Equivariant multiplications and idempotent splittings of G-spectra

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Abstract. This PhD thesis consists of two research papers, background material and perspectives for future research. In G-equivariant homotopy theory, there are many possible notions of an E_∞ ring spectrum, made precise by Blumberg and Hill's N_∞ rings. My main results are explicit descriptions of the maximal N_∞ ring structures of the idempotent summands of certain equivariant commutative ring spectra in terms of the subgroup lattice and conjugation in G. Algebraically, my results characterize the extent to which multiplicative induction on the level of homotopy groups is compatible with the idempotent splitting. Here, G always denotes a finite group.
In the first paper "Multiplicativity of the idempotent splittings of the Burnside ring and the G-sphere spectrum", the above program is carried out for the G-equivariant sphere spectrum. As an application, I obtain an explicit description of the multiplicativity of the idempotent splitting of the equivariant stable homotopy category.
In the second paper "Idempotent characters and equivariantly multiplicative splittings of K-theory", the above is established in the case of G-equivariant topological K-theory. The main new ingredient is a classification of the primitive idempotents of the p-local complex representation ring. It implies that all of these idempotents come from primitive idempotents of the Burnside ring, which is used to reduce the solution for K-theory to that for the sphere given in the first paper.

Resumé. Denne PhD afhandling består af to forskningsartikler, indledende materiale og perspektiver for fremtidig forskning. I G-ækvivariant homotopiteori er der mange mulige versioner af E_∞ ringspektre som kan gøres præcis via Blumberg og Hills N_∞ ringe. Mine hovedresultater er eksplicite beskrivelser af de maksimale N_∞ ringstrukturer på de idempotente summander af visse ækvivariante kommutative ringspektre med hensyn til undergrupper og konjugation i G. Algebraisk set karakteriserer mine resultater i hvilket omfang multiplikativ induktion på homotopigrupper er kompatibel med den idempotente opsplitning. Her betegner G altid en endelig gruppe.
I den første artikel "Multiplicativity of the idempotent splittings of the Burnside ring and the G-sphere spectrum" er det ovennævnte program realiséret for det G-ækvivariante sfærerspektrum. Som anvendelse opnår jeg en eksplicit beskrivelse af multiplikativiteten af den idempotente opsplitning af den stabile homotopikategori.
I den anden artikel "Idempotent characters and equivariantly multiplicative splittings of K-theory" er det ovennævnte etableret for G-ækvivariant K-teori. Den primære nye ingrediens er en klassifikation af den p-lokale komplekse repræsentationsrings primitive idempotenter. Det medfører at alle disse idempotenter kommer fra primitive idempotenter i Burnsideringen, hvilket er brugt for at reducére løsningen for K-teori til løsningen for sfæren fra den første artikel.

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Contents

Part 1

General introduction

My PhD research concerns multiplicative phenomena in equivariant stable homotopy theory. Loosely speaking, equivariant homotopy theory is the study of symmetry-preserving deformations. It is often convenient to analyze geometric objects with symmetries by assigning to them algebraic objects, equipped with binary operations such as addition and multiplication, in a deformation-invariant fashion. Non-equivariantly, ignoring symmetries, one way to make this precise is via the notion of a *commutative ring spectrum*. Similar to classical algebra, one can construct new ring spectra from others, e.g. by taking products or by inverting elements, a process known as "localization".

Equivariantly with respect to a finite group G, there are several possible notions of a commutative ring spectrum, the strongest being that of a *G-E_∞ ring spectrum*. Such an object R comes equipped not only with the usual homotopy-coherent n-ary multiplication maps $R^{\wedge n} \to R$, but additionally with *Hill-Hopkins-Ravenel norm maps* [**HHR16**, §2.3.2] that can be thought of as "equivariantly twisted multiplications". Hill and Hopkins observed that this kind of structure is not compatible with localization: even inverting a single element in the homotopy ring $\pi_*^G(R)$ might yield a G-spectrum that does not admit all possible norm maps anymore, see [**HH14**] and [**HH16**, §6].

An interesting class of examples arises from product decompositions: given a G-E_∞ ring spectrum R and a decomposition of $1 \in \pi_0^G(R)$ into a sum of idempotents e_i, R splits into a product of G-spectra of the form $R[e_i^{-1}]$. These are known to admit at least the weakest possible G-equivariant E_∞ ring structure and one can ask about the best possible structure available on each of these idempotent summands. My PhD research gives a detailed answer in the case of two of the most important G-equivariant ring spectra, the equivariant sphere spectrum $\mathbb{S}$ and equivariant topological K-theory in its complex version KU_G and its real version KO_G. See §1.6 for a more detailed description of the main objectives of my PhD project.

Organization: This PhD thesis is an amalgamation of two research papers, structured as follows: Part 1 is a general introduction to my thesis project. I provide some background material and state the main problem in §1 and summarize the contents of the two papers in §2 and §3, respectively. Finally, possible directions of future research are outlined in §4.

Part 2 consists of a copy of the first research paper,

> Paper A: *Multiplicativity of the idempotent splittings of the Burnside ring and the G-sphere spectrum*, cited as [**Böh18b**].

Part 3 consists of a copy of the second research paper,

> Paper B: *Idempotent characters and equivariantly multiplicative splittings of K-theory*, cited as [**Böh18a**].

Publication information: A previous, mostly identical version of Paper A is available as the electronic preprint [**Böh18c**] and has been submitted to "Advances in Mathematics". The updated version included here provides a simpler formulation of Theorem A suggested by Malte Leip, a slightly improved proof of that result, as well as small cosmetic changes. Paper B is identical with the arXiv preprint [**Böh18a**].

1. Background material

This section provides some background material. I recall equivariant operads and the Hill-Hopkins-Ravenel norms in §1.2, the induced norms on homotopy groups in §1.3, review the more general notions of N_∞ ring spectra and incomplete Tambara functors in §1.4 and discuss the behavior of norms under localization in §1.5. Finally, §1.6 collects some facts about idempotent splittings and states the main problem to be solved in this thesis.

The reader is assumed to be familiar with the notion of a *genuine G-spectrum*, see e.g. [**GM95**, §3], [**MM02**], [**Sch**] and the appendices of [**HHR16**].

1.1. Conventions. Unless otherwise stated, G will always a denote a finite group. Most results of my two thesis papers are model-independent; for some parts of Paper A, though, it was convenient to work in the Mandell-May category of orthogonal G-spectra [**MM02**, §II.2], equipped with the positive complete stable model structure [**HHR16**, §B.4.1]. Cofibrancy conditions are only stated informally in Part 1; the reader is referred to the two papers for details.

Restriction-to-subgroup functors are denoted as Res^G_H or as Res_H when G is clear from the context. For brevity, I write R^G_H or R_H for the induced restriction operations on homotopy groups.

When working locally, P denotes a collection of prime numbers. I write $\mathbb{Z}_{(P)} := \mathbb{Z}\left[p^{-1} \mid p \notin P\right]$ for the P-local integers, and $A_{(P)} := A \otimes \mathbb{Z}_{(P)}$ for the P-localization of a commutative ring A, cf. [**Böh18b**, §3.1].

1.2. Equivariant operads and norm maps. By a *G-operad* I mean an operad in the category of G-spaces endowed with its symmetric monoidal structure given by cartesian product. Call a map of G-operads $f\colon \mathcal{O} \to \mathcal{O}'$ an equivalence if $f(n)^K$ is an equivalence for all $n \geq 0$ and all subgroups $K \leq G \times \Sigma_n$. As usual, we let G-operads act on G-spectra via the (symmetric monoidal) infinite suspension functor $\Sigma^\infty_+\colon \mathrm{Top}^G \to \mathrm{Sp}^G$ from G-spaces to genuine G-spectra.

Definition 1.1. A subgroup $\Gamma \leq G \times \Sigma_n$ is called a *graph subgroup* if $\Gamma \cap (1 \times \Sigma_n) = 1$, or equivalently, if Γ is the graph of some group homomorphism $H \to \Sigma_n$ for some subgroup $H \leq G$.

Definition 1.2. A G-operad $\mathcal{O}$ is called a G-E_∞ *operad* if the following holds: for every $n \geq 0$, the n-th level has fixed points $(\mathcal{O}(n))^K$ a contractible space if $K \leq G \times \Sigma_n$ is a graph subgroup, and is empty otherwise.

It is easy to see that any two G-E_∞ operads are equivalent. By a G-E_∞ *ring spectrum* we mean an algebra in G-spectra over any G-E_∞ operad.

Example 1.3. Let $\mathcal{U}$ be a universe for the group G (i.e., $\mathcal{U} \cong \bigoplus_{\mathbb{N}}(\mathbb{R} \oplus V)$ for some finite-dimensional G-representation V). The *linear isometries operad* on $\mathcal{U}$ is a G-operad as follows: its n-th level is the G-space of linear isometric embeddings $\mathcal{U}^{\oplus n} \to \mathcal{U}$, with G acting by conjugation, and the structure maps are induced by orthogonal direct sum of maps and composition.

For any choice of G-E_∞ operad $\mathcal{O}$, the homotopy category of $\mathcal{O}$-algebras in G-spectra is equivalent to the homotopy category of strict commutative monoids in orthogonal G-spectra, see [**GW**, Thm. 6.3], [**BH15a**, Thm. 3.16] and cf. [**EKMM97**, Thm. II.4.6]. In particular, any G-E_∞ ring spectrum R comes equipped with multiplication maps $R^{\wedge n} \to R$ for $n \geq 0$ which are unique up to homotopy and are unital, associative and commutative up to coherent higher homotopy. In contrast to the non-equivariant setting, there is even more structure: for any nested subgroups $K \leq H \leq G$, the restriction functor from H-E_∞ rings to K-E_∞ rings admits a left adjoint $\bigwedge_{H/K}(-)$, the *Hill-Hopkins-Ravenel norm functor* [**HHR16**, §2.3.2]. The counits N_K^H of these (Quillen) adjunctions then give rise to *Hill-Hopkins-Ravenel norm maps*

$$G_+ \wedge_H N_K^H \colon G_+ \wedge_H \bigwedge_{H/K} \mathrm{Res}^G_K(R) \to R$$

which can be thought of as "equivariantly twisted" multiplication maps that satisfy similar coherence conditions.

More generally, for any finite H-set T, one can define a norm map

$$G_+ \wedge_H N_T \colon G_+ \wedge_H \bigwedge_T \mathrm{Res}^G_T(R) \to R$$

that specializes to the norm map $G_+ \wedge_H N_K^H$ if $T = H/K$ and to the usual "untwisted" multiplication if the action on T is trivial. Any choice of orbit decomposition

$$T \cong \coprod_i H/K_i$$

yields a natural equivalence between $G_+ \wedge_H N_T$ and the functor

$$G_+ \wedge_H \bigwedge_i N_{K_i}^H \colon G_+ \wedge_H \bigwedge_i \bigwedge_{H/K_i} \mathrm{Res}^G_{K_i}(R) \to R,$$

but it is also possible to define $G_+ \wedge_H N_T$ in a coordinate-free fashion. I refer to **[HHR16**, §B.5.1] and [**BH15b**] for details.

Following [**BH15b**], I now explain how norm maps $G_+ \wedge_H N_T$ arise from the operad action on R.

NOTATION 1.4. Fix a subgroup $H \leq G$. For a finite H-set T of cardinality n, write $\varphi_T \colon H \to \Sigma_n$ for the action homomorphism and $\Gamma_T \leq G \times \Sigma_n$ for its graph subgroup, i.e., the set $\{(h, \varphi(h)) \mid h \in H\}$.

Now let R be an algebra in G-spectra over some fixed G-E_∞ operad $\mathcal{O}$. Every finite H-set T gives rise to a norm map

$$G_+ \wedge_H N_T \colon G_+ \wedge_H \bigwedge_T \mathrm{Res}_T(R) \to R,$$

unique up to homotopy, as follows. Since Γ_T is a graph subgroup, the space $\mathcal{O}(n)^{\Gamma_T} \cong map_{G\times\Sigma_n}((G \times \Sigma_n)/\Gamma_T, \mathcal{O}(n))$ is contractible. Applying the functor $(-) \wedge_{\Sigma_n} R^{\wedge n}$ then yields a contractible space of maps

$$(G \times \Sigma_n)/\Gamma_T \wedge_{\Sigma_n} R^{\wedge n} \to \mathcal{O}(n) \wedge_{\Sigma_n} R^{\wedge n}.$$

The left hand side is known to be one possible construction of the norm functor $G_+ \wedge_H \bigwedge_T \mathrm{Res}_T(R)$, so post-composing the above map with the operadic structure map $\mathcal{O}(n) \wedge_{\Sigma_n} R^{\wedge n} \to R$ yields a norm map $G_+ \wedge_H N_T$ as desired, unique up to homotopy. See [**BH15b**, §6.1] for further details.

1.3. Norms on homotopy groups. We keep the notation of the previous subsection. Let $K \leq H \leq G$ be subgroups. The norm map N_K^H for R induces a multiplicative transfer

$$N_K^H \colon \pi_V^K(R) \to \pi_{\mathrm{Ind}(V)}^H(R),$$

on the equivariant homotopy groups of R, where $\mathrm{Ind}(V) = \mathrm{Ind}_K^H(V)$ is the induced representation. It is also called the *norm* and was first studied by Greenless and May [**GM97**]. It can be obtained from the Hill-Hopkins-Ravenel norm map as follows: given a class in $\pi_V^K(R)$ represented by a map of K-spectra $f \colon \mathbb{S}^V \to R$, define the norm $N_K^H([f])$ to be the class of the composite of maps of H-spectra

$$\mathbb{S}^{\mathrm{Ind}_K^H(V)} \simeq \bigwedge_{H/K} \mathrm{Res}_K^G(\mathbb{S}^V) \to \bigwedge_{H/K} \mathrm{Res}_K^G(R) \to \mathrm{Res}_H^G(R),$$

where the first identification uses the monoidality of the norm functor, the second map is induced by f and the third is the norm map.

It is well-known that the collection of equivariant homotopy groups $\{\pi_0^H(X) \mid H \leq G\}$ of any G-spectrum X together with restriction maps and transfers forms a *Mackey* *functor* $\underline{\pi}_0(X)$, see e.g. [**May96**, §XII.6], [**HHR16**, §3.1]. If $X = R$ is an equivariant E_∞ ring spectrum in a naive sense, then $\underline{\pi}_0(R)$ is even a *Green ring*, i.e., all groups $\pi_0^H(R)$ come equipped with commutative ring structures that are compatible with restrictions and transfers. We have just seen that in the case of a G-E_∞ ring spectrum, $\underline{\pi}_0(R)$ is moreover endowed with multiplicative norms. Brun [**Bru07**] showed that these make $\underline{\pi}_0(X)$ into a *Tambara functor*. See [**Böh18b**, §2.3] for definitions and further details and references.

1.4. **N_∞ ring spectra and incomplete Tambara functors.** In [**BH18**], Blumberg and Hill generalized the notion of a G-E_∞ operad in order to describe G-equivariant ring spectra that only come equipped with a partial collection of norm maps for certain equivariant sets. As we saw in the last subsection that norm maps arise from the fixed points of the operad, it is natural to expect that operads with fewer fixed points structure ring spectra with fewer norms.

DEFINITION 1.5. Let $\mathcal{F}$ be a family of subgroups of G, i.e., a collection of subgroups that is closed under conjugation and under taking subgroups. A *universal space* for $\mathcal{F}$ is a G-space $E\mathcal{F}$ such that the fixed points $(E\mathcal{F})^H$ are empty if $H \notin \mathcal{F}$ and weakly contractible if $H \in \mathcal{F}$.

By definition, the n-th space in a G-E_∞ operad is a universal space for the family of all graph subgroups of $G \times \Sigma_n$.

DEFINITION 1.6 ([**BH15b**], Def. 1.1). A G-operad $\mathcal{O}$ is called an N_∞ *operad* if the following holds: Each $\mathcal{O}(n)$ is a universal space for a family $\mathcal{F}(n)$ of subgroups of $G \times \Sigma_n$ such that $\mathcal{F}(n)$ is contained in the family of all graph subgroups, and contains at least all graph subgroups corresponding to equivariant sets with trivial action.
Call an H-set T of cardinality n an *admissible set* for $\mathcal{O}$ if Γ_T is contained in $\mathcal{F}(n)$.

Thus, algebras over arbitrary N_∞ operads interpolate between G-equivariant commutative ring spectra that only have multiplication maps in the non-equivariant sense and those that admit a full collection of norm maps for all equivariant sets. The former are often called *naive* E_∞ ring spectra, whereas the latter are the G-E_∞ rings defined above, sometimes also called *complete* E_∞ ring spectra. Examples of N_∞ operads that are neither naive nor complete arise from localization, see §1.5 below.

Write $\mathcal{S}et^H$ for the category of finite H-sets with its monoidal structure given by disjoint union. The data of the admissible sets of an N_∞ operad $\mathcal{O}$ can be stored conveniently as a coefficient system

$$\mathcal{O}rb_G^{op} \to \text{Symmetric monoidal categories}$$

that assigns to an object G/H of the orbit category $\mathcal{O}rb_G$ the full symmetric monoidal subcategory of $\mathcal{S}et^H$ spanned by the admissible H-sets of $\mathcal{O}$. The axioms of an operad impose certain closure conditions on these collections of admissible sets, axiomatized by Blumberg and Hill in the notion of an *indexing system*. The reader is referred to [**Böh18b**, Def. 2.5] or [**BH15b**, Def. 3.22] for a precise definition. Note that the collection of all possible indexing system forms a poset under inclusion.

The homotopy type of an N_∞ operad is completely determined by its associated indexing system. Conversely, any indexing system is the collection of admissible sets of some N_∞ operad. This has been made precise as follows:

Theorem 1.7 (Blumberg-Hill et al.). *The functor from the homotopy category of N_∞ operads (with respect to the above notion of weak equivalence) to the poset of indexing systems which assigns to each N_∞ operad its collection of admissible sets is an equivalence of categories.*

The fully faithfulness was proven in [**BH15b**, Thm. 3.24]. The essential surjectivity was conjectured by Blumberg and Hill and proven independently by Gutiérrez-White [**GW**, Thm. 4.7], Rubin [**Rub17**, Thm. 3.3] and Bonventre-Pereira [**BP17**, Cor. IV].

As an algebraic counterpart of N_∞ rings, Blumberg and Hill [**BH18**] introduced the notion of an *incomplete $\mathcal{I}$-Tambara functor* structured by an indexing system $\mathcal{I}$. It is a Green ring equipped with additional multiplicative transfers, or norms, N_T for all admissible sets T. Thus, $\mathcal{I}$-Tambara functors for varying $\mathcal{I}$ interpolate between Green rings and (complete) Tambara functors. If R is an algebra over an N_∞ operad with indexing system $\mathcal{I}$, then $\underline{\pi}_0(R)$ is an $\mathcal{I}$-Tambara functor [**BH18**, Thm. 1.6]. See also [**Böh18b**, §2.3] for more details.

1.5. Norms and localization. Let R be a naive E_∞ ring spectrum and let $x \in \pi_V^G(R)$ be an element of its homotopy ring. Using a choice of multiplication map $\mu\colon R \wedge R \to R$, we obtain a "multiplication by x" map

$$\cdot x\colon R \simeq R \wedge \mathbb{S} \xrightarrow{\mathrm{id}\wedge x} R \wedge R \wedge \mathbb{S}^{-V} \xrightarrow{\mu} R \wedge \mathbb{S}^{-V},$$

unique up to homotopy.

Definition 1.8. The *localization* $R[x^{-1}]$ is the sequential homotopy colimit

$$\operatorname{hocolim}(R \xrightarrow{\cdot x} R \wedge \mathbb{S}^{-V} \xrightarrow{\cdot x} R \wedge \mathbb{S}^{-(V\oplus V)} \to \dots).$$

On homotopy groups, $\underline{\pi}_0(R[x^{-1}])$ recovers the levelwise localization of the Green ring $\underline{\pi}_0(R)$, whence the name.

Hill and Hopkins [**HH14**] observed that inverting homotopy elements can destroy the structure of a G-E_∞ operad, cf. [**HH16**, §6]: the localization $R[x^{-1}]$ does not typically admit all norm maps for all equivariant sets anymore. In other words, the indexing system of the maximal N_∞ operad acting on $R[x^{-1}]$ might be strictly smaller than the maximal indexing system structuring R. A precursor to their result is McClure's observation [**McC96**] that the Tate construction preserves naive E_∞ ring structures, but not G-E_∞ ring structures. Hill and Hopkins gave the following necessary and sufficient conditions for localization to be compatible with norms:

THEOREM 1.9 ([**HH14**], §4). *Let R be a G-E_∞ ring spectrum. Fix $x \in \pi_*^G(R)$. Then the following are equivalent:*

(i) The localization $R[x^{-1}]$ is a G-E_∞ ring spectrum under R.
(ii) For all $K \leq H \leq G$, the element $N_K^H(R_K^G(x))$ divides a power of $R_H^G(x)$ in $\pi_*^H(R)$.

I will refer to the conditions given in (ii) as the *Hill-Hopkins conditions* for x.

Building on recent work of Gutiérrez and White [**GW**, Cor. 7.10], I show in [**Böh18b**, Prop. 2.30] that Theorem 1.9 holds for any N_∞ structure on the sphere spectrum $R = \mathbb{S}$ in the expected way, by testing condition (ii) only for those $K \leq H$ such that H/K is admissible. Bachmann and Hoyois gave an ∞-categorical proof of a similar, but more general theorem [**BH17**, Prop. 12.6] in the setting of motivic homotopy theory, which might be adapted to the equivariant world.

If x is an idempotent, Lemma [**Böh18b**, Lemma 4.11] translates the Hill-Hopkins conditions into equations that are a bit easier to check in practice:

LEMMA 1.10. *Let $e \in \pi_0^G(R)$ be idempotent. Then $N_K^H(R_K^G(e))$ divides $R_H^G(e)$ in $\pi_0^H(R)$ if and only if $N_K^H(R_K^G(e)) \cdot R_H^G(e) = R_H^G(e)$.*

1.6. Idempotent splittings. Recall that a commutative ring R splits as a product of rings if and only it contains an idempotent element e other than zero and one. Call e *primitive* if it cannot be written as a sum of non-zero idempotents. Assume that R only has finitely many primitive idempotents $\{e_i\}$. Then there is an idempotent splitting

$$R \cong \prod_i e_i \cdot R.$$

Note that since $e_i = e_i^2$, the ideal $e_i \cdot R$ is a ring with unit e_i which identifies with the localization $R[e_i^{-1}]$. The following is well-known:

LEMMA 1.11. *The primitive idempotents of R are in canonical bijection with the subsets of* $\mathrm{Spec}(R)$ *that are both open and closed. If there are only finitely many of these, then they agree with the connected components of the space* $\mathrm{Spec}(R)$.

