} - \frac{1}{a+n} \right) \binom{b-1}{n} (-1)^n e^{-n\xi} \frac{1}{b} - \sum_{n=0}^{\infty} \frac{1}{a+n} \binom{b-1}{n} (-1)^n e^{-n\xi} \xi \\
&\quad + \sum_{n=0}^{\infty} \sum_{k\neq n} \binom{b-1}{n} (-1)^n e^{-n\xi} \left( \frac{1}{a+k+b} \frac{1}{k+b-n} - \frac{1}{a+k} \frac{1}{k-n} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{1}{a+n}\binom{b-1}{n}(-1)^n e^{-n\xi}\left(\sum_{k\neq n}\left(\frac{1}{k-n}-\frac{1}{k+b-n}\right)\right)\\
&\quad + \sum_{n=0}^{\infty}\left(\frac{1}{a+n+b}-\frac{1}{a+n}\right)\binom{b-1}{n}(-1)^n e^{-n\xi}\frac{1}{b} - \sum_{n=0}^{\infty}\frac{1}{a+n}\binom{b-1}{n}(-1)^n e^{-n\xi}\xi\\
&\quad + \sum_{n=0}^{\infty}\sum_{k\neq n}\binom{b-1}{k}(-1)^k e^{-k\xi}\left(\frac{1}{a+n+b}\frac{1}{n+b-k}-\frac{1}{a+n}\frac{1}{n-k}\right)\\
&= \sum_{n=0}^{\infty} \frac{1}{a+n}\binom{b-1}{n}(-1)^n e^{-n\xi}\left(\sum_{k\neq n}\left(\frac{1}{k-n}-\frac{1}{k+b-n}\right)-\frac{1}{b}\right)\\
&\quad + \sum_{n=0}^{\infty}\frac{1}{a+n}\left(\sum_{k\neq n}\binom{b-1}{k}\frac{(-1)^k e^{-k\xi}}{k-n}-\binom{b-1}{n}(-1)^n e^{-n\xi}\xi\right)\\
&\quad + \sum_{n=0}^{\infty}\frac{1}{a+n+b}\sum_{k\neq n}\binom{b-1}{k}\frac{(-1)^k e^{-k\xi}}{n+b-k} + \sum_{n=0}^{\infty}\frac{1}{a+n+b}\binom{b-1}{n}(-1)^n e^{-n\xi}\frac{1}{b}\\
&= \sum_{n=0}^{\infty} \frac{1}{a+n}\binom{b-1}{n}(-1)^n e^{-n\xi}\left(\sum_{k\neq n}\left(\frac{1}{k-n}-\frac{1}{k+b-n}\right)-\frac{1}{b}\right)\\
&\quad + \sum_{n=0}^{\infty}\frac{1}{a+n}\left(\sum_{k\neq n}\binom{b-1}{k}\frac{(-1)^k e^{-k\xi}}{k-n}-\binom{b-1}{n}(-1)^n e^{-n\xi}\xi\right)\\
&\quad + \sum_{n=0}^{\infty}\frac{1}{a+n+b}\sum_{k=0}^{\infty}\binom{b-1}{k}\frac{(-1)^k e^{-k\xi}}{n+b-k}
\end{aligned}
$$

Thus we have

$$\xi' = \sum_{n=0}^{\infty}\frac{1}{a+n}X_n(\xi) + \sum_{n=0}^{\infty}\frac{1}{a+b+n}Z_n(\xi)$$

where

$$
\begin{aligned}
X_n(\xi) := \Bigg[&\binom{b-1}{n}(-1)^n e^{-n\xi}\left(\sum_{k\neq n}\left(\frac{1}{k-n}-\frac{1}{k+b-n}\right)-\frac{1}{b}\right)\\
&+\sum_{k\neq n}\binom{b-1}{k}(-1)^k e^{-k\xi}\frac{1}{k-n}-\binom{b-1}{n}(-1)^n e^{-n\xi}\xi\Bigg]/(1-e^{-\xi})^{b-1}
\end{aligned}
$$

and

$$Z_n(\xi) := \left(\sum_{k=0}^{\infty}\binom{b-1}{k}(-1)^k e^{-k\xi}\frac{1}{n+b-k}\right)/(1-e^{-\xi})^{b-1}$$

By Lemma 5.2 and

$$\partial_\xi\left(\sum_{k=0}^{\infty}\binom{b-1}{k}(-1)^k e^{(n+b-k)\xi}\frac{1}{n+b-k}\right) = e^{(n+b)\xi}(1-e^{-\xi})^{b-1}$$

we have that

$$\sum_{k=0}^{\infty}\binom{b-1}{k}(-1)^k e^{(n+b-k)\xi}\frac{1}{n+b-k} = \int_0^{\xi} e^{(n+b)t}(1-e^{-t})^{b-1}\mathrm{d}t$$

and hence we get

$$\begin{aligned}
Z_n(\xi) =& e^{-(n+b)\xi}\left(\sum_{k=0}^{\infty}\binom{b-1}{k}(-1)^k e^{(n+b-k)\xi}\frac{1}{n+b-k}\right)/(1-e^{-\xi})^{b-1}\\
=& e^{-(n+b)\xi}\int_0^{\xi} e^{(n+b)t}(1-e^{-t})^{b-1}\mathrm{d}t/(1-e^{-\xi})^{b-1}\\
=& \frac{\int_0^{\xi} e^{(n+b)t}(1-e^{-t})^{b-1}\mathrm{d}t}{e^{(n+b)\xi}(1-e^{-\xi})^{b-1}} = Y_{n+b}(\xi)
\end{aligned}$$

Similarly, Lemma 5.2 and

$$\partial_\xi\left(\sum_{k\neq n}\binom{b-1}{k}(-1)^k e^{-(k-n)\xi}\frac{1}{k-n} - \binom{b-1}{n}(-1)^n\xi\right) = -e^{n\xi}(1-e^{-\xi})^{b-1}$$

give

$$X_n(\xi) = -\frac{\int_0^{\xi} e^{nt}(1-e^{-t})^{b-1}\mathrm{d}t}{e^{n\xi}(1-e^{-\xi})^{b-1}} = -Y_n(\xi)$$

hence (21) is proved. □

LEMMA 5.2. *For $n \in \mathbb{N}$ and $b > 0$, we have*

$$\sum_{k=0}^{\infty}\binom{b-1}{k}\frac{(-1)^k}{n+b-k} = 0 \tag{23}$$

*and*

$$\sum_{k\neq n}\binom{b-1}{k}(-1)^k\frac{1}{k-n} = -\binom{b-1}{n}(-1)^n\left(\sum_{k\neq n}\left(\frac{1}{k-n}-\frac{1}{k+b-n}\right)-\frac{1}{b}\right) \tag{24}$$

*Proof.* For $z \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}$ we have, applying [3, Theorem 2.2.2],

$$\begin{aligned}
\sum_{k=0}^{\infty}\binom{b-1}{k}\frac{(-1)^k}{k+z} =& \frac{1}{z}\sum_{k=0}^{\infty}\frac{(1-b)_k(z)_k}{k!(z+1)_k} = \frac{{}_2F_1(1-b,z;z+1;1)}{z}\\
=& \frac{\Gamma(z+1)\Gamma(b)}{z\Gamma(z+b)} = \frac{\Gamma(z)\Gamma(b)}{\Gamma(z+b)}
\end{aligned}$$

Hence, we get

$$\sum_{k=0}^{\infty}\binom{b-1}{k}\frac{(-1)^k}{n+b-k} = \lim_{z\to n}\sum_{k=0}^{\infty}\binom{b-1}{k}\frac{(-1)^k}{z+b-k} = -\lim_{z\to n}\frac{\Gamma(b)\Gamma(-z-b)}{\Gamma(-z)} = 0$$

proving (23). For (24), assume $z \in \mathbb{C}\backslash\mathbb{N}$ and let

$$\begin{aligned}
&\binom{b-1}{n}(-1)^n\left(\sum_{k\neq n}\left(\frac{1}{k-z}-\frac{1}{k+b-z}\right)-\frac{1}{n+b-z}\right)+\sum_{k\neq n}\binom{b-1}{k}\frac{(-1)^k}{k-z}\\
=&\binom{b-1}{n}(-1)^n\left(\sum_{k=0}^{\infty}\left(\frac{1}{k-z}-\frac{1}{k+b-z}\right)-\frac{1}{n-z}\right)+\sum_{k=0}^{\infty}\binom{b-1}{k}\frac{(-1)^k}{k-z}-\binom{b-1}{n}\frac{(-1)^n}{n-z}\\
=&\binom{b-1}{n}(-1)^n\left(\psi(b-z)-\psi(-z)-\frac{1}{n-z}\right)+\frac{\Gamma(b)\Gamma(-z)}{\Gamma(b-z)}-\binom{b-1}{n}\frac{(-1)^n}{n-z}\\
=&\binom{b-1}{n}(-1)^n\left(\psi(b-z)-\psi(1+z)-\pi\frac{\cos(\pi z)}{\sin(\pi z)}-\frac{1}{n-z}\right)+\frac{\Gamma(b)\Gamma(-z)}{\Gamma(b-z)}-\binom{b-1}{n}\frac{(-1)^n}{n-z}
\end{aligned}$$

where we have used the reflection formula for the digamma function. We have, using that $\mathrm{Res}\left(\frac{\pi\cos\pi z}{\sin\pi z}, n\right) = 1$, that

$$\lim_{z\to n}\left(\psi(b-z) - \psi(1+z) - \pi\frac{\cos(\pi z)}{\sin(\pi z)} - \frac{1}{n-z}\right) = \psi(b-n) - \psi(1+n)$$

Furthermore, using de L'Hôpital's rule, we get

$$\begin{aligned}
&\lim_{z\to n}\left(\frac{\Gamma(b)\Gamma(-z)}{\Gamma(b-z)} - \binom{b-1}{n}\frac{(-1)^n}{n-z}\right) = \\
&= \lim_{z\to n}\left(-\frac{\Gamma(b)}{\Gamma(b-z)\Gamma(1+z)}\frac{\pi}{\sin\pi z} - \binom{b-1}{n}\frac{(-1)^n}{n-z}\right) \\
&= \lim_{z\to n}\frac{\frac{\Gamma(b)}{\Gamma(b-z)\Gamma(1+z)}(n-z) - \binom{b-1}{n}(-1)^n\frac{\sin\pi z}{\pi}}{\frac{\sin\pi z}{\pi}(n-z)} \\
&= \lim_{z\to n}\frac{\left(\frac{\Gamma(b)}{\Gamma(b-z)\Gamma(1+z)}\right)'(n-z) + \frac{\Gamma(b)}{\Gamma(b-z)\Gamma(1+z)} + \binom{b-1}{n}(-1)^n\cos\pi z}{\frac{\sin\pi z}{\pi} - (n-z)\cos\pi z} \\
&= \lim_{z\to n}\frac{\left(\frac{\Gamma(b)}{\Gamma(b-z)\Gamma(1+z)}\right)''(n-z) + \left(\frac{\Gamma(b)}{\Gamma(b-z)\Gamma(1+z)}\right)' + \pi\binom{b-1}{n}(-1)^n\sin\pi z}{\cos\pi z + (n-z)\pi\sin\pi z + \cos\pi z} \\
&= \frac{\frac{\Gamma(b)}{\Gamma(b-n)\Gamma(1+n)}(\psi(1+n) - \psi(b-n))}{2\cos\pi n} \\
&= (-1)^n\binom{b-1}{n}(\psi(1+n) - \psi(b-n))
\end{aligned}$$

hence getting (24). □

LEMMA 5.3. *Let $b > 1$ and $c > 0$. Then, $Y_c$ is increasing on $(0, \infty)$. Moreover, $Y_c(x)$, $Y_c^{'}(x)$ are decreasing wrt $c$ for fixed $x$.*

*Proof.* We rewrite

$$\begin{aligned}
\frac{\int_0^x e^{ct}(1-e^{-t})^{b-1}\mathrm{d}t}{e^{cx}(1-e^{-x})^{b-1}} &= \int_0^x e^{c(t-x)}\left(\frac{1-e^{-t}}{1-e^{-x}}\right)^{b-1}\mathrm{d}t \\
&= \int_0^x e^{c(t-x)}\left(\frac{e^x - e^{x-t}}{e^x - 1}\right)^{b-1}\mathrm{d}t \\
&= \int_0^x e^{-cv}\left(\frac{e^x - e^v}{e^x - 1}\right)^{b-1}\mathrm{d}v
\end{aligned}$$

Differentiating, we get

$$\begin{aligned}
\left(\int_0^x e^{-cv}\left(\frac{e^x - e^v}{e^x - 1}\right)^{b-1}\mathrm{d}t\right)' &= \int_0^x e^{-cv}\partial_x\left(\frac{e^x - e^v}{e^x - 1}\right)^{b-1}\mathrm{d}v \\
&= \int_0^x e^{-cv+x}(b-1)\left(\frac{e^x - e^v}{e^x - 1}\right)^{b-2}\frac{e^v - 1}{(e^x - 1)^2}\mathrm{d}v
\end{aligned}$$

and this completes the proof. □

*Proof of Theorem 1.5* We shall show the convexity of $\xi = -\log q$, which is equivalent to logarithmic concavity of $q$. The case $b < 1$ is given by Theorem 1.4, as $a\xi'' = \phi'' - 2\xi' > 0$. For $b = 1$, we have $\xi = \frac{\log(1/p)}{a}$ hence $\xi'' = 0$. For $b > 1$, differentiating

(21) we get

$$\xi'' = \sum_{n=0}^{\infty} \frac{1}{(a+n)^2} Y_n(\xi) - \sum_{n=0}^{\infty} \frac{1}{(a+b+n)^2} Y_{n+b}(\xi) + \left( \sum_{n=0}^{\infty} \frac{1}{a+b+n} Y'_{n+b}(\xi) - \sum_{n=0}^{\infty} \frac{1}{a+n} Y'_n(\xi) \right) \xi' > 0$$

using that $\xi' < 0$ and Lemma 5.3. □

Remark 5.4. We notice that (21) also gives

$$q' = \sum_{n=0}^{\infty} \frac{1}{a+b+n} \frac{\int_q^1 t^{-n-b-1}(1-t)^{b-1} \mathrm{d}t}{q^{-n-b-1}(1-q)^{b-1}} - \sum_{n=0}^{\infty} \frac{1}{a+n} \frac{\int_q^1 t^{-n-1}(1-t)^{b-1} \mathrm{d}t}{q^{-n-1}(1-q)^{b-1}} \tag{25}$$

## Appendix

Finally, we want to see how the $p$-quantile depends on the second parameter of the beta distribution. For clarity, from now on we denote the $p$-quantile of the beta distribution with parameters $a$ and $b$ by $q_p(a,b)$. We shall consider $a$ constant, and try to relate $q$ as a function of $b$ with the previous results.

