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REGULARITY OF C*-ALGEBRAS AND CENTRAL SEQUENCE ALGEBRAS

PhD Thesis
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Regularity of Villadsen algebras and characters on their central sequence algebras,
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Abstract

The main topic of this thesis is regularity properties of C^*-algebras and how these regularity properties are reflected in their associated central sequence algebras. The thesis consists of an introduction followed by four papers [A], [B], [C], [D].

In [A], we show that for the class of simple Villadsen algebra of either the first type with seed space a finite dimensional CW complex, or the second type, tensorial absorption of the Jiang-Su algebra is characterized by the absence of characters on the central sequence algebra. Additionally, in a joint appendix with Joan Bosa, we show that the Villadsen algebra of the second type with infinite stable rank fails the corona factorization property.

In [B], we consider the class of separable C^*-algebras which do not admit characters on their central sequence algebra, and show that it has nice permanence properties. We also introduce a new divisibility property, that we call local divisibility, and relate Jiang-Su stability of unital, separable C^*-algebras to the local divisibility property for central sequence algebras. In particular, we show that a unital, simple, separable, nuclear C^*-algebra absorbs the Jiang-Su algebra if, and only if, there exists $k \geq 1$ such that the central sequence algebra is k-locally almost divisible.

In [C], we show that for a substantial class of unital, separable and $\mathcal{Z}$-stable C^*-algebras, there exists a closed 2-sided ideal in the central sequence algebra which is not a σ-ideal.

In [D], we give a characterization of asymptotic regularity in terms of the Cuntz semigroup for simple, separable C^*-algebras, and show that any simple, separable C^*-algebra which is neither stably finite nor purely infinite is not asymptotically regular either.

Resumé

Hovedemnet for denne afhandling er regularitet af C^*-algebraer samt hvordan disse regularitetsegenskaber afspejles i den tilhørende C^*-algebra bestående af aymptotisk centrale følger. Afhandlingen består af en indledning efterfulgt af fire artikler [A], [B], [C], [D].

I [A] viser vi, at for klassen af simple Villadsen algebraer af enten den første type, som tillader en standard dekomposition, hvor basisrummet er et endeligt CW complex, eller af den anden type, er tensoriel absorption af Jiang-Su algebraen karakteriseret ved at C^*-algebraen bestående af asymptotisk centrale følger ikke har karakterer. I et fælles appendiks med Joan Bosa viser vi desuden at Villadsen algebraen af den anden type, med uendelig stabil rank, ikke besidder korona faktoriseringsegenskaben.

I [B] betragter vi klassen af separable C^*-algebraer, der ikke har karakterer på deres associerede C^*-algebra bestående af asymptotisk centrale følger, og viser at den har pæne stabilitetsegenskaber. Vi introducerer desuden en ny divisibilitetsegenskab, som vi kalder lokal divisibilitet, og relaterer Jiang-Su stabilitet til den lokale divisibilitetsegenskab for C^*-algebraer bestående af asymptotisk centrale følger. Specielt viser vi at en unital, simpel, separabel, nukleær C^*-algebra er Jiang-Su stabil hvis, og kun hvis, der ekisterer $k \geq 1$ så den associerede C^*-algebra bestående af asymptotisk centrale følger er k-lokalt næsten divisibel

I [C] finder vi, for en substantiel klasse af unitale, separable C^*-algebraer, et ideal i den associerede C^*-algebra bestående af asymptotisk centrale følger, der ikke er et σ-ideal.

I [D] giver vi en karakterisering af asymptotisk regularitet for simple, separable C^*-algebraer via Cuntz semigruppen og viser at hvis en simpel, separabel C^*-algebra hverken er stabilt endelig eller rent uendelig, da er den heller ikke asymptotisk regulær.

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Contents

Part I

Thesis overview and background

In this part of the thesis we introduce the background for the subjects investigated and provide an overview of the articles [A], [B], [C], [D] contained in Part II. We also explain the connection between the different articles and provide common motivation for them.

Sections 2 and 3 are expository in nature, and introduce the core objects of this thesis; central sequence algebras and the Cuntz semigroup. Section 4 motivates the comparison properties introduced in Section 3, by considering commutative C^*-algebras and AH algebras. Sections 5 and 6 describe the results of this thesis, and, finally, Section 7 discusses some possibilities for further research.

1. Introduction

The classification programme was initiated by George Elliott in [19], written in 1989, in which he proved that a substantial class of AH-algebras of real rank 0 are classified by their K-theory. A classification result had been obtained earlier by Glimm, in [25], but while the invariant used by Glimm, the supernatural number, is only defined for UHF algebras, Elliott's invariant is defined for all C^*-algebras. In [19], Elliott also suggested that his classification result might cover all separable, nuclear C^*-algebras with real rank 0, stable rank 1, and torsion free K_0- and K_1-groups. This was the first formulation of what became known as the Elliott Conjecture. This conjecture was subsequently modified to cover all unital, simple, separable, nuclear C^*-algebras with an augmented invariant known as the Elliott invariant colloquially referred to as 'K-theory and traces'. Henceforth, we refer to this modified conjecture as (EC), in the interest of brevity. The classification programme, that is, the effort to prove (EC), enjoyed remarkable success throughout the 1990's, with the most complete milestone being the classification of unital, simple, separable, nuclear, purely infinite C^*-algebras in the UCT class, obtained independently by Kirchberg and Phillips.

The Elliott Conjecture was met with considerable scepticism by the Operator Algebra community. By way of example, Effros wrote about (EC): "This was regarded as ridiculous by many (including myself), and we waited for the counter-examples to appear. We are still waiting." Well, we are no longer waiting: in [55] Rørdam constructed a unital, simple, separable and nuclear C^*-algebra in the UCT class which contains a non-zero finite projection while the unit is an infinite projection. Since (EC) predicts that any unital, simple, separable, nuclear C^*-algebra in which the unit is infinite must be purely infinite, this was the first counter-example to (EC). Other examples followed, and amongst these, the C^*-algebra constructed by Toms in [63] stands out, since it demonstrates that no 'reasonable' functor (meaning homotopy-invariant and continuous, in the sense of commuting with inductive limits) can act as classifying functor for the class of unital, simple, separable, nuclear C^*-algebras in the UCT class.

The examples above necessitated a revision of (EC), and two approaches appear natural: restrict (EC) to a suitably well-behaved subclass or augment the invariant. The latter approach requires a rather delicate touch. Indeed, as demonstrated by the example of Toms, the addition of any 'reasonable' functor will not suffice. Hence, the impact of the completeness of any such augmented invariant might be severely diminished. In the present thesis, we focus on the former approach, which seems more tractable and has proved remarkably successful in the last decade.

Three regularity properties have emerged as leading candidates for ensuring confirmation of (EC): Jiang-Su stability, finite nuclear dimension and strict comparison of positive elements. The Jiang-Su algebra $\mathcal{Z}$ is a unital, separable, infinite-dimensional, stably finite, strongly self-absorbing C^*-algebra with the same Elliott invariant as $\mathbb{C}$, introduced by Jiang and Su in [32] (see also [58] for an introduction). The class of unital, simple, separable, nuclear C^*-algebras which absorb the Jiang-Su algebra, that is $A \otimes \mathcal{Z} \cong A$, is the largest class for which (EC) can be expected to hold. Indeed, if A is a unital, simple, separable and nuclear C^*-algebra such that the ordered group $K_0(A)$ is weakly unperforated, then A and $A \otimes \mathcal{Z}$ have the same Elliott invariant (see [26]). The nuclear dimension of a C^*-algebra A, denoted $\dim_{\mathrm{nuc}}(A)$, is a measure of the 'non-commutative topological dimension' of A, and as such has a strong flavour of non-commutative topology. On the other hand, strict comparison of A is a comparison property of the Cuntz semigroup $\mathrm{Cu}(A)$ of A, which ensures that the order on $\mathrm{Cu}(A)$ is reasonable. The conjecture that these three regularity properties are equivalent for all unital, simple, separable, nuclear, non-elementary C^*-algebras is known as the Toms–Winter conjecture, and is based on the fact that for a wide range of simple, nuclear C^*-algebras they *are* equivalent (see for instance [66]). Remarkably, it was recently proven that finite nuclear dimension suffices for classification by the Elliott invariant, under the additional assumption of UCT, thus making the task of proving or disproving the Toms–Winter conjecture all the more relevant. The complete proof of this classification theorem has a long history and is the work of many hands, but the final steps were carried out in [20], [27] and [60].

A common approach to the regularity properties discussed above is to view them as C^*-algebraic analogues of von Neumann algebraic properties. In this view, the Jiang-Su algebra $\mathcal{Z}$ plays the role of the hyperfinite II_1-factor $\mathcal{R}$, and finite nuclear dimension is viewed as a 'coloured' version of being almost finite dimensional. Strict comparison can be thought of as a strong version of the fact that, in a finite von Neumann algebra, the order of projections, up to Murray-von-Neumann equivalence, is determined by the trace simplex. The philosophy behind this approach is that the techniques applied in the classification of injective von Neumann algebras (see [11], [29] and [49]) may be pulled back to the world of C^*-algebras with suitable modifications. This approach has inspired enormous progress in the last five years, especially following the papers of Matui and Sato ([43, 44]) and the introduction of property (SI). This property allows one to deduce information about the C^*-algebraic central sequence algebra from structural properties of the tracial central sequence algebra, which is of a more von Neumann algebraic nature.

The introduction of central sequence algebras in the study of regularity of von Neumann algebras was initiated by McDuff in [45]. In this paper, she proved that a separable II_1-factor $\mathcal{M}$ absorbs $\mathcal{R}$, i.e., $\mathcal{M} \overline{\otimes} \mathcal{R} \cong \mathcal{M}$, if, and only if, $\mathcal{R}$ embeds unitally into the von Neumann algebraic central sequence algebra $\mathcal{M}^\omega \cap \mathcal{M}'$. This result has a C^*-algebraic analouge, namely if $\mathcal{D}$ is a strongly self-absorbing C^*-algebra and A is a unital, separable C^*-algebra then $A \otimes \mathcal{D} \cong A$ if, and only if, there is a unital embedding of $\mathcal{D}$ into the C^*-algebraic central sequence algebra $A_\omega \cap A'$. This fact was crucial in the proof that every unital, simple, separable, nuclear, purely infinite C^*-algebra absorbs the Cuntz algebra

$\mathcal{O}_\infty$, see [34]. This, in turn is crucial to the classification of this class of C^*-algebras by the Elliott invariant.

Central sequence algebras pervade the literature on C^*-regularity, often appearing as a technical tool. By way of example, Winters seminal paper [74] passes through the central sequence algebra to prove that finite nuclear dimension implies $\mathcal{Z}$-stability. However, in [43] and [44], Matui and Sato considered actual structural properties of the central sequence algebra in a more direct way. They did this in order to prove that strict comparison implies $\mathcal{Z}$-stability, and that $\mathcal{Z}$-stability implies finite decomposition rank, which in turn implies finite nuclear dimension, under certain restrictions on the trace simplex of the C^*-algebra in question. They introduced property (SI), and showed that this property facilitates the translation of information about the *tracial* central sequence algebra to information about the *C^*-algebraic* central sequence algebra. In particular, if A has property (SI) and there exists a unital embedding of $\mathcal{R}$ into the tracial central sequence algebra of A, then $A \otimes \mathcal{Z} \cong A$, thus providing a tangible link between the study of C^*-regularity and the classification of injective von Neumann algebra factors. This approach has since been successfully applied in [8], [36] and [67] to prove that $\mathcal{Z}$-stability implies finite nuclear dimension and strict comparison implies $\mathcal{Z}$-stability, respectively, again under certain restrictions on the trace simplex.

As mentioned above, McDuff proved in [45] that a separable II_1-factor $\mathcal{M}$ absorbs $\mathcal{R}$ if, and only if, there is a unital embedding $\mathcal{R} \to \mathcal{M}^\omega \cap \mathcal{M}'$, but she also proved that this is equivalent to $\mathcal{M}^\omega \cap \mathcal{M}'$ being non-commutative. If this is the case, then it is automatically a II_1 von Neumann algebra. Inspired by the analogy between C^*-algebras and von Neumann algebras outlined above, one might wonder if a unital, simple and separable C^*-algebra A is $\mathcal{Z}$-stable if, and only if, $A_\omega \cap A'$ is non-commutative. This is too much to hope for though. As demonstrated in [1], the central sequence algebra $A_\omega \cap A'$ is almost always non-commutative. Seeking to remedy this, Kirchberg and Rørdam asked the following question in [37].

Question 1.1. *Let A be a unital, separable C^*-algebra. Does it follow that $A \otimes \mathcal{Z} \cong A$ if, and only if, $A_\omega \cap A'$ admits no characters.*

This question is the central subject of the present thesis and will be motivated in the sections to come. The main results of the thesis are briefly described below.

In [A] we prove that the above question has an affirmative when A is either a unital, simple Villadsen algebra of the first type, admitting a standard decomposition with seed space a finite CW-complex, or a Villadsen algebra of the second type. Since Villadsen algebras are a common source of counter-examples to statements about unital, simple AH algebra, this result indicates that an affirmative answer to the above question is possible. Additionally, in [A, Appendix A], written jointly with Joan Bosa, we show that the Villadsen algebra $\mathcal{V}_\infty$ of the second type with infinite stable rank fails the corona factorization property. This is the first example of a unital, simple, separable, nuclear C^*-algebra with a unique tracial state failing this property. Additionally, this result led the author to consider whether the corona factorization property implies ω-comparison for all simple and separable C^*-algebras. While no progress was made on this question, it did lead to a characterization of asymptotic S-regularity in [D], which demonstrates, with minimal effort, that any simple, separable and asymptotically S-regular C^*-algebra is either stably finite or purely infinite.

In [B], a new divisibility property is introduced, that we call local divisibility. This divisibility property is weaker than the k–almost divisibility property considered by Winter in [74], and stronger than the weak divisibility property considered by Robert and Rørdam in [51], which characterizes an absence of finite-dimensional, irreducible representations. It is shown that $A \otimes \mathcal{Z} \cong A$, if A is a unital, simple, separable, nuclear C^*-algebra such that $F(A)$ is k-locally almost divisible, for some $k \geq 1$. Looking beyond the simple case, permanence properties of the class of separable C^*-algebras A for which the central sequence algebra $F(A)$ admits no characters was also considered in [B]. It is shown that this class has good permamence: the class os closed un arbitrary tensor products, taking hereditary subalgebras, quotients and extensions. We also consider inductive limits and provide a neccesary and sufficient criterion for when the central sequence algebra of the limit admits no characters, in terms of the inductive limit structure. Finally, in [C] we show that for a substantial class of $\mathcal{Z}$-stable C^*-algebras, there is an ideal in $F(A)$ which is not a σ-ideal.

1.1. Notation and conventions. The letters A, B, C, D are reserved for C^*-algebras. If A is a unital C^*-algebra and $B \subseteq A$ is a C^*-subalgebra, then we say that B is a unital C^*-subalgebra, if B contains the unit of A. Given a C^*-algebra A, we let A_+ denote its positive cone and let $(A)_1$ denote the closed unit ball of A. Given a C^*-algebra A, an ideal $I \subseteq A$ will always mean a closed, 2-sided ideal. We use $\mathcal{H}$ to denote a Hilbert space, $B(\mathcal{H})$ to denote the bounded linear operators on $\mathcal{H}$ and $\mathbb{K}(\mathcal{H})$ to denote the compact operators on $\mathcal{H}$. We let $\mathbb{K}$ denote the compact operators on $\ell^2(\mathbb{N})$. Furthermore, we let $T(A)$ denote the Choquet simplex of tracial states on a unital C^*-algebra A and let $\partial_e T(A)$ denote the extremal boundary of $T(A)$.

2. Central sequence algebras

In this section we introduce the reader to central sequence algebras. As these objects are the core of the present thesis, the introduction will be fairly thorough. We will consider both the metric and tracial central sequence algebras of C^*-algebras. Although the tracial versions are rarely considered explicitely in Part II, they motivate many of the properties of metric central sequence algebra we discuss, and they have played a crucial role in the study of C^*-regularity in recent years, following the papers of Matui and Sato ([43, 44]).

For the purposes of this thesis, the most important central sequence algebra is the algebra consisting of bounded sequences in a C^*-algebra A, which are asymptotically central *in norm*. To be more precise, fix a free ultrafilter ω on $\mathbb{N}$ and a C^*-algebra A, and let $\ell^\infty(A)$ denote the set of uniformly bounded sequences in A, i.e.,

$$\ell^\infty(A) = \{(a_n)_n \subseteq A \mid \sup_n \|a_n\| < \infty\}.$$

Let $c_\omega(A) := \{(a_n)_n \in \ell^\infty(A) \mid \lim_{n\to\omega} \|a_n\| = 0\}$. It is easy to check that $c_\omega(A)$ is an ideal in $\ell^\infty(A)$ and we let A_ω denote the quotient algebra, i.e.,

$$A_\omega = \ell^\infty(A)/c_\omega(A).$$

Similarly, given a sequence $(A_n)_{n\geq 1}$ of C^*-algebras, we let

$$\prod_\omega A_n := \big(\prod_{n\geq 1} A_n / \big\{(a_n)_n \in \textstyle\prod_{n\geq 1} A_n\big) \bigm| \lim_{n\to\omega} \|a_n\| = 0\big\}.$$

Given a sequence $(a_n)_n \in \ell^\infty(A)$, we let $[(a_n)_n] \in A_\omega$ denote its image under the quotient map $\ell^\infty(A) \to A_\omega$. Note that the *-homomorphism $\iota\colon A \to A_\omega$, given by $\iota(a) = [(a, a, \dots)]$, is injective. Hence, we usually omit reference to ι and simply consider A to be a C^*-subalgebra of A_ω. Given a C^*-subalgebra $B \subseteq A_\omega$, let $\mathrm{Ann}(B, A_\omega)$ denote the annihilator of B in A_ω, i.e., $\mathrm{Ann}(B, A_\omega) := \{x \in A_\omega \mid xB = Bx = \{0\}\}$. Note that $\mathrm{Ann}(B, A_\omega) \subseteq A_\omega \cap B'$ is an ideal, and let

$$F(B, A) := (A_\omega \cap B')/\mathrm{Ann}(B, A_\omega). \tag{2.1}$$

We simply write $F(A)$ for $F(A, A)$, and refer to $F(A)$ as the *central sequence algebra of* A. Note that, if A is unital and $B \subseteq A_\omega$ is a unital C^*-subalgebra, then $\mathrm{Ann}(B, A_\omega) = \{0\}$, whence $F(B, A) \cong A_\omega \cap B'$. This definition is due to Kirchberg, see [33, Definition 1.1]

Note that we have suppressed reference to ω in the notation $F(A)$. In fact, depending on which axioms of set theory one is willing to accept, $F(A)$ either does or does not depend on the choice of ω. To be precise, if one assumes the Continuum Hypothesis, then $A_\omega \cap A'$ *does not* depend on ω, by [24], but this is false without the Continuum Hypothesis, see [22, Theorem 5.1]. However, whether or not the Continuum Hypothesis holds, the answers to the questions considered in this thesis *do not* depend on ω. To be precise, let ω and ω' be free ultrafilters on $\mathbb{N}$ and assume that A is separable and unital. Then, given any unital and separable C^*-subalgebra $B \subseteq A_\omega \cap A'$, it is straightforward to check that there exists a unital and injective *-homomorphism $B \to A_{\omega'} \cap A'$. In other words, the isomorphism class of (unital) separable C^*-subalgebras of $A_\omega \cap A'$ does not depend on ω. Using [33, Corollary 1.8 and Proposition 1.9 (3)] it is easy to see that the same applies to $F(A)$, when A is separable but not necessarily unital.

It may not be apparent why it is useful to consider $F(A)$ rather than $A_\omega \cap A'$, when A is non-unital. To motivate this we, briefly consider the case $A = \mathbb{K}$. Let $(f_n)_n \subseteq \mathbb{K}$ be a sequence of mutually orthogonal projections of rank 1 such that $e_k := \sum_{n=1}^k f_n \to \mathrm{id}_{\mathcal{H}}$ in the strong operator topology. It is easy to see that the sequence $(f_k)_k \subseteq \mathbb{K}$ is an approximate unit for $\mathbb{K}$ and one might hope that $[(f_k)_k] \in \mathbb{K}_\omega \cap \mathbb{K}'$ would define a unit. However, for each $x \in \mathbb{K}_\omega \cap \mathbb{K}'$, there exists a non-zero element $y \in \mathbb{K}_\omega \cap \mathbb{K}'$ such that $xy = 0$; in particular, $\mathbb{K}_\omega \cap \mathbb{K}'$ is non-unital. We sketch a proof of this fact: for any $K \in \mathbb{K}$ and $\varepsilon > 0$ there exist elements $E, F \in \mathbb{K}$ such that $\|K - E\| < \varepsilon$ and $\|F\| = 1$, while $EF = FE = 0$. In fact, one can choose $E := e_n K e_n$ and $F := f_{n+1}$, for some sufficiently large $n \geq 1$. Hence, supposing $(K_n)_n \subseteq \mathbb{K}$ is a bounded sequence such that $[(K_n)_n] \in \mathbb{K}_\omega \cap \mathbb{K}'$, there exist bounded sequences $(E_n)_n, (F_n)_n \subseteq \mathbb{K}$ such that $[(K_n)_n] = [(E_n)_n]$, $\|F_n\| = 1$ and $E_n F_n = 0$ for all $n \geq 1$. It follows that $[(E_n)_n] \in \mathbb{K}_\omega \cap \mathbb{K}'$, and with a little extra effort, one can ensure that $[(F_n)_n] \in \mathbb{K}_\omega \cap \mathbb{K}'$, thus proving the claim. Similarly, one checks that $[(f_n)_n] \in \mathbb{K}_\omega \cap \mathbb{K}'$ defines a non-zero element in $\mathrm{Ann}(\mathbb{K}, \mathbb{K}_\omega)$, whence there is no isomorphism $\mathbb{K}_\omega \cap \mathbb{K}' \to \mathbb{C}_\omega \cap \mathbb{C}' \cong \mathbb{C}$ (the latter can also be realized from the statement that $\mathbb{K}_\omega \cap \mathbb{K}'$ is non-unital), and hence the assignment $A \mapsto A_\omega \cap A'$ is not a stable invariant.

In contrast, with the central sequence algebra given as in (2.1), it follows that $F(A)$ is unital whenever A is σ-unital, and that the assignment $A \mapsto F(A)$ *is* a stable invariant for the class of σ-unital C^*-algebras (see [33, Corollary 1.10]). Crucially, $F(A)$ still has the property that the map $\rho_A\colon A \otimes_{\max} F(A) \to A_\omega$, given by

$$a \otimes \big(x + \mathrm{Ann}(A, A_\omega)\big) \mapsto ax,$$

is a well-defined *-homomorphism such that $\rho_A(a \otimes \mathbf{1}_{F(A)}) = a$ for all $a \in A$.

Now, let A be a separable and unital C^*-algebra such that $T(A)$, the set of tracial states on A, is non-empty. Given a non-empty subset $S \subseteq T(A)$ and $p \geq 1$, let

$$\|a\|_{p,S} := \sup_{\tau \in S} \tau\big((a^*a)^{p/2}\big)^{1/p}, \quad a \in A$$

Similarly, given a sequence $\mathcal{S} = (S_1, S_2, \dots)$ of non-empty subsets $S_n \subseteq T(A)$, let

$$\|[(a_n)_n]\|_{p,\mathcal{S}} := \lim_{n \to \omega} \|a_n\|_{p,S_n}, \quad (a_n)_n \in \ell^\infty(A).$$

In the special case where $S_n = T(A)$, for all $n \geq 1$, we let $\|[(a_n)_n]\|_{p,\omega}$ denote the left-hand side above. It is well-known that $\|\cdot\|_{p,\mathcal{S}}$ is a semi-norm on A_ω, for each p and $\mathcal{S}$, and that the following properties are satisfied for all $x, y \in A_\omega$,

$$\|x\|_{p,\mathcal{S}} = \|x^*\|_{p,\mathcal{S}}, \quad \|yx\|_{p,\mathcal{S}} \leq \|y\|\|x\|_{p,\mathcal{S}}, \quad \|x\|_{p,\mathcal{S}} \leq \|x\|,$$
$$\|x\|_{1,\mathcal{S}} \leq \|x\|_{p,\mathcal{S}} \leq \|x\|_{1,\mathcal{S}}^{1/p} \cdot \|x\|^{1-1/p}.$$

In particular, $\|x\|_{1,\mathcal{S}} = 0$ if, and only if, $\|x\|_{p,\mathcal{S}} = 0$ for some $p \geq 1$.

Given a sequence $\mathcal{S} = (S_1, S_2, \dots)$ of non-empty subsets of $T(A)$, let $J_\mathcal{S} := \{x \in A_\omega \mid \|x\|_{1,\mathcal{S}} = 0\}$ and note that $J_\mathcal{S} \subseteq A_\omega$ is an ideal. We let

$$A^\omega_\mathcal{S} := A_\omega / J_\mathcal{S}.$$

As before, there is a canonical *-homomorphism $\iota_\mathcal{S} \colon A \to A^\omega_\mathcal{S}$, given by $\iota_\mathcal{S} := \rho_\mathcal{S} \circ \iota$, where $\rho_\mathcal{S} \colon A_\omega \to A^\omega_\mathcal{S}$ denotes the quotient map. In contrast with the situation before, the map $\iota_\mathcal{S}$ is not always injective. In fact, it is easy to check that $\iota_\mathcal{S}$ is injective if, and only if, $\|\cdot\|_{p,\mathcal{S}}$ restricts to a norm on A. This is always the case if A is simple, but in general it need not be true. Despite this, we shall often suppress $\iota_\mathcal{S}$ in our notation, and simply write $A \subseteq A^\omega_\mathcal{S}$. We refer to $A^\omega_\mathcal{S} \cap A'$ as the tracial central sequence of A with respect to $\mathcal{S}$. Finally, when $\mathcal{S} = (T(A), T(A), \dots)$, we write A^ω for $A^\omega_\mathcal{S}$, and simply refer to $A^\omega \cap A'$ as *the* tracial central sequence algebra of A. We will usually consider the full tracial central sequence algebra $A^\omega \cap A'$, but the 'partial' tracial central sequence algebras $A^\omega_\mathcal{S} \cap A'$ are occasionally useful to consider as well, in particular when $\partial_e T(A)$ is not compact. When $\mathcal{S} = (S, S, \dots)$ for some fixed non-empty subset $S \subseteq T(A)$, we shall abuse terminology and notation slightly, by writing A^ω_S rather than $A^\omega_\mathcal{S}$, J_S rather than $J_\mathcal{S}$, and referring to the tracial central sequence algebra with respect to S rather than $\mathcal{S}$.

2.1. Key properties. In this section, we describe key properties of the central sequence algebra $F(A)$ of a C^*-algebra A. We start with a proposition which really concerns ultrafilters rather than central sequence algebras. However, the result is a wonderful illustration of the philosophy behind the study of central sequence algebras, and it is therefore worth stating at the start of this section. A proof may be found in [33] or, alternatively, in [36]. Recall that ω denotes a fixed free ultrafilter on $\mathbb{N}$.

Proposition 2.2 (Kirchberg's ε-test). *Let $X_1, X_2, \dots$ be any sequence of non-empty sets and suppose that, for each $k \in \mathbb{N}$, we are given a sequence $(f_n^{(k)})_{n \geq 1}$ of functions $f_n^{(k)} \colon X_n \to [0, \infty)$.*

For each $k \in \mathbb{N}$, define a new function $f_\omega^{(k)} \colon \prod_{n \geq 1} X_n \to [0, \infty]$ by

$$f_\omega^{(k)}(s_1, s_2, \dots) := \lim_{n \to \omega} f_n^{(k)}(s_n), \quad (s_n)_{n \geq 1} \in \prod_{n \geq 1} X_n.$$

Suppose that, for each $m \in \mathbb{N}$ and $\varepsilon > 0$, there exists $s = (s_1, s_2, \dots) \in \prod_{n\geq 1} X_n$ such that

$$f_\omega^{(k)}(s) < \varepsilon, \quad \text{for } k = 1, \dots, m.$$

It follows that there exists $t = (t_1, t_2, \dots) \in \prod_{n\geq 1} X_n$ with

$$f_\omega^{(k)}(t) = 0, \quad \text{for all } k \in \mathbb{N}.$$

The beauty of this proposition lies, primarily, in its generality, i.e., that there are no restrictions on the sets $X_1, X_2, \dots$ or the 'test-functions' $f_n^{(k)}$. It demonstrates that, for a wide range properties, if A_ω satisfies the 'ε-version' of that property, then A_ω also satisfies the exact version. By way of example, if B is separable, A is unital and $\varphi, \psi\colon B \to A_\omega$ are approximately unitarily equivalent *-homomorphisms, then they are in fact unitarily equivalent. Similarly, if $B \subseteq A_\omega$ is a separable C^*-subalgebra and $b_0 \in B$ is a strictly positive element such that, for every $\varepsilon > 0$, there exists a positive contraction $x_\varepsilon \in A_\omega$ with $\|x_\varepsilon b_0 - b_0\| < \varepsilon$, then there exists a positive contraction $x \in A_\omega$ such that $xb = b$, for all $b \in B$. When A is separable, the same technique can be applied to $A_\omega \cap A'$ and $F(A)$.

The study of central sequence algebras carries a distinct flavour of infinite tensor powers (see for instance [37]). This flavour can be traced back to the following theorem.

Theorem 2.3 (Kirchberg, [33])**.** *Suppose A is a separable C^*-algebra, $B \subseteq A_\omega$ is a separable C^*-subalgebra and $D \subseteq F(A)$ is a separable C^*-subalgebra with $\mathbf{1} \in D$. Then there exists a *-homomorphism $\psi\colon C_0((0,1]) \otimes D \to A_\omega \cap B'$ such that $\psi(\iota \otimes \mathbf{1})b = b$ for all $b \in B$, where $\iota\colon [0,1] \to [0,1]$ denotes the identity map. In particular, the map*

$$[\psi]\colon D \ni d \mapsto \psi(\iota \otimes d) + \mathrm{Ann}(B, A_\omega) \in F(B, A)$$

is a unital *-homomorphism.

In particular, one deduces the following result.

Corollary 2.4 (Kirchberg, [33])**.** *Suppose A is a separable C^*-algebra and that C and $B_1, B_2, \dots$ are separable and unital C^*-subalgebras of $F(A)$. Then there exists a unital *-homomorphism*

$$\psi\colon C \otimes \bigotimes_{i\geq 1} B_i \to F(A),$$

from the infinite maximal tensor product to $F(A)$, such that $\psi(c \otimes \mathbf{1} \otimes \mathbf{1} \otimes \cdots) = c$ for all $c \in C$.

Dadarlat and Toms show in [16] that, if A is a unital and separable C^*-algebra such that the infinite minimal tensor power $A^{\otimes\infty}$ contains, unitally, a subhomogeneous algebra without characters, then $A^{\otimes\infty} \otimes \mathcal{Z} \cong A^{\otimes\infty}$. Combining this theorem with the above corollary, one obtains the following.

Theorem 2.5 (Kirchberg, [33], Dadarlat–Toms, [16])**.** *Suppose A is a unital and separable C^*-algebra such that $F(A)$ contains, unitally, a subhomogeneous C^*-algebra without characters. Then $A \otimes \mathcal{Z} \cong A$.*

Given an inductive limit $A \cong \varinjlim(A_i, \varphi_i)$, there is a useful relationship between $F(A)$ and the sequence of central sequence algebras $(F(A_i))_{i\geq 1}$. While the former cannot necessarily be recovered from the latter, one *can* recover information about which separable C^*-algebras are unitally contained in $F(A)$. To be more precise:

Proposition 2.6 (Kirchberg, [33]). *Suppose A is a separable C^*-algebra and $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$ is a sequence of C^*-subalgebras with dense union in A. Then, for every separable and unital C^*-subalgebra D of the ultraproduct $\prod_\omega F(A_n)$, there is a unital *-homomorphism $D \to F(A)$.*

It should be noted that the converse statement is not true, i.e., given a separable and unital C^*-subalgebra $D \subseteq F(A)$ there is not necessarily a unital *-homomorphism $D \to \prod_\omega F(A_n)$. Indeed, let M_{2^∞} denote the CAR algebra and let $\{A_n\}_{n\geq 1}$ denote an increasing sequence of unital C^*-subalgebras of M_{2^∞} such that $A_n \cong M_{2^n}$ and $\bigcup_n A_n \subseteq M_{2^\infty}$ is dense. Then, since the closed unit ball $(A_n)_1$ is compact, for all $n \geq 1$, it follows that

$$F(A_n) = (A_n)_\omega \cap A_n' \cong A_n \cap A_n' = \mathbb{C}.$$

Since $F(M_{2^\infty})$ contains a unital copy of M_2, the desired statement follows.

Finally, one might wonder how the assignment $F(-)$ behaves with respect to short exact sequences. As ever, any information we can conceivably obtain will be on the containment of separable C^*-subalgebras, but before we state the next results, we need to introduce some terminology.

Definition 2.7 (Kirchberg, [33]). Let D be a C^*-algebra and $I \subseteq D$ be an ideal.

(i) We say that I is a *σ-ideal*, if for every separable C^*-subalgebra $B \subseteq D$, there exists a positive contraction $e \in I \cap B'$ such that $eb = b$ for all $b \in B \cap I$.
(ii) Let $\pi\colon D \to D/I$ denote the quotient map. We say that the sequence $0 \to I \to D \to D/I \to 0$ is *strongly locally semi-split*, if for every separable C^*-subalgebra $C \subseteq D/I$ there exists a *-homomorphism $\psi\colon C_0((0,1]) \otimes C \to D$ such that $\pi \circ \psi(\iota \otimes c) = c$, for every $c \in C$, where $\iota\colon [0,1] \to [0,1]$ denotes the identity map.

The reason for introducing σ-ideals is given in the next proposition

Proposition 2.8 (Kirchberg, [33]). *Let D be a C^*-algebra, $I \subseteq D$ an ideal and $\pi\colon D \to D/I$ the quotient map. If I is a σ-ideal in D, then, for every separable C^*-subalgebra $C \subseteq D$, the sequence*

$$0 \to I \cap C' \to D \cap C' \to (D/I) \cap \pi(C)' \to 0$$

is exact and strongly locally semi-split.

Of course, this does not immediately apply to central sequence algebras, but that is remedied by the next result.

Proposition 2.9 (Kirchberg, [33]). *Let A be a C^*-algebra and suppose that $J \subseteq A$ is an ideal and $B \subseteq A_\omega$ is a separable C^*-subalgebra. Then J_ω is a σ-ideal in A_ω, and both $J_\omega \cap (A_\omega \cap B')$ and $\mathrm{Ann}(B, A_\omega)$ are σ-ideals in $A_\omega \cap B'$.*

Given an ideal $I \subseteq A$ in a C^*-algebra A, it is not difficult to check that, with $\pi\colon A \to A/I$ denoting the quotient map, the induced *-homomorphism $\pi_\omega\colon A_\omega \to (A/I)_\omega$ is surjective, with $\ker(\pi) = I_\omega$. Combining this with the propositions above one obtains the following:

Corollary 2.10 (Kirchberg, [33]). *Let A be a C^*-algebra and $I \subseteq A$ an ideal. Then the sequences*

$$0 \to I_\omega \cap A' \to A_\omega \cap A' \to (A/I)_\omega \cap (A/I)' \to 0$$

and

$$0 \to \operatorname{Ann}(A, A_\omega) \to A_\omega \cap A' \to F(A) \to 0$$

are exact and strongly locally semi-split.

The above proposition together with Theorem 2.3 allows one to infer properties of $F(I)$ and $F(A/I)$ from properties of $F(A)$. For example, if D is a unital and separable C^*-subalgebra of $F(A)$, then there exist unital *-homomorphisms from D to both $F(I)$ and $F(A/I)$. Passing from properties of $F(I)$ and $F(A/I)$ to properties of $F(A)$ is less straightforward, but something can still be said. Given unital C^*-algebras D_0 and D_1, let

$$\mathcal{E}(D_0, D_1) := \big\{ f \in C([0,1], D_0 \otimes_{\max} D_1) \;\big|\; f(0) \in D_0 \otimes \mathbf{1},\ f(1) \in \mathbf{1} \otimes D_1 \big\}.$$

Proposition 2.11 (Kirchberg, [33])**.** *Let A be a unital and separable C^*-algebra, $I \subseteq A$ an ideal and $B \subseteq A$ a unital, separable C^*-subalgebra. If $D_1 \subseteq F(\pi_\omega(B), A/I)$ and $D_0 \subseteq F(I)$ are unital C^*-subalgebras, then there is a unital *-homomorphism $\mathcal{E}(D_0, D_1) \to F(B, A)$.*

We now turn our attention to the properties of A that can be inferred from properties of $F(A)$. First of all, we say that a completely positive map $T\colon A \to A_\omega$ is *ω-nuclear* if there exists a bounded sequence of nuclear, completely positive maps $T_n\colon A \to A$ such that $T = T_\omega|_A$, where T_ω denotes the completely positive map $A_\omega \to A_\omega$ given by $T([(a_1, a_2, \dots)]) = [(T_1(a_1), T_2(a_2), \dots)]$. Furthermore, we let $\rho_A\colon A \otimes_{\max} F(A) \to A_\omega$ denote the *-homomorphism given by

$$\rho_A\big(a \otimes (b + \operatorname{Ann}(A, A_\omega))\big) = ab,$$

for $a \in A$ and $b \in A_\omega \cap A'$.

Theorem 2.12 (Kirchberg, [33])**.** *Let A be a separable C^*-algebra. Then the following holds:*

(i) Let H denote the set of positive elements $b \in F(A)$ such that the completely positive map

$$A \ni a \mapsto \rho_A(a \otimes b) \in A_\omega$$

is ω-nuclear. Then H is the positive part of an essential *ideal $J_{\mathrm{nuc}} \subseteq F(A)$. In particular, J_{nuc} is always non-zero.*

(ii) A is nuclear if, and only if, $J_{\mathrm{nuc}} = F(A)$.

It follows that A is nuclear, if $F(A)$ is simple. In fact, much more can be inferred from simplicity of $F(A)$. We say that a C^*-algebra A is *elementary*, if $A \cong \mathbb{K}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$.

Theorem 2.13 (Kirchberg, [33])**.** *Let A be a separable C^*-algebra. Then the following holds*

(i) A is elementary, if $F(A) \cong \mathbb{C}$.

(ii) A is simple, purely infinite and nuclear, if $F(A)$ is simple and $F(A) \ncong \mathbb{C}$.

Finally, let us observe that $F(A)$ remembers whether or not A absorbs $\mathcal{Z}$. In the formulation below, the theorem is due to Kirchberg, but the genesis of the proof is the approximate intertwining argument of Elliott, which first appeared in [18], and was refined in [19]. The 'only if' statement follows from the fact that $\mathcal{Z}$ is strongly self-absorbing, and hence satisfies $\mathcal{Z} \cong \bigotimes_{i=1}^\infty \mathcal{Z}$.

Theorem 2.14 (Kirchberg, [33]). *Let A be a separable C^*-algebra. Then $A \otimes \mathcal{Z} \cong A$ if, and only if, there is a unital *-homomorphism $\mathcal{Z} \to F(A)$.*

The above theorem was established earlier for unital C^*-algebras, and similar characterizations have been given elsewhere for non-unital C^*-algebras, see for instance [54, Theorem 7.2.2] and [68, Theorem 2.3].

2.2. Property (SI). While the C^*-algebra $F(A)$ seems to contain detailed information about A, sometimes in surprising ways, it often guards its secrets well, and it is therefore desirable to consider a more manageable object which retains sufficient information. In recent years, following the papers of Matui and Sato ([43, 44]), the tracial central sequence algebra has emerged as a strong candidate for such an object.

The following result was originally proved by Sato for nuclear C^*-algebras in [59], and later substantially generalized by Kirchberg and Rørdam in [36].

Theorem 2.15 (Kirchberg–Rørdam, [36]). *Let A be a unital C^*-algebra such that $T(A)$ is non-empty. Then, for any sequence $\mathcal{S} = (S_1, S_2, \dots)$ of non-empty subsets $S_n \subseteq T(A)$, the ideal $J_{\mathcal{S}}$ is a σ-ideal. In particular, the quotient map $A_\omega \to A^\omega_{\mathcal{S}}$ restricts to a surjection $F(A) \to A^\omega_{\mathcal{S}} \cap A'$.*

Let $\tau \in \partial_e T(A)$ be an extremal tracial state, $\pi_\tau \colon A \to B(\mathcal{H}_\tau)$ denote the GNS-representation of A with respect to τ, and $\mathcal{N}_\tau$ denote the von Neumann algebra generated by π_τ. Using Kaplanski's Density Theorem, one can show that $\pi_\tau \colon A \to \mathcal{N}_\tau$ induces an isomorphism $A^\omega_\tau \cap A' \cong (N_\tau)^\omega \cap \mathcal{N}_\tau'$, whence the above theorem implies the existence of a surjection $F(A) \to (\mathcal{N}_\tau)^\omega \cap \mathcal{N}_\tau'$. In fact, if $\partial_e T(A)$ is finite, say $\partial_e T(A) = \{\tau_1, \dots, \tau_n\}$, one can use the Chinese Remainder Theorem to show that

$$A^\omega \cap A' \cong \bigoplus_{i=1}^{n} (\mathcal{N}_{\tau_i})^\omega \cap \mathcal{N}_{\tau_i}'.$$

In particular, if A is unital, simple, nuclear and infinite dimensional, deep theorems from the theory of von Neumann algebras imply the existence of a unital *-homomorphism $\mathcal{R} \to A^\omega \cap A'$. The question remains how much information about A this provides, or, in the first instance, how much information about $F(A)$ one obtains from this.

One of the major contributions of Matui and Sato was to introduce a property which they called property (SI), the presence of which means that a substantial amount of information about $F(A)$ can be deduced from structural properties of $A^\omega \cap A'$. The definition of property (SI) is only given for simple, unital C^*-algebras, since it is not clear what the definition should be in the absence of simplicity. We state the definition given by Kirchberg–Rørdam. See [36, Lemma 5.2] for a proof that this definition is equivalent to the original definition.