Now let R be a G-E_∞ ring spectrum such that the commutative ring $\pi_0^G(R)$ has only finitely many primitive idempotents $\{e_i\}$. Then the idempotent splitting of $\pi_0^G(R)$

gives rise to a splitting of Green rings

$$\underline{\pi}_0(R) \cong \prod_i \underline{\pi}_0(R)[e_i^{-1}] \tag{1.12}$$

!here localization is taken levelwise. Consequently, the canonical maps $R \to R[e_i^{-1}]$ assemble into an equivalence of G-equivariant naive E_∞ ring spectra

$$R \simeq \prod_i R[e_i^{-1}].$$

But R is a G-E_∞ ring, so $\underline{\pi}_0(R)$ is a Tambara functor, and it is natural to ask whether the above splitting is just a splitting of Green functors or preserves some additional structure of non-trivial norm maps.

Main question (algebraic formulation): What is the maximal incomplete Tambara functor structure that the idempotent summand $\underline{\pi}_0(R)[e_i^{-1}]$ inherits from $\underline{\pi}_0(R)$? What is the maximal incomplete Tambara functor structure preserved by the splitting?

Main question (homotopical formulation): What is the maximal N_∞ ring structure that the idempotent summand $R[e_i^{-1}]$ inherits from R? What is the maximal N_∞ ring structure preserved by the splitting?

Theorem 1.9 and Lemma 1.10 reduce the question to understanding certain multiplicative relations involving norms in the equivariant homotopy ring of R. In general, the computation of norms and of such relations is far from being easy.
The goal of my thesis project is to find explicit answers to the main question in some of the most fundamental examples of G-E_∞ ring spectra. Paper A covers the case of the G-equivariant sphere spectrum $\mathbb{S}$, while Paper B covers G-equivariant topological K-theory in both its complex variant KU_G and its real variant KO_G. We summarize the content of these two papers in the next section.

2. Summary of Paper A

The first article [**Böh18b**] answers the main question in the case of the G-equivariant sphere spectrum $\mathbb{S}$. We quickly summarize its contents.

2.1. Idempotents in the Burnside ring. The 0-th homotopy Tambara functor $\underline{\pi}_0(\mathbb{S})$ can be identified with the Burnside ring Tambara functor $A(-)$, which goes back to Segal [**Seg71**, Cor. of Prop. 1]. Dress [**Dre69**, Prop. 2] proved that the primitive idempotent elements $e_L \in A(G)$ are in bijection with the conjugacy classes of perfect subgroups $L \leq G$. He also derived an analogous statement in the P-local case. All of this is discussed in detail in [**Böh18b**, §3].

2.2. Understanding the Hill-Hopkins conditions via group theory. The technical heart of the paper is an analysis of norms in the Burnside ring Tambara functor $A(-)$. More precisely, we derive a non-obvious reformulation of the Hill-Hopkins conditions that only depends on the subgroup structure and conjugation in G and can be stated without reference to norms or the multiplication in $A(-)$.

THEOREM 2.1 ([**Böh18b**], Thm. 4.1). *Fix a P-perfect subgroup $L \leq G$ and arbitrary subgroups $K \leq H \leq G$. Then the norm N_K^H for $A(-)_{(P)}$ descends to a norm $\tilde{N}_K^H$ for the idempotent summand $A(-)_{(P)}[e_L^{-1}]$ if and only if the following holds: whenever $L' \leq H$ is conjugate in G to L, then L' is contained in K.*

DEFINITION 2.2. Call H/K *admissible* for e_L if K and H satisfy the equivalent statements of Theorem 2.1.

REMARK 2.3. The condition given in Theorem 2.1 is not quite the condition given in the arXiv version [**Böh18c**, Thm. 4.1], but rather an easy and equivalent reformulation suggested by Malte Leip, cf. [**Böh18a**, Lemma 3.4]

The key idea in the proof of Theorem 2.1 is the following: for an idempotent $e \in A(G)$, we can translate the Hill-Hopkins conditions for e,

$$N_K^H R_K^G(e) \text{ divides } R_H(e) \text{ in } A(H),$$

into various equations of integers, using the injectivity of the homomorphism of marks $\phi\colon A(H) \to \prod \mathbb{Z}$ and [**Böh18b**, Lemma 4.11]. Since e is idempotent, these integers can only be 0 or 1. I compute them explicitly using Dress' description of the marks of e, a multiplicative version of the double coset formula, and some well-known properties of the norm. Note that this strategy of proof applies without changes in the P-local case. See [**Böh18b**, §4.2] for details.

2.3. Further results. The other main results of the paper besides Theorem 2.1 can be summarized as follows. In all statements, the "maximality" refers to the poset of all indexing systems for G, cf. Theorem 1.7.

(1) The admissible sets for e_L form an indexing system $\mathcal{I}_L$ ([**Böh18b**, Thm. 4.20]).
(2) The idempotent summand $A(-)_{(P)}[e_L^{-1}]$ is an $\mathcal{I}_L$-Tambara functor, maximally so ([**Böh18b**, Thm. 4.20]).
(3) The splitting
$$A(-)_{(P)} \cong \prod A(-)_{(P)}[e_L^{-1}]$$
is an isomorphism of $\mathcal{I}$-Tambara functors, maximally so, where $\mathcal{I} = \bigcap \mathcal{I}_L$ ([**Böh18b**, Cor. 4.24]).

(4) For any (suitably cofibrant) N_∞ operad $\mathcal{O}_L$ realizing $\mathcal{I}_L$, the idempotent summand $\mathbb{S}_{(P)}[e_L^{-1}]$ is an $\mathcal{O}_L$-algebra, maximally so ([**Böh18b**, Cor. 4.26]).
(5) The splitting
$$\mathbb{S}_{(P)} \simeq \prod \mathbb{S}_{(P)}[e_L^{-1}]$$
is an equivalence of $\mathcal{O}$-algebras, maximally so, where $\mathcal{O}$ is a (suitably cofibrant) N_∞ operad realizing $\mathcal{I}$ ([**Böh18b**, Cor. 4.29]).

The operads $\mathcal{O}_L$ and $\mathcal{O}$ always exist, see [**Böh18b**, Thm. 2.10] and the references given there, and the cofibrancy assumption can always be guaranteed, see [**Böh18b**, Rem. 2.11].

2.4. The idempotent splitting of the category of genuine G-spectra. I give an application of my results: Since $\mathbb{S}$ is the monoidal unit in the category of G-spectra, every G-spectrum is naturally a module over $\mathbb{S}$, so the idempotent splitting of $\mathbb{S}$ induces a splitting of the entire category of G-spectra by breaking it up into categories of modules over the idempotent summands $\mathbb{S}[e_L^{-1}]$:
$$\mathrm{Sp}^G \simeq \prod_{(L)} \mathrm{Mod}(\mathbb{S}[e_L^{-1}])$$
Upon restriction, the idempotents e_L also induce splittings of the categories of H-spectra for all $H \leq G$. Building on work by Blumberg and Hill [**BH15a**], I show in [**Böh18b**, Cor. 6.1] that all the norm functors $\bigwedge_T$ parametrized by the admissible sets T of the indexing system $\mathcal{I}$ from §2.3 (3) are compatible with this splitting.

3. Summary of Paper B

The second article [**Böh18a**] answers the main question for G-equivariant complex topological K-theory KU_G and G-equivariant real topological K-theory KO_G. This can be seen as a sequel to the first paper, as the solution here reduces to the one given there.

In this summary, I will only state the "complex" version my results, but all of what follows holds without changes for real K-theory KO_G and the real representation ring $RO(G)$.

3.1. Idempotents in the representation ring. The 0-th homotopy Tambara functor $\underline{\pi}_0((KU_G)_{(P)})$ of P-local complex K-theory is the complex representation ring Tambara functor $RU(G)_{(P)}$. Write $lin\colon A(G)_{(P)} \to RU(G)_{(P)}$ for the *linearization map* given by sending a finite G-set to its associated permutation representation. Recall from §2.1 that Dress gave a bijection between the conjugacy classes of P-perfect subgroups $L \leq G$ and the primitive idempotents $e_L \in A(G)_{(P)}$. As a first main result, I show in [**Böh18a**, Thm. 1.2] that the elements $lin(e_C) \in RU(G)_{(P)}$ are precisely the primitive idempotents in the complex representation ring, where C runs over the conjugacy classes of cyclic P-perfect subgroups $C \leq G$ (whereas the other e_L for non-cyclic L are in the kernel). Parts of the statement of [**Böh18a**, Thm. 1.2] were already known, see Remark [**Böh18a**, Rem. 1.3].

I use my classification of idempotents to reduce the analysis of the Hill-Hopkins conditions in the case of the representation ring to the one for the Burnside ring, as explained in detail in [**Böh18a**, §3.2].

3.2. Splitting results. The main results regarding the multiplicative properties of the idempotent splittings of the representation ring and K-theory can be summarized as follows. As before, the "maximality" refers to the poset of all indexing systems for G. Let $C \leq G$ denote a cyclic P-perfect subgroup and let $\mathcal{I}_C$ be the indexing system introduced in §2.3 (1).

(1) The idempotent summand $RU(-)_{(P)}[e_C^{-1}]$ is an $\mathcal{I}_C$-Tambara functor, maximally so ([**Böh18a**, Thm. 3.8]).
(2) The splitting
$$RU(-)_{(P)} \cong \prod RU(-)_{(P)}[e_C^{-1}]$$
is an isomorphism of $\mathcal{I}_{cyc}$-Tambara functors, maximally so, where $\mathcal{I}_{cyc} = \bigcap \mathcal{I}_C$ ([**Böh18a**, Prop. 3.11]).
(3) For any (suitably cofibrant) N_∞ operad $\mathcal{O}_C$ realizing $\mathcal{I}_C$, the idempotent summand $\mathbb{S}_{(P)}[e_C^{-1}]$ is an $\mathcal{O}_C$-algebra, maximally so ([**Böh18a**, Thm. 4.3]).
(4) The splitting
$$(KU_G)_{(P)} \simeq \prod (KU_G)_{(P)}[e_C^{-1}]$$
is an equivalence of $\mathcal{O}_{cyc}$-algebras, maximally so, where $\mathcal{O}_{cyc}$ is a (suitably cofibrant) N_∞ operad realizing $\mathcal{I}_{cyc}$ ([**Böh18a**, Cor. 4.6]).

4. Perspectives

I briefly sketch some possible future directions that extend my thesis project in a natural way.

4.1. Compact Lie groups. Much of G-equivariant homotopy theory can be carried out for any compact Lie group G. It is natural to ask whether my thesis results also hold in this generality. However, there are two caveats: It seems that it is only possible to define norm maps N_K^H when the index of K in H is finite, and moreover, the topology on G has to be taken into account when studying the idempotents in $A(G)$, as I now explain.

Tom Dieck [**tD78**] classified the primitive idempotents in the Burnside ring of a compact Lie group. They can be obtained from a certain space of conjugacy classes of perfect closed subgroups of G [**tD78**, Thm. 1], where the topology is induced from that of the underlying space of G. If the number of idempotents is finite, then there is a product decomposition similar to the one in Equation 1.12 above. In many cases though, there are infinitely many idempotents and hence the element 1 cannot be written as a sum of the primitive idempotents. One can ask the following:

QUESTION 4.1. Do Theorem 2.1 and the statements (1), (2) and (4) of §2.3 hold more generally for any compact Lie group G and any norm $\tilde{N}_K^H$ along a finite index inclusion $K \leq H$? Do (3) and (5) hold for all compact Lie groups G such that the splitting exists?

My proof of [**Böh18a**, Thm. 1.2] builds on work by Atiyah [**Ati61**, Prop. 6.4] on the representation ring of a finite group. Segal [**Seg68**] generalized some of Atiyah's results to compact Lie groups. One might hope that my strategy of proof given in [**Böh18a**, §2] can be generalized to compact Lie groups as well. It should then be possible to answer the following:

QUESTION 4.2. For a compact Lie group G, are the primitive idempotents of $RU(G)_{(P)}$ precisely the elements $lin(e_C)$, where $e_C \in A(G)_{(P)}$ is tom Diecks's primitive idempotent associated to a P-perfect Cartan subgroup $C \leq G$ in the sense of Segal [**Seg68**, Def. 1.1]?

Here, $lin\colon A(G)_{(P)} \to RU(G)_{(P)}$ can be taken to be the map induced by the unit map $\mathbb{S}_{(P)} \to (KU_G)_{(P)}$ on 0-th homotopy groups, but it should also be possible to describe it more directly in terms of equivariant Euler characteristics, cf. e.g. [**tD79**, §5.3] or [**LMS86**, V.§1]. Assuming Question 4.2 can be answered positively, one can again study splitings of K-theory:

QUESTION 4.3. Do the statements (1) and (3) of §3.2 hold more generally for any compact Lie group G and any norm $\tilde{N}_K^H$ along a finite index inclusion $K \leq H$? Moreover, do (2) and (4) hold for all compact Lie groups G such that the splitting exists?

4.2. Other idempotent splittings. There are many other G-E_∞ rings and Tambara functors and it might be possible to obtain explicit descriptions of the multiplicativity of their idempotent splittings as well. Interesting candidates might include appropriate G-equivariant versions of complex cobordism or of algebraic K-theory. Their homotopy rings are only known in a few special cases, though.
Algebraically, it would be interesting to study the idempotent splittings of other repre–sentation-theoretic gadgets such as the Brauer character ring for representations in positive characteristic and analyze potential implications for modular representation theory.

4.3. Explicit constructions of N_∞ operads. Typical examples of N_∞ operads include the linear isometries operads on G-universes and G-equivariant versions of the classical little disks and Steiner operads, but it is known that not all homotopy types of N_∞ operads arise in such a way [**BH15b**, Def. 3.11, §4.3]. The various proofs of the essential surjectivity part of Theorem 1.7 are not very explicit and it would be desirable to find geometrically defined operads that realize all possible indexing systems.

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Part 2

Paper A

MULTIPLICATIVITY OF THE IDEMPOTENT SPLITTINGS OF THE BURNSIDE RING AND THE G-SPHERE SPECTRUM

BENJAMIN BÖHME

ABSTRACT. We provide a complete characterization of the equivariant commutative ring structures of all the factors in the idempotent splitting of the G-equivariant sphere spectrum, including their Hill-Hopkins-Ravenel norms, where G is any finite group. Our results describe explicitly how these structures depend on the subgroup lattice and conjugation in G. Algebraically, our analysis characterizes the multiplicative transfers on the localization of the Burnside ring of G at any idempotent element, which is of independent interest to group theorists. As an application, we obtain an explicit description of the incomplete sets of norm functors which are present in the idempotent splitting of the equivariant stable homotopy category.

1. INTRODUCTION

Let G be a finite group and recall that the zeroth G-equivariant homotopy group $\pi_0^G(\mathbb{S})$ of the G-sphere spectrum identifies with the Burnside ring $A(G)$ [Seg71]. Dress' classification [Dre69] of the primitive idempotent elements $e_L \in A(G)$ in terms of perfect subgroups $L \leq G$ gives rise to a splitting of G-spectra

$$\mathbb{S} \simeq \prod_{(L)\leq G} \mathbb{S}[e_L^{-1}] \tag{1.1}$$

where the localization $\mathbb{S}[e_L^{-1}]$ is the sequential homotopy colimit

$$\text{hocolim}(\mathbb{S} \xrightarrow{e_L} \mathbb{S} \xrightarrow{e_L} \dots)$$

along countably many copies of (a representative of) e_L. The present paper investigates the multiplicative nature of this splitting.

The sphere is a commutative monoid in any good symmetric monoidal category of G-spectra and hence admits the structure of a G-E_∞ *ring spectrum*, i.e., it comes equipped with a full set of *Hill-Hopkins-Ravenel norm maps*

$$N_K^H\colon \bigwedge_{H/K} \text{Res}_K \mathbb{S} \to \text{Res}_H \mathbb{S}$$

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for all $K \leq H \leq G$. These are equivariantly commutative multiplication maps which feature prominently in the solution to the Kervaire invariant problem [HHR16]. The resulting norms on homotopy groups first appeared in [GM97]. They are multiplicative transfer maps

$$N_K^H\colon \pi_0^K(\mathbb{S}) \cong A(K) \to A(H) \cong \pi_0^H(\mathbb{S})$$

which equip $\underline{\pi}_0(\mathbb{S}) \cong A(-)$ with the structure of a *Tambara functor* [Tam93] (and agree with the multiplicative transfers of $A(-)$ induced by co-induction of finite G-sets, see Section 3).

It is known that norm maps behave badly with respect to Bousfield localization of spectra and levelwise localization of Tambara functors, see Example 2.23. Thus, it is natural to ask about the equivariant multiplicative behavior of the idempotent splitting (1.1). Throughout the paper, we will decorate the norms of a localization with a tilde to distinguish them from the norms of the original object.

Question 1.2 (Main question, homotopy-theoretic formulation)**.** For which nested subgroups $K \leq H \leq G$ does the norm map N_K^H of $\mathbb{S}$ descend to a norm map

$$\tilde{N}_K^H\colon \bigwedge_{H/K} \mathrm{Res}_K \mathbb{S}[e_L^{-1}] \to \mathrm{Res}_H \mathbb{S}[e_L^{-1}]$$

on the idempotent localization $\mathbb{S}[e_L^{-1}]$, and which norms are preserved by the idempotent splitting (1.1)?

Question 1.3 (Main question, algebraic formulation)**.** For which nested subgroups $K \leq H \leq G$ does the Green ring $\underline{\pi}_0\mathbb{S}[e_L^{-1}] \cong A(-)[e_L^{-1}]$ inherit a norm map $\tilde{N}_K^H$ from that of $A(-)$, and which norms are preserved by the idempotent splitting

$$A(-) \cong \prod_{(L)\leq G} A(-)[e_L^{-1}]?$$

We now state our main results which provide an explicit and exhaustive answer to both questions. All of our results hold locally for any collection of primes inverted. For simplicity, we only include the integral statements in the introduction.

1.1. **Statement of algebraic results.** The following result will be restated as Theorem 4.1, including the local variants.

Theorem A. *Let $L \leq G$ be a perfect subgroup and let $e_L \in A(G)$ be the corresponding primitive idempotent given by Dress' classification of idempotents in $A(G)$ (see Theorem 3.4). Fix subgroups $K \leq H \leq G$. Then the norm map $N_K^H\colon A(K) \to A(H)$ descends to a well-defined map of multiplicative monoids*

$$\tilde{N}_K^H\colon A(K)[e^{-1}] \to A(H)[e^{-1}]$$

if and only if the following holds:

($\star$) *Whenever $L' \leq H$ is conjugate in G to L, then L' is contained in K.*

Theorem A builds on previous work by Hill-Hopkins [HH14] and Blumberg-Hill [BH18, Section 5.4] which reduced the question to understanding certain division relations between norms and restrictions of the elements $e_L \in \pi_0^G(\mathbb{S})$, but did not make explicit the relationship with the subgroup structure of G. The proof of Theorem A is entirely algebraic and can be found in Section 4.2.

We now record some immediate consequences of Theorem A that will be restated as Corollary 4.2 and Corollary 4.3.

Corollary B. *Let $L \leq G$ be perfect. Then L is normal in G if and only if the summand $A(-)[e_L^{-1}]$ admits all norms of the form $\tilde{N}_K^H$ such that K contains a subgroup conjugate in G to L.*

Corollary C. *The Green ring $A(-)[e_L^{-1}]$ admits all norms $\tilde{N}_K^H$ for all $K \leq H$ if and only if $L = 1$ is the trivial group. In this case, the norm maps equip $A(-)[e_L^{-1}]$ with the structure of a Tambara functor.*

For an arbitrary perfect subgroup $L \leq G$, we explain how the levelwise localization $A(-)[e_L^{-1}]$ fits into Blumberg-Hill's framework of *incomplete Tambara functors* [BH18], the basics of which we recall in Section 2.3. For $K \leq H \leq G$, call the H-set H/K *admissible* for e_L if $K \leq H$ satisfy the condition ($\star$) of Theorem A. Call a finite H-set admissible if all of its orbits are admissible. Theorem A is complemented by the following two structural results.

Theorem D (see Theorem 4.20)**.** *Let $L \leq G$ be a perfect subgroup and let $e_L \in A(G)$ be the corresponding primitive idempotent. Then the following hold:*

i) The admissible sets assemble into an indexing system $\mathcal{I}_L$ (in the sense of [BH18, Def. 1.2], *see Section 2.1) such that $A(-)[e_L^{-1}]$ is an $\mathcal{I}_L$-Tambara functor under $A(-)$.*
ii) In the poset of indexing systems, $\mathcal{I}_L$ is maximal among the elements that satisfy i).
iii) The map $A(-) \to A(-)[e_L^{-1}]$ is a localization at e_L in the category of $\mathcal{I}_L$-Tambara functors.

Corollary E (see Corollary 4.24)**.** *The localization maps $A(-) \to A(-)[e_L^{-1}]$ assemble into a canonical isomorphism of $\mathcal{I}$-Tambara functors*

$$A(-) \to \prod_{(L) \leq G \text{ perfect}} A(-)[e_L^{-1}],$$

where $\mathcal{I}$ is the intersection

$$\mathcal{I} = \bigcap_{(L)\leq G} \mathcal{I}_L$$

of the indexing systems given by Theorem D.

Together, Theorem A, Theorem D and Corollary E answer Question 1.3. A simple characterization of the norms parametrized by $\mathcal{I}$ can be found in Lemma 4.23.

1.2. **Statement of homotopical results.** It was conjectured by Blumberg-Hill [BH15b, Section 5.2] and proven in [GW, Rub17, BP17] that any indexing system can be realized by an N_∞ operad which encodes norms precisely for the admissible sets of that indexing system. In particular, for any of the indexing systems $\mathcal{I}_L$ of Theorem D, we can choose a corresponding Σ-cofibrant N_∞ operad $\mathcal{O}_L$. See Section 2.2 for details.
We use general preservation results for N_∞ algebras under localization [HH14, GW] to lift our algebraic results about $\mathcal{I}_L$-Tambara functor structures on homotopy groups to a homotopical statement about $\mathcal{O}_L$-algebra structures on G-spectra. The following result is restated as Corollary 4.26.

Corollary F. *Let $\mathcal{O}_L$ be any Σ-cofibrant N_∞ operad whose associated indexing system is $\mathcal{I}_L$. Then:*

i) The G-spectrum $\mathbb{S}[e_L^{-1}]$ is an $\mathcal{O}_L$-algebra under $\mathbb{S}$.
ii) In the poset of homotopy types of N_∞ operads, $\mathcal{O}_L$ is maximal among the elements that satisfy i).
iii) The map $\mathbb{S} \to \mathbb{S}[e_L^{-1}]$ is a localization at e_L in the category of $\mathcal{O}_L$-algebras.