A simple change of variables $s = 1-t$ in (3) gives the functional relation

$$I(x;a,b) = 1 - I(1-x;b,a) \tag{26}$$

which implies

$$p = I(q_p(a,b);a,b) = 1 - I(q_p(a,b);b,a) \Rightarrow I(q_p(a,b);b,a) = 1-p = I(q_{1-p}(a,b);b,a)$$

and, using the uniqueness of the $p$-quantile, we get

$$q_p(a,b) = 1 - q_{1-p}(b,a) \tag{27}$$

Hence, by Proposition 1.2, we get that $q_p$ is decreasing in $b$ and

$$\lim_{b\to 0} q_p(a,b) = 1$$

$$\lim_{b\to\infty} q_p(a,b) = 0$$

Moreover, we have

$$(1 - q_p(a,b))^b = q_{1-p}(b,a)^b = e^{-\varphi_{1-p}(b)} \tag{28}$$

where $\varphi_{1-p}(b) = -b \log q_{1-p}(b,a)$, hence the behaviour of $q_p(a,b)$ as a function of $b$ can again be studied similarly through the function $\varphi_p$. We also easily see that $b \mapsto 1 - q_p(a,b)$ is log-concave. We remark that numerical evidence shows that $b \mapsto q_p(a,b)$ itself is not (log-)concave/convex. However, the function $b \mapsto \varphi_p(b)$ seems to be convex.

Acknowledgements. I would like to thank H.L.Pedersen for careful reading of the original manuscript and useful suggestions.

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# ASYMPTOTIC EXPANSIONS OF THE INVERSE INCOMPLETE BETA FUNCTION WITH RESPECT TO THE FIRST PARAMETER

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Abstract. In this work in progress, we study the asymptotic behaviour of the $p$-quantile of the Beta distribution, i.e. the quantity $q$ defined implicitly by $\int_0^q t^{a-1}(1-t)^{b-1}\mathrm{d}t = pB(a,b)$, as a function of the first parameter $a$. In particular, we derive asymptotic expansions of and $q$ and its logarithm at 0 and $\infty$. Moreover, we provide some relations between Bell and Nørlund Polynomials, a generalisation of Bernoulli numbers. Finally, we provide Maple and Sage algorithms for computing the terms of the asymptotic expansions.

*2010 Mathematics Subject Classification*: Primary 41A60; Secondary 33B15, 60E05, 11B68
*Keywords*: median, beta distribution, asymptotic expansion

## 1. Introduction

1.1. **Background.** Granted a probability distribution on $\mathbb{R}$, its median is defined as the value $m \in \mathbb{R}$ that leaves exactly half of the "mass" of the distribution on its left and half on its right. Instead of requiring that $m$ splits the mass exactly in two equal parts, one may choose a $p \in [0,1]$ and define the more general notion of the $p$-quantile value of the probability distribution:[4]

Definition 1. Let $F$ be a cumulative distribution function on some subset $I \subset \mathbb{R}$. Let $p \in [0,1]$. A $p$-quantile of $F$ is a point $q \in I$ such that $F(q) = p$. If $p = 1/2$, a $1/2$-quantile is called median.

For an arbitrary probability distribution on $\mathbb{R}$, not always do $p$-quantiles exist, neither do they have to be unique, but for a distribution with density wrt to Lebesgue measure $p$-quantile values always exist, as then the distribution function is continuous and increasing, and if furthermore the density is a.e. non-zero, they are also unique, as the distribution function shall be strictly increasing.

One point of interest has been the study of the $p$-quantiles, including medians, of a parametrised family of probability distributions as a function of the parameter, given a fixed value of $p$. Such a function is well defined if the distribution has density wrt to the Lebesgue measure which is a.e. non-zero. Questions that may arise in this context have to do with analyticity, monotonicity, geometric properties and approximations, in particular asymptotic expansions, of the implicit function $q(a)$ defined by an equation of the form $F_a(q(a)) = p$, where $F_a$ is a family of commulative distribution functions. Because of the implicit definition, the study of its properties can be challenging. An example is the median of the gamma distribution, which has been studied in several occasions, for example in [7], [5], and many connections have been found, for example with the Ramanujan's rational approximation of $e^x$, see [2], [8] and [1], while in [6] it was also proved that it is is a convex function.

In this paper, considering $p$ fixed in $(0,1)$, we focus on studying the $p$-quantile of the beta distribution, i.e. the distribution on $[0,1]$ with the density function

$t \mapsto t^{a-1}(1-t)^{b-1}$, as a function of the parameter $a$ considering $b$ fixed. The $p$-quantile of the beta distribution has been considered by Temme in [12], who studied the asymptotic behaviour of the $p$-quantile (or in his notation, the inverse of the normalised beta incomplete function) under restrictions over relations between the two parameters of the beta distribution. Also, see [11] for some inequalities on the median. This preprint is to be a continuation of our work in [4], which deals with convexity/concavity properties.

The $p$-quantile of the beta distribution, as a function of the first parameter, is defined as:

Definition 2. Fix $p \in (0,1)$ and $b \in (0,+\infty)$. The function $q:(0,+\infty) \to (0,1)$ defined implicitly by

$$\int_0^{q(a)} t^{a-1}(1-t)^{b-1}dt = p\int_0^1 t^{a-1}(1-t)^{b-1}dt \tag{1.1}$$

is called the $p$-quantile of the beta distribution with parameters $a$ and $b$.

As in [5] for the case of the median of the gamma distribution, to study the $p$-quantile we consider and study an auxilliary function related to its logarithm

$$\varphi(a) := -a\log q(a) \tag{1.2}$$

and it will become clear that studying the logarithm gives more information on the behaviour of the $p$-quantile. One may also consider $\varphi$ itself as the $(1-p)$-quantile of some distribution. Indeed, using change of variables in (1.1)

$$\int_0^{\varphi(a)} e^{-s}(1-e^{-s/a})^{b-1}\mathrm{d}s = (1-p)\int_0^{\infty} e^{-s}(1-e^{-s/a})^{b-1}\mathrm{d}s \tag{1.3}$$

Later, Bernoulli numbers and a generalisation of them known as Nørlund polynomials will become useful. The Bernoulli numbers $B_n$ are classically defined through their generating function

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n\frac{x^n}{n!} \tag{1.4}$$

They can be generalised to the Bernoulli polynomials $B_n(t)$, defined similarily through the generating function

$$\frac{xe^{tx}}{e^x-1} = \sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!} \tag{1.5}$$

Another generalisation of Bernoulli numbers are the Nørlund polynomials $B_n^{(s)}$ defined through the generating function

$$\left(\frac{x}{e^x-1}\right)^s = \sum_{n=0}^{\infty} B_n^{(s)}\frac{x^n}{n!} \tag{1.6}$$

They are polynomials in $s$. If $s \in \mathbb{N}$, then $B_n^{(s)}$ is the $s$-fold convolution of Bernoulli numbers. An account on Nørlund polynomials can be found in [9, 24.16] and references within. Bernoulli and Nørlund appear often when we consider asymptotic expansions of the gamma and related functions (see e.g. [13].

1.2. **Main results.** We state the following propositions regarding first order asymptotics. They are proved in [4]. In the rest, $\gamma_b$ denotes the $(1-p)$-quantile of the gamma distribution with parameter $b$.

PROPOSITION 1. [4, Proposition 1.2] *The $p$-quantile of the beta distribution $q(a)$ is a real analytic, increasing function of $a$. It has limits*

$$\lim_{a\to 0} q(a) = 0$$

*and*

$$\lim_{a\to\infty} q(a) = 1$$

PROPOSITION 2. [4, Proposition 1.3] *The function $\varphi(a) = -a\log q(a)$ is real analytic and increasing for $b > 1$, constant for $b = 1$ and decreasing for $b < 1$. It has limits*

$$\lim_{a\to 0} \varphi(a) = \log p \tag{1.7}$$

*and*

$$\lim_{a\to\infty} \varphi(a) = \gamma_b \tag{1.8}$$

To study the asymptotic behaviour of $q$ and $\varphi$ in more depth, we shall try to find the asymptotic expansions of $\varphi$ at 0 and $\infty$. Studying asymptotic expansions of implicit functions can be highly non-trivial, as the method and the obstacles arising depend much on the form of the defining implicit relation. For the $p$-quantile of the beta distribution, we consider the cases of asymptotic expansions of $\varphi$ centered at 0 and at $\infty$. In both cases, we shall combine differentiation and Faà di Bruno's formula (2.10), and the existence of the expansion has to be proved inductively.

For the case of 0, we shall compute the limits of the derivatives. For the case of $\infty$, for the same purpose, we shall introduce the differential operator $D$ defined by

$$Df(x) = x^2 \partial f(x)$$

where $\partial$ denotes the common differentiation operator. The calculus of $D$ is studied in subsection 3.1.

This operator has the importance that it can give, under certain conditions, the asymptotic expansion of a suitably smooth function at infinity, which is summarized in the following lemma, which is proved in subsection 3.1:

LEMMA 1.1. *Let $f \in C^n(0,\infty)$ for some $n \in \mathbb{N}$. Then, the following hold:*

*i. If $\lim_{x\to\infty} D^m f(x)$ exists in $\mathbb{R}$ for all $m \leqslant n$, we have the asymptotic expansion*

$$f(x) \sim \sum_{k=0}^{n-1} \frac{c_k}{x^k} + \mathcal{O}\left(\frac{1}{x^n}\right)$$

*where*

$$c_k = \frac{(-1)^k}{k!} \lim_{a\to\infty} D^k f(a), \quad m < n$$

*ii. Assume, conversely, that $f$ has asymptotic expansion of order $n$, i.e. $f(x) \sim \sum_{k=0}^{n} \frac{c_k}{x^k} + \mathcal{O}\left(\frac{1}{x^{n+1}}\right)$ ,as well as that its derivatives $f^{(m)}$ admit asymptotic expansions of orders $m+n$, for $m \leqslant n$. Then, we have*

$$c_k = \frac{(-1)^k}{k!} \lim_{a\to\infty} D^k f(a)$$

We note that, if conditions in *i.* hold, we may apply the lemma to $D^k f$ and get asymptotic expansions of higher derivatives, hence the expansion in *i.* can be differentiated. Also, if in the previous lemma $f \in C^\infty(0,\infty)$ and its conditions hold for all $n$, then we may get the whole asymptotic expansion of $f$.

Regarding the functions $\varphi$ and $q$, we have the following two pairs of theorems and Corollaries on their asymptotic expansions, which are proved in sections 2 and 3 respectively. In the following, $\Psi(n,z) := \partial^{n+1} \operatorname{Log}\Gamma(z)$ denotes the polygamma function. Also, $(m)_n$ denotes the Pochhammer symbol of $m$, i.e. $(m)_n = m(m+$

$1)\dots(m+n-1)$. If $m \in \mathbb{N}$, then we have $(m)_n = \frac{(m+n-1)!}{(m-1)!}$, and $(-m)_n = (-1)^n \frac{m!}{(m-n)!}$ if $n \leqslant m$, and $(-m)_n = 0$ if $n > m$. These identities will be widely used in this paper.

THEOREM 1. *The function $\varphi$ admits the asymptotic expansion $\varphi(a) \sim \sum_{n=0}^{\infty} c_n a^n$ at 0, with $c_0 = \log p$ and*

$$c_n = \frac{\Psi(n-1,b) - \Psi(n-1,1)}{n!} = \frac{(-1)^{n+1}}{n!} \int_0^\infty u^{n-1} \left( \frac{e^{-u} - e^{-bu}}{1 - e^{-u}} \right) \mathrm{d}u, \quad n \geqslant 1$$

*For $b \in \mathbb{N}$, we have in particular*

$$c_n = (-1)^{n+1}(n-1)! \sum_{k=1}^{b-1} \frac{1}{k^n} \tag{1.9}$$

COROLLARY 1. *An approximation for $\varphi$ for values of $a$ close to 0 is*

$$\varphi(a) \sim \log \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} - \log p$$

*and for $q$*

$$\frac{q(a)}{p^{1/a}} \sim \left( \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} \right)^{1/a}$$

*each having a remainder term vanishing faster than $a^n$ at 0, $\forall n \in \mathbb{N}$. Hence, we have the asymptotic expansion*

$$\frac{q(a)}{p^{1/a}} \sim e^{-\gamma - \Psi(0,b)} \left( \sum_{n=0}^{\infty} \frac{\mathcal{B}_n(c_1, c_2, \dots, c_n)}{n!} a^n \right) \tag{1.10}$$

*where $\gamma$ is the Euler constant, $c_n = \frac{\Psi(n-1,b) - \Psi(n-1,1)}{n!}$ and $\mathcal{B}_n$ denotes the nth complete Bell polynomial (see Remark 2.1).*

THEOREM 2. *The function $\varphi$ admits the asymptotic expansion*

$$\varphi(a) \sim \sum_{n=0}^{\infty} \varphi_n \frac{(-1)^n}{n! a^n}, \quad a \to \infty$$

*at $\infty$, with $\varphi_n$ satisfying the system of recursive relations*

$$\varphi_n = -\sum_{j=1}^{n-1} \binom{n-1}{j} \varphi_{n-j} \delta(0,j,0) - \sum_{k=0}^{n-2} \sum_{j=0}^{k} \binom{k}{j} \varphi_{k-j+1} \delta(0,j,n-k-1)$$
$$+ B_n^{(1-b)} \sum_{k=0}^{n-1} (b+n-k)_k \gamma_b^{n-k} \tag{1.11}$$

$$\delta(k,m,n) = \delta(k,m-1,n+1) + \sum_{j=0}^{m-1} \binom{m-1}{j} \varphi_{m-j} \delta(k+1,j,n) \tag{1.12}$$

*and the initial conditions*

$$\varphi_0 = \gamma_b \tag{1.13}$$

$$\delta(k,0,n) = B_n^{(1-b)} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (b+n-j)_j \gamma_b^{n-j} \tag{1.14}$$

The recursive relations (1.11) and (1.12) in the foregoing lemma work inductively. We know $\varphi_0$ and once we have computed $\varphi_0, \dots, \varphi_{n-1}$, in order to compute $\varphi_n$ we use (1.11), where the maximum of the second argument of $\delta$ that is at most $n-1$, and we can compute these terms using (1.12) and the initial conditions, as $\varphi_k$ appears there in orders at most equal to the second argument of $\delta$, and that we already have computed. This algorithm can give us the first terms of the asymptotic expansion:

$$\begin{aligned}\varphi(a) =&\gamma_b - \frac{\gamma_b(b-1)}{2a} + \frac{\gamma_b(-1+b)\,(7\,b+\gamma_b-5)}{24a^2} \\ &- \frac{\gamma_b(-1+b)^2\,(3\,b+\gamma_b-1)}{16a^3} + \mathcal{O}\left(\frac{1}{a^4}\right)\end{aligned} \tag{1.15}$$

Also, for $q$ we then get:

COROLLARY 2. *For $a \to \infty$, an asymptotic expansion for $q$ is*

$$q(a) \sim \sum_{n=0}^{\infty} \frac{\mathcal{B}_n(-\varphi_0, 2\varphi_1, -3\varphi_2, \ldots, (-1)^n n\varphi_{n-1})}{n!}\frac{1}{a^n} \tag{1.16}$$

*where $\varphi_n$ is the sequence defined in Theorem 2.*

In section 4 we state some relations between Nørlund, Bernoulli and Bell polynomials that we came upon and we could not find in the literature. These relations come out by considering the coefficients of Bernoulli generating functions as taylor coefficients, i.e. as limits of derivatives, and using Faà di Bruno's formula, and its relation to Bell polynomials, to compute these derivatives. Finally, in the appendix we implement the recursive relations of Theorem 2 as Maple and Sage algorithms and give coefficients of asymptotic expansions for some specific values.