Definition 2.16 (Matui–Sato). A unital, simple C^*-algebra A is said to have *property (SI)* if, for all positive contractions $e, f \in F(A)$ with $e \in J_{T(A)}$ and $\sup_n \|1 - f^n\|_{2,\omega} < 1$, there exists $s \in F(A)$ such that $fs = s$ and $s^*s = e$.

In [37], a third characterization of property (SI) was given. To state it, we need a small amount of notation: let $T_\omega(A) \subseteq T(A_\omega)$ denote the set of tracial states τ on the form

$$\tau([(a_1, a_2, \dots)]) = \lim_{n\to\omega} \tau_n(a_n), \quad (a_1, a_2, \dots) \in \ell^\infty(A),$$

where $(\tau_n)_n$ is a sequence of tracial states on A.

Proposition 2.17 (Kirchberg–Rørdam, [37])**.** *Let A be a unital, separable, simple, and exact C^*-algebra. Then A has property (SI) if, and only if, $a \precsim b$ in $F(A)$, whenever $a, b \in F(A)$ are positive contractions for which there exists $\delta > 0$ satisfying $\tau(a) = 0$ and $\tau(b) \geq \delta$, for all $\tau \in T_\omega(A)$.*

The relation $\precsim$ is defined in Definition 3.1. Stated as above, property (SI) may appear to be fairly harmless, but the introduction of this property has had a big impact on the study of the regularity properties mentioned in the introduction. Let us state an application of property (SI), which is implicitly contained in [43] (see also [36, Proposition 5.12]).

Theorem 2.18 (Matui–Sato, [43])**.** *Let A be a unital, simple, separable, exact, stably finite C^*-algebra such that A has property (SI). Then the following are equivalent:*

(i) $A \otimes \mathcal{Z} \cong A$.
(ii) *There exists a unital *-homomorphism $\mathcal{R} \to A^\omega \cap A'$.*
(iii) *There exists a unital *-homomorphism $M_k \to A^\omega \cap A'$ for some $k \geq 2$.*

The implications (i)⇒(ii)⇒(iii) are true for any unital and separable C^*-algebra with $T(A) \neq \emptyset$, and the full range of assumptions is only necessary for (iii)⇒(i). However, all the assumptions *are* necessary for this implication. Indeed, as demonstrated by Villadsen in [72], there exists a sequence of unital, simple, separable and nuclear C^*-algebras $\mathcal{V}_n$, such that each $\mathcal{V}_n$ has a unique tracial state, while $\mathcal{V}_n \otimes \mathcal{Z} \not\cong \mathcal{V}_n$, for all $n \geq 1$. It follows that there are a unital *-homomorphisms $\mathcal{R} \to \mathcal{V}_n^\omega \cap \mathcal{V}_n'$. Thus, without property (SI), there is a real difference in the information contained in $F(A)$ and $A^\omega \cap A'$. Finally, let us note how property (SI) relates to the Toms–Winter conjecture. See Definition 3.16 for the definition of strict comparison.

Proposition 2.19 (Matui–Sato, [43], Kirchberg–Rørdam, [36])**.** *Let A be a unital, simple, separable, nuclear, stably finite C^*-algebra with strict comparison of positive elements. Then A has property (SI).*

In fact, a weaker comparison property than strict comparison will suffice for the above to hold, for instance local weak comparison, see [36]

2.3. W^*-bundles. As indicated above, the study of $A^\omega \cap A'$ has a distinct flavour of von Neumann algebras. When $\partial_e T(A)$ is finite, this is apparent from the comments below Proposition 2.15. However, Ozawa made this flavour tangible for a much bigger class of C^*-algebras, namely those for which $\partial_e T(A)$ is non-empty and compact, with the introduction of W^*-bundles in [48].

Definition 2.20 (Ozawa, [48])**.** Let K be a compact Hausdorff space and M a unital C^*-algebra. We say that M is a *tracial continuous W^*-bundle* over K if the following axioms holds:

(i) There is a unital, positive, faithful and tracial map $E\colon M \to C(K)$.
(ii) The closed unit ball of M is complete with respect to the uniform 2-norm:

$$\|x\|_{2,u} := \|E(x^*x)^{1/2}\|, \quad x \in M.$$

(iii) $C(K)$ is unitally contained in the center of M and E is a conditional expectation onto $C(K)$.

When M is given as in the definition above, we simply refer to M as a *W^*-bundle* over K, rather than a tracial continuous W^*-bundle.

Let M be a W^*-bundle over K. For each point $\lambda \in K$, we let τ_λ denote the tracial state $\tau_\lambda := \mathrm{ev}_\lambda \circ E$, and π_λ denote the GNS-representation of M corresponding to τ_λ. We refer to $\pi_\lambda(M)$ as the *fibre of M over* λ, and denote it M_λ. Note that a W^*-bundle over a one-point space is nothing more than a W^*-algebra with a faithful tracial state.

A few observations on this definition are in order. First, clearly M is a $C(K)$-algebra (as defined in [30]) whenever M is a W^*-bundle over K, but the notion of fibre depends on the view-point. To be precise, in general, the $C(K)$-algebra fibre over M corresponding to λ, i.e., $M/(C_0(K\backslash\{\lambda\})M)$, is not isomorphic to M_λ. Indeed, it is easy to check that

$$\ker(\pi_\lambda) = \overline{C_0(K\backslash\{\lambda\})M},$$

where the closure is taken with respect to the uniform 2-norm, see for instance [C, Lemma 2.4]. Hence there is a surjection $M/(C_0(K\backslash\{\lambda\})M) \to M_\lambda$, but this map is not injective in general. Second, the terminology W^*-bundle is derived from the fact that all the fibres are W^*-algebras (see [48, Theorem 11]), and the word *continuous* refers to the fact that, for any $x \in M$, the map $K \ni \lambda \mapsto \|x\|_{2,\tau_\lambda}$ is continuous.

As for $C(K)$-algebras, there is a notion of triviality for W^*-bundles.

Definition 2.21 (Ozawa, [48])**.** Suppose that K is a compact Hausdorff space and $\mathcal{N}$ is a finite factor. Let $C_\sigma(K, \mathcal{N})$ denote the W^*-bundle

$$C_\sigma(K, \mathcal{N}) := \big\{f \colon K \to \mathcal{N} \;\big|\; f \text{ is norm-bounded and continuous in } \|\cdot\|_{2,\tau}\big\},$$

where τ denotes the unique tracial state on $\mathcal{N}$.

Let M and N be W^*-bundles over K and L respectively, with conditional expectations E_M and E_N. A morphism $\theta\colon M \to N$ of W^*-bundles consists of a *-homomorphism $\theta\colon M \to N$ and a *-homomorphism $\rho\colon C(K) \to C(L)$ such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\;\theta\;} & N \\ {\scriptstyle E_M}\downarrow & & \downarrow{\scriptstyle E_n} \\ C(K) & \xrightarrow[\;\rho\;]{} & C(L). \end{array}$$

Definition 2.22 (Ozawa, [48])**.** Let M be a W^*-bundle over K and $\mathcal{N}$ be a finite factor such that $M_\lambda \cong \mathcal{N}$, for all λ. Then we say that M is a *trivial W^*-bundle* with fibres $\mathcal{N}$, if there exists an isomorphism of W^*-bundles $\theta\colon M \to C_\sigma(K, \mathcal{N})$.

There is also a natural notion of central sequence algebra for a W^*-bundle. Given a compact Hausdorff space K, let K^ω denote the compact Hausdorff space such that $C(K^\omega) \cong C(K)_\omega$ (this is unique up to homeomorphism). This space can also be constructed explicitly (see [2]), but the precise construction is not important in this context.

Definition 2.23 (Bosa–Brown–Tikuisis–Sato–White–Winter, [8])**.** Let M be a W^*-bundle over a compact Hausdorff space K. Define the *ultrapower M^ω of M* as follows: the underlying C^*-algebra of the W^*-bundle M^ω is given by

$$M^\omega := \ell^\infty(M)/\{(x_n)_n \mid \lim_{n\to\omega} \|x_n\|_{2,u} = 0\}.$$

Furthermore, under the identification of $C(K^\omega)$ with its image under the natural inclusion $C(K^\omega) \cong C(K)_\omega \to M^\omega$, let $E^\omega \colon M^\omega \to C(K^\omega)$ denote the conditional expectation given by $E^\omega([(x_1, x_2, \dots)]) = [(E(x_1), E(x_2), \dots)]$.

See [8, Proposition 3.9] for a proof that M^ω is a W^*-bundle over K^ω. It is straightforward to check that the natural unital embedding $M \to M^\omega$ is an embedding of W^*-bundles, whence the relative commutant $M^\omega \cap M'$ is a W^*-bundle.

Of special interest are the trivial W^*-bundles with fibres $\mathcal{R}$, and there is a characterization of these bundles which closely resembles the characterization of McDuff II_1-factors (see [8, Proposition 3.11]). However, note the absence of a McDuff-type dichotomy. The definition of the tensor product $M \overline{\otimes} N$ of W^*-bundles M and N may be found in [8, Definition 3.4]

Theorem 2.24 (Bosa–Brown–Tikuisis–Sato–White–Winter, [8], Ozawa, [48])**.** *Let M be a $\|\cdot\|_{2,u}$-separable W^*-bundle over a compact Hausdorff space K. Then the following are equivalent:*

- *(i) $M \overline{\otimes} \mathcal{R} \cong M$ as W^*-bundles.*
- *(ii) There is a unital embedding $\mathcal{R} \to M^\omega \cap M'$.*
- *(iii) There is a unital embedding $M_2 \to M^\omega \cap M'$.*
- *(iv) For any $k \geq 2$ and any $\|\cdot\|_{2,u}^\omega$-separable and self-adjoint subset $S \subseteq M^\omega$ there is a unital embedding $M_k \to M^\omega \cap S'$.*
- *(v) $M \cong C_\sigma(K, \mathcal{R})$ as W^*-bundles.*

Let us describe the construction of Ozawa ([48]) which connects the worlds of C^*-algebras and W^*-bundles. Suppose A is a unital, separable C^*-algebra and $K \subseteq \partial_e T(A)$ is a non-empty, compact subset. Let $S \subseteq T(A)$ denote the weak*-closed convex hull of K. Note that, since A is separable, the weak* topology on $T(A)$ is metrizable, whence $S \subseteq T(A)$ is a closed, metrizable face. Let $\mathcal{N}_S := (\bigoplus_{\tau \in S} \pi_\tau)(A)''$ and $\mathrm{ctr}_S \colon \mathcal{N}_S \to \mathrm{cent}(\mathcal{N}_S)$ denote the centre-valued trace on $\mathcal{N}_S$. Although the *-homomorphism $\bigoplus_{\tau \in S} \pi_\tau \colon A \to \mathcal{N}_S$ may not be injective, we consider A to be a C^*-subalgebra of $\mathcal{N}_S$, to simplify notation. Define a semi-norm $\|\cdot\|_{2,S}$ on A by $\|a\|_{2,S} = \|\mathrm{ctr}_S(a^*a)\|^{1/2}$, for $a \in A$, and let B_S denote the C^*-algebra of norm-bounded sequences, which are Cauchy with respect to $\|\cdot\|_{2,S}$, modulo the ideal of $\|\cdot\|_{2,S}$-null sequences. Given an element $a \in A$, let $\hat{a} \colon K \to [0, \infty)$ denote the continuous map given by $\hat{a}(\tau) = \tau(a)$. The theorem below is a modified version of [48, Theorem 3].

Theorem 2.25 (Ozawa, [48])**.** *Let A, K, S, B_S and $\mathcal{N}_S$ be given as above. Then there exists a unital *-homomorphism $\theta_S \colon C(K) \to \mathrm{cent}(\mathcal{N}_S)$ such that $\theta(\hat{a}) = \mathrm{ctr}(a)$, for all $a \in A$, and*

$$B_S \cap \mathrm{cent}(\mathcal{N}_S) = \{\theta_S(f) \mid f \in C(K)\}.$$

Moreover, $\pi_\tau(B_S) = \pi_\tau(\mathcal{N}_S) = \pi_\tau(A)''$, for every $\tau \in S$.

In other words, B_S is a W^*-bundle over K, and henceforth we simply write M_K for B_S whenever A is implied by the context. In particular, if A is a unital, separable C^*-algebra such that $\partial_e T(A)$ is non-empty and compact, and $\pi_\tau(A)''$ is a McDuff factor for each $\tau \in \partial_e T(A)$, then $M := M_{\partial_e T(A)}$ is a W^*-bundle over $\partial_e T(A)$ such that $M_\lambda \cong \mathcal{R}$, for all $\lambda \in \partial_e T(A)$. The final ingredient in connecting W^*-bundles to C^*-algebras is the following (slightly modified version of a) result from [8].

Proposition 2.26 (Bosa–Brown–Tikuisis–Sato–White–Winter, [8])**.** *Let A be a unital, separable C^*-algebra, $K \subseteq \partial_e T(A)$ be a non-empty and compact, and M_K be the associated W^*-bundle over K. Then the canonical map $\iota_K \colon A \to M_K$ induces an isomorphism*

$$A_K^\omega \cap A' \cong M_K^\omega \cap M_K'.$$

In particular, if $A \otimes \mathcal{Z} \cong A$, there is a unital embedding $\mathcal{R} \to M_K^\omega \cap M_K'$.

The following theorem summarizes the connection between C^*-algebras and W^*-bundles.

Theorem 2.27. *Let A be a unital, simple and separable C^*-algebra with property (SI) and suppose $\partial_e T(A)$ is non-empty and compact. Let M denote the associated W^*-bundle over $\partial_e T(A)$. Then the following are equivalent.*

- *(i)* $A \otimes \mathcal{Z} \cong A$.
- *(ii) There is a unital embedding* $\mathcal{Z} \to F(A)$.
- *(iii) There is a unital embedding* $\mathcal{R} \to A^\omega \cap A'$.
- *(iv)* $M \overline{\otimes} \mathcal{R} \cong M$ *as W^*-bundles.*
- *(v)* $M \cong C_\sigma(K, \mathcal{R})$ *as W^*-bundles.*

It is well-known that, whenever A is unital, simple, separable, nuclear and infinite-dimensional, then the GNS-representation π_τ of A generates an injective II_1-factor, for every $\tau \in \partial_e T(A)$. Hence, if $\partial_e T(A)$ is non-empty and compact, then the associated W^*-bundle M over K satisfies $M_\lambda \cong \mathcal{R}$, for all $\lambda \in K$. Given the above, it is therefore natural to consider the following question (posed in [8]).

Question 2.28 (Bosa–Brown–Tikuisis–Sato–White–Winter, [8])**.** *Suppose that M is a $\|\cdot\|_{2,u}$-separable W^*-bundle over a compact Hausdorff space K, such that $M_\lambda \cong \mathcal{R}$, for each $\lambda \in K$. Does it follow that $M \cong C_\sigma(K, \mathcal{R})$ as W^*-bundles?*

Note that an affirmative answer to this question will imply that if A is a unital, simple, separable, nuclear C^*-algebra with stict comparison such that $\partial_e T(A)$ is non-empty and compact, then $A \otimes \mathcal{Z} \cong A$, by Theorem 2.27 and Proposition 2.19. Ozawa proved that the question has an affirmative answer when K has finite topological dimension, and Evington and Pennig showed in [21] that any W^*-bundle as above, which is *locally* trivial, is also *globally* trivial. This parallels results for $C(X)$-algebras A, whose fibres are isomorphic to a fixed strongly self-absorbing C^*-algebra, see [17, Theorem 1.1] and [30, Proposition 4.11]. It follows from [73], that the assumption that the fibres are K_1-injective, in the cited results, is superfluous. While it is known that there exists a non-trivial $C(X)$-algebra, whose fibres are isomorphic to the CAR-algebra M_{2^∞}, see [30, Example 4.8], the general case is still open for W^*-bundles.

3. The Cuntz semigroup

In this section, we introduce the Cuntz semigroup, both the original definition due to Cuntz, see [14], and the modern version introduced by Coward, Elliott and Ivanescu, see [12]. Note that the original and the modern Cuntz semigroup, while closely related, do not coincide in general.

Definition 3.1 (Cuntz, [14], [13])**.** Let D be a C^*-algebra and let a, b be positive elements in D. Then we say that a is *Cuntz dominated* by b, and write $a \precsim b$, if there exists $(x_n)_n \subseteq D$ such that $x_n^* b x_n \to a$. If $a \precsim b$ and $b \precsim a$, then we say that a and b are *Cuntz equivalent* and write $a \sim b$.

For a positive element a in a C^*-algebra, we let $(a-\varepsilon)_+ := f_\varepsilon(a)$, where $f_\varepsilon\colon \mathbb{R}_+ \to \mathbb{R}_+$ is given by $f_\varepsilon(t) = \max\{t-\varepsilon, 0\}$.

Proposition 3.2 (Rørdam, [52])**.** *Let D be a C^*-algebra and a, b be positive elements in D. Then the following are equivalent:*

(i) $a \precsim b$.
(ii) $(a-\varepsilon)_+ \precsim b$ *for all* $\varepsilon > 0$.
(iii) For all $\varepsilon > 0$ there exist $\delta > 0$ and $x \in D$ such that $(a-\varepsilon)_+ = x^*(b-\delta)_+ x$.

Now, let A be a C^*-algebra and let M_n denote the set of $n \times n$ matrices. For each $n \in \mathbb{N}$, let $\iota_n\colon M_n \to M_{n+1}$ denote the *-homomorphism given by

$$\iota_n(X) = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}.$$

With this notation, let $M_\infty(A)$ denote the *algebraic* limit of the inductive system

$$A \xrightarrow{\mathrm{id}\otimes\iota_1} A \otimes M_2 \xrightarrow{\mathrm{id}\otimes\iota_2} A \otimes M_3 \longrightarrow \cdots \longrightarrow M_\infty(A)\ . \tag{3.3}$$

Note that any finite set of elements $\{a_1, \dots, a_n\} \subseteq M_\infty(A)$ may be viewed as a subset of some matrix algebra $A \otimes M_k$ over A. In particular, it makes sense to apply the functional calculus to elements of $M_\infty(A)$. Similarly, an element $a \in M_\infty(A)$ is positive if it is positive in some finite matrix algebra over A, and, for positive elements $a, b \in M_\infty(A)$, we write $a \precsim b$ if this relation holds in $A \otimes M_k$, for some integer $k \geq 1$.

Given elements $a, b \in A \otimes \mathbb{K}$ let $a \oplus b := \mathrm{diag}(a, b) \in A \otimes \mathbb{K}$. It is worth noting that, even though there is no canonical choice of isomorphism $\mathbb{K} \otimes M_2 \to \mathbb{K}$, any two isomorphisms are approximately unitarily equivalent. Hence, the Cuntz equivalence class of $a \oplus b$ does not depend on the choice of isomorphism. Note that $a \oplus b \in M_\infty(A)$, if $a, b \in M_\infty(A)$.

Definition 3.4 (Cuntz, [14], Coward–Elliott–Ivanescu, [12])**.** Let A be a C^*-algebra. Let the *pre-complete Cuntz semigroup* $W(A)$ of A be defined as follows:

$$W(A) := M_\infty(A)_+/\sim\ .$$

Similarly, let the *Cuntz semigroup* $\mathrm{Cu}(A)$ of A be defined as follows:

$$\mathrm{Cu}(A) := (A \otimes \mathbb{K})_+/\sim\ .$$

Given a in $M_\infty(A)_+$, and b in $(A \otimes \mathbb{K})_+$, let $\langle a \rangle$ and $\langle b \rangle$ denote the equivalence class of a and b in $W(A)$ and $\mathrm{Cu}(A)$, respectively. Both $W(A)$ and $\mathrm{Cu}(A)$ are ordered abelian semigroups when equipped with the addition given by

$$\langle a \rangle + \langle b \rangle := \langle a \oplus b \rangle$$

and the order given by $\langle a \rangle \leq \langle b \rangle$ if, and only if, $a \precsim b$.

We will mainly concentrate on $\mathrm{Cu}(A)$, the modern version of the Cuntz semigroup, but we will also consider the original Cuntz semigroup $W(A)$ occasionally. Note that the inclusion $W(A) \to \mathrm{Cu}(A)$ is an order embedding, i.e., for $x, y \in W(A)$ we have $x \leq y$ in $W(A)$ if, and only if, $x \leq y$ in $\mathrm{Cu}(A)$. We identify A with its image under the inclusion $A \to M_\infty(A) \subseteq A \otimes \mathbb{K}$.

In addition to being an ordered abelian semigroup, $\mathrm{Cu}(A)$ admits supremum of increasing sequences, which gives rise to extra structure on $\mathrm{Cu}(A)$, compared to $W(A)$. Given x, y in an ordered abelian semigroup S, which admits suprema of increasing sequences, we write $x \ll y$ if, whenever $y_1, y_2, \dots$ is an increasing sequence in S, with $\sup y_n \geq y$, there exists $k \in \mathbb{N}$ such that $x \leq y_k$. We say that x is *compact*, if $x \ll x$. Note that, if $x, x', y \in S$ are given such that $x' \ll x \leq y$ or $x' \leq x \ll y$ it follows that $x' \ll y$. Furthermore, if $a, b \in (A \otimes \mathbb{K})_+$, then $\langle a \rangle \ll \langle b \rangle$ in $\mathrm{Cu}(A)$ if, and only if, there exists $\varepsilon > 0$ such that $\langle a \rangle \leq \langle (b - \varepsilon)_+ \rangle$. In particular, it follows from [35, Lemma 2.5], that for any $x', x \in \mathrm{Cu}(A)$ such that $x' \ll x$, there exists $y \in W(A)$ satisfying $x' \leq y \leq x$.

Proposition 3.5 (Coward–Elliott–Ivanescu, [12], Robert, [50], Rørdam–Winter, [58])**.** *Let S denote the Cuntz semigroup of a C^*-algebra. Then S has the following properties:*

- *(O1) Every increasing sequence admits a supremum.*
- *(O2) For every $x \in S$, there exists a sequence $(x_i)_i$ in S such that $x_i \ll x_{i+1}$ and $x = \sup x_i$.*
- *(O3) If $x' \ll x$ and $y' \ll y$, then $x' + y' \ll x + y$.*
- *(O4) If $(x_i)_i$ and $(y_i)_i$ are increasing sequences, then $\sup_i x_i + \sup_i y_i = \sup_i (x_i + y_i)$.*
- *(O5) If $x' \ll x \leq y + z$, then there exists $y', z' \in S$ such that*
$$y' \leq x, y, \quad z' \leq x, z, \quad \textit{and} \quad x' \ll y' + z'.$$
- *(O6) If $x' \ll x \leq y$, then there exists $z \in S$ such that $x' + z \leq y \leq x + z$.*

Note that it follows from (O1) and [35, Proposition 2.7] that there exists a maximal element $\infty \in \mathrm{Cu}(A)$, whenever A is a σ-unital C^*-algebra. In [12] it was shown that the functor $\mathrm{Cu}(-)$ preserves inductive limits. To be more precise:

Proposition 3.6 (Coward–Elliott–Ivanescu, [12])**.** *Let $A = \varinjlim (A_i, \varphi_i)$ be a sequential inductive limit of C^*-algebras.*

- *(i) For every $x \in \mathrm{Cu}(A)$, there exists an increasing sequence $(x_i)_i$, with supremum x, such that each x_i belongs to $\bigcup_j \mathrm{Im}(\mathrm{Cu}(\varphi_{j,\infty}))$.*
- *(ii) If $x, y \in \mathrm{Cu}(A_i)$ are given such that $\mathrm{Cu}(\varphi_{i,\infty})(x) \leq \mathrm{Cu}(\varphi_{i,\infty})(y)$ then, for every $x' \ll x$, there exists $j \geq i$ such that $\mathrm{Cu}(\varphi_{i,j})(x') \leq \mathrm{Cu}(\varphi_{i,j})(y)$.*

3.1. Divisibility and comparison. In this section, we discuss divisibility and comparison properties of C^*-algebras in terms of their Cuntz semigroups.

Definition 3.7 (Robert–Rørdam, [51])**.** Let A be a C^*-algebra, $m, n \geq 1$ be integers and fix $u \in \mathrm{Cu}(A)$.

- (i) We say that u is *(m,n)-divisible* if, for every $u' \in \mathrm{Cu}(A)$ with $u' \ll u$, there exists $x \in \mathrm{Cu}(A)$ with $mx \leq u$ and $u' \leq nx$. The least n for which $u \in \mathrm{Cu}(A)$ is (m,n)-divisible is denoted $\mathrm{Div}_m(u, A)$, with $\mathrm{Div}_m(u, A) = \infty$ if no such n exists. We let $\mathrm{Div}_*(u, A) \in [0, \infty]$ denote the number
$$\mathrm{Div}_*(u, A) := \liminf_{m \to \infty} \frac{\mathrm{Div}_m(u, A)}{m}.$$
- (ii) We say that u is *weakly (m,n)-divisible* if, for every $u' \in \mathrm{Cu}(A)$ with $u' \ll u$, there exists $x_1, \ldots, x_n \in \mathrm{Cu}(A)$ with $mx_j \leq u$, for $j = 1, \ldots, n$, and $u' \leq x_1 + \cdots + x_n$. The least n for which $u \in \mathrm{Cu}(A)$ is weakly (m,n)-divisible is denoted $\text{w-Div}_m(u, A)$, with $\text{w-Div}_m(u, A) = \infty$ if no such n exists. We let $\text{w-Div}_*(u, A) \in [0, \infty]$ denote the number
$$\text{w-Div}_*(u, A) := \liminf_{m \to \infty} \frac{\text{w-Div}_m(u, A)}{m}.$$

Suppose that A is σ-unital C^*-algebra, i.e., A contains a strictly positive element, say $e \in A_+$. Then $\langle A\rangle := \langle e\rangle \in \mathrm{Cu}(A)$ is independent of the choice of strictly positive element. In this case, we simply write $\mathrm{Div}_m(A)$ and $\mathrm{Div}_*(A)$ for $\mathrm{Div}_m(\langle A\rangle, A)$ and $\mathrm{Div}_*(\langle A\rangle, A)$, respectively. Similarly, we simply write $\text{w-Div}_m(A)$ and $\text{w-Div}_*(A)$. We refer to $\mathrm{Div}_*(A)$ and $\text{w-Div}_*(A)$ as the *asymptotic divisibility number of* A and the *asymptotic weak divisibility number of* A, respectively.

Proposition 3.8 (Robert–Rørdam, [51])**.** *Suppose* A *and* B *are unital* C^**-algebras such that there exists a unital* **-homomorphism from* B *to* A*. Then* $\mathrm{Div}_m(A) \leq \mathrm{Div}_m(B)$ *and* $\text{w-Div}_m(A) \leq \text{w-Div}_m(B)$*, for any integer* $m \geq 2$*.*

Proposition 3.9 (Robert–Rørdam, [51])**.** *Let* A *be a unital* C^**-algebra. Then, for all integers* $m \geq 2$*, we have*

$$\mathrm{Div}_m(A) \leq m \cdot \mathrm{Div}_*(A) + 1 \quad \textit{and} \quad \text{w-Div}_m(A) \leq m \cdot \text{w-Div}_*(A) + 1.$$

When A is unital, there are nice interpretations of when the numbers $\mathrm{Div}_m(A)$ and $\text{w-Div}_m(A)$ are finite in terms of the C^*-algebra A, rather than $\mathrm{Cu}(A)$. Before stating this, let us first define the rank of a C^*-algebra.

Definition 3.10 (Robert–Rørdam, [51])**.** Let A be a C^*-algebra. The *rank* of A, denoted $\mathrm{rank}(A)$, is the smallest integer $n \geq 1$ for which A has an irreducible representation on a Hilbert space of dimension n, and set $\mathrm{rank}(A) = \infty$ if A has no finite-dimensional irreducible representation.

Given an integer $m \geq 1$, we let CM_m denote the *cone over* M_m, that is,

$$CM_m = C_0((0,1], M_m) \cong C_0((0,1]) \otimes M_m.$$

Proposition 3.11 (Robert–Rørdam, [51])**.** *Let* A *be a unital* C^**-algebra and* $m \geq 1$ *an integer. Then the following are equivalent:*

(i) $\text{w-Div}_m(A) < \infty$.
(ii) $\mathrm{rank}(A) \geq m$.
(iii) There exist **-homomorphisms* $\varphi_1, \dots, \varphi_n \colon CM_m \to A$*, for some* $n \in \mathbb{N}$*, such that the union of their images is full in* A*.*

Moreover, $\mathrm{Div}_m(A) < \infty$ *if, and only if, this happens for* $n = 1$*.*

In particular, it follows that a unital C^*-algebra A admits no characters if, and only if, $\text{w-Div}_2(A) < \infty$.

Proposition 3.12 (Robert–Rørdam, [51])**.** *Let* D *be a unital* C^**-algebra such that* $\text{w-Div}_2(D) < \infty$*. Then, for any* $m \geq 2$*, we have* $\text{w-Div}_m(\bigotimes_{\max}^{(\infty)} D) < \text{w-Div}_2(D)^n$*, where* n *is the least integer such that* $m \leq 2^n$*.*

We now turn our attention to comparability properties of C^*-algebras, stated in terms of their Cuntz semigroups. There are many comparability properties in the literature, and we do not pretend that the list presented here is exhaustive.

Definition 3.13 (Ortega–Perera–Rørdam, [47])**.** Let A be a C^*-algebra and $x, y \in \mathrm{Cu}(A)$. Then we say that x is *stably dominated by* y, and write $x <_s y$, if there exists an integer $k \geq 1$ such that $(k+1)x \leq ky$.

It follows from [47, Proposition 2.1] that the relation $<_s$ is transitive.

Definition 3.14 (Rørdam, [52], Ortega–Perera–Rørdam, [47])**.** Let A be a C^*-algebra.

(i) We say that $\mathrm{Cu}(A)$ is *almost unperforated*, if $x \leq y$, whenever $x, y \in \mathrm{Cu}(A)$ satisfy $x <_s y$.
(ii) Let $n \geq 1$. We say that $\mathrm{Cu}(A)$ has *the n-comparison property*, if $x \leq y_0 + y_1 + \cdots + y_n$ whenever $x, y_0, y_1, \ldots, y_n \in \mathrm{Cu}(A)$ satisfy $x <_s y_j$, for all j.
(iii) We say that $\mathrm{Cu}(A)$ has *the ω-comparison property*, if $x \leq \sum_{i=0}^{\infty} y_i$ whenever $x, y_0, y_1, y_2, \ldots$ in $\mathrm{Cu}(A)$ satisfy $x <_s y_j$, for all j.

It is clear that the comparability properties in the above definition get progressively weaker, i.e., almost unperforation implies n-comparison which, in turn, implies ω-comparison.

Focusing on simple C^*-algebras for the moment, one can restate the above properties in terms of tracial comparison properties. Let $\tau \in T(A)$ be a tracial state on A. Letting τ denote the extension of τ to a lower semicontinuous, densely defined tracial functional on $A \otimes \mathbb{K}$ as well, we let $d_\tau \colon \mathrm{Cu}(A) \to [0, \infty]$ denote the map given by

$$d_\tau(\langle a \rangle) = \lim_{n \to \infty} \tau(a^{1/n}).$$

Part (i) of the proposition below was essentially proved by Rørdam in [52], and the same techniques yield part (ii). Part (iii) was proven by Bosa and Petzka in [9]

Proposition 3.15 (Rørdam, [52], Bosa–Petzka, [9])**.** *Let A be a unital, simple and exact C^*-algebra. Then*

(i) $\mathrm{Cu}(A)$ *is almost unperforated if, and only if, $x \leq y$ whenever $x, y \in \mathrm{Cu}(A)$ satisfy $d_\tau(x) < d_\tau(y)$, for all $\tau \in T(A)$.*
(ii) $\mathrm{Cu}(A)$ *has n-comparison if, and only if, $x \leq y_0 + y_1 + \cdots + y_n$ whenever $x, y_0, \ldots, y_n \in \mathrm{Cu}(A)$ satisfy $d_\tau(x) < d_\tau(y_j)$, for all j and all $\tau \in T(A)$.*
(iii) $\mathrm{Cu}(A)$ *has ω-comparison if, and only if, $y = \infty$, the maximal element in* $\mathrm{Cu}(A)$*, whenever $d_\tau(y) = \infty$, for all $\tau \in T(A)$.*

The assumptions that A is unital and exact are not, strictly speaking, necessary, but are there to simplify the statement. Despite these restrictions, we shall often refer to almost perforation, n-comparison and ω-comparison as *tracial* comparison properties. The tracial comparison property in part (i) is commonly referred to as strict comparison of positive elements. We define this separately for reference.

Definition 3.16 (Blackadar, [3])**.** Let A be a unital, simple and exact C^*-algebra. We say the A has *strict comparison of positive elements* (or simply *strict comparison*, for brevity), if $x \leq y$ whenever $x, y \in \mathrm{Cu}(A)$ satisfy $d_\tau(x) < d_\tau(y)$, for all $\tau \in T(A)$.

There is another tracial comparison property, introduced by Toms, which has a different form than those above, namely:

Definition 3.17 (Toms, [61])**.** Let A be a unital and exact C^*-algebra such that every quotient of A is stably finite. The *radius of comparison of A*, denoted $\mathrm{rc}(A)$, is the infimum of the set of real numbers $r > 0$ with the property that $x \leq y$ whenever $x, y \in \mathrm{Cu}(A)$ satisfy

$$d_\tau(x) + r < d_\tau(y), \quad \text{for all } \tau \in T(A).$$

The seemingly arbitrary assumption that all quotients of A are stably finite is there to ensure compatibility with the more algebraic (and recent) definition given in [7]. While there are technical advantages to the latter definition, the above is more readily applicable in the motivating examples given in Section 4.

The corona factorization property, defined for Cuntz semigroups below, was originally defined as a property of C^*-algebras rather than Cuntz semigroups. We return to the original definition in Section 5.

Definition 3.18 (Ortega–Perera–Rørdam, [47], C., [A])**.** Let A be a C^*-algebra.

(i) We say that $\mathrm{Cu}(A)$ has the *strong corona factorization property* (the strong CFP for short), if, whenever $x, y_1, y_2, \dots$ are elements in $\mathrm{Cu}(A)$ and $m \geq 1$ is an integer such that $x \leq m y_j$, for all j, then $x \leq \sum_{i=1}^{\infty} y_i$.

(ii) Let $n \geq 1$ be an integer, and S denote either $\mathrm{Cu}(A)$ or $W(A)$. Let $\mathrm{UCFP}_n(S)$ denote the smallest integer m such that $x \leq y_1 + \cdots + y_m$, whenever $x, y_1, \dots, y_m$ are elements in S satisfying $x \leq n y_j$, for $j = 1, \dots, m$, with $\mathrm{UCFP}_n(S) = \infty$ if no such integer exists. We say that S has *the Uniform Corona Factorization Property* (the UCFP for short), if $\mathrm{UCFP}_n(S) < \infty$ for all $n \geq 1$.

As indicated by the name *strong* CFP, there is a weaker version. However, the weak version is not relevant in this thesis. Note that the uniform corona factorization property, though not mentioned by name, was first considered in [A]. Although not explicitly mentioned, it is clearly the comparison property in [A, Proposition 2.1]. We refer to $\mathrm{UCFP}_n(\mathrm{Cu}(A))$, for $n \geq 2$, as the uniform corona factorization constants of A.

Proposition 3.19. *Let A be a C^*-algebra, S denote either $\mathrm{Cu}(A)$ or $W(A)$, and $n \geq 2$ an integer. If $\mathrm{UCFP}_2(S) = m < \infty$, then $\mathrm{UCFP}_n(S) \leq m^k$, where k is the least integer such that $n \leq 2^k$.*

Proof. Suppose $x, y_1, \dots, y_{m^k} \in S$ are elements such that $x \leq n \cdot y_j \leq 2^k \cdot y_j$, for all j. Then, since $\mathrm{UCFP}_2(S) = m$, it follows that $x \leq \sum_{s=(l-1)m+1}^{lm} 2^{k-1} y_s$, for all $l = 1, \dots, m^{k-1}$. Therefore, letting $z_l := \sum_{s=(l-1)m+1}^{lm} 2^{k-2} y_s$, for $l = 1, \dots, m^{k-1}$, it follows that $x \leq 2 \cdot z_l$, for each l. Using, once again, that $\mathrm{UCFP}_2(S) = m$, we find that

$$x \leq \sum_{l=(r-1)m+1}^{rm} z_l = \sum_{l=(r-1)m+1}^{rm} \sum_{s=(l-1)m+1}^{lm} 2^{k-2} y_s = \sum_{p=(r-1)m^2+1}^{rm^2} 2^{k-2} y_p,$$

for $r = 1, \dots, m^{k-2}$. Proceeding inductively, we obtain the desired result. ☐

The strong CFP and the UCFP are both weaker comparison properties than their tracial counterparts. We include the proof of this fact, as it is fairly short.

Proposition 3.20 (Ortega–Perera–Rørdam, [47])**.** *Let A be a C^*-algebra.*

(i) If $\mathrm{Cu}(A)$ has n-comparison, then

$$\mathrm{UCFP}_m(\mathrm{Cu}(A)) \leq (n+1)(m+1),$$

for all $m \geq 1$.

(ii) If $\mathrm{Cu}(A)$ has ω-comparison, then $\mathrm{Cu}(A)$ has the strong CFP.

Proof. We only prove (i), as (ii) is very similar. Let $m \geq 1$ be given and suppose that $x, y_1, y_2, \dots, y_{(n+1)(m+1)}$ are elements in $\mathrm{Cu}(A)$ satisfying $x \leq m y_j$, for all j. If, for each $i = 0, 1, \dots, n$, we let $z_i := y_{i(m+1)+1} + \cdots + y_{i(m+1)+(m+1)}$, then

$$(m+1)x \leq m \cdot y_{i(m+1)+1} + \cdots + m \cdot y_{i(m+1)+(m+1)} = m z_i.$$

Hence, $x <_s z_i$, for all i, and therefore n-comparison of $\mathrm{Cu}(A)$ implies

$$x \leq z_0 + z_1 + \cdots z_n = y_1 + y_2 + \cdots + y_{(n+1)(m+1)}. \qquad \square$$

Finally, we discuss the connection between divisibility and comparability properties. Given C^*-algebras A and B, and any tensor product $A \otimes B$, there is a bi-additive map $\mathrm{Cu}(A) \times \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes B)$, given by $(\langle a\rangle, \langle b\rangle) \mapsto \langle a \otimes b\rangle$, for $a, b \in (A \otimes \mathbb{K})_+$. Here we have identified $(A \otimes \mathbb{K}) \otimes (B \otimes \mathbb{K})$ and $(A \otimes B) \otimes \mathbb{K}$. We let $x \otimes y \in \mathrm{Cu}(A \otimes B)$ denote the image of $(x, y) \in \mathrm{Cu}(A) \times \mathrm{Cu}(B)$ under this map. In the proposition below, the tensor product can be taken to be any tensor product.

Proposition 3.21 (Robert–Rørdam, [51])**.** *Let A and B be unital C^*-algebras and $1 \leq m < n$ be integers.*

- *(i) Let $x, y \in \mathrm{Cu}(A)$ be given such that $nx \leq my$. If $\mathrm{Div}_m(B) \leq n$, then $x \otimes \langle \mathbf{1}_B\rangle \leq y \otimes \langle \mathbf{1}_B\rangle$ in $\mathrm{Cu}(A \otimes B)$.*
- *(ii) Let $x, y_1, \ldots, y_n \in \mathrm{Cu}(A)$ be given such that $x \leq my_j$, for all j. If $\text{w-Div}_m(B) \leq n$, then $x \otimes \langle \mathbf{1}_B\rangle \leq (y_1 + \cdots + y_n) \otimes \langle \mathbf{1}_B\rangle$ in $\mathrm{Cu}(A \otimes B)$.*

In particular, it follows from the above proposition that, if $\mathrm{Div}_*(B) = 1$ and $x, y \in \mathrm{Cu}(A)$ satisfies $x <_s y$, then $x \otimes \langle \mathbf{1}_B\rangle \leq y \otimes \langle \mathbf{1}_B\rangle$ in $\mathrm{Cu}(A \otimes B)$.

4. Dimension

In this section, we relate the comparison properties discussed above to the notion of topological dimension. To do this, we consider C^*-algebras of the form

$$p(C(X) \otimes \mathbb{K})p,$$

where X is a compact Hausdorff space and $p \in C(X) \otimes \mathbb{K}$ is a projection. We will restrict our attention to metrizable spaces, and will, in addition, often assume that X is connected or a finite CW complex, i.e., constructed from a finite number of cells. First, let us define topological dimension.

Definition 4.1 (Kirchberg–Winter, [38])**.** Suppose that X is a compact, metrizable topological space.

- (i) Given covers $\mathcal{U}$ and $\mathcal{U}'$ of X, we say that $\mathcal{U}'$ *refines* $\mathcal{U}$, if for every $U' \in \mathcal{U}'$, there exists $U \in \mathcal{U}$ such that $U' \subseteq U$.
- (ii) We say that a cover $\mathcal{U}$ of X is *n-decomposable*, if there is a partition $\mathcal{U}_0, \ldots, \mathcal{U}_n$ of $\mathcal{U}$ such that, for all j, we have
$$U, V \in \mathcal{U}_j \text{ and } U \neq V \Rightarrow U \cap V = \emptyset.$$
- (iii) We say that the *topological dimension of* X, or just the dimension of X, does not exceed n, and write $\dim(X) \leq n$, if every finite open cover of X has a n-decomposable, finite, open refinement. We let $\dim(X)$ denote the smallest integer n satisfying $\dim(X) \leq n$, with $\dim(X) = \infty$ if no such integer exists.

There are many different notions of topological dimension in the literature, but they all agree for compact, metrizable spaces X.