A homotopical reformulation of Corollary C shows that the idempotent splitting of $\mathbb{S}$ is far from being a splitting of G-E_∞ ring spectra.

Corollary G (see Corollary 4.27)**.** *The G-spectrum $\mathbb{S}[e_L^{-1}]$ is a G-E_∞ ring spectrum if and only if $L = 1$ is the trivial group.*

Locally at a prime p, this recovers a (currently unpublished) result of Grodal [Gro, Cor. 5.5], which we state as Theorem 4.28.
There is a homotopical analogue of Corollary E.

Corollary H (see Corollary 4.29)**.** *Let $\mathcal{O}$ be any Σ-cofibrant N_∞ operad whose associated indexing system is $\mathcal{I}$. Then the idempotent splitting*

$$\mathbb{S} \simeq \prod_{(L)\leq G} \mathbb{S}[e_L^{-1}]$$

is an equivalence of $\mathcal{O}$-algebras, where the product is taken over conjugacy classes of perfect subgroups.

Together, Corollary F and Corollary H answer Question 1.2.

1.3. **Examples.** In Section 5, we use our results to explicitly calculate the multiplicative structure of the idempotent splittings of the sphere in the case of the alternating group A_5 and the symmetric group Σ_3 (working 3-locally). Moreover, for arbitrary G, we deduce that the rational idempotent splitting of $\mathbb{S}_{\mathbb{Q}}$ cannot preserve any non-trivial norm maps. The latter is not a new insight, cf. e.g. [BGK17, Section 7].

1.4. **Applications to modules.** Corollary F, together with the theory of modules of [BH15a], also characterizes the norm functors which arise on the level of modules over the N_∞ ring $\mathbb{S}[e_L^{-1}]$ and its restrictions to subgroups.

The following result will be restated as Corollary 6.1.

Corollary 1.4. *Let $L \leq G$ be perfect and let $\mathcal{O}_L$ as in Corollary F. Assume furthermore that $\mathcal{O}_L$ has the homotopy type of the linear isometries operad on some (possibly incomplete) universe U. For all admissible sets H/K of $\mathcal{I}_L$, there are norm functors*

$$_{\mathrm{Res}_H(\mathbb{S}[e_L^{-1}])}N^{H,\mathrm{Res}_H(U)}_{K,\mathrm{Res}_K(U)} : \mathit{Mod}(\mathrm{Res}^G_K(\mathbb{S}[e_L^{-1}])) \to \mathit{Mod}(\mathrm{Res}^G_H(\mathbb{S}[e_L^{-1}]))$$

built from the smash product relative to $\mathbb{S}_{(P)}[e_L^{-1}]$ which satisfy a number of relations analogous to those for the norm functor $\mathrm{Sp}^H \to \mathrm{Sp}^G$, stated in [BH15a, Thm. 1.3].

Any G-spectrum is a module over $\mathbb{S}$, hence the idempotent splitting (1.1) of $\mathbb{S}$ induces a splitting of the category of G-spectra. Corollary 1.4 then says that this does not give rise to a splitting of G-symmetric monoidal categories in the sense of [HH16]. Indeed, the categories of modules over (restrictions to subgroups of) $\mathbb{S}[e_L^{-1}]$ will only admit an incomplete set of norm functors, which then can be read off from Theorem A.

1.5. **Topological K-theory spectra.** We will answer the analogues of our main questions for G-equivariant complex and real topological K-theory in the sequel [Böh18], see Section 6.

Organization: Section 2 provides some background material on N_∞ operads and their algebras in G-spectra, (incomplete) Tambara functors, indexing systems and their behavior under localization. In Section 3, we recall Dress' classification of idempotent elements in the Burnside ring and explain how to obtain the splitting (1.1) of the G-equivariant sphere spectrum. We state and prove our results (including the local variants) in Section 4 and discuss examples in Section 5. Finally, applications are discussed in Section 6.

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2. Preliminaries

We briefly recall some background material on N_∞ operads and N_∞ ring spectra, incomplete Tambara functors and localizations. Most of this section follows [BH15b, BH18].

2.1. N_∞ **operads and indexing systems.** Recall that a subgroup $\Gamma \leq G \times \Sigma_n$ is a *graph subgroup* if it is the graph of a group homomorphism $H \to \Sigma_n$ for some $H \leq G$, or equivalently, if $\Gamma \cap (\{1\} \times \Sigma_n)$ is trivial. By a *G-operad* we mean an operad in the category of (unbased) G-spaces.

Definition 2.1 ([BH15b], Def. 1.1)**.** A G-operad $\mathcal{O}$ is called an N_∞ *operad* if each G-space $\mathcal{O}(n)$ is a universal space for a family $\mathcal{F}_n$ of graph subgroups of $G \times \Sigma_n$ which contains all graphs of trivial homomorphisms, i.e., all subgroups of the form $H \times \{\mathrm{id}\}$.

The following properties are immediate from the definition.

Lemma 2.2. *For an N_∞ operad $\mathcal{O}$, the following holds:*

(i) The G-spaces $\mathcal{O}(0)$ and $\mathcal{O}(1)$ are G-equivariantly contractible.
(ii) The action of Σ_n on $\mathcal{O}(n)$ is free.
(iii) The underlying non-equivariant operad is always an E_∞ operad.

Example 2.3 ([BH15b], Lemma 3.15)**.** Let U be a (not necessarily complete) G-universe, and let $\mathcal{L}(U)$ be the associated operad of linear isometric embeddings. Then it is a G-operad under the conjugation action, and it is always an N_∞ operad.

Definition 2.4. An H-set X of cardinality n is called *admissible* for $\mathcal{O}$ if the graph of the corresponding action homomorphism $H \to \Sigma_n$ is contained in $\mathcal{F}_n$.

Algebras R over an N_∞ operad $\mathcal{O}$ are G-equivariant E_∞ ring spectra which in addition admit coherent equivariant multiplications given by Hill-Hopkins-Ravenel norm maps [BH15b, Thm. 6.11]

$$N_K^H\colon \bigwedge_{H/K} \mathrm{Res}^G_K(R) \to \mathrm{Res}^G_H(R)$$

for those nested subgroups $K \leq H \leq G$ such that H/K is an admissible set for $\mathcal{O}$. (More generally, there is a norm map N_f associated to a map of G-sets $f\colon X \to Y$

provided that for all $y \in Y$, the preimage $f^{-1}(y)$ is an admissible G_y-set, where G_y denotes the stabilizer group of y.) Here, $\bigwedge_{H/K}$ denotes the *indexed smash product* or *Hill-Hopkins-Ravenel norm functor* [HHR16, Section 2.2.3], and the maps N_K^H arise as the counits of the adjunctions [HHR16, Prop. 2.27]

$$\bigwedge_{H/K}(-): \mathbf{Comm^K} \rightleftarrows \mathbf{Comm^H}: \mathrm{Res}^H_K(-)$$

between categories of commutative monoids in equivariant spectra.

The data of admissible H-sets for all $H \leq G$ can be organized in the following way: For fixed H, the collection of admissible H-sets forms a symmetric monoidal subcategory of the category $\mathcal{S}et^H$ of finite H-sets under disjoint union. Together, these assemble into a subfunctor $\mathcal{I}$ of the coefficient system $\underline{\mathcal{S}et}$ whose value at G/H is the symmetric monoidal category $\mathcal{S}et^H$. The operad structure of $\mathcal{O}$ forces $\mathcal{I}$ to be closed under certain operations, as captured in the following definition.

Definition 2.5 ([BH18], Def. 1.2)**.** An *indexing system* is a contravariant functor

$$\mathcal{I}\colon \mathcal{O}rb_G^{op} \to \mathcal{S}ym,\ G/H \mapsto \underline{C}(H)$$

from the orbit category of G to the category of symmetric monoidal categories and strong symmetric monoidal functors, such that the following holds:

(i) The value $\mathcal{I}(H)$ of $\mathcal{I}$ at G/H is a full symmetric monoidal subcategory of the category $\mathcal{S}et^H$ of finite H-sets and H-equivariant maps which contains all trivial H-sets.
(ii) Each $\mathcal{I}(H)$ is closed under finite limits.
(iii) The functor $\mathcal{I}$ is closed under "self-induction": If $H/K \in \mathcal{I}(H)$ and $T \in \mathcal{I}(K)$, we require that $\mathrm{Ind}_K^H(T) = H \times_K T \in \mathcal{I}(H)$.

The collection of all indexing systems (for a fixed group G) forms a poset under inclusion. N_∞ operads give rise to indexing systems.

Definition 2.6. Let $\mathcal{I}$ be an indexing system. Call an H-set X *admissible* if $X \in \mathcal{I}(H)$. Call a map $f\colon Y \to Z$ of finite G-sets *admissible* if the orbit $G_{f(y)}/G_y$ obtained from stabilizer subgroups is admissible for all $y \in Y$.

Proposition 2.7 ([BH18], Thm. 4.14)**.** *The admissible sets of any N_∞ operad $\mathcal{O}$ form an indexing system.*

2.2. **The poset of N_∞ ring structures.** Two extreme cases of N_∞ operads arise:

Definition 2.8 ([BH15b], Section 3.1)**.** If for all $n \in \mathbb{N}$, $\mathcal{F}_n$ is the family of all graph subgroups of $G \times \Sigma_n$, then $\mathcal{O}$ is called a *G-E_∞ operad* or *complete N_∞ operad*. If for all n, $\mathcal{F}_n$ is the family of trivial graphs $H \times \{\mathrm{id}\}$, then $\mathcal{O}$ is called a *naive N_∞ operad*.

Algebras over G-E_∞ operads are equivariant E_∞ ring spectra which admit all norm maps and form a category which is Quillen equivalent to that of strict commutative monoids in G-spectra. Naive N_∞ operads are non-equivariant E_∞ operads equipped with the trivial G-action. Their algebras are all G-spectra that are underlying E_∞ ring spectra, but do not necessarily possess any non-trivial norms. The N_∞ operads with other collections of admissible sets interpolate between those two extremes. We refer to [BH15b, Section 6] for proofs and further details.

The collection of homotopy classes of N_∞ operads forms a poset that only depends on the combinatorial data of the admissible sets, as we recall now.

Definition 2.9 ([BH15b], Def. 3.9)**.** A morphism of N_∞ operads $\mathcal{O} \to \mathcal{O}'$ is a *weak equivalence* if it induces a weak equivalence of spaces $\mathcal{O}(n)^\Gamma \to \mathcal{O}'(n)^\Gamma$ for all $n \geq 0$ and all subgroups $\Gamma \leq G \times \Sigma_n$.

Blumberg-Hill conjectured the following equivalence of categories and proved the "fully faithful" part [BH15b, Thm. 3.24]. Different proofs for the essential surjectivity were given by Gutierrez-White [GW, Thm. 4.7], Rubin [Rub17, Thm. 3.3] and Bonventre-Pereira [BP17, Cor. IV], and it should be possible to extract an ∞-categorical proof from [BDG$^+$17] and its sequels.

Theorem 2.10 (Blumberg-Hill et al.)**.** *The functor from the homotopy category of N_∞ operads (with respect to the above notion of weak equivalence) to the poset of indexing systems which assigns to each N_∞ operad its collection of admissible sets is an equivalence of categories.*

Remark 2.11. We record a technical detail for later reference: [GW, Thm. 4.10] guarantees that for each indexing system $\mathcal{I}$, we can find a corresponding N_∞ operad $\mathcal{O}$ which is Σ-*cofibrant*, i.e., each $\mathcal{O}(n)$ has the homotopy type of a (necessarily Σ_n-free) $(G \times \Sigma_n)$-CW complex. This will be used in Section 2.4.

2.3. **Mackey functors, Green rings and (incomplete) Tambara functors.** Recall that a *Mackey functor* $\underline{M}$ (with respect to an ambient group G which we leave implicit in the notation) consists of an abelian group $\underline{M}(T)$ for each finite G-set, equipped with a structure map $\underline{M}(X) \to \underline{M}(Z)$ for each span

$$X \xleftarrow{r} Y \xrightarrow{t} Z,$$

subject to a list of axioms. In particular, $\underline{M}$ is additive in the sense that $\underline{M}(S \sqcup T) \cong \underline{M}(S) \times \underline{M}(T)$. Thus, it is determined on objects by the values $\underline{M}(H) := \underline{M}(G/H)$ for subgroups $H \leq G$. We refer to [Str12, Section 3] for details.

A Mackey functor $\underline{R}$ is a *Green ring* if $\underline{R}(X)$ is a commutative ring for all G-sets X such that all restriction maps are ring homomorphisms and all transfers are homomorphisms of modules over the target.

Many naturally occuring examples of Green rings such as the Burnside ring $A(-)$ or the complex representation ring $RU(-)$ come equipped with additional multiplicative transfers, called *norms*. Green rings with compatible norms are known as *Tambara functors* (originally defined as "TNR functors" [Tam93]) and were generalized in [BH18] to cases where only some of the norm maps are available. We quickly review these *incomplete Tambara functors*.

Let $bispan^G$ denote the category of *bispans of G-sets*. It has objects the finite G-sets and morphisms the isomorphism classes of bispans of finite G-sets

$$X \xleftarrow{r} Y \xrightarrow{n} Z \xrightarrow{t} W.$$

We refer to [Str12, Section 6] for the definition of composition and further details. Blumberg-Hill observed that one can restrict the class of maps n which are allowed at the central position of a bispan to encode Tambara functors with incomplete collections of norms, as we recall now.

Definition 2.12 ([BH18], Sections 2.2, 3.1)**.** A subcategory D of $\mathcal{S}et^G$ is called

1) *wide* if it contains all objects,
2) *pullback-stable* if any base-change of a map in D is again in D, and
3) *finite coproduct-complete* if it has all finite coproducts and they are created in $\mathcal{S}et^G$.

Theorem 2.13 ([BH18], Thm. 2.10)**.** *Let $D \subseteq \mathcal{S}et^G$ be a wide, pullback-stable subcategory, then the wide subgraph $bispan^G_D$ of the category of bispans that only contains morphisms of the form*

$$X \leftarrow Y \rightarrow Z \rightarrow W$$

where $Y \rightarrow Z$ is in D, forms a subcategory.

Definition 2.14 ([BH18], Def. 3.9)**.** For an indexing system $\mathcal{I}$, let $\mathcal{S}et^G_{\mathcal{I}} \subseteq \mathcal{S}et^G$ be the wide subgraph which contains a morphism $f\colon X \rightarrow Y$ if and only if for all $y \in Y$, the quotient of stabilizers $G_{f(y)}/G_y$ is in $\mathcal{I}(G_{f(y)})$.

Theorem 2.15 ([BH18], Thm. 3.18)**.** *The assignment $\mathcal{I} \mapsto \mathcal{S}et^G_{\mathcal{I}}$ gives rise to an isomorphism between the poset of indexing systems and the poset of wide, pullback-stable, finite coproduct-complete subcategories $D \subseteq \mathcal{S}et^G$.*

Definition 2.16 ([BH18], Def. 4.1)**.** Let $D \subseteq \mathcal{S}et^G$ be a wide, pullback-stable symmetric monoidal subcategory.

1) A *D-semi Tambara functor* is a product-preserving functor $bispan_D^G \to \mathcal{S}et$.
2) A *D-Tambara functor* is a D-Tambara functor that is abelian group valued on objects.
3) For an indexing system $\mathcal{I}$ and $D = \mathcal{S}et_{\mathcal{I}}^G$, define an *$\mathcal{I}$-Tambara functor* to be a D-Tambara functor.
4) If $D = \mathcal{S}et^G$, then D-Tambara functors are simply called Tambara functors.

Remark 2.17. We did not require that D be finite coproduct-complete in the definition. If this also holds, i.e., if D corresponds to an indexing system $\mathcal{I}$, then it can be shown that every $\mathcal{I}$-Tambara functor has an underlying Green ring and all norm maps are maps of multiplicative monoids, see [BH18, Prop. 4.6 and Cor. 4.8].

The condition that any D-Tambara functor $\underline{R}$ be product-preserving means that

$$\underline{R}(S \sqcup T) \cong \underline{R}(S) \times \underline{R}(T)$$

for all finite G-sets S and T. Hence, on the level of objects, $\underline{R}$ is determined by the groups $\underline{R}(H) := \underline{R}(G/H)$ for all $H \leq G$.

Notation 2.18. We will use the following special cases of the structure maps frequently in the present paper: Spans of the form $(Y \xleftarrow{f} X \xrightarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X)$ give rise to *restrictions* $R_f \colon \underline{R}(Y) \to \underline{R}(X)$ and spans of the form $(X \xleftarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X \xrightarrow{f} Y)$ induce *transfers* $T_f \colon \underline{R}(X) \to \underline{R}(Y)$. Moreover, spans of the form $(X \xleftarrow{\mathrm{id}} X \xrightarrow{f} Y \xrightarrow{\mathrm{id}} Y)$ give rise to *norms* $N_f \colon \underline{R}(X) \to \underline{R}(Y)$. If $f \colon X \to Y$ is the canonical surjection $G/K \to G/H$ arising from nested subgroup inclusions $K \leq H \leq G$, then we write $R_K^H := R_f$, $T_K^H := T_f$ and $N_K^H := N_f$, respectively.

Example 2.19. The Burnside ring $A(G)$ is a Tambara functor. Restrictions, transfers and norms are given by restriction, induction and co-induction of G-sets, respectively. Similarly, the complex representation ring $RU(G)$ is a Tambara functor with restrictions, transfers and norms given by restriction, induction and tensor induction of G-representations, respectively. The "linearization map" $A(G) \to RU(G)$ that sends a finite G-set to its associated permutation representation is a map of Tambara functors, i.e., it is compatible with all of the structure maps.

Another class of examples arises from equivariant stable homotopy theory: The norm maps N_K^H of an N_∞ ring spectrum R give rise to multiplicative transfers on equivariant homotopy groups

$$N_K^H \colon \pi_V^K(R) \to \pi_{\mathrm{Ind}_K^H(V)}^H\Big(\bigwedge_{H/K} R\Big)$$

given by sending the K-equivariant homotopy class of $f\colon \mathbb{S}^V \to R$ to the H-equivariant homotopy class of the composite

$$\mathbb{S}^{\mathrm{Ind}_K^H(V)} \cong \bigwedge_{H/K} \mathbb{S}^V \xrightarrow{\wedge f} \bigwedge_{H/K} R \xrightarrow{N_K^H} R,$$

see [HHR16, Section 2.3.3].

Theorem 2.20 ([Bru07], [BH18], Thm. 4.14)**.** *Let R be an algebra over an N_∞ operad $\mathcal{O}$, then $\underline{\pi}_0(R)$ is an $\mathcal{I}$-Tambara functor structured by the indexing system $\mathcal{I}$ corresponding to $\mathcal{O}$ under the equivalence of categories from Theorem 2.10.*

The structure on the entire homotopy ring $\{\pi_V^H(R)\}_{H\leq G,\, V\in RO(H)}$ is described in [AB18]. For the purpose of the present paper, it suffices to consider the zeroth equivariant homotopy groups.

2.4. **Localization and N_∞ rings.** We record some preservation results for algebraic structure under localizations of G-spectra which invert a single element $x \in \pi_*^G(\mathbb{S})$. For definiteness, we work in the category of orthogonal G-spectra equipped with the positive complete model structure [HHR16, Thm. B.63].

Notation 2.21. By abuse of notation, let x be a map representing the homotopy class $x \in \pi_0^G(\mathbb{S})$. We write $\mathcal{C}_x$ for the set of morphisms of G-spectra

$$\mathcal{C}_x = \{G_+ \wedge_H S^n \wedge x \mid H \leq G,\, n \in \mathbb{Z}\}.$$

Proposition 2.22. *Bousfield localization at $\mathcal{C}_x$ has the following properties:*

i) It is given by smashing with $\mathbb{S}[x^{-1}]$, hence recovers (orbitwise) x-localization on the level of equivariant homotopy groups.
ii) It is a monoidal *localization in the sense that the resulting local model structure is again a monoidal model category.*

Proof. Since the map $ho(\mathrm{Sp}^G)(G/H_+ \wedge \mathbb{S}^n \wedge x, X)$ is just the action of x on $\pi_n^H(X)$, we see that an object X is $\mathcal{C}_x$-local if and only if its equivariant homotopy groups are x-local. But x-localization is given by smashing with $\mathbb{S}[x^{-1}]$. By [Whi14, Thm. 4.5], the localization is monoidal if and only if $\mathcal{C}_x$ is closed under all functors $G_+ \wedge_H S^n \wedge (-)$, which holds by definition. □

Even such a seemingly innocent localization need not preserve any of the (non-trivial) norm maps of an N_∞ ring spectrum, as the following example illustrates.

Example 2.23 ([HH16], Prop. 6.1)**.** The inclusion of 0 into the reduced regular representation $\tilde{\rho}$ of G defines an essential map $S^0 \to S^{\tilde{\rho}}$ of G-spaces all of whose restrictions to proper subgroups are equivariantly contractible because they necessarily have fixed points along which the two points of S^0 can be connected by an equivariant path. The resulting map gives rise to an element $\alpha \in \pi^G_{-\tilde{\rho}}(\mathbb{S})$ such that the resulting G-spectrum $\mathbb{S}[\alpha^{-1}]$ is non-trivial but all of its restrictions to proper subgroups are equivariantly contractible. Thus, it cannot admit any norms

$$\bigwedge_{G/H} \mathrm{Res}^G_H \mathbb{S}[\alpha^{-1}] \to \mathbb{S}[\alpha^{-1}]$$

because on homotopy rings, they would induce ring maps from zero rings to non-trivial rings.

Remark 2.24. Strictly speaking, the element α is not an element of the $\mathbb{Z}$-graded homotopy groups $\pi^G_*(\mathbb{S})$, but only of the $RO(G)$-graded homotopy groups $\pi^G_\star(\mathbb{S})$. However, our results in Section 4 show that even when we restrict attention to elements $x \in \pi^G_0(\mathbb{S})$, we can construct many other examples of the loss of N_∞ structure under localization in terms of elementary group theory. Indeed, the A_5-spectra $\mathbb{S}[e_{A_5}^{-1}]$ and $\mathbb{S}[\alpha^{-1}]$ are very similar in terms of their equivariant multiplicative behavior, see Section 5.

We now present a preservation result for N_∞ ring structures due to Gutierrez and White. A similar result first appeared in the special case of G-E_∞ rings in [HH14, Cor. 4.11] for G-E_∞ ring spectra and goes back at least to [EKMM97, Thm. VIII.2.2].