## 2. Asymptotics at 0

For computing the asymptotic expansion of $\varphi$ at 0, our method consists of iterated differentiation of relations that implicitely contain the $p$-quantile and use Faà di Bruno's formula. Then, taking limits for $a \to 0$ and computing the limits of all the terms, we compute the limits of the derivatives which then wields the asymptotic expansion, as, if $f \in C^\infty(0, \varepsilon)$, for some $\varepsilon > 0$, and $\lim_{x\to 0} f^{(n)}(x)$ exists in $\mathbb{R}$ for all $n$, denoting this limit by $f^{(n)}(0)$ we have $f \sim \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$. The converse is not necessarily valid: if $f$ admits asymptotic expansion at 0 it is not necessary that the limits of the derivatives exist, as there may be oscillations. The limits of the derivatives of $\varphi$ will be computed then inductively.

First, we use integration by parts in (1.1) getting

$$e^{-\varphi(a)}(1-q(a))^{b-1} + (b-1)\int_0^{q(a)} t^a(1-t)^{b-2}dt = ap\frac{\Gamma(b)\Gamma(a)}{\Gamma(a+b)} = p\Gamma(b)W(a) \tag{2.1}$$

where $W(a) = \Gamma(a+1)/\Gamma(a+b)$. This function $W$ is studied in [3, C4], and in a generalised form in [10], where several properties, such as complete monotonicity, are proved. We consider the logarithmic derivative of $W$ and we note that, as $\log W(a) = \log\Gamma(a+1) - \log\Gamma(a+b)$, by the integral representation of the digamma function $\Psi(0, z)$ (see [3, Theorem 1.6.1]), we get that

$$(\log W(a))' = \Psi(0, a+b) - \Psi(0, a+1) = -\int_0^{\infty} e^{-au}\left(\frac{e^{-u}-e^{-bu}}{1-e^{-u}}\right)\mathrm{d}u \tag{2.2}$$

We define the function $\psi$ by

$$\psi(a) := -\log\frac{\Gamma(a+1)}{\Gamma(a+b)} - \log p\Gamma(b) = -\log W(a) - \log p\Gamma(b) \tag{2.3}$$

which implies that $e^{-\psi(a)} = p\Gamma(b)W(a)$ and (2.1) can be rewritten as

$$e^{-\varphi(a)}(1-q(a))^{b-1} + (b-1)\int_0^{q(a)} t^a(1-t)^{b-2}\mathrm{d}t = e^{-\psi(a)} \tag{2.4}$$

Hence, as $\psi \in C^\infty(-1,\infty)$, denoting the limit of the $k$th derivative of $\psi$ at 0 by $\psi^{(k)}(0)$, by (2.2) we have

$$\psi^{(k)}(0) = \Psi(k-1,b) - \Psi(k-1,1) = (-1)^{k-1}\int_0^\infty u^{k-1}\left(\frac{e^{-u}-e^{-bu}}{1-e^{-u}}\right)\mathrm{d}u \tag{2.5}$$

Let denote by $\varphi^{(n)}(0)$ the right limit of $\varphi^{(n)}$ at 0, supposing it exists. We already have, combining (1.7) and (2.3), that $\varphi(0) = \psi(0) = -\log p$. Our goal is to prove that for the limits of all the derivatives of $\varphi$ and $\psi$ at 0 are the same, i.e. we have $\varphi^{(k)}(0) = \psi^{(k)}(0)$. Differentiating (2.4) we get

$$-\psi'(a)e^{-\psi(a)} + \varphi'(a)e^{-\varphi(a)}(1-q(a))^{b-1} = (b-1)\int_0^{q(a)} t^a(1-t)^{b-2}\log t\mathrm{d}t$$

We define the functions

$$\rho(a) := (1-q(a))^{b-1} \tag{2.6}$$

$$\sigma(a) := \int_0^{q(a)} t^a(1-t)^{b-2}\log t\mathrm{d}t \tag{2.7}$$

and hence the last equation can be rewritten as

$$-\psi'(a)e^{-\psi(a)} + \varphi'(a)e^{-\varphi(a)}\rho(a) = (b-1)\sigma(a) \tag{2.8}$$

We will use this equality to find the limits of the derivatives of $\varphi$. This will be done inductively, differentiating (2.8) at each step. Our strategy is, at the $k$th step, where we will want to compute the limit of the $k+1$ derivative, that we use the results from the previous steps about the asymptotic behaviour of $\varphi$ up to the $k$th derivative to find the asymptotic behaviour of the derivatives of $q$ up to $k$, and then use this result to find the behaviour of the derivatives of $g$ and $h$ up to $k$, so that we finally compute the limit of the $k+1$ derivative of $\varphi$. The first part will be done in the next lemmas, and the inductive proof will be given in the end of the section.

We state the following well known differentiation formulas that we will be constantly using, see (1.4.12) and (1.4.13) in [9]: The product formula for derivation,

$$\left(\prod_{i=1}^k f_i(x)\right)^{(n)} = \sum_{\{\boldsymbol{j}\in\mathbb{N}^k \mid \sum_{i=1}^k j_i = n\}} \binom{n}{j_1, j_2, \ldots, j_k} \prod_{i=1}^k f_i^{(j_i)}(x) \tag{2.9}$$

and the Faà di Bruno formula, for the derivatives of composite functions,

$$(f\circ g)^{(n)}(x) = \sum_{\{\boldsymbol{m}\in\mathbb{N}^n \mid \sum_{j=1}^n jm_j = n\}} \frac{n!}{m_1!m_2!\ldots m_n!} f^{(\sum_{j=1}^n m_j)}(g(x)) \prod_{j=1}^n \left(\frac{g^{(j)}(x)}{j!}\right)^{m_j} \tag{2.10}$$

The latter, in case $f(x) = \log(x)$, can take the simpler form

$$(\log g(x))^{(n)} = \sum_{\{\boldsymbol{m}\in\mathbb{N}^n \mid \sum_{j=1}^n jm_j = n\}} C_{\boldsymbol{m}} \prod_{j=1}^n \left(\frac{g^{(j)}(x)}{g(x)}\right)^{m_j} \tag{2.11}$$

where

$$C_{\boldsymbol{m}} = (-1)^{1+\sum_{j=1}^n m_j} \frac{n!\left(\sum_{j=1}^n m_j - 1\right)!}{m_1!m_2!\ldots m_n!} \prod_{j=1}^n \frac{1}{j!^{m_j}}$$

and for $f(x) = e^x$,

$$(e^{g(x)})^{(n)} = e^{g(x)} \sum_{\{\boldsymbol{m}\in\mathbb{N}^n \mid \sum_{j=1}^n jm_j = n\}} \frac{n!}{m_1!m_2!\ldots m_n!} \prod_{j=1}^n \left(\frac{g^{(j)}(x)}{j!}\right)^{m_j} \tag{2.12}$$

REMARK 2.1. Faà di Bruno formula (2.10) is related to the polynomials that are known as Bell polynomials. The (complete) Bell polynomials are defined by the relation

$$\mathcal{B}_n(x_1, x_2, \ldots, x_n) = \sum_{\{\boldsymbol{\kappa} \in \mathbb{N}^n \mid \sum_{j=1}^n j\kappa_j = n\}} \frac{n!}{\kappa_1! \kappa_2! \cdots \kappa_n!} \prod_{j=1}^n \left( \frac{x_j}{j!} \right)^{\kappa_j} \tag{2.13}$$

We can express the special case (2.12) of Faà di Bruno's formula for the exponential in terms of these Bell polynomials

$$\left( e^{g(x)} \right)^{(n)} = e^{g(x)} \mathcal{B}_n(g'(x), g''(x), \ldots, g^{(n)}(x))$$

LEMMA 2.1. *Let $k, l \in \mathbb{N}$. Then,*

$$\lim_{a \to 0} \frac{q(a) \log^k q(a)}{a^l} = 0 \tag{2.14}$$

*Proof.* We have

$$\log \left( \frac{q(a)}{a^m} \right) = \log q(a) - m \log a \to -\infty$$

for $a \to 0$, as, by (1.7),

$$a(\log q(a) - m \log a) = a \log q(a) - ma \log a \to \log p$$

This implies that

$$\lim_{a \to 0} \frac{q(a)}{a^m} = 0$$

Also (1.7) gives

$$\lim_{a \to 0} a^k \log^k q(a) = \log^k p$$

Hence

$$\lim_{a \to 0} \frac{q(a) \log^k q(a)}{a^l} = \lim_{a \to 0} \frac{q(a)}{a^{l-k}} \frac{\log^k q(a)}{a^k} = 0$$

$\square$

LEMMA 2.2. *Let $N \in \mathbb{N}^*$ and assume that $\lim_{a \to 0} \varphi^{(k)}(a)$ exists in $\mathbb{R}$, $\forall k \leqslant N$. Then, $\forall k \leqslant N$,*

$$\lim_{a \to 0} \frac{a^{2k} q^{(k)}(a)}{q(a)} \textit{exists in } \mathbb{R} \tag{2.15}$$

*In particular, we have that*

$$\lim_{a \to 0} \frac{q^{(k)}(a)}{a^m} = 0, \quad m \geqslant 0 \tag{2.16}$$

*Proof.* For $k = 1$, as $\varphi'(a) = -\log q(a) - aq'(a)/q(a)$, we have that

$$\frac{a^2 q'(a)}{q(a)} = -a\varphi'(a) - a \log q(a) \to -\log p$$

so (2.15) holds. Assume that $1 \leqslant n < N$ and that (2.15) holds $\forall k \leqslant n$. We will prove that (2.15) holds for $k = n + 1$. Indeed, using (2.11), we get, for some coefficients $c_{\boldsymbol{k}}$ and $d_{\boldsymbol{k}}$,

$$\begin{aligned}
-\varphi^{(n+1)}(a) =& a(\log q(a))^{(n+1)} + (n+1)(\log q(a))^{(n)} \\
=& a \sum_{\{\boldsymbol{k} \mid \sum_{j=1}^{n+1} jk_j = n+1\}} \left[ c_{\boldsymbol{k}} \prod_{j=1}^{n+1} \left( \frac{q^{(j)}(a)}{q(a)} \right)^{k_j} \right] \\
& + (n+1) \sum_{\{\boldsymbol{k} \mid \sum_{j=1}^{n} jk_j = n\}} \left[ d_{\boldsymbol{k}} \prod_{j=1}^{n} \left( \frac{q^{(j)}(a)}{q(a)} \right)^{k_j} \right]
\end{aligned}$$

But one can write

$$\sum_{\{\boldsymbol{k} \mid \sum_{j=1}^{n+1} jk_j = n+1\}} c_{\boldsymbol{k}} \prod_{j=1}^{n+1} \left( \frac{q^{(j)}(a)}{q(a)} \right)^{k_j} = \frac{q^{(n+1)}(a)}{q(a)} + \sum_{\{\boldsymbol{k} \mid \sum_{j=1}^{n} jk_j = n+1\}} c_{\boldsymbol{k}} \prod_{j=1}^{n} \left( \frac{a^{2j} q^{(j)}(a)}{q(a)} \right)^{k_j}$$

hence, rearranging the equation above and multiplying each side by $a^{2n+1}$, we get

$$\begin{aligned}
a^{2(n+1)} \frac{q^{(n+1)}(a)}{q(a)} =& - a^{2n+1} \varphi^{(n+1)}(a) - \sum_{\{\boldsymbol{k} \mid \sum_{j=1}^{n} jk_j = n+1\}} c_{\boldsymbol{k}} \prod_{j=1}^{n} \left( \frac{a^{2j} q^{(j)}(a)}{q(a)} \right)^{k_j} \\
& - a(n+1) \sum_{\{\boldsymbol{k} \mid \sum_{j=1}^{n} jk_j = n\}} d_{\boldsymbol{k}} \prod_{j=1}^{n} \left( \frac{a^{2j} q^{(j)}(a)}{q(a)} \right)^{k_j}
\end{aligned}$$

and the right hand side converges in $\mathbb{R}$ as $a \to 0$ by our induction hypothesis, proving (2.15). To prove (2.16), we see that combining this result with Lemma 2.1 gives

$$\lim_{a \to 0} \frac{q^{(k)}(a)}{a^m} = \lim_{a \to 0} \frac{a^{2k} q^{(k)}(a)}{q(a)} \frac{q(a)}{a^{m-2k}} = 0$$

□

LEMMA 2.3. *Let $N \in \mathbb{N}^*$ and assume that $\lim_{a \to 0} \varphi^{(k)}(a)$ exists in $\mathbb{R}$, $\forall k \leqslant N$. Then, $\forall k \leqslant N$,*

$$\lim_{a \to 0} \rho^{(k)}(a) = 0, \quad k \neq 0$$

$$\lim_{a \to 0} \rho(a) = 1$$

*Proof.* As $q(a) \to 0$, then $\rho(a) \to 1$. The $n$th derivative of $\rho$ can be expressed using (2.10) as

$$\rho^{(n)}(a) = \sum_{\{\boldsymbol{k} \mid \sum_{j=1}^{n} jk_j = n\}} c_{\boldsymbol{k}} (1 - q(a))^{b-1-\sum_{j=1}^{n} k_j} \prod_{j=1}^{n} (q^{(j)}(a))^{k_j}$$

which, by Lemma 2.2 tends to 0 as $a \to 0$, as $q^{(j)}(a) \to 0$. □

LEMMA 2.4. *Let $N \in \mathbb{N}^*$ and assume that $\lim_{a \to 0} \varphi^{(k)}(a)$ exists in $\mathbb{R}$, $\forall k \leqslant N$. Then, $\forall k \leqslant N$,*

$$\lim_{a \to 0} \sigma^{(k)}(a) = 0$$

*Proof.* We have

$$\int_0^{q(a)} t^a (1-t)^{b-2} \log^m t \mathrm{d}t \to 0$$

as $q(a) \to 0$ and $(1-t)^{b-2}\log^m t$ is integrable near 0. Hence, $\sigma(a) \to 0$. For $n > 0$ we have

$$\sigma^{(n)}(a) = \int_0^{q(a)} t^a(1-t)^{b-2}\log^{n+1} t\mathrm{d}t + \sum_{k=1}^{n}[e^{-\varphi(a)}(1-q(a))^{b-2}q'(a)\log^k q(a)]^{(n-k)} \tag{2.17}$$