The nuclear dimension of a C^*-algebra A, denoted $\dim_{\mathrm{nuc}}(A)$, is a measure of the 'non-commutative topological' dimension of A, in the sense that it is an $\mathbb{N}_0$-valued invariant such that $\dim_{\mathrm{nuc}}(C(X)) = \dim(X)$, whenever X is a compact, metrizable space (see

[76, Proposition 2.4]). Before giving the definition of nuclear dimension, we need to introduce the notion of an order zero map.

Definition 4.2 (Winter–Zacharias, [75])**.** Let A and B be C^*-algebras. A completely positive map $\varphi\colon A \to B$ is said to have *order zero*, if $\varphi(x)\varphi(y) = 0$ whenever $x, y \in A_+$ satisfy $xy = 0$.

Definition 4.3 (Winter–Zacharias, [76])**.** A C^*-algebra A has *nuclear dimension* at most n, denoted $\dim_{\mathrm{nuc}}(A) \leq n$, if there exists a net $(F_\lambda, \varphi_\lambda, \psi_\lambda)_{\lambda\in\Lambda}$, where the F_λ's are finite-dimensional C^*-algebras, and both $\psi_\lambda\colon A \to F_\lambda$ and $\varphi_\lambda\colon F_\lambda \to A$ are completely positive maps satisfying

(i) $\|\psi_\lambda\| \leq 1$ for all $\lambda \in \Lambda$.
(ii) $\lim_\lambda \|a - \varphi_\lambda \circ \psi_\lambda(a)\| = 0$, for all $a \in A$.
(iii) F_λ decomposes into a sum $F_\lambda = F_\lambda^{(0)} \oplus F_\lambda^{(1)} \oplus \cdots \oplus F_\lambda^{(n)}$ such that $\varphi_\lambda|_{F_\lambda^{(i)}}$ is a completely positive, contractive order zero map, for each λ and $i = 0, 1, \ldots, n$.

We let $\dim_{\mathrm{nuc}}(A)$ denote the smallest integer n satisfying $\dim_{\mathrm{nuc}}(A) \leq n$, and set $\dim_{\mathrm{nuc}}(A) = \infty$ if no such integer exists.

As mentioned in the introduction, one can view finite nuclear dimension as 'coloured' version of being almost finite-dimensional. Another interpretation is to view finite nuclear dimension as an abstraction of bounded dimension growth for AH algebras (see Definition 4.10). Indeed, by the permanence properties established in [76], it is not difficult to see that

$$\dim_{\mathrm{nuc}}\Big(\bigoplus_{l=1}^{n} p_l(C(X_l) \otimes \mathbb{K})p_l\Big) \leq \max_{1\leq l\leq n} \dim(X_l)$$

whenever $n \geq 1$ is an integer, each X_l is a compact, metrizable space, and $p_l \in C(X_{i,l}) \otimes \mathbb{K}$ is a non-zero projection of constant rank. It therefore follows from [76] that $\dim_{\mathrm{nuc}}(A) < \infty$, whenever A is a unital, simple AH algebra of bounded dimension growth. For the remainder of this section, we aim to justify that the condition $\mathrm{UCFP}_n(\mathrm{Cu}(A)) < \infty$, for all $n \geq 2$, might be viewed as an abstraction of bounded dimension growth for AH algebras. While doing so, we also relate the radius of comparison and strict comparison to dimension properties of unital, simple AH algebras. This is done to demonstrate that $\mathrm{Cu}(A)$ *does* capture certain aspects of dimension of AH algebras.

4.1. Dimension and the Cuntz semigroup. For the remainder of this section, X will denote a compact, connected and metrizable space such that $\dim(X) < \infty$. We will sometimes need a few extra assumptions, but these will be added as we go along. Let $p \in C(X) \otimes \mathbb{K}$ be a non-zero projection, and D denote the unital C^*-algebra $p(C(X) \otimes \mathbb{K})p$. Note that, since X is connected, the rank of $p(x) \in \mathbb{K}$ is constant (and finite). For a point $x \in X$, we let $\mathrm{ev}_x\colon C(X) \otimes \mathbb{K} \to \mathbb{K}$ denote the *-homomorphism $\mathrm{ev}_x(f) = f(x)$. Here we have identified $C(X) \otimes \mathbb{K}$ with $C(X, \mathbb{K})$, the C^*-algebra of continuous functions $f\colon X \to \mathbb{K}$. We also use $\mathrm{ev}_x\colon D \to p(x)\mathbb{K}p(x) \cong M_{\mathrm{rank}(p)}$ to denote the restriction of ev_x to $D = p(C(X) \otimes \mathbb{K})p$, and rely on context to differentiate between these applications of the notation.

We aim to relate the radius of comparison and the UCFP to the topological dimension of X. We stick to the original Cuntz semigroup here, as this approach is more intuitive, and no information is lost.

For each n, let $\mathrm{Tr}_n\colon M_n(\mathbb{C}) \to \mathbb{C}$ denote usual *unnormalized* trace, that is,

$$\mathrm{Tr}_n\big((c_{ij})_{i,j=1}^n\big) = \sum_{i=1}^n c_{ii}, \quad (c_{ij})_{i,j=1}^n \in M_n(\mathbb{C}).$$

Note that $\lim_{k\to\infty} \mathrm{Tr}_n(a^{1/k}) = \mathrm{rank}(a)$ for any $a \in M_n(\mathbb{C})$. With $M_\infty(\mathbb{C}) \subseteq \mathbb{K}$ denoting the algebraic limit from (3.3), the sequence of tracial functionals $\mathrm{Tr}_n\colon M_n(\mathbb{C}) \to \mathbb{C}$ can be extended to a positive tracial functional $\mathrm{Tr}\colon M_\infty(\mathbb{C}) \to \mathbb{C}$, satisfying

$$\lim_{k\to\infty} \mathrm{Tr}(a^{1/k}) = \mathrm{rank}(a), \quad \text{for all } a \in M_\infty(\mathbb{C})_+.$$

Let tr denote the normalized tracial state on $M_{\mathrm{rank}(p)}$, and, for each $x \in X$, let τ_x denote the tracial state on $D = p(C(X) \otimes \mathbb{K})p$, given by $\tau_x := \mathrm{tr} \circ \mathrm{ev}_x$. Note that $\tau_x(f) = \mathrm{Tr}_{\mathrm{rank}(p)}(f(x)) \cdot \mathrm{rank}(p)^{-1}$ for any $x \in X$, whence it follows that, for any $g \in M_\infty(D)_+$, we have

$$d_{\tau_x}(g) = \lim_{k\to\infty} \tau_x(g^{1/k}) = \frac{\mathrm{rank}(g(x))}{\mathrm{rank}(p)}.$$

Hence, for $f, g \in M_\infty(D)_+$, the condition that $d_\tau(f) + r < d_\tau(g)$, for some $r > 0$ and all $\tau \in T(D)$, should be considered as an abstraction of the condition

$$\mathrm{rank}(f(x)) + r \cdot \mathrm{rank}(p) < \mathrm{rank}(g(x)), \quad \text{for all } x \in X.$$

The following result was obtained by Toms, see [65, Corollary 5.2] and [61, Theorem 6.6].

Theorem 4.4 (Toms, [61, 65])**.** *Let X be a compact, connected, metrizable space such that $\dim(X) = d < \infty$, and let $p \in C(X) \otimes \mathbb{K}$ be a non-zero projection.*

(i) The radius of comparison satisfies

$$\mathrm{rc}\big(p(C(X) \otimes \mathbb{K})p\big) \leq \frac{d-1}{2 \cdot \mathrm{rank}(p)}.$$

(ii) If, additionally, X is a CW complex, then

$$\mathrm{rc}\big(p(C(X) \otimes \mathbb{K})p\big) \geq \frac{d-2}{2 \cdot \mathrm{rank}(p)}.$$

Inspired by the above theorem, we consider the UCFP for C^*-algebras of the form $p(C(X) \otimes \mathbb{K})p$. The estimates given here are, in all probability, not sharp, but they are sufficient for the purposes of this thesis. To ease notation, let $W(X)$ denote the pre-complete Cuntz semigroup of $C(X)$, i.e., $W(C(X))$.

Proposition 4.5. *Let X be a finite CW complex. Then, the uniform corona factorization constant satisfies*

$$\left\lfloor \frac{\dim(X)}{3} \right\rfloor \leq \mathrm{UCFP}_2(W(X))$$

and

$$\mathrm{UCFP}_2(W(X)) \leq 18 \cdot \dim(X) + 2$$

Proof. We only prove the first estimate for $X = (S^2)^n$, for arbitrary $n \geq 1$. The estimate for general, finite CW-complexes can then be derived exactly as in the first part of the proof of [A, Theorem 3.5]. Note that the proof below actually yields

$$\left\lfloor \frac{\dim((S^2)^n)}{2} \right\rfloor \leq \mathrm{UCFP}_2(W((S^2)^n)),$$

but the estimate for general finite CW complexes, in the statement above, remains the best estimate known to the author. We use the properties of the Euler class, see [55] or [72] for an introduction.

Let $n \geq 1$ be given, and fix some line bundle ζ over S^2 with non-zero Euler class $e(\zeta) \in H^*(S^2)$. For each $1 \leq k \leq n$, let $\rho_k \colon (S^2)^n \to S^2$ denote the k'th coordinate projection and let ξ_k denote the line bundle $\rho_k^*(\zeta)$ over $(S^2)^n$. Since the Euler class

$$e\big(\bigoplus_{k=1}^n \xi_k\big) = \prod_{k=1}^n \rho_k^*\big(e(\zeta)\big)$$

is non-zero by the Künneth formula, it follows that $\bigoplus_{k=1}^n \xi_k$ cannot dominate a trivial bundle. Letting θ_1 denote a trivial line bundle over S^2, it follows from dimension considerations that $\theta_1 \precsim \zeta \oplus \zeta$, see for instance [31, Proposition 9.1.2]. In particular, letting $x := \langle \theta_1 \rangle$ and $y_k := \langle \xi_k \rangle$, for $k = 1, \ldots, n$, it follows that $x \leq 2y_k$, for each k, while $x \not\leq y_1 + y_2 + \cdots + y_n$.

For the second inequality, let $d := \dim(X)$ and $n := 18d + 2$. Suppose $x, y_1, \ldots, y_n \in W(X)$ satisfy $x \leq 2y_j$, for all j. Choose $N \in \mathbb{N}$ and $a, b_1, \ldots, b_n \in M_N(C(X))_+$ such that $x = \langle a \rangle$ and $y_j = \langle b_j \rangle$, for $j = 1, \ldots, n$. Let $U \subseteq X$ denote the open set consisting of all points $x \in X$ such that $\|a(x)\| > 0$. For each $m \in \mathbb{N}$, let $V_m \subseteq X$ denote the set $\{x \in X \mid \|a(x)\| > 1/m\}$ and K_m denote the closure of V_m. Suppose, for convenience, that the set V_m is non-empty, for each $m \geq 1$, and choose $f_m \colon X \to [0,1]$ such that $f_m|_{K_m} \equiv 1$ and $\mathrm{supp}(f_m) \subseteq V_{m+1}$. Let $a^{(m)} := f_m \cdot a$ and $b_l^{(m)} := f_m \cdot b_l$ for $m \geq 1$ and $l = 1, \ldots, n$. Finally, let $\pi_m \colon C(X, M_N) \to C(K_{m+1}, M_N)$ denote the restriction map.

Since $a \precsim b_l \oplus b_l$, for each l, it follows that $\pi_m(a) \precsim \pi_m(b_l) \oplus \pi_m(b_l)$, for each m. In particular, $\mathrm{rank}(a(x)) \leq 2 \cdot \mathrm{rank}(b_l(x))$, for every point $x \in K_{m+1}$, whence

$$\mathrm{rank}\big(\bigoplus_{l=1}^n b_l(x)\big) \geq \frac{n}{2} \cdot \mathrm{rank}(a(x)) = \mathrm{rank}(a(x)) + 9d \cdot \mathrm{rank}(a(x)),$$

for every $x \in K_m$. Since $\mathrm{rank}(a(x)) \geq 1$, for every $x \in K_{m+1}$, it follows that

$$\mathrm{rank}\big(\bigoplus_{l=1}^n \pi_m(b_l)(x)\big) \geq \mathrm{rank}\big(\pi_m(a(x))\big) + 9d.$$

Using that $\dim(K_m) \leq \dim(X)$ for every m, [64, Theorem 3.15] implies that $\pi_m(a) \precsim \bigoplus_{l=1}^n \pi_m(b)$, for every m. Since each $f_m \in C(X, M_n)$ is central, it follows that $a^{(m)} \precsim \bigoplus_{l=1}^n b_l^{(m)} \precsim \bigoplus_{l=1}^n b_l$, for every m. Finally, since $\|a - a^{(m)}\| < 1/m$, for all m, it follows that $a \precsim \bigoplus_{l=1}^n b_l$, i.e., $x \leq y_1 + y_2 + \cdots + y_n$ in $W(X)$. $\square$

While we considered the pre-complete Cuntz semigroup $W(X)$, rather than $\mathrm{Cu}(C(X))$, the estimates for $\mathrm{Cu}(C(X))$ are exactly the same, as demonstrated below.

Lemma 4.6. *Let A be a C^*-algebra. Then, for any integer $n \geq 1$,*

$$\mathrm{UCFP}_n(\mathrm{Cu}(A)) = \mathrm{UCFP}_n(W(A)).$$

Proof. We only show that $\mathrm{UCFP}_n(\mathrm{Cu}(A)) \leq \mathrm{UCFP}_n(W(A))$ since the other inequality is clear. Thus, let us suppose that $\mathrm{UCFP}_n(W(A)) = m$ and let $x, y_1, \ldots, y_m \in \mathrm{Cu}(A)$ be given such that $x \leq ny_j$, for all j. Let $x' \in W(A)$ be an arbitrary element satisfying $x' \ll x$. Since each $y_j \in \mathrm{Cu}(A)$ is the supremum of a sequence of elements in $W(A)$, there exists $y_j' \in W(A)$ such that $x' \leq ny_j'$ in $\mathrm{Cu}(A)$, for each j. Since the inclusion $W(A) \to \mathrm{Cu}(A)$ is an order embedding, it follows that $x' \leq ny_j'$ in $W(A)$. By assumption, this implies that $x' \leq y_1' + \cdots + y_m' \leq y_1 + \cdots + y_m$. Since $x' \ll x$ was arbitrary, and $x \in \mathrm{Cu}(A)$ is the supremum of an increasing sequence in $W(A)$, it follows that $x \leq y_1 + \cdots + y_m$. ☐

Finally, we consider the behaviour of UCFP_n under direct sums.

Proposition 4.7. *Let A and B be C^*-algebras. Then*

$$\mathrm{UCFP}_n(\mathrm{Cu}(A \oplus B)) = \max\big\{\mathrm{UCFP}_n(\mathrm{Cu}(A)), \mathrm{UCFP}_n(\mathrm{Cu}(B))\big\}$$

Proof. Identifying $(A \oplus B) \otimes \mathbb{K}$ with $(A \otimes \mathbb{K}) \oplus (B \otimes \mathbb{K})$ we let

$$\pi_1 \colon (A \oplus B) \otimes \mathbb{K} \to A \otimes \mathbb{K} \quad \text{ and } \quad \pi_2 \colon (A \oplus B) \otimes \mathbb{K} \to (B \otimes \mathbb{K})$$

denote the quotient maps. Let $\varphi \colon \mathrm{Cu}(A \oplus B) \to \mathrm{Cu}(A) \times \mathrm{Cu}(B)$ be given by $\varphi(\langle d \rangle) = \big(\langle \pi_1(d) \rangle, \langle \pi_2(d) \rangle\big)$, for $d \in ((A \oplus B) \otimes \mathbb{K})_+$. It is easy to check that φ is bijective, additive map, and an order isomorphism, when $\mathrm{Cu}(A) \times \mathrm{Cu}(B)$ is equipped with the product order. The result then follows from a straightforward computation. ☐

Observing that $\mathrm{UCFP}_n(\mathrm{Cu}(A)) = \mathrm{UCFP}_n(\mathrm{Cu}(B))$, for all $n \geq 1$, when $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$, since the functor $\mathrm{Cu}(-)$ is a stable invariant, the results above combine to produce the following:

Corollary 4.8. *Suppose D is a C^*-algebra of the form*

$$D := \bigoplus_{i=1}^{n} p_i(C(X_i) \otimes \mathbb{K})p_i,$$

where each X_i is a finite CW complex and $p_i \in C(X_i) \otimes \mathbb{K}$ is a non-zero projection. Then

$$\mathrm{UCFP}_2(\mathrm{Cu}(D)) \leq 18 \cdot \max_{1 \leq i \leq n} \dim(X_i) + 2.$$

4.2. AH algebras.

We extend the discussion above to AH algebras, which necessitates a definition of these objects. A unital, separable C^*-algebra A is said to be a unital *AH algebra* if can written as a sequential limit $A \cong \varinjlim(A_i, \varphi)$, where each $\varphi_i \colon A_i \to A_{i+1}$ is unital and each A_i is *semi-homogeneous*, that is, of the form

$$A_i = \bigoplus_{l=1}^{n_i} p_{i,l}(C(X_{i,l}) \otimes \mathbb{K})p_{i,l}, \tag{4.9}$$

where $n_i \geq 1$ is an integer, each $X_{i,l}$ is a compact, metrizable space and $p_{i,l} \in C(X_{i,l}) \otimes \mathbb{K}$ is a non-zero projection of constant rank. It follows from an argument in the proof of [28, Proposition 3.4] that we may always assume that each $X_{i,l}$ is a finite CW complex.

Given a unital AH algebra A and an inductive system (A_i, φ_i) such that $A \cong \varinjlim(A_i, \varphi_i)$ with each A_i of the form (4.9), we say that (A_i, φ_i) is an *AH decomposition* for A.

Definition 4.10 (Toms, [61], Blackadar–Dadarlat–Rørdam, [4])**.** Let A be a unital AH algebra.

(i) The dimension-rank ratio of A (denoted $\mathrm{drr}(A)$) is the infimum of the set of numbers $0 < c \in \mathbb{R}$ such that A, with the notation of (4.9), admits an AH decomposition satisfying

$$\limsup_{i\to\infty} \max_{1\le l\le n_i} \left\{ \frac{\dim(X_{i,l})}{\mathrm{rank}(p_{i,l})} \right\} = c$$

whenever this set is non-empty, and $\mathrm{drr}(A) = \infty$ otherwise. We say that A has *flat dimension growth* if $\mathrm{drr}(A) < \infty$.

(ii) If, additionally, A is simple, we say that A has *slow dimension growth* if A admits an AH decomposition such that

$$\limsup_{i\to\infty} \max_{1\le l\le n_i} \left\{ \frac{\dim(X_{i,l})}{\mathrm{rank}(p_{i,l})} \right\} = 0.$$

(iii) If, again, A is simple, we say the A has *bounded dimension growth*, if A admits an AH decomposition satisfying

$$\sup_i \max_{1\le l\le n_i} \dim(X_{i,l}) < \infty,$$

The concepts of dimension-rank ratio and flat dimension growth were introduced by Toms in [61], while the definition of slow dimension growth was given in [4]. The radius of comparison was introduced in [61] as an abstraction of drr for AH algebras, a viewpoint that we briefly discuss below. First we consider slow dimension growth.

When slow dimension growth was introduced, no examples of unital, simple AH algebras *without* slow dimension growth were known, but this changed with the breakthrough of Villadsen in [71]. Gong, Jiang and Su proved in [26] that if A is a unital and simple C^*-algebra, then the inclusion $A \to A \otimes \mathcal{Z}$ induces an isomorphism of the Elliott invariants, if, and only if, the ordered group $K_0(A)$ is weakly unperforated. In particular, the examples of Villadsen does not absorb the Jiang-Su algebra. Deep results of Lin, Toms and Winter demonstrate that this always happens. To be precise, Winter proved in [74] that if A is a unital, simple, separable, infinite-dimensional C^*-algebra with locally finite nuclear dimension (e.g., any separable AH algebra) then $A \otimes \mathcal{Z} \cong A$ if, and only if, A has strict comparison and $W(A)$ is almost divisibile (in fact, weaker conditions suffice). Toms proved in [64] that any unital, simple and infinite-dimensional AH-algebra with slow dimension growth has strict comparison, and hence automatically has almost divisible Cuntz semigroup, as demonstrated in [66], building substantially on [10]. Lin proved, in [42], that any unital, simple and infinite-dimensional AH-algebra A satisfying $A \otimes \mathcal{Z} \cong A$ has bounded dimension growth, as a consequence of a deep classification result. Combining all these results, one obtains the following theorem.

Theorem 4.11. *Let A be a unital, simple and infinite-dimensional AH algebra. Then the following are equivalent:*

(i) $A \otimes \mathcal{Z} \cong A$.
(ii) A has strict comparison.
(iii) A has slow dimension growth.

(iv) A has bounded dimension growth.

This theorem can be applied to show that if A is a unital, simple, separable, infinite-dimensional AH algebra satisfying $\mathrm{drr}(A) = 0$, then A has slow dimension growth. More generally, the radius of comparison is bounded by the dimension-rank ratio, as demonstrated by the proposition below.

Proposition 4.12 (Blackadar–Robert–Tikuisis–Toms–Winter, [7])**.** *Let A be a simple and unital C^*-algebra, and suppose that A is the limit of a sequence of stably finite C^*-algebras (A_i, φ_i), with both A_i and φ_i unital. Then*

$$\mathrm{rc}(A) \leq \liminf_{i\to\infty} \mathrm{rc}(A_i).$$

As a consequence, $\mathrm{rc}(A) \leq \mathrm{drr}(A)/2$ when A is a unital AH algebra, by Theorem 4.4, but the other inequality is, in all likelihood, very difficult to prove.

In analogy with Proposition 4.12, we show that UCFP_n behaves as well as can be expected under inductive limits.

Proposition 4.13. *Suppose $A \cong \varinjlim(A_i, \varphi_i)$ is a sequential inductive limit of C^*-algebras A_i. Then*

$$\mathrm{UCFP}_n(\mathrm{Cu}(A)) \leq \sup_{i\geq 1} \mathrm{UCFP}_n(\mathrm{Cu}(A_i)).$$

Proof. Let m denote the supremum on the right hand side and suppose that $m < \infty$ (otherwise, there is nothing to prove). Let $x, y_1, \ldots, y_m \in \mathrm{Cu}(A)$ be given such that $x \leq ny_j$, for all j, and let $x' \in \bigcup_{i\geq 1} \mathrm{Im}(\mathrm{Cu}(\varphi_{i,\infty}))$ be an arbitrary element satisfying $x' \ll x$. Then, using Proposition 3.6, choose elements $y_j' \in \bigcup_{i\geq 1} \mathrm{Im}(\mathrm{Cu}(\varphi_{i,\infty}))$, for each j, such that $y_j' \ll y_j$ and $x' \leq ny_j$. Fix $k \geq 1$ so that $x', y_1', \ldots, y_n' \in \mathrm{Im}(\mathrm{Cu}(\varphi_k))$. For notational convenience, we will consider $x', y_1', \ldots, y_m'$ to be elements of $\mathrm{Cu}(A_k)$. Once again using Proposition 3.6, we get that there exists $k_0 \geq k$ such that $\mathrm{Cu}(\varphi_{k,k_0})(x') \leq n \cdot \mathrm{Cu}(\varphi_{k,k_0})(y_j')$, for each j. Since $\mathrm{UCFP}_n(\mathrm{Cu}(A_{k_0})) \leq m$, it follows that

$$\mathrm{Cu}(\varphi_{k,\infty})(x') \leq \sum_{l=1}^{m} \mathrm{Cu}(\varphi_{k,\infty})(y_l') \leq \sum_{l=1}^{m} y_l.$$

Since x is the supremum of elements of the form $\mathrm{Cu}(\varphi_{k,\infty})(x')$, by Proposition 3.6, the desired result follows. □

At this point, it follows from the proposition above, along with Corollary 4.8, that any AH algebra A of bounded dimension growth satisfies $\mathrm{UCFP}_2(A) < \infty$, and hence $\mathrm{UCFP}_n(A) < \infty$, for any $n \geq 1$. In view of Proposition 4.5, the author humbly suggests that the condition $\mathrm{UCFP}_n(A) < \infty$ for any $n \geq 1$, should be considered an abstract analogue of bounded dimension growth for AH algebras. However, given that a unital, simple AH algebra A with slow dimension growth has strict comparison, and therefore satisfies $\mathrm{UCFP}_n(\mathrm{Cu}(A)) \leq n + 1$, by Proposition 3.20, it may be necessary to consider the condition

$$\limsup_{n\to\infty} \frac{\mathrm{UCFP}_n(\mathrm{Cu}(A))}{n} < \infty,$$

rather than the condition $\mathrm{UCFP}_n(\mathrm{Cu}(A)) < \infty$, for all n.

There is some evidence in favour of the idea above. Indeed, it is clear that the proof of [A, Theorem 3.5] yields the following result: If A is a unital, simple Villadsen algebra of the first type which admits a standard decomposition with seed space a finite-dimensional CW complex (see [70, Definition 3.2]) and satisfies $\mathrm{UCFP}_2(\mathrm{Cu}(A)) < \infty$, then A has strict comparison of positive elements. It therefore follows from [70, Proposition 5.2] that A has bounded dimension growth. The proof of [70, Proposition 5.2] unfortunately passes through fairly heavy classification results. However, the proof of [70, Proposition 5.2] yields *very* slow dimension growth, without using these classification results.

5. The CFP and S-regularity

In this section, we relate the strong CFP and the ω-comparison to stability properties of C^*-algebras. Note that, historically, this is going the wrong way around in the sense that the corresponding properties of C^*-algebra were studied first and only later formulated in terms of comparison properties of the Cuntz semigroup. However, the narrative presented here is more well-suited to the needs of the present thesis. We only consider simple C^*-algebras in this section, and therefore simply speak of the corona factorization property, rather than the *strong* corona factorization property, for reasons that will be made clear below.

The study of the corona factorization property was initiated in [39], and originated in a study of KK-theory and the question of when extensions are absorbing, and it was only later, in [40], that the corona factorization property was related to stability properties of C^*-algebras. We focus on the latter viewpoint, and refer readers interested in the former to [23] and [39].

The following definition appears in [40]. Given a C^*-algebra B, we let $\mathcal{M}(B)$ denote the multiplier algebra of B.

Definition 5.1 (Kucerovsky–Ng, [39])**.** A C^*-algebra A is said to have the *corona factorization property* (abbreviated the CFP) if every norm-full projection in $\mathcal{M}(A\otimes\mathbb{K})$ is Murray–von Neumann equivalent to the unit in $M(A\otimes\mathbb{K})$.

Several characterizations of the CFP are given in [40], but the following is the most useful for our purposes. Recall that a C^*-algebra D is said to be *stable* if $D\otimes\mathbb{K}\cong D$.

Theorem 5.2 (Kucerovsky–Ng, [40])**.** *Suppose A is a σ-unital C^*-algebra. Then the following are equivalent:*

(i) A has the corona factorization property.
(ii) A full hereditary C^*-subalgebra $D\subseteq A\otimes\mathbb{K}$ is stable if, and only if, $D\otimes M_n$ is stable, for some $n\geq 1$.

As an immediate consequence of this, the simple, separable, nuclear C^*-algebras constructed in [53] must fail to have the CFP.

As the astute reader may have guessed, there is relationship between the CFP for C^*-algebras and the CFP for Cuntz semigroups.

Theorem 5.3 (Ortega–Perera–Rørdam, [47])**.** *Let A be a separable C^*-algebra. Then* $\mathrm{Cu}(A)$ *has the strong corona factorization property if, and only if, every ideal of A has the corona factorization property.*

From this we conclude that a simple C^*-algebra A has the CFP if, and only if, $\mathrm{Cu}(A)$ has the CFP. Now, to connect ω-comparison to properties of C^*-algebras, we consider the notion of S-regularity. Unfortunately, the term *regular* is an ubiquitous term in the literature and it is common to refer to the C^*-algebras in the definition below as regular, rather than S-regular. However, the term regular has a broader meaning in the present thesis, and we therefore felt compelled to change the terminology slightly. Unfortunately, the term S-regular was already used in [40] in a different, albeit related, context. We hope that no confusion will arise from this. The reader is referred to [5] for the definition and properties of 2-quasitraces.

Definition 5.4 (Rørdam, [57])**.** Let A and B be separable C^*-algebras.

(i) We say that B has *property (S)*, if B has no non-zero unital quotients and admits no bounded lower semicontinuous 2-quasitrace.
(ii) We say that A is *S-regular*, if any full hereditary C^*-subalgebra $D \subseteq A \otimes \mathbb{K}$ with property (S) is stable.

Note that any separable, stable C^*-algebra has property (S). It is possible to give an intrinsic characterization of property (S), and to do so we need to introduce some notation. Given a C^*-algebra D, let $L(D)$ denote the set of positive elements $d \in D$ for which there exists a positive element $e \in D$ satisfying $ed = de = d$. Usually, the set $L(D)$ is denoted $F(D)$, but in the present thesis $F(D)$ denotes the central sequence algebra of D, hence we have altered the notation.

Proposition 5.5 (Ortega–Perera–Rørdam, [47])**.** *Let D be a separable C^*-algebra. Then D has property (S) if, and only if, for every $a \in L(D)$ there exists $b \in D_+$ such that $ab = ba = 0$, and $\langle a \rangle <_s \langle b \rangle$ in $\mathrm{Cu}(D)$.*

It was proven in [47, Proposition 4.7] that if a simple, separable C^*-algebra A is given such that $\mathrm{Cu}(A)$ has ω-comparison, then A is S-regular. The converse was shown in [7, Theorem 4.2.1]. Hence, we obtain the following characterization of S-regularity for simple, separable C^*-algebras.

Theorem 5.6 (Ortega–Perera–Rørdam, [47], Blackadar–Robert–Tikuisis–Toms–Winter, [7])**.** *Let A be a simple, separable C^*-algebra. Then A is S-regular if, and only if, $\mathrm{Cu}(A)$ has the ω-comparison property.*

Thus, one can either apply Proposition 3.20 to show that a simple, separable, S-regular C^*-algebra has the CFP, or one can apply Theorem 5.3, by observing that a separable C^*-algebra D has property (S) if, and only if, $D \otimes M_n$ has property (S), for some $n \geq 1$ (see [47, Corollary 4.6]). However, the question of whether the CFP implies ω-comparison for Cuntz semigroups, or whether the CFP implies S-regularity for C^*-algebras, remains unanswered. We discuss what is known about this question below.

Definition 5.7 (Ortega–Perera–Rørdam, [47], Bosa–Petzka, [9])**.** Let A be a simple C^*-algebra.

(i) We say that $\mathrm{Cu}(A)$ has *property (QQ)*, if $x = \infty$ whenever $x \in \mathrm{Cu}(A)$ satisfies $n \cdot x = \infty$, for some $n \in \mathbb{N}$.
(ii) We say that $\mathrm{Cu}(A)$ has *cancellation of small elements at infinity*, if $y = \infty$ whenever $x, y \in \mathrm{Cu}(A)$ satisfy $x \ll \infty$, $x + y = \infty$ and $y \neq 0$.

Property (QQ) was introduced in [47], while cancellation of small elements at infinity was introduced [9]. It is known that Property (QQ) implies both cancellation of small elements at infinity and the CFP (see [9, Proposition 4.27] and [9, Proposition 4.24], respectively) for the Cuntz semigroup of a simple C^*-algebra. In fact, the converse is also true, as demonstrated in [9, Theorem 5.9]. To sum up:

Theorem 5.8 (Bosa–Petzka, [9]). *Let A be a simple, separable C^*-algebra. Then* $\mathrm{Cu}(A)$ *has property (QQ) if, and only if,* $\mathrm{Cu}(A)$ *has the corona factorization property and cancellation of small elements at infinity.*

In [9], Bosa and Petzka asked whether the Cuntz semigroup of any simple, separable, stably finite C^*-algebra has cancellation of small elements at infinity. It is tempting to think that the Cuntz semigroup of any such C^*-algebra has property (QQ), which in particular implies cancellation of small elements at infinity. It is certainly true that, if A is a simple and stably finite and $x \in \mathrm{Cu}(A)$ satisfies $x \ll \infty$, then $nx \neq \infty$, for any $n \in \mathbb{N}$. Hence the implication $nx = \infty \Rightarrow x = \infty$ is true, for trivial reasons. However, in [A, Appendix A], written jointly with Joan Bosa, we show that the Villadsen algebra $\mathcal{V}_\infty$ of the second type with infinite stable rank fails the corona factorization property, and hence also fails property (QQ). It seems likely that, using similar techniques, one can show that any unital, simple Villadsen algebra A of the first type which admits a standard decomposition with seed space a finite CW complex, has the corona factorization property if, and only if, $\mathrm{Cu}(A)$ has ω-comparison. It is also tempting to think that if $x \in \mathrm{Cu}(A)$ satisfies $\lambda(x) = \infty$, for all functionals λ on $\mathrm{Cu}(A)$, then there exists $n \in \mathbb{N}$ such that $nx = \infty$. See [7] for the definition of a functional. However, this approach is ruled out by [9, Example 4.11].

If one believes that the CFP does *not* imply ω-comparison, a candidate for a C^*-algebra witnessing this would be a simple, separable C^*-algebra which is neither purely infinite nor stably finite and has the CFP. However, the only current example of a simple, separable C^*-algebra which is neither purely infinite nor stably finite, is known to fail the CFP (see [37, Remark 4.4(i)] or [9, Theorem 5.8]).

5.1. Asymptotic regularity. In this section we consider the notion of asymptotic S-regularity, introduced by Ng in [46], and state the main result of [D]. We recall the definition below.

Definition 5.9 (Ng, [46]). Let A be a separable C^*-algebra. We say that A is *asymptotically S-regular*, if for any full hereditary C^*-subalgebra $D \subseteq A \otimes \mathbb{K}$ with property (S) there exists an integer $n \geq 1$ such that $M_n(D)$ is stable.

The following result follows immediately from the definition and Theorem 5.2.

Proposition 5.10 (Ng, [46]). *Let A be a separable C^*-algebra. Then A is S-regular if, and only if, A is asymptotically S-regular and has the corona factorization property.*

Given the above proposition, one naturally wonders whether the corona factorization implies asymptotic S-regularity, since an affirmative answer would imply that S-regularity and the CFP are equivalent for simple, separable C^*-algebras. In order to investigate this, we gave a characterization of asymptotic S-regularity for simple, separable C^*-algebras A in terms of the Cuntz semigroup $\mathrm{Cu}(A)$, in [D, Proposition 3.8]. The main result of [D] is given below.

Proposition 5.11. *Let A be a simple and separable C^*-algebra. Then A is asymptotically S-regular if, and only if, the following holds: for any sequence $y_1, y_2, \ldots$ of non-zero elements in* $\mathrm{Cu}(A)$ *satisfying $y_i <_s y_{i+1}$ and $y_i \ll \infty$, for all $i \geq 1$, there exists $n \in \mathbb{N}$ such that $n \cdot \sum_{j=m}^{\infty} y_j = \infty$, for all $m \geq 1$.*

While no progress was made on the question that motivated [D], it follows from this result, without too much effort, that any simple, separable, asymptotically S-regular C^*-algebra A is either stably finite or purely infinite, see [D, Proposition 3.9]. It should also be noted that [D] represents work in progress and thus interesting results may yet be derived from the effort.

6. The main question

Recall that the main question which this thesis seeks to answer is the following:

Question 6.1 (Kirchberg–Rørdam, [37])**.** *Let A be a unital, separable C^*-algebra. Does it follow that $A \otimes \mathcal{Z} \cong A$ if, and only if, $A_\omega \cap A'$ admits no characters?*

In this section we seek to provide an overview of the progress on this question, and in subsection 6.2, we also discuss some possible approaches, which turned out to be flawed. First, we connect divisibility properties of $F(A)$ to divisibility and comparability properties of the Cuntz semigroup of a C^*-algebra A. The following result can be found in [37].

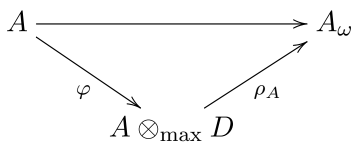
Proposition 6.2 (Kirchberg–Rørdam, [37])**.** *Let A and P be C^*-algebras such that $A \subseteq P \subseteq A_\omega$. For each pair $x, y \in \mathrm{Cu}(A)$,*

(i) $x \leq y$ in $\mathrm{Cu}(A)$ if, and only if, $x \leq y$ in $\mathrm{Cu}(P)$.
(ii) $x \ll y$ in $\mathrm{Cu}(A)$ if, and only if, $x \ll y$ in $\mathrm{Cu}(P)$.

In other words, if $A \subseteq P \subseteq A_\omega$, then the map $\mathrm{Cu}(A) \to \mathrm{Cu}(P)$ induced by the inclusion $A \to P$ is an order inclusion. A particularly useful application of this fact is the following: Suppose A is s separable C^*-algebra, and let $\rho_A \colon A \otimes_{\max} F(A) \to A_\omega$ denote the *-homomorphism given by

$$\rho_A(a \otimes (b + \mathrm{Ann}(A, A_\omega))) = ab, \quad b \in A_\omega \cap A'.$$

It is easily seen that the diagram



commutes whenever $D \subseteq F(A)$ is a unital C^*-subalgebra, where φ denotes the *-homomorphism given by $\varphi(a) = a \otimes \mathbf{1}$. Letting $P = \mathrm{Im}(\rho_A)$, and noting that $A \subseteq P \subseteq A_\omega$, it follows that the inclusion $A \to P$ is an order inclusion at the level of Cuntz semigroups, and since the above diagram commutes, the map $\mathrm{Cu}(\varphi) \colon \mathrm{Cu}(A) \to \mathrm{Cu}(A \otimes D)$ is also an order inclusion. Thus, by Proposition 3.21, it follows that good divisibility properties of $F(A)$ implies good comparability (and divisibility) properties of A. In fact, it also implies good comparability properties of $F(A)$, at least when A is unital, as demonstrated by the next result. To state it, we need a definition.

Definition 6.3 (Kirchberg–Rørdam, [36, 37])**.** Let D be a unital C^*-algebra and $\alpha \geq 1$.

(i) We say that D has *α-comparison* if $x \leq y$, whenever $x, y \in \mathrm{Cu}(A)$ are given such that there exist $n, m \in \mathbb{N}$ satisfying $nx \leq my$ and $n > \alpha m$.
(ii) We say that D is *α-divisible* if, for all $x \in \mathrm{Cu}(A)$ and all integers $n, m \geq 1$ such that $n > \alpha m$, there exists $y \in \mathrm{Cu}(A)$ such that $my \leq x \leq ny$.

Proposition 6.4 (Kirchberg–Rørdam, [37]). *Let A be a unital and separable C^*-algebra such that the asymptotic divisibility constant $\alpha := \mathrm{Div}_*(F(A))$ is finite. Then A and $F(A)$ are both α-divisible and have α-comparison.*

A weaker, but still very interesting, result holds when $F(A)$ does not admit characters.

Proposition 6.5 (Kircberg–Rørdam, [37]). *Let A be a unital, separable C^*-algebra such that $F(A)$ does not admit characters. Then* $\mathrm{Cu}(A)$ *has the strong CFP.*

It was noted in [A] that the proof of the above result actually yields a stronger result. We include a proof of this, since it is fairly short.

Proposition 6.6. *Let A be a unital, separable C^*-algebra such that $F(A)$ does not admit characters. Then* $\mathrm{Cu}(A)$ *has the UCFP.*

PROOF. By [37, Lemma 3.5], there exists a unital, separable C^*-subalgebra $D \subseteq F(A)$ such that D admits no characters. Let B denote the infinite maximal tensor product $\bigotimes_{k\geq 1} D$. By Proposition 2.4, there exists a unital $*$-homomorphism $B \to F(A)$, and by Proposition 3.12 there exists, for each $m \geq 2$, an integer $n \geq 1$ such that $\text{w-Div}_m(B) \leq n$. Suppose $x, y_1, \ldots, y_n \in \mathrm{Cu}(A)$ are given such that $x \leq my_i$, for all i. Then it follows from Proposition 3.21 that

$$x \otimes \langle \mathbf{1} \rangle \leq \sum_{j=1}^{n} y_j \otimes \langle \mathbf{1} \rangle$$

in $\mathrm{Cu}(A \otimes B)$. By the comments below Proposition 6.2, this implies that $x \leq \sum_{j=1}^{n} y_j$ in $\mathrm{Cu}(A)$. □

Unfortunately, the proof of this proposition does not provide any useful estimates on the constants $\mathrm{UCFP}_n(\mathrm{Cu}(A))$. Despite of this deficiency, the result is still very useful. Furthermore, although it is not immediately clear, it is also strictly stronger than Proposition 6.5. Indeed, for each Villadsen algebra $\mathcal{V}_n$ of the second type with finite stable rank, we show in [A, Corollary 4.7] that $\mathrm{rc}(\mathcal{V}_n) < \infty$, and hence $\mathcal{V}_n$ has the strong CFP by [7, Theorem 4.2.1] and Proposition 3.20. However, $\mathcal{V}_n$ does not have the uniform UCFP, as witnessed by [A, Proposition 4.3]. Similarly, whenever A is a unital, simple Villadsen algebra of the first type which admits a standard decomposition with seed space a finite-dimensional CW complex satisfying $0 < \mathrm{rc}(A) < \infty$, it follows that A has the strong CFP, but A fails the UCFP by [A, Theorem 3.5]. Examples of such Villadsen algebras were constructed by Toms in [65, Theorem 5.11]. In fact, for each $0 < r \in \mathbb{R}$, a C^*-algebra as described above is constructed with $\mathrm{rc}(A) = r$.

6.1. Partial answers. Here, we provide an overview on the progress that has been made in answering Question 6.1. We also put the results of [B] into the context of this question. Although the relation is indirect, this question certainly was the motivation for [B].