Definition 2.25 ([GW], Def. 7.3)**.** For a G-operad $\mathcal{P}$, let U denote the forgetful functor from $\mathcal{P}$-algebras to G-spectra. A Bousfield localization $L_{\mathcal{C}}$ is said to *preserve* $\mathcal{P}$-algebras if the following two conditions hold:

(1) If E is a $\mathcal{P}$-algebra, then there is some $\mathcal{P}$-algebra $\tilde{E}$ which is weakly equivalent as a G-spectrum to $L_{\mathcal{C}}(E)$.
(2) In addition, if E is a cofibrant $\mathcal{P}$-algebra, then there is a choice of $\tilde{E}$ in the category of $\mathcal{P}$-algebras with $U(\tilde{E})$ local in G-spectra, there is a $\mathcal{P}$-algebra homomorphism $r_E \colon E \to \tilde{E}$ that lifts the localization map $l_E \colon E \to L_{\mathcal{C}}(E)$ up to homotopy, and there is a weak equivalence $\beta_E \colon L_{\mathcal{C}}(UE) \to U(\tilde{E})$ such that $\beta_E \circ l_{UE} \cong Ur_E$ in $ho(\mathrm{Sp}^G)$.

Recall that a G-operad $\mathcal{P}$ is called Σ-*cofibrant* if all of its spaces $\mathcal{P}(n)$ have the homotopy type of $(G \times \Sigma_n)$-CW complexes. The following preservation result is a direct translation of [GW, Cor. 7.10] to the positive complete model structure on orthogonal G-spectra.

Theorem 2.26 ([GW], Cor. 7.10). *Let $\mathcal{P}$ be a Σ-cofibrant N_∞ operad. Let $L_{\mathcal{C}}$ be a monoidal left Bousfield localization. Then $L_{\mathcal{C}}$ preserves $\mathcal{P}$-algebras in G-spectra if and only if the functors*

$$G_+ \wedge_H \bigwedge_{H/K} \operatorname{Res}^G_K(-) \colon \mathrm{Sp}^G \to \mathrm{Sp}^G$$

preserve $\mathcal{C}$-local equivalences between cofibrant objects for all $H \leq G$ and all transitive H-sets H/K which are admissible for $\mathcal{P}$.

Remark 2.27. The statement of [GW, Cor. 7.10] is actually phrased in terms of the functors $G_+ \wedge_H \bigwedge_T \operatorname{Res}^G_K(-)$ for all $H \leq G$ and all admissible H-sets T. Both formulations are easily seen to be equivalent, using that $\bigwedge_{T_1 \coprod T_2}(-) \simeq \bigwedge_{T_1}(-) \wedge \bigwedge_{T_2}(-)$ and that the smash product of two equivalences between cofibrant objects is an equivalence.

Corollary 2.28 ([GW], Cor. 7.5). *Let $\mathcal{P}$ be any N_∞ operad. Then any monoidal left Bousfield localization $L_{\mathcal{C}}$ takes $\mathcal{P}$-algebras in Sp^G to G-spectra which are at least algebras over some naive N_∞ operad.*

If the localization is given by inverting a single element $x \in \pi_0^G(\mathbb{S})$, the condition in Theorem 2.26 can be verified on homotopy groups, as we explain now. The following results generalize [HH14, Thm. 4.11] to the N_∞ setting in the case of the (p-local) sphere spectrum.

Lemma 2.29. *Fix a transitive H-set H/K and an element $x \in \pi_0^G(\mathbb{S})$. Then:*

i) The functor $\bigwedge_{H/K} \operatorname{Res}^G_K$ admits a left derived functor $\mathbb{L}(\bigwedge_{H/K} \operatorname{Res}^G_K)$ which commutes with sifted homotopy colimits. In particular, it commutes with sequential homotopy colimits.

ii) If $\mathbb{S}_c \to \mathbb{S}$ is a cofibrant replacement in the positive complete model structure on orthogonal G-spectra, then so is $\mathbb{S}_c[x^{-1}] \to \mathbb{S}[x^{-1}]$. Moreover, $\bigwedge_{H/K} \operatorname{Res}^G_K \mathbb{S}_c \to \bigwedge_{H/K} \operatorname{Res}^G_K \mathbb{S}$ and $\bigwedge_{H/K} \operatorname{Res}^G_K \mathbb{S}_c[x^{-1}] \to \bigwedge_{H/K} \operatorname{Res}^G_K \mathbb{S}[x^{-1}]$ are cofibrant replacements in orthogonal H-spectra.

iii) The map

$$\operatorname{Res}^G_H(\mathbb{S}) \cong \bigwedge_{H/K} \operatorname{Res}^G_K(\mathbb{S}) \to \bigwedge_{H/K} \operatorname{Res}^G_K(\mathbb{S}[x^{-1}])$$

induced from the canonical map $\mathbb{S} \to \mathbb{S}[x^{-1}]$ induces an equivalence

$$\operatorname{Res}^G_H(\mathbb{S})[(N_K^H R_K^G(x))^{-1}] \to \bigwedge_{H/K} \operatorname{Res}^G_K(\mathbb{S}[x^{-1}]).$$

Proof. i): This is well-known for the restriction functor; for the norm it follows from [HHR16, Prop. B.104] combined with [HHR16, Prop. A.27, A.53].

ii): The first statement is easy. The sphere is the initial commutative monoid, hence

cofibrant as a commutative monoid, and so the other two statements follow from [HHR16, Prop. 2.30] applied to the maps $\mathbb{S}_c \to \mathbb{S}$ and $\mathbb{S}_c[x^{-1}] \to \mathbb{S}[x^{-1}]$.
iii): Consider the following diagram:

$$\begin{array}{ccc} \operatorname{hocolim}(\bigwedge_{H/K} \operatorname{Res}^G_K(\mathbb{S}_c) \xrightarrow{\bigwedge \operatorname{Res}(\cdot x)} \bigwedge_{H/K} \operatorname{Res}^G_K(\mathbb{S}_c) \to \ldots) & \longrightarrow & (\bigwedge_{H/K} \operatorname{Res}^G_K)(\mathbb{S}_c[x^{-1}]) \\ \downarrow & & \downarrow \\ \operatorname{hocolim}(\bigwedge_{H/K} \operatorname{Res}^G_K(\mathbb{S}) \xrightarrow{\bigwedge \operatorname{Res}(\cdot x)} \bigwedge_{H/K} \operatorname{Res}^G_K(\mathbb{S}) \to \ldots) & \dashrightarrow & (\bigwedge_{H/K} \operatorname{Res}^G_K)(\mathbb{S}[x^{-1}]) \end{array}$$

The vertical maps are equivalences by part ii). The dashed horizontal map is induced by the map

$$\operatorname{Res}^G_H(\mathbb{S}) \cong \bigwedge_{H/K} \operatorname{Res}^G_K(\mathbb{S}) \to \bigwedge_{H/K} \operatorname{Res}^G_K(\mathbb{S}[x^{-1}]),$$

and similar for the solid horizontal one. The solid horizontal map is an equivalence by part i) and the fact that left derived functors can be computed by passing to cofibrant replacements. Hence the dashed arrow is an equivalence. It now suffices to see that the domain of the dashed arrow computes the localization $\operatorname{Res}^G_H(\mathbb{S})[(N^H_K R^G_K(x))^{-1}]$. This holds because $N^H_K R^G_K(x)$ is given as the composite of $\bigwedge_{H/K} \operatorname{Res}^G_K(x)$ with the norm map $\bigwedge_{H/K} \operatorname{Res}^G_K \mathbb{S} \to \operatorname{Res}^G_H(\mathbb{S})$, and the latter is an isomorphism since the sphere is the monoidal unit. □

Proposition 2.30. *Let $\mathcal{P}$ be a Σ-cofibrant (see Remark 2.11) N_∞ operad. Fix $x \in \pi_0^G(\mathbb{S})$. Then $L_{\mathcal{C}_x}$ preserves $\mathcal{P}$- algebras in G-spectra if and only if for all $H \leq G$ and all transitive admissible H-sets H/K, the element $N^H_K R^G_K(x)$ divides a power of $R^G_H(x)$ in the ring $\pi_0^H(\mathbb{S})$.*

Proof. We have to show that for admissible such H/K, the functors $G_+ \wedge_H \bigwedge_{H/K} \operatorname{Res}^G_K(-)$ preserve $\mathcal{C}_x$-local equivalences between cofibrant objects if and only if the elements $N^H_K R^G_K(x)$ divide powers of $R^G_H(x)$.
If $\mathcal{C}_x$-local equivalences are preserved, then in particular the map of G-spectra

$$G_+ \wedge_H \bigwedge_{H/K} \operatorname{Res}^G_K(x) \colon G_+ \wedge_H \bigwedge_{H/K} \operatorname{Res}^G_K(\mathbb{S}) \to G_+ \wedge_H \bigwedge_{H/K} \operatorname{Res}^G_K(\mathbb{S})$$

is an x-local equivalence. Under the standard isomorphism $\pi_*^G(G_+ \wedge_H -) \cong \pi_*^H(-)$, the induced map on $\pi_*^G(-)$ agrees with multiplication by the element $N^H_K R^G_K(x)$ and becomes a unit after inverting $R^G_H(x)$, hence the element $N^H_K R^G_K(x)$ must divide a power of $R^G_H(x)$.
Conversely, assume the division relation holds and let $f \colon X \to Y$ be a $\mathcal{C}_x$-local equivalence between cofibrant objects. Since induction is a left Quillen functor, it suffices to

show that the map $\bigwedge_{H/K} \mathrm{Res}^G_K(f)$ becomes an equivalence of H-spectra upon smashing with $\mathbb{S}[R^G_H(x)^{-1}]$. We are going to show that it is an equivalence upon smashing with $\mathbb{S}[N^H_K R^G_K(x)^{-1}]$. Since the element $N^H_K R^G_K(x)$ divides $R^G_H(x)$ by assumption, the claim then follows.
The map $f \wedge \mathbb{S}[x^{-1}]$ is an equivalence by assumption, so for any cofibrant replacement $\mathbb{S}_c \to \mathbb{S}$, the map $f \wedge \mathbb{S}_c[x^{-1}]$ is an equivalence between cofibrant G-spectra. Then $\bigwedge_{H/K} \mathrm{Res}^G_K(f \wedge \mathbb{S}_c[x^{-1}])$ is an equivalence of H-spectra by [HHR16, Prop. B.103]. By part ii) of Lemma 2.29, the map $\bigwedge_{H/K} \mathrm{Res}^G_K(f \wedge \mathbb{S}[x^{-1}])$ must be an equivalence. But the norm and restriction functors commute with smash products, so

$$(\bigwedge_{H/K} \mathrm{Res}^G_K)(f) \wedge (\bigwedge_{H/K} \mathrm{Res}^G_K)(\mathbb{S}[x^{-1}])$$

is an equivalence. Finally, part iii) of Lemma 2.29 implies that

$$(\bigwedge_{H/K} \mathrm{Res}^G_K)(f) \wedge \mathbb{S}[(N^H_K R^G_K(x))^{-1}]$$

is an equivalence, which finishes the proof. □

Corollary 2.31 ([HH14], §4; [BH17], Lemma 12.8)**.** *Let $n \in \mathbb{Z}$, viewed as the element $n \cdot [G/G] \in A(G)$. Then $\mathbb{S}[\frac{1}{n}]$ is a complete G-E_∞ ring spectrum.*

Consequently, for any collection P of primes, $\mathbb{S}_{(P)} := \mathbb{S}[q^{-1}, q \notin P]$ is a complete G-E_∞ ring spectrum, or equivalently, a commutative monoid in Sp^G. One can now mimick the proof of Proposition 2.30 in the P-local case.

Proposition 2.32. *Let $\mathcal{P}$ be a Σ-cofibrant N_∞ operad. Fix $x \in \pi^G_0(\mathbb{S}_{(P)})$. Then $L_{\mathcal{C}_x}$ preserves $\mathcal{P}$-algebras in P-local G-spectra if and only for all $H \leq G$ and all transitive admissible H-sets H/K, the element $N^H_K R^G_K(x)$ divides a power of $R^G_H(x)$ in the ring $\pi^H_0(\mathbb{S}_{(P)})$.*

2.5. **Localization and incomplete Tambara functors.** There are analogous preservation results for incomplete Tambara functors under localization. Given an $\mathcal{I}$-Tambara functor $\underline{R}$ and an element $x \in \underline{R}(G)$, consider the levelwise localization $\underline{R}[x^{-1}](H) := \underline{R}(H)[R^G_H(x)^{-1}]$. By [Str12, Lemma 10.2], this agrees with the sequential colimit along countably many copies of multiplication by x, taken in the category of Mackey functors. Multiplication by x is typically not a map of Tambara functors, and the levelwise localization is usually not a Tambara functor. An alternative notion of localization which enjoys a universal property in the category of Tambara functors is discussed in [BH18, Section 5.4]. The two notions agree if and only if the Hill-Hopkins conditions are satisfied.

Theorem 2.33 ([BH18], Thm. 5.26)**.** *Let $\underline{R}$ be an $\mathcal{I}$-Tambara functor structured by an indexing system $\mathcal{I}$. Let $x \in \underline{R}(G)$. Then the orbit-wise localization $\underline{R}[x^{-1}]$ is a localization in the* *category of $\mathcal{I}$-Tambara functors if and only if for all admissible sets H/K of $\mathcal{I}$, the element $N_K^H R_K^G(x)$ divides a power of $R_H^G(x)$.*

Blumberg and Hill do not give a detailed proof in [BH18], but assert that the proof strategy of [HH14] can be mimicked in the setting of incomplete Tambara functors. For completeness, we include a (different and more elementary) proof here.

Proof. As before, we decorate the structure maps of the localization with a tilde. In order to simplify notation, write $x_K := R_K^G(x)$ and similar for H. Fix an admissible set $H/K \in \mathcal{I}(H)$. If $\tilde{N} := \tilde{N}_K^H$ exists, it must necessarily be given as

$$\tilde{N}\left(\frac{a}{x_K^n}\right) = \frac{N(a)}{N(x_K)^n}. \tag{2.34}$$

This expression is well-defined if and only $N(x_K) \in \underline{R}(H)$ becomes a unit after inverting x_H, i.e., if and only if it divides a power of x_H.
Thus, $\underline{R}[x^{-1}]$ is a Green ring equipped with norms $\tilde{N}_K^H$ for all admissible sets $H/K \in \mathcal{I}(H)$ and all $H \leq G$. From (2.34) we see that the reciprocity relations [BH18, Prop. 4.10, Prop. 4.11] satisfied by the norms of $\underline{R}$ imply the reciprocity relations for the norms of $\underline{R}[x^{-1}]$. Thus by [BH18, Thm. 4.13], $\underline{R}[x^{-1}]$ is a $\mathcal{I}$-Tambara functor. Moreover, the canonical map $\underline{R} \to \underline{R}[x^{-1}]$ is a map of $\mathcal{I}$-Tambara functors. One readily verifies that the unique ring maps out of $\underline{R}(H)[x_H^{-1}]$ given by the universal properties for varying $H \leq G$ assemble into a map of $\mathcal{I}$-Tambara functors which exhibits $\underline{R}[x^{-1}]$ as the localization of $\underline{R}$ at x.
For the “only if" direction, observe that the division relations are also necessary because the norms and restrictions of the incomplete Tambara functor $\underline{R}[x^{-1}]$ are multiplicative maps. □

As before, this always applies to localizations which invert natural numbers.

Corollary 2.35. *Let $n \in \mathbb{Z}$, viewed as the element $n \cdot [G/G] \in A(G)$. Then $A(-)[\frac{1}{n}]$ is a complete Tambara functor.*

In particular, Question 1.3 also makes sense for the local variants of the Burnside ring.

3. Idempotent splittings of the Burnside ring and the G-sphere spectrum

We review Dress' classification of idempotents in the (P-local) Burnside ring and describe the resulting product decompositions of the Burnside Mackey functor and the G-equivariant sphere spectrum. All of the statements in this section are easy consequences of Dress' result and are probably well-known to the experts. The author does not claim any originality for these results.

3.1. **Idempotents in the Burnside ring.** Let P be a collection of prime numbers and set $\mathbb{Z}_{(P)} := \mathbb{Z}\left[p^{-1} \mid p \notin P\right]$. If P is the collection of all primes, nothing is inverted and hence $\mathbb{Z}_{(P)} = \mathbb{Z}$. If P is the empty set, then all primes are inverted, hence $\mathbb{Z}_{(P)} = \mathbb{Q}$. For $P = \{p\}$, we obtain the usual p-localization $\mathbb{Z}_{(P)} = \mathbb{Z}_{(p)}$, which justifies the notation. Write $A(G)_{(P)} := A(G) \otimes_{\mathbb{Z}} \mathbb{Z}_{(P)}$ for the P-local Burnside ring.

Lemma 3.1 ([tD78], Prop. 1)**.** *Every finite group G has a unique minimal normal subgroup $O^P(G)$ such that the quotient $G/O^P(G)$ is a solvable P-group, i.e. a solvable group whose order is only divisible by primes in P.*

Definition 3.2. The group $O^P(G) \leq G$ is called the *P-residual subgroup* of G. A group G is called *P-perfect* if $G = O^P(G)$. If P contains all primes, we will write $O^{\text{solv}}(G) := O^P(G)$ for the minimal normal subgroup with solvable quotient.

Remark 3.3. The following statements are easily verified.

i) For $P = \{\text{all primes}\}$, this agrees with the usual definition of a perfect group.
ii) For $P = \{p\}$, the group $O^P(G)$ is known to group theorists as the p-residual subgroup $O^p(G)$ and the condition that the quotient be solvable is redundant since every finite p-group is solvable.
iii) For $P = \emptyset$, every finite group G is P-perfect because the trivial group is the only P-group.

The following classification result is due to Dress. Recall that the assignment $S \mapsto |S^H|$ given by taking the cardinality of the H-fixed points of a finite G-set S extends to an injective ring homomorphism

$$\phi^H \colon A(G) \to \prod_{(H) \leq G} \mathbb{Z}$$

where the product is taken over conjugacy classes of subgroups $H \leq G$ [Dre69, (4), (5), Lemma 1]. The same is true after inverting primes since $\mathbb{Z}_{(P)}$ has no torsion. The number $\phi^H(x)$ is called the *mark* of x at H.

Theorem 3.4 ([Dre69], Prop. 2)**.** *There is a bijection between the conjugacy classes of P-perfect subgroups $L \leq G$ and the set of primitive idempotent elements of $A(G)_{(P)}$ which sends L to the element $e_L \in A(G)_{(P)}$ whose marks $\phi^H(e_L)$ at a subgroup $H \leq G$ are one if $O^P(H) \sim L$ are conjugate in G, and zero otherwise.*

Remark 3.5. It follows immediately that G is solvable if and only if $A(G)$ does not have any non-trivial idempotents. This originally motivated Dress' work in [Dre69].

Remark 3.6. Note that if p does not divide the order G, then all subgroups $L \leq G$ are p-perfect, hence all idempotents of $A(G) \otimes \mathbb{Q}$ are contained in the subring $A(G)_{(p)}$. For the other extreme case, if G is a p-group, then only the trivial subgroup is p-perfect, hence the only idempotents in $A(G)_{(p)}$ are zero and one.

3.2. **Idempotent splittings of the Burnside ring.** For any commutative ring, decomposing 1 into a sum of idempotents yields a product decomposition.

Corollary 3.7. *There is an isomorphism of rings*

$$A(G)_{(P)} \cong \prod_{(L)\leq G} A(G)_{(P)}[e_L^{-1}]$$

where the product is taken over conjugacy classes of perfect subgroups of $L \leq G$.

One readily verifies that the statement above can be upgraded to a splitting of Green rings, where for any subgroup $H \leq G$, we view e_L as an element of $A(G)_{(P)}$ via the restriction map $R_H^G\colon A(G)_{(P)} \to A(H)_{(P)}$.

Notation 3.8. For brevity, we will write $A(-)_{(P)}[e_L^{-1}]$ for the levelwise localization $A(-)_{(P)}[R_{(-)}^G(e_L)^{-1}]$, see Section 2.5.

Proposition 3.9. *There is an isomorphism of Green rings*

$$A(-)_{(P)} \cong \prod_{(L)\leq G} A(-)_{(P)}[e_L^{-1}].$$

The left hand side is even a Tambara functor. Question 1.3 asks whether the factors on the right hand side inherit norms from $A(-)_{(P)}$, and whether the splitting preserves these norms.

Remark 3.10. The value of the Green ring $A(-)_{(P)}[e_L^{-1}]$ at a subgroup $K \leq G$ is non-zero if and only if L is subconjugate to K, as follows from the description of e_L in terms of marks in Theorem 3.4.

Remark 3.11. Note that for any idempotent $e \in A(G)_{(P)}$, the localization $A(G)_{(P)}[e^{-1}]$ is canonically isomorphic to the submodule $e \cdot A(G)_{(P)}$. The restriction maps $\tilde{R}_K^H$ and transfer maps $\tilde{T}_K^H$ of $A(-)_{(P)}[e^{-1}]$ are given by the formulae

$$\tilde{R}_K^H(R_H^G(e)\cdot a) := R_K^G(e)\cdot R_K^H(a)$$

and

$$\tilde{T}_K^H(R_K^G(e)\cdot b) := R_H^G(e)\cdot T_K^H(b)$$

for all $a \in A(H)$ and $b \in A(K)$, where R and T denote the restrictions and transfers of $A(-)_{(P)}$ (cf. [LMS86, Thm. V.4.6]). The equations that go into verifying the proposition can easily be read off from these formulae. The analogous P-local statements hold.

Remark 3.12. We warn the reader that even though any restriction of e_L to a proper subgroup $H \leq G$ is still an idempotent, it will in general not be primitive. More precisely, it splits as an n-fold sum of primitive idempotents of $A(H)_{(P)}$ where n is the number of H-conjugacy classes contained in the G-conjugacy class of L.

3.3. **Idempotent splittings of the sphere spectrum.** We now turn to the homotopical consequences of the above splitting. First recall the following theorem which goes back to Segal [Seg71, Cor. of Prop. 1].

Theorem 3.13 (See [Sch], Thm. 6.14, Ex. 10.11)**.** *For all $H \leq G$, there is a ring isomorphism $A(H) \to \pi_0^H(\mathbb{S})$ which sends the class represented by H/K to the element $T_K^H(\mathrm{id})$. For varying H, these maps assemble into an isomorphism of Tambara functors $A(-) \cong \underline{\pi}_0(\mathbb{S})$.*

Remark 3.14. The isomorphism $A(-) \cong \underline{\pi}_0(\mathbb{S})$ is completely determined by the requirements that it be unital and respect transfers.

Dress' classification of idempotent elements then immediately implies the next statement.

Proposition 3.15. *The product of the canonical maps to the localizations is a weak equivalence of P-local G-spectra*

$$\mathbb{S}_{(P)} \simeq \prod_{(L)\leq G} \mathbb{S}_{(P)}[e_L^{-1}]$$

where the product is taken over conjugacy classes of P-perfect subgroups. For any naive N_∞-operad $\mathcal{O}$, i.e., any N_∞ operad whose homotopy type is the unique minimal element in the poset of N_∞ ring structures, this is a splitting of $\mathcal{O}$-algebras (up to equivalence of G-spectra).