So, it suffices to prove that

$$[e^{-\varphi(a)}(1-q(a))^{b-2}q'(a)\log^k q(a)]^{(l)} \to 0, \quad \forall k,l \leqslant N$$

By (2.9) we can write

$$[e^{-\varphi(a)}(1-q(a))^{b-2}q'(a)\log^k q(a)]^{(l)} = \sum_{\{\boldsymbol{m}|\sum_{j=1}^3 m_j = l\}} c_{\boldsymbol{m}}[e^{-\varphi(a)}]^{(m_1)}[(1-q(a))^{b-2}]^{(m_2)}[q'(a)\log^k q(a)]^{(m_3)}$$

By our assumptions, $\lim_{a\to 0}[e^{-\varphi(a)}]^{(m_1)} \in \mathbb{R}$, and as in Lemma 2.3, $((1-q(a))^{b-2})^{(m_2)}$ also converges. Finally, by (2.9), (2.10) and Lemma 2.2

$$[q'(a)\log^k q(a)]^{(m)} = \sum_{\{\boldsymbol{n}|\sum_{j=1}^{k+1} n_j = m\}} c_{\boldsymbol{n}} q^{(n_1+1)}(a)\prod_{j=2}^{k+1}[\log q(a)]^{(n_j)} =$$

$$\sum_{\{\boldsymbol{n}|\sum_{j=1}^{k+1} n_j = m\}} c_{\boldsymbol{n}} q^{(n_1+1)}(a)\prod_{j=2}^{k+1}\sum_{\{r|\sum_{s=1}^{n_j} sr_s = n_j\}} d_{\boldsymbol{r}}\prod_{s=1}^{n_j}\left(\frac{q^{(s)}(a)}{q(a)}\right)^{r_s} =$$

$$\sum_{\{\boldsymbol{n}|\sum_{j=1}^{k+1} n_j = m\}} c_{\boldsymbol{n}} \frac{q^{(n_1+1)}(a)}{a^{2^k\prod_{j=2}^{k+1} n_j}}\prod_{j=2}^{k+1}\sum_{\{r|\sum_{s=1}^{n_j} sr_s = n_j\}} d_{\boldsymbol{r}}\prod_{s=1}^{n_j}\left(\frac{q^{(s)}(a)a^{2s}}{q(a)}\right)^{r_s} \to 0$$

which completes the proof of the Lemma. □

*Proof of theorem 1.* By Proposition 2 we have that $\varphi(0) = -\log p$. For the first derivative, as $\rho(0) = 1$ and $\sigma(0) = 0$, and $\varphi(0) = \psi(0) = -\log p$, we get from (2.8) that the limit $\lim_{a\to 0}\varphi'(a) = \varphi'(0)$ exists and $\varphi'(0) = \psi'(0)$. We proceed inductively. Let $n \in \mathbb{N}^*$ and assume that $\lim_{a\to 0}\varphi^{(k)}(a)$ exists and $\varphi^{(k)}(0) = \psi^{(k)}(0)$ $\forall k \leqslant n$. Differentiating (2.8) $n$ times we get

$$(e^{-\psi(a)})^{(n+1)} - (e^{-\varphi(a)})^{(n+1)}\rho(a) - \sum_{k=0}^{n-1}(e^{-\varphi(a)})^{(k+1)}\rho(a)^{(n-k)} = (b-1)\sigma^{(n)}(a)$$

and by Lemmas 2.3 and 2.4 we get

$$\lim_{a\to 0}(e^{-\psi(a)})^{(n+1)} = \lim_{a\to 0}(e^{-\varphi(a)})^{(n+1)}$$

which, by formula (2.10) and the induction hypothesis, gives that the limit $\lim_{a\to 0}\varphi^{(n+1)}(a) =: \varphi^{(n+1)}(0)$ exists in $\mathbb{R}$ and

$$\sum_{\{\boldsymbol{k}|\sum_{j=1}^{n+1} jm_j = n+1\}} c_{\boldsymbol{k}}\, e^{-\psi(0)}\prod_{j=1}^{n+1}\left(\frac{\psi^{(j)}(0)}{j!}\right)^{m_j}$$

$$= \sum_{\{\boldsymbol{k}|\sum_{j=1}^{n+1} jm_j = n+1\}} c_{\boldsymbol{k}} e^{-\varphi(0)}\prod_{j=1}^{n+1}\left(\frac{\varphi^{(j)}(0)}{j!}\right)^{m_j}$$

and as by the induction hypothesis $\varphi^{(j)}(0) = \psi^{(j)}(0)$ for $j \leqslant n$, it gives

$$\varphi^{(n+1)}(0) = \psi^{(n+1)}(0)$$

which completes the induction. To prove (1.9), the fact that

$$\log\Gamma(x+1)-\log\Gamma(x)=\log x$$

gives the functional relation for the polygamma function

$$\Psi(k,x+1)-\Psi(k,x)=\frac{(-1)^k k!}{x^{k+1}} \tag{2.18}$$

hence

$$\varphi^{(k+1)}(0)=\Psi(k,b)-\Psi(k,1)=\sum_{n=1}^{b-1}(\Psi(k,n+1)-\Psi(k,n))=\sum_{n=1}^{b-1}\frac{(-1)^k k!}{n^{k+1}}$$

□

*Proof of Corollary 1.* The fact that $\varphi$ and $\psi$ have the same asymptotic expansion at 0 implies that an approximation of $\varphi$ is

$$\varphi(a)\sim\log\frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)}-\log p \quad \text{as}\, a\to 0$$

and the error decreases faster than any positive power of $a$. This also implies that

$$q(a)\sim\left(\frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b)}\right)^{1/a}p^{1/a} \quad \text{as}\, a\to 0$$

in the sense that $\forall n\in\mathbb{N},\varepsilon>0,\exists a_{n,\varepsilon}>0$ such that $\forall a<a_{n,\varepsilon}$

$$e^{-\varepsilon a^n}\left(\frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b)}\right)^{1/a}p^{1/a}<q(a)<e^{\varepsilon a^n}\left(\frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b)}\right)^{1/a}p^{1/a}$$

hence

$$\lim_{a\to 0}\frac{q(a)}{p^{1/a}}=e^{-\gamma-\Psi(0,b)} \tag{2.19}$$

$\gamma$ being the Euler's constant. The RHS of the above inequality may be rewritten as

$$\frac{q(a)}{p^{1/a}}<\left(\frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b)}\right)^{1/a}+\varepsilon' a^n\left(\frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b)}\right)^{1/a}$$

close to 0 and for an $\varepsilon'>\varepsilon$, and the LHS

$$\left(\frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b)}\right)^{1/a}-\varepsilon a^n\left(\frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b)}\right)^{1/a}<\frac{q(a)}{p^{1/a}}$$

Hence

$$\frac{q(a)}{p^{1/a}}\sim\left(\frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b)}\right)^{1/a}$$

with a remainder term vanishing faster than any power of $a$ at 0. The rest comes from considering

$$\left(\frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b)}\right)^{1/a}=\exp\left(\frac{1}{a}\log\frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b)}\right)$$

along with Faà di Bruno formula. □

## 3. Asymptotics at ∞

3.1. **The operator $D$.** To find the asymptotic expansion at infinity, the previous technique has to be adjusted accordingly. First, we introduce the differential operator $D$ defined by

$$Df(a) = a^2 \partial f(a) \tag{3.1}$$

It satisfies the product rule

$$D(fg)(a) = g(a)Df(a) + f(a)Dg(a) \tag{3.2}$$

and the composition rule

$$D(f \circ g)(a) = f'(g(a))Dg(a)$$

The last two relations combined give us the Faa di Bruno formula for $D$

$$D^n(f \circ g)(a) = \sum_{\{\boldsymbol{m} \in \mathbb{N}^n | \sum_{j=1}^n jm_j = n\}} \frac{n!}{m_1!m_2!\dots m_n!} f^{(|\boldsymbol{m}|)}(g(a)) \prod_{j=1}^n \left( \frac{D^j g(a)}{j!} \right)^{m_j} \tag{3.3}$$

where $|\boldsymbol{m}| = \sum_{j=1}^n m_j$. Also, we have the two-arguments composition rule

$$Df(a, \varphi(a)) = D_1 f(a, \varphi(a)) + D\varphi(a) \partial_2 f(a, \varphi(a)) \tag{3.4}$$

where $D_1 f(a, b) = a^2(\partial_1 f)(a, b)$, $\partial_1$ denoting differentiation wrt the first variable of a multivariate function, i.e. in our case $D_1 f(a, \varphi(a)) = a^2(\partial_1 f)(a, \varphi(a))$. Furthermore, we remark that it acts on monomials, for $m \in \mathbb{Z}$, by

$$Da^m = na^{m+1}$$

and by induction

$$D^n a^m = (m)_n a^{m+n}$$

The operator $D$ can be used to deal with asymptotic expansions at infinity. To see this, intuitively, starting from the formal power series

$$f(x) = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \dots$$

one can get

$$D^n f(x) = \sum_{k=n}^{\infty} (-1)^n \frac{k!}{(n-k)!} \frac{c_k}{x^{k-n}}$$

If certain conditions apply and it is possible to take limits to $\infty$, all but the first term of the sum vanish and we get

$$\lim_{x \to \infty} D^n f(x) = (-1)^n n! c_n$$

This is rigorously treated in Lemma 1.1, which is proved below:

*Proof of Lemma 1.1.* To show i), we notice that if for a function $f$ we have $\lim_{x \to \infty} f(x) = a_0 \in \mathbb{R}$ and $Df(x) = a_1 + a_2/x + a_3/x^2 + \dots + a_{k+1}/x^k + \mathcal{O}(1/x^{k+1})$, then by integrating we get that $f(x) = a_0 - a_1/x - a_2/2x^2 - a_3/3x^3 + \dots + a_{k+1}/kx^{k+1} + \mathcal{O}(1/x^{k+2})$. Next, we see that, under the assumptions of the first part of the lemma, we have that $\lim_{x \to \infty} D^{n-1} f(x) = a \in \mathbb{R}$ and $\lim_{x \to \infty} D^n f(x) = b \in \mathbb{R}$. This implies that $D^{n-1} f(x) = a + \mathcal{O}(1/x)$. Applying this observation inductively to find the asymptotic expansions of lower powers of $D$ proves the first part of the Lemma. For the second part, we notice that as the derivatives admit asymptotic expansions, these can be obtained by differentiating the asymptotic expansion of the original function. In the same way, we may apply the operator $D$ to the original asymptotic expansion, as $D^k$ can be expressed as a combination of operators $\partial^l$ for $l \leqslant k$, and take limits to $\infty$ to prove the second part. □

In the following subsections we shall compute the asymptotic expansion of $\varphi$ using the operator $D$. We start with the equation

$$\int_0^{\varphi(a)} \tau(a;s)\mathrm{d}s = (1-p)\frac{\Gamma(b)\Gamma(a)a^b}{\Gamma(a+b)} \tag{3.5}$$

Where

$$\tau(a;s) = e^{-s}(a - ae^{-s/a})^{b-1} \tag{3.6}$$

Our method consists of acting and iterating the operator $D$ on (3.5) and taking the limits to $\infty$ on both sides. So we have to see how $D$ acts on $\tau$ and on the right hand side.

3.2. **Asymptotics of the RHS.** To study the right hand side of the equation (3.5), we study the asymptotics of the ratio

$$\frac{\Gamma(a)a^b}{\Gamma(a+b)} \tag{3.7}$$

In [13], Tricomi and Erdelyi derived an asymptotic expansion for such ratios of Gamma functions, in terms of a generalisation of Nørlund Polynomials, which in our special case it may be expressed as

$$\frac{\Gamma(a)a^b}{\Gamma(a+b)} \sim \sum_{n\geqslant 0} \frac{\Gamma(1-b)}{\Gamma(1-(b+n))} \frac{B_n^{(1-b)}}{n!a^n}, \quad x \to \infty$$

which by the reflection formula for the Gamma function can be rewritten as

$$\frac{\Gamma(a)a^b}{\Gamma(a+b)} \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(b)_n \frac{B_n^{(1-b)}}{a^n} \tag{3.8}$$

We shall prove the following Lemma:

Lemma 3.1. *For $n \in \mathbb{N}$, we have*

$$\lim_{a\to\infty} D^n\left((1-p)\frac{\Gamma(b)\Gamma(a)a^b}{\Gamma(a+b)}\right) = (1-p)\Gamma(b+n)B_n^{(1-b)} \tag{3.9}$$

*Proof.* The coefficients of the asymptotic expansion (3.8), by Lemma 1.1, can be used to give the limit in (3.9), if the derivatives of the ratio also admit asymptotic expansions. Hence we shall find these asymptotic expansions of the derivatives, and also a different expression for the coefficients in the asymptotic expansion of the ratio (3.7) on the way.