The main result of [A] is that for a substantial class of unital, simple, separable AH-algebras, which contains counter-examples to many long-standing questions about simple AH-algebras, Question 6.1 has an affirmative answer. To be more precise, combining [A, Theorem 3.5] and [A, Corollary 4.4], one obtains the following theorem:

Theorem 6.7. *If A is either a unital, simple Villadsen algebra of the first type, admitting a standard decomposition with seed space a finite-dimensional CW complex or* *a Villadsen algebra of the second type, then $A \otimes \mathcal{Z} \cong A$ if, and only if, $F(A)$ admits no characters.*

Examples of such Villadsen algebras of the first type include the original construction of Villadsen in [71], which were the first examples of unital and simple AH algebras without slow dimension growth, and the examples of Toms in [62] and [63] as well as those in Theorem 5.11 in [65]. The Villadsen algebras of the second type were constructed by Villadsen in [72], and were the first examples of unital, simple AH algebras with stable rank higher than one. In fact, for each $k \geq 1$, Villadsen constructted a simple, unital AH algebra $\mathcal{V}_k$ with stable rank equal to $k+1$. The Villadsen algebra $\mathcal{V}_1$ also served as the first example of a unital, simple C^*-algebra satisfying the strong CFP while failing to have strict comparison, see [41].

The proof of Theorem 6.7 works by proving that any C^*-algebra A which fails to absorb the Jiang-Su algebra and satisfies the hypotheses of the theorem also fails the UCFP, and hence, by Proposition 6.6, the central sequence algebra $F(A)$ admits a character. In fact, the techniques used to prove this seems to apply equally well to any AH algebra constructed using the techniques of Villadsen. For instance, in [B] we show that the examples of [30, Example 4.8] and [15] also yield to the same analysis, and thus they admit characters on their central sequence algebras. One can show that the same techniques apply to the AH-algebras in [51, Theorem 7.17].

Looking beyond the simple case in [B], we examined the permanence properties of the class of separable (but not necessarily unital) C^*-algebras A such that $F(A)$ admits no characters. We show that this class is as well-behaved as can be expected. Indeed, it is stable under arbitrary tensor products, quotients, extensions, and taking hereditary C^*-subalgebras. Finally, if $A \cong \varinjlim(A_i, \varphi_i)$ is a sequential inductive limit of unital C^*-algebras with unital connecting maps such that $\sup_i \text{w-Div}_2(F(A_i)) < \infty$, then $F(A)$ does not admit characters. Although it would be desirable to remove the uniformly bounded condition on the weak divisibility constants, it does not appear feasible, barring a proof that this condition is automatically satisfied. For instance, if Question 6.1 *does* have an affirmative answer, then $\text{w-Div}_2(F(A)) \leq 3$ whenever $F(A)$ admits no characters. However, results of this nature still appear to be well out of reach.

Additionally, we show that if a separable C^*-algebra A satisfies that $F(A)$ does not admit characters, then an obvious obstruction to $\mathcal{Z}$-stability is removed. More precisely, we proved that no hereditary C^*-subalgebra of A admits an irreducible representation on a finite-dimensional Hilbert space. It was shown in [6, Proposition 4.1] that a unital, separable AF algebra A is approximately divisible if, and only if, no quotient of A contains an abelian projection. Since any separable, approximately divisible C^*-algebra is $\mathcal{Z}$-stable by [69], we obtain a characterization of the class of unital, separable AF algebras which absorb the Jiang-Su algebra $\mathcal{Z}$: it is precisely the class of AF algebras A such that $F(A)$ does not admit characters.

In [B] we also introduce a new divisibility property, that we call k-local divisibility, see [B, Definition 5.1]. This divisibility property is closely related to the covering number of a C^*-algebra which was introduced by Kirchberg in [33]. It is weaker than the k-almost divisibility property considered by Winter in [74], and the α-divisibility property considered by Kirchberg and Rørdam in [37], but stronger than the weak divisibility property. We show that if A is a unital, simple, separable, nuclear C^*-algebra such that $F(A)$ is k-locally almost divisible, for some $k \geq 1$, then A has strong tracial m-comparison in the sense of [74, Definition 3.1 (iii)], for some $m \geq 1$. In particular, it follows from [36, Lemma 2.4] that A has local weak comparison in the sense of [36, Definition 2.1], whence A has property (SI). Relying heavily on [67], we show that if $F(A)$ is k-locally almost divisible, for some $k \geq 1$, and $T(A) \neq \emptyset$, then there exists a unital *-homomorphism $M_2 \to A^\omega \cap A'$. Hence, to summarize, it follows that if A is a unital, simple, separable and nuclear C^*-algebra, such that $F(A)$ is k-locally almost divisible, for some $k \geq 1$, then $A \otimes \mathcal{Z} \cong A$.

In general, if A is a unital, separable C^*-algebra such that $F(A)$ is k-locally almost divisible, for some $k \geq 1$, then $F(A)$ admits no characters, but it is not clear if the converse statement holds. However, if A has finite nuclear dimension, the converse *does* hold. To be more precise, Winter and Zacharias proved in [76] that if A is a separable C^*-algebra with $\dim_{\mathrm{nuc}}(A) \leq n < \infty$, and no hereditary C^*-subalgebra of A admits a finite-dimensional, irreducible representation, then $\mathrm{cov}(F(A)) \leq (n+1)^2$, where $\mathrm{cov}(F(A))$ denotes Kirchberg's covering number of $F(A)$. In particular, if $F(A)$ admits no characters, then no hereditary C^*-subalgebra of A admits a finite-dimensional irreducible representation, whence $\mathrm{cov}(F(A)) \leq (n+1)^2$, and it therefore follows from [B], that $F(A)$ is $2(n+1)^2$-locally almost divisible. By combining this with the results mentioned above, we recover Winter's seminal result (see [74]) that if A is a unital, simple, separable, infinite-dimensional C^*-algebra with finite nuclear dimension, then $A \otimes \mathcal{Z} \cong A$, at least in the stably finite case. While the proof obtained from this approach does avoid the more technical parts of [74], it should be noted that in many ways, both explicit and implicit, it relies on the results and techniques of [74], and it does require the full force of the approach to establishing property (SI) used in [36], along with the results of [67]. Thus, the main achievement of this approach is conceptual, since it gives a common framework for the two distinct flavours of finite dimension used to establish $\mathcal{Z}$-stability in [74] and [67], respectively. It also indicates that divisibility properties of the central sequence algebra of a unital, separable C^*-algebra reflects certain dimensional aspects of the C^*-algebra itself.

6.2. Failed approaches. In [C], we provide, for a substantial class of C^*-algebras, an example of an ideal in the central sequence algebra which is not a σ-ideal in the sense of [33, Definition 1.5]. While this may naturally be considered a success, it actually grew out of a failed attempt to prove that all ideals in $F(A)$ *are* σ-ideals. This failure extended to an idea for proving that if $F(A)$ has no characters, then $\mathrm{Div}_2(F(A)) < \infty$. This is an attractive line reasoning, since, if it is true, one would be able to prove the following statement: any unital, simple, separable, exact C^*-algebra with property (SI) which admits no characters on its central sequence algebra absorbs the Jiang-Su algebra.

We provide a proof of the fact that if all ideals of $F(A)$ are σ-ideals, then $\mathrm{Div}_2(F(A)) < \infty$ if, and only if, $F(A)$ admits no characters, in the hopes that some ideas in the proof may be of use.

Proposition 6.8. *Let A be a unital and separable C^*-algebra such that all ideals of $F(A)$ are σ-ideals. If $F(A)$ admits no characters, then $\mathrm{Div}_2(F(A)) < \infty$, i.e., there exists a *-homomorphism $CM_2 \to F(A)$ with full image.*

Proof. The strategy is to prove the following: if there exist *-homomorphisms $\varphi_1, \dots, \varphi_k \colon CM_2 \to F(A)$ such that the union of their images is full in $F(A)$, then we may reduce the number of *-homomorphisms by 1, i.e., there exist *-homomorphisms $\psi_1, \ldots, \psi_{k-1} \colon CM_2 \to F(A)$ such that the union of their images is full in $F(A)$. Applying Proposition 3.11, and an induction argument, this will complete the proof.

So suppose that $\varphi_1, \ldots, \varphi_k \colon CM_2 \to F(A)$ are given as above, and, for each $1 \leq n \leq k$, let $I_n \subseteq F(A)$ denote the ideal generated by $\mathrm{Im}(\varphi_n)$. Note that I_n is generated, as an ideal, by $\varphi_n(\iota \otimes \mathbf{1}_2)$, whence, for some $l \geq 1$, there exist elements s_{ij}, for $i = 1, \ldots, l$ and $j = 1, \ldots, k$, such that

$$\mathbf{1}_{F(A)} = \sum_{j=1}^{k} \sum_{i=1}^{l} s_{ij}^* \varphi_j(\iota \otimes \mathbf{1}_2) s_{ij}.$$

Let $D \subseteq F(A)$ denote the separable C^*-algebra generated by

$$\bigcup_{i=1}^{k-1} \mathrm{Im}(\varphi_i) \cup \{ s_{ij} \mid i = 1, \ldots, l \text{ and } j = 1, \ldots, k \}.$$

Since I_k is a σ-ideal, there exists $e \in I_k \cap D'$ such that $ec = c$, for every $c \in I_k \cap D$. In particular, $e\varphi_k(x) = \varphi_k(x)$ for every $x \in CM_2$. For each $m = 1, \ldots, k-1$, let $\rho_m \colon M_2 \to F(A)$ denote the contractive, completely positive order zero map, given by

$$\rho_m(x) = (\mathbf{1} - e)\varphi_m(\iota \otimes x) + \varphi_k(\iota \otimes x), \quad x \in M_2.$$

Since $(1-e)$ commutes with the images of the φ_m's and $(1-e)\varphi_k(\iota \otimes x) = 0$ for all $x \in M_2$, it is easy to verify that each ρ_m is completely positive and order zero. Furthermore, since $\rho_m(\mathbf{1}_2) \leq (1 - e) + e = \mathbf{1}$, it follows that ρ_m is contractive. By [75, Corollary 4.1], there exists a *-homomorphism $\psi_m \colon CM_2 \to F(A)$ such that $\psi_m(\iota \otimes \mathbf{1}_2) = \rho_m(\mathbf{1}_2)$, for each $m = 1, \ldots, k-1$. Since $\varphi_k(\iota \otimes \mathbf{1}_2) \leq \psi_m(\iota \otimes \mathbf{1}_2)$, it follows that the ideal J_m generated by $\mathrm{Im}(\psi_m)$, contains I_k, for all m. Similarly, $(1 - e)\varphi_m(\mathbf{1}_2) \in J_m$. Recalling that e commutes with D, we get

$$\mathbf{1} - e = (\mathbf{1} - e) \sum_{j=1}^{k} \sum_{i=1}^{l} s_{ij}^* \varphi_j(\iota \otimes \mathbf{1}_2) s_{ij}.$$

In conclusion, $\mathbf{1} - e \in \bigcup_{m=1}^{k-1} J_m$ and $e \in I_n \subseteq \bigcup_{m=1}^{k-1} J_m$, whence $\mathbf{1}$ belongs to the ideal generated by $\bigcup_{m=1}^{k-1} \psi_m(CM_2)$. □

As noted above, the hypotheses of this proposition are rarely satisfied, especially for the class of C^*-algebras of interest in the present thesis, i.e., the unital, separable C^*-algebras satisfying $A \otimes \mathcal{Z} \cong A$. However, it may be possible to weaken the hypothesis that all ideals are σ-ideals suitably.

Looking for a way to salvage something useful from the results in [C], one might consider the following approach: suppose M is a W^*-bundle over a compact Hausdorff space K. Viewing $M^\omega \cap M'$ as a $C(K^\omega)$-algebra, with fibres $(M^\omega \cap M')^\lambda$, for $\lambda \in K^\omega$, it follows that $M^\omega \cap M'$ admits a character if, and only if, $(M^\omega \cap M')^\lambda$ admits a character, for some $\lambda \in K^\omega$. Thus, if M is a strictly separable W^*-bundle such that $M_\sigma \cong \mathcal{R}$, for every $\sigma \in K$, can one conclude that $(M^\omega \cap M')^\lambda$ does not admit characters for any $\lambda \in K^\omega$? We elaborate on this idea below.

Let K^ω denote the spectrum of the commutative C^*-algebra $\prod_\omega C(K)$. Then $C(K^\omega) \cong \prod_\omega C(K)$. Given a sequence of points $(\lambda_n)_n \in \prod_{n \geq 1} K$, there is an associated point

$\mu = [(\lambda_n)_n] \in K^\omega$, given by the character $\mathrm{ev}_\mu \colon \prod_{n\in\omega} C(K) \to \mathbb{C}$, defined by

$$\mathrm{ev}_\mu([(f_n)_n] = \lim_{n\to\omega} f_n(\lambda_n), \quad (f_n)_n \in \ell^\infty(C(K)).$$

Let $K_\omega \subseteq K^\omega$ denote the set of points that arise from sequences in K in this way. On can show that $K_\omega \subseteq K^\omega$ is a dense subset, for instance by checking that, for any $f \in \prod_{n\in\omega} C(K)$, there exists $\mu \in K_\omega$ such that $f(\mu) = \|f\|$. However, the ε-test yields a stronger statement.

Lemma 6.9. *Suppose that K is a compact Hausdorff space. Then, for every separable C^*-subalgebra $D \subseteq \prod_{n\in\omega} C(K)$ and any point $\lambda \in K^\omega$, there exists a point $\mu \in K_\omega$ such that $g(\lambda) = \mathrm{ev}_\lambda(g) = \mathrm{ev}_\mu(g) = g(\mu)$, for every $g \in D$.*

Proof. Let $(g^{(k)})_{k\geq 1} \subseteq D$ be a dense countable set and let $\lambda \in K^\omega$ be an arbitrary point. For every k, let $(g_n^{(k)})_n \in \ell^\infty(C(K))$ be a lift of $g^{(k)}$. It suffices to prove that there exists a point $\mu \in K_\omega$ such that $g^{(k)}(\lambda) = g^{(k)}(\mu)$, for every $k \geq 1$.

For each $k \geq 1$, let $f_n^{(k)} \colon K \to [0, \infty)$ be given by

$$f_n^{(k)}(\sigma) = \big|g^{(k)}(\lambda) - g_n^{(k)}(\sigma)\big|.$$

Since $K_\omega \subseteq K^\omega$ is dense, for each $m \geq 1$ and every $\varepsilon > 0$, there exists a sequence of points $\mu_\varepsilon = (\sigma_n^{(\varepsilon)})_n \in \prod_{n\geq 1} K$ such that

$$f_\omega^{(l)}(\mu_\varepsilon) = \lim_{n\to\omega} \big|g^{(k)}(\lambda) - g_n^{(k)}(\sigma_n^{(\varepsilon)})\big| = \big|g^{(k)}(\lambda) - g^{(k)}(\mu_\varepsilon)\big| < \varepsilon.$$

Therefore, Proposition 2.2 implies the existence of a sequence of points $(\lambda_n)_n \in \prod_{n\geq 1} K$ such that $\mu := [(\lambda)_n] \in K_\omega$ satisfies $g^{(k)}(\lambda) = g^{(k)}(\mu)$, for every $k \geq 1$. ☐

The link between the $C(K^\omega)$-algebra structure on $M^\omega \cap M'$ and the W^*-bundle structure, is, essentially, given in the lemma below.

Lemma 6.10. *Let K be a compact Hausdorff space and $\mu = [(\lambda_n)_n] \in K_\omega$ be given. Then the ideal $C_0(K^\omega \backslash \{\mu\}) \subseteq C(K^\omega)$ equals the ultraproduct*

$$C_0(K^\omega \backslash \{\mu\}) = \prod_{n\in\omega} C_0(K \backslash \{\lambda_n\})$$

of ideals in $C(K)$. In particular, if M is a W^*-bundle over K, then $C_0(K^\omega \backslash \{\mu\})M^\omega$ is a σ-ideal of M^ω.

Proof. Suppose $f \in C(K^\omega \backslash \{\mu\})$, and let $(f_n)_n \in \ell^\infty(C(K))$ be any lift of f. Then

$$|\mathrm{ev}_\mu(f)| = \lim_{n\to\omega} |f_n(\lambda_n)| = 0.$$

In particular, it follows that

$$\lim_{n\to\omega} \big\|f_n - \big(f_n - f_n(\lambda_n)\big)\big\| = \lim_{n\to\omega} |f_n(\lambda_n)| = 0.$$

Hence, $[(f_n - f_n(\lambda_n))_n] = [(f_n)_n]$, and $f_n - f_n(\lambda(n)) \in C_0(K \backslash \{\lambda_n\})$, for all $n \geq 1$, whence the first statement follows.

From the first statement we can conclude that the ideal $C_0(K^\omega \backslash \{\mu\})M^\omega$ in M^ω is of the form $\prod^\omega I_n$, for a sequence of ideals I_n in M. We show that such an ideal is a σ-ideal. Essentially, this follows from the proof of [33, Corollary 1.7], but we include a proof regardless.

It is easy to see that $J := \prod^\omega I_n \subseteq M^\omega$ is a norm-closed, 2-sided ideal. Let $B \subseteq M^\omega$ be a norm-separable C^*-subalgebra, let $d \in B \cap J$ be a strictly positive element, and let $\{b^{(k)}\}_{k\geq 1} \subseteq B$ be a dense countable set. For each $k \geq 1$, let $(b_n^{(k)})_n \in \ell^\infty(M)$ be a lift of $b^{(k)}$, and let $(d_n)_n \in \ell^\infty(M)$ be a lift of d. For each n, let $X_n \subseteq I_n$ denote the set of positive contractions and, for $l \geq 1$, let functions $f_n^{(l)} \colon X_n \to [0, \infty)$ be given by

$$f_n^{(1)}(x_n) = \|x_n d_n - d_n\|_{2,u},$$
$$f_n^{(l+1)}(x_n) = \|x_n b_l - b_l x_n\|_{2,u}.$$

Choose a quasi-central approximate unit $(e_\alpha)_\alpha \subseteq J$, and, for each $m \geq 1$, choose α_m such that $e_m := e_{\alpha_m}$ satisfies $\|e_m d - e_m\|_{2,u} < 1/m$ and $\|e_m b_l - b_l e_m\|_{2,u} < 1/m$, for $l = 1, \dots, m$. Letting $e^{(m)} = (e_1^{(m)}, e_2^{(m)}, \dots) \in \prod_{n\geq 1} I_n$ denote a positive contractive lift of e_m, it follows that $e_n^{(m)} \in X_n$, for all n, and $f_\omega^{(l)}(e^{(m)}) < 1/m$, for $l = 1, \dots, m+1$. An application of Proposition 2.2 yields a sequence $\overline{e} = (e_1, e_2, \dots) \in \prod_{n\geq 1} I_n$ satisfying $f_\omega^{(l)}(\overline{e}) = 0$, for all $l \geq 1$. It is straightforward to check that $e = [\overline{e}] \in M^\omega$ satisfies $eb = b$, for all $b \in J \cap B$ and $e \in J \cap B'$. In other words, $J \subseteq M^\omega$ is a σ-ideal. □

Before stating the next result, we need to introduce some notation. Let M be a W^*-bundle over a compact Hausdorff space K. For each $\lambda \in K^\omega$, let $I_\lambda \subseteq M^\omega \cap M'$ denote the ideal $C_0(K^\omega \backslash \{\lambda\})(M^\omega \cap M')$, and $(M^\omega \cap M')^\lambda$ denote the quotient $(M^\omega \cap M')/I_\lambda$. Recall that $(M^\omega \cap M')_\lambda$ denotes the fibre over λ when $M^\omega \cap M'$ is viewed as a W^*-bundle rather than a $C(K^\omega)$-algebra, and that $\pi_\lambda \colon M^\omega \cap M' \to (M^\omega \cap M')_\lambda$ denotes the quotient map.

Proposition 6.11. *Let M be a strictly separable W^*-bundle over a compact Hausdorff space K. Then, for every point $\mu \in K_\omega \subseteq K^\omega$, we have*

$$(M^\omega \cap M')^\mu \cong (M^\omega \cap M')_\mu = (M^\omega)_\mu \cap \pi_\mu(M)'.$$

In particular, if $M_\sigma \cong \mathcal{R}$, for every $\sigma \in K$, there is a unital *-homomorphism $M_2 \to (M^\omega \cap M')^\mu$, for every $\mu \in K_\omega$.

Proof. Let $\mu \in K_\omega$ be arbitrary. It follows from Lemma 6.10 that $I_\mu \cap M'$ is a σ-ideal in $M^\omega \cap M'$. By [C, Lemma 2.4] and the proof of [C, Proposition 2.5], we conclude that $(M^\omega \cap M')^\mu = (M^\omega \cap M')_\mu$. A straightforward modification of [36, Remark 4.7], shows that the kernel of the quotient map $\pi_\mu \colon M^\omega \to (M^\omega)_\mu$ is a σ-ideal, whence

$$(M^\omega \cap M')_\mu = (M^\omega)_\mu \cap \pi_\mu(M)'.$$

Assume now that $M_\sigma \cong \mathcal{R}$, for each $\sigma \in K$, and suppose that $\mu \in K_\omega$ arises from the sequence $(\lambda_n)_n$ in K. Since there is a unital *-homomorphism $M_2 \to (M_{\lambda_n})^\omega \cap M'_{\lambda_n}$, for each $n \geq 1$, a diagonal argument shows that there exists a unital *-homomorphism $M_2 \to (M^\omega \cap M')_\mu \cong (M^\omega \cap M')^\mu$. □

It follows from the above proposition that, viewing $M^\omega \cap M'$ as a $C(K^\omega)$-algebra, rather than a W^*-bundle, the fibre $(M^\omega \cap M')^\mu$ admits no characters, whenever $\mu \in K_\omega$.

Prompted by this observation one might ask: if X is a compact Hausdorff space and A is a unital $C(X)$-algebra such that A_x does not admit characters for a dense set of points $x \in X_0 \subseteq X$, does it follows that A admits no characters? It follows from [B, Remark 3.3] that things are not quite so simple. Indeed, there is a $C(X)$-algebra A

such that the *metric* central sequence algebra $F(A)$ satisfies that $F(A)_x$ admits a unital embedding of M_2 for a dense set of points $X_0 \subseteq X^\omega$, while $F(A)$ *does* admit a character. Of course, this does not automatically imply that $A^\omega \cap A'$ admits a character, but it does demonstrate that if the questions considered above have affirmative answers, then the W^*-bundle structure is essential, i.e., simply viewing $M^\omega \cap M'$ as a $C(K^\omega)$-algebra does not provide an answer.

7. Further research

In this section we sketch some problems to be investigated in the future.

The Toms–Winter conjecture has not been mentioned often so far in this thesis, but it is obviously a motivation for Question 6.1. Recall that the conjecture states that finite nuclear dimension, $\mathcal{Z}$-stability and strict comparison are equivalent for the class of unital, simple, separable, nuclear, non-elementary C^*-algebras. At this point, it is known that finite nuclear dimension implies $\mathcal{Z}$-stability, see [74], and that $\mathcal{Z}$-stability implies strict comparison, see [56]. Furthermore, it is known that, if A additionally satisfies that $\partial_e T(A)$ is non-empty and compact, then $\mathcal{Z}$-stability also implies finite nuclear dimension. The general feeling seems to be that this result will, in time, be extended to C^*-algebras with arbitrary trace simplex. However, concerning whether strict comparison implies $\mathcal{Z}$-stability, the general opinion appears to be more ambiguous. While it was proved in [36] and [67], independently, that strict comparison *does* imply $\mathcal{Z}$-stability when $\partial_e T(A)$ is compact *and finite-dimensional*, neither of the strategies applied in the cited papers appear to generalize well to the general case. Below, we outline a few ideas on how to approach the question of whether strict comparison implies $\mathcal{Z}$-stability.

As noted in subsection 2.3, if any $\|\cdot\|_{2,u}$-separable W^*-bundle over a compact Hausdorff space K satisfying $M_\lambda \cong \mathcal{R}$ is automatically trivial, then strict comparison will imply $\mathcal{Z}$-stability for any unital, simple, separable, nuclear C^*-algebra for which $\partial_e T(A)$ is compact. Looking for a counter-example to this, the example of a non-$\mathcal{Z}$-stable $C(X)$-algebra A whose fibres are the CAR algebra in [30], might be a good place to start. Indeed, as noted in [B, Remark 4.3], the central sequence algebra $F(A)$ is a $C(X^\omega)$-algebra such that at least one fibre of $F(A)$ admits a character. Now, each fibre A_x carries a unique tracial state, which induces a tracial state τ_x on A. Hence, there is an injective map $X \to \partial_e T(A)$, given by $x \mapsto \tau_x$. First of all, is this map continuous? If it is, then there exists a W^*-bundle M_X over X, such that $A^\omega_X \cap A' \cong (M_X)^\omega \cap M'_X$. Letting $y \in X^\omega$ denote some point such that $F(A)_y$ admits a character, does it follow that the corresponding fibre $\big((M_X)^\omega \cap M'_X\big)_y$ admits a character? If this is true, then M_x cannot be trivial W^*-bundle, while each fibre of M_X is isomorphic to the von Neumann algebra generated by the GNS representation of the CAR algebra with respect to its unique tracial state, i.e., the hyperfinite II_1-factor $\mathcal{R}$. The C^*-algebra A is clearly not simple, and, therefore, will not constitute a counter-example to the Toms–Winter conjecture regardless of the answers to the above questions. It seems likely, though, that the construction in [15] will yield to the same analysis, and thus, possibly, provide the sought after counter-example.

Supposing that strict comparison *does not* suffice to conclude $\mathcal{Z}$-stability, for instance if the strategy sketched above proves successful, it will be necessary to consider strict comparison alongside other regularity properties in order to ensure $\mathcal{Z}$-stability. Requiring suitable divisibility properties seems reasonable since $\mathcal{Z}$-stability implies almost divisibility, see [74, Proposition 3.7]. For instance, one may wonder whether the following question has an affirmative answer: if A is a unital, simple, separable, nuclear C^*-algebra with strict comparison such that every element $x \in \mathrm{Cu}(A)$ is k-locally almost divisible, does it follow that $F(A)$ is k-locally almost divisible? Inspired by [67], a related question may be more tractable: if, additionally, $\partial_e T(A)$ is non-empty and compact, and M denotes the associated W^*-bundle over $\partial_e T(A)$, does it follow that there exist $\alpha > 0$ such that, for every $m \geq 2$, there exist k order zero maps $\varphi_1, \dots, \varphi_k \colon M_m \to M^\omega \cap M'$ satisfying

$$\tau_\lambda\Big(\sum_{i=1}^{k} \varphi_i(\mathbf{1})x\Big) \geq \alpha\tau_\lambda(x),$$

for every $x \in M$ and every $\lambda \in K^\omega$? Here τ_λ denotes the tracial state on $M^\omega \cap M'$ corresponding to a point $\lambda \in K^\omega$. At least in this case, each fibre M_λ is approximately finite-dimensional, which may provide a route towards adapting the techniques developed by Winter in [74] for handling C^*-algebras with *locally* finite nuclear dimension.

Looking in another direction, the McDuff inspired characterization in [8] of trivial W^*-bundles does not include a McDuff-type dichotomy result. Thus, given a $\|\cdot\|_{2,u}$-separable W^*-bundle over a compact Hausdorff space K whose fibres are $\mathcal{R}$, one may wonder whether any of the following conditions suffice to conclude that M is trivial:

(i) No fibre of $M^\omega \cap M'$ is commutative.
(ii) No fibre of $M^\omega \cap M'$ admits a character.
(iii) $M^\omega \cap M'$ does not admit characters.

The conditions above get progressively stronger, i.e., (iii)⇒(ii)⇒(i), and if M is the W^*-bundle associated with a unital, separable, simple C^*-algebra A with property (SI), then the last condition is equivalent to the condition that $F(A)$ admits no characters, see [37, Proposition 3.19]. However, in general, it is not clear whether any of the implications can be reversed, or whether they suffice to conclude triviality of M.

Finally, one may wonder whether, if A is a unital, simple, separable, nuclear C^*-algebra which is 'k-coloured equivalent' to $A \otimes \mathcal{Z}$, does it follow that $F(A)$ is k-locally almost divisible? An elaboration of this question was, at the time of writing, somewhat hampered by the fact that no definitive definition of k-coloured equivalence of C^*-algebras was available.

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Part II

Included articles

The author of this thesis is also the sole author of the four articles listed below, except for the appendix to Article A, which is joint work with Joan Bosa.

REGULARITY OF VILLADSEN ALGEBRAS AND CHARACTERS ON THEIR CENTRAL SEQUENCE ALGEBRAS

MARTIN S. CHRISTENSEN

Abstract. We show that if A is a simple Villadsen algebra of either the first type with seed space a finite dimensional CW complex, or of the second type, then A absorbs the Jiang-Su algebra tensorially if and only if the central sequence algebra of A does not admit characters.

Additionally, in a joint appendix with Joan Bosa, we show that the Villadsen algebra of the second type with infinite stable rank fails the Corona Factorization Property, thus providing the first example of a unital, simple, separable and nuclear C^*-algebra with a unique tracial state which fails to have this property.

1. Introduction

Villadsen algebras, introduced by Villadsen in [37] and [38], respectively, fall into two types and both display properties not previously observed for simple AH algebras. Together they form a class of unital, simple and separable AH algebras exhibiting a wide range of exotic behaviour; arbitrary stable and real rank, arbitrary radius of comparison, and perforation in their ordered K_0 groups and Cuntz semigroups.

The first type of Villadsen algebras was introduced in [37] as the first examples of unital, simple AH algebras with perforation in their ordered K_0 groups. In particular, they were the first examples of simple AH algebras without slow dimension growth. Modifying the construction, Toms exhibited for each positive real number $r > 0$ a unital, simple AH algebra with rate of growth r (in the sense that the radius of comparison is r); see [35]. The techniques introduced by Villadsen also played a crucial role in Rørdam's construction in [29] of a simple, separable and nuclear C^*-algebra in the UCT class containing an infinite and a non-zero finite projection, the first counterexample to the Elliott conjecture in its previous incarnation. In [34] Toms used a modification of the AH algebras in [37] to provide a particularly egregious counterexample to the previous Elliott conjecture. Toms and Winter gave a formal definition of Villadsen algebras of the first type in [36], which includes Villadsen's original constructions, and the subsequent modifications of Toms in [34] and [35]. In the same paper they confirmed what has later been named the Toms–Winter conjecture for this class of C^*-algebras, i.e., they showed that for a simple Villadsen algebra of the first type with seed space a finite dimensional CW complex (see Definition 3.2), the regularity properties Jiang-Su stability, strict comparison of positive elements, and finite decomposition rank are equivalent. The latter regularity

This work was completed as a PhD-student at the University of Copenhagen.

property, or even the weaker requirement of finite nuclear dimension, has since been proven to suffice for classification, under the additional assumption of UCT (the complete proof of this has a long history and is the work of many hands, but the final steps were carried out in [13],[7] and [31]).

The second type of Villadsen algebras was introduced in [38] as the first examples of simple AH algebras with stable rank higher than one. In fact, every possible value of the stable rank is achieved, i.e., for each $1 \leq k \leq \infty$ a unital, simple AH algebra $\mathcal{V}_k$ is constructed such that $\mathrm{sr}(\mathcal{V}_k) = k+1$, and the real rank satisfies $k \leq RR(\mathcal{V}_k) \leq k+1$. In addition, each C^*-algebra $\mathcal{V}_k$ has a unique tracial state and perforation in the ordered K_0 group, in particular $\mathcal{V}_k \otimes \mathcal{Z} \ncong \mathcal{V}_k$. Ng and Kucerovsky showed in [20] that $\mathcal{V}_2$ has the Corona Factorization Property, thus providing the first example of a simple C^*-algebra satisfying this property while having perforation in the ordered K_0 group. The construction also formed the basis for Toms' counterexample to the previous Elliott conjecture in [32].

As indicated in the preceding paragraphs, the class of Villadsen algebras form a rich class containing examples of both regular C^*-algebras and C^*-algebras displaying a wide range of irregularity, while still remaining amenable to analysis. As such, they form a good 'test class' for statements concerning simple and nuclear C^*-algebras.

The central sequence algebra of a unital separable C^*-algebra A (see Section 2.1 for a definition), which we denote $F(A)$, was studied extensively by Kirchberg in [16], wherein the notation $F(A)$ was introduced, and the definition of $F(A)$ was extended to not necessarily unital C^*-algebras in a meaningful way (for instance, $F(A)$ is unital whenever A is σ-unital, and the assignment $A \mapsto F(A)$ is a *stable* invariant). In analogy with the *von Neumann* central sequence algebra of II_1-factors, the central sequence algebra detects absorption of certain well-behaved C^*-algebras. More precisely, if B is a unital, separable C^*-algebra with approximately inner half-flip (i.e., the two factor embeddings $B \to B \otimes B$ are approximately unitarily equivalent), then $A \otimes B \cong A$ if there exists a unital embedding $B \to F(A)$. If, moreover, $B \cong \bigotimes_{n=1}^{\infty} B$, e.g., when B is the Jiang-Su algebra $\mathcal{Z}$, then $A \otimes B \cong A$ if *and only if* such an embedding exists. Significant progress in our understanding of the central sequence algebra of stably finite C^*-algebras was obtained by Matui and Sato in [21, 22]. In these papers they introduced property (SI), a regularity property which facilitates liftings of certain properties of a tracial variant of the central sequence algebra to the central sequence algebra itself (see for instance [17, Proposition 3.9]). Furthermore, they prove that whenever A is a unital, simple, separable and nuclear C^*-algebra with strict comparison, then A has property (SI) and as a consequence, if A has only finitely many extremal tracial states, then $\mathcal{Z}$ embeds unitally in $F(A)$ hence $A \otimes \mathcal{Z} \cong A$.

Prompted by the analogy with von Neumann II_1 factors one might hope that the McDuff dichotomy (see [23]) carries over to C^*-algebras. However, as proven by Ando and Kirchberg in [1], the central sequence algebra $F(A)$ is non-abelian whenever A is separable and not type I. In addition, it can happen that $F(A)$ is non-abelian and contains no simple, unital C^*-algebra other than $\mathbb{C}$ (see [16, Corollary 3.14]). Hence, non-commutativity of $F(A)$

does not suffice to conclude regularity. Addressing this issue, Kirchberg and Rørdam asked the following question in [17].

Question 1.1. Let A be a unital and separable C^*-algebra. Does it follows that $A \otimes \mathcal{Z} \cong A$ if and only if $F(A)$ has no characters?

Another question under consideration in the present paper is the following: given a unital, simple C^*-algebra A with a unique tracial state, when can one conclude that A is regular? In certain situations, a unique tracial state is sufficient to conclude regularity and even classifiability by the Elliott invariant. For instance, Elliott and Niu showed in [8] that if X is a compact metrizable Hausdorff space and σ is a minimal homeomorphism of X such that the dynamical system (X, σ) is uniquely ergodic, i.e., $C(X) \rtimes_\sigma \mathbb{Z}$ has a unique tracial state, then $C(X) \rtimes_\sigma \mathbb{Z}$ is $\mathcal{Z}$-stable and classifiable (this is not automatic, see [11]). Similarly, as proven by Niu (see [25, Theorem 1.1]) if A is a unital, simple AH algebra with diagonal maps such that the set of extremal tracial states is countable, then A is without dimension growth. In particular, any AH algebra of this type with a unique tracial state has real rank zero (cf. [3]). On the other hand, as demonstrated in [38], a unique tracial state does not suffice to conclude either real rank zero or $\mathcal{Z}$ stability for general AH algebras. It is therefore natural to ask what (if any) regularity properties are implied by the existence of a unique tracial state.

The Corona Factorization Property was introduced by Kucerovsky and Ng in [19] and is related to both the theory of extensions and the question of when extensions are automatically absorbing (see for instance [18]). It is a very mild regularity condition, which nonetheless does exclude the most exotic behaviour. For instance, if A is a separable C^*-algebra satisfying the Corona Factorization Property and $M_n(A)$ is stable for some $n \in \mathbb{N}$ then A must also be stable (see [26, Proposition 4.7]). Under the additional assumption that A is simple and has real rank zero it also follows that A is either stably finite or purely infinite. Examples of C^*-algebras failing the Corona Factorization Property have been provided in the literature. For instance, the C^*-algebras constructed in [29] and [28] fail the Corona Factorization Property

The main result of the present paper is that question 1.1 has an affirmative answer when A is either a simple Villadsen algebra of the first type with seed space a finite dimensional CW complex or a Villadsen algebra of the second type (see Theorem 3.5 and Corollary 4.4 respectively).

Additionally, in a joint appendix with Joan Bosa, we show that the Villadsen algebra of the second type with infinite stable rank fails to have the Corona Factorization Property, thus providing an example of a unital, simple, separable and nuclear C^*-algebra with a unique tracial state which fails this property (see Theorem A.1). While examples of unital, simple, separable and nuclear C^*-algebras without the Corona Factorization Property are already known, as noted above, the example provided here is to the best of the authors' knowledge the first of its kind with a unique tracial state.

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2. Background

2.1. **The Central Sequence Algebra.** Let A be a unital C^*-algebra, ω be a free ultrafilter on $\mathbb{N}$ and $\ell^\infty(A)$ denote the sequences $(a_n)_n \subseteq A$ such that $\sup_n \|a_n\| < \infty$. The ultrapower A_ω of A with respect to ω is defined by

$$A_\omega := \ell^\infty(A)/\{(a_n)_n \in \ell^\infty(A) \mid \lim_{n\to\omega} \|a_n\| = 0\}.$$

Given a sequence $(a_n)_n \in \ell^\infty(A)$ let $[(a_n)_n] \in A_\omega$ denote the image under the quotient map. There is a natural embedding $\iota\colon A \to A_\omega$ given by $\iota(a) = [(a, a, a, \dots)]$. Since ι is injective it is often suppressed and A is considered to be a subalgebra of A_ω, a convention we shall follow here. The central sequence algebra $F(A)$ of A is defined by $F(A) := A_\omega \cap A'$. The notation $F(A)$ was introduced by Kirchberg in [16], wherein the definition of the central sequence algebra was extended to (possibly non-unital) σ-unital C^*-algebras in a meaningful way. We retain this notation, although only unital C^*-algebras are considered here, to emphasize the connection with Kirchberg's work. Furthermore, the ultrafilter is suppressed in the notation, since the isomorphism class of (unital) separable sub-C^*-algebras $B \subseteq F(A)$ is independent of the choice of free ultrafilter. More precisely, if B is a separable C^*-algebra and there exists a (unital) injective *-homomorphism $B \to A_\omega \cap A'$ for some free ultrafilter ω on $\mathbb{N}$, then there exists a (unital) injective *-homomorphism $B \to A_{\omega'} \cap A'$ for any other free ultrafilter ω' on $\mathbb{N}$. In particular, the question of whether $F(A)$ has characters is independent of the choice of free ultrafilter (see [17, Lemma 3.5]). Whether $A_\omega \cap A' \cong A_{\omega'} \cap A'$ for arbitrary free ultrafilters ω and ω' on $\mathbb{N}$ depends on the Continuum Hypothesis (see [10] and [9, Theorem 5.1]).

As described in [17], building on results from [27], there is a useful relationship between divisibility properties of $F(A)$ and comparability properties of $\mathrm{Cu}(A)$. We rely on an elaboration of this technique to obtain our results.

2.2. **Vector Bundles and Characteristic Classes.** Readers who are unfamiliar with the theory of characteristic classes of (complex) vector bundles may wish to consult [24] for a general textbook on the subject. Alternatively, the papers [29] and [37] also contains good summaries of (the relevant parts of) the theory.

In order to access the machinery of characteristic classes within the framework of C^*-algebras we need the following observation: Let $\mathbb{K}$ denote the compact operators acting on a separable, infinite-dimensional Hilbert space $\mathcal{H}$, let $p \in C(X) \otimes \mathbb{K}$ be a projection and let ξ_p denote vector bundle over X given by

$$\xi_p := \{(x, v) \in X \times \mathcal{H} \mid v \in p(x)(\mathcal{H})\}.$$

It is a consequence of Swan's Theorem that the assignment $p \mapsto \xi_p$ induces a one-to-one correspondence of Murray-von Neumann equivalence classes of projections in $C(X) \otimes \mathbb{K}$ with isomorphism classes of vector bundles over X, in such a way that $q \precsim p$ if and only if there exists a vector bundle η over X such that $\xi_q \oplus \eta \cong \xi_p$. We shall be concerned with the ordering of vector bundles according to the above described pre-order. For this purpose we employ the machinery of characteristic classes of vector bundles described below, a technique pioneered by Villadsen in [37] and [38].

Given a compact Hausdorff space X and vector bundle ω of (complex) fibre dimension k, the *total Chern class* $c(\omega) \in H^*(X)$ is

$$c(\omega) = 1 + \sum_{i=1}^{\infty} c_i(\omega),$$

where $c_j(\omega) \in H^{2j}(X)$ is the j'th Chern class for each $1 \leq j \leq k$, and $c_j(\omega) = 0$ whenever $j > k$. Furthermore, the top Chern class $c_k(\omega)$ is the Euler class $e(\omega)$ of ω. We will simply refer to $c(\omega)$ as the Chern class of ω, rather than the *total* Chern class. The Chern class has the following properties:

(i) If θ_k denotes the trivial vector bundle of fibre dimension $k \in \mathbb{N}$, then $c(\theta_k) = 1 \in H^0(X)$ for any $k \in \mathbb{N}$.
(ii) For arbitrary vector bundles ω, η over X we have $c(\omega \oplus \eta) = c(\omega)c(\eta)$, where the product is the cup product in the cohomology ring $H^*(X)$.
(iii) If Y is another compact Hausdorff space and $f\colon Y \to X$ is continuous then $c(f^*(\omega)) = f^*(c(\omega))$.

Properties (ii) and (iii) above also holds for the Euler class, while the first property instead becomes $e(\theta_k) = 0$ for all $k \in \mathbb{N}$. This can be deduced from the above description of the Chern class.