Proof. The fact that the e_L form a complete set of orthogonal idempotents implies that the map induces isomorphisms on all equivariant homotopy groups. Moreover, for any naive N_∞ operad $\mathcal{O}$, the G-homotopy equivalence $\mathcal{O}(0) \to *$ induces a canonical equivalence of G-spectra $(\Sigma_+^\infty \mathcal{O}(0))_{(P)} \to \mathbb{S}_{(P)}$, so we can view the latter as an $\mathcal{O}$-algebra. Under this identification, the canonical maps $\mathbb{S}_{(P)} \to \mathbb{S}_{(P)}[e_L^{-1}]$ are all maps of $\mathcal{O}$-algebras, as follows from the fact that localization always preserves naive N_∞ rings, see Corollary 2.28. □

Question 1.2 asks about the maximal N_∞ ring structures on the localizations $\mathbb{S}_{(P)}[e_L^{-1}]$, and about the maximal N_∞ ring structure preserved by the splitting. The answer is given in Corollary 4.26 (Corollary F) and Corollary 4.29 (Corollary H).

4. Norms in the idempotent splittings

We state and prove the results which answer Question 1.3 and Question 1.2, including the local variants where any collection of primes P is inverted.

4.1. **Theorem A and consequences.** The main combinatorial result of this paper is the following version of Theorem A, stated in full P-local generality:

Theorem 4.1. *Let P be a collection of primes. Let $L \leq G$ be a P-perfect subgroup and let $e_L \in A(G)_{(P)}$ be the corresponding primitive idempotent under the bijection from Theorem 3.4. Fix subgroups $K \leq H \leq G$. Then the norm map $N_K^H\colon A(K)_{(P)} \to A(H)_{(P)}$ descends to a well-defined map of multiplicative monoids*

$$\tilde{N}_K^H\colon A(K)_{(P)}[e^{-1}] \to A(H)_{(P)}[e^{-1}]$$

if and only if the following holds:

($\star$) *Whenever $L' \leq H$ is conjugate in G to L, then L' is contained in K.*

The characterization of e_L in terms of marks in Theorem 3.4 implies that $R_H^G(e_L) = 0$ whenever L is not subconjugate in G to H. From this, it is clear that the norm $\tilde{N}_K^H$ exists for trivial reasons if K is not super-conjugate in G to L: it is just the zero morphism between zero rings. Similarly, there cannot be a norm map $\tilde{N}_K^H$ inherited from N_K^H if K is not super-conjugate to L, but H is. Indeed, it would have to be a map of multiplicative monoids from the zero ring to a non-trivial ring, hence would satisfy $\tilde{N}_K^H(0) = 1$. But $N_K^H(0) = [\mathrm{map_K(H, \emptyset)}] = 0$ before localizing, which is a contradiction. The other cases are not obvious. We defer the proof of Theorem 4.1 to Section 4.2 and first state and prove the locally enhanced versions of Corollary B and Corollary C.

Corollary 4.2. *Assume that $L \leq G$ is P-perfect. Then L is normal in G if and only if the summand $A(-)_{(P)}[e_L^{-1}]$ inherits from $A(-)_{(P)}$ all norms of the form $\tilde{N}_K^H$ such that K contains a subgroup conjugate in G to L.*

Proof. If L is normal, it is the only group in its G-conjugacy class, hence the condition ($\star$) of Theorem 4.1 is satisfied for such $K \leq H$. Conversely, if the condition holds for the groups $K := L$ and $H := G$, then any G-conjugate of L is contained in L, hence L is normal in G. □

Corollary 4.3. *The Green ring $A(-)_{(P)}[e_L^{-1}]$ admits norms $\tilde{N}_K^H$ for all $K \leq H$ if and only if $L = 1$ is the trivial group. In this case, the norm maps equip $A(-)_{(P)}[e_1^{-1}]$ with the structure of a Tambara functor.*

Proof. If $L = 1$, then all groups are supergroups of L and all subgroup inclusions give rise to norm maps by Corollary 4.2. It then follows from [BH18, Thm. 4.13] that $A(-)_{(P)}[e_1^{-1}]$ is a Tambara functor, cf. the proof of Theorem 2.33. Conversely, if L is non-trivial P-perfect, the inclusion $1 \to G$ does not give rise to a well-defined norm on $A(-)_{(P)}[e_L^{-1}]$. □

Remark 4.4. The "only if" part is implicit in work of Blumberg and Hill, at least integrally: All idempotents e_L different from e_1 lie in the augmentation ideal of $A(G)$. If inverting such an element yielded a Tambara functor, then it would have to be the zero Tambara functor, see [BH18, Example 5.25]. But $A(-)[e_L^{-1}]$ is always non-zero.

Remark 4.5. It is also implicit in Nakaoka's work on ideals of Tambara functors [Nak12] that the idempotent summands of the (P-local) Burnside ring Mackey functor cannot all be Tambara functors, for if they were, then the idempotent splitting would be a splitting of Tambara functors. But by [Nak12, Prop. 4.15], this implies that $A(1) \cong \mathbb{Z}$ splits non-trivially, which is absurd. (Note that there is a minor error in statements (2)–(4) of loc. cit.: the requirement that the respective ideals and elements be non-zero is missing.)

Remark 4.6. When working p-locally, the ring $A(G)_{(p)}[e_1^{-1}]$ can be described in two different ways: It agrees with the p-local Burnside ring with p-isotropy, i.e., the p-localization of the Grothendieck ring of finite G-sets all of whose isotropy groups are p-groups. Moreover, it can be identified with the p-localization of the Burnside ring of the p-fusion system of the group G. We refer the reader to [Gro, Section 5] for details.

As an illustration of Theorem 4.1, we will discuss the idempotent splittings of $A(A_5)$ (integrally) and $A(\Sigma_3)$ (locally at the primes 2 and 3) in detail in Section 5. There, we also spell out what happens in the rational splitting ($P = \emptyset$) for any finite group G.

4.2. **The proof of Theorem A.** The main idea of the proof is that we can check the hypotheses for preservation of norm maps from Theorem 2.33 on marks. As norm maps in the Burnside ring are given by co-induction functors of equivariant sets, we need to understand how they interact with taking fixed points. To that end, we first record some technical statements before giving the proof of Theorem 4.1 (Theorem A).

Lemma 4.7. *For subgroups $K, H \leq N \leq G$, let P be the pullback in the category of G-sets of the canonical surjections $G/H \to G/N$ and $G/K \to G/N$.*

$$\begin{array}{ccc} P & \longrightarrow & G/H \\ \big\downarrow & & \big\downarrow \\ G/K & \longrightarrow & G/N \end{array}$$

Then P has an orbit decomposition given by

$$P \cong \coprod_{n \in K\backslash N/H} G/(K \cap {}^nH)$$

where the summation is over representatives of double cosets.

This implies a multiplicative double coset formula for norm maps of $A(-)_{(P)}$.

Corollary 4.8. *For K, H, N and G as before and all $x \in A(H)_{(P)}$, the following identity holds in $A(K)_{(P)}$:*

$$R^N_K N^N_H(x) = \prod_{n \in K\backslash N/H} N^K_{K\cap {}^nH} \cdot c_n \cdot R_{n^{-1}K\cap H}(x)$$

where we wrote c_n for the map induced from conjugation by $n \in N$.

Lemma 4.9. *The norms of $A(-)_{(P)}$ satisfy $\phi^H(N^H_K(a)) = \phi^K(a)$ for all $a \in A(K)_{(P)}$ and all nested subgroups $K \leq H \leq G$.*

Proof. Let $Q \leq G$ be any subgroup. Under the isomorphism $A(G)_{(P)} \cong \pi^G_0(\mathbb{S}_{(P)})$ of Theorem 3.13, the homomorphism of marks $\phi^Q\colon A(G)_{(P)} \to \mathbb{Z}_{(P)}$ identifies with the map

$$\pi^G_0(\mathbb{S}_{(P)}) \to \mathbb{Z}_{(P)}, \ \left[f\colon \mathbb{S}_{(P)} \to \mathbb{S}_{(P)}\right] \mapsto \deg(\phi^Q(f))$$

which sends a class represented by f to the degree of the map $\phi^Q(f)$ induced on geometric fixed points, see [Seg71, p. 60]. Thus, it suffices to prove that the degrees of the two maps of non-equivariant spectra

$$\phi^H\left(\bigwedge_{H/K} \mathrm{Res}_K\mathbb{S}_{(P)} \to \bigwedge_{H/K} \mathrm{Res}_K\mathbb{S}_{(P)} \xrightarrow{\cong} \mathrm{Res}_H\mathbb{S}_{(P)}\right)$$

and $\phi^K(f)$ coincide. This follows immediately from [HHR16, B.209]. □

Corollary 4.10 (Cf. [Oda14], Lemma 2.2)**.** *For $Q, K \leq H$, we can compute the marks ϕ^Q of a norm as follows:*

$$\phi^Q N^H_K(x) = \prod_{h \in Q\backslash H/K} \phi^{Q\cap {}^hK}(x)$$

Proof. In the following computation, the second equality is the multiplicative double coset formula of Corollary 4.8, and the third uses that ϕ^Q is a ring homomorphism. The fourth equality is an application of Lemma 4.9.

$$\phi^Q N^H_K(x) = \phi^Q R^H_Q N^H_K(x) = \phi^Q\left(\prod_h N^Q_{Q\cap {}^hK} c_h R^K_{h^{-1}Q\cap K}(x)\right)$$

$$= \prod_h \phi^Q N^Q_{Q\cap {}^hK} c_h R^K_{h^{-1}Q\cap K}(x) = \prod_h \phi^{Q\cap {}^hK} c_h R^K_{h^{-1}Q\cap K}(x) = \prod_h \phi^{Q\cap {}^hK}(x)$$

□

Lemma 4.11. *Let $e, e' \in R$ be idempotents in a commutative ring. Then e divides e' if and only if $e \cdot e' = e'$.*

Proof. Assume that e divides e'. Then $e' \in eR$, hence $e \cdot e' = e'$, since multiplication by e is projection onto the idempotent summand eR of R. The other direction is obvious. □

Lemma 4.12. *For $H \leq G$ and $g \in G$, the following holds:*

a) $O^P(H) \subseteq O^P(G)$

b) $O^P({}^gH) = {}^g(O^P(H))$

The author learned the proof of part a) from Joshua Hunt.

Proof. Since $O^P(G)$ is normal in G, we know that $H \cap O^P(G)$ is normal in H. Now the group $H/(H \cap O^P(G)) \cong (H \cdot O^P(G))/O^P(G) \leq G/O^P(G)$ is isomorphic to a subgroup of a solvable P-group, hence is a solvable P-group itself. By minimality, $O^P(H) \leq H \cap O^P(G) \leq O^P(G)$, which proves a).
The assertion b) follows from the fact that conjugation by g induces a bijection between the subgroup lattices of H and ${}^gH$ which preserves normality. □

Proposition 4.13. *In the situation of Theorem 4.1, the following are equivalent:*

($\star$) *Every subgroup $L' \leq H$ that is conjugate in G to L is contained in K.*

($\Diamond$) *For all $Q \leq H$ such that $O^P(Q) \sim_G L$, we have $\phi^Q(N_K^H(R_K(e_L))) = 1$.*

Proof. The proof proceeds in three steps. Step 1 and 2 simplify the condition ($\Diamond$), whereas Step 3 shows that the resulting reformulation of ($\Diamond$) is equivalent to ($\star$).
Step 1: Let $Q \leq Q' \leq H$ such that $O^P(Q) \sim_G L \sim_G O^P(Q')$. We claim that if the statement of ($\Diamond$) holds for Q, then it does so for Q'.
Indeed, if ($\Diamond$) holds for Q, then Theorem 3.4 together with Corollary 4.10 implies that for all $h \in H$, we have $O^P(Q \cap {}^hK) \sim_G L$. From Lemma 4.12, we see that

$$L \sim_G O^P(Q \cap {}^hK) \leq O^P(Q' \cap {}^hK) \leq O^P(Q') \sim_G L,$$

so $O^P(Q' \cap {}^hK)$ is conjugate to L, and hence the statement ($\Diamond$) holds for Q'.
The above claim shows that when verifying ($\Diamond$), we need not take into account all elements of the set $\{Q \leq H \,|\, O^P(Q) \sim_G L\}$ but can restrict attention to its minimal elements under inclusion, i.e., to the groups $L'' \leq H$ such that $L'' \sim_G L$. In other words, ($\Diamond$) is equivalent to:

($\Diamond a$) For all $L'' \leq H$ such that $L'' \sim_G L$, we have $\phi^{L''}(N_K^H(R_K(e_L))) = 1$.

Step 2: Let L'' be as in ($\Diamond a$). As we have seen, the equation $\phi^{L''}(N_K^H(R_K(e_L))) = 1$ holds if and only for all $h \in H$, we have $O^P(L'' \cap {}^hK) \sim_G L$. But

$$O^P(L'' \cap {}^hK) = O^P({}^{h^{-1}}L'' \cap K),$$

so substituting L' for ${}^{h^{-1}}L''$ shows that ($\Diamond$) is equivalent to:

($\Diamond b$) For all $L' \leq H$ such that $L' \sim_G L$, we have $O^p(L' \cap K) \sim_G L$.

Step 3: We are left to show that for L' as in ($\Diamond b$), L' is in K if and only $O^p(L' \cap K) \sim_G L$. For the "only if" part, assume that $L' \leq K$, then $O^p(L' \cap K) = L' \sim_G L$. For the "if" part, observe that

$$L \sim_G O^p(L' \cap K) \leq L' \cap K \leq L'.$$

The conjugate copy of L contained in $L' \cap K$ must be L', so $L' \leq K$. □

Proof of Theorem 4.1. We know from Theorem 2.33 that the norm N_K^H descends to a well-defined map $\tilde{N}_K^H$ if and only if the element $N_K^H(R_K(e_L))$ divides $R_H(e_L)$ in $A(H)_{(P)}$. By Lemma 4.11, this division relation is equivalent to the equation

$$N_K^H(R_K(e_L)) \cdot R_H(e_L) = R_H(e_L)$$

and holds if and only if for all $Q \leq H$, we have

$$\phi^Q(N_K^H(R_K(e_L))) \cdot \phi^Q(R_H(e_L)) = \phi^Q(R_H(e_L)).$$

Here, we used that the homomorphism of marks

$$\phi = \prod_{(Q) \leq H} \phi^Q \colon A(H)_{(P)} \to \prod_{(Q) \leq H} \mathbb{Z}_{(P)}$$

is an injective ring homomorphism.

All three integers in the last equation are idempotents, hence can only be 0 or 1, and the equation holds in all cases except when $\phi^Q(N_K^H(R_K(e_L)))$ is zero, but $\phi^Q(R_H(e_L))$ is one. The formula for marks given in Theorem 3.4 then implies that the equation is equivalent to the condition ($\Diamond$) of Proposition 4.13. The latter is equivalent to ($\star$), and Theorem 4.1 follows. □

4.3. **The incomplete Tambara functor structure.** It still remains to see how the collection of norm maps described by Theorem 4.1 fits into the framework of [BH18]. First of all, we describe the norm maps in $A(-)_{(P)}[e_L^{-1}]$ arising from arbitrary maps of G-sets. This is the special case $\underline{R} = A(-)_{(P)}$, $x = e_L$ of the following result:

Proposition 4.14. *Let $x \in \underline{R}(G)$ be an idempotent. Let f be an arbitrary map of finite G-sets. Choose orbit decompositions of X and Y such that f is the sum of canonical surjections*

$$f \colon X = \coprod_{i,j} G/K_{ij} \to Y = \coprod_i G/H_i$$

induced by subgroup inclusions $K_{ij} \leq H_i$. Then the levelwise localization $\underline{R}[x^{-1}]$ inherits a norm map $\tilde{N}_f$ from $\underline{R}$ if and only if each restriction to orbits $f_{ij} \colon G/K_{ij} \to G/H_i$ does.

Proof. The proof proceeds in two steps.

Step 1: By the universal property of the product (of underlying multiplicative monoids), a potential norm map defined by f is given componentwise by the potential norms induced by the restricted maps $f_i\colon \coprod_j G/K_{ij} \to G/H_i$. Consequently, $\tilde{N}_f$ exists if and only if $\tilde{N}_{f_i}$ exists for all i.

Step 2: We are left to show that a map $f_i\colon \coprod_j G/K_{ij} \to G/H_i$ gives rise to a norm map if and only if all of the maps $f_{ij}\colon G/K_{ij} \to G/H_i$ do. But under the identification $\underline{R}(\coprod_j G/K_{ij}) \cong \prod_j \underline{R}(K_{ij})$, the norm N_f is of the form

$$\prod_j \underline{R}(K_{ij}) \to \underline{R}(H_i), \quad (a_j)_j \mapsto \prod_j N_{f_{ij}}(a_j).$$

The analogous statement holds for the norms of $\underline{R}[x^{-1}]$, provided they exist. Thus, $\tilde{N}_{f_i}$ exists if and only if $\tilde{N}_{f_{ij}}$ exists for all j. □

We would like to use Theorem 2.33 in order to show that $A(-)_{(P)}[e_L^{-1}]$ is an incomplete Tambara functor with norms as described in Theorem 4.1. However, Theorem 2.33 is a statement about Tambara functors structured by indexing systems, or equivalently (see Theorem 2.15), structured by wide, pullback-stable, finite coproduct-complete subcategories $D \subseteq \mathcal{S}et^G$. Thus, we first need to see that the maps f which give rise to norm maps form such a category D.

Definition 4.15. Let $D_L \subseteq \mathcal{S}et^G$ be the wide subgraph consisting of all the maps of finite G-sets $f\colon X \to Y$ such that the orbit $G_{f(x)}/G_x$ obtained from stabilizer subgroups satisfies the conditions of Theorem 4.1 for all $x \in X$.

Proposition 4.16. *The subgraph D_L is a wide, pullback-stable, finite coproduct-complete subcategory of $\mathcal{S}et^G$, hence corresponds to an indexing system $\mathcal{I}_L$ under the equivalence of posets of Theorem 2.15.*

Explicitly, the admissible H-sets in $\mathcal{I}_L$ are the objects over G/H in D_L, see [BH18, Lemma 3.19]. The three lemmas below constitute the proof.

Lemma 4.17. *The graph D_L is a wide subcategory of $\mathcal{S}et^G$.*

Proof. It is wide by definition and clearly contains all identities. Once we have shown that it is closed under composition, associativity follows from associativity in $\mathcal{S}et^G$. Let $f\colon S \to T$ and $g\colon T \to U$ be admissible maps of G-sets. By Proposition 4.14, we may assume that $S = G/A, T = G/B$ and $U = G/C$ are transitive G-sets for nested subgroups $A \le B \le C \le G$, and f, g are the canonical surjections. Thus, it suffices to show that if C/B and B/A are admissible, so is C/A. This is immediate from the condition ($\star$) given in Theorem 4.1. □

Lemma 4.18. *The subcategory D_L is finite coproduct-complete.*

Proof. This follows directly from Proposition 4.14. □

Lemma 4.19. *The subcategory D_L is pullback-stable.*

Proof. The problem reduces to canonical surjections between orbits by Proposition 4.14. We have to show that if the canonical surjection $G/K \to G/H$ in the following pullback diagram is admissible, then so is its pullback along the canonical map $G/A \to G/H$, where $A, K \leq H$ are subgroups.

$$\begin{array}{ccc} P & \longrightarrow & G/K \\ \downarrow & & \downarrow \\ G/A & \longrightarrow & G/H \end{array}$$

This in turn amounts to verifying the condition $(\star)$ of Theorem 4.1 for all summands of

$$R^H_A(H/K) \cong \coprod_{[h] \in A \backslash H/K} A/(A \cap {}^hK).$$

Note that since H/K is admissible, so are the isomorphic H-sets $H/{}^hK$ for all $h \in H$. Fix $L' \leq A$ such that $L' \sim_G L$. We have to show that $L' \leq A \cap {}^hK$. But L' is in H and $H/{}^hK$ is admissible, so $L' \leq {}^hK$ and hence $L' \leq A \cap {}^hK$. □

We obtain (the locally enhanced) Theorem D:

Theorem 4.20. *Let P be a collection of primes. Let $L \leq G$ be a P-perfect subgroup and let $e_L \in A(G)_{(P)}$ be the corresponding primitive idempotent. Then the following hold:*

i) The admissible sets for e_L assemble into an indexing system $\mathcal{I}_L$ such that $A(-)_{(P)}[e_L^{-1}]$ is an $\mathcal{I}_L$-Tambara functor under $A(-)_{(P)}$.
ii) In the poset of indexing systems, $\mathcal{I}_L$ is maximal among the elements that satisfy i).
iii) The map $A(-)_{(P)} \to A(-)_{(P)}[e_L^{-1}]$ is the localization of $A(-)_{(P)}$ at e_L in the category of $\mathcal{I}_L$-Tambara functors.

Proof. Proposition 4.16 shows that $\mathcal{I}_L$ is an indexing system. Then $A(-)_{(P)}[e_L^{-1}]$ is an $\mathcal{I}_L$-Tambara functor by [BH18, Thm. 4.13], see the proof of Theorem 2.33 for details. Theorem 2.33 also implies part iii). Part ii) follows from Theorem 4.1 together with Proposition 4.14. □

Finally, we describe the maximal incomplete Tambara functor structure which is preserved by the idempotent splitting of the Green ring $A(-)_{(P)}$ stated in Proposition 3.9.

Lemma 4.21. *The (levelwise) intersection of a finite number of indexing systems is an indexing system.* □

Notation 4.22. Write $\mathcal{I}$ for the indexing system

$$\mathcal{I} := \bigcap_{(L)\leq G} \mathcal{I}_L$$

where the intersection is over all conjugacy classes of P-perfect subgroups of G, and the indexing systems $\mathcal{I}_L$ are the ones given by Theorem 4.20.

For each P-perfect $L \leq G$, the $\mathcal{I}_L$-Tambara functor $A(-)_{(P)}[e_L^{-1}]$ is an $\mathcal{I}$-Tambara functor by forgetting structure. Theorem 4.1 provides a very explicit description of the admissible sets of $\mathcal{I}$.

Lemma 4.23. *Let $K \leq H \leq G$, then H/K is an admissible set for $\mathcal{I}$ if and only if for all perfect $L \leq H$, L is contained in K.* □

We can now restate Corollary E.