The tool we shall work with is the operator $D$ and its Faà di Bruno formula eq3.3. We denote the logarithmic derivative of the ratio (3.7) by

$$V(a) := \log\frac{\Gamma(a)a^b}{\Gamma(a+b)} = b\log a + \log\Gamma(a) - \log\Gamma(a+b) \tag{3.10}$$

A classic result on the asymptotic expansion of $\log\Gamma$ is the following, see [9, 5.11.8], for fixed $h \in \mathbb{C}$,

$$\log\Gamma(x+h) \sim \log\sqrt{2\pi} + \left(x + h - \frac{1}{2}\right)\log x - x + \sum_{n\geqslant 2}\frac{B_n(h)}{n(n-1)}x^{1-n}, \quad x \to +\infty \tag{3.11}$$

which has the nice property that it can also be differentiated, and give us asymptotic expansions of polygamma functions. This implies also that the derivatives of $V$ admit asymptotic expansions. We have, asymptotically,

$$V(a) \sim \textstyle\sum_{n\geqslant 2}\frac{B_n - B_n(b)}{n(n-1)}\frac{1}{a^{n-1}} = \sum_{n\geqslant 1}\frac{B_{n+1}-B_{n+1}(b)}{n(n+1)}\frac{1}{a^n}$$

We have, then, by Lemma 1.1, that

$$\lim_{a\to\infty} D^n V(a) = (-1)^n n! \frac{B_{n+1} - B_{n+1}(b)}{n(n+1)} \tag{3.12}$$

Acting $D$ $n$ times on ratio (3.7) we get

$$\begin{aligned} D^n\left(\frac{\Gamma(a)a^b}{\Gamma(a+b)}\right) &= D^n e^{V(a)} \\ &= e^{V(a)} \sum_{\{\boldsymbol{m}\in\mathbb{N}^n \mid \sum_{j=1}^n jm_j = n\}} \frac{n!}{m_1!m_2!\dots m_n!} \prod_{j=1}^n \left(\frac{D^j V(a)}{j!}\right)^{m_j} \end{aligned} \tag{3.13}$$

and taking limits we end up with

$$\lim_{a\to\infty} D^n\left(\frac{\Gamma(a)a^b}{\Gamma(a+b)}\right) = \sum_{\{\boldsymbol{m}\in\mathbb{N}^n \mid \sum_{j=1}^n jm_j = n\}} \frac{(-1)^n n!}{m_1!m_2!\dots m_n!} \prod_{j=1}^n \left(\frac{B_{j+1} - B_{j+1}(b)}{j(j+1)}\right)^{m_j}$$

Hence, by Lemma 1.1, the derivatives of the ratio (3.7) admit asymptotic expansions at infinity, and these can be given by differentiating the asymptotic expansion (3.8). □

Remark 3.1. In the proceeding proof, we find two different ways to express the asymptotic expansion of the ratio of gamma functions, which implies a relation between Nørlund, Bernoulli and Bell polynomials we could not trace in the literature,

$$(b)_n B_n^{(1-b)} = \mathcal{B}_n(B_2(b) - B_2, B_3(b) - B_3, \dots, B_{n+1}(b) - B_{n+1}) \tag{3.14}$$

and using the fact that

$$\sum_{k=1}^{j+1} B_{j-k+1} \frac{(j-1)!}{k!(j+1-k)!} b^k = B_{j+1}(b) - B_{j+1}$$

we get

$$(b)_n B_n^{(1-b)} = \sum_{\{\boldsymbol{m}\in\mathbb{N}^n \mid \sum_{j=1}^n jm_j = n\}} \frac{n!}{m_1!m_2!\dots m_n!} \prod_{j=1}^n \left(\sum_{k=1}^{j+1} B_{j-k+1} \frac{(j-1)!}{k!(j-k+1)!} b^k\right)^{m_j}$$

3.3. **Asymptotics of the LHS.** We shall first study the asymptotic behaviour of $\tau$, defined in (3.6), through the following Lemma.

Lemma 3.2. *We have the limits*

$$\lim_{a\to\infty} D^n \tau(a;s) = B_n^{(1-b)} e^{-s} s^{b-1+n} \tag{3.15}$$

*Proof.* We have that

$$\tau(a;s) = e^{-s}(a - ae^{-s/a})^{b-1} = e^{-s}\left(\frac{\frac{1}{a}}{1 - e^{-s/a}}\right)^{1-b} = e^{-s}s^{b-1}\left(\frac{-\frac{s}{a}}{e^{-s/a} - 1}\right)^{1-b}$$

We may write, in terms of Nørlund polynomials, by (1.6),

$$\tau(a;s) = e^{-s}s^{b-1} \sum_{k=0}^{\infty} B_k^{(1-b)} \frac{(-1)^k s^k}{k! a^k} \tag{3.16}$$

and

$$D^n \tau(a;s) = e^{-s}s^{b-1} \sum_{k=n}^{\infty} B_k^{(1-b)} \frac{(-1)^{k+n} s^k}{(k-n)! a^{k-n}}$$

and thus we get

$$\lim_{a\to\infty} D^n\tau(a;s) = B_n^{(1-b)} e^{-s} s^{b-1+n}$$

□

Acting $D$ on the left hand side of (3.5) gives the expression

$$D\int_0^{\varphi(a)} \tau(a;s)\mathrm{d}s = D\varphi(a)\tau(a;s) + \int_0^{\varphi(a)} D\tau(a;s)\mathrm{d}s$$

and hence by induction, iterating $D$ totally $n$ times,

$$D^n \int_0^{\varphi(a)} \tau(a;s)\mathrm{d}s = \sum_{k=0}^{n-1} D^k(D\varphi(a)D_1^{n-k-1}\tau(a;\varphi(a))) + \int_0^{\varphi(a)} D^n\tau(a;s)\mathrm{d}s \quad (3.17)$$

We shall study the terms

$$D^k(D\varphi(a)D_1^{n-k-1}\tau(a;\varphi(a))) = \sum_{j=0}^{k} \binom{k}{j} D^{k-j+1}\varphi(a)D^j[D_1^{n-k-1}\tau(a;\varphi(a))]$$

and as

$$D^j[D_1^{n-k-1}\tau(a;\varphi(a))] = D^{j-1}[D_1^{n-k}\tau(a;\varphi(a)) + D\varphi(a)D_1^{n-k-1}\partial_2\tau(a;\varphi(a))]$$

it is important to study the terms defined as

$$d(k,m,n) := \lim_{a\to\infty} D^m[D_1^n\partial_2^k\tau(a;\varphi(a))] \quad (3.18)$$

In other words, we will compute, recursively, the limits of these terms for $a\to\infty$. We note that, as $D^n\tau(a;s)$ is an analytic function of $s$ in some disc around 0, as seen by its power series, we can interchange differentiation wrt the second variable and the limit for $a\to\infty$, as we know that $\varphi(a)$ converges to a finite limit, provided that the convergence for $a\to\infty$ is locally uniform, which indeed is (an argument: as $a\to\infty$, the radius of convergence of the power series increase, so taking a compact set and assuming $a$ large enough, we can use the convergence of the sequence of power series to prove this result). We have

$$D^m[D_1^n\partial_2^k\tau(a;\varphi(a))] = D^{m-1}[D_1^{n+1}\partial_2^k\tau(a;\varphi(a)) + D\varphi(a)D_1^n\partial_2^{k+1}\tau(a;\varphi(a))]$$

hence we get the recursive relation

$$d(k,m,n) = d(k,m-1,n+1) + \sum_{j=0}^{m-1}\binom{m-1}{j}\varphi_{m-j}d(k+1,j,n) \quad (3.19)$$

where $\varphi_l = \lim_{a\to\infty} D^l\varphi(a)$, assuming that the limit is already known, and the boundary conditions

$$\begin{aligned} d(k,0,n) &= \lim_{a\to\infty} D_1^n\partial_2^k\tau(a;\varphi(a)) = \lim_{a\to\infty}\partial_2^k D_1^n\tau(a;\varphi(a)) \\ &= B_n^{(1-b)}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(b+n-j)_j e^{-\gamma_b}\gamma_b^{b-1+n-j} \end{aligned}$$

As for the integral term, we have

$$\lim_{a\to\infty}\int_0^{\varphi(a)} D^n\tau(a;s)\mathrm{d}s = B_n^{(1-b)}\int_0^{\gamma_b} e^{-s}s^{b-1+n}\mathrm{d}s \quad (3.20)$$

and

$$\int_0^{\gamma_b} e^{-s}s^{b-1+n}\mathrm{d}s = -\sum_{k=0}^{n-1}(b+n-k)_k e^{-\gamma_b}\gamma_b^{b-1+n-k} + (b)_n(1-p)\Gamma(b) \quad (3.21)$$

by repeated integrations by parts and the fact that $\int_0^{\gamma_b} e^{-s} s^{b-1} \mathrm{d}s = (1-p)\Gamma(b)$. We have got, then, for the left hand side that

$$\begin{aligned}
&\lim_{a\to\infty} D^n \int_0^{\varphi(a)} \tau(a;s)\mathrm{d}s \\
&= \lim_{a\to\infty} \sum_{k=0}^{n-1} D^k(D\varphi(a) D_1^{n-k-1} f(a;\varphi(a))) + \lim_{a\to\infty} \int_0^{\varphi(a)} D^n f(a;s)\mathrm{d}s \\
&= \lim_{a\to\infty} \sum_{k=0}^{n-1} \sum_{j=0}^{k} \binom{k}{j} D^{k-j+1}\varphi(a) D^j [D_1^{n-k-1} f(a;\varphi(a))] \\
&\quad - B_n^{(1-b)} \sum_{k=0}^{n-1} (b+n-k)_k e^{-\gamma_b} \gamma_b^{b-1+n-k} + (b)_n (1-p)\Gamma(b) B_n^{(1-b)} \\
&= \sum_{k=0}^{n-1} \sum_{j=0}^{k} \binom{k}{j} \varphi_{k-j+1} d(0,j,n-k-1) \\
&\quad - B_n^{(1-b)} \sum_{k=0}^{n-1} (b+n-k)_k e^{-\gamma_b} \gamma_b^{b-1+n-k} + (1-p)\Gamma(b+n) B_n^{(1-b)} \\
&= \varphi_n d(0,0,0) + \sum_{j=1}^{n-1} \binom{n-1}{j} \varphi_{n-j} d(0,j,0) \\
&\quad + \sum_{k=0}^{n-2} \sum_{j=0}^{k} \binom{k}{j} \varphi_{k-j+1} d(0,j,n-k-1) \\
&\quad - B_n^{(1-b)} \sum_{k=0}^{n-1} (b+n-k)_k e^{-\gamma_b} \gamma_b^{b-1+n-k} + (1-p)\Gamma(b+n) B_n^{(1-b)}
\end{aligned}$$

We notice that the term $(1-p)\Gamma(b+n)B_n^{(1-b)}$ cancels exactly with the right hand side.

3.4. **Conclusion.** *Proof of Theorem 2.* Summing up, using the normalisation $\delta = \frac{d}{e^{-\gamma_b}\gamma_b^{b-1}}$, we are left with

$$\begin{aligned}
\varphi_n = -\sum_{j=1}^{n-1} \binom{n-1}{j} \varphi_{n-j} \delta(0,j,0) - \sum_{k=0}^{n-2} \sum_{j=0}^{k} \binom{k}{j} \varphi_{k-j+1} \delta(0,j,n-k-1) \\
+ B_n^{(1-b)} \sum_{k=0}^{n-1} (b+n-k)_k \gamma_b^{n-k}
\end{aligned}$$

and $\delta$ and $\varphi$ also satisfying the recursive relation, by (3.19),

$$\delta(k,m,n) = \delta(k,m-1,n+1) + \sum_{j=0}^{m-1} \binom{m-1}{j} \varphi_{m-j} \delta(k+1,j,n)$$

We have the initial conditions

$$\varphi_0 = \gamma_b$$

$$\delta(k,0,n) = B_n^{(1-b)} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (b+n-j)_j \gamma_b^{n-j}$$

□

To prove Corollary 2 we need the following lemma.

LEMMA 3.3. *Let $f$ have asymptotic expansion*

$$f(x) = \sum_{k=0}^{N} \frac{a_k}{k!x^k} + r(x)$$

*where $r(x) = \mathcal{O}(1/x^{N+1})$. Then,*

$$e^{f(x)} = e^{a_0} + e^{a_0} \sum_{k=1}^{N} \frac{\mathcal{B}_k(a_1, a_2, ..., a_k)}{k!x^k} + \mathcal{O}(1/x^{N+1})$$

*Proof.* We have

$$\begin{aligned} f(x) = \sum_{k=0}^{N} \frac{a_k}{k!x^k} + r(x) &\Rightarrow e^{f(x) - \sum_{k=0}^{N} \frac{a_k}{k!x^k}} = e^{r(x)} = 1 + \mathcal{O}(1/x^{N+1}) \\ \Rightarrow e^{f(x)} &= e^{\sum_{k=0}^{N} \frac{a_k}{k!x^k}} + \mathcal{O}(1/x^{N+1}) = e^{a_0} \prod_{k=1}^{N} e^{\frac{a_k}{k!x^k}} + \mathcal{O}(1/x^{N+1}) \\ &= e^{a_0} \prod_{k=1}^{N} \left( 1 + \sum_{m=1}^{\lceil \frac{N+1}{k} - 1 \rceil} \frac{a_k^m}{m!k!^m x^{km}} + \mathcal{O}(1/x^{N+1}) \right) + \mathcal{O}(1/x^{N+1}) \\ &= e^{a_0} \sum_{n=0}^{N} \frac{\mathcal{B}_n(a_1, a_2, ..., a_n)}{n!x^n} + \mathcal{O}(1/x^{N+1}) \end{aligned}$$

where the last equality is derived by a combinatorial argument, the coefficient of $1/x^n$ being the sum of products of the form $\prod_{k=1}^{n} \frac{a_k^{m_k}}{(k!)^{m_k} m_k!}$ such that $\sum_{k=1}^{n} k m_k = n$, which defines the complete Bell polynomials. □

*Proof of Corollary 2.* The proof is an immediate consequence of Theorem 2 and the foregoing Lemma. □

## 4. Relations between Bell, Bernoulli and Nørlund Polynomials

In the course of trying to find the asymptotic expansion of $\varphi$ at $\infty$, using Faà di Bruno formulas, we encountered identities between Bell polynomials and Nørlund polynomials, that we have not been able to trace in the literature, hence we state them in this section as a separate result.