In the following sections it will suffice to find a reasonably good method for determining which Chern classes of a vector bundle are non-zero. Such a method is provided by the following observation. Given a finite number of sets $X_1, \ldots, X_n$, let $\rho_j\colon X_1 \times \cdots \times X_n \to X_j$ denote the j'th coordinate projection. If each of the spaces $X_1, \ldots, X_n$ is a finite CW-complex such that $H^i(X_j)$ is a free $\mathbb{Z}$-module for each i and j, it follows from the Künneth formula (see [24, Theorem A.6]) that the map

$$\mu\colon H^{i_1}(X_1) \otimes H^{i_2}(X_2) \otimes \cdots \otimes H^{i_n}(X_n) \to H^i(X_1 \times X_2 \times \cdots \times X_n),$$

where $i = \sum_{k=1}^{n} i_k$, given by

$$a_1 \otimes a_2 \otimes \cdots \otimes a_n \mapsto \rho_1^*(a_1)\rho_2^*(a_2) \cdots \rho_n^*(a_n),$$

is injective. A particular application of this observation is the following: suppose that $X_1, \ldots, X_n$ satisfies the hypothesis above and, for each $i = 1, \ldots, n$, that ξ_i is a vector bundle over X_i such that $e(\xi_i) \in H^*(X_i)$ is non-zero for $i = 1, \ldots, n$. Since each $H^i(X_j)$ is without torsion, the element $e(\xi_1) \otimes \cdots \otimes e(\xi_n)$ is also non-zero, whence it follows from naturality of the Euler class and the product formula above that

$$\begin{aligned} e\big(\rho_1^*(\xi_1) \oplus \rho_2^*(\xi_2) \oplus \cdots \oplus \rho_n^*(\xi_n)\big) &= \rho_1^*(e(\xi_1))\rho_2^*(e(\xi_2)) \cdots \rho_n^*(e(\xi_n)) \\ &= \mu(e(\xi_1) \otimes \cdots \otimes e(\xi_n)) \neq 0. \end{aligned}$$

We will apply this observation only to the situation where each X_i is either of the form $(S^2)^k$ for some k or a complex projective space $\mathbb{C}P^k$, in which case the hypothesis' are satisfied.

2.3. **The Cuntz Semigroup, Comparison and Divisibility.** We give a brief introduction to the Cuntz semigroup as defined in [6]. We restrict our attention to the properties needed in the current exposition, and interested readers should consult [6] or [2] for a fuller exposition.

Let A be a C^*-algebra and let $a, b \in A_+$. We say that a is Cuntz dominated by b, and write $a \precsim b$, if there exists a sequence $(x_n)_n \subseteq A$ such that $\|a - x_n^* b x_n\| \to 0$. We say that a is Cuntz equivalent to b, and write $a \sim b$, if $a \precsim b$ and $b \precsim a$. Let $\mathbb{K}$ denote the compact operators on a separable, infinite-dimensional Hilbert space and define

$$\mathrm{Cu}(A) := (A \otimes \mathbb{K})_+ / \sim .$$

We write $\langle a \rangle$ for the equivalence class of an element $a \in (A \otimes \mathbb{K})_+$. Then $\mathrm{Cu}(A)$ becomes an ordered abelian semgroup when equipped with the operation

$$\langle a \rangle + \langle b \rangle := \langle a \oplus b \rangle, \quad a, b \in (A \otimes \mathbb{K})_+$$

and order defined by $\langle a \rangle \leq \langle b \rangle$ if and only if $a \precsim b$. Additionally, any upwards directed countable set $S \subseteq \mathrm{Cu}(A)$ admits a supremum. Given $x, y \in \mathrm{Cu}(A)$ we say that x is *compactly contained* in y, and write $x \ll y$, if for any increasing sequence $(y_k)_k \subseteq \mathrm{Cu}(A)$ with $\sup_k y_k = y$ there exists $k_0 \in \mathbb{N}$ such that $x \leq y_{k_0}$. Equivalently, if $a, b \in (A \otimes \mathbb{K})_+$ then $\langle a \rangle \ll \langle b \rangle$ if and only if there exists $\varepsilon > 0$ such that $a \precsim (b - \varepsilon)_+$. An element $x \in \mathrm{Cu}(A)$ satisfying $x \ll x$ is said to be *compact*. Note that $\langle p \rangle$ is compact whenever $p \in (A \otimes \mathbb{K})_+$ is a projection.

The following proposition is a strengthening of [17, Theorem 4.9] with essentially the same proof. Although the strengthening is minor, it is crucial to Theorem 3.5 and Corollary 4.4.

Proposition 2.1. *Let A be a unital, separable C^*-algebra. If $F(A)$ has no characters, then for each $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that the following holds: given $x, y_1, \dots, y_n \in \mathrm{Cu}(A)$ such that $x \leq m y_i$ for all $i = 1, \dots, n$, then $x \leq \sum_{i=1}^n y_i$.*

Proof. It follows from [17, Lemma 3.5] that there exists a unital, separable sub-C^*-algebra $B \subseteq F(A)$ such that B has no characters. Hence, [27, Corollary 5.6 (i) and Lemma 6.2] imply that for each $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that the infinite maximal tensor product C^*-algebra $D := \bigotimes_{k \in \mathbb{N}} B$ is weakly (m, n)-divisible, i.e., there exist elements $y_1, \dots, y_n \in \mathrm{Cu}(D)$ satisfying $m y_i \leq \langle 1_D \rangle$, for all $i = 1, \dots, n$, and $\langle 1_D \rangle \leq \sum_{j=1}^n y_j$. Note that since $B \subseteq F(A)$ is unital and separable, it follows from [16, Corollary 1.13] that there exists a unital *-homomorphism $\varphi : D \to F(A)$. Let $P \subseteq A_\omega$ denote the image under the natural map $A \otimes_{\max} D \to A_\omega$. By [17, Lemma 4.1] the induced map $\mathrm{Cu}(A) \to \mathrm{Cu}(P)$ is an order embedding, and therefore the result finally follows from [27, Lemma 6.1]. □

3. Villadsen Algebras of the first type

In this section we study Villadsen algebras of the first type, as defined by Toms and Winter in [36] based on the construction by Villadsen in [37]. We prove that for a simple Villadsen algebra A of the first type with seed space a finite dimensional CW complex, $F(A)$ has no characters if and only if A has strict comparison of positive elements (Theorem 3.5). We also note in passing that if A is not an AF algebra, then A has real rank zero if and only if it has a unique tracial state (Proposition 3.6).

For the readers convenience we recall the definition of a Villadsen algebra of the first type (see also [36]).

Definition 3.1. Let X, Y be a compact Hausdorff spaces and $n, m \in \mathbb{N}$ be given such that $n \mid m$. A *-homomorphism $\varphi : M_n \otimes C(X) \to M_m \otimes C(Y)$ is said to be *diagonal* if it has the form

$$f \mapsto \begin{pmatrix} f \circ \lambda_1 & 0 & \cdots & 0 \\ 0 & f \circ \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & f \circ \lambda_{m/n} \end{pmatrix},$$

where each $\lambda_i \colon Y \to X$ is a continuous map for $i = 1, \dots, m/n$. The maps $\lambda_1, \dots, \lambda_{m/n}$ are called the *eigenvalue maps* of φ.

The map φ above is said be a *Villadsen map of the first type* (a $\mathcal{V}$I-map) if $Y = X^k$ for some $k \in \mathbb{N}$ and each eigenvalue map is either a coordinate projection or constant.

Note that, in contrast with the construction in [37], given a $\mathcal{V}$I map $\varphi \colon C(X) \otimes M_n \to C(X^k) \otimes M_m$ as above, it is not necessary that the coordinate projections that occur as eigenvalue maps for φ are distinct, nor that every possible coordinate projection $X^k \to X$ occurs as an eigenvalue map for φ.

Definition 3.2. Let X be a compact Hausdorff space and let $(n_i)_{i \in \mathbb{N}}$ and $(m_i)_{i \in \mathbb{N}}$ be sequences of natural numbers with $n_1 = 1$ and such that $m_i | m_{i+1}$ and $n_i | n_{i+1}$ for all $i \in \mathbb{N}$. Put $X_i = X^{n_i}$. A unital C^*-algebra A is said to be a *Villadsen algebra of the first type* (a $\mathcal{V}$I algebra) if it can be written as an inductive limit

$$A \cong \varinjlim (M_{m_i} \otimes C(X_i), \varphi_i),$$

where each φ_i is a $\mathcal{V}$I map. We refer to the above inductive system as a *standard decomposition* for A with *seed space* X.

Although not required in the above definition, we shall only consider *simple* $\mathcal{V}$I algebras in the present paper. Additionally, we require that the seed space is a finite-dimensional CW complex. This is a particularly tractable class of C^*-algebras, as demonstrated by the following theorem due to Toms and Winter.

Theorem 3.3 (See [36]). *Let A be a simple $\mathcal{V}I$ algebra admitting a standard decomposition with seed space a finite-dimensional CW complex. The following are equivalent:*

(i) *A has finite decomposition rank.*
(ii) *A is $\mathcal{Z}$-stable.*
(iii) *A has strict comparison of positive elements.*
(iv) *A has slow dimension growth as an AH algebra.*

It follows directly from Definition 3.2 that if X is a zero-dimensional CW complex, i.e., is a finite discrete space, then the corresponding $\mathcal{V}$I algebra is a unital AF algebra. In the interest of the fluency of this exposition we shall henceforth assume that $\dim(X) > 0$, since the case $\dim(X) = 0$ often requires separate consideration, and unital, simple AF algebras are already well-understood. We proceed to introduce some notation.

For each $j \geq i$ let $\pi_{i,j}^{(s)}$ denote the s'th coordinate projection $X_j = X_i^{(n_j/n_i)} \to X_i$. Following standard notation, we set $\varphi_{i,j} := \varphi_{j-1} \circ \cdots \circ \varphi_i$, when $j > i$, set $\varphi_{i,i}$ to be the identity map on $M_{m_i} \otimes C(X_i)$, and $\varphi_{i,j}$ to be the zero map when $j < i$. It is easy to check that $\varphi_{i,j} \colon M_{m_i} \otimes C(X_i) \to M_{m_j} \otimes C(X_j)$ is a $\mathcal{V}$I map whenever $j > i$. For each $j > i$ let $E_{i,j}$ denote the set of eigenvalue maps of $\varphi_{i,j}$, and for each $\lambda \in E_{i,j}$ let $m(\lambda)$ denote the multiplicity of λ, i.e., the number of times λ occurs as an eigenvalue map of $\varphi_{i,j}$. Furthermore, let

$$E_{i,j}^{(1)} := \{\lambda \in E_{i,j} \mid \lambda \text{ is a coordinate projection}\},$$
$$E_{i,j}^{(2)} := \{\lambda \in E_{i,j} \mid \lambda \text{ is constant}\}.$$

We will refer to the eigenvalue maps $\lambda \in E_{i,j}^{(2)}$ as point evaluations. For each $i < j$ write $\varphi_{i,j} = \psi_{i,j} \oplus \chi_{i,j}$, where $\psi_{i,j}$ is the diagonal *-homomorphism corresponding to the eigenvalue maps of $\varphi_{i,j}$, which are contained in $E_{i,j}^{(1)}$, and $\chi_{i,j}$ is the diagonal *-homomorphism corresponding to the eigenvalue maps of $\varphi_{i,j}$, which are contained in $E_{i,j}^{(2)}$. Finally, we define the following numbers

$$N(i,j) := |E_{i,j}^{(1)}|, \quad \alpha(i,j) := \sum_{\lambda \in E_{i,j}^{(1)}} m(\lambda), \quad M(i,j) := \sum_{\lambda \in E_{i,j}} m(\lambda).$$

In other words, $M(i,j)$ denotes the multiplicity (number of eigenvalue maps) of $\varphi_{i,j}$, $\alpha(i,j)$ denotes the number of coordinate projections occurring in $\varphi_{i,j}$, while $N(i,j)$ denotes the number of *different* coordinate projections occurring in $\varphi_{i,j}$. Note that when $j > i$ we have

$$M(i,j) = M(i,j-1)M(j-1,j), \quad N(i,j) = N(i,j-1)N(j-1,j),$$
$$\alpha(i,j) = \alpha(i,j-1)\alpha(j-1,j),$$

and that $0 \leq \frac{N(i,j)}{M(i,j)} \leq \frac{\alpha(i,j)}{M(i,j)} \leq 1$. In particular, the sequences

$$\left(\frac{N(i,j)}{M(i,j)}\right)_{j>i} \quad \text{and} \quad \left(\frac{\alpha(i,j)}{M(i,j)}\right)_{j>i}$$

are decreasing and convergent. Furthermore, setting $c_i = \lim_{j\to\infty} \frac{N(i,j)}{M(i,j)}$ and $d_i = \lim_{j\to\infty} \frac{\alpha(i,j)}{M(i,j)}$, the sequences $(c_i)_i$ and $(d_i)_i$ are both increasing and $c_i \leq d_i$ for all $i \in \mathbb{N}$. In fact, it is easy to check that either $c_i = 0$ for all $i \in \mathbb{N}$ or $\lim_{i\to\infty} c_i = 1$. Similarly, either $d_i = 0$ for all i or $\lim_{i\to\infty} d_i = 1$ (see the proof of [36, Lemma 5.1]).

During the proof of Theorem 3.5 we need the following Chern class obstruction, essentially due to Villadsen, and later refined by Toms in [34],[35] and Toms–Winter in [36]. In the statement (and proof) of the lemma, we will use the following notation: given a finite cartesian power of spheres $(S^2)^n$, and $1 \leq j \leq n$, let $\rho_j \colon (S^2)^n \to S^2$ denote the j'th coordinate projection.

Lemma 3.4. *Let A be a Villadsen algebra which admits a standard decomposition (A_i, φ_i) with seed space a finite-dimensional CW-complex X of non-zero dimension. Furthermore, assume that, for some $i \in \mathbb{N}$, there exist $n \in \mathbb{N}$, a closed subset $X_i \supseteq K \cong (S^2)^n$ and a positive element $a \in A_i \otimes \mathbb{K}$,* *such that $a|_K$ is a projection for which the corresponding vector bundle ξ is of the form $\xi \cong \rho_1^*(\eta) \oplus \cdots \oplus \rho_n^*(\eta)$, where η is a (complex) line bundle over S^2 with non-zero Euler class $e(\eta)$. For each $j > i$ define a closed subset $K_{i,j} \subseteq X_j$ by*

$$K_{i,j} := \times_{s=1}^{n_j/n_i} K_{i,j}^{(s)},$$

where

$$K_{i,j}^{(s)} = \begin{cases} K, & \text{if } \pi_{i,j}^{(s)} \in E_{i,j}^{(1)}, \\ \{x_j\}, & \text{otherwise.} \end{cases}$$

and $x_j \in X_i$. Let ξ_j denote the vector bundle over $K_{i,j}$ corresponding to $\psi_{i,j}(a)|_{K_{i,j}}$. Then the $nN(i,j)$'th Chern class $c_{nN(i,j)}(\xi_j)$ is non-zero.

Proof. Note that $K_{i,j} \cong K^{N(i,j)} \cong (S^2)^{nN(i,j)}$. Since $a|_K$ is a projection, it follows from the definition of $\psi_{i,j}$, that $\psi_{i,j}(a)|_{K_{i,j}}$ is a projection. As in the statement above, let ξ denote the vector bundle corresponding to $a|_K$ and ξ_j the vector bundle corresponding to $\psi_{i,j}(a)|_{K_{i,j}}$. We easily deduce that

$$\xi_j \cong \bigoplus_{\lambda \in E_{i,j}^{(1)}} \bigoplus_{j=1}^{m(\lambda)} \lambda^*(\xi).$$

Applying the Chern class to this equation, and using the product formula, we obtain

$$c(\xi_j) = \prod_{\lambda \in E_{i,j}^{(1)}} \prod_{j=1}^{m(\lambda)} c(\lambda^*(\xi)) = \prod_{\lambda \in E_{i,j}^{(1)}} \lambda^*\big(c(\xi)\big)^{m(\lambda)}.$$

Write $E_{i,j}^{(1)} = \{\lambda_1, \lambda_2, \ldots, \lambda_{N(i,j)}\}$. For $l = 1, \ldots, N(i,j)$ and $s = 1, \ldots, n$ set $z_{l,s} := \lambda_l^*\big(\rho_s^*(e(\eta))\big)$. Since $e(\eta)^2 = 0$ (recall that $H^j(S^2) = 0$ for all $j > 2$), we find that $z_{l,s}^m = 0$ for l, s and $m > 1$. By assumption, $\xi \cong \rho_1^*(\eta) \oplus \cdots \oplus \rho_n^*(\eta)$, whence

$$c(\xi_j) = \prod_{l=1}^{N(i,j)} \prod_{s=1}^{n} (1 + z_{l,s})^{m(\lambda_l)} = \prod_{l=1}^{N(i,j)} \prod_{s=1}^{n} (1 + m(\lambda_l) z_{l,s}).$$

Given a subset $S \subseteq \{1, \ldots, n\}$ let $z_{l,S} := \prod_{s \in S} m(\lambda_l) z_{l,s}$ when $S \neq \emptyset$ and $z_{l,\emptyset} := 1$ for all $1 \leq l \leq N(i,j)$. It follows from the above computation that, for $1 < q \leq \operatorname{rank}(\xi_j)$, the q'th Chern class $c_q(\xi_j)$ can be computed as $\sum \prod_{l=1}^{N(i,j)} z_{l,S_l}$, where the sum ranges over all families $\{S_l\}_l$ of subsets $S_l \subseteq \{1, \ldots, n\}$ such that $\sum_{l=1}^{N(i,j)} |S_l| = q$. Now, supposing that $\{S_l\}_l$ is a family of subsets $S_l \subseteq \{1, \ldots, n\}$ such that $S_{l_0} \neq \{1, \ldots, n\}$ for some l_0, it follows that $\sum_{l=1}^{N(i,j)} |S_l| < nN(i,j)$. In particular, we find that

$$c_{nN(i,j)}(\xi_j) = \prod_{l=1}^{N(i,j)} z_{l,\{1,\ldots,n\}} = \prod_{l=1}^{N(i,j)} \prod_{s=1}^{n} m(\lambda_l) z_{l,s}.$$

It therefore follows from the Künneth formula that $c_{nN(i,j)}(\xi_j) \neq 0$. □

The following theorem is the main result of this section. The proof is based on the proof of [36, Lemma 4.1]. However, since the statement of the following theorem is different, the proof needs to be modified, and in the interest of clarity of the exposition, we include a full proof.

Theorem 3.5. *Let A be a simple Villadsen algebra of the first type which admits a standard decomposition (A_i, φ_i) with seed space a finite-dimensional CW-complex. Then A has strict comparison (and hence $A \otimes \mathcal{Z} \cong A$) if and only if $F(A)$ has no characters.*

Proof. Assume A has strict comparison. Then it follows from Theorem 3.3 that $A \otimes \mathcal{Z} \cong A$, whence there exists a unital embedding $\mathcal{Z} \to F(A)$. Since $\mathcal{Z}$ has no characters it follows that $F(A)$ does not admit a character either. We show, using Proposition 2.1, that $F(A)$ has at least one character if A does not have strict comparison.

Fix $n \geq 2$. Since A does not have strict comparison it follows from [36, Lemma 5.1] that

$$\lim_{i\to\infty} \lim_{j\to\infty} \frac{N(i,j)}{M(i,j)} = 1. \tag{1}$$

Note that since $\dim(X) > 0$ and A is simple, the number of point evaluations occurring as eigenvalue maps in $\varphi_{i,j}$ is unbounded as $j \to \infty$ for any $i \in \mathbb{N}$. In particular, $M(i,j) \to \infty$ as $j \to \infty$, whence (1) implies $\dim(X_i) \to \infty$ as $i \to \infty$. Hence, we may choose $i \in \mathbb{N}$ such that $\dim(X_i) \geq 3n$ and

$$\frac{N(i,j)}{M(i,j)} \geq \frac{2n-1}{2n}, \quad \text{for all } j > i. \tag{2}$$

Choose an open subset $O \subseteq X_i$ such that $O \cong (-1,1)^{\dim(X_i)} =: D$. Let

$$\overline{Y} := \{x \in (-1,1)^3 \mid \operatorname{dist}\big(x, (0,0,0)\big) = 1/2\}$$

and

$$\overline{Z} := \{x \in (-1,1)^3 \mid 1/3 \leq \operatorname{dist}\big(x, (0,0,0)\big) \leq 2/3\}.$$

Furthermore, define closed subsets

$$K := \overline{Y}^{\times n} \times \{0\}^{\dim(X_i)-3n} \subseteq D$$

and

$$Z := \overline{Z}^{\times n} \times [-4/5, 4/5]^{\dim(X_i)-3n} \subseteq D.$$

Let Z_0 denote the interior of Z and note that $K \subseteq Z_0$. We identify K and Z with their homeomorphic images in X_i and note that $K \cong (S^2)^n$. For each $l = 1, \ldots, n$, let $\rho_l : (S^2)^n \to S^2$ denote the l'th coordinate projection. Choose some line bundle η over S^2 with non-zero Euler class $e(\eta)$ (for instance the Hopf bundle), and set $\eta_l := \rho_l^*(\eta)$. We consider each η_l to be a vector bundle over K. Furthermore, let θ_2 denote the trivial vector bundle of fibre dimension 2 over K. It follows from [15, Proposition 9.1.2] that $\theta_2 \precsim \eta_l \oplus \eta_l \oplus \eta_l$, for each $l = 1, \ldots, n$, while $\theta_2 \not\precsim \bigoplus_{l=1}^n \eta_l$, since the Euler class of the right hand vector bundle is non-zero. We aim to construct positive elements in A such that the above relationships between vector bundles persist in $\mathrm{Cu}(A)$.

Let $\mathrm{pr}\colon \overline{Z} \to \overline{Y}$ be the projection along rays emanating from the origin and let $f\colon X_i \to \mathbb{C}$ be a continuous map satisfying $f|_K \equiv 1$ and $f|_{X_i \setminus Z_0} \equiv 0$. Let $P\colon Z \to K$ be given by

$$P = \underbrace{\mathrm{pr} \times \cdots \times \mathrm{pr}}_{n \text{ times}} \times \underbrace{\mathrm{ev}_0 \times \cdots \times \mathrm{ev}_0}_{\dim(X_i)-3n \text{ times}},$$

where $\mathrm{ev}_0(z) = 0$ for any $z \in (-1, 1)$. For each $l = 1, \ldots, n$, let $p_l \in C(Z, \mathbb{K})$ denote the projection corresponding to $P^*(\eta_l)$ and let $p' \in C(Z, \mathbb{K})$ denote the projection corresponding to $P^*(\theta_2)$. Define elements $b_l, a \in A_i$, for $l = 1, \ldots, n$, by $b_l := f \cdot p_l$ and $a := f \cdot p'$. Since $f \in A_i$ is central, and $p' \precsim p_l \oplus p_l \oplus p_l$ for each $l = 1, \ldots, n$, it easily follows that $a \precsim b_l \oplus b_l \oplus b_l$, for each $l = 1, \ldots, n$. Let

$$x := \langle \varphi_{i,\infty}(a) \rangle \in \mathrm{Cu}(A), \quad y_l := \langle \varphi_{i,\infty}(b_l) \rangle \in \mathrm{Cu}(A), \text{ for } l = 1, \ldots, n.$$

Clearly $x \leq 3y_l$ for $l = 1, \ldots, n$. To finish the proof we need to show $x \not\leq y_1 + y_2 + \cdots + y_n$, and then Proposition 2.1 (with $m = 3$) will yield the desired result.

Letting a be given as above and $b = \bigoplus_{l=1}^n b_l \in (A_i \otimes \mathbb{K})_+$, we aim to show that $\varphi_{i,\infty}(a) \not\precsim \varphi_{i,\infty}(b)$ in $A \otimes \mathbb{K}$. It suffices to prove that

$$\|v^* \varphi_{i,j}(b) v - \varphi_{i,j}(a)\| \geq \frac{1}{2},$$

for each $j > i$ and $v \in A_j \otimes \mathbb{K}$. Note that $\chi_{i,j}(b)$ is a constant, positive, matrix valued function, whence $q := \lim_{n\to\infty} \chi_{i,j}(b)^{1/n} \in A_j \otimes \mathbb{K}$ is a constant projection such that $\chi_{i,j}(b) q = \chi_{i,j}(b)$. Setting $Q := \psi_{i,j}(\mathbf{1}) \oplus \chi_{i,j}(b)^{1/2}$, we have

$$\varphi_{i,j}(b) = \psi_{i,j}(b) \oplus \chi_{i,j}(b) = Q(\psi_{i,j}(b) \oplus q)Q. \tag{3}$$

Now, let $j > i$ be given and suppose for a contradiction, that there exists $v \in A_j \otimes \mathbb{K}$ such that $\|v^* \varphi_{i,j}(b) v - \varphi_{i,j}(a)\| < 1/2$. Then, setting $w := Qv\psi_{i,j}(\mathbf{1}_{A_i})$, it follows from (3) that

$$\frac{1}{2} > \|v^* Q(\psi_{i,j}(b) \oplus q) Q v - \varphi_{i,j}(a)\| \geq \|w^*(\psi_{i,j}(b) \oplus q) w - \psi_{i,j}(a)\|. \tag{4}$$

This estimate remains valid upon restriction to any closed subset of X_j.

Let ξ denote the vector bundle over K corresponding to $b|_K$. Plug A, X, X_i, b, K and ξ into Lemma 3.4 to get $K_{i,j} \subseteq X_j$ and ξ_j. Note that $b|_K = (b_1|_K) \oplus \cdots \oplus (b_n|_K)$, whence $\xi \cong \rho_1^*(\eta) \oplus \cdots \oplus \rho_n^*(\eta)$, and therefore the hypothesis of Lemma 3.4 are satisfied. It is easily deduced that $q|_{K_{i,j}}$ corresponds to a trivial vector bundle θ_{nr}, where $0 \leq r \leq M(i,j) - \alpha(i,j)$, and since $a|_K \in C(K) \otimes \mathbb{K}$ is a constant projection valued function of rank 2 it follows that $\psi_{i,j}(a)|_{K_{i,j}}$ corresponds to the trivial vector bundle $\theta_{2\alpha(i,j)}$. It therefore follows from (4) and [34, Lemma 2.1] that there exists a vector bundle ζ of fibre dimension $(n-2)\alpha(i,j) + nr$ and $t \in \mathbb{N}$ such that

$$\zeta \oplus \theta_{2\alpha(i,j)+t} \cong \xi_j \oplus \theta_{nr+t}.$$

Applying the Chern class to both sides of the above expression, we obtain that $c(\zeta) = c(\xi_j)$. In particular, $c_{nN(i,j)}(\zeta) = c_{nN(i,j)}(\xi_j)$, whence Lemma 3.4

implies that $c_{nN(i,j)}(\zeta)$ is non-zero. Hence $\mathrm{rank}(\zeta) \geq nN(i,j)$, and therefore

$$\begin{aligned} nN(i,j) &\leq (n-2)\alpha(i,j) + nr \\ &\leq (n-2)\alpha(i,j) + n(M(i,j) - \alpha(i,j)) \\ &\leq nM(i,j) - 2N(i,j). \end{aligned}$$

Thus, dividing both sides by $nM(i,j)$ we obtain

$$\frac{N(i,j)}{M(i,j)} \leq 1 - \frac{2}{n} \cdot \frac{N(i,j)}{M(i,j)}.$$

Hence (2) implies

$$\frac{2n-1}{2n} \leq 1 - \frac{2(2n-1)}{n(2n)} = \Big(\frac{n-1}{n}\Big)^2 < \frac{n-1}{n},$$

which is the desired contradiction. □

Before considering Villadsen algebras of the second type let us record the following proposition, which is an aggregation of results by other authors. However, it does serve to illustrate the added complexity of Villadsen algebras of the second type (compare with Theorem 4.2), which are less studied than those of the first type.

Proposition 3.6. *Suppose A is a simple Villadsen algebra which admits a standard decomposition with seed space a finite dimensional CW-complex of non-zero dimension. Then A has real rank zero if and only if A has a unique tracial state. Furthermore, in this case, $A \otimes \mathcal{Z} \cong A$.*

Proof. The proof that real rank zero implies unique tracial state is essentially contained in [36, Proposition 7.1]. Indeed, replacing every instance of $N(i,j)$ in the cited proof with $\alpha(i,j)$, it follows that if $RR(A) = 0$, then $\lim_{j\to\infty} \frac{\alpha(i,j)}{M(i,j)} = 0$ for all $i \in \mathbb{N}$. It is easy to check that this implies that A has a unique tracial state. Furthermore, the statement that A is $\mathcal{Z}$ stable follows from [36, Proposition 7.1] and a series of results summarized in [36, Theorem 3.4].

On the other hand, assuming A has a unique tracial state, it follows from [25, Theorem 1.1] that A has slow dimension growth. There is a simpler proof for $\mathcal{V}$I algebras, which we omit to keep the exposition at a reasonable length. Therefore, [3, Theorem 2] implies that A has real rank zero. □

4. Villadsen Algebras of the second type

In this section we study the Villadsen algebras of the second type. We prove that for each Villadsen algebra A of the second type, $F(A)$ has at least one character. For the convenience of the reader we recall the construction from [38]

Definition 4.1. Let X, Y be compact Hausdorff spaces. A *-homomorphism $\varphi\colon C(X) \otimes \mathbb{K} \to C(Y) \otimes \mathbb{K}$ is said to be a *diagonal map of the second type* if there exists $k \in \mathbb{N}$, continuous maps $\lambda_1, \ldots, \lambda_k\colon Y \to X$, and mutually orthogonal projections $p_1, \ldots, p_k \in C(Y) \otimes \mathbb{K}$ such that

$$\varphi = (\mathrm{id}_{C(Y)} \otimes \alpha) \circ (\tilde{\varphi} \otimes \mathrm{id}_{\mathbb{K}}),$$

where $\alpha\colon \mathbb{K}\otimes\mathbb{K}\to\mathbb{K}$ is some isomorphism and $\tilde{\varphi}\colon C(X)\to C(Y)\otimes\mathbb{K}$ is given by

$$\tilde{\varphi}(f)=\sum_{i=1}^{k}(f\circ\lambda_i)p_i.$$

In this case, we say φ arises from the tuple $(\lambda_i,p_i)_{i=1}^k$, and the maps λ_i, $i=1,\dots,k$, are referred to as the *eigenvalue maps* of φ.

Note that in the above definition we have implicitly used that the C^*-algebra $C(X)\otimes\mathbb{K}$ has a natural $C(X)$-module structure. Since all diagonal maps appearing from this point on will be of the second type defined above, we simply refer to them as diagonal maps.

For each $l\in\mathbb{N}$, let $\mathbb{C}P^l$ denote the l'th complex projective space, let γ_l denote the universal line bundle over $\mathbb{C}P^l$, and let $\mathbb{D}^l$ denote the l-fold cartesian product of the unit disc $\mathbb{D}\subseteq\mathbb{C}$. It is well-known that the l-fold cup product $e(\gamma_l)^l$ of the Euler class $e(\gamma_l)$ is non-zero for all $l\in\mathbb{N}$. For each integer $n\geq 1$, let $\sigma(n):=n(n!)$ and $\sigma(0):=1$. Furthermore, let $\mathbb{N}_\infty=\mathbb{N}\cup\{\infty\}$ and let $\kappa:\mathbb{N}_\infty\times\mathbb{N}\to\mathbb{N}$ be given by

$$\kappa(k,n)=\begin{cases}k\sigma(n), & \text{if } k<\infty,\\ n\sigma(n), & \text{if } k=\infty.\end{cases}$$

For all integers $k\geq 1$ and $n\geq 0$, define compact Hausdorff spaces $X_n^{(k)}$ by $X_0^{(k)}:=\mathbb{D}^k$ and

$$X_n^{(k)}:=\mathbb{D}^k\times\mathbb{C}P^{\kappa(k,1)}\times\mathbb{C}P^{\kappa(k,2)}\times\cdots\times\mathbb{C}P^{\kappa(k,n)},$$

when $n\geq 1$. Also, for $k=\infty$, we set $X_0^{(k)}:=\mathbb{D}$ and

$$X_n^{(k)}:=\mathbb{D}^{n\sigma(n)^2}\times\mathbb{C}P^{\kappa(k,1)}\times\mathbb{C}P^{\kappa(k,2)}\times\cdots\times\mathbb{C}P^{\kappa(k,n)}.$$

Thus $X_n^{(k)}=X_{n-1}^{(k)}\times\mathbb{C}P^{k\sigma(n)}$, whenever $k<\infty$ and $n\geq 1$, and

$$X_1^{(\infty)}:=X_0^{(\infty)}\times\mathbb{C}P^1;$$
$$X_n^{(\infty)}:=\mathbb{D}^{n\sigma(n)^2-(n-1)\sigma(n-1)^2}\times X_{n-1}^{(\infty)}\times\mathbb{C}P^{n\sigma(n)},\quad n\geq 2.$$

For each $k\in\mathbb{N}_\infty$ and $n\in\mathbb{N}$, let

$$\pi_{k,n}^1:X_n^{(k)}\to X_{n-1}^{(k)},\qquad \pi_{k,n}^2:X_n^{(k)}\to\mathbb{C}P^{\kappa(k,n)},$$

denote the coordinate projections, and set $\zeta_n^{(k)}:=\pi_{k,n}^{2*}(\gamma_{\kappa(k,n)})$. If $y_0\in X_n^{(k)}$ is a point, we also let y_0 denote the constant map $f\colon X_{n+1}^{(k)}\to X_n^{(k)}$ with $f(x)=y_0$ for all $x\in X_{n+1}^{(k)}$.

For each $k\in\mathbb{N}_\infty$ and integer $n\geq 0$, let $\tilde{\varphi}_n^{(k)}:C(X_n^{(k)})\otimes\mathbb{K}\to C(X_{n+1}^{(k)})\otimes\mathbb{K}$ be the diagonal map arising from the tuple $(\pi_{k,n+1}^1,\theta_1)\cup(y_{n,j}^{(k)},\zeta_{n+1}^{(k)})_{j=1}^{n+1}$, where the points $\{y_{n,j}^{(k)}\}_{j=1}^{n+1}\subseteq X_n^{(k)}$ are chosen such that the resulting C^*-algebra is simple (see [38] for more details) and θ_1 denotes the trivial line bundle. Let $p_0^{(k)}\in C(X_0^{(k)})\otimes\mathbb{K}$ denote a constant projection of rank 1 and $p_n^{(k)}:=\tilde{\varphi}_{n,0}^{(k)}(p_0^{(k)})$. Furthermore, let

$$A_n^{(k)}:=p_n^{(k)}\big(C(X_n^{(k)})\otimes\mathbb{K}\big)p_n^{(k)},$$

and $\varphi_n^{(k)} := \tilde{\varphi}_n^{(k)}|_{A_n^{(k)}}$. Define $\mathcal{V}_k$ to be the inductive limit of the system $(A_n^{(k)}, \varphi_n^{(k)})$. The following results about the C^*-algebras $\mathcal{V}_k$ may be found in [38].

Theorem 4.2 (Villadsen)**.** *For each $k \in \mathbb{N}_\infty$, let $\mathcal{V}_k$ be defined as above.*

(i) *The C^*-algebra $\mathcal{V}_k$ has a unique tracial state τ, for each $k \in \mathbb{N}_\infty$.*
(ii) *The stable rank* $\mathrm{sr}(\mathcal{V}_k)$ *of $\mathcal{V}_k$ is $k+1$, when $k < \infty$, and infinite, when $k = \infty$.*
(iii) *The real rank* $\mathrm{RR}(\mathcal{V}_k)$ *of $\mathcal{V}_k$ satisfies $k \leq \mathrm{RR}(\mathcal{V}_k) \leq k+1$, when $k < \infty$, and is infinite, when $k = \infty$.*

It is easy to check that, if η is an arbitrary vector bundle over $X_i^{(k)}$, then

$$\big(\varphi_i^{(k)}\big)^*(\eta) \cong \pi_{k,i+1}^{1*}(\eta) \oplus (i+1)\mathrm{rank}(\eta)\zeta_{i+1}^{(k)}, \tag{5}$$

where $(\varphi_i^{(k)})^*$ denotes the map from (isomorphism classes of) vector bundles over $X_i^{(k)}$ to (isomorphism classes of) vector bundles over $X_{i+1}^{(k)}$ induced by $\varphi_i^{(k)}$. For each $k, n \in \mathbb{N}$ let $\xi_i^{(k)}$ denote the vector bundle over $X_i^{(k)}$ corresponding to $p_i^{(k)}$. Then (5) implies that

$$\xi_i^{(k)} \cong \theta_1 \times \sigma(1)\gamma_{\kappa(k,1)} \times \cdots \times \sigma(i)\gamma_{\kappa(k,i)}. \tag{6}$$

A brief word on notation: as before, for each $i < j$ and $k \in \mathbb{N}_\infty$, we let $\varphi_{i,j}^{(k)} \colon A_i^{(k)} \to A_j^{(k)}$ and $\varphi_{i,\infty}^{(k)} \colon A_i^{(k)} \to \mathcal{V}_k$ denote the induced maps from the inductive limit decomposition. We will often omit the superscript (k) in the following (whenever k is implied by the context).

Proposition 4.3. *Let $k \in \mathbb{N}_\infty$ be given. For each $n \in \mathbb{N}$ there exist projections $e_n, q_1^{(n)}, \ldots, q_n^{(n)} \in \mathcal{V}_k \otimes \mathbb{K}$ such that*

(i) *$e_n \precsim q_i^{(n)} \oplus q_i^{(n)}$, for all $i = 1, \ldots, n$.*
(ii) *$e_n \not\precsim q_1^{(n)} \oplus \cdots \oplus q_n^{(n)}$.*
(iii) *$\tau(q_1^{(n)} \oplus q_2^{(n)} \oplus \cdots \oplus q_n^{(n)}) \to k$ and $\tau(e_n) \to 0$ as $n \to \infty$.*

Proof. We fix an arbitrary $k \in \mathbb{N}_\infty$, and omit k from our notation. For each $l \in \mathbb{N}$ and $j = 1, \ldots, l$, let $\rho_j^l \colon X_l = X_l^{(k)} \to \mathbb{C}P^{\kappa(k,j)}$ denote the coordinate projection. Note that $\rho_l^l = \pi_{k,l}^2$ and $\rho_j^l \circ \pi_{k,l+1}^1 = \rho_j^{l+1}$ for all $l \geq 1$ and $1 \leq j \leq l$. For each $n \in \mathbb{N}$ and $i = 1, \ldots, n$, let $\overline{q}_i^{(n)} \in A_n \otimes \mathbb{K}$ denote the projection corresponding to the vector bundle $\eta_{n,i} := \rho_i^{n*}(\kappa(k,i) \cdot \gamma_{\kappa(k,i)})$ over X_n, where $\gamma_{\kappa(k,i)}$ is as defined above, and $r_n \in A_n \otimes \mathbb{K}$ denote the projection corresponding to the trivial line bundle θ_1. Let $q_i^{(n)} := \varphi_{n,\infty}(\overline{q}_i^{(n)})$ and $e_n := \varphi_{n,\infty}(r_n)$. We prove that the projections $e_n, q_1^{(n)}, \ldots, q_n^{(n)}$ have the properties claimed in the above statement. In the interest of brevity, let

$$\eta_n := \eta_{n,1} \oplus \eta_{n,2} \oplus \cdots \oplus \eta_{n,n}.$$

It follows from the Künneth formula, and the fact $e(\gamma_l)^l \neq 0$ for all $l \in \mathbb{N}$, that the Euler class $e(\eta_n) \in H^*(X_n)$ is non-zero for each $n \in \mathbb{N}$.

(i): It suffices to prove that $2\kappa(k,i) \cdot \gamma_{\kappa(k,i)}$ dominates a trivial line bundle for each $i \in \mathbb{N}$. However, this follows from straightforward dimension considerations. Indeed, since

$$2 \cdot \mathrm{rank}(2\kappa(k,i) \cdot \gamma_{\kappa(k,i)}) - 1 \geq 2\kappa(k,i) = \dim(\mathbb{C}P^{\kappa(k,i)}),$$

the desired result follows (see for instance [15, Proposition 9.1.1]).

(ii): Note that it follows from (5) that

$$\varphi_l^*(\eta_l) \cong \Big(\bigoplus_{j=1}^{l} \kappa(k,j) \cdot \rho_j^{(l+1)*}(\gamma_{\kappa(k,j)})\Big) \oplus (l+1)\mathrm{rank}(\eta_l) \cdot \rho_{l+1}^{(l+1)*}(\gamma_{\kappa(k,l+1)}).$$

Since $(l+1)\mathrm{rank}(\eta_l) = (l+1)\sum_{i=1}^{l} \kappa(k,i) \leq \kappa(k,l+1)$, it follows that $\varphi_l^*(\eta_l) \precsim \eta_{l+1}$. By induction, $\varphi_{l,m}^*(\eta_l) \precsim \eta_m$, for all $m \geq l$. Furthermore, again by (5), we have that $\theta_1 \precsim \varphi_{l,m}^*(\theta_1)$.

Now, assume that $e_n \precsim q_1^{(n)} \oplus \cdots \oplus q_n^{(n)}$. Since e_n is compact in $\mathrm{Cu}(\mathcal{V}_k)$ it follows from continuity of $\mathbf{Cu}(-)$ that there exists some $m > n$ such that

$$\theta_1 \precsim \varphi_{n,m}^*(\theta_1) \precsim \varphi_{n,m}^*(\eta_n) \precsim \eta_m.$$

But since the Euler class of the right hand side is non-zero, this is a contradiction.