Corollary 4.24. *The localization maps $A(-)_{(P)} \to A(-)_{(P)}[e_L^{-1}]$ assemble into an isomorphism of $\mathcal{I}$-Tambara functors*

$$A(-)_{(P)} \to \prod_{(L)\leq G\ P-\text{perfect}} A(-)_{(P)}[e_L^{-1}].$$

Proof. It is an isomorphism of Green rings by Proposition 3.9. Moreover, each of the localization maps $A(-)_{(P)} \to A(-)_{(P)}[e_L^{-1}]$ is a map of $\mathcal{I}$-Tambara functors, and the product in the category of $\mathcal{I}$-Tambara functors is computed levelwise, see [Str12, Prop. 10.1]. □

Remark 4.25. We point out a possible alternative to our proof of Corollary 4.24. Blumberg and Hill generalized parts of Nakaoka's theory of ideals of Tambara functors [Nak12] to the setting of incomplete Tambara functors, see [BH18, Section 5.2]. The author is confident that one could similarly generalize Nakaoka's splitting result [Nak12, Prop. 4.15] to the incomplete setting. It would state that an $\mathcal{I}$-Tambara functor $\underline{R}$ splits non-trivially as a product of $\mathcal{I}$-Tambara functors if and only if for each admissible set X of $\mathcal{I}$, there are non-zero elements $a, b \in \underline{R}(X)$ such that $a + b = 1$ and $\langle a\rangle \cdot \langle b\rangle = 0$. Such a result would reprove our Corollary 4.24, using that the restrictions of the primitive idempotents e_L along admissible maps never become zero. We leave the details to the interested reader.

4.4. **The N_∞ ring structure.** We return to the situation of Question 1.2, lift our algebraic results to the category of G-spectra and prove the locally enhanced versions of Corollary F, Corollary G and Corollary H.

Observe that for any N_∞ operad $\mathcal{P}$, the object $\mathbb{S}_{(P)}$ admits the structure of a commutative monoid in orthogonal G-spectra, hence admits a natural $\mathcal{P}$-algebra action that factors through the commutative operad.

Corollary 4.26. *Let $L \leq G$ be a P-perfect subgroup and let $e_L \in \underline{\pi}_0^G(\mathbb{S})$ be the associated idempotent. For any Σ-cofibrant N_∞ operad $\mathcal{O}_L$ whose associated indexing system is $\mathcal{I}_L$, the following hold:*

i) The G-spectrum $\mathbb{S}_{(P)}[e_L^{-1}]$ is an $\mathcal{O}_L$-algebra under $\mathbb{S}_{(P)}$.
ii) In the poset of homotopy types of N_∞ operads, $\mathcal{O}_L$ is maximal among the elements that satisfy i).
iii) The map $\mathbb{S}_{(P)} \to \mathbb{S}_{(P)}[e_L^{-1}]$ is a localization at e_L in the category of $\mathcal{O}_L$-algebras.

The cofibrancy assumption does not impose an obstruction to the existence of $\mathcal{O}_L$, see Remark 2.11.

Proof. It is clear from Theorem 2.33 that $\mathcal{O}_L$ satisfies the hypothesis of Proposition 2.26, which proves part i) and iii). For part ii), assume that there is an N_∞ operad $\mathcal{O}'$ whose homotopy type is strictly greater than that of $\mathcal{O}_L$ such that $\mathbb{S}_{(P)}[e_L^{-1}]$ is an $\mathcal{O}'$-algebra. Then, by Theorem 2.20, its 0-th equivariant homotopy forms a $\mathcal{I}'$-Tambara functor for the indexing system $\mathcal{I}'$ corresponding to $\mathcal{O}'$. But this contradicts the maximality proved in Corollary 4.20. □

The following local enhancement of Corollary G is a homotopical reformulation of Corollary 4.3 (Corollary C).

Corollary 4.27. *The G-spectrum $\mathbb{S}_{(P)}[e_L^{-1}]$ is a G-E_∞ ring spectrum if and only if $L = 1$ is the trivial group.*

In particular, we see that the idempotent splitting of $\mathbb{S}$ is far from being a splitting of G-E_∞ ring spectra. Locally at the prime p, Corollary 4.27 recovers a (yet unpublished) result of Grodal.

Theorem 4.28 ([Gro], Cor. 5.5)**.** *The G-spectrum $\mathbb{S}_{(p)}[e_1^{-1}]$ is a G-E_∞ ring spectrum.*

Finally, we state the homotopy-theoretic analogue of Corollary 4.24 in order to describe the maximal N_∞-ring structure preserved by the P-local idempotent splitting of the sphere. It is the local reformulation of Corollary H.

Corollary 4.29. *Let $\mathcal{O}$ be a Σ-cofibrant N_∞ operad realizing the indexing system $\mathcal{I} = \cap_{(L)}\mathcal{I}_L$. Then the idempotent splitting*

$$\mathbb{S}_{(P)} \simeq \prod_{(L)\leq G} \mathbb{S}_{(P)}[e_L^{-1}]$$

is an equivalence of $\mathcal{O}$-algebras, where the product is taken over conjugacy classes of P-perfect subgroups.

Proof. The splitting is an equivalence of G-spectra by Proposition 3.15. Moreover, all of the maps to the localizations are maps of $\mathcal{O}$-algebras, as can be seen from 4.26. □

Together, Corollary 4.26 and Corollary 4.29 answer Question 1.2 completely, for any family of primes inverted.

5. Examples

We illustrate our results in the rational case, in the case of the alternating group A_5, working integrally, and that of the symmetric group Σ_3, working 3-locally.

5.1. **The rational case.** In the case when $P = \emptyset$ and hence $\mathbb{Z}_{(P)} = \mathbb{Q}$, the rational Burnside ring $A(G)_\mathbb{Q}$ has exactly one primitive idempotent e_L for each conjugacy class of subgroups $L \leq G$. The incomplete Tambara functor structures of the idempotent summands $A(-)_\mathbb{Q}[e_L^{-1}]$ depend on the subgroup structure of G as described by Theorem 4.1. However, it is immediately clear from Lemma 4.23 that the idempotent splitting is only a splitting of Green rings, but not a splitting of $\mathcal{I}'$-Tambara functors for any indexing system $\mathcal{I}'$ greater than the minimal one. This phenomenon is also discussed in [BGK17, Section 7], and it is precisely the reason why their approach only provides an algebraic model for the rational homotopy theory of naive N_∞ ring spectra, but cannot possibly account for any non-trivial Hill-Hopkins-Ravenel norms.

5.2. **The alternating group A_5.** It is well-known that A_5 is the smallest non-trivial perfect group. Thus, it is the smallest example of a group whose Burnside ring admits a non-trivial idempotent splitting when working integrally. Indeed, the only perfect subgroups are 1 and A_5, and these give rise to idempotent elements $e_1, e_{A_5} \in A(A_5)$. Theorem 3.4 implies that their marks are given by

$$\phi^H(e_{A_5}) = \begin{cases} 1, & H = A_5 \\ 0, & H \neq A_5 \end{cases}$$

and vice versa for e_1. We know from Corollary 4.3 that $A(A_5)[e_1^{-1}]$ is a complete Tambara functor. On the other hand, $A(H)[e_{A_5}^{-1}]$ is trivial unless $H = A_5$, hence there cannot be any norm maps $N_H^{A_5}$ for proper subgroups $H \leq A_5$. Moreover, by Corollary 4.24, the idempotent splitting of $A(-)$ is a splitting of $\mathcal{I}_{A_5}$-Tambara functors, i.e., it only preserves norms between proper subgroups.

By Corollary 4.26 and Corollary 4.29, the analogous statements hold for the N_∞ ring structures on $\mathbb{S}[e_1^{-1}]$ and $\mathbb{S}[e_{A_5}^{-1}]$. Just like Example 2.23, this provides another instance of the phenomenon that inverting a single homotopy element does not preserve any of the Hill-Hopkins-Ravenel norm maps from proper subgroups to the ambient group.

Of course, all of this holds for any perfect group G whose only perfect subgroup is the trivial group.

5.3. **The symmetric group Σ_3 at the prime 3.** Since Σ_3 is solvable, its Burnside ring $A(\Sigma_3)$ does not have any idempotents other than zero and one. We can obtain interesting idempotent splittings by working locally at primes p dividing the group order. All 2-perfect subgroups of Σ_3 are normal, hence the case $p = 2$ is completely covered by Corollary 4.2 and we only discuss the more interesting case $p = 3$ in detail.

Any map in the orbit category can be factored as an isomorphism followed by a canonical surjection, hence the admissibilty of Σ_3/H just depends on the conjugacy class of H and we can just write C_2 for any of the three conjugate subgroups of order two. Note that the 3-residual subgroups $O^3(H)$ for $H \leq \Sigma_3$ are given as follows:

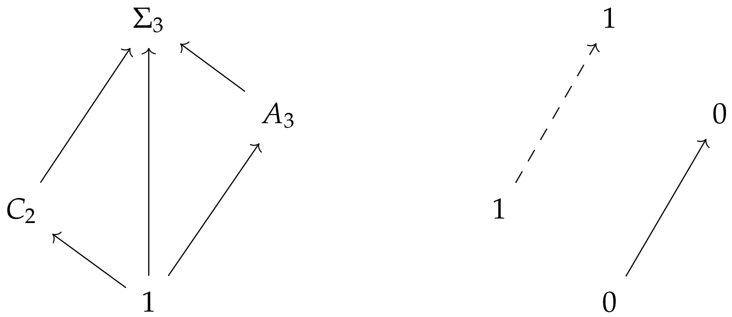
$$\mathrm{O}^3(H) = \begin{cases} \Sigma_3, & H = \Sigma_3 \\ 1, & H = A_3 \\ C_2, & H = C_2 \\ 1, & H = 1 \end{cases}$$

Thus, all subgroups of Σ_3 except for A_3 are 3-perfect. All subgroups of order two are conjugate in Σ_3, so there are three idempotent elements in $A(\Sigma_3)_{(3)}$, corresponding to the conjugacy classes of the 3-perfect subgroups $1, C_2$ and Σ_3. In terms of marks, they are given as

Subgroup $H \leq \Sigma_3$	$\phi^H(e_1)$	$\phi^H(e_{C_2})$	$\phi^H(e_{\Sigma_3})$
1	1	0	0
C_2	0	1	0
A_3	1	0	0
Σ_3	0	0	1

The localization $A(-)_{(3)}[e_1^{-1}]$ admits all norms by Corollary 4.3. The norm maps of $A(-)_{(3)}[e_{\Sigma_3}^{-1}]$ are described by Corollary 4.2. In detail, this Mackey functor is zero at all proper subgroups, but non-trivial at Σ_3. Consequently, there are norm maps $\tilde{N}_K^H$ if and only if H and hence K is a proper subgroup of Σ_3, but all of these norms are maps

between trivial rings.
It remains to describe the idempotent localization $A(-)_{(3)}[e_{C_2}^{-1}]$. The left of the following two diagrams depicts the subgroups $H \leq \Sigma_3$ (up to conjugacy) and their inclusions. The right hand diagram displays the ranks (as free $\mathbb{Z}_{(3)}$-modules) of the corresponding values of $A(-)_{(3)}[e_{C_2}^{-1}]$ at the subgroup H.



There is a norm map from 1 to A_3 for trivial reasons (indicated by the solid arrow) and the only other norm maps which could potentially exist would be the maps $\tilde{N}_{C_2}^{\Sigma_3}$ where C_2 is any subgroup of order two (indicated by the dashed arrow). However, if we choose $K = L = (12)$ and let $L' = (13)$, then the condition ($\star$) of Theorem 4.1 is not satisfied. Indeed, L' is conjugate to L, but not contained in K. Consequently, there is no norm map $\tilde{N}_{C_2}^{\Sigma_3}$.
We see from Lemma 4.23 that $\mathcal{I} = \mathcal{I}_{C_2}$, so in this case the idempotent splittings of $A(-)_{(3)}$ and hence $\mathbb{S}_{(3)}$ only preserve the norm map $N_1^{A_3}$.

6. Applications

6.1. **Norm functors in the idempotent splitting of** Sp^G**.** Any G-spectrum X is a module over the sphere spectrum, hence admits an idempotent splitting

$$X \simeq \prod_{(L) \leq G} X[e_L^{-1}]$$

where $X[e_L^{-1}]$ is the sequential homotopy colimit along countably many copies of the map $X \cong X \wedge \mathbb{S} \xrightarrow{\mathrm{id} \wedge e_L} X \wedge \mathbb{S} \cong X$. Thus, the idempotent elements of $A(G)$ induce a product decomposition of the category of G-spectra Sp^G by breaking it up into categories of modules over the idempotent summands $\mathbb{S}[e_L^{-1}]$. Similar statements hold in the local cases, cf. e.g. [Bar09, Thm. 4.4, Section 6]. While this only depends on the additive splitting of Proposition 3.15, some additional multiplicative structure is present.

It is useful to consider not just the category of (P-local) G-spectra, but rather the symmetric monoidal categories of (P-local) H-spectra for all subgroups $H \leq G$ together with their restriction and norm functors. This kind of structure has been studied in [HH16, BH15a] under the name of *G-symmetric monoidal categories*. From this perspective, Theorem A measures the failure of the idempotent splitting of Sp^G to give rise to a splitting of G-symmetric monoidal categories. Indeed, the factors only admit some of the Hill-Hopkins-Ravenel norm functors and hence form "incomplete G-symmetric monoidal categories":

Corollary 6.1. *Let $L \leq G$ be P-perfect and let $\mathcal{O}_L$ as in Corollary 4.26. Assume furthermore that $\mathcal{O}_L$ has the homotopy type of the linear isometries operad on a (possibly incomplete) universe U. For all admissible sets H/K of $\mathcal{I}_L$, there are norm functors*

$$_{\mathrm{Res}_H(\mathbb{S}_{(P)}[e_L^{-1}])}N_{K,\mathrm{Res}_K(U)}^{H,\mathrm{Res}_H(U)} : \mathit{Mod}(\mathrm{Res}_K^G(\mathbb{S}_{(P)}[e_L^{-1}])) \to \mathit{Mod}(\mathrm{Res}_H^G(\mathbb{S}_{(P)}[e_L^{-1}]))$$

built from the smash product relative to $\mathbb{S}_{(P)}[e_L^{-1}]$ which satisfy a number of relations analogous to those for the norm functor $\mathrm{Sp}^H \to \mathrm{Sp}^G$, stated in [BH15a, Thm. 1.3].

This is an immediate application of [BH15a, Thm. 1.1, Thm. 1.3] to Corollary 4.26. We refer to [BH15a] for a detailed discussion of modules over N_∞ ring spectra.

The reason for the "linear isometries" hypothesis is explained in the introduction to [BH15a]. It is expected that it is not necessary, and that the ∞-categorical tools developed in [BDG$^+$17] and its sequels will remove this technical assumption.

6.2. **Idempotent splittings of equivariant topological K-theory.** Our main questions, Question 1.2 and Question 1.3, can be asked for any G-E_∞ ring spectrum and its idempotent splitting, assuming there are only finitely many primitive idempotents and that these admit a suitably explicit description. In the sequel [Böh18], we will answer the analogues of our main questions for the G-equivariant complex topological K-theory spectrum KU_G and its real analogue KO_G. It turns out that the solution can be reduced to the one given here, but in order to see this, a careful analysis of the complex representation ring and its relationship with the Burnside ring is required.

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Part 3

Paper B

IDEMPOTENT CHARACTERS AND EQUIVARIANTLY MULTIPLICATIVE SPLITTINGS OF K-THEORY

BENJAMIN BÖHME

ABSTRACT. We classify the primitive idempotents of the p-local complex representation ring of a finite group G in terms of the cyclic subgroups of order prime to p and show that they all come from idempotents of the Burnside ring. Our results hold without adjoining roots of unity or inverting the order of G, thus extending classical structure theorems. We then derive explicit group-theoretic obstructions for tensor induction to be compatible with the resulting idempotent splitting of the representation ring Mackey functor.

Our main motivation is an application in homotopy theory: we conclude that the idempotent summands of G-equivariant topological K-theory and the corresponding summands of the G-equivariant sphere spectrum admit exactly the same flavors of equivariant commutative ring structures, made precise in terms of Hill-Hopkins-Ravenel norm maps.

This paper is a sequel to the author's earlier work on multiplicative induction for the Burnside ring and the sphere spectrum, see arXiv:1802.01938v1.

1. INTRODUCTION

The purpose of this paper is twofold: We first classify the primitive idempotents in the real and complex representation rings $RO(G)$ and $RU(G)$ of a finite group G and their local variants, as summarized in §1.1, extending various classical results. We then study the compatibility of tensor induction with the splittings of $RO(G)$ and $RU(G)$ into idempotent summands, and as a consequence obtain an explicit description of the G-equivariant commutative ring spectrum structures occuring as idempotent summands of real and complex G-equivariant topological K-theory. See §1.2 for a summary of these results.

We begin with some motivation. Multiplicative induction is a familiar tool in representation theory and group cohomology. In the wake of Hill, Hopkins and Ravenel's ground-breaking solution to the Kervaire invariant one problem [HHR16], it has also

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received much interest in equivariant homotopy theory. Starting from the observation that localization can destroy some of the structure of an equivariant commutative ring spectrum, Hill and Hopkins [HH14] gave a necessary and sufficient criterion (cf. Proposition 4.4) for the localization

$$R[x^{-1}] := \operatorname{hocolim}\left(R \xrightarrow{x} S^{-V} \wedge R \xrightarrow{x} S^{-(V\oplus V)} \wedge R \xrightarrow{x} \dots\right)$$

of a G-E_∞ ring spectrum R at an element $x \in \pi_V^G(R)$ to admit a G-E_∞ ring structure. The critical part is that $R[x^{-1}]$ might not admit Hill-Hopkins-Ravenel *norm maps*

$$N_K^H \colon G_+ \wedge_H \bigwedge_{H/K} \operatorname{Res}_K^G(R) \to R$$

for all nested subgroups $K \leq H \leq G$. Subsequently, more general notions of equivariant commutative ring spectra equipped with incomplete collections of norm maps, called N_∞ *ring spectra*, were studied by Blumberg and Hill in [BH15b], [BH18] and [BH15a].

Interesting examples of equivariant localizations arise from primitive[1] idempotent elements $e \in \pi_0^G(R)$. These induce a decomposition of the homotopy Mackey functor $\underline{\pi}_*(R)$ into indecomposable summands (also called *blocks*) of the form

$$e \cdot \underline{\pi}_*(R) \cong \underline{\pi}_*(R)[e^{-1}]$$

and hence yield a block decomposition of R as a wedge of G-spectra $R[e^{-1}]$. One can now ask about the possible N_∞ ring structures on these blocks. Hill and Hopkins' aforementioned criterion involves checking relations involving multiplicative induction in $\pi_0^G(R)$, which in general are hard to access.

Problem 1.1. Determine the nested subgroups $K \leq H \leq G$ such that

(1) the norm map N_K^H for R descends to a well-defined norm map[2]

$$\tilde{N}_K^H \colon G_+ \wedge_H \bigwedge_{H/K} \operatorname{Res}_K^G(R[e^{-1}]) \to R[e^{-1}]$$

on the block of R defined by the primitive idempotent $e \in \pi_0^G(R)$

(2) the induced norm operation on homotopy groups $N_K^H \colon \pi_0^K(R) \to \pi_0^H(R)$ descends to a well-defined norm operation

$$\tilde{N}_K^H \colon \pi_0^K(R[e^{-1}]) \to \pi_0^H(R[e^{-1}]).$$

[1] An idempotent is *primitive* if it cannot be written as a sum of non-zero idempotents.

[2] Throughout the paper, we write $\tilde{N}$ for the norms of a localization to distinguish them from the norms of the original object.

In the prequel [Böh], the author gave an explicit group-theoretical answer in the fundamental example of the G-equivariant sphere spectrum $\mathbb{S}$. It built on an analysis of multiplicative induction in the *Burnside ring* $A(G)$ and Segal's identification $\pi_0^G(\mathbb{S}) \cong A(G)$ [Seg71].

In the present paper, we present a complete solution to Problem 1.1 for *G-equivariant complex topological K*-theory KU_G and its real analogue KO_G. The homotopy groups

$$\pi_0^G(KU_G) \cong RU(G), \quad \pi_0^G(KO_G) \cong RO(G)$$

identify with the *complex* and *real representation ring* $RU(G)$ and $RO(G)$, respectively, see e.g. [Seg68, §2].

1.1. **Primitive idempotents in representation rings.** Dress' classification of primitive idempotents in the Burnside ring and its local variants [Dre69] was the starting point for the investigation of the idempotent splittings of $A(G)$ and $\mathbb{S}$ in [Böh]. Given a collection P of prime numbers, write $A(G)_{(P)} := A(G) \otimes \mathbb{Z}_{(P)}$ for the P-local Burnside ring, where $\mathbb{Z}_{(P)} := \mathbb{Z}\left[p^{-1} \mid p \notin P\right]$. Dress showed that the primitive idempotent elements $e_L \in A(G)_{(P)}$ are in canonical bijection with the conjugacy classes of P-perfect subgroups $L \leq G$. See § 2.1 for further details.

It is known that the complex representation ring $RU(G)$ has no idempotents other than zero or one, see [Ser77, §11.4, Corollary]. We extend this result to a classification of the primitive idempotents in the P-local representation ring $RU(G)_{(P)} := RU(G) \otimes \mathbb{Z}_{(P)}$ as follows. Consider the "linearization" map

$$lin\colon A(G)_{(P)} \to RU(G)_{(P)}$$

given by sending a finite G-set to its associated permutation representation.

Theorem 1.2. *The assignment $C \mapsto lin(e_C)$ defines a bijection between the conjugacy classes of cyclic subgroups $C \leq G$ of order not divisible by any prime in P and the primitive idempotent elements of the ring $RU(G)_{(P)}$. Here, $e_C \in A(G)_{(P)}$ denotes Dress' idempotent associated to C, see Theorem 2.2.*

Theorem 1.2 is an instance of the phenomenon that one passes from the Burnside ring to the representation ring by restricting attention to cyclic subgroups. The proof is given in §2.

Remark 1.3. Theorem 1.2 extends classical work in the following way: Building on work by Solomon [Sol67], Gluck [Glu81a] studies the idempotents $lin(e_C)$ and their character values in the rational and the p-local case for a single prime p, but does not show that they are primitive. He also observes that Dress' idempotent $e_L \in A(G)_{(P)}$

is in the kernel of the linearization map if L is not a cyclic group; we prove this in the general P-local case in Corollary 2.4.

We record an immediate consequence of Theorem 1.2. Write $RO(G)_{(P)}$ for the P-local real representation ring and $R\mathbb{Q}(G) \otimes \mathbb{Z}_{(P)}$ for the ring of $\mathbb{Z}_{(P)}$-linear combinations of G-representations over the rational numbers. It is well-known that these embed into $RU(G)_{(P)}$ as subrings.