PROPOSITION 3. *Let $c \in \mathbb{C}$. Then, the Nørlund polynomial $B_n^{(c)}$ can be expressed as*

$$B_n^{(c)} = \sum_{\{\boldsymbol{m} \in \mathbb{N}^n | \sum_{j=1}^{n} j m_j = n\}} \frac{n!}{m_1! m_2! \dots m_n!} \prod_{j=1}^{n} \left( \frac{(-1)^{j+1} c B_j}{j! j} \right)^{m_j} \tag{4.1}$$

*or, phrased in terms of Bell polynomials $\mathcal{B}_n$,*

$$B_n^{(c)} = \mathcal{B}_n(cB_1, -cB_2/2, 0, -cB_4/4, 0, \dots, -cB_n/n), \quad n > 1 \tag{4.2}$$

*Moreover, we have that*

$$(c-n)_n B_n^{(c)} = (-1)^n \mathcal{B}_n(B_2(c) - B_2, -B_3(c) - B_3, \dots, (-1)^{n+1} B_{n+1}(c) - B_{n+1}) \tag{4.3}$$

*Proof.* The last equation is derived by Remark 3.1, and the symmetries $B_n(1-x) = (-1)^n B_n(x)$ and $(1-c)_n = (-1)^n(c-n)_n$. For the rest, by (1.6) we have

$$B_n^{(c)} = \lim_{z\to 0} \partial^n \left[ \left( \frac{z}{e^z-1} \right)^c \right]$$

By using Faà di Bruno formula we get

$$\begin{aligned} \partial^n \left[ \left( \frac{z}{e^z-1} \right)^c \right] &= \partial^n \left( e^{c \log \frac{z}{e^z-1}} \right) \\ &= \left( \frac{z}{e^z-1} \right)^c \sum_{\{\boldsymbol{m}\in\mathbb{N}^n | \sum_{j=1}^n jm_j=n\}} \frac{n!}{m_1!m_2!\ldots m_n!} \prod_{j=1}^n \left( \frac{c}{j!} \partial^j \left( \log \frac{z}{e^z-1} \right) \right)^{m_j} \end{aligned}$$

and we have the limit

$$\lim_{z\to 0} \left( \frac{z}{e^z-1} \right)^c = 1$$

and

$$\begin{aligned} -z\left( \log \frac{e^z-1}{z} \right)' = -\frac{ze^z}{e^z-1} + 1 &= \sum_{n=1}^{\infty} (-1)^{n+1} B_n \frac{z^n}{n!} \\ \Rightarrow \log \frac{z}{e^z-1} &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{B_n}{n} \frac{z^n}{n!} \end{aligned}$$

hence

$$\lim_{z\to 0} \partial^j \left( \log \frac{z}{e^z-1} \right) = (-1)^{j+1} \frac{B_j}{j}$$

and thus, summing up,

$$\begin{aligned} B_n^{(c)} &= \lim_{a\to 0} \partial^n \left[ \left( \frac{z}{e^z-1} \right)^c \right] \\ &= \sum_{\{\boldsymbol{m}\in\mathbb{N}^n | \sum_{j=1}^n jm_j=n\}} \frac{n!}{m_1!m_2!\ldots m_n!} \prod_{j=1}^n \left( \frac{(-1)^{j+1} c B_j}{j!j} \right)^{m_j} \end{aligned}$$

which concludes the proposition. □

## Appendix

In this appendix, we provide code in Maple and Sage for computing the terms of asymptotic expansion of $\varphi$ and $q$ at infinity recursively.

## Appendix A. Maple code

In the first algorithm, the procedure `phiinf(n)` computes what we define as $\varphi_n$ in Theorem 2.

`Algorithm A.1:`

```
norl:=proc(n,c);
            if n=0 then
            return 1;
    else
        return (-1)^n*CompleteBellB(
n,seq(-c*bernoulli(j)/j,j=1..n)
        );
    end if;
```

```
end proc;
phiinf:=proc(n) option remember;
        if n=0 then
               return gamma[b];
        end if;
    for k from 1 to n-1 do
       phiinf(k);
    od;
    return expand(
-add(binomial(n-1,j)*phiinf(n-j)*delta(0,j,0),j=1..n-1)
-add(add(binomial(k,j)*phiinf(k-j+1)*delta(0,j,n-k-1)
,j=0..k),k=0..n-2)+norl(n,1-b)*add(pochhammer(b+n-k,k)
*phiinf(0)^(n-k),k=0..n-1);
end proc;
delta:=proc(k,m,n) option remember;
        if m=0 then
               return simplify(
norl(n,1-b)*add(binomial(k,j)*(-1)^(k-j)*
pochhammer(b+n-j,j)*phiinf(0)^(n-j),j=0..k)
        );
        end if;
    return simplify(delta(k,m-1,n+1)+
add(binomial(m-1,j)*phiinf(m-j)*delta(k+1,j,n),j=0..m-1));
end proc;
```

In the second algorithm, the procedure `qinf(n)` computes the nth coefficient of the asymptotic expansion of $q$ in Corollary 2.

`Algorithm A.2:`

```
qinf:=proc(b,n);
   if n=0 then
      return 1;
   else
      return CompleteBellB(
      n,seq((-1)^(k+1)*phiinf(k),k=0..n-1))/n!;
   end if;
end proc;
```

## Appendix B. Sage code

The function `phiinf(n)` computes what we define as $\varphi_n$ in Theorem 2.

`Algorithm B.1:`

```
gamma_b=var('gamma_b')
    b=var('b')
    def norlund(n,c):
       if n==0:
           return 1
       else:
```

```
        return
(-1)^n*sum(bell_polynomial(n,k)([-c*bernoulli(j)/j for j in
[1..n-k+1]]) for k in [1..n])
@CachedFunction
def phiinf(n):
   if n==0:
        return gamma_b
   else:
        return expand(-sum(binomial(n-1,j)*phiinf(n-j)*
delta(0,j,0) for j in [1..n-1])-sum(sum(binomial(k,j)*
phiinf(k-j+1)*delta(0,j,n-k-1) for j in [0..k]) for k
in [0..n-2])+norlund(n,1-b)*sum(rising_factorial(b+n-k,k)*
phiinf(0)^(n-k) for k in [0..n-1]))
@CachedFunction
def delta(k,m,n):
          if m==0:
        return
simplify(norlund(n,1-b)*sum(binomial(k,j)*(-1)^(k-j)*
rising_factorial(b+n-j,j)*phiinf(0)^(n-j) for j in [0..k]))
         else:
        return
simplify(delta(k,m-1,n+1)+sum(binomial(m-1,j)
*phiinf(m-j)*delta(k+1,j,n)
for j in [0..m-1]))
```

The function `qinf(n)` computes the nth coefficient of the asymptotic expansion of $q$ in Corollary 2.

`Algorithm B.2:`

```
def qinf(n):
         return sum(bell_polynomial(n,j)(
         [(-1)^(k+1)*phiinf(k)
for k in [0..n-j]]) for j in [1..n])/factorial(n)
```

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# COMPLETE MONOTONICITY IN RATIOS OF PRODUCTS OF ENTIRE FUNCTIONS

DIMITRIS ASKITIS

Abstract. Ratios of gamma functions have been studied with respect to complete monotonicity in several occasions. Here, we extend these results to ratios of entire functions of finite genus. To do so, we have to take into account the exact order of the entire function, as well as the order and the density of its zero sequence. We apply these results to multiple gamma functions.

## 1. Introduction

Complete monotonicity properties of ratios of gamma functions in the form

$$\frac{\prod_{j=1}^{p}\Gamma(x+a_j)}{\prod_{j=1}^{p}\Gamma(x+b_j)}$$

have attracted extensive interest in literature. A function $f:\mathbb{R}_+\to\mathbb{R}_+$ is said to be completely monotonic when for all $x>0$ and $n\in\mathbb{N}$ we have $(-1)^n f^{(n)}(x)\geq 0$. A function $f:\mathbb{R}_+\to\mathbb{R}_+$ is said to be logarithmically completely monotonic when $(-\log f)'$ is completely monotonic. Logarithmic completely monotonic functions are an important subset of the class of completely monotonic functions. Completely monotonic functions are characterised as the Laplace transforms of non-negative Borel measures on the positive half-line, what is known as Bernstein-Widder theorem.

**Bernstein's representation.** *A function $f\in C^\infty(0,\infty)$ is completely monotonic if and only if it is the Laplace transform of a non-negative Borel measure on $[0,\infty]$, i.e. if and only if*

$$f(x)=\int_0^\infty e^{-xt}\mathrm{d}\mu t$$

*for a non-negative measure $\mu$ on $[0,\infty]$ such that*

$$\int_0^\infty e^{-t}\mathrm{d}\mu t<\infty$$

.

The first result in this direction was by Bustoz and Ismail [3] who showed that the above ratio is logarithmically completely monotonic for $p=2$, $a_1=0$ and $a_2=b_1+b_2$. Several extensions of this result were found by Ismail and Muldoon [5], Alzer [1], and Grinshpan and Ismail [4]. Karp and Prilepkina [6] consider the more general, weighted case

$$W(x)=\frac{\prod_{j=1}^{n}\Gamma(A_jx+a_j)}{\prod_{j=1}^{m}\Gamma(B_jx+b_j)}$$

and, using properties specific to the gamma function, they prove that the above ration is logarithmically completely monotonic if and only if $\sum_j B_j=\sum_i A_i$, $\sigma=\prod_j B_j^{B_j}\prod_i A_i^{A_i}\leq 1$ and $\sum_i \frac{e^{-a_iu/A_i}}{1-e^{-u/A_i}}-\sum_j\frac{e^{-b_ju/B_j}}{1-e^{-u/B_j}}\geq 0,\forall u>0$.

In a different context, Pedersen [8] had showed the representation

$$(-1)^p \partial_x^{p+1} \log f(x) = \int_0^\infty e^{-sx} s^p h(s)\, ds,$$

where $f$ entire function of finite genus with exclusively non-positive zeros, and

$$h(s) = \sum_{k=1}^\infty e^{-\lambda_k s}. \tag{1}$$

This generalised prior results on gamma function, as $1/\Gamma$ is entire of genus 1 with zeros on the non-positive integers. In particular, in the case where $1/f$ is Eulers gamma function, the function $h$ can be rewritten as

$$h(s) = \frac{1}{1-e^{-s}}.$$

A question then arises if one can also generalise the previous results of Karp and Prelepkina to ratios of entire functions. The present manuscript proceeds as follows. In the second section, we state the definitions we need and the basic setting. In the second section, we study the case where the order of growth of the entire function is integer. The main results are Theorems 1,2 and 3. In the fourth section we study the case when the order it is non-integer. In general, we always shall assume that the sequence of zeros has asymptotic density. In the fifth section, we apply the results of the third section to multiple gamma functions.

## 2. Entire functions of finite order $\rho$ with only non-positive zeros

Let $f$ be an entire function of finite order $\rho$, having only non-positive zeros, the negative ones being $\{-\lambda_k\}$. Setting $p = \lfloor \rho \rfloor$, i.e. having $\rho \in [p, p+1)$, we can represent $f$ by the Hadamard theorem as

$$f(z) = z^\kappa e^{Q(z)} \prod_{k=1}^\infty \left(1 + \frac{z}{\lambda_k}\right) e^{\sum_{j=1}^p (-1)^j \frac{z^j}{j\lambda_k^j}} \tag{2}$$

where $Q$ is a polynomial of degree less or equal to $\lfloor \rho \rfloor$ and $\kappa$ is the multiplicity of the (possible) root at 0. The logarithm of $f$ on $(0,\infty)$ is

$$\log f(x) = \kappa \log x + Q(x) + \sum_{k=1}^\infty \left( \log\left(1 + \frac{x}{\lambda_k}\right) + \sum_{j=1}^p (-1)^j \frac{x^j}{j\lambda_k^j} \right).$$

Differentiating $p$ times we get

$$\frac{(-1)^{p-1}}{(p-1)!} \partial^p \log f(x) = \frac{\kappa}{x^p} + (-1)^{p-1} qp - \sum_{k=1}^\infty \left( \frac{1}{\lambda_k^p} - \frac{1}{(x+\lambda_k)^p} \right). \tag{3}$$

The following lemma will be useful.

Lemma 2.1. *Suppose that $g : (0,\infty) \to \mathbb{R}$ is a $C^\infty$-function and that its derivative is a completely monotonic function. Then there exists a positive measure $\mu$ on $[0,\infty)$ such that $\int_0^\infty e^{-xs}\, d\mu(s)$ converges for all $x > 0$, and such that, for $a, b \geq 0$,*

$$g(x+a) - g(x+b) = \int_0^\infty e^{-xs} \frac{1}{s} \left(e^{-bs} - e^{-as}\right) d\mu(s).$$

*Proof.* This follows immediately by combining

$$g(x+a) - g(x+b) = \int_b^a g'(t+x)\, \mathrm{d}t$$

and Bernstein's representation of $g'$. $\square$

From [8, Proposition 2.1] we have the representation

$$(-1)^p \partial_x^{p+1} \log f(x) = \int_0^\infty e^{-sx} s^p h(s)\,\mathrm{d}s, \tag{4}$$

where

$$h(s) = \kappa + \sum_{k=1}^\infty e^{-\lambda_k s}\,. \tag{5}$$

Definition 2.1. *For $f$ as above and for sequences $a_1, \ldots, a_N$, $b_1, \ldots, b_M$ of non-negative numbers, $A_1, \ldots, A_N$, $B_1, \ldots, B_M$ of positive numbers define $W_f$ as the function*

$$W_f(x) = \frac{\prod_{j=1}^N f(A_j x + a_j)}{\prod_{j=1}^M f(B_j x + b_j)}, \quad x > 0. \tag{6}$$

Proposition 2.2. *The function $(-1)^{p+1}\partial_x^{p+1} \log W_f(x)$ is completely monotonic if and only if*

$$\sum_{j=1}^N e^{-b_j s} h(s/B_j) - \sum_{j=1}^M e^{-a_j s} h(s/A_j) \geq 0 \tag{7}$$

*Proof.* Relation (4) gives us that

$$(-1)^{p+1} \partial_x^{p+2} \log f(x) = \int_0^\infty e^{-sx} s^{p+1} h(s)\,\mathrm{d}s,$$

and hence we get, using Lemma 2.1,

$$\begin{aligned}
&(-1)^{p+1}\partial_x^{p+1}(\log f)(A_j x + a_j) \\
&= \int_0^{a_j}\int_0^\infty e^{-s(A_j x + t)} s^{p+1} h(s)\,\mathrm{d}s\,\mathrm{d}t + (-1)^{p+1}\partial_x^{p+1}(\log f)(A_j x) \\
&= \int_0^\infty e^{-sA_j x} s^p h(s)(1 - e^{-sa_j})\,\mathrm{d}s - \int_0^\infty e^{-sA_j x} s^p h(s)\,\mathrm{d}s \\
&= -\int_0^\infty e^{-sA_j x} s^p h(s) e^{-sa_j}\,\mathrm{d}s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(-1)^{p+1}\partial_x^{p+1}\log W_f(x) &= \sum_{j=1}^N A_j^{p+1}(-1)^{p+1}\partial_x^{p+1}(\log f)(A_j x + a_j) \\
&\quad - \sum_{j=1}^M B_j^{p+1}(-1)^{p+1}\partial_x^{p+1}(\log f)(B_j x + b_j) \\
&= -\sum_{j=1}^N A_j^{p+1}\int_0^\infty e^{-sA_j x} s^p h(s) e^{-sa_j}\,\mathrm{d}s \\
&\quad + \sum_{j=1}^M B_j^{p+1}\int_0^\infty e^{-sB_j x} s^p h(s) e^{-sb_j}\,\mathrm{d}s \\
&= \int_0^\infty e^{-xs} s^p \left(\sum_{j=1}^N h(s/B_j) e^{-b_j s} - \sum_{j=1}^M h(s/A_j) e^{-a_j s}\right)\mathrm{d}s.
\end{aligned}$$