(iii): Recall that ξ_n denotes the vector bundle over X_n corresponding to the unit $p_n \in A_n$. Since each $q_i^{(n)}$ is a projection and $\varphi_{i,\infty}$ is unital, we have

$$\begin{aligned}\tau(q_1^{(n)} \oplus q_2^{(n)} \oplus \cdots \oplus q_n^{(n)}) &= \frac{\mathrm{rank}(\eta_n)}{\mathrm{rank}(\xi_n)} = \frac{\sum_{l=1}^{n} \kappa(k,l)}{\sum_{l=0}^{n} \sigma(l)} \\ &= \frac{\sum_{l=1}^{n} \kappa(k,l)}{(n+1)!}.\end{aligned}$$

Hence, when $k < \infty$,

$$\tau(q_1^{(n)} \oplus q_2^{(n)} \oplus \cdots \oplus q_n^{(n)}) = \frac{\big(k\sum_{l=0}^{n} \sigma(l)\big) - k}{(n+1)!} = \frac{k(n+1)! - k}{(n+1)!} \to k,$$

while the case $k = \infty$ follows from the observation that

$$\frac{\sum_{l=1}^{n} l\sigma(l)}{(n+1)!} \geq \frac{n\sigma(n)}{(n+1)!} = \frac{n^2}{(n+1)} \to \infty.$$

Similarly, for arbitrary $1 \leq k \leq \infty$ we find that $\tau(e_n) = \frac{1}{(n+1)!} \to 0$. □

Corollary 4.4. *For each $1 \leq k \leq \infty$, the central sequence algebra $F(\mathcal{V}_k)$ has at least one character.*

Proof. This is a straightforward consequence of Proposition 4.3 parts (i) and (ii) and Proposition 2.1 (with $m = 2$). □

Remark 4.5. An alternative, albeit slightly artificial, statement of the above corollary is that for each $1 \leq k \leq \infty$, the k'th Villadsen algebra of the second type $\mathcal{V}_k$ absorbs $\mathcal{Z}$ if and only if $F(\mathcal{V}_k)$ has no characters. Indeed, it follows from [38, Proposition 11] that $K_0(\mathcal{V}_k)$ is not weakly unperforated for any $1 \leq k < \infty$, and essentially the same proof applies to $k = \infty$. Hence [12, Theorem 1] implies that $\mathcal{V}_k \otimes \mathcal{Z} \ncong \mathcal{V}_k$ for each $1 \leq k \leq \infty$. As stated in the above corollary, $F(\mathcal{V}_k)$ has at least one character for each $1 \leq k \leq \infty$, whence the desired result follows.

Remark 4.6. It was proven in [17] that if A is a unital C^*-algebra with $T(A) \neq \emptyset$ and property (SI), then $F(A)$ has a character if and only if $F(A)/(F(A) \cap J(A))$ has a character (see [17] for a definition of $J(A)$). It follows from the above corollary that this is no longer true, if the assumption of property (SI) is removed. Indeed, let $1 \leq k \leq \infty$ be arbitrary and $\mathscr{N}_k$ denote the weak closure of $\pi_\tau(\mathcal{V}_k) \subseteq B(\mathcal{H}_\tau)$, where π_τ denotes the GNS representation of $\mathcal{V}_k$ with respect to the tracial state τ. Since $\mathcal{V}_k$ has a unique tracial state, it is a straightforward consequence of [30, Lemma 2.1] that

$$F(\mathcal{V}_k)/(F(\mathcal{V}_k) \cap J(\mathcal{V}_k)) \cong \mathscr{N}_k^\omega \cap \mathscr{N}_k'.$$

Here $\mathscr{N}_k^\omega$ denotes the von Neumann ultrapower of $\mathscr{N}_k$ with respect to the tracial state τ. Since $\mathscr{N}_k$ is an injective II_1-factor, it follows that $\mathscr{N}_k \cong \mathcal{R}$, where $\mathcal{R}$ denotes the hyperfinite II_1 factor. In particular, there exists a unital embedding $\mathcal{R} \to F(\mathcal{V}_k)/(F(\mathcal{V}_k) \cap J(\mathcal{V}_k))$ whence $F(\mathcal{V}_k)/(F(\mathcal{V}_k) \cap J(\mathcal{V}_k))$ does not have any characters. Hence, the above corollary shows that the assumption of property (SI) in [17, Proposition 3.19] is indeed necessary.

Proposition 4.3 (iii) allows us to compute the radius of comparison for each $\mathcal{V}_k$ (the radius of comparison was originally defined by Toms in [33], and an extended definition was given in [4] and shown to agree with the original definition for all sufficiently finite C^*-algebras, e.g., unital, simple and stably finite C^*-algebras).

Corollary 4.7. $\mathrm{rc}(\mathcal{V}_k) = k$ *for each* $1 \leq k < \infty$.

Proof. Fix $1 \leq k < \infty$. By [35, Corollary 5.2] and [4, Proposition 3.2.4]

$$\mathrm{rc}(\mathcal{V}_k) \leq \lim_{n\to\infty} \frac{\dim(X_n^{(k)})}{2 \cdot \mathrm{rank}(p_n^{(k)})} = \lim_{n\to\infty} \frac{k(n+1)!}{(n+1)!} = k.$$

Fix arbitrary $\varepsilon > 0$. By Proposition 4.3 parts (ii) and (iii) we may choose projections $e, q \in \mathcal{V}_k \otimes \mathbb{K}$ such that $\tau(e) < \varepsilon/2$, $\tau(q) > k - \varepsilon/2$, while $e \not\precsim q$. In particular $d_\tau(e) + (k - \varepsilon) = \tau(e) + k - \varepsilon < k - \varepsilon/2 < d_\tau(q)$, while $e \not\precsim q$. Since $\varepsilon > 0$ was arbitrary, it follows that $\mathrm{rc}(\mathcal{V}_k) \geq k$. □

The proof of the above corollary can easily be modified to show that $\mathrm{rc}(\mathcal{V}_\infty) = \infty$ (or even that $r_{\mathcal{V}_\infty,\infty} = \infty$; see [4] for a definition of $r_{\mathcal{V}_\infty,\infty}$), but as evidenced by Theorem A.1, in this case a stronger statement holds

Appendix A. The failure of the Corona Factorization Property for the Villadsen algebra $\mathcal{V}_\infty$

By Joan Bosa[1] and Martin S. Christensen

In this appendix we prove that the Villadsen algebra $\mathcal{V}_\infty$ does not satisfy the Corona Factorization Property (CFP), thereby improving the result, from an earlier version of this paper, that $\mathcal{V}_\infty$ does not satisfy the ω-comparison property.

Both ω-comparison and the CFP may be regarded as comparison properties of the Cuntz semigroup invariant, and both properties are related to the question of when a given C*-algebra is stable (see for instance [26, Proposition 4.8]). In particular, a simple, separable C^*-algebra A has the *CFP* if and only if, whenever $x, y_1, y_2, \dots$ are elements in $\mathrm{Cu}(A)$ and $m \geq 1$ is an integer satisfying $x \leq m y_j$ for all $j \geq 1$, then $x \leq \sum_{i=1}^{\infty} y_i$ ([26, Theorem 5.13]). On the other hand, given a simple C^*-algebra A, $\mathrm{Cu}(A)$ has *ω-comparison* if and only if $\infty = x \in \mathrm{Cu}(A)$ whenever $f(x) = \infty$ for all functionals f on $\mathrm{Cu}(A)$ ([5, Proposition 5.5]). Recall that a *functional* f on the Cuntz semigroup $\mathrm{Cu}(A)$ of a C^*-algebra A is an ordered semigroup map $f\colon \mathrm{Cu}(A) \to [0, \infty]$ which preserves suprema of increasing sequences. In particular, the latter comparability condition is satisfied for all unital C^*-algebras A with finite radius of comparison by [4].

From the above characterization it follows that any separable C^*-algebra A whose Cuntz semigroup $\mathrm{Cu}(A)$ has the ω-comparison property also has the CFP (see [26, Proposition 2.17]). Whether the converse implication is true remains an open question. This question was considered by the first author of this appendix and Petzka in [5], where the failure of the converse implication was shown just in the algebraic framework of the category Cu. However, it was emphasized there that a more analytical approach will be necessary in order to verify (or disprove) the converse implication for any (simple) C^*-algebra A.

The Villadsen algebras have been used several times to certify bizarre behaviour in the theory of C*-algebras; hence, after Ng and Kucerosvky showed in [20] that $\mathcal{V}_2$ satisfies the CFP, one wonders whether it satisfies the ω-comparison or not. From Corollary 4.7 (together with [4, Theorem 4.2.1]) one gets that, for all $1 \leq n < \infty$, the Cuntz semigroups $\mathrm{Cu}(\mathcal{V}_n)$ have the ω-comparison property and hence the CFP. But this is not the case for the C^*-algebra $\mathcal{V}_\infty$. As demonstrated below, it does not have the CFP (and hence $\mathrm{Cu}(\mathcal{V}_\infty)$ does not have ω-comparison). Notice that although $\mathcal{V}_\infty$ has a different structure than $\mathcal{V}_n$, it does not witness the potential non-equivalence of ω-comparison and the CFP for unital, simple and stably finite C^*-algebras.

Theorem A.1. *Let $\mathcal{V}_\infty$ be given as above. Then $\mathcal{V}_\infty$ is a unital, simple, separable and nuclear C^*-algebras with a unique tracial state such that the Cuntz semigroup $\mathrm{Cu}(\mathcal{V}_\infty)$ does not have the Corona Factorization Property for semigroups.*

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Proof. We use the notation introduced above, with $k=\infty$ fixed and omitted. Additionally, for each $n\geq 1$, let $\lambda(n) := \kappa(\infty,n) = n\sigma(n) = n^2(n!)$, and $Y_n := \mathbb{C}P^{\lambda(1)}\times\cdots\times\mathbb{C}P^{\lambda(n)}$. Note that $X_n = \mathbb{D}^{n\sigma(n)^2}\times Y_n$, let $\overline{\pi}_n\colon X_n\to Y_n$ denote the coordinate projection and $\overline{\psi}_n\colon C(Y_n)\otimes\mathbb{K}\to C(X_n)\otimes\mathbb{K}\cong A_n\otimes\mathbb{K}$ denote the *-homomorphism induced by $\overline{\pi}$.

For each $n\geq 1$ and $1\leq j\leq n$, let $\rho_{n,j}\colon Y_n\to\mathbb{C}P^{\lambda(j)}$ denote the projection map and let $\overline{\zeta}_{n,j}$ denote the vector bundle $\rho^*_{n,j}(\gamma_{\lambda(j)})$ over Y_n. To avoid overly cumbersome notation, we simply write $\overline{\zeta}_j$ for $\overline{\zeta}_{n,j}$ whenever $j\leq n$. Furthermore, for each $n\geq 1$, let $\overline{\xi}_n$ denote the vector bundle over Y_n given by $\theta_1\oplus\sigma(1)\overline{\zeta}_1\oplus\cdots\oplus\sigma(n)\overline{\zeta}_n$. Recall that, for each $j\geq 1$, ζ_j denotes the vector bundle $\pi_j^{2*}(\gamma_{\lambda(j)})$ over X_j. To avoid overly cumbersome notation we also let ζ_j denote the vector bundle $\pi_n^{1*}\circ\cdots\circ\pi_{j+1}^{1*}(\zeta_j)$, whenever $n>j$. With this notation, the vector bundle ξ_n over X_n corresponding to the unit $p_n\in A_n$ may be written $\xi_n\cong\theta_1\oplus\sigma(1)\zeta_1\oplus\cdots\oplus\sigma(n)\zeta_n$. It is immediately verified that $\overline{\psi}_n(\overline{\zeta}_j)\cong\zeta_j$ for all $j\leq n$, and in particular $\psi_n^*(\overline{\xi}_n)\cong\xi_n$. Hence, if $q\in C(Y_n)\otimes\mathbb{K}$ is a projection corresponding to a vector bundle η satisfying $\overline{\xi}_n\precsim\eta$, then $p_n\precsim\psi_n(q)$.

Note that,

$$\lim_{n\to\infty}\frac{\dim(Y_n)}{2\lambda(n)}\leq\lim_{n\to\infty}\frac{n^2(n!)+n\big(\sum_{i=0}^{n-1}\sigma(i)\big)}{n^2(n!)} = 1+\lim_{n\to\infty}\frac{1}{n} = 1. \tag{7}$$

Furthermore, it follows from (5), by induction, that for any $1\leq n<m$ and an arbitary vector bundle η over X_n, we have

$$\varphi^*_{n,m}(\eta)\cong\mu^*_{m,n}(\eta)\oplus(n+1)\mathrm{rank}(\eta)\zeta_{n+1}\oplus\cdots\oplus\frac{\sigma(m)}{(n+1)!}\mathrm{rank}(\eta)\zeta_m, \tag{8}$$

where $\mu_{m,n} := \pi^1_{n+1}\circ\cdots\circ\pi^1_m\colon X_m\to X_n$. Moreover, we find that

$$\lim_{m\to\infty}\frac{\sigma(m)}{(n+1)!\lambda(m)} = \lim_{m\to\infty}\frac{1}{(n+1)!m} = 0. \tag{9}$$

Choose $l(1)\geq 1$ large enough that $\lambda(k)$ is divisible by 4 and $\frac{5}{4}\lambda(k)\geq\dim(Y_k)/2>\mathrm{rc}(C(Y_k)\otimes\mathbb{K})$ for all $k\geq l(1)$, which is possible by (7). Set $k(1) := \frac{1}{2}\lambda(l(1))$. Define sequences $(l(n))_{n\geq 1}$ and $(k(n))_{n\geq 1}$ as follows: given $n\geq 2$ and $l(1),\dots,l(n-1)$ choose $l(n)>l(n-1)$ such that

$$\frac{\big(\sum_{j=1}^{l(n-1)}\lambda(j)\big)\sigma(l(n))}{(l(n-1)+1)!\lambda(l(n))}\leq\frac{1}{2},\ \text{i.e.,}\ \frac{\big(\sum_{j=1}^{l(n-1)}\lambda(j)\big)\sigma(l(n))}{(l(n-1)+1)!}\leq\frac{\lambda(l(n))}{2}, \tag{10}$$

which is possible by (9), and set $k(n) := \frac{1}{2}\lambda(l(n))$. Finally, for each $n\geq 1$, let $\overline{q}_n\in A_{l(n)}\otimes\mathbb{K}$ be the projection corresponding to the vector bundle $\zeta_{l(n)}$ over $X_{l(n)}$ and $x_n := k(n)\langle\varphi_{l(n),\infty}(q_n)\rangle\in\mathrm{Cu}(\mathcal{V}_\infty)$. We aim to show that the sequence $(x_n)_{n\geq 1}$ in $\mathrm{Cu}(\mathcal{V}_\infty)$ witnesses the failure of the Corona Factorization Property in $\mathrm{Cu}(\mathcal{V}_\infty)$.

First, we show that $5x_n\geq\langle\mathbf{1}_{\mathcal{V}}\rangle =: e$ for all $n\geq 1$. As noted above, it suffices to show that $5k(n)\overline{\zeta}_{\lambda(l(n))}\geq\overline{\xi}_{\lambda(l(n))}$. But, by choice of $k(n)$ and $l(n)$ we have that

$$\mathrm{rank}(5k(n)\overline{\zeta}_{\lambda(l(n))}) = \frac{5}{2}\lambda(l(n))\geq\frac{\dim(Y_{l(n)})}{2}+\mathrm{rank}(\overline{\xi}_{l(n)}),$$

since $\mathrm{rank}(\overline{\xi}_{l(n)}) = (l(n)+1)! \leq \frac{\dim(Y_{l(n)})}{2}$. The desired result therefore follows from [14, Theorem 2.5].

Next, we show that $e \not\precsim \sum_{i=1}^{\infty} x_i$. Proceeding as in the proof of Proposition 4.3 part (ii), it suffices to prove that

$$\langle p_j \rangle \not\leq \langle \bigoplus_{i=1}^{n} \varphi_{l(i),j}(q_i) \rangle$$

for all $j \geq l(n)$ (recall that $p_j \in A_j$ denotes the unit, i.e., the projection corresponding to ξ_j). Since ξ_j dominates a trivial line bundle for each j, it suffices to prove that the vector bundle corresponding to the right hand side above does not. We do this by proving that

$$\bigoplus_{i=1}^{n} \varphi^*_{l(i),j}(k(i)\zeta_{l(i)}) \precsim \bigoplus_{s=1}^{j} \lambda(s)\zeta_s. \tag{11}$$

Since the right hand side does not dominate any trivial bundle, by the proof of Proposition 4.3 part (ii), this will complete the proof. Note that it also follows from the proof of Proposition 4.3 part (ii) that $\varphi^*_{j,m}(\bigoplus_{s=1}^{j} \lambda(s)\zeta_s) \precsim \bigoplus_{s=1}^{m} \lambda(s)\zeta_s$ for all $m \geq j$. Thus, it suffices to prove that

$$\bigoplus_{i=1}^{n-1} \varphi^*_{l(i),l(n)}(k(i)\zeta_{l(i)}) \oplus k(n)\zeta_{l(n)} \precsim \bigoplus_{s=1}^{l(n)} \lambda(s)\zeta_s$$

for all $n \geq 1$. We proceed by induction. Clearly the statement is true for $n = 1$, so suppose it is true for $n-1$ with $n \geq 2$. Then

$$\bigoplus_{i=1}^{n-1} \varphi^*_{l(i),l(n)}(k(i)\zeta_{l(i)}) \oplus k(n)\zeta_{l(n)} \precsim \varphi^*_{l(n-1),l(n)}\big(\bigoplus_{s=1}^{l(n-1)} \lambda(s)\zeta_s\big) \oplus k(n)\zeta_{l(n)}.$$

by induction hypothesis.

Now, letting $N := \sum_{s=1}^{l(n-1)} \lambda(s) = \mathrm{rank}(\bigoplus_{s=1}^{l(n-1)} \lambda(s)\zeta_s)$, it follows by the choice of $l(n)$ and $k(n)$ (see (10)) that $k(n) + \frac{N\sigma(l(n))}{(l(n-1)+1)!} \leq \lambda(l(n))$. Hence, combining the above induction step with (8), one has:

$$\begin{aligned}
&\bigoplus_{i=1}^{n-1} \varphi^*_{l(i),l(n)}(k(i)\zeta_{l(i)}) \oplus k(n)\zeta_{l(n)} \\
&\precsim \varphi^*_{l(n-1),l(n)}\big(\bigoplus_{s=1}^{l(n-1)} \lambda(s)\zeta_s\big) \oplus k(n)\zeta_{l(n)} \\
&\overset{(8)}{\precsim} \big(\bigoplus_{s=1}^{l(n-1)} \lambda(s)\zeta_s\big) \oplus (l(n-1)+1)N\zeta_{l(n-1)+1} \\
&\qquad \oplus \cdots \oplus \frac{N\sigma(l(n))}{(l(n-1)+1)!}\zeta_{l(n)} \oplus k(n)\zeta_{l(n)} \\
&\precsim \bigoplus_{s=1}^{l(n)} \lambda(s)\zeta_s.
\end{aligned}$$

Thus, the desired result follows. $\square$

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DIVISIBILITY PROPERTIES OF CENTRAL SEQUENCE ALGEBRAS

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Abstract. We consider the class of separable C^*-algebra which do not admit characters on their central sequence algebras, and show that it has nice permanence properties. Additionally, we consider concrete examples of separable C^*-algebras which do not absorb the Jiang-Su algebra, which are of independent interest, and show that their central sequence algebras admit characters.

Finally, we introduce a new divisibility property, that we call local divisibility, and relate Jiang-Su-stability of unital, separable C^*-algebras to the local divisibility property for central sequence algebras. In particular, we show that a unital, simple, separable, nuclear C^*-algebra absorbs the Jiang-Su algebra if, and only if, the central sequence algebra is k-locally almost divisible.

1. Introduction

Dusa McDuff proved in [17] that the von Neumann algebraic central sequence algebra $\mathcal{M}^\omega \cap \mathcal{M}'$, with respect to a free ultrafilter ω on $\mathbb{N}$, of a separable II_1-factor $\mathcal{M}$, satisfies the following dichotomy: either $\mathcal{M}^\omega \cap M'$ is abelian or a II_1 von Neumann algebra. In the latter case, $\mathcal{M}^\omega \cap \mathcal{M}'$ admits a unital embedding of the hyperfinite II_1-factor $\mathcal{R}$, in which case $\mathcal{M}\overline{\otimes}\mathcal{R} \cong \mathcal{M}$. In analogy, if A is a unital and separable C^*-algebra and $\mathcal{D}$ is a strongly self-absorbing C^*-algebra (see [25]), then $A \otimes \mathcal{D} \cong A$ if, and only if, the C^*-algebraic central sequence algebra $A_\omega \cap A'$ admits a unital embedding of $\mathcal{D}$. In particular, A absorbs the Jiang-Su algebra if, and only if, $A_\omega \cap A'$ admits a unital embedding of $\mathcal{Z}$. Henceforth, we use $F(A)$ to denote the central sequence algebra of A, in keeping with the notation introduced by Kirchberg in [12].

The McDuff dichotomy does not immediately carry over to C^*-algebras. As demonstrated by Ando and Kirchberg in [1], $F(A)$ is non-abelian, whenever A is unital, separable and not of type I, but it does not necessarily contain a unital copy of the Jiang-Su algebra in that case. Hence, a stronger condition than non-commutativity of $F(A)$ is required in order to conclude that A is well-behaved. In [15], Kirchberg and Rørdam asked whether absence of characters on $F(A)$ was the correct analogy of being non-abelian in the von Neumann algebra case. To be more precise.

Question 1.1 (Kirchberg–Rørdam, [15]). Let A be a unital and separable C^*-algebra. Does it follow that $A \otimes \mathcal{Z} \cong A$ if, and only if, $F(A)$ admits no characters?

In the present paper, we obtain no definitive answers to Question 1.1, but we do show that an absence of characters on $F(A)$ removes an obvious obstruction to $\mathcal{Z}$-stability; no hereditary subalgebra of A admits an irreducible, finite-dimensional representation. In [4], it was shown that, if A is a unital and separable AF algebra, for which no qoutient contains an abelian projection, then A is approximately divisible. In particular, if $F(A)$ admits no characters, then $A \otimes \mathcal{Z} \cong A$. Additionally, we consider the C^*-algebra constructed in [10, Example 4.8], of a $C(X)$-algebra whose fibres are isomorphic to the CAR algebra while the C^*-algebra itself does not absorb the Jiang-Su algebra, and the modification of this construction in [7] which yields a unital, simple AH algebra which does not admit a unital embedding of the Jiang-Su-algebra. We show that both C^*-algebras admit characters on their central sequence algebras. This supports an affirmative answer to question 1.1, since neither C^*-algebra absorbs $\mathcal{Z}$.

We consider permanence properties of the class of separable (but not necessarily unital) C^*-algebras A, for which $F(A)$ admits no characters, and show that it has nice permanence properties: the class is closed under arbitrary tensor products, quotients and extensions, and hereditary C^*-subalgebras of C^*-algebras in this class remain within. We also consider inductive limits and provide a sufficient criterion for when the central sequence algebra of the limit admits no characters, in terms of the central sequence algebras of the building blocks. We also provide a necessary and sufficient criterion for an absence of characters on $F(A)$ in terms of the inductive limit structure, see Theorem 3.8.

Finally, we introduce a new divisibility, which is closely related to Kirchbergs covering number, see Proposition 5.6, that we call local divisibility. We show, relying on [14], [16] and [24], that, if A is a unital, simple, separable and nuclear C^*-algebra such that $F(A)$ is k-locally almost divisible, for some $k \geq 1$, then $A \otimes \mathcal{Z} \cong A$, see Theorem 6.16. In [29, Proposition 4.3], Winter and Zacharias showed that if A is a sufficiently non-commutative C^*-algebra, meaning no hereditary C^*-subalgebra of A admits an irreducible, finite-dimensional representation, with finite nuclear dimension, then the covering number of $F(A)$ is finite. Applying this result, along with Theorem 6.16, we recover Winter's seminal result that any unital, simple, separable and infinite-dimensional C^*-algebra with finite nuclear dimension is $\mathcal{Z}$-stable, see Corollary 6.17. However, it should be noted that this result in may ways depend, both explicitly and implicitly, on results and techniques from [27], see the comments preceding Corollary 6.17.

The paper is organized as follows: in Section 3 we review the definition of the central sequence algebra $F(A)$ and the Cuntz semigroup $\mathrm{Cu}(A)$ of a C^*-algebra A, and prove that if $F(A)$ admits no characters, then no hereditary C^*-subalgebra of A admits an irreducible, finite-dimensional representation. In Section 3, we examine the permanence properties of the class of separable C^*-algebras for which $F(A)$ admits no characters. In Section 4, we consider the examples [10, Example 4.8] and the modification in [7], mentioned above. Furthermore, we show that a unital, separable (but not necessarily simple) AF algebra A absorbs $\mathcal{Z}$ if, and only if, $F(A)$ admits no characters. Furthermore, this occurs precisely when no hereditary C^*-subalgebra of A admits an irreducible, finite-dimensional representation, a property which may be viewed as a strong version of being anti-liminal. In Section 5, we introduce the local divisibility property, and examine its basic properties. Finally, in Section 6, we prove our main result: if A is a unital, simple, separable and nuclear C^*-algebra such that $F(A)$ is k-locally almost divisible, for some $k \geq 1$, then $A \otimes \mathcal{Z} \cong A$.

2. Preliminaries

2.1. **The central sequence algebra.** Throughout this paper, ω will denote a fixed free ultrafilter on $\mathbb{N}$. Given a C^*-algebra A, let $\ell^\infty(A)$ denote the C^*-algebra consisting of sequences $(a_n)_n \subseteq A$ such that $\sup_n \|a_n\| < \infty$. The *ultrapower* of A, with respect to ω, is defined as

$$A_\omega := \ell^\infty(A)/\{(a_n)_n \in \ell^\infty(A) \mid \lim_{n\to\omega} \|a_n\| = 0\}.$$

Given $(a_n)_n \in \ell^\infty(A)$, let $[(a_n)_n] \in A_\omega$ denote the image under the quotient map, and let $\iota\colon A \to A_\omega$ denote the embedding $\iota(a) = [(a, a, \dots)]$. We suppress ι in notation and simply consider A to be a C^*-subalgebra of A_ω.

The definition of the central sequence algebra of A, stated below, is due to Kirchberg (see [12, Definition 1.1]).

Definition 2.1. Let A be a C^*-algebra, let $\mathcal{M}(A)$ denote the multiplier algebra of A and let $B \subseteq \mathcal{M}(A)_\omega$ be a C^*-subalgebra. Let $\mathrm{Ann}(B, A_\omega) \subseteq A_\omega \cap B' \subseteq \mathcal{M}(A)_\omega \cap B'$ denote the *annihilator* of B in $A_\omega \cap B'$, i.e.,

$$\mathrm{Ann}(B, A_\omega) := \{x \in A_\omega \cap B' \mid xB = Bx = \{0\}\},$$

and let $F(B, A)$ denote the C^*-algebra

$$F(B, A) := (A_\omega \cap B')/\mathrm{Ann}(B, A_\omega)$$

When $A = B$, we simply write $F(A)$ for $F(A, A)$.

Note that if A is unital and $\mathbf{1}_A \in B \subseteq A_\omega$ then $F(B, A) = A_\omega \cap B'$. The advantage of defining $F(A)$ as above is that $F(A)$ is a *unital* C^*-algebra, whenever A is σ-unital, and the assignment $A \mapsto F(A)$ is a *stable* invariant for the class of σ-unital C^*-algebras (see [12, Corollary 1.10]). Crucially, $F(A)$ retains the property that there is a well-defined *-homomorphism $\rho_A\colon A \otimes_{\max} F(A) \to A_\omega$, given by

$$a \otimes \big(b + \mathrm{Ann}(A, A_\omega)\big) \mapsto ab.$$

Using techniques from [19] and [15], it follows that divisibility properties of $F(A)$ yield divisibility and comparability properties of A.

Observe that we have suppressed ω in the notation $F(A)$. The reason for this is that the isomorphism class of (unital) separable C^*-subalgebras $B \subseteq F(A)$ does not depend on ω. More precisely, if B is a separable C^*-algebra and there exists a (unital) injective *-homomorphism $B \to A_\omega \cap A'$ for some free ultrafilter ω on $\mathbb{N}$, then there exists a (unital) injective *-homomorphism $B \to A_{\omega'} \cap A'$, for any other free ultrafilter ω' on $\mathbb{N}$. Using [12, Proposition 1.12] it follows that the same observation applies to $F(A)$. In particular, the question of whether $F(A)$ has characters is independent of the choice of free ultrafilter (see [15, Lemma 3.5]). Whether the isomorphism class of the entire C^*-algebra $A_\omega \cap A'$ depends on ω depends on the Continuum Hypothesis (see [9] and [8, Theorem 5.1]).

2.2. **The Cuntz semigroup.** We give a brief introduction to the Cuntz semigroup as defined in [6], primarily to fix notation. We refer readers to [6] and [2] for fuller expositions.

Let D be a C^*-algebra and let $a, b \in D_+$. We say that a is Cuntz dominated by b, and write $a \precsim b$, if there exists a sequence $(x_n)_n \subseteq D$ such that $\|a - x_n^* b x_n\| \to 0$. We say that a is Cuntz equivalent to b, and write $a \approx b$, if $a \precsim b$ and $b \precsim a$. We write $a \sim b$, and say that a and b are equivalent, if there exists $x \in D$ such that $a = x^*x$ and $b = xx^*$. Let $\mathbb{K}$ denote the compact operators on a separable, infinite-dimensional Hilbert space and define

$$\mathrm{Cu}(A) := (A \otimes \mathbb{K})_+ / \approx .$$

We write $\langle a \rangle$ for the equivalence class of an element $a \in (A \otimes \mathbb{K})_+$. Then $\mathrm{Cu}(A)$ becomes an ordered abelian semgroup, when equipped with the operation

$$\langle a \rangle + \langle b \rangle := \langle a \oplus b \rangle, \quad a, b \in (A \otimes \mathbb{K})_+$$

and order defined by $\langle a \rangle \leq \langle b \rangle$ if, and only if, $a \precsim b$. Additionally, any upwards directed countable set $S \subseteq \mathrm{Cu}(A)$ admits a supremum. Given $x, y \in \mathrm{Cu}(A)$ we say that x is *compactly contained* in y, and write $x \ll y$, if, for any increasing sequence $(y_k)_k \subseteq \mathrm{Cu}(A)$ with $\sup_k y_k \geq y$ there exists $k_0 \in \mathbb{N}$ such that $x \leq y_{k_0}$. Equivalently, if $a, b \in (A \otimes \mathbb{K})_+$ then $\langle a \rangle \ll \langle b \rangle$ if, and only if, there exists $\varepsilon > 0$ such that $a \precsim (b - \varepsilon)_+$. An element $x \in \mathrm{Cu}(A)$ satisfying $x \ll x$ is said to be *compact.* Note that $\langle p \rangle$ is compact, whenever $p \in (A \otimes \mathbb{K})_+$ is a projection.

For the remainder of this section, we focus on weak (m, n)-divisibility of C^*-algebras, as defined below.

Definition 2.2. Let D be a C^*-algebra and $u \in \mathrm{Cu}(D)$ be an element. Then we say that u is weakly (m, n)-divisible in $\mathrm{Cu}(D)$ if, for every $u' \in \mathrm{Cu}(D)$ with $u' \ll u$, there exists $x_1, \ldots, x_n \in \mathrm{Cu}(D)$ such that $mx_j \leq u$, for every j, and $u' \leq x_1 + \cdots + x_n$.

If D is σ-unital and $e \in D$ is a strictly positive element, we let $\text{w-Div}_m(D)$ denote the least integer $n \geq 1$ for which $\langle e \rangle \in \mathrm{Cu}(D)$ is weakly (m, n)-divisible, with $\text{w-Div}_m(D) = \infty$, if no such n exists.

Note that $\text{w-Div}_m(D)$ is well-defined by [13, Proposition 2.7]. For a unital C^*-algebra D, the weak divisibility constant $\text{w-Div}_2(D)$ measures how far D is from having characters, in a suitable sense ([19, Theorem 8.10]). In general, there is a non-trivial relationship between weak divisibility of $\mathrm{Cu}(A)$ and irreducible representations of A, determined by Robert–Rørdam (see [19, Theorem 5.3]).

Proposition 2.3 ([19])**.** *Let D be a unital C^*-algebra. Then D admits no characters if, and only if,* $\text{w-Div}_2(D) < \infty$.

More generally, if D is σ-unital and $\text{w-Div}_m(D) < \infty$, *then, for every irreducible representation $\pi \colon D \to B(\mathcal{H})$, we have* $\dim(\mathcal{H}) \geq m$. *If, additionally, D is unital, then the reverse implication also holds.*

There is a unital, separable C^*-algebra $A(n, 2)$ which is universal with respect to the property that $\text{w-Div}_2(A(n, 2)) = n$. To be more precise: for each $n \geq 2$, let $A(n,2)$ denote the universal unital C^*-algebra generated by elements $a_1, \dots, a_n, b_1, \dots, b_n$, subject to the relations

$$\mathbf{1} = \sum_{i=1}^{n} a_i^* a_i, \quad a_j^* a_j = b_j^* b_j, \quad b_j^* a_j = 0. \tag{1}$$

for $j = 1, \dots, n$. The following result is an elaboration of [15, Proposition 3.3]. Given an integer $m \geq 1$, we let CM_m denote the cone over the matrix algebra M_m, that is, $CM_m := C_0((0,1]) \otimes M_m$, and $\iota \in C_0((0,1])$ denote the identity map $(0,1] \to (0,1]$.

Lemma 2.4. *Let D be a unital C^*-algebra. Then the following are equivalent:*

(i) w-Div$_2(D) \leq n$
(ii) *There are n *-homomorphisms $\varphi_1, \dots, \varphi_n \colon CM_2 \to D$, and elements $s_1, \dots, s_n \in D$, such that*

$$\mathbf{1} = \sum_{i=1}^{n} s_i^* \varphi_i(\iota \otimes e_{11}) s_i.$$

(iii) *There is a unital *-homomorphism $A(n,2) \to A$.*

Proof. (i)⇒(ii): Suppose that w-Div$_2(D) = n < \infty$, and choose $x_1, \dots, x_j \in (D \otimes \mathbb{K})_+$ such that $\langle \mathbf{1} \rangle \leq \sum_{i=1}^{n} \langle x_i \rangle$ and $2\langle x_j \rangle \leq \langle \mathbf{1} \rangle$, for $j = 1, \dots, n$. Using compactness of $\langle \mathbf{1} \rangle \in \mathrm{Cu}(D)$, there exists $\delta > 0$ such that $\langle \mathbf{1} \rangle \leq \sum_{i=1}^{n} \langle (x_i - \delta)_+ \rangle$. Since $2\langle x_j \rangle \leq \langle \mathbf{1} \rangle$ it follows from [19, Lemma 2.5] that there is a *-homomorphism $\varphi_j \colon CM_2 \to D$ such that $\langle \varphi_j(\iota \otimes e_{11}) \rangle = \langle (x_j - \delta)_+ \rangle$, whence [19, Lemma 2.4 (i)] implies the existence of $s_1, \dots, s_n \in D$ such that

$$\mathbf{1} = \sum_{i=1}^{n} s_i^* \varphi_i(\iota \otimes e_{11}) s_i.$$

(ii)⇒(iii): For $j = 1, \dots, n$, set $a_j' := \varphi_j(\iota^{1/2} \otimes e_{11}) s_j$ and $b_j' := \varphi_j(\iota^{1/2} \otimes e_{21}) s_j$. It is easy to check that the elements $a_1', \dots, a_n', b_1', \dots, b_n' \in D$ satisfies the relations (1), whence there exists a unital *-homomorphism $A(n,2) \to D$.

(iii)⇒(i): Suppose there is a unital *-homomorphism $A(n,2) \to D$. We show that w-Div$_2(A(n,2)) \leq n$, which will imply w-Div$_2(D) \leq n$. Let $a_1, \dots, a_n, b_1, \dots, b_n \in A(n,2)$ be the generators. Then

$$a_j^* a_j = b_j^* b_j \sim b_j b_j^* \perp a_j a_j^* \sim a_j^* a_j.$$

Hence $2\langle a_j^* a_j \rangle = \langle a_j a_j^* \rangle + \langle b_j b_j^* \rangle = \langle a_j a_j^* + b_j b_j^* \rangle \leq \langle \mathbf{1}_{A(n,2)} \rangle$. On the other hand

$$\langle \mathbf{1}_{A(n,2)} \rangle = \langle \sum_{j=1}^{n} a_j^* a_j \rangle \leq \sum_{j=1}^{n} \langle a_j^* a_j \rangle,$$

which shows that w-Div$_2(A(n,2)) \leq n$. ☐

Corollary 2.5. *Let D be a unital C^*-algebra. Then* w-Div$_2(D)$ *is the least natural number, for which part (ii) or (iii) of Lemma 2.4 is satisfied.*

In [15], Kirchberg and Rørdam showed that, if A is a unital, separable C^*-algebra such that $F(A)$ does not admit characters, then $\text{w-Div}_m(A) < \infty$ for all $m \geq 2$. In particular, A admits no finite-dimensional irreducibel representation. We elaborate on this result in Proposition 2.8 below. The intermediate lemmas are slight elaborations of results in [19].

Lemma 2.6. *Let A be a C^*-algebra and $a \in A$ be a positive contraction. Then $\langle a \rangle \in \mathrm{Cu}(A)$ is weakly (m,n)-divisible in $\mathrm{Cu}(A)$ if, and only if, for every $\varepsilon > 0$, there exist positive contractions b_{ij} and contractions y_j in $\overline{aAa}$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, such that $b_{1j}, \ldots, b_{mj}$ are pairwise orthogonal and pairwise equivalent, for all j, and $(a - \varepsilon)_+ = \sum_{j=1}^n y_j^* b_{1j} y_j$.*

Proof. 'Only if': Let $\varepsilon > 0$ be arbitrary and let $\delta > 0$ be a small positive number (to be decided upon later). Choose $x_1, \ldots, x_n \in \mathrm{Cu}(A)$ such that $mx_j \leq \langle a \rangle$ for every j and $\langle (a-\delta)_+ \rangle \leq x_1 + \cdots + x_n$. Choose $x_j' \ll x_j$, for each j, such that $\langle (a - 2\delta)_+ \rangle \leq x_1' + \cdots x_n'$. Since $mx_j \leq \langle a \rangle$, it follows from [19, Lemma 2.4 (ii)] that there exists pairwise orthogonal and pairwise equivalent positive elements $c_{1j}, \ldots, c_{mj} \in B := \overline{aAa}$, such that $x_j' \leq \langle c_{ij} \rangle \leq x_j$, for all j and $i = 1, \ldots, m$. In particular, $\langle (a - 2\delta)_+ \rangle \leq \sum_{j=1}^n \langle c_{1j} \rangle$ in $\mathrm{Cu}(B)$, whence it follows from [19, Lemma 2.4 (i)] that there exists $z_1, \ldots, z_j \in B$, and $\gamma > 0$, such that $(a - 3\delta)_+ = \sum_{j=1}^n z_j^* (c_{1j} - \gamma)_+ z_j$. Let $h_\gamma \colon \mathbb{R}_+ \to [0,1]$ be a continuous function such that $h_\gamma(0)$ and $h_\gamma(t) = 1$, for $t \geq \gamma$, and let $y_j := (c_{1j} - \gamma)_+^{1/2} z_j$ and $b_{ij} := h_\gamma(c_{ij})$. With $\delta = \varepsilon/3$, it follows that the contractions y_j and the positive contractions b_{ij} have the desired properties.

'if': Fix $\varepsilon > 0$ and let the b_{ij}'s and y_j's be given as above. Put $x_j := \langle b_{1j} \rangle$. Then, for every j, we have

$$mx_j = \sum_{i=1}^m \langle b_{ij} \rangle = \Big\langle \sum_{i=1}^m b_{ij} \Big\rangle \leq \langle a \rangle,$$

while

$$\langle (a - \varepsilon)_+ \rangle = \Big\langle \sum_{j=1}^n y_j^* b_{1j} y_j \Big\rangle \leq \sum_{j=1}^n \langle y_j^* b_{1j} y_j \rangle \leq \sum_{j=1}^n x_j. \qquad \square$$

Note that, as a consequence of the above, $\langle a \rangle$ is weakly (m,n)-divisible in $\mathrm{Cu}(A)$ if and only if $\langle a \rangle$ is weakly (m,n)-divisible in $\mathrm{Cu}(\overline{aAa})$.

Lemma 2.7. *Let A be a C^*-algebra and a be a positive contraction in A. Then $\langle a \rangle$ is weakly (m,n)-divisible in $\mathrm{Cu}(\overline{aAa})$ if, and only if, $\langle \iota(a) \rangle$ is weakly (m,n)-divisible in $\mathrm{Cu}(A_\omega)$.*

Proof. The 'if' is clear, so we only prove the 'only if'. Let $B := \overline{aAa} \subseteq A$. Note that $B_\omega \subseteq A_\omega$ is a hereditary C^*-subalgebra and that $\iota(B) \subseteq B_\omega$. In particular, $\iota(a) \in B_\omega$, whence $D \subseteq B_\omega$, where $D \subseteq A_\omega$ denotes the hereditary C^*-subalgebra generated by $\iota(a)$. By Lemma 2.6, there exist contractions y_j and positive contractions b_{ij} in D, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, such that the elements $b_{1j}, \ldots, b_{mj}$ are pairwise equivalent and pairwise orthogonal, for each j, and $(\iota(a) - \varepsilon/2)_+ = \iota\big((a - \varepsilon/2)_+\big) = \sum_{j=1}^n y_j^* b_{1j} y_j$. For each i and j, let $(y_j^{(n)})_{n \geq 1}$ and $(b_{ij}^{(n)})_{n \geq 1}$ be sequences of contractions in B (recall that $D \subseteq B_\omega$) which lift the y_j's and b_{ij}'s, such that, for each n and j, the elements $b_{1j}^{(n)}, \dots, b_{mj}^{(n)}$ are pairwise equivalent and pairwise orthogonal positive contractions in B (this is possible because the CM_m is a projective C^*-algebra). Now, choose $k \in \mathbb{N}$ such that $c_{ij} := b_{ij}^{(k)}$ and $z_j := y_j^{(k)}$, for $i = 1, \dots, m$ and $j = 1, \dots, n$, satisfy

$$\Big\| (a - \varepsilon/2)_+ - \sum_{j=1}^{n} z_j^* c_{1j} z_j \Big\| < \varepsilon/2.$$

It follows from [13, Lemma 2.5] that

$$\langle (a-\varepsilon)_+ \rangle \leq \sum_{j=1}^{n} \langle z_j^* c_{1j} z_j \rangle \leq \sum_{j=1}^{n} \langle c_{1j} \rangle.$$

Thus, with $x_j := \langle c_{1j} \rangle$ for each j, it follows that $m x_j \leq \langle a \rangle$ and $\langle (a-\varepsilon)_+ \rangle \leq x_1 + \cdots + x_n$ in $\mathrm{Cu}(A)$, as desired. □

Proposition 2.8. *Let A be a separable C^*-algebra and suppose that $\mathbf{1} \in F(A)$ is weakly (m,n)-divisible. Then $\langle a \rangle \in \mathrm{Cu}(\overline{aAa})$ is weakly (m,n)-divisible whenever $a \in A$ is a non-zero positive element. In particular, $\overline{aAa}$ admits no finite-dimensional irreducible representation.*

Proof. It suffices to prove the proposition for contractions, so fix a positive, non-zero contraction $a \in A$. By Lemma 2.7, it suffices to prove that $\langle a \rangle$ is weakly (m,n)-divisible in $\mathrm{Cu}(A_\omega)$.