Corollary 1.4. *The primitive idempotents of $RU(G)_{(P)}$ all lie in the subrings $RO(G)_{(P)}$ and $R\mathbb{Q}(G) \otimes \mathbb{Z}_{(P)}$. Hence, they are precisely the primitive idempotents of these subrings.*

In the special case of $R\mathbb{Q}(G) \otimes \mathbb{Q}$, this result appeared as [Sol67, Thm. 3].

1.2. **Multiplicativity of idempotent summands.** We now turn to the multiplicative properties of the idempotent splittings of the complex and real representation rings and equivariant K-theory spectra. Since the block $RU(G)_{(P)}[lin(e_C)^{-1}]$ agrees with the $A(G)_{(P)}$-module localization

$$RU(G)_{(P)}[e_C^{-1}] \cong RU(G)_{(P)} \otimes_{A(G)_{(P)}} A(G)_{(P)}[e_C^{-1}],$$

we obtain an identification

$$(KU_G)_{(P)}[lin(e_C)^{-1}] \simeq (KU_G)_{(P)} \wedge \mathbb{S}_{(P)}[e_C^{-1}]$$

of the blocks of P-local G-equivariant K-theory with an e_C-localization in genuine G-spectra. By Corollary 1.4, the same is true for $RO(G)_{(P)}$ and $(KO_G)_{(P)}$. This enables us to reduce the solution to Problem 1.1 for equivariant K-theory to the one for the sphere given in the prequel [Böh]. The resulting classification of the maximal N_∞ ring structures of the idempotent summands of $(KU_G)_{(P)}$ can be summarized as follows:

Theorem 1.5. *Let $C \leq G$ be a cyclic group of order not divisible by any prime in P and let e_C be the corresponding primitive idempotent in $A(G)_{(P)}$. Let $K \leq H \leq G$ be nested subgroups. Then the following are equivalent:*

(a) The G-spectrum $\mathbb{S}_{(P)}[e_C^{-1}]$ inherits a norm map $\tilde{N}_K^H$ from the norm map N_K^H of $\mathbb{S}_{(P)}$.
(b) The G-spectrum $(KU_G)_{(P)}[e_C^{-1}]$ inherits a norm map $\tilde{N}_K^H$ from that of $(KU_G)_{(P)}$.
(c) The Mackey functor $A(-)_{(P)}[e_C^{-1}]$ inherits a norm map $\tilde{N}_K^H$ from that of $A(-)_{(P)}$.
(d) The Mackey functor $RU(-)_{(P)}[e_C^{-1}]$ inherits a norm map $\tilde{N}_K^H$ from that of $RU(-)_{(P)}$.
(e) Any subgroup $C' \leq H$ conjugate in G to C lies in K.

All of the above holds with $(KU_G)_{(P)}$ and $RU(-)_{(P)}$ replaced by their real variants $(KO_G)_{(P)}$ and $RO(-)_{(P)}$.

The equivalence of (a), (c) and (e) was already proven in [Böh], up to a rephrasing of statement (e) explained in Lemma 3.4. Theorem 1.5 is made more precise in Theorem 3.3, Theorem 3.8 and Corollary 4.3 in terms of Blumberg and Hill's framework of incomplete Tambara functors and N_∞ operads.

Remark 1.6. If H does not contain a group conjugate in G to C, then the norms $\tilde{N}_K^H$ exist for trivial reasons: It can be seen from Theorem 2.2 that the restriction of e_C to H vanishes, and so $A(H)_{(P)}[e_C^{-1}]$ and $RU(H)_{(P)}[e_C^{-1}]$ must be zero. In other cases, these groups are always non-zero.

An immediate consequence of Theorem 1.5 is the following:

Corollary 1.7. *The summand $(KU_G)_{(p)}[e_C^{-1}]$ is a G-E_∞ ring spectrum if and only if $C \leq G$ is the trivial group. The same is true for real K-theory.*

1.3. **Organization.** In §2, we recall Dress' work on idempotents in the Burnside ring and give a proof of Theorem 1.2. The algebraic and homotopical parts of Theorem 1.5 are discussed in §3 and §4, respectively.

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2. Idempotent elements in representation rings

The goal of this section is to prove Theorem 1.2. In §2.1, we show how some parts of the theorem follow easily from the classification of idempotents in the Burnside ring. The difficult part is to prove that the images of the Burnside ring idempotents are indeed primitive. We recall Atiyah's description [Ati61] of the prime ideal spectrum $\operatorname{Spec}(RU(G)_{(P)} \otimes \mathcal{O}_\mathbb{F})$ in §2.2, where $\mathcal{O}_\mathbb{F}$ is obtained from $\mathbb{Z}$ by adjoining sufficiently many roots of unity, classify the idempotents of $RU(G)_{(P)} \otimes \mathcal{O}_\mathbb{F}$ in §2.3, and deduce the primitivity part of Theorem 1.2 in §2.4. In the rational and in the p-local case, it is possible to prove the primitivity in an easier way, as we explain in §2.5.

2.1. **Idempotents in the Burnside ring.** We recall Dress' classification of idempotents of $A(G)_{(P)}$. For a group H, let $O^P(H) \leq H$ denote its *P-residual subgroup*, i.e., its unique minimal normal subgroup such that the quotient is a solvable group of order not divisible by any of the primes in P. Recall that H is called *P-perfect* if $O^P(H) = H$.

Lemma 2.1. *A cyclic group is P-perfect if and only if its order is not divisible by any element of P.* □

Recall that for a subgroup $H \leq G$, the *mark homomorphism* $\phi^H\colon A(G)_{(P)} \to \mathbb{Z}_{(P)}$ is extended additively from the assignment $X \mapsto |X^H|$ for finite G-sets X.

Theorem 2.2 ([Dre69], Prop. 2)**.** *There is a canonical bijection between the conjugacy classes of P-perfect subgroups $L \leq G$ and the set of primitive idempotent elements of $A(G)_{(P)}$. It sends L to the element $e_L \in A(G)_{(P)}$ whose marks $\phi^H(e_L)$ at a subgroup $H \leq G$ are one if $O^P(H)$ and L are conjugate in G, and zero otherwise.*

Write $\chi(V)(g)$ for the value of the character of $V \in RU(G)_{(P)}$ at the element $g \in G$. The linearization map $lin\colon A(G)_{(P)} \to RU(G)_{(P)}$ satisfies the following simple identity:

Lemma 2.3. *For $X \in A(G)_{(P)}$, we have $\chi(lin(X))(g) = \phi^{\langle g \rangle}(X) \in \mathbb{Z}$, where ϕ^H is the homomorphism of marks $\phi^H(X) = |(lin(X))^H|$ associated to the subgroup $H \leq G$.*

Recall that by the Chinese remainder theorem, each $g \in G$ can be written uniquely as a product $(g)_{P'} \cdot h$, where $(g)_{P'}$ is a power of g of order prime to P and h is a power of g of order divisible only by primes in P. The element $(g)_{P'}$ is called the *P-prime part* of g.

Corollary 2.4. *Let $L \leq G$ be a P-perfect subgroup. Then the virtual representation $lin(e_L)$ has character values*

$$\chi(lin(e_L))(g) = \phi^{\langle g \rangle}(e_L) = \begin{cases} 1 & \text{if } \langle (g)_{P'} \rangle \sim_G L \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $lin(e_L)$ is zero if L is not cyclic. The elements $lin(e_C)$ are mutually orthogonal idempotents summing to one, where C ranges over a set of representatives for the conjugacy classes of cyclic P-perfect subgroups.

Proof. The statement follows from Theorem 2.2 and Lemma 2.3, using the fact that $O^P(\langle g \rangle) = \langle (g)_{P'} \rangle$. □

This proves all the statements of Theorem 1.2 except for the primitivity of the idempotents $lin(e_C)$. Note that the rational case ($P = \emptyset$) of Corollary 2.4 is stated in [Glu81a, Theorem] and goes back to a similar result by Solomon [Sol67, Thm. 3].

The following observation is not part of the proof of Theorem 1.2, but we record it for later reference.

Lemma 2.5. *The P-local Burnside ring splits as*

$$A(G)_{(P)} \cong e_{cyc} \cdot A(G)_{(P)} \times e_{ker} \cdot A(G)_{(P)}$$

where e_{cyc} (respectively e_{ker}) is defined to be the sum of all primitive idempotents e_L with L cyclic (respectively non-cyclic). Moreover, the summand $e_{ker} \cdot A(G)_{(P)}$ is precisely the kernel of the linearization map $lin\colon A(G)_{(P)} \to R(G)_{(P)}$.

Proof. The first part follows from Theorem 2.2 by writing $1 = e_{cyc} + e_{ker}$. Lemma 2.3 implies that the kernel of *lin* consists of those virtual G-sets whose marks vanish at all cyclic subgroups. By Corollary 2.4, these are precisely the elements of the ideal $e_{ker} \cdot A(G)_{(P)}$. □

2.2. **Prime ideals in the splitting field case.** Let $\exp(G)$ be the exponent[3] of G and write $\mathbb{F}$ for the $\exp(G)$-th cyclotomic extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_\mathbb{F}$ and Galois group $\Gamma := Gal(\mathbb{F}\colon \mathbb{Q})$. All characters of G-representations over the complex numbers take values in $\mathcal{O}_\mathbb{F}$, and therefore can be viewed as class functions $G/\!\sim \to \mathcal{O}_\mathbb{F}$, where $G/\!\sim$ is the set of conjugacy classes of G. When working P-locally, the elements of $RU(G)_{(P)}$ are $\mathbb{Z}_{(P)}$-linear combinations of irreducible representations of G over the complex numbers, hence their characters take values in $\mathcal{O}_{\mathbb{F},(P)} := \mathcal{O}_\mathbb{F} \otimes \mathbb{Z}_{(P)}$.

Notation 2.6. Any element $V \in RU(G)_{(P)} \otimes \mathcal{O}_\mathbb{F}$ can be written as an $\mathcal{O}_{\mathbb{F},(P)}$-linear combination $V = \sum_i \lambda_i \cdot V_i$ of irreducible G-representations V_i. We write

$$\hat{\chi}(V)(g) := \sum_i \lambda_i \cdot \chi(V_i)(g)$$

for the value of the $\mathcal{O}_{\mathbb{F},(P)}$-linear character of V at $g \in G$.

Atiyah [Ati61] described the structure of the prime ideal spectrum $\operatorname{Spec}(RU(G) \otimes \mathcal{O}_\mathbb{F})$. His proof applies without changes to the open subscheme $\operatorname{Spec}(RU(G)_{(P)} \otimes \mathcal{O}_\mathbb{F})$ cut out by P-localization.

Proposition 2.7 (Cf. [Ati61], Prop. 6.4)**.** *The topological space* $\operatorname{Spec}(RU(G)_{(P)} \otimes \mathcal{O}_\mathbb{F})$ *can be described as follows:*

(1) Every prime ideal of $RU(G)_{(P)} \otimes \mathcal{O}_\mathbb{F}$ is of the form

$$Q(\mathfrak{p}, g) := (\hat{\chi}(-)(g))^{-1}(\mathfrak{p}) = \{V \in RU(G)_{(P)} \otimes \mathcal{O}_\mathbb{F} \mid \hat{\chi}(V)(g) \in \mathfrak{p}\}$$

for some element $g \in G$ and some prime ideal $\mathfrak{p} \trianglelefteq \mathcal{O}_{\mathbb{F},(P)}$.

(2) Let $\mathfrak{p}, \mathfrak{q} \trianglelefteq \mathcal{O}_{\mathbb{F},(P)}$ be prime ideals such that $\mathbb{Z} \cap \mathfrak{q} = q\mathbb{Z}$ for a prime $q \in \mathbb{Z}$. There is an inclusion $Q(\mathfrak{p}, g) \subseteq Q(\mathfrak{q}, h)$ if and only if $\mathfrak{p}$ is contained in $\mathfrak{q}$ and $(g)_{q'}$ is conjugate in G to $(h)_{q'}$.

[3]The *exponent* of a finite group is the least common multiple of the orders of all group elements.

(3) The prime ideals $Q(\mathfrak{p}, g)$ with $\mathfrak{p} = (0)$ are minimal and the ones with $\mathfrak{p} \neq (0)$ are maximal. In particular, the Krull dimension of $RU(G)_{(P)} \otimes \mathcal{O}_{\mathbb{F}}$ is one.

2.3. **Idempotents in the splitting field case.** We can deduce a classification of the idempotent elements of $RU(G)_{(P)} \otimes \mathcal{O}_{\mathbb{F}}$ from Proposition 2.7. Our proof is inspired by Dress' approach [Dre69, Prop. 2] to the idempotents in the Burnside ring.

Theorem 2.8. *The map*

$$G \to \pi_0(\mathrm{Spec}(RU(G)_{(P)} \otimes \mathcal{O}_{\mathbb{F}}))$$

that sends $x \in G$ to the connected component of $Q(0, x)$ induces a bijection between the set of conjugacy classes of P-prime elements of G and the set of connected components of $\mathrm{Spec}(RU(G)_{(P)} \otimes \mathcal{O}_{\mathbb{F}})$*. In particular, the prime ideal spectrum of $RU(G) \otimes \mathcal{O}_{\mathbb{F}}$ is connected.*

This follows directly from:

Proposition 2.9. *For any (not necessarily P-prime) elements $x, y \in G$, the prime ideals $Q(\mathfrak{p}, x)$ and $Q(\mathfrak{q}, y)$ lie in the same connected component of* $\mathrm{Spec}(RU(G) \otimes \mathcal{O}_{\mathbb{F}})$ *if and only if $(x)_{P'}$ and $(y)_{P'}$ are conjugate in G.*

Proof. First observe that for $\mathfrak{p} \neq (0)$, the height one ideal $Q(\mathfrak{p}, x)$ lies in the closure of the height zero ideal $Q((0), x)$, so without loss of generality we may assume that $\mathfrak{p} = \mathfrak{q} = (0)$. Since $RU(G)_{(P)} \otimes \mathcal{O}_{\mathbb{F}}$ has Krull dimension one, two points $Q((0), x)$ and $Q((0), y)$ lie in the same component if and only if there is a zig-zag of inclusions of prime ideals

$$\begin{array}{ccccccc} & & Q(\mathfrak{p}_0, x_0) = Q(\mathfrak{p}_0, x_1) & & & \cdots & \\ & \nearrow & & \nwarrow & \nearrow & & \nwarrow \\ Q((0), x_0) & & & & Q((0), x_1) & & & Q((0), x_r) \end{array}$$

for some elements $x = x_0, x_1, \ldots, x_r = y \in G$ and some prime ideals $\mathfrak{p}_i \trianglelefteq \mathcal{O}_{\mathbb{F},(P)}$. By part (2) of Theorem 2.7, we have an equality $Q(\mathfrak{p}_i, x_i) = Q(\mathfrak{p}_i, x_{i+1})$ if and only $(x_i)_{(p_i')}$ is conjugate in G to $(x_{i+1})_{(p_i')}$, where p_i is given by $\mathbb{Z} \cap \mathfrak{p}_i = p_i\mathbb{Z}$.
For the "only if" part of the proposition, given a zig-zag as above, it follows that

$$(x)_{P'} = ((x_0)_{p_0'})_{P'} \sim_G ((x_1)_{p_0'})_{P'} = ((x_1)_{p_1'})_{P'} \sim_G \ldots \sim_G ((x_r)_{p_{r-1}'})_{P'} = (y)_{P'}$$

where $\sim_G$ indicates being conjugate in G.
For the "if" part, assume that $(x)_{P'} \sim_G (y)_{P'}$. Since the prime ideals $Q((0), g)$ only depend on the conjucagy class of g, it follows that $Q((0), (x)_{P'})$ and $Q((0), (y)_{P'})$ agree. Thus, it suffices to show that for any $g \in G$, the prime ideals $Q((0), g)$ and

$Q((0), (g)_{P'})$ lie in the same component. We will construct an explicit zig-zag as above. Let $p_0, p_1, \dots, p_r$ be all primes in P that divide the order of G. By the going-up theorem, we can find prime ideals $\mathfrak{p}_i \trianglelefteq \mathcal{O}_{\mathbb{F},(P)}$ such that $\mathfrak{p}_i \cap \mathbb{Z} = p_i\mathbb{Z}$. Then $(g)_{P'}$ may be computed as

$$(g)_{P'} = (\cdots((g)_{p_0'})_{p_1'}\cdots)_{p_r'}.$$

Inductively, define $g_0 := g$ and $g_i := (g_{i-1})_{p_{i-1}'}$ so that we have $(g_i)_{p_i'} = (g_{i+1})_{p_i'}$. Then these choices of elements g_i and prime ideals $\mathfrak{p}_i$ give rise to a zig-zag between $Q((0), g)$ and $Q((0), (g)_{P'})$, which completes the proof. □

Corollary 2.10. *The conjugacy classes of P-prime elements (x) of G are in canonical bijection with the primitive idempotents e_x of $RU(G)_{(P)} \otimes \mathcal{O}_{\mathbb{F}}$. The character of the element e_x is given as follows:*

$$\hat{\chi}(e_x)(g) = \begin{cases} 1 & \text{if } (g)_{P'} \sim_G x \\ 0 & \text{otherwise} \end{cases}$$

Proof. It is a standard fact of algebraic geometry that for any commutative ring R, the subsets $V \subseteq \mathrm{Spec}(R)$ that are both open and closed are in canonical bijection with the idempotent elements of R, by assigning to V the global section[4] which is constant one on V and constant zero on the complement of V. Under this identification, the primitive idempotents correspond to the minimal non-empty open and closed subsets. The latter agree with the connected components of $\mathrm{Spec}(RU(G)_{(P)} \otimes \mathcal{O}_{\mathbb{F}})$ since there are only finitely many of them. The first claim now follows from Theorem 2.8. For the description of characters, note that $\hat{\chi}(e_x)(g) = 1$ if and only if the corresponding global section e_x evaluates to one at the point $Q((0), g)$ if and only if $Q((0), x)$ and $Q((0), g)$ are in the same connected component. □

Remark 2.11. Roquette [Roq52] shows that the classification given in Corollary 2.10 also holds for the primitive idempotents in the p-adic representation ring after adjoining all e-th roots of unity.

Remark 2.12. Using Schur's orthogonality relations, it follows from Corollary 2.10 that e_x is given explicitly as

$$e_x = \frac{1}{|C_G(x)|}\sum_V \chi(V)(x^{-1}) \cdot V$$

where V runs over a system of representatives of the irreducible representations of G and $C_G(x)$ denotes the centralizer of x in G. This observation goes back at least to Brauer [Bra47, (7)]. The coefficients can also be expressed in terms of Möbius functions, see [Sol67, Thm. 4], [Glu81b, Prop.] and [Yos83, §3].

[4] Here we use that by definition, the global sections of $\mathrm{Spec}(R)$ agree with the ring R.

2.4. **Idempotents of** $RU(G)_{(P)}$**.** Recall that $\Gamma \cong (\mathbb{Z}/\exp(G))^\times$ denotes the Galois group of the cyclotomic extension $\mathbb{F}/\mathbb{Q}$. The left Γ-action on $\mathbb{F}$ restricts to an action on $\mathcal{O}_\mathbb{F}$. Let Γ act on $RU(G)_{(P)} \otimes \mathcal{O}_\mathbb{F}$ via its action on the right factor. Then clearly $RU(G)_{(P)} = (RU(G)_{(P)} \otimes \mathcal{O}_\mathbb{F})^\Gamma$. The group Γ then acts from the right on $\mathrm{Spec}(RU(G)_{(P)} \otimes \mathcal{O}_\mathbb{F})$ and we have $\mathrm{Spec}(RU(G)_{(P)}) \cong (\mathrm{Spec}(RU(G)_{(P)} \otimes \mathcal{O}_\mathbb{F}))/\Gamma$, cf. [Ser77, §11.4, Exerc. 11.4]. We will now describe these Γ-orbits in terms of the prime ideals $Q(\mathfrak{p}, x)$.

First recall that the left Γ-action on $\mathcal{O}_\mathbb{F}$ induces a right Γ-action on $\mathrm{Spec}(\mathcal{O}_\mathbb{F})$ that is given by $\mathfrak{p}.\gamma = \gamma^{-1}(\mathfrak{p})$.

Definition 2.13 ([Ser77], §12.4)**.** Define a right Γ-action on the underlying set of G as follows: If $\gamma \in \Gamma$ corresponds to the unit $m \in (\mathbb{Z}/\exp(G))^\times$, let $g.\gamma := g^{m^{-1}}$, where m^{-1} is (any integer representing) the inverse of m in the group Γ.

This action is well-defined since the order of $g \in G$ divides $\exp(G)$. Moreover, it is compatible with conjugation in G. We can describe the Γ-orbits in G easily:

Lemma 2.14 ([Ser77], §13.1, Cor.)**.** *Two elements $x, y \in G$ lie in the same Γ-orbit if and only if they generate the same cyclic subgroup of G.*

Proof. Let n divide $\exp(G)$. Then $\Gamma \cong (\mathbb{Z}/\exp(G))^\times$ permutes the generators of $\mathbb{Z}/\exp(G)$ transitively, and the same is true for the generators of the cyclic group $\mathbb{Z}/n$, viewed as a subgroup of $\mathbb{Z}/\exp(G)$. □

Proposition 2.15. *The left Γ-action on $RU(G)_{(P)} \otimes \mathcal{O}_\mathbb{F}$ induces a right Γ-action on the space $\mathrm{Spec}(RU(G)_{(P)} \otimes \mathcal{O}_\mathbb{F})$ which coincides with the action defined by $Q(\mathfrak{p}, x).\gamma = Q(\mathfrak{p}.\gamma, x.\gamma)$.*

Proof. As in Notation 2.6, write $\sum_i \lambda_i \cdot V_i$ for a generic element of $RU(G)_{(P)} \otimes \mathcal{O}_\mathbb{F}$. Then

$$\begin{aligned} Q(\mathfrak{p}, x).\gamma = \gamma^{-1}(Q(\mathfrak{p}, g)) &= \{\sum_i \lambda_i \cdot V_i \mid \sum_i \gamma(\lambda_i) \cdot \chi(V_i)(g) \in \mathfrak{p}\} \\ &= \{\sum_i \lambda_i \cdot V_i \mid \sum_i \lambda_i \cdot \gamma^{-1}(\chi(V_i)(g)) \in \mathfrak{p}.\gamma\} \\ &= \{\sum_i \lambda_i \cdot V_i \mid \sum_i \lambda_i \cdot \chi(V_i)(g.\gamma) \in \mathfrak{p}.\gamma\} \\ &= Q(\mathfrak{p}.\gamma, g.\gamma) \end{aligned}$$
□

Corollary 2.16. *The map*

$$G \to \pi_0(\mathrm{Spec}(RU(G)_{(P)}))$$

that sends an element x to the component of the orbit $Q((0), x).\Gamma$ induces a bijection between the Γ-orbits of conjugacy classes of P-prime elements $x \in G$ and the set of components of the topological space $\mathrm{Spec}(RU(G)_{(P)})$. In particular, the spectrum of $RU(G)$ is connected.