This completes the proof. □

Taking logarithms and differentiating (6) $p$ times, and using (3) we get

$$\frac{(-1)^{p-1}}{(p-1)!}\partial^p \log W(x) = \tag{8}$$

$$=\kappa\left(\sum_{j=1}^{N}\frac{A_j^p}{(A_jx+a_j)^p}-\sum_{j=1}^{M}\frac{B_k^p}{(B_jx+b_j)^p}\right)+(-1)^{p-1}\left(\sum_{j=1}^{N}A_j^p-\sum_{j=1}^{M}B_j^p\right)qp$$

$$-\sum_{k=1}^{\infty}\left(\sum_{j=1}^{N}A_j^p\left(\frac{1}{\lambda_k^p}-\frac{1}{(A_jx+a_j+\lambda_k)^p}\right)-\sum_{j=1}^{M}B_j^p\left(\frac{1}{\lambda_k^p}-\frac{1}{(B_jx+b_j+\lambda_k)^p}\right)\right)$$

Our goal is to find necessary and sufficient conditions so that the function $(-1)^p\partial^p \log W(x)$ is completely monotonic, and for this it suffices to find when it is non-negative, respectively. We will see that much depends on the density of the zeros of $f$. Let $n(t)$ be the counting function of the negative zeros, counting also multiplicities, i.e.

$$n(t) = \#\{\lambda_k \le t \mid k \in \mathbb{N}\}$$

The order of the zero counting function is denoted by $\rho_0$. We have that $\rho_0 \le \rho$ and $p \ge \lfloor \rho_0 \rfloor$. Moreover, $\rho_0$ is the convergence exponent of the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$, i.e.

$$\rho_0 = \inf\left\{a \in \mathbb{R} \;\middle|\; \sum_{k=1}^{\infty}\frac{1}{\lambda_k^a} < \infty\right\}$$

We call such a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ of *divergent type* if

$$\sum_{k=1}^{\infty}\frac{1}{\lambda_k^{\rho_0}} = \infty\,.$$

We can have $\rho_0 < \rho$ only if $\rho$ is an integer, i.e. $\rho = p$. Else, $\rho_0 = \rho$. Also, if $\rho_0 = p$, then also $\rho = \rho_0 = p$. In general, we have $\rho = \max\{\rho_0, p\}$. Also, a quantity that will play an important role is

$$C_\rho := \left(\sum_{j=1}^{N}A_j^\rho - \sum_{j=1}^{M}B_j^\rho\right) \tag{9}$$

Our purpose is to find analogues of Thereom 4 in [Karp], for more general entire functions. In particular, we are more interested to find, for a given order $\rho$, necessary/sufficient conditions such that for entire functions of that order, $(-1)^p\partial^p \log W(x)$ is completely monotonic, and the conditions are indepedent of the specific choice of the function. We manage it in the case of integer order, that $\rho = \rho_0 \in \mathbb{N}$. In the cases that the order is non-integer or the order of the zeros is smaller than the order of the functions, we prove that only such sufficient conditions exist, but not necessary. Our strategy can be summarized as: In Proposition 2.2, we have seen than

$$(-1)^{p+1}\partial^{p+1}\log W_f(x) = \int_0^\infty e^{-xs}s^p\left(\sum_{j=1}^{N}h(s/B_j)e^{-b_js}-\sum_{j=1}^{M}h(s/A_j)e^{-a_js}\right)\mathrm{d}s.$$

This gives that

$$(-1)^{p}\partial^{p}\log W_f(x) = \int_0^\infty e^{-xs}s^{p-1}\left(\sum_{j=1}^{N}h(s/B_j)e^{-b_js}-\sum_{j=1}^{M}h(s/A_j)e^{-a_js}\right)\mathrm{d}s+\Lambda$$

where $\Lambda := \lim_{x\to+\infty}(-1)^p\partial^p \log W_f(x)$. Hence, if $(-1)^{p+1}\partial^{p+1}\log W_f(x)$ is completely monotonic, then $(-1)^p\partial^p\log W_f(x)$ is completely monotonic if and only if $\Lambda \geq 0$, and then we have the representation

$$(-1)^p\partial^p \log W_f(x) = \int_0^\infty e^{-xs}\mathrm{d}\mu(s) \tag{10}$$

where

$$\mathrm{d}\mu(s) = \Lambda \mathrm{d}\delta_0(s) + s^{p-1}\left(\sum_{j=1}^N h(s/B_j)e^{-b_j s} - \sum_{j=1}^M h(s/A_j)e^{-a_j s}\right)\mathrm{d}s \tag{11}$$

Hence, our strategy is to investigate the value of the limit $\Lambda := \lim_{x\to+\infty}(-1)^p\partial^p \log W_f(x)$.

## 3. The case $\rho = p$

The equality $\rho = p$ may happen in two cases: if $\rho_0 = p$ or if $\rho_0 < p$. In the course of this chapter, we shall treat the two cases separately.

It can be useful to write part of (8) in the following way:

$$\begin{aligned}
S(x) :=& \sum_{k=1}^\infty \left(\sum_{j=1}^N \left(\frac{A_j^p}{\lambda_k^p} - \frac{A_j^p}{(A_j x + a_j + \lambda_k)^p}\right) - \sum_{j=1}^M \left(\frac{B_j^p}{\lambda_k^p} - \frac{B_j^p}{(B_j x + b_j + \lambda_k)^p}\right)\right) \\
=& \sum_{k=1}^\infty \left(\frac{C}{\lambda_k^p} - \sum_{j=1}^N \frac{A_j^p}{(A_j x + a_j + \lambda_k)^p} + \sum_{j=1}^M \frac{B_j^p}{(B_j x + b_j + \lambda_k)^p}\right) \\
=& C\sum_{k=1}^\infty \left(\frac{1}{\lambda_k^p} - \frac{1}{(x+\lambda_k)^p}\right) \\
&+ \sum_{j=1}^N A_j^p \sum_{k=1}^\infty \left(\frac{1}{(x+\lambda_k)^p} - \frac{1}{(A_j x + a_j + \lambda_k)^p}\right) \\
&- \sum_{j=1}^M B_j^p \sum_{k=1}^\infty \left(\frac{1}{(x+\lambda_k)^p} - \frac{1}{(B_j x + b_j + \lambda_k)^p}\right)
\end{aligned}$$

hence

$$\begin{aligned}
\Lambda =& \lim_{x\to\infty}(-1)^p\partial^p \log W(x) \\
=& (p-1)!(-1)^p \left(\sum_{j=1}^N A_j^p - \sum_{j=1}^M B_j^p\right) qp + (p-1)! \lim_{x\to\infty} S(x)
\end{aligned} \tag{12}$$

hence the asymptotic behaviour of $S$ determines the value of $\Lambda$. To study the asymptotic behaviour of $S$, it becomes clear that we have to study the function

$$\begin{aligned}
g_{A,a}(x) :=& \sum_{k=1}^\infty \left(\frac{1}{(x+\lambda_k)^p} - \frac{1}{(Ax + a + \lambda_k)^p}\right) \\
=& \int_0^\infty \left(\frac{1}{(x+t)^p} - \frac{1}{(Ax+a+t)^p}\right)\mathrm{d}n(t)\,.
\end{aligned}$$

Under some assumptions on the zero counting function having asymptotic density, we have the following lemma.

Lemma 3.1. *Assume that*

$$\lim_{t\to\infty}\frac{n(t)}{t^p} = \Delta \in [0, +\infty)$$

*Then,*

$$\lim_{x\to\infty} g_{A,a}(x) = p\Delta \log A\,.$$

*Proof.* We have that

$$\begin{aligned} g_{A,a}(x) =& \int_0^\infty \left(\frac{1}{(x+t)^p} - \frac{1}{(Ax+a+t)^p}\right)\mathrm{d}n(t) \\ =& p\int_0^\infty \left(\frac{1}{(x+t)^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}}\right)n(t)\mathrm{d}t \end{aligned}$$

We have that $\forall\epsilon>0, \exists R>0 : \forall t>R, (\Delta-\epsilon)t^p < n(t) < (\Delta+\epsilon)t^p$. Hence we can write

$$\begin{aligned} &(\Delta-\epsilon)p\int_R^\infty \left(\frac{1}{(x+t)^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}}\right)t^p\mathrm{d}t \\ &< p\int_R^\infty \left(\frac{1}{(x+t)^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}}\right)n(t)\mathrm{d}t \\ &< (\Delta+\epsilon)p\int_R^\infty \left(\frac{1}{(x+t)^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}}\right)t^p\mathrm{d}t \end{aligned}$$

Hence we shall study the function

$$h(x) := \int_R^\infty \left(\frac{1}{(x+t)^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}}\right)t^p\mathrm{d}t\,.$$

After repeated integrations by parts we reach

$$\begin{aligned} h(x) =& u(x) + \int_R^\infty \left(\frac{1}{x+t} - \frac{1}{Ax+a+t}\right)\mathrm{d}t \\ =& u(x) + \log\left(\frac{Ax+a+R}{x+R}\right) \end{aligned}$$

where $u(x) = \mathcal{O}(1/x)$. Finally, as $(\Delta-\epsilon)ph(x) < g_{A,a}(x) < (\Delta+\epsilon)ph(x)$,

$$\lim_{x\to\infty} g_{A,a}(x) = p\Delta \log A\,.$$

□

We can derive the following Lemma about $S(x)$:

LEMMA 3.2. *Assume that $\rho_0 = p$ and the sequence of zeros is of divergent type. Moreover, assume*

$$\lim_{t\to\infty}\frac{n(t)}{t^p} = \Delta \in [0,+\infty)$$

*Then, the limit of $S(x)$ at infinity exists if and only if $C_\rho = 0$. In this case,*

$$\lim_{x\to\infty} S(x) = p\Delta\left(\sum_{j=1}^N A_j^p \log A_j - \sum_{j=1}^M B_j^p \log B_j\right) \tag{13}$$

*Proof.* By the previous Lemma, we have

$$\lim_{x\to\infty} S(x) = \lim_{x\to\infty}\left[C_\rho\sum_{k=1}^\infty\left(\frac{1}{\lambda_k^p} - \frac{1}{(x+\lambda_k)^p}\right)\right] + p\Delta\left(\sum_{j=1}^N A_j^p \log A_j - \sum_{j=1}^M B_j^p \log B_j\right)$$

and by monotone convergence,

$$\lim_{x\to\infty}\sum_{k=1}^\infty\left(\frac{1}{\lambda_k^p} - \frac{1}{(x+\lambda_k)^p}\right) = +\infty$$

as $\{\lambda_k\}_{k\in\mathbb{N}}$ is of divergent type, which gives that the limit exists if and only if $C = 0$, and the result. $\square$

We proceed to the main results, which depend on the density of the zeros.

THEOREM 1. *Assume that*

$$\lim_{t\to\infty} \frac{n(t)}{t^p} = \Delta \in (0, \infty)$$

*and the sequence of zeros is of divergent type. Then, the function* $(-1)^p \partial^p \log W(x)$ *is completely monotonic if and only if condition* (7) *holds,* $C_\rho = 0$ *and*

$$\sum_{i=1}^{N} A_i^p \log A_i - \sum_{j=1}^{M} B_j^p \log B_j \geq 0 \tag{14}$$

*In the affirmative case, we have*

$$(-1)^p \partial^p \log W_f(x) = \int_0^\infty e^{-xs} s^{p-1} \left( \sum_{j=1}^{N} h(s/B_j) e^{-b_j s} - \sum_{j=1}^{M} h(s/A_j) e^{-a_j s} \right) \mathrm{d}s + \Lambda$$

*where*

$$\Lambda = p\Delta \left( \sum_{j=1}^{N} A_j^p \log A_j - \sum_{j=1}^{M} B_j^p \log B_j \right) \tag{15}$$

*Proof.* Relation (7) is necessary and sufficient for $(-1)^{p+1}\partial^{p+1} \log W(x)$ to be completely monotonic. Then, $(-1)^p \partial^p \log W(x)$ is completely monotonic if and only if $\lim_{x\to\infty} (-1)^p \partial^p \log W(x) \geq 0$, and by the previous lemma this happens if and only if the conditions $C_\rho = 0$ and (14) are satisfied. $\square$

THEOREM 2. *Assume that* $\rho_0 = p$ *and* $n(t)$ *has zero density w.r.t.* $p$*, i.e.*

$$\lim_{t\to\infty} \frac{n(t)}{t^p} = 0$$

*and the sequence of zeros is of divergent type. Then, the function* $(-1)^p \partial^p \log W(x)$ *is completely monotonic if and only if condition* (7) *holds and* $C_\rho = 0$*. In the affirmative case,*

$$(-1)^p \partial^p \log W_f(x) = \int_0^\infty e^{-xs} s^{p-1} \left( \sum_{j=1}^{N} h(s/B_j) e^{-b_j s} - \sum_{j=1}^{M} h(s/A_j) e^{-a_j s} \right) \mathrm{d}s$$

*i.e. in this case* $\Lambda = 0$.