Choose contractions b_{ij} and y_j in $F(A)$ from Lemma 2.6, such that $\mathbf{1} = \sum_{j=1}^{n} y_j^* b_{1j} y_j$ (this is possible because $\mathbf{1}$ is a projection). Let $D := C^*(\{\mathbf{1}, y_j, b_{ij} \mid i = 1, \dots, m,\ j = 1, \dots, n\}) \subseteq F(A)$, and, using [12, Proposition1.12] let $\psi \colon C_0((0,1]) \otimes D \to A_\omega \cap A'$ be a *-homomorphism such that $\psi(\iota \otimes \mathbf{1})c = c$ for all $c \in A$. Let $B \subseteq A_\omega$ denote the hereditary C^*-subalgebra generated by a, and $d_{ij} := a \cdot \psi(\iota^{1/2} \otimes b_{ij})$, $z_j := \psi(\iota^{1/4} \otimes y_j)$ for each i and j. Note that each d_{ij} belongs to B, whence, setting $x_j := \langle d_{1j} \rangle \in \mathrm{Cu}(A_\omega)$, we find that

$$m x_j = \sum_{i=1}^{m} \langle d_{ij} \rangle = \Big\langle \sum_{i=1}^{m} d_{ij} \Big\rangle \leq \langle a \rangle$$

in $\mathrm{Cu}(A_\omega)$. Furthermore,

$$\sum_{j=1}^{n} z_j^* d_{ij} z_j = a \cdot \sum_{i=1}^{n} \psi(\iota \otimes y_j^* b_{ij} y_j) = a \cdot \psi(\iota \otimes \mathbf{1}) = a,$$

whence it follows that $\langle a \rangle \leq x_1 + \cdots + x_n$. The last statement then follows from [19, Theorem 5.3]. □

3. Permanence properties

In this section we investigate the permanence properties for the class of separable C^*-algebras A for which $F(A)$ admits no characters. All the results are quantitative, in the sense that we give obtain bounds on the weak divisibility constant. Throughout this section, whenever A is a C^*-algebra, we let $(A)_1$ denote the closed unit ball of A

3.1. **Hereditary subalgebras.** The result below may be viewed as a strong version of Proposition 2.8.

Proposition 3.1. *Let A be a separable C^*-algebra and $B \subseteq A$ be a hereditary C^*-subalgebra. Then* $\text{w-Div}_2(F(B)) \leq \text{w-Div}_2(F(A))$.

Proof. (i): Note that $B_\omega \subseteq A_\omega$ is a hereditary C^*-subalgebra. Suppose that $\text{w-Div}_2(F(A)) = n < \infty$ (if $\text{w-Div}_2(F(A)) = \infty$ there is nothing to prove), let $e \in B_\omega \cap B'$ be any positive, contractive lift of $\mathbf{1} \in F(B)$, and set $D := C^*(B, e) \subseteq A_\omega$. It follows from Lemma 2.4 and [12, Proposition 1.12] that there exists a *-homomorphism $\varphi\colon C_0((0,1]) \otimes A(n,2) \to A_\omega \cap D'$ such that $\varphi(\iota \otimes \mathbf{1})x = x$ for all $x \in D$. For $i = 1, \dots, n$ let $c_1, \dots, c_n, d_1, \dots, d_n \in B_\omega \cap B'$ be given by

$$c_i := e^{1/2}\varphi(\iota^{1/2} \otimes a_i), \quad d_i := e^{1/2}\varphi(\iota^{1/2} \otimes b_i).$$

Since $e\varphi(y) = \varphi(y)e$, for all $y \in C_0((0,1]) \otimes A(n,2)$, it is easy to see that $c_i^* c_i = d_i^* d_i$ and $d_i^* c_i = 0$ for $i = 1, \dots, n$. Furthermore,

$$\sum_{j=1}^n c_j^* c_j = e \sum_{i=1}^n \varphi(\iota \otimes a_j^* a_j) = e\varphi(\iota \otimes \mathbf{1}) = e.$$

Since $e + \text{Ann}(B, B_\omega) = \mathbf{1} \in F(B)$, it follows that $a_1', \dots, a_n', b_1', \dots, b_n' \in F(B)$, defined by $a_i' := c_i + \text{Ann}(B, B_\omega)$ and $b_i' := d_i + \text{Ann}(B, B_\omega)$, satisfies the relations (1), whence $\text{w-Div}_2(F(B)) \leq n$. □

3.2. **Short exact sequences.** We show that if A is a separable C^*-algebra and $I \subseteq A$ is an ideal, then $F(I)$ and $F(A/I)$ do not admit characters, if $F(A)$ does not admit characters. Conversely, if neither $F(I)$ or $F(A/I)$ admits characters, then the same holds for $F(A)$.

Proposition 3.2. *Suppose A is a separable C^*-algebra, $B \subseteq A_\omega$ is a separable C^*-subalgebra and $I \subseteq A$ is an ideal. Let $\pi\colon A_\omega \to (A/I)_\omega$ denote the natural quotient map. Then*

$$\max\{\text{w-Div}_2(F(I)), \text{w-Div}_2(F(\pi(B), A/I))\} \leq \text{w-Div}_2(F(A)).$$

Proof. Proposition 3.1 implies that $\text{w-Div}_2(F(I)) \leq \text{w-Div}_2(F(A))$. Since $F(\pi(B), A/I)$ is a quotient of $F(B, A)$ (see [12, Remark 1.15 (2)]) it follows that $\text{w-Div}_2(F(\pi(B), A/I)) \leq \text{w-Div}_2(F(B, A))$. Lemma 2.4, along with [12, Proposition 1.12 (1)], shows that $\text{w-Div}_2(F(B, A)) \leq \text{w-Div}_2(F(A))$, whence we obtain the desired estimate. □

A bit of notation is required before we proceed: given unital C^*-algebras D_0 and D_1, let $\mathcal{E}(D_0, D_1)$ denote the unital C^*-algebra

$$\mathcal{E}(D_0, D_1) := \{f \in C([0,1], D_0 \otimes_{\max} D_1) \mid f(0) \in D_0 \otimes \mathbf{1}, f(1) \in \mathbf{1} \otimes D_1\}.$$

Lemma 3.3. *If D_0 and D_1 are unital C^*-algebras, such that* $\text{w-Div}_2(D_0) \leq n$ *and* $\text{w-Div}_2(D_1) \leq m$, *then* $\text{w-Div}_2(\mathcal{E}(D_0, D_1)) \leq n + m$.

Proof. For any unital C^*-algebra D, let $\text{cone}(D) \subseteq C([0,1], D)$ be the unitization of $C_0((0,1], D)$. It is well-known that $\mathcal{E}(D_0, D_1)$ is a quotient of $\text{cone}(D_0) \otimes_{\max} \text{cone}(D_1)$, such that $\iota \otimes \mathbf{1} + \mathbf{1} \otimes \iota \mapsto \mathbf{1}$. Choose $a_i^{(0)}, b_i^{(0)} \in D_0$,

$i = 1, \dots, n$ and $a_j^{(1)}, b_j^{(1)} \in D_1$, $j = 1, \dots, m$ satisfying the relations in (1). Define elements $c_l, d_l \in \mathrm{cone}(D_0) \otimes_{\max} \mathrm{cone}(D_1)$, for $l = 1, \dots, n+m$, by

$$c_l = \begin{cases} (\iota^{1/2} a_l^{(0)}) \otimes \mathbf{1}, & \text{if } 1 \le l \le n \\ \mathbf{1} \otimes (\iota^{1/2} a_l^{(1)}), & \text{if } n+1 \le l \le m, \end{cases}$$

and

$$d_l = \begin{cases} (\iota^{1/2} b_l^{(0)}) \otimes \mathbf{1}, & \text{if } 1 \le l \le n \\ \mathbf{1} \otimes (\iota^{1/2} b_l^{(1)}), & \text{if } n+1 \le l \le m, \end{cases}$$

Finally, let $a_l := q(c_l) \in \mathcal{E}(D_0, D_1)$ and $b_l := q(d_l) \in \mathcal{E}(D_0, D_1)$, for $l = 1, \dots, n+m$. It is easy to check that $a_l^* a_l = b_l^* b_l$ and $b_l^* a_l = 0$, for every l. Furthermore,

$$\sum_{l=1}^{n+m} a_l^* a_l = \sum_{i=1}^{n} q(c_i^* c_i) + \sum_{j=n+1}^{m} q(c_j^* c_j) = q(\iota \otimes \mathbf{1} + \mathbf{1} \otimes \iota) = \mathbf{1}.$$

Hence $\text{w-Div}_2(\mathcal{E}(D_0, D_1)) \le n + m$. □

Theorem 3.4. *Let A be a separable C^*-algebra, and $I \subseteq A$ be an ideal. Then* $\text{w-Div}_2(F(A)) \le \text{w-Div}_2(F(I)) + \text{w-Div}_2(F(A/I)) + 1$. *If A is unital, then* $\text{w-Div}_2(F(A)) \le \text{w-Div}_2(F(I)) + \text{w-Div}_2(F(A/I))$.

Proof. Suppose $\text{w-Div}_2(F(I)) = n$ and $\text{w-Div}_2(F(A/I)) = m$. Then there are unital *-homomorphisms $A(n,2) \to F(I)$ and $A(m,2) \to F(A/I)$. Hence [12, Corollary 1.18] implies the existence of a unital *-homomorphism

$$\mathcal{E}\big(A(n,2), \mathcal{E}(A(m,2), \mathcal{O}_2)\big) \to F(A/I),$$

where $\mathcal{O}_2$ denote the Cuntz algebra on two generators. Noting that $\mathbf{1} \in \mathcal{O}_2$ is properly infinite, whence $\text{w-Div}_2(\mathcal{O}_2) = 1$, and applying Lemma 3.3 (twice), we find that

$$\text{w-Div}_2\big(\mathcal{E}(A(n,2), \mathcal{E}(A(m,2), \mathcal{O}_2))\big) \le n + m + 1,$$

which was the desired conclusion. If A is unital, then [12, Proposition 1.17] implies the existence of a unital *-homomorphism $\mathcal{E}(A(n,2), A(m,2)) \to F(A)$, whence the desired conclusion follows from Lemma 3.3. □

3.3. **Tensor products.** We show that, if neither $F(A)$ or $F(B)$ admits characters, then $F(A \otimes_{\max} B)$ and $F(A \otimes_{\min} B)$ does not admit characters.

Lemma 3.5. *If A and B be separable C^*-algebras, then there is a unital *-homomorphism*

$$F(A) \otimes_{\max} F(B) \to F(A \otimes_{\max} B).$$

Proof. Any tensor product appearing in this proof will be the maximal tensor product. Obviously there is a *-homomorphism $\varphi\colon A_\omega \otimes B_\omega \to (A \otimes B)_\omega$ given by

$$\varphi([(a_n)_n] \otimes [(b_n)_n]) = [(a_n \otimes b_n)_n].$$

A straightforward calculation shows that

$$\varphi((A_\omega \cap A') \otimes (B_\omega \cap B')) \subseteq (A \otimes B)_\omega \cap (A \otimes B)'.$$

Furthermore, if $x = [(x_n)_n] \in \mathrm{Ann}(A, A_\omega)$ and $y = [(y_n)_n] \in B_\omega \cap B'$, then, for arbitrary $a \in A$ and $b \in B$, we find that

$$\|\varphi(x \otimes y)(a \otimes b)\| = \big\|[(x_n a \otimes y_n b)_n]\big\| \leq \lim_{n\to\omega} \|x_n a\| \cdot \lim_{n\to\omega} \|y_n b\| = 0.$$

Similarly, $\varphi((A_\omega \cap A') \otimes \mathrm{Ann}(B, B_\omega)) \subseteq \mathrm{Ann}(A \otimes B, (A \otimes B)_\omega)$. Since the kernel of the quotient map $q\colon (A_\omega \cap A') \otimes (B_\omega \cap B') \to F(A) \otimes F(B)$ equals $(A_\omega \cap A') \otimes \mathrm{Ann}(B, B_\omega) + \mathrm{Ann}(A, A_\omega) \otimes (B_\omega \cap B')$, it follows that φ induces a *-homomorphism $\overline{\varphi}\colon F(A) \otimes F(B) \to F(A \otimes B)$. It is easy to check that $\overline{\varphi}$ is unital. □

Proposition 3.6. *Suppose that A and B are separable C^*-algebras. Then for any tensor product $A \otimes_\alpha B$*

$$\text{w-Div}_2(F(A \otimes_\alpha B) \leq \min\{\text{w-Div}_2(F(A)), \text{w-Div}_2(F(B))\}.$$

Proof. By Lemma 2.4 and [19, Lemma 6.2],

$$\text{w-Div}_2(F(A \otimes_{\max} B)) \leq \min\{\text{w-Div}_2(F(A)), \text{w-Div}_2(F(B))\},$$

whence the result follows from Lemma 3.5. □

3.4. **Inductive limits.** Suppose that (A_i, φ_i) is an inductive system of unital C^*-algebras with unital connecting maps. For each $i \in \mathbb{N}$, let $\varphi_{i,\omega}\colon A_i \to \prod_\omega A_n$ denote the *-homomorphism given by

$$\varphi_{i,\omega}(a) = [(\varphi_{i,1}(a), \varphi_{i,2}(a), \dots)],$$

where, as usual,

$$\varphi_{i,j} = \begin{cases} 0, & \text{if } i > j, \\ \mathrm{id}_{A_i}, & \text{if } i = j, \\ \varphi_{j-1} \circ \cdots \circ \varphi_{i+1} \circ \varphi_i & \text{if } j > i. \end{cases}$$

Note that $A \cong \overline{\bigcup_{i \geq 1} \varphi_{i,\omega}(A_i)}$. For each $i \in \mathbb{N}$, let

$$F\big((A_j, \varphi_j)_{j\geq 1}, A_i\big) := \big(\prod_\omega A_n\big) \cap \varphi_{i,\omega}(A_i)'.$$

Finally, let $F\big((A_j, \varphi_j)_{j\geq 1}\big) = \big(\prod_\omega A_n\big) \cap A'$, and

$$\big(\prod_\omega F\big)\big((A_j, \varphi_j)_{j\geq 1}\big) := \prod_\omega F\big((A_j, \varphi_j)_{j\geq 1}, A_n\big).$$

Since $F\big((A_j, \varphi_j)_{j\geq 1}\big) \subseteq F\big((A_j, \varphi_j)_{j\geq 1}, A_i\big)$ is a unital C^*-subalgebra, for each $i \geq 1$, there is an induced unital *-homomorphism

$$F\big((A_j, \varphi_j)_{j\geq 1}\big) \to \big(\prod_\omega F\big)\big((A_j, \varphi_j)_{j\geq 1}\big)$$

Proposition 3.7. *Suppose $A = \varinjlim(A_i, \varphi_i)$ is a separable, unital sequential inductive limit with unital connecting maps. Then, for every unital and separable C^*-subalgebra $B \subseteq \big(\prod_\omega F\big)\big((A_j, \varphi_j)_{j\geq 1}\big)$, there is a unital *-homomorphism $\rho\colon B \to F(A)$. Conversely, for every unital and separable C^*-subalgebra $D \subseteq F(A)$, there is a unital *-homomorphism $\lambda\colon D \to F\big((A_j, \varphi_j)_{j\geq 1}\big)$.*

Proof. Throughout the proof, let $\mathcal{P}$ denote the unital universal C^*-algebra generated by a countable set $\{x_n\}_{n\geq 1}$ of contractions, and note that $\mathcal{P}$ is projective among unital C^*-algebras.

First, let $B \subseteq \big(\prod_\omega F\big)\big((A_j, \varphi_j)_{j\geq 1}\big)$ be a unital, separable C^*-subalgebra, and let $\{y_n\}_{n\geq 1} \subseteq (A)_1$ be a dense sequence such that $\{y_n\}_{n\geq 1}$ is contained in $\bigcup_{i=1}^\infty \varphi_{i,\omega}(A_i)$. By definition of $\mathcal{P}$, there is a unital and surjective *-homomorphism $\psi\colon \mathcal{P} \to B \subseteq \big(\prod_\omega F\big)\big((A_j, \varphi_j)_{j\geq 1}\big)$. Let $h_0 \in \ker\psi \subseteq \mathcal{P}$ be a strictly positive element. Since $\mathcal{P}$ is projective, there exists a sequence of unital *-homomorphisms $\psi^{(k)}\colon \mathcal{P} \to F\big((A_j, \varphi_j)_{j\geq 1}, A_k\big)$ such that $(\psi^{(1)}, \psi^{(2)}, \dots)_\omega = \psi$, and, for each $k \geq 1$, a sequence of *-homomorphisms $\psi_n^{(k)}\colon \mathcal{P} \to A_n$, such that $(\psi_1^{(k)}, \psi_2^{(k)}, \dots)_\omega = \psi^{(k)}$.

For each $k, n \geq 1$, let $\rho_n^{(k)} := \varphi_{n,\omega} \circ \psi_n^{(k)}\colon \mathcal{P} \to A$ and $X := \{\rho_n^{(k)} \mid k, n \geq 1\}$. Define a sequence of functions $f^{(m)}\colon X \to [0, 2]$ by

$$f^{(1)}(\rho) := \|\rho(h_0)\|$$
$$f^{(m)}(\rho) := \max\{\big\|[\rho(x_i), y_j]\big\| \mid 1 \leq i, j \leq m\}, \quad m \geq 2.$$

We want to show that, for each $m \in \mathbb{N}$ and $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\lim_{n\to\omega} f^{(l)}(\rho_n^{(k)}) < \varepsilon$, for $l = 1, \dots, m$. Kirchberg's ε-test (see [14, Lemma 3.1]) will then complete the proof.

Fix $m \geq 1$ and $\varepsilon > 0$. Since $\psi(h_0) = 0$, there is a set $Y_1 \in \omega$ such that $\|\psi^{(s)}(h_0)\| < \varepsilon$ for all $s \in Y_1$. Choose some $k \in \mathbb{N}$ and $z_1, \dots, z_m \in A_k$ such that $y_j = \varphi_{k,\omega}(z_j)$ for $j = 1, \dots, m$, and note that, since ω is a free filter, we may assume that $k \in Y_1$. Since

$$\psi^{(k)}(\mathcal{P}) \subseteq F\big((A_j, \varphi_j)_{j\geq 1}, A_k\big) = \big(\prod_\omega A_n\big) \cap \varphi_{k,\omega}(A_k)',$$

and $(\psi_1^{(k)}, \psi_2^{(k)}, \dots)_\omega = \psi^{(k)}$, it follows that for each $1 \leq i, j \leq m$:

$$\lim_{n\to\omega} \big\|[\psi_n^{(k)}(x_i), \varphi_{k,n}(z_j)]\big\| = 0$$

By choice of k, and the fact that ω is free, it follows that there exists $n > k$ such that

$$\|\psi_n^{(k)}(h_0)\| < \varepsilon \quad \text{and} \quad \max\{\big\|[\psi_n^{(k)}(x_i), \varphi_{k,n}(z_j)]\big\| \mid 1 \leq i, j \leq m\} < \varepsilon.$$

It easily follows that $f^{(l)}(\rho_n^{(k)}) < \varepsilon$ for $l = 1, \dots, m$. Hence, Kirchberg's ε-test yields a sequence $(\rho_n)_{n\geq 1} \subseteq X$, such that $\lim_{n\to\omega} f^{(m)}(\rho_n) = 0$, for all $m \in \mathbb{N}$. Therefore, $\overline{\rho} = (\rho_1, \rho_2, \dots)_\omega\colon \mathcal{P} \to A_\omega$ is a unital *-homomorphism such that $\overline{\rho}(\mathcal{P}) \subseteq F(A)$ and $\overline{\rho}(\ker\psi) = \overline{\rho}\big(\overline{h_0\mathcal{P}h_0}\big) = \{0\}$. Hence, $\overline{\rho}$ induces a unital *-homomorphism $\rho\colon B \cong \mathcal{P}/\ker(\psi) \to F(A)$.

We now prove the other claim. Note that, as before, there exists a surjective and unital *-homomorphism $\psi\colon \mathcal{P} \to D \subseteq F(A)$. As above, let $h_0 \in \ker\psi \subseteq \mathcal{P}$ be a strictly positive element. We want to show that there exists a unital *-homomorphism $\overline{\lambda}\colon \mathcal{P} \to F\big((A_j, \varphi_j)_{j\geq 1}\big)$ such that $\overline{\lambda}(h_0) = 0$. As above, this will imply the existence of a unital *-homomorphism $\lambda\colon D \to F\big((A_j, \varphi_j)_{j\geq 1}\big)$.

let $\{y_n\}_{n\geq 1} \subseteq (A)_1$ be a dense sequence, such that $\{y_n\}_{n\geq 1}$ is contained in $\bigcup_{i=1}^\infty \varphi_{i,\omega}(A_i)$. As above, there exists a sequence of unital *-homomorphisms

$\psi^{(k)}\colon \mathcal{P} \to A$ such that $(\psi^{(1)}, \psi^{(2)}, \dots)_\omega = \psi$. Identifying A with a C^*-subalgebra of $\prod_\omega A_n$, we find that, for each k, there exists a sequence of unital *-homormophisms $\psi_n^{(k)}\colon \mathcal{P} \to A_n$ such that for all $y \in \mathcal{P}$ we have

$$\lim_{n\to\omega} \|\psi^{(k)}(y) - \varphi_{n,\omega} \circ \psi_n^{(k)}(y)\| = 0.$$

For $k, n \in \mathbb{N}$, and $1 \le l \le n$, let $\lambda_{l,n}^{(k)} := \varphi_{l,n} \circ \psi_l^{(k)}\colon \mathcal{P} \to A_n$ and $X_n := \{\lambda_{l,n}^{(k)} \mid 1 \le k, 1 \le l \le n\} \cup \{0\}$. For each $j \ge 1$, let $l(j) \in \mathbb{N}$ be given such that $y_j \in A_{l(j)}$ and for $m, n \ge 1$, let functions $f_n^{(m)}\colon X_n \to [0, 2]$ be given by

$$f_n^{(1)}(\lambda) := \|\lambda(h_0)\|$$
$$f_n^{(m)}(\lambda) := \max\{\big\|[\lambda(x_i), \varphi_{l(j),n}(y_j)]\big\| \mid 1 \le i, j \le m\}, \quad m \ge 2.$$

We aim to apply Kirchberg's ε-test again, so fix $m \in \mathbb{N}$ and $\varepsilon > 0$. Choose $k \in \mathbb{N}$, such that $\|\psi^{(k)}(h_0)\| < \varepsilon/2$, and

$$\max\{\big\|[\psi^{(k)}(x_i), \varphi_{l(j),\omega}(y_j)]\big\| \mid 1 \le i, j \le m\} < \varepsilon/3.$$

Now, choose $N \in \mathbb{N}$ such that

$$\|\psi^{(k)}(h_0) - \varphi_{N,\omega} \circ \psi_N^{(k)}(h_0)\| < \varepsilon/2,$$
$$\max\{\|\psi^{(k)}(x_i) - \varphi_{N,\omega} \circ \psi_N^{(k)}(x_i)\| \mid 1 \le i \le m\} < \varepsilon/3.$$

Let $\lambda_n^{(\varepsilon)} := \varphi_{N,n} \circ \psi_N^{(k)}$ and $\lambda^{(\varepsilon)} := (\lambda_n^{(\varepsilon)})_{n\ge1} \in \prod_{n\ge1} X_n$. Since $\|\varphi_{N,\omega}(x)\| = \lim_{n\to\omega} \|\varphi_{N,n}(x)\|$, for all $x \in A_N$, it follows that

$$\begin{aligned} f_\omega^{(1)}(\lambda^\varepsilon) &= \lim_{n\to\omega} \|\lambda_n^\varepsilon(h_0)\| \\ &= \|\varphi_{N,\omega} \circ \psi_N^{(k)}(h_0)\| \\ &< \|\psi^{(k)}(h_0)\| + \varepsilon/2 < \varepsilon. \end{aligned}$$

Similarly, it follows that $f_\omega^{(l)}(\lambda^{(\varepsilon)}) < \varepsilon$, for $l = 2, \dots, m$, whence the desired result follows, as above. □

Theorem 3.8. *Suppose $A \cong \varinjlim(A_i, \varphi_i)$ is unital, separable and sequential inductive limit, with unital connecting maps. Then*

$$\begin{aligned} \text{w-Div}_2(F(A)) &= \text{w-Div}_2\big(F\big((A_j, \varphi_j)_{j\ge1}\big)\big) \\ &= \lim_{i\to\omega} \text{w-Div}_2\big(F\big((A_j, \varphi_j)_{j\ge1}, A_i\big)\big) \\ &\le \lim_{i\to\omega} \lim_{j\to\omega} \text{w-Div}_2(F(\varphi_{i,j}(A_i), A_j)) \\ &\le \lim_{i\to\omega} F(A_i). \end{aligned}$$

Proof. The first two equalities follows by combining Proposition 3.7, Lemma 2.4 and [19, Proposition 8.4 (iii)].

For the first inequality above, note that, for any $i \ge 1$ and $j \ge i$, the unital *-homomorphism $\iota_j\colon (A_j)_\omega \to \prod_\omega(A_1, A_2, \dots)$ given by $\iota_j\big([(a_1, a_2, \dots)]\big) = (\varphi_{j,1}(a_1), \varphi_{j,2}(a_2), \dots)$ restricts to a unital *-homomorphism

$$F(\varphi_{i,j}(A_i), A_j) \to \prod_\omega (A_1, A_2, \dots) \cap \varphi_{i,\omega}(A_i)'.$$

In particular, it follows that

$$\text{w-Div}_2\big(\prod_\omega (A_1, A_2, \dots) \cap \varphi_{i,\omega}(A_i)'\big) \le \text{w-Div}_2\big(F(\varphi_{i,j}(A_i), A_j)\big)$$

for all $i \in \mathbb{N}$ and $j \ge i$, whence the desired inequality follows. Finally, for any $i \ge 1$ and $j \ge i$, the map $\varphi_{i,j}\colon A_i \to A_j$ induces a unital *-homomorphism $\varphi^\omega_{i,j}\colon F(A_i) \to F(\varphi_{i,j}(A_i), A_j)$ whence $\text{w-Div}_2(F(\varphi_{i,j}(A_i), A_j)) \le F(A_i)$ and the desired result follows. □

4. Examples

We consider the example [10, Example 4.8], and the modification in [7], of unital and separable C^*-algebras A such that $A \otimes \mathcal{Z} \ncong A$, and we show that $F(A)$ *does* admit a character. Furthermore, building on results of Blackadar–Kumjian–Rørdam and Toms–Winter we characterize when a unital, separable (but not necessarily simple) AF-algebra A absorbs $\mathcal{Z}$. In particular, we show that this happens precisely when $F(A)$ does not admit characters.

Before proceeding with the examples, we fix some common notation.

Definition 4.1. Let $p \in C(S^2) \otimes \mathbb{K}$ denote a projection corresponding to a (complex) line bundle ζ, with non-zero Euler class $e(\zeta)$ (one could for instance choose the Hopf bundle, in which case p can be realized in $C(S^2) \otimes M_2$). Now, for given $d \in \mathbb{N}$, let $\pi_1, \dots, \pi_d\colon (S^2)^d \to S^2$ denote the coordinate projections, and, for each $1 \le i \le d$, let $p_i \in C((S^2)^d) \otimes \mathbb{K}$ denote the projection $p_i := p \circ \pi_i$. Given a subset $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, d\}$, let $p_I \in C((S^2)^d) \otimes \mathbb{K}$ denote the projection $p_I := p_{i_1} \otimes \cdots \otimes p_{i_k}$.

For Examples 4.2 and 4.4, we use [5, Proposition 2.1] to show that the C^*-algebras admit characters on their central sequence algebras.

Example 4.2. We consider here the example of Hirshberg, Rørdam and Winter of a $C(X)$-algebra A (see [10, Definition 1.1]) such that each fiber A_x is isomorphic to the CAR algebra M_{2^∞}, while $A \otimes \mathcal{Z} \ncong A$ ([10, Example 4.8]). First, we recall the construction.

For each natural number m, identify the C^*-algebras $\bigotimes_{n=1}^m M_2(C(S^2))$ and $M_{2^m}(C((S^2)^m))$ and find in $M_{2^m+1}(C((S^2)^m))$ orthogonal projections e and $p^{\otimes m}$, such that e is a trivial one-dimensional projection and $p^{\otimes m}$ is (equivalent to) $p_{\{1,\dots,m\}}$, in the notation of Definition 4.1. Put

$$m(1) = m(2) = 1, \qquad \text{and} \quad m(j) = 2^{j-2}, \text{ when } j \ge 3.$$

and

$$B_j := (e + p^{\otimes m(j)}) M_{2^{m(j)}+1}\big(C\big((S^2)^{m(j)}\big)\big)(e + p^{\otimes m(j)}), \qquad A := \bigotimes_{j=1}^\infty B_j.$$

We aim to demonstrate that $F(A)$ admits a character by showing that, for each $n \ge 1$, there exist projections $\overline{q}_0, \overline{q}_1, \dots, \overline{q}_{2^n} \in A \otimes \mathbb{K}$, such that $\langle \overline{q}_0 \rangle \le 2\langle \overline{q}_j \rangle$, for each $j = 1, \dots, 2^n$, while $\langle \overline{q}_0 \rangle \not\le \sum_{j=1}^{2^n} \langle \overline{q}_j \rangle$, in $\mathrm{Cu}(A)$.

Fix $n \in \mathbb{N}$, and put

$$q_0 = e \otimes e \otimes e \otimes \cdots \otimes e \otimes e \in \big(\bigotimes_{i=1}^{n+2} B_i\big) \otimes \mathbb{K}$$

and, for $j = 1, \dots, 2^n$,

$$q_j := e \otimes e \otimes e \otimes \cdots \otimes e \otimes p_j \in (\bigotimes_{i=1}^{n+2} B_i) \otimes \mathbb{K},$$

where $p_j \in B_{n+2} \otimes \mathbb{K} \cong C\big((S^2)^{m(n+2)}\big) \otimes \mathbb{K} = C\big((S^2)^{2^n}\big) \otimes \mathbb{K}$ is given as in Definition 4.1. For each $l > n+2$, let $q_{j,l} \in (\bigotimes_{i=1}^{m} B_l) \otimes \mathbb{K}$ denote the projection $q_j \otimes \mathbf{1}_{B_{n+3}} \otimes \cdots \otimes \mathbf{1}_{B_l}$, and $\overline{q}_j \in A \otimes \mathbb{K}$ denote the corresponding projection. Since $e \precsim p \oplus p$, it follows that $q_0 \precsim p_j \oplus p_j$, for each $j = 1, \dots, 2_n$. Since each q_j is a projection, it suffices to show that $q_{0,l} \not\precsim \bigoplus_{j=1}^{2^n} q_{j,l}$, for each $l > n$. Since $q_{0,l}$ dominates a trivial one-dimensional projection, for each $l > n+2$, it is sufficient to show that the Euler class of $\bigoplus_{j=1}^{2^n} p_{j,l}$ is non-zero, for all $l > n+2$.

Fix $l > n+2$. For each integer $s \geq 1$ let $M(s) := m(1) + m(2) + \cdots + m(s)$ and note that $\bigotimes_{j=1}^{s} B_j \otimes \mathbb{K} \cong C((S^2)^{M(s)}) \otimes \mathbb{K}$. For each $n+3 \leq i \leq l$, let H_i be the set of integers l such that $M(i-1)+1 \leq l \leq M(i)$, and for a subset $I \subseteq \{n+3, \dots, l\}$, let $H_I := \bigcup_{i \in I} H_i$. Note that,

$$\begin{aligned} \bigoplus_{j=1}^{2^n} q_{j,l} &= \bigoplus_{j=1}^{2^n} p_j \otimes (e + p^{\otimes m(n+3)}) \otimes \cdots \otimes (e + p^{\otimes m(l)}) \\ &\sim \bigoplus_{j=1}^{2^n} \bigoplus_{I \subseteq \{n+3,\dots,l\}} p_j \otimes p_{H_I}. \end{aligned}$$

For each subset $I \subseteq \{n+3, \dots, l\}$ and $j = 1, \dots, 2^n$, let $J_j(I) := \{M(n+1)+j\} \cup H_I$. Then, $p_j \otimes p_{H_I} \sim p_{J_j(I)}$, whence

$$\bigoplus_{j=1}^{2^n} \bigoplus_{I \subseteq \{n+3,\dots,l\}} p_j \otimes p_J \sim \bigoplus_{j=1}^{2^n} \bigoplus_{I \subseteq \{n+3,\dots,l\}} p_{J_j(I)}.$$

Thus, by [22, Proposition 3.2], in order to conclude that the Euler class of $\bigoplus_{j=1}^{2^n} q_{j,l}$ is non-zero, it suffices to prove that the family

$$\{J_j(I) \mid j = 1, \dots, 2^n,\ I \subseteq \{n+3, \dots, l\}\}$$

admits a matching.

If $I = \emptyset$, then $J_j(I) = \{M(n+1)+j\}$, and it is therefore clear how to match these sets. If $I \neq \emptyset$ choose the matching elements for the sets $J_j(I)$ in $H_{\max I}$. This is possible, since there are 2^{k-n-3} subsets $I\{n+3, \dots, l\}$ with $\max I = k$ and $m(k) = 2^{k-2} > 2^n \cdot 2^{k-n-3}$ elements in H_k.

Remark 4.3. Note that, with A as in the example above, A_ω and $F(A)$ are $C(X^\omega)$-algebras, where X^ω denotes the spectrum of the abelian and unital C^*-algebra $C(X)_\omega$. Given a sequence $(x_n)_n \subseteq X$, we may associate a point $x = [(x_n)_n] \in X^\omega$, by letting the character $\mathrm{ev}_x\colon C(X)_\omega \to \mathbb{C}$ being given by $\mathrm{ev}_x([(f_n)_n]) := \lim_{n \to \omega} f_n(x_n)$, for $(f_n)_n \in \ell^\infty(C(X))$. Let $X_0 \subseteq X^\omega$ denote the set of points that arise this way, and for each $y \in X^\omega$, let $I_y \subseteq F(A)$ denote the ideal $I_y := C_0(X^\omega \backslash \{y\})F(A)$. Given a point $x = [(x_n)_n] \in X_0$, it is straightforward to check that

$$C_0(X^\omega \backslash \{x\}) A_\omega = \prod_\omega C_0(X \backslash \{x_n\}) A,$$

and hence $C(X^\omega\backslash\{x\})A_\omega$ is a σ-ideal in A_ω (see [12, Definition 1.5 and Corollary 1.7]). Since $C(X^\omega) \subseteq A_\omega$ is central, it follows that $\big(C_0(X^\omega\backslash\{x\})A_\omega\big) \cap A' = C_0(X^\omega\backslash\{x\})F(A)$ and, by [12, Proposition 1.6],

$$F(A)/I_x \cong \prod_\omega (A_{x_n}) \cap \varphi_{x,\omega}(A)',$$

where $\varphi_{x,\omega}\colon A \to \prod_\omega(A_{x_n})$ is given by $\varphi_{x,\omega}(a) = [(\mathrm{ev}_{x_1}(a), \mathrm{ev}_{x_2}(a), \dots)]$. Since $A_{x_n} \otimes M_{2^\infty} \cong A_{x_n}$ for every $n \geq 1$, it follows that there exists a unital *-homomorphism $M_2 \to F(A)/I_x$. On the other hand, since $F(A)$ is a $C(X^\omega)$-algebra, and admits a character, it follows $F(A)/I_y$ admits a character, for some $y \in X^\omega$ (see [10, 1.4]).

Example 4.4. We consider the modification of Example 4.2 constructed in [7]. We briefly recall the construction.

For each integer $i \geq 1$, let $D_i := \bigotimes_{j=1}^i B_j$, and $e_i \in B_i$ denote the projection e, and $f_i \in B_i$ denote the projection $p^{\otimes m(i)} \in B_i$ in Example 4.2. Observe that the canonical embedding $\psi_i\colon D_i \to D_{i+1}$ is of the form $\psi_i := \psi_i^{(1)} \oplus \psi_i^{(2)}$, where the non-unital embeddings $\psi_i^{(1)}, \psi_i^{(2)}\colon D_i \to D_i \otimes B_{i+1} = D_{i+1}$ are given by

$$\psi_i^{(1)} := \mathrm{id} \otimes e_{i+1}, \qquad \psi_i^{(2)} := \mathrm{id} \otimes f_{i+1}.$$

Now each D_i can be realized as a corner in $M_{N_i}(C(X_i))$, for a sufficiently large N_i, and a suitable compact Hausdorff space X_i. Given a homeomorphism $\alpha\colon X_i \to X_i$, let $\alpha^*\colon M_{N_i}(C(X_i)) \to M_{N_i}(C(X_i))$ denote the isomorphism given by $\alpha^*(g) = g \circ \alpha$. If α is homotopic to id_{X_i}, then there exists a unitary $u \in M_{N_i}(C(X_i))$ such that $u^*\alpha(r_i)u = r_i$, where $r_i \in D_i$ denotes the unit. In particular, the *-isomorphism $\overline{\alpha}\colon M_{N_i}(C(X_i)) \to M_{N_i}(C(X_i))$, given by $\overline{\alpha}(g) = u^*\alpha^*(g)u$, restricts to an isomorphism $D_i \to D_i$.

Now, the construction in Example 4.2 is modified by setting $\varphi_i^{(1)} := \psi_i^{(1)}$ and $\varphi_i^{(2)} := \overline{\alpha_i} \otimes f_{i+1}$, where $\overline{\alpha_i}\colon D_i \to D_i$ is induced by a suitable homeomorphism $\alpha_i\colon X_i \to X_i$ homotopic to id_{X_i}, in the manner described above. Let $\varphi_i := \varphi_i^{(1)} \oplus \varphi_i^{(2)}\colon D_i \to D_{i+1}$ and $D := \varinjlim(D_i, \varphi_i)$. With an appropriate choice of homeomorphism $\alpha_i\colon X_i \to X_i$ it follows that D is simple (see [7]) for details).

Now, it is easily follows from the construction of φ_i, that for any $i \geq 1$ and any projection $q \in D_i \otimes \mathbb{K}$ the projections $\psi_i(q)$ and $\varphi_i(q)$ are Murray-von Neumann equivalent. Hence, $\varphi_{i,j}(q)$ and $\psi_{i,j}(q)$ are equivalent projections, for all $j \geq i$. Thus, with $q_l \in D_{n+2} \otimes \mathbb{K}$ and $\overline{q}_l := \varphi_{i,\infty}(q_l) \in D \otimes \mathbb{K}$, for $l = 0, 1, \dots, 2^n$, as in Example 4.2, the relations $\langle q_0\rangle \leq 2\langle \overline{q}_l\rangle$ and $\langle \overline{q}_0\rangle \not\leq \sum_{k=1}^{2^n} \langle \overline{q}_k\rangle$ still hold in $\mathrm{Cu}(D)$. In particular, it follows that $F(D)$ admits a character.

Finally, we consider unital and separable AF-algebras. Recall the following definition.

Definition 4.5. A separable, unital C^*-algebra A is said to be *approximately divisible* if, for all integers $N \geq 1$, there is a unital *-homomorphism $M_N \oplus M_{N+1} \to F(A)$.

Given a C^*-algebra A, and a representation $\pi\colon A \to B(\mathcal{H})$, we say that π *meets the compacts*, if $\pi(A) \cap \mathbb{K}(\mathcal{H}) \neq \{0\}$.

Theorem 4.6. *Let A be a unital, separable AF algebra. Then the following are equivalent.*

(i) *No non-zero hereditary subalgebra admits a finite-dimensional, irreducible representation.*
(ii) *No irreducible representation $\pi\colon A \to B(\mathcal{H})$ meets the compacts.*
(iii) *No quotient of A contains an abelian projection.*
(iv) *A is approximately divisible.*
(v) *$A \otimes \mathcal{Z} \cong A$.*
(vi) *$F(A)$ admits no characters.*

Proof. The implication (iii)⇒(iv) was shown in [4, Proposition 4.1], the implication (iv)⇒(v) was shown in [26, Theorem 2.3], for general separable and unital C^*-algebras, and (v)⇒(vi) is well-known (see for instance [21, Theorem 7.2.2]). The implication (vi)⇒(i) follows from Proposition 2.8 along with [15, Proposition 3.6]. We show (i)⇒(ii)⇒(iii), or rather, the contrapositive of these statements.

Suppose B is a quotient of A which contains an abelian projection p. Let $\rho\colon B \to B(\mathcal{H})$ be an irreducible representation, such that $\rho(p) \neq 0$. Then $\mathrm{rank}(\rho(p)) = 1$, whence $\rho(B) \cap \mathbb{K}(\mathcal{H}) \ni \rho(p) \neq 0$. Thus, letting $\pi := \rho \circ \varphi$, where $\varphi\colon A \to B$ denotes the quotient map, it follows that π is an irreducible representation of A which meets the compacts.

Suppose, that $\pi\colon A \to B(\mathcal{H})$ is an irreducible representation, which meets the compacts. Then $\mathbb{K}(\mathcal{H}) \subseteq \pi(A)$, whence there exists $a \in A$ such that $\pi(a) =: p \in B(\mathcal{H})$ is a rank one projection. Hence, letting $B := \overline{aAa} \subseteq A$ we find that $\pi|_B\colon B \to B(p(\mathcal{H}))$ is a one-dimensional, and therefore irreducible, representation. □

5. Local divisibility

In this section we introduce a new divisibility property, that we call local divisibility. It is closely related to Kirchberg's covering number, introduced in [12]. We develop the basic properties of local divisibility, with emphasis applications to the central sequence algebra.