Corollary 2.17. *There is a canonical bijection between the Γ-orbits of conjugacy classes of P-prime elements $x \in G$ and the primitive idempotents in $RU(G)_{(P)}$. The idempotent $e_{x.\Gamma}$ associated to the orbit of (x) has character given by*

$$\chi(e_{x.\Gamma})(g) = \begin{cases} 1 & \text{if } (g)_{P'} \sim_G x.\gamma \text{ for some } \gamma \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

Proof. This follows from Corollary 2.16 in the same way that Corollary 2.10 follows from Theorem 2.8, see the proof of Corollary 2.10. □

Remark 2.18. In particular, we have $e_{x.\Gamma} = \sum_{\gamma \in \Gamma} e_{x.\gamma}$ in $RU(G)_{(P)}$. A simple calculation shows that $e_{x.\gamma} = \gamma^{-1}(e_x)$ in $RU(G)_{(P)} \otimes \mathcal{O}_{\mathbb{F}}$. Therefore $e_{x.\Gamma} = tr_{\mathbb{F}/\mathbb{Q}}(e_x)$ is the field trace of e_x. We will not use this fact.

By Lemma 2.14, we can write $e'_{\langle x \rangle} := e_{x.\Gamma}$ and rephrase Corollary 2.17 in terms of cyclic subgroups. At this point, there is no dependence on the field extension $\mathbb{F}/\mathbb{Q}$ anymore.

Corollary 2.19. *There is a canonical bijection between the conjugacy classes of cyclic P-perfect subgroups $C \in G$ and the primitive idempotents in $RU(G)_{(P)}$. The primitive idempotent e'_C has character given by*

$$\chi(e'_C)(g) = \begin{cases} 1 & \text{if } \langle (g)_{P'} \rangle \sim_G C \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the character of e'_C agrees with that of $lin(e_C)$ given in Corollary 2.4 and hence we have $e'_C = lin(e_C)$.

Theorem 1.2 follows.

Remark 2.20. It is clear from Corollary 2.19 that the primitive idempotents of $RU(G)_{(P)}$ only depend on those primes $p \in P$ that divide the order of G.

2.5. **Quick proofs of special cases.** In the rational and p-local case, we can give short ad-hoc proofs of the primitivity of the elements $lin(e_C)$ stated as part of Theorem 1.2.

Lemma 2.21. *Let $x, y \in G$ generate the same subgroup. If all character values of the virtual representation $V \in RU(G)_{(P)}$ lie in $\mathbb{Z}_{(P)}$, then $\chi(V)(x) = \chi(V)(y)$.*

Proof. By Lemma 2.14, we can find $\gamma \in \Gamma$ such that $y = x.\gamma$. Then

$$\chi(V)(y) = \chi(V)(x.\gamma) = \gamma^{-1}(\chi(V)(x)) = \chi(V)(x)$$

because $\chi(V)(x) \in \mathbb{Z}_{(P)} = (\mathcal{O}_{\mathbb{F},(P)})^{\Gamma}$. □

Corollary 2.22. *For any cyclic $C \leq G$, the idempotent $lin(e_C) \in RU(G) \otimes \mathbb{Q}$ is primitive.*

Proof. Recall that the character of $lin(e_C)$ is one on elements that generate subgroups conjugate to C and zero otherwise. But Lemma 2.21 shows that any integer-valued character must be constant on the set where $lin(e_C)$ is one, hence $lin(e_C)$ cannot decompose as a sum of idempotents. □

For the p-local case, we need another lemma. It was used in Atiyah's proof of Theorem 2.7.

Lemma 2.23 ([Ati61], proof of Lemma 6.3)**.** *Let $V \in RU(G)_{(p)}$ and let $\mathfrak{p}$ be a prime of $\mathcal{O}_{\mathbb{F},(p)} = \mathcal{O}_{\mathbb{F}} \otimes \mathbb{Z}_{(p)}$. Then $\chi(V)(g) \equiv \chi(V)((g)_{p'}) \mod \mathfrak{p}$.*

Proof. Without loss of generality, we may assume that G is cyclic and V one-dimensional, hence its character is multiplicative. Write $g = (g)_{p'} \cdot h$ where the order of h is p^r, then $(\chi(V)(h))^{p^r} = 1$. But $\mathcal{O}_{\mathbb{F},(P)}/\mathfrak{p}$ is a finite field of characteristic p, so

$$\chi(V)(h) \equiv 1 \mod \mathfrak{p},$$

and consequently

$$\chi(V)(g) \equiv \chi(V)((g)_{p'}) \cdot \chi(V)(h) \equiv \chi(V)((g)_{p'}) \mod \mathfrak{p}.$$ □

Definition 2.24. For $C \leq G$ cyclic of order prime to p, let

$$S_C := \{g \in G \mid \langle (g)_{p'} \rangle \sim_G C\}.$$

Combining Lemma 2.21 and Lemma 2.23 gives:

Corollary 2.25. *Let $\mathfrak{p}$ be any prime ideal in $\mathcal{O}_{\mathbb{F},(p)}$. If all character values of $V \in RU(G)_{(p)}$ lie in $\mathbb{Z}_{(p)}$, then the character of V is constant modulo $\mathfrak{p}$ on the set S_C.*

Finally, a proof similar to that of Corollary 2.22 shows:

Corollary 2.26. *For any cyclic p-perfect $C \leq G$, the idempotent $lin(e_C) \in RU(G)_{(p)}$ is primitive.*

Lemma 2.23 does not hold in the general P-local case, as the next example shows. However, it follows from Theorem 1.2 that the statement becomes true under the additional assumption that the character of V be zero outside of S_C. We do not know how to use this assumption to give a quick proof of the primitivity of the elements $lin(e_C)$ that applies to all choices of P.

Example 2.27. Let $G = C_2 \times C_3$ be the cyclic group of order 6 and $P = \{2, 3\}$. Write $\mathbb{1}$ for the trivial representation and let $V \in RU(G)_{(P)}$ be given as the tensor product of the sign representation of C_2 with the sum of the two non-trivial irreducible C_3-representations. Let $g \in G$ be a generator and observe that $(g)_{P'} = 1$. However,

$$\chi(V - \mathbb{1})(g) = 0 \not\equiv 1 = \chi(V - \mathbb{1})(1) \mod \mathfrak{p}$$

for any prime ideal $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{F},(P)}$.

3. Idempotent splittings of representation rings

As before, let P be a fixed collection of prime numbers, and let $R(G)_{(P)}$ denote one of the rings $RO(G)_{(P)}$ or $RU(G)_{(P)}$. The goal of this section is to describe the multiplicativity of the idempotent splitting

$$R(-)_{(P)} \cong \prod_{(C)} R(-)_{(P)}[e_C^{-1}].$$

We start by briefly recalling the notion of an (incomplete) Tambara functor in §3.1. In §3.2, we study the multiplicativity of the idempotent splitting of $R(-)_{(P)}$: we characterize the norms which are compatible with e_C-localization in Theorem 3.3 and describe the incomplete Tambara functor structure of each idempotent summand in Theorem 3.8. It is then easy to read off the structure that is preserved by the entire splitting, as we explain in §3.3.

3.1. **Incomplete Tambara functors.** Recall that many naturally arising Mackey functors have additional multiplicative structure.

Definition 3.1. A *Green functor* is a Mackey functor $\underline{R}$ equipped with commutative ring structures on the values $\underline{R}(H)$ for all $H \leq G$ such that all restrictions maps $R^H_K \colon \underline{R}(H) \to \underline{R}(K)$ become ring homomorphisms and all transfer maps $T^H_K \colon \underline{R}(K) \to \underline{R}(H)$ are morphisms of $\underline{R}(G)$-modules.

Often, Green functors come equipped with additional *multiplicative transfer maps* or *norms* $N^H_K \colon \underline{R}(K) \to \underline{R}(H)$ for all subgroup inclusions $K \leq H \leq G$, satisfying a number of compatibility relations for norms, additive transfers and restrictions. Tambara [Tam93] axiomatized the structure of these objects and called them *TNR-functors*; nowadays they are referred to as *Tambara functors*.

Blumberg and Hill [BH18] introduced the more general notion of an *(incomplete) $\mathcal{I}$-Tambara functor* that only admits a partial collection of norms for certain subgroup inclusions $K \leq H \leq G$, parametrized by well-behaved collections $\mathcal{I}$ of *admissible H-sets H/K*. These *indexing systems* form a poset under inclusion. Thus, $\mathcal{I}$-Tambara functors for varying $\mathcal{I}$ interpolate between the notion of a Green functor (which doesn't necessarily admit any norms) and that of a Tambara functor (which admits all possible norms). We refer to the above sources for precise definitions and further details.

Example 3.2. The Mackey functors defined by the Burnside ring $A(-)$ and the representation rings $R(-)$ are examples of Tambara functors. The multiplicative norms N_K^H in the Burnside ring are induced by the co-induction functor $map_K(H, -)$ from finite K-sets to finite H-sets. Those of the representation ring are induced by tensor induction of representations. The linearization maps $lin\colon A(-) \to R(-)$ are maps of Tambara functors.
More generally, Brun [Bru07] showed that the zeroth equivariant homotopy groups of a G-E_∞ ring spectrum naturally form a Tambara functor.

3.2. **Multiplicativity of the idempotent summands.** Observe that the canonical localization maps $R(-)_{(P)} \to R(-)_{(P)}[e_C^{-1}]$ are levelwise ring homomorphisms that are compatible with the Mackey functor structure, hence the idempotent splitting of $R(G)_{(P)}$ induces a splitting of the underlying Green functor of $R(-)_{(P)}$. Our next goal is to describe the idempotent summands $R(-)_{(P)}[e_C^{-1}]$ by proving the equivalence of the statements (c), (d) and (e) of Theorem 1.5. For convenience of the reader, we record this in the following theorem.

Theorem 3.3. *Let $C \leq G$ be a cyclic P-perfect subgroup and let $e_C \in A(G)_{(P)}$ be the corresponding primitive idempotent element. Fix subgroups $K \leq H \leq G$. Then the following are equivalent:*

(c) The norm map $N_K^H\colon A(K)_{(P)} \to A(H)_{(P)}$ descends to a well-defined map of multiplicative monoids

$$\tilde{N}_K^H\colon A(K)_{(P)}[e_C^{-1}] \to A(H)_{(P)}[e_C^{-1}].$$

(d) The norm map $N_K^H\colon R(K)_{(P)} \to R(H)_{(P)}$ descends to a well-defined map of multiplicative monoids

$$\tilde{N}_K^H\colon R(K)_{(P)}[e_C^{-1}] \to R(H)_{(P)}[e_C^{-1}].$$

(e) Any subgroup $C' \leq H$ conjugate in G to C lies in K.

In the prequel [Böh], statement (e) appeared in a slightly more complicated form as the statement (e′) of the next lemma. The author is grateful to Malte Leip for pointing out this simplification.

Lemma 3.4. *The statement (e) is equivalent to*

(e′) One of the following holds:
- *i) Neither K nor H are super-conjugate in G to L.*
- *ii) Both K and H are super-conjugate in G to L and satisfy the following: If $L' \leq G$ is conjugate in G to L and is contained in H, then it is contained in K.* □

It was proven in [Böh, Thm. 4.1] that the statements (c) and (e′) are equivalent, so Theorem 3.3 reduces to showing that (c) and (d) are equivalent.

We recall the following fact due to Blumberg and Hill (see [Böh, Thm. 2.33] for an elementary proof):

Theorem 3.5 ([BH18], Thm. 5.25)**.** *Let $\underline{R}$ be an $\mathcal{I}$-Tambara functor structured by an indexing system $\mathcal{I}$. Let $x \in \underline{R}(G)$. Then the orbit-wise localization $\underline{R}[x^{-1}]$ is a localization in the category of $\mathcal{I}$-Tambara functors if and only if for all admissible sets H/K of $\mathcal{I}$, the element $N_K^H R_K^G(x)$ divides a power of $R_H^G(x)$.*

If the element x is idempotent, then checking the above division relation amounts to checking an equation:

Lemma 3.6. *Let $e, e' \in R$ be idempotents in a commutative ring. Then e divides e' if and only if $e \cdot e' = e'$.*

Proof. Assume that e divides e'. Then $e' \in eR$, hence $e \cdot e' = e'$, since multiplication by e is projection onto the idempotent summand eR of R. The other direction is obvious. □

Proof of Theorem 3.3. We only need to show the equivalence (c) ⇔ (d). By Theorem 3.5 and Lemma 3.6, the statement (c) (respectively (d)) holds if and only if the equation

$$N_K^H R_K^G(x) \cdot R_H^G(x) = R_H^G(x)$$

holds in $A(H)_{(P)}$ for $x = e_C$ (respectively in $R(H)_{(P)}$ for $x = lin(e_C)$). The linearization map $lin\colon A(-)_{(P)} \to R(-)_{(P)}$ is a map of Tambara functors, hence preserves norms, restrictions and multiplication. By Lemma 2.5, lin is injective on the ideal summand $e_{cyc} \cdot A(G)_{(P)}$ and that summand contains the element e_C. It follows that the above equation holds for $x = e_C$ if and only if it holds for $x = lin(e_C)$. □

We can use the language of incomplete Tambara functors [BH18] to describe the algebraic structure of $R(G)_{(P)}[e_C^{-1}]$ in terms of certain indexing systems.

Proposition 3.7 ([Böh], Prop. 4.16)**.** *Let $L \leq G$ be P-perfect. There is an indexing system $\mathcal{I}_L$ given as follows: for all $H \leq G$, $\mathcal{I}_L(H)$ is the full subcategory of finite H-sets spanned by all coproducts of the orbits H/K such that the groups $K \leq H \leq G$ satisfy statement (e) of Theorem 1.5 with respect to L.*

Theorem 3.8. *Let $C \leq G$ be a cyclic P-perfect subgroup, and denote by $R(-)_{(P)}$ one of the Tambara functors $RU(-)_{(P)}$ or $RO(-)_{(P)}$. Then the following hold:*

i) The Green functor $R(-)_{(P)}[e_C^{-1}]$ admits the structure of an $\mathcal{I}_C$-Tambara functor under $R(-)_{(P)}$.

ii) The indexing system $\mathcal{I}_C$ is maximal among the indexing systems that satisfy i).
iii) The canonical map $R(-)_{(P)} \to R(-)_{(P)}[e_C^{-1}]$ is an e_C-localization in the category of $\mathcal{I}_C$-Tambara functors. □

We record two easy consequences of our characterization of norm maps in the idempotent summands.

Corollary 3.9. *The summand $R(-)_{(P)}[e_C^{-1}]$ is a Tambara functor (i.e., has a complete set of norms) if and only if C is the trivial group.* □

Corollary 3.10. *The subgroup C is normal in G if and only if the summand $R(-)_{(P)}[e_C^{-1}]$ admits all norms of the form $\tilde{N}_K^H$ such that K contains a subgroup conjugate in G to C.* □

3.3. **Multiplicativity of the idempotent splittings.** We can now describe the multiplicativity of the idempotent splitting of $R(-)_{(P)}$ in terms of the indexing system

$$\mathcal{I}_{cyc} := \bigcap_{(C)} \mathcal{I}_C$$

arising as the intersection of the indexing systems $\mathcal{I}_C$ defined in Prop. 3.7.

Proposition 3.11. *The localization maps $R(-)_{(P)} \to R(-)_{(P)}[e_C^{-1}]$ assemble into an isomorphism of $\mathcal{I}_{cyc}$-Tambara functors*

$$R(-)_{(P)} \to \prod_{(C)\leq G} R(-)_{(P)}[e_C^{-1}]$$

where the product is taken over conjugacy classes of cyclic P-perfect subgroups. Moreover, $\mathcal{I}_{cyc}$ is maximal among all indexing sets with this property.

Proof. Each of the canonical maps $R(-)_{(P)} \to R(-)_{(P)}[e_C^{-1}]$ is a map of $\mathcal{I}_C$-Tambara functors by 3.8, hence their product is a map of $\mathcal{I}_{cyc}$-Tambara functors. It is a levelwise isomorphism by construction. The maximality also follows from Theorem 3.8: it implies that $\mathcal{I}_{cyc}$ is maximal among the indexing systems $\mathcal{J}$ such that each summand $R(-)_{(P)}[e_C^{-1}]$ is a $\mathcal{J}$-Tambara functor. □

The admissible sets of $\mathcal{I}_{cyc}$ can be characterized as follows.

Lemma 3.12 ([Böh], Lemma 4.23). *Let $K \leq H \leq G$, then H/K is an admissible set for $\mathcal{I}_{cyc}$ if and only if for all cyclic P-perfect $C \leq H$, C is contained in K.*

4. Idempotent splittings of equivariant K-theory

Let K_G denote one of the genuine G-spectra KU_G or KO_G, i.e., either complex or real equivariant K-theory. We will determine the multiplicativity of the P-local idempotent splitting

$$(K_G)_{(P)} \simeq \prod_{(C)} (K_G)_{(P)}[e_C^{-1}],$$

i.e., we will explicitly describe the maximal N_∞ algebra structure on each of the factors, as well as the maximal N_∞ algebra structure preserved by the splitting. Recall that as a consequence of Theorem 1.2, the blocks of $(K_G)_{(P)}$ are given as the e_C-localizations

$$(K_G)_{(P)} \wedge \mathbb{S}_{(P)}[e_C^{-1}]$$

of $(K_G)_{(P)}$ in the category of G-spectra.

4.1. **Preliminaries.** The N_∞ operads of [BH15b] structure G-equivariant ring spectra with incomplete sets of norm maps parametrized by their associated indexing systems. According to [GW, Thm. 4.7], any given indexing system can be realized as the indexing system of a Σ-cofibrant[5] N_∞ operad. Similar existence results were given in [Rub17, Thm. 3.3] and [BP17, Cor. IV].

Notation 4.1. For each conjugacy class of cyclic P-perfect subgroups $C \leq G$, let $\mathcal{O}_C$ be a Σ-cofibrant N_∞ operad whose associated indexing system is $\mathcal{I}_C$. Let $\mathcal{O}_{cyc}$ be a Σ-cofibrant N_∞ operad whose associated indexing system is $\mathcal{I}_{cyc}$.

Note that by definition, an N_∞ operad $\mathcal{P}$ is a certain operad in the category of unbased G-spaces. By the usual abuse of notation, we refer to an algebra over the operad $\Sigma_+^\infty \mathcal{P}$ in G-spectra as a *$\mathcal{P}$-algebra*.

Remark 4.2. For any choice of the operad $\mathcal{O}_C$, both $\mathbb{S}$ and K_G are naturally algebras over $\mathcal{O}_C$: both spectra can be modelled as strictly commutative monoids in orthogonal G-spectra, and hence admit an action by $\mathcal{O}_C$ that factors through the action of the commutative operad.

4.2. **Multiplicativity of the idempotent summands.** We are now ready to state our main homotopical result.

Theorem 4.3. *Let $C \leq G$ be a cyclic P-perfect subgroup. Then:*

i) The G-spectrum $(K_G)_{(P)}[e_C^{-1}]$ is an $\mathcal{O}_C$-algebra under $(K_G)_{(P)}$.
ii) The operad $\mathcal{O}_C$ is maximal among the N_∞-operads that satisfy i).
iii) The canonical map $(K_G)_{(P)} \to (K_G)_{(P)}[e_C^{-1}]$ is an e_C-localization in the category of $\mathcal{O}_C$-algebras in G-spectra.

[5] An operad $\mathcal{O}$ in G-spaces is Σ-*cofibrant* if each space $\mathcal{O}(n)$ is of the homotopy type of a $(G \times \Sigma_n)$-CW complex.

The key to the proof is the following preservation result for N_∞ algebras given in [Böh]. It extends previous work of Hill and Hopkins [HH14] and uses a result of Gutiérrez and White [GW, Cor. 7.10].

Proposition 4.4 ([Böh], Prop. 2.32)**.** *Let $\mathcal{P}$ be a Σ-cofibrant N_∞ operad. Fix $x \in \pi_0^G(\mathbb{S}_{(P)})$. Then the Bousfield localization L_x given by smashing with*

$$\mathbb{S}_{(P)}[x^{-1}] = \text{hocolim}\left(\mathbb{S}_{(P)} \xrightarrow{x} \mathbb{S}_{(P)} \xrightarrow{x} \dots\right)$$

preserves[6] *$\mathcal{P}$-algebras in P-local G-spectra if and only if for all $H \leq G$ and all transitive admissible H-sets H/K, the element $N_K^H R_K^G(x)$ divides a power of $R_H^G(x)$ in the ring $\pi_0^H(\mathbb{S}_{(P)})$.*

Proof of Theorem 4.3. Ad i): We know from Theorems 3.5 and 3.3 that for each of the admissible sets of $\mathcal{I}_C$, hence of $\mathcal{O}_C$, the division relation of Prop. 4.4 holds, so $(K_G)_{(P)}[e_C^{-1}]$ is an $\mathcal{O}_C$-algebra under $(K_G)_{(P)}$.
Ad ii): Assume that $\mathcal{P}$ is an element strictly greater than $\mathcal{O}_C$ in the poset of (homotopy types of) N_∞ operads. Then any norm that comes from $\mathcal{P}$ but not from $\mathcal{O}_C$ induces a corresponding norm on homotopy groups that does not correspond to an admissible set of $\mathcal{I}_C$, thus contradicting the maximality statement included in Theorem 3.8.
Ad iii): It is an e_C-localization in G-spectra and a map of $\mathcal{O}_C$-algebras. □

We obtain the homotopical analogue of Corollary 3.9, stated as Corollary 1.7 in the introduction. There is also a homotopical version of Corollary 3.10:

Corollary 4.5. *The group C is normal in G if and only if $(K_G)_{(P)}[e_C^{-1}]$ admits all norm maps of the form $\tilde{N}_K^H$ such that K and H both contain a subgroup conjugate in G to C.* □

4.3. **Multiplicativity of the idempotent splitting.** We can also describe the multiplicativity of the entire idempotent splitting:

Corollary 4.6. *Let $\mathcal{O}_{cyc}$ be a Σ-cofibrant N_∞ operad realizing the indexing system $\mathcal{I}_{cyc} = \bigcap_{(C)} \mathcal{I}_C$. Then the idempotent splitting*

$$(K_G)_{(P)} \simeq \prod_{(C)} (K_G)_{(P)}[e_C^{-1}]$$

is an equivalence of $\mathcal{O}_{cyc}$-algebras. Here, the product is taken over all conjugacy classes of cyclic P-perfect subgroups of G.

[6] in the sense of [GW, Def. 7.3]

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