*Proof.* The proof is in the same spirit. As $\Delta = 0$, $\lim_{x\to\infty} (-1)^p \partial^p \log W(x) = 0$ if and only if $C = 0$, and the limit is infinite if and only if $C_\rho \neq 0$. $\square$

REMARK 3.3. *In case the sequence of zeros is not of divergent type, i.e.*

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty$$

*then* $\Delta = 0$ *by* [7, 3.2 Lemma 1]. *In case* $\rho_0 < p$*, we again have* $\Delta = 0$*. In these cases, the conditions involve a possibly non-zero value of* $C_\rho$*.*

Theorem 3. *Assume $\rho_0 = p$ and the sequence of zeros is not of divergent type, or $\rho_0 < p$ . Then, the function $(-1)^p \partial^p \log W(x)$ is completely monotonic if and only if condition* (7) *holds and*

$$C_\rho \left( \sum_{k=1}^{\infty} \frac{1}{\lambda_k^p} - (-1)^{p-1} qp \right) \geq 0 \, .$$

*In the affirmative case,*

$$(-1)^p \partial^p \log W_f(x) = \int_0^\infty e^{-xs} s^{p-1} \left( \sum_{j=1}^{N} h(s/B_j) e^{-b_j s} - \sum_{j=1}^{M} h(s/A_j) e^{-a_j s} \right) \mathrm{d}s + \Lambda$$

*where*

$$\Lambda = C_\rho \left( \sum_{k=1}^{\infty} \frac{1}{\lambda_k^p} - (-1)^{p-1} qp \right) . \tag{16}$$

*Proof.* By the previous remark, the sum $\sum_{k=1}^{\infty} \frac{1}{\lambda_k^p}$ converges. Hence, $C = 0$ is no longer required for the whole limit to converge. Rather, now, by dominated convergence we have that $\lim\limits_{x\to\infty} g_{A,a}(x) = 0$ and

$$\lim_{x\to\infty} (-1)^p \partial^p \log W(x) = C_\rho \left( \sum_{k=1}^{\infty} \frac{1}{\lambda_k^p} - (-1)^{p-1} qp \right)$$

hence, the result. ☐

## 4. The case $\rho > p$

If $\rho > p$, then $\rho \in (p, p+1)$ and $\rho = \rho_0$. We shall study the asymptotic behaviour of $S(x)$. We denote

$$C_p := \sum_{j=1}^{N} A_j^p - \sum_{j=1}^{M} B_j^p, \quad C_\rho := \sum_{j=1}^{N} A_j^\rho - \sum_{j=1}^{M} B_j^\rho \, .$$

We assume

$$\lim_{t\to\infty} \frac{n(t)}{t^\rho} = \Delta \in \mathbb{R} \, .$$

We set

$$h_{A,a}(x) := \int_0^\infty \left( \frac{1}{t^p} - \frac{1}{(Ax+a+t)^p} \right) \mathrm{d}n(t) \, . \tag{17}$$

Then, the sum $S$ becomes

$$S(x) = \sum_{j=1}^{N} A_j^p h_{A_j,a_j}(x) - \sum_{j=1}^{M} B_j^p h_{B_j,b_j}(x) \tag{18}$$

We prove the following Lemma:

Lemma 4.1. *Let*

$$\lim_{t\to\infty} \frac{n(t)}{t^\rho} = \Delta \in [0, \infty)$$

*and $r := \rho - p$. Then,*

$$\lim_{x\to\infty} \frac{h_{A,a}(x)}{x^r} = A^r \Delta \frac{(r+1)_{p-1}}{(p-1)!} \frac{\pi}{\sin \pi r} \, . \tag{19}$$

*Proof.* We have, choosing $R$ and $\epsilon$ such that for $t \geq R$, $(\Delta - \epsilon)t^\rho \leq n(t) \leq (\Delta + \epsilon)t^\rho$,

$$\begin{aligned}
h_{A,a}(x) = &\int_0^\infty \left(\frac{1}{t^p} - \frac{1}{(Ax+a+t)^p}\right) \mathrm{d}n(t)\\
=&p\int_0^\infty \left(\frac{1}{t^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}}\right) n(t)\mathrm{d}t\\
=&p\int_0^R \left(\frac{1}{t^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}}\right) n(t)\mathrm{d}t + p\int_R^\infty \left(\frac{1}{t^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}}\right) n(t)\mathrm{d}t
\end{aligned}$$

with the bounds

$$\begin{gathered}
(\Delta - \epsilon)p\int_R^\infty \left(\frac{1}{t^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}}\right) t^\rho \mathrm{d}t \leq\\
p\int_R^\infty \left(\frac{1}{t^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}}\right) n(t)\mathrm{d}t \leq\\
(\Delta + \epsilon)p\int_R^\infty \left(\frac{1}{t^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}}\right) t^\rho \mathrm{d}t\,.
\end{gathered}$$

Hence we need to study the integral

$$I := p\int_R^\infty \left(\frac{1}{t^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}}\right) t^\rho \mathrm{d}t\,.$$

By repeated integrations by parts, and as $r = \rho - p \in (0,1)$, we get

$$I = u(x) + \frac{(r+1)_{p-1}}{(p-1)!}\int_R^\infty \left(\frac{1}{t} - \frac{1}{Ax+a+t}\right) t^r \mathrm{d}t$$

where $u(x) = \mathcal{O}(1)$ and the integral on the right hand side, after some changes of variables, becomes

$$\begin{aligned}
\int_R^\infty \left(\frac{1}{t} - \frac{1}{Ax+a+t}\right) t^r \mathrm{d}t =& \int_1^\infty \left(\frac{1}{tR} - \frac{1}{Ax+a+tR}\right) t^r R^{r+1}\mathrm{d}t\\
=&\int_1^\infty \left(\frac{1}{t} - \frac{1}{\frac{Ax+a}{R}+t}\right) t^r R^r\mathrm{d}t\\
=&\int_0^1 \left(s - \frac{1}{\frac{Ax+a}{R}+\frac{1}{s}}\right) \frac{R^r}{s^{r+2}}\mathrm{d}s\\
=&\int_0^1 \frac{\frac{Ax+a}{R}}{s\frac{Ax+a}{R}+1}\frac{R^r}{s^r}\mathrm{d}s\\
=&(Ax+a)^r\int_0^{\frac{Ax+a}{R}} \frac{1}{1+s}\frac{\mathrm{d}s}{s^r}\,.
\end{aligned}$$

Hence we end up with

$$\int_R^\infty \left(\frac{1}{t} - \frac{1}{Ax+a+t}\right) t^r \mathrm{d}t = (Ax+a)^r\int_0^{\frac{Ax+a}{R}} \frac{1}{1+s}\frac{\mathrm{d}s}{s^r}\,. \tag{20}$$

Note that by contour integration we have

$$\lim_{x\to\infty}\int_0^{\frac{Ax+a}{R}} \frac{1}{1+s}\frac{\mathrm{d}t}{s^r} = \int_0^\infty \frac{1}{1+s}\frac{\mathrm{d}t}{s^r} = \frac{\pi}{\sin \pi r}\,.$$

Summarizing, we have for $h_{A,a}(x)$:

$$(\Delta - \epsilon)\frac{(r+1)_{p-1}}{(p-1)!}\frac{\pi}{\sin \pi r} \leq \lim_{x\to\infty}\frac{h_{A,a}(x)}{(Ax+a)^r} \leq (\Delta + \epsilon)\frac{(r+1)_{p-1}}{(p-1)!}\frac{\pi}{\sin \pi r}\,.$$

Hence the asymptotic result

$$\lim_{x\to\infty} \frac{h_{A,a}(x)}{x^r} = A^r \Delta \frac{(r+1)_{p-1}}{(p-1)!} \frac{\pi}{\sin \pi r} .$$

□

For the asymptotic behaviour of $S(x)$ we get

$$\begin{aligned}\lim_{x\to\infty} \frac{S(x)}{x^r} =& \Delta \frac{(r+1)_{p-1}}{(p-1)!} \frac{\pi}{\sin \pi r} \left( \sum_{j=1}^{N} A_j^\rho - \sum_{j=1}^{M} B_j^\rho \right) \\ =& C_\rho \Delta \frac{(r+1)_{p-1}}{(p-1)!} \frac{\pi}{\sin \pi r} .\end{aligned} \tag{21}$$

This shows the following result.

Proposition 4.2. *A necessary condition for the limit of $S$ to exist in $\mathbb{R}$, is $C_\rho$ or $\Delta$ to be* 0.

To get sufficient conditions, we need some more restrictions on the growth of the zero counting function $n$:

Proposition 4.3. *Let $\Delta \in (0, +\infty)$ and, for some $\epsilon > 0$,*

$$|n(t) - \Delta t^\rho| = o(t^{p-\epsilon})$$

*Then, $(-1)^p \partial^p \log W(f)$ is c.m. if and only if $C_\rho = 0$, (7) holds and*

$$C_p \left( \Delta \left( \sum_{k=1}^{p} \frac{1}{\lambda_1^k} - \frac{1}{r} \right) + \int_{\lambda_1}^{\infty} \frac{n(t) - \Delta t^\rho}{t^{p+1}} \mathrm{d}t + (-1)^{p-1} qp \right) \geq 0 . \tag{22}$$

*In this case,*

$$\Lambda = C_p \left( \Delta \left( \sum_{k=1}^{p} \frac{1}{\lambda_1^k} - \frac{1}{r} \right) + \int_{\lambda_1}^{\infty} \frac{n(t) - \Delta t^\rho}{t^{p+1}} \mathrm{d}t + (-1)^{p-1} qp \right) . \tag{23}$$

*Proof.* The condition $C_\rho = 0$ is obtained by the proceeding proposition, as $\Delta \neq 0$. If we set $\phi(t) = n(t) - \Delta t^\rho$, we have

$$\int_{\lambda_1}^{\infty} \left( \frac{1}{t^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}} \right) (n(t) - \Delta t^\rho) \mathrm{d}t = \int_{\lambda_1}^{\infty} \left( \frac{1}{t^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}} \right) \phi(t) \mathrm{d}t$$

and $\phi(t) = o(t^{p-\epsilon})$. We have

$$\int_{\lambda_1}^{\infty} \left( \frac{1}{t^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}} \right) n(t) \mathrm{d}t = I_{A,a}(x) + \Phi_{A,a}(x)$$

where

$$\begin{aligned} I_{A,a}(x) &:= \int_{\lambda_1}^{\infty} \left( \frac{1}{t^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}} \right) \Delta t^\rho \mathrm{d}t , \\ \Phi_{A,a}(x) &:= \int_{\lambda_1}^{\infty} \left( \frac{1}{t^{p+1}} - \frac{1}{(Ax+a+t)^{p+1}} \right) \phi(t) \mathrm{d}t . \end{aligned}$$

We have

$$\lim_{x\to\infty} \Phi_{A,a}(x) = \int_{\lambda_1}^{\infty} \frac{\phi(t)}{t^{p+1}} \mathrm{d}t$$

and

$$
\begin{aligned}
I_{A,a}(x) &= \Delta \sum_{k=1}^{p} \frac{1}{\lambda_1^k} + u(x) + \Delta (Ax+a)^r \int_0^{\frac{Ax+a}{\lambda_1}} \frac{1}{1+s} \frac{\mathrm{d}s}{s^r} \\
&= \Delta \sum_{k=1}^{p} \frac{1}{\lambda_1^k} + u(x) + \Delta \frac{\pi}{\sin \pi r} (Ax+a)^r - \Delta (Ax+a)^r \int_{\frac{Ax+a}{\lambda_1}}^{\infty} \frac{1}{1+s} \frac{\mathrm{d}s}{s^r}
\end{aligned}
$$

where

$$
u(x) = \mathcal{O}\left(\frac{1}{x}\right) .
$$

We furthermore have

$$
\begin{aligned}
&\lim_{x\to\infty} (Ax+a)^r \int_{\frac{Ax+a}{\lambda_1}}^{\infty} \frac{1}{1+s} \frac{\mathrm{d}s}{s^r} = \frac{1}{r} , \\
&\lim_{x\to\infty} \Big( I_{A,a}(x) - \Delta \frac{\pi}{\sin \pi r} A^r x^r \Big) = \Delta \sum_{k=1}^{p} \frac{1}{\lambda_1^k} - \frac{\Delta}{r} .
\end{aligned}
$$

Hence

$$
\lim_{x\to\infty} S(x) = C_p \left( \Delta \left( \sum_{k=1}^{p} \frac{1}{\lambda_1^k} - \frac{1}{r} \right) + \int_{\lambda_1}^{\infty} \frac{n(t) - \Delta t^\rho}{t^{p+1}} \mathrm{d}t \right)
$$

which completes the proof. □

## 5. Multiple gamma functions

We shall investigate ratios of the form

$$
W_N(x) = \frac{\prod_{j=1}^{m} \Gamma_N(B_j x + b_j)}{\prod_{j=1}^{n} \Gamma_N(A_j x + a_j)}
$$

for non-negative parameters $a_1, b_1, \ldots, a_n, b_m$. Here $\Gamma_N$ denotes the so-called $N$-multiple gamma function with parameters $(1,1)$. See [9] and [2]. They can be defined through the recursive relations:

(1) $\Gamma_N(1) = 1$
(2) $\Gamma_1(z) = \Gamma(z)$
(3) $\Gamma_{N+1}(z+1) = \dfrac{\Gamma_{N+1}(z)}{\Gamma_N(z)}$

It is easy to see by the above definition that the reciprocal of $\Gamma_N(z)$ is an entire function of genus $N$, having zeros exactly at $z = -k$, $k \in \{0, 1, \ldots\}$. The third property gives $\Gamma_{N+1}(z) = \Gamma_{N+1}(z+1)\Gamma_N(z)$ and, as multiplication of the zeros implies addition of the multiplicities, we can see that the multiplicities of the zeros form a Pascal triangle, and the sequence of the multiplicities of the zeros of the $N$-multiple gamma function are its diagonals. Thus, the multiplicity of the zero at $-k$ for the $N$-multiple gamma function is $\binom{k+N-1}{N-1}$.

Corollary 1. *For*

$$
W_N(x) = \frac{\prod_{j=1}^{m} \Gamma_N(B_j x + b_j)}{\prod_{j=1}^{n} \Gamma_N(A_j x + a_j)}
$$

$\partial^N \log W_N$ *is completely monotonic if and only if*

$$
\sum_{j=1}^{m} B_j = \sum_{i=1}^{n} A_i \ , \ \sum_{j=1}^{n} A_j^p \log A_j - \sum_{j=1}^{m} B_j^p \log B_j \geq 0
$$

*and*

$$\sum_{i=1}^{n} \frac{e^{-a_i u/A_i}}{(1-e^{-u/A_i})^N} - \sum_{j=1}^{m} \frac{e^{-b_j u/B_j}}{(1-e^{-u/B_j})^N} \geq 0, \forall u > 0$$

*In the affirmative case, we have*

$$(-1)^{N+1}\partial^N \log W_N(x) = \int_0^\infty e^{-sx} s^{N-1} \left( \sum_{i=1}^{n} \frac{e^{-a_i u/A_i}}{(1-e^{-u/A_i})^N} - \sum_{j=1}^{m} \frac{e^{-b_j u/B_j}}{(1-e^{-u/B_j})^N} \right) \mathrm{d}s + \Lambda$$

*where*

$$\Lambda = \frac{1}{(N-1)!} \left( \sum_{j=1}^{n} A_j^p \log A_j - \sum_{j=1}^{m} B_j^p \log B_j \right)$$

*Proof.* This is a straightforward application of Theorem 1. First of all, the corresponding $h$ function for $\Gamma_N$ as in that theorem is summable, and in particular

$$h(s) = \sum_{k=1}^{\infty} e^{-\lambda_k s} = \sum_{k=0}^{\infty} \binom{k+N-1}{N-1} e^{-ks} = \frac{1}{(1-e^{-s})^N}$$

We have for $t > 0$ that

$$n(t) = \sum_{k=0}^{\lfloor t \rfloor} \binom{k+N-1}{N-1} = \binom{\lfloor t \rfloor + N}{N}$$

and

$$\lim_{t\to\infty} \frac{n(t)}{t^N} = \lim_{t\to\infty} \frac{\binom{\lfloor t \rfloor + N}{N} - 1}{t^N} = \lim_{t\to\infty} \frac{(\lfloor t \rfloor + N)!}{N!(\lfloor t \rfloor)! t^N} = \frac{1}{N!}$$

using Sterling's formula. Hence, we get

$$\Lambda = \frac{1}{(N-1)!} \left( \sum_{j=1}^{n} A_j^p \log A_j - \sum_{j=1}^{m} B_j^p \log B_j \right)$$

and hence the result by Theorem 1. □

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