Definition 5.1. Let D be unital C^*-algebra and $k \geq 1$ be an integer. We say that $\langle \mathbf{1}_D \rangle \in \mathrm{Cu}(D)$ is k-locally (m, n)-divisible, if there exist $x_1, \ldots, x_k \in \mathrm{Cu}(D)$ such that $mx_j \leq \langle \mathbf{1}_D \rangle$, for all j, and $\langle \mathbf{1}_D \rangle \leq \sum_{i=1}^k n \cdot x_i$.

Let $\text{L-Div}_k(D, m)$ denote the least integer $n \geq 1$, such that $\langle \mathbf{1}_D \rangle \in \mathrm{Cu}(D)$ is k-locally (m, n)-divisible. We say that D is *k-locally almost divisible*, if $\langle \mathbf{1} \rangle \in \mathrm{Cu}(D)$ is k-locally $(m, m+1)$ divisible, for all $m \geq 2$.

Immediately, we can relate this divisibility propety to Kirchberg's covering number $\mathrm{cov}(D, m)$, see [12, Definition 3.1].

Proposition 5.2. *Let D be a unital C^*-algebra.*

(i) *If $\mathrm{cov}(D, m) = k$, then $\text{L-Div}_k(D, m) \leq 2m$.*
(ii) *If $\text{L-Div}_k(D, m) = n$, then $\mathrm{cov}(D, m) \leq k \cdot \left\lceil \frac{n}{m} \right\rceil$.*

Proof. Before the proof, let us record the following observation: by [19, Proposition 3.17 (i)], $\mathrm{cov}(D, m)$ is the least integer k, for which there exist elements $x_1, \ldots, x_k \in \mathrm{Cu}(D)$ satisfying

$$x_i \leq \langle \mathbf{1} \rangle \leq x_1 + \cdots + x_k, \quad x_i = \sum_{j=1}^{l_i} m_{ij} y_{ij}, \tag{2}$$

for some integers $m \geq m_{ij}$, some integers $l_i \geq 1$, and some $y_{ij} \in \mathrm{Cu}(D)$. Replacing y_{ij} with an integral multiple of y_{ij}, we may assume that $m \leq m_{ij} < 2m$, for all i, j.

(i): Let x_i, y_{ij} and m_{ij} be given as in (2), with $m \leq m_{ij} < 2m$, for all i, j. For each i, set $z_i := \sum_{j=1}^{l_i} y_{ij}$. It follows that $mz_i \leq x_i \leq \langle \mathbf{1} \rangle$, for all i, while

$$\langle \mathbf{1} \rangle \leq \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} \sum_{j=1}^{l_i} 2my_{ij} = \sum_{i=1}^{k} 2mz_i.$$

(ii): Choose $z_1, \ldots, z_k \in \mathrm{Cu}(D)$, such that $mz_i \leq \langle \mathbf{1} \rangle \leq nz_1 + \cdots + nz_k$. Let $l := \lceil \frac{n}{m} \rceil$, and note that $\langle \mathbf{1} \rangle \leq lmz_1 + \cdots lmz_k$. Let elements $y_1, \ldots, y_{kl} \in \mathrm{Cu}(D)$ be given by $y_{(r-1)l+s} := z_r$, for $r = 1, \ldots, k$ and $s = 1, \ldots, l$. Note that $\sum_{j=(r-1)l+1}^{rl} y_j = lz_r$. Hence, letting $x_j = my_j$, for $j = 1, \ldots, lk$, we find that $x_i \leq \langle \mathbf{1} \rangle$, for all i, and

$$\sum_{j=1}^{kl} x_j = \sum_{j=1}^{kl} my_j = \sum_{r=1}^{k} (lm)z_r \geq \langle \mathbf{1} \rangle. \qquad \square$$

We also consider the asymptotic local divisibility constants.

Definition 5.3. Given a unital C^*-algebra D, let

$$\text{L-Div}_k(D) := \liminf_{m\to\infty} \frac{\text{L-Div}_k(D, m)}{m}.$$

The proof of the following is almost verbatim the proof of [19, Proposition 4.1]. A full proof is included in the interest of completeness

Proposition 5.4. *Let D be a unital C^*-algebra and $\alpha := \text{L-Div}_k(D)$. If $\alpha > 0$, then $\text{L-Div}_k(D, m) \leq n$, for every integer $m \geq 2$ and every $n > \alpha \cdot m$.*

Proof. If $\text{L-Div}_k(D) = \infty$, there is nothing to prove, so assume that $0 < \text{L-Div}_k(D) < \infty$, and fix arbitrary $m \geq 2$. Let M be the smallest integer strictly greater than αm. We show that $\text{L-Div}_k(D, m) \leq M$.

Choose $\beta > 1$, and an integer r_0, such that

$$\beta \frac{r_0 + 1}{r_0} m\text{L-Div}_k(D) = \beta \frac{r_0 + 1}{r_0} m\alpha \leq M.$$

By definition of $\text{L-Div}_k(D)$, there exists an integer $p \geq r_0 m$ such that $l := \text{L-Div}_k(D, p) \leq \beta p \cdot \text{L-Div}_k(D)$. Hence, let $x_1, \ldots, x_k \in \mathrm{Cu}(D)$ be given such that $px_i \leq \langle \mathbf{1} \rangle$ and $\langle \mathbf{1} \rangle \leq \sum_{j=1}^{k} lx_j$. Write $p = rm + d$, with $0 \leq d < m$ and $r \geq r_0$, and write $l = tr - d'$, with $0 \leq d' < r$ and $t \geq 1$. Put $y_i = rx_i$, for $i = 1, \ldots, k$, and note that $my_i \leq \langle \mathbf{1} \rangle$, and $\langle \mathbf{1} \rangle \leq \sum_{j=1}^{k} ty_j$. Therefore, $\text{L-Div}_k(D, m) \leq t$, and, using that

$$\frac{p}{p - d} = \frac{rm + d}{rm} \leq \frac{r + 1}{r},$$

we find that

$$\begin{aligned}
\text{L-Div}_k(D,m) \le t &= \left\lceil \frac{l}{r} \right\rceil \\
&= \left\lceil \frac{l}{p-d} m \right\rceil \\
&\le \left\lceil \beta \frac{p}{p-d} m \text{L-Div}_k(D) \right\rceil \\
&\le \left\lceil \beta \frac{r+1}{r} m \text{L-Div}_k(D) \right\rceil \\
&\le \left\lceil \beta \frac{r_0+1}{r_0} m \text{L-Div}_k(D) \right\rceil \le M. \qquad \square
\end{aligned}$$

Lemma 5.5. *Suppose D is a unital C^*-algebra such that $0 < \text{L-Div}_k(D) = \alpha < \infty$, for some $k \ge 1$. Then, with $k' := k\lceil \alpha \rceil$, we have $\text{L-Div}_{k'}(D) \le 1$.*

Proof. Let $l := \lceil \alpha \rceil$. For arbitrary $m \ge 2$, using Proposition 5.4, we may choose elements $x_1, \dots, x_k$ such that $mx_i \le \langle \mathbf{1}_D \rangle$ and $\langle \mathbf{1}_D \rangle \le \sum_{j=1}^k nx_j$, with $n := \lfloor \alpha m \rfloor + 1$, i.e., the smallest integer strictly greater than αm. Define elements $z_1, \dots, z_{lk} \in \text{Cu}(D)$ as follows: for $r = 1, \dots, k$ and $s = 1, \dots, l$, let $z_{(r-1)l+s} := x_r$. Hence $\sum_{j=(r-1)l+1}^{rl} z_j = lx_r$, for each $r = 1, \dots, k$, and obviously, $mz_i \le \langle \mathbf{1}_D \rangle$. Noting that $(m+1)l > \alpha m$, it follows that $(m+1)l \ge n$, whence

$$\sum_{j=1}^{kl} (m+1) z_j = \sum_{r=1}^{k} (m+1) l x_r \ge \sum_{r=1}^{k} n x_r \ge \langle \mathbf{1}_D \rangle.$$

Therefore, $\text{L-Div}_{k'}(F(A), m) \le m+1$, for every $m \ge 2$, whence the desired result follows. □

Proposition 5.6. *Let D be a unital C^*-algebra. If $\text{cov}(D) = k < \infty$, then $\text{L-Div}_k(D) \le 2$. Conversely, if $0 < \text{L-Div}_k(D) = \alpha < \infty$, then $\text{cov}(D) \le 2 \cdot \lceil \alpha \rceil \cdot k$.*

Proof. This follows from Propositions 5.2 and 5.4, and Lemma 5.5. □

In the following proposition, we use the fact that, given C^*-algebras B and D, there exists a bi-additive map $\text{Cu}(B) \times \text{Cu}(D) \to \text{Cu}(B \otimes_{\max} D)$, $(x, y) \mapsto x \otimes y$, such that $x_1 \otimes x_2 \le y_1 \otimes y_2$, if $x_1 \le x_2$ and $y_1 \le y_2$, see [19, Section 6].

Proposition 5.7. *Let B, D be C^*-algebras, with D unital, and let m, n be integers. If $x, y_1, \dots, y_k \in \text{Cu}(B)$ satisfy $nx \le my_i$ for all i, and $\langle \mathbf{1}_D \rangle \in \text{Cu}(D)$ is k-locally (m,n)-divisible, then*

$$x \otimes \langle \mathbf{1}_D \rangle \le (y_1 + \cdots + y_k) \otimes \langle \mathbf{1}_D \rangle$$

in $\text{Cu}(B \otimes D)$.

Proof. Choose elements $z_1, \dots, z_k \in \mathrm{Cu}(D)$ such that $mz_i \le \langle \mathbf{1}_D \rangle$, for all i, and $\langle \mathbf{1} \rangle \le \sum_{j=1}^{k} nz_j$. Then

$$\begin{aligned} x \otimes \langle \mathbf{1}_D \rangle &\le x \otimes nz_1 + \cdots + x \otimes nz_k \\ &= nx \otimes z_1 + \cdots + nx \otimes z_k \\ &\le my_1 \otimes z_1 + \cdots + my_k \otimes z_k \\ &\le (y_0 + y_1 + \cdots + y_n) \otimes \langle \mathbf{1}_D \rangle. \end{aligned}$$

□

We now focus on local divisibility for central sequence algebras. First, we need the following lemma, which follows from [15, Lemma 6.3].

Lemma 5.8. *Let C be a (possibly non-separable) unital C^*-algebra.*

(i) *If $x, y_0, y_1, \dots, y_n \in \mathrm{Cu}(C)$ satisfy $nx \le my_i$, for $i = 0, 1, \dots, n$, then there exists a unital and separable C^*-subalgebra $D \subseteq C$ such that $x, y_0, y_1, \dots, y_n$ belongs to the image of the induced map $\mathrm{Cu}(D) \to \mathrm{Cu}(C)$ and $nx \le my_i$, for $i = 0, 1, \dots, n$, in $\mathrm{Cu}(D)$.*

(ii) *If $\langle \mathbf{1}_C \rangle \in \mathrm{Cu}(C)$ is k-locally (m, n)-divisible, then there exists a unital and separable C^*-subalgebra $D \subseteq C$, such that $\langle \mathbf{1}_D \rangle \in \mathrm{Cu}(D)$ is k-locally (m, n)-divisible.*

Recall the following definition from [18].

Definition 5.9. Let D be a C^*-algebra.

(i) Given elements $x, y \in \mathrm{Cu}(D)$, write $x <_s y$, if there exists an integer $n \ge 1$ such that $(n+1)x \le ny$.

(ii) $\mathrm{Cu}(D)$ is said to have *n-comparison*, if $x \le y_0 + y_1 + \cdots + y_n$, whenever $x, y_0, y_1, \dots, y_n \in \mathrm{Cu}(D)$ satisfy $x <_s y_i$, for $i = 0, 1, \dots, n$.

A similar, but different, definition appears in [27, Definition 3.1 (i)]. While the two definitions do not agree in general, it follows from the proof of [20, Theorem 5.2 (i)] and [3], that the two notions coincide for unital and simple C^*-algebras.

Proposition 5.10. *Let A be a unital and separable C^*-algebra and suppose there exists an integer $k \ge 1$ such that $0 < \text{L-Div}_k(F(A)) =: \alpha < \infty$.*

(i) *For every positive element $a \in A$, respectively $a \in F(A)$, the element $\langle a \rangle$ is k-locally (m, n)-divisible in $\mathrm{Cu}(A)$, respectively $\mathrm{Cu}(F(A))$, whenever $n > \alpha m$.*

(ii) *Letting k' denote the integer $\lceil \alpha \rceil k$, both $\mathrm{Cu}(A)$ and $\mathrm{Cu}(F(A))$ have $(k' - 1)$-comparison.*

Proof. (i): Before proceeding with the proof we record the following observation. Suppose B and D are C^*-algebras, such that D is unital and $\text{L-Div}_k(D) = \alpha$. Then, by Proposition 5.4, whenever $n > \alpha m$, there exist $z_1, \dots, z_k \in \mathrm{Cu}(D)$ such that $mz_i \le \langle \mathbf{1}_D \rangle$, for all i, and $\langle \mathbf{1}_D \rangle \le nz_1 + \cdots + nz_k$. In particular, for all $x \in \mathrm{Cu}(B)$, we have that $m(x \otimes z_i) \le x \otimes \langle \mathbf{1}_D \rangle$, for all i, and $x \otimes \langle \mathbf{1}_D \rangle \le n(x \otimes z_1) + \cdots + n(x \otimes z_k)$. In other words, $x \otimes \langle \mathbf{1}_D \rangle$ is k-locally divisible (m, n)-divisible in $\mathrm{Cu}(B \otimes_{\max} D)$.

First, let $a \in A$ be a positive element, and let $n, m \ge 1$ be integers such that $n > \alpha m$. Then, since $\langle \mathbf{1} \rangle \in \mathrm{Cu}(F(A))$ is k-locally (m, n)-divisible, it follows from the observation above that $\langle a \rangle \otimes \langle \mathbf{1}_{F(A)} \rangle$ is k-locally (m, n)-divisible in $\mathrm{Cu}(A \otimes_{\max} F(A))$. Letting $\varphi \colon A \otimes_{\max} F(A) \to A_\omega$ denote the natural *-homomorphism, it follows that $\mathrm{Cu}(\varphi)(\langle a\rangle\otimes\langle \mathbf{1}_{F(A)}\rangle) = \langle a\rangle$, whence $\langle a\rangle$ is k-locally (m,n)-divisible in $\mathrm{Cu}(A_\omega)$. It is easy to modify the proof of [19, Proposition 8.4] to show that this implies $\langle a\rangle$ is k-locally (m,n)-divisible in $\mathrm{Cu}(A)$ (see also the proof of Lemma 2.7).

Second, let $a \in F(A)$ be a positive element, and let $n, m \geq 1$ be integers such that $n > \alpha m$. Choose a unital and separable C^*-subalgebra $D \subseteq F(A)$, such that $\mathrm{Cu}(D)$ is k-locally (m,n)-divisible. By [12, Proposition 1.12], there is a *-homomorphism $\varphi\colon C^*(a)\otimes D \to F(A)$ such that $\varphi(a\otimes \mathbf{1}_D) = a$. By the same arguments as above, it follows that $\langle a\rangle$ is k-locally (m,n)-divisible in $\mathrm{Cu}(F(A))$.

(ii): We first prove the statement for $F(A)$. Suppose that $x, y_1, \ldots, y_{k'}$ are given such that $x <_s y_i$, for all i, and choose $n \geq$ such that $(n+1)x \leq ny_i$, for all i, using [18, Proposition 2.1]. Let $B \subseteq F(A)$ be a separable C^*-subalgebra such that $x, y_1, \ldots, y_{k'}$ belongs to the image of the induced map $\mathrm{Cu}(B) \to \mathrm{Cu}(F(A))$, and $(n+1)x \leq ny_i$ in $\mathrm{Cu}(B)$, for all i. Choose a separable C^*-subalgebra $D \subseteq F(A)$ which is k'-locally $(n, n+1)$-divisible, using Lemma 5.5 and Lemma 5.8. Then, by Proposition 5.7, we have that $x\otimes\langle \mathbf{1}_D\rangle \leq (y_1+\cdots+y_k)\otimes\langle \mathbf{1}_D\rangle$ in $\mathrm{Cu}(B\otimes_{\max}D)$. By [12, Proposition 1.12], there exists a *-homomorphism $\varphi\colon B\otimes_{\max}D \to F(A)$, such that $\varphi(b\otimes\mathbf{1}) = b$, for all $b \in B$. In particular, $\mathrm{Cu}(\varphi)(x\otimes\langle \mathbf{1}_D\rangle) = x \in \mathrm{Cu}(F(A))$, and

$$\mathrm{Cu}(\varphi)((y_1+\cdots+y_k)\otimes\langle \mathbf{1}_D\rangle) = y_1+\cdots+y_k.$$

Hence $x \leq y_0 + y_1 + \cdots + y_n$ in $\mathrm{Cu}(F(A))$.

Second, let $x, y_1, \ldots, y_k \in \mathrm{Cu}(A)$ and $n > \alpha m$ be given such that $nx \leq my_i$, for all i. Using the same arguments as above, it follows that $x \leq y_1 + \cdots + y_k$ in $\mathrm{Cu}(A_\omega)$, whence [15, Lemma 4.1] implies that the same relation holds in $\mathrm{Cu}(A)$. □

6. Local divisibility and $\mathcal{Z}$-stability

We examine the relation between local k-divisibility of $F(A)$ and $\mathcal{Z}$-stability of A. First we consider what can be said for general unital and separable C^*-algebras.

Recall that, for each integer $m \geq 1$, we let $CM_m := C_0((0,1])\otimes M_m$ denote the cone over M_m, and $\iota \in C_0((0,1])$ denote the identity map, i.e., $\iota(t) = t$, for all $t \in (0,1]$. We use, without mention, that for any pair of C^* algebras A, B, there is a canonical bijective correspondence between ccp. order zero maps $A \to B$ and *-homomorphisms $C_0((0,1])\otimes A \to B$ (see [28, Corollary 4.1]).

Lemma 6.1. *Let D be a unital C^*-algebra and $k, n, m \geq 1$ be integers. Then $\text{L-Div}_k(D,m) \leq n$ if, and only if, there exist ccp. order zero maps $\varphi_1, \ldots, \varphi_k\colon M_m \to D$, and elements $s_{ij} \in D$, for $i = 1, \ldots, n$ and $j = 1, \ldots, k$, such that*

$$\mathbf{1}_D = \sum_{j=1}^{k}\sum_{i=1}^{n} s_{ij}^*\varphi_j(e_{11})s_{ij}.$$

Proof. Suppose $\text{L-Div}_k(D,m) = n < \infty$. Choose $a_1, \ldots, a_k \in (D\otimes\mathbb{K})_+$ such that $\langle \mathbf{1}_D\rangle \leq \sum_{j=1}^k n\langle a_j\rangle$ and $m\langle a_i\rangle \leq \langle \mathbf{1}_D\rangle$, for $i = 1, \ldots, k$. Using compactness of $\langle \mathbf{1}_D\rangle \in \mathrm{Cu}(D)$, there exists $\delta > 0$ such that $\langle \mathbf{1}_D\rangle \leq \sum_{j=1}^k n\langle (a_j-\delta)_+\rangle$.

Since $m\langle a_i\rangle \leq \langle \mathbf{1}_D\rangle$, it follows from [19, Lemma 2.5] that there is a *-homomorphism $\psi_i\colon CM_m \to D$ such that $\langle \psi_i(\iota\otimes e_{11})\rangle = \langle (a_i-\delta)_+\rangle$. Let $\varphi_i\colon M_m \to D$ denote the order zero map given by $\varphi_i(x) = \psi_i(\iota\otimes x)$, for $x\in M_m$. It follows from [19, Lemma 2.4 (i)] that there exist $s_{ij}\in D$, for $i=1,\dots,n$ and $j=1,\dots,k$, such that

$$\mathbf{1}_D = \sum_{j=1}^{k}\sum_{i=1}^{n} s_{ij}^*\varphi_i(e_{11})s_{ij}. \qquad \square$$

Remark 6.2. Let D be a unital C^*-algebra. Suppose that $\text{L-Div}_k(D,m)\leq n$ and let $l := \lceil n/m\rceil$. We show, using techniques from [12, Remark 3.2], that there exist ccp. order zero maps $\psi_1,\dots,\psi_k\colon M_m\to D$ and elements t_{rj}, for $r=1,\dots,l$ and $j=1,\dots,k$, such that

$$\psi_i(\mathbf{1})t_{rj} = t_{rj}, \quad \mathbf{1} = \sum_{q=1}^{k}\sum_{p=1}^{l} t_{pq}^* t_{pq}$$

for all r,j. In particular, it follows that each t_{rj} is a contraction.

Let $\varphi_1,\dots,\varphi_k\colon M_m\to D$ be the ccp. order zero maps, and $s_{i,j}\in D$ be the elements, obtained from Lemma 6.1. Using [13, Lemma 2.2], we may assume that

$$\mathbf{1}_D = \sum_{j=1}^{k}\sum_{i=1}^{n} s_{i,j}^*\varphi_j(e_{11})^3 s_{i,j}.$$

Fix $j\geq 1$. For each $r=1,\dots,l-1$ let

$$\overline{s}_{rj} := \sum_{d=1}^{m}\varphi_j(e_{d1})s_{(r-1)m+d,j},$$

and let

$$\overline{s}_{lj} := \sum_{d=1}^{n-(l-1)m}\varphi_j(e_{t1})s_{(l-1)m+d,j}.$$

Since each φ_j is ccp. and order zero, it follows that $\varphi_j(\mathbf{1}_m)$ commutes with the image of φ_j. Furthermore, it follows from [28, Corollary 4.1], that $\varphi_j(e_{1d'})\varphi_j(e_{d1}) = 0$, when $d\neq d'$, and

$$\varphi_j(\mathbf{1}_m)\varphi_j(e_{1d})\varphi_j(e_{d1}) = \varphi_j(e_{11})^3.$$

Hence, it follows that

$$\sum_{r=1}^{l}\overline{s}_{rj}^*\varphi_j(\mathbf{1}_m)\overline{s}_{rj} = \sum_{i=1}^{n} s_{i,j}^*\varphi_j(e_{11})^3 s_{i,j},$$

and therefore

$$\sum_{j=1}^{k}\sum_{r=1}^{l}\overline{s}_{rj}^*\varphi_j(\mathbf{1}_n)s_{rj} = \mathbf{1}.$$

Let $\hat{\varphi}_j\colon CM_m\to F(A)$ denote the *-homomorphism induced by φ_j, for each $j=1,\dots,k$, i.e., $\hat{\varphi}_j(\iota\otimes x) = \varphi_j(x)$ for all $x\in M_m$. Note that

$\hat{\varphi}_j((\iota-\varepsilon)_+\otimes\mathbf{1})\to\hat{\varphi}_j(\iota\otimes\mathbf{1})$, as $\varepsilon\to 0$, whence there exists $\delta>0$ such that the element

$$b := \sum_{j=1}^{k}\sum_{r=1}^{l}\overline{s}_{rj}^*\hat{\varphi}_j\big((\iota-\delta)_+\otimes\mathbf{1}\big)\overline{s}_{rj}\geq 0$$

is invertible. Choose some continuous function $f_\delta\colon[0,1]\to[0,1]$ such that $f_\delta(0)=0$ and $f_\delta(t)=1$ whenever $t\geq\delta$. Let $\overline{\psi}_j\colon CM_m\to D$ be a *-homomorphism such that $\overline{\psi}_j(\iota\otimes\mathbf{1})=\hat{\varphi}_j(f_\delta\otimes\mathbf{1})$, and $\psi_j\colon M_m\to D$ denote the associated order zero map, i.e., $\psi_j(x)=\overline{\psi}_j(\iota\otimes x)$, for $x\in M_m$. Set

$$t_{rj} := \hat{\varphi}_j\big((\iota-\delta)_+^{1/2}\otimes\mathbf{1}\big)\overline{s}_{rj}b^{-1/2},$$

for each r,j. It follows from straightforward calculations that the maps $\psi_1,\ldots,\psi_k\colon M_m\to D$, and elements t_{rj}, have the desired properties.

The proof of the following lemma is contained in the proof of [14, Lemma 7.6].

Lemma 6.3. *Suppose D is a unital C^* algebra, and $\varphi_0\colon M_p\to D$ and $\varphi_1\colon M_q\to D$ are ccp. order zero maps with commuting images, that is, $\varphi_0(x)\varphi_1(y)=\varphi_1(y)\varphi_0(x)$ for all $x\in M_p$ and $y\in M_q$, and such that $\|\varphi_0(\mathbf{1})+\varphi_1(\mathbf{1})\|\leq 1$. Then there exists an order zero map $\psi\colon I(p,q)\to D$ such that $\psi(\mathbf{1})=\varphi_0(\mathbf{1})+\varphi_1(\mathbf{1})$ and $\mathrm{Im}(\psi)$ is contained in the C^*-subalgebra generated $\mathrm{Im}(\varphi_0)$ and $\mathrm{Im}(\varphi_1)$.*

Before proving the next proposition, we record the following observation about ccp. order zero maps.

Lemma 6.4. *Suppse A and B are C^*-algebras, with A unital, and that $\varphi\colon A\to B$ is a ccp. map. Then φ is order zero if and only if $\|\varphi(x)\varphi(y)\|\leq\|\varphi(xy)\|$, for all $x,y\in A$.*

Proof. The 'if' statement is obvious. Hence, assume that φ is order zero, and let $\rho\colon C_0((0,1])\otimes A\to B$ be the associated *-homomorphism. Then, for every $x,y\in A$, we have

$$\varphi(x)\varphi(y)=\rho(\iota\otimes x)\rho(\iota\otimes y)=\rho(\iota\otimes\mathbf{1})\rho(\iota\otimes xy)=\varphi(\mathbf{1})\varphi(xy).$$

Since φ is contractive, it follows that

$$\|\varphi(x)\varphi(y)\|=\|\varphi(\mathbf{1})\varphi(xy)\|\leq\|\varphi(xy)\|. \qquad \square$$

The proof of the following is inspired by [26, Proposition 2.2].

Proposition 6.5. *Let A is a unital, separable C^*-algebra, and $\mathcal{D}$ be a unital and nuclear C^*-algebra. Suppose $\mathcal{D}$ can be written as the closure of an increasing union of unital and nuclear C^*-subalgebras $\mathcal{D}_i\subseteq\mathcal{D}$. If there exist integers $k,m\geq 1$ such that, for every $i\geq 1$, there are order zero ccp. maps $\psi_1^{(i)},\ldots,\psi_k^{(i)}\colon\mathcal{D}_i\to F(A)$ and elements $s_{jl}^{(i)}\in F(A)$, for $j=1,\ldots,m$ and $l=1,\ldots,k$, such that*

$$\mathbf{1}=\sum_{l=1}^{k}\sum_{j=1}^{m}\big(s_{jl}^{(i)}\big)^*\psi_l^{(i)}(\mathbf{1})s_{jl}^{(i)},$$

then there exist order zero maps $\rho_1, \dots, \rho_k \colon \mathcal{D} \to F(A)$, and elements $t_{jl} \in F(A)$, for $j = 1, \dots, m$ and $l = 1, \dots, k$, such that

$$\mathbf{1} = \sum_{j=1}^{k} \sum_{l=1}^{m} t_{jl}^* \rho_l(\mathbf{1}) t_{jl}.$$

Proof. Since each $\mathcal{D}_i$ is nuclear, it follows, from the Choi–Effros lifting theorem, that we may lift each $\psi_l^{(i)}$ to a ccp. map $\overline{\psi}_l^{(i)} \colon \mathcal{D}_i \to \prod_{n\geq 1} A$. For each i and l, let $\overline{\psi}_{l,n}^{(i)} \colon \mathcal{D}_i \to A$, $n \geq 1$, be a sequence of ccp. maps which implements $\overline{\psi}_l^{(i)}$, i.e., $\overline{\psi}_l^{(i)}(d) = (\overline{\psi}_{l,1}^{(i)}(d), \overline{\psi}_{l,2}^{(i)}(d), \dots)$.

Since each $\overline{\psi}_{l,n}^{(i)}$ is nuclear, it may be approximated, in the point-norm topology, by maps of the form $\kappa \circ \sigma$, where $\sigma \colon \mathcal{D}_i \to M_q$ and $\sigma \colon M_q \to A$ are ccp. maps. Since $\mathcal{D}_i$ is separable, we may therefore assume that each $\overline{\psi}_{l,n}^{(i)}$ is of this form. It follows from Arveson's extension theorem, that any ccp. map $\mathcal{D}_i \to M_q$ extends to a ccp. map $\mathcal{D} \to M_q$. Hence, each of the maps $\overline{\psi}_{l,n}^{(i)} \colon \mathcal{D}_i \to A$ extend to ccp. maps $\rho_{l,n}^{(i)} \colon \mathcal{D} \to A$.

Now, let $\{d_s\}_{s\geq 1} \subseteq \mathcal{D}$ and $\{a_t\}_{t\geq 1} \subseteq A$ be dense countable sets. Note that we may assume $\{d_s\}_{s\geq 1} \subseteq \bigcup_{i\geq 1} \mathcal{D}_i$. For each i, j and l, let $(s_{jl,n}^{(i)})_{n\geq 1} \in \prod_{n\geq 1} A$ be a lift of $s_{jl}^{(i)}$, and, for each $r \geq 1$, let $F_r := \{d_1, \dots, d_r\}$. We show that there exist sequences $(i_r)_{r\geq 1}$ and $(n_r)_{r\geq 1}$, such that $\big(\rho_{l,n_r}^{(i_r)}\big)_{r\geq 1}$ implements ccp. order zero maps $\rho_l \colon \mathcal{D} \to F(A)$, and $\big(s_{jl,n_r}^{(i_r)}\big)_{r\geq 1} \in \prod_{r\geq 1} A$ gives rise to elements $t_{jl} \in F(A)$, with the desired properties.

Let $r \geq 1$ be given, and choose $i_r \geq 1$ such that $F_r \subseteq \mathcal{D}_{i_r}$. By the choices above, and lemma 6.4, we may choose $n_r \geq 1$ such that

$$\Big\| \mathbf{1} - \sum_{l=1}^{k} \sum_{j=1}^{m} \big(s_{jl,n_r}^{(i_r)}\big)^* \rho_{l,n_r}^{(i_r)}(\mathbf{1}) s_{jl,n_r}^{(i_r)} \Big\| < \frac{1}{r},$$

$$\max_{1\leq l\leq k} \max_{x,y\in F_r} \big(\|\rho_{l,n_r}^{(i_r)}(x)\rho_{l,n_r}^{(i_r)}(y)\| - \|\rho_{l,n_r}^{(i_r)}(xy)\| \big) < \frac{1}{r}$$

$$\max_{1\leq l\leq k} \max_{x\in F_r} \max_{1\leq t\leq r} \Big\| \big[\rho_{l,n_r}^{(i_r)}(x), a_t\big] \Big\| < \frac{1}{r},$$

$$\max_{1\leq j\leq m} \max_{1\leq l\leq k} \max_{1\leq t\leq r} \Big\| \big[s_{jl,n_r}^{(i_r)}, a_t\big] \Big\| < \frac{1}{r}.$$

It is straightforward to check that, with this choice of n_r, the desired relations are satisfied. □

Theorem 6.6. *Let A be a unital, separable C^*-algebra. If $F(A)$ is k-locally almost divisible, for some $k \geq 1$, then there exist order zero maps $\psi_1, \dots, \psi_k \colon \mathcal{Z} \to F(A)$ and elements $t_{0,1}, \dots, t_{0,k}, t_{1,1}, \dots, t_{1,k} \in F(A)$ such that*

$$\mathbf{1} = \sum_{j=1}^{k} \sum_{i=0}^{1} t_{i,j}^* \psi_j(\mathbf{1}) t_{i,j}.$$

Proof. It follows from [11, Proposition 2.5], that $\mathcal{Z}$ is the closure of an increasing sequence of prime dimension drop algebras, i.e., C^*-algebras of the form $I(p,q)$, where $p,q \geq 1$ are relatively prime integers. By Proposition 6.5, it therefore suffices to show that, for each pair of such integers, there exist k ccp. order zero maps $\varphi_1,\dots,\varphi_k\colon I(p,q) \to F(A)$, and elements s_{ij}, for $i = 0,1$ and $j = 1,\dots,k$, such that

$$\mathbf{1} = \sum_{j=1}^{k}\sum_{i=0}^{1} s_{ij}^{*}\varphi_j(\mathbf{1})s_{ij}.$$

Fix relatively prime integers $p,q \geq 1$. It follows, from Remark 6.2, that there exist order zero maps $\rho_1^{(0)},\dots,\rho_k^{(0)}\colon M_p \to F(A)$ and elements $r_{ij}^{(0)} \in F(A)$, for $i = 0,1$ and $j = 1,\dots,k$, such that

$$\sum_{j=1}^{k}\sum_{i=0}^{1} \big(r_{ij}^{(0)}\big)^{*}\big(\rho_j^{(0)}(\mathbf{1}_p)\big)r_{ij}^{(0)}.$$

Let B denote the separable C^*-algebra generated by the images of the $\rho_j^{(0)}$'s and the $r_{ij}^{(0)}$'s. Using [12, Proposition 1.12], it follows, as above, that there exist ccp. order zero maps $\rho_1^{(1)},\dots,\rho_k^{(1)}\colon M_q \to F(B,A)$ and elements $r_{ij}^{(1)} \in F(B,A)$, for $i = 0,1$ and $j = 1,\dots,k$, such that

$$\sum_{j=1}^{k}\sum_{i=0}^{1} \big(r_{ij}^{(1)}\big)^{*}\rho_j^{(1)}(\mathbf{1}_q)r_{ij}^{(1)} = \mathbf{1}.$$

By Lemma 6.3, there exists, for each $j = 1,\dots,k$, a ccp. order zero map $\varphi_j\colon I(p,q) \to F(A)$, such that $\varphi_j(\mathbf{1}) = (\rho_j^{(0)}(\mathbf{1}_p) + \rho_j^{(1)}(\mathbf{1}_q))/2$. Note that

$$2\cdot\sum_{j=1}^{k}\sum_{i=0}^{1} \big(r_{ij}^{(0)}\big)^{*}\varphi_j(\mathbf{1})r_{ij}^{(0)} \geq \sum_{j=1}^{k}\sum_{i=0}^{1} \big(r_{ij}^{(0)}\big)^{*}\rho_j^{(0)}(\mathbf{1}_p)r_{ij}^{(0)} = \mathbf{1}.$$

Hence, the element $\sum_{j=1}^{k}\sum_{i=0}^{1}\big(r_{ij}^{(0)}\big)^{*}\varphi_j(\mathbf{1})r_{ij}^{(0)}$ is invertible, and the claim therefore easily follows. □

Suppose A is a unital and separable C^*-algebra for which there exist ccp. order zero maps $\psi_1,\dots,\psi_k\colon \mathcal{Z} \to F(A)$, such that $\sum_{j=1}^{k}\psi_j(\mathbf{1}) = \mathbf{1}$. It is not difficult to check, using [23, Lemma 4.2], that $F(A)$ is k-locally almost divisible. The above theorem may be viewed as a partial converse to this statement.

6.1. **Simple C^*-algebras.** We now turn out attention to unital, simple and separable C^*-algebras. First, we show that the condition $\text{L-Div}_k(F(A)) < \infty$ ensures the existence of a uniformly tracially large ccp. order zero map $M_2 \to F(A)$, using techniques from [24].

Lemma 6.7. *Given integers $k,m,n \geq 1$, there exists $\beta_{k,m,n} > 0$ such that the following statement holds: For any unital and separable C^*-algebra A with $T(A) \neq \emptyset$ and $\text{L-Div}_k(F(A),m) \leq n$, and any separable subset $X \subseteq A_\omega$, there exist ccp. order zero maps $\psi_1,\dots,\psi_k\colon M_m \to A_\omega \cap C^*(A,X)'$ satisfying*

$$\tau\Big(\sum_{j=1}^{k}\psi_j(1_m)b\Big) \geq \beta_{k,m,n}\tau(b),$$

for all $\tau \in T(A_\omega)$ and all positive elements $b \in C^*(A, X)$.

Proof. Set $l := \lceil n/m \rceil$, $\beta_{k,m,n} := 1/(k^2 l)$, and let $X \subseteq A_\omega$ be an arbitrary separable subset. By Remark 6.2 and [12, Proposition 1.12], there exist ccp. order zero maps $\psi_1, \dots, \psi_k \colon M_m \to A_\omega \cap C^*(A, X)'$ and contractions $t_{ij} \in A_\omega \cap C^*(A, X)'$, for $i = 1, \dots, l$ and $j = 1, \dots, k$, satisfying

$$\psi_i(\mathbf{1}) t_{ij} = \frac{1}{k} t_{ij}, \quad \mathbf{1} = \sum_{q=1}^{k} \sum_{p=1}^{l} t_{pq}^* t_{pq}, \quad \| \sum_{q=1}^{k} \psi_q(\mathbf{1}) \| \leq 1$$

for every i, j. Note that we have scaled the maps ψ_j, to ensure that the sum $\sum_{j=1}^{k} \psi_j(\mathbf{1})$ is a contraction, which means that the conditions above are slightly different from those in Remark 6.2.

Suppose, towards a contradiction, that there exists some positive element $b \in C^*(A, X)$ and a tracial state $\tau \in T(A_\omega)$, such that $\tau(\psi_j(\mathbf{1})b) < 1/(k^2 l)\tau(b)$ for each $j = 1, \dots, k$. Since each t_{ij} is a contraction, it follows that

$$\tau(t_{ij}^* t_{ij} b) = k \cdot \tau(t_{ij}^* \psi_i(\mathbf{1}) t_{ij} b) = k \cdot \tau(\psi_i(\mathbf{1})^{1/2} t_{ij} t_{ij}^* \psi_i(\mathbf{1})^{1/2} b) < \frac{1}{kl}$$

for each i, j. Hence,

$$\tau(b) = \tau\Big(\sum_{q=1}^{k} \sum_{p=1}^{l} t_{pq}^* t_{pq} b \Big) = \sum_{q=1}^{k} \sum_{p=1}^{l} \tau(t_{pq}^* t_{pq} b) < \sum_{q=1}^{k} \sum_{p=1}^{l} \frac{1}{kl} \tau(b) = \tau(b),$$

which is clearly a contradiction. It follows that

$$\tau\Big(\sum_{j=1}^{k} \psi_j(\mathbf{1}_m) b \Big) \geq \frac{1}{k^2 l} \tau(b) = \beta_{k,m,n} \tau(b), \tag{3}$$

for every $\tau \in T(A_\omega)$ and every positive element $b \in C^*(A, X)$. □

Given a unital and separable C^*-algebra A with $T(A) \neq \emptyset$, let $T_\omega(A)$ denote the set of tracial states τ on A_ω of the form

$$\tau([(a_n)_n]) = \lim_{n \to \omega} \tau_n(a_n), \quad (a_n)_n \in \ell^\infty(A),$$

where $(\tau_n)_n$ is a sequence of tracial states on A.

Lemma 6.8. *Let A be a unital and separable C^*-algebra with $T(A) \neq \emptyset$. Let $(a_n)_n, (b_n)_n$ and $(d_n)_n$ be sequences of positive elements in $\ell^\infty(A)$, let $a := [(a_n)_n] \in A_\omega$, $b := [(b_n)_n] \in A_\omega$ and $d := [(d_n)_n] \in A_\omega$, and $\beta > 0$ be an arbitrary positive number. Then $\sigma(da) \geq \beta\sigma(b)$, for all $\sigma \in T_\omega(A)$ if, and only if,*

$$\lim_{n \to \omega} \inf_{\tau \in T(A)} \tau(d_n a_n - \beta b_n) \geq 0.$$

Proof. 'If': Let $(\tau_n)_n$ be an arbitrary sequence of tracial states on A. Then

$$\lim_{n \to \omega} \tau_n(d_n a_n - \beta b_n) \geq \lim_{n \to \omega} \inf_{\tau \in T(A)} \tau(d_n a_n - \beta b_n) \geq 0.$$

Hence, the tracial state $\sigma \in T_\omega(A)$ associated with $(\tau_n)_n$ satisfies

$$\sigma(da - \beta b) = \sigma(da) - \beta\sigma(b) \geq 0,$$

i.e., $\sigma(da) \geq \beta\sigma(b)$.

'Only if': Suppose $\lim_{n\to\omega}\inf_{\tau\in T(A)}\tau(d_na_n-\beta b_n)<0$. For each natural number $n\in\mathbb{N}$, choose $\tau_n\in T(A)$, such that $\inf_{\tau\in T(A)}\tau(d_na_n-\beta b_n)=\tau_n(d_na_n-\beta b_n)$. Then, with $\sigma\in T_\omega(A)$ denoting the tracial state associated with the sequence $(\tau_n)_n$, we find that

$$\sigma(da-\beta b)=\lim_{n\to\omega}\tau_n(d_na_n-\beta b_n)=\lim_{n\to\omega}\inf_{\tau\in T(A)}\tau(d_na_n-\beta b_n)<0.$$

Hence, $\sigma(da)<\beta\sigma(b)$. □

Lemma 6.9. *Given integers $k,l,n\geq 1$, there exists $\gamma_{k,l,n}>0$, such that the following statement holds: If A is a unital and separable C^*-algebra such that $T(A)\neq\emptyset$ and $\text{L-Div}_k(F(A),2k)\leq n$, then, for every separable subset $X\subseteq A_\omega$, there exist pairwise orthogonal and positive contractions $d^{(1)},\dots,d^{(l)}\in A_\omega\cap C^*(A,X)'$ satisfying*

$$\tau(d^{(i)}b)\geq\gamma_{k,l,n}\tau(b),$$

for every $i=1,\dots,l$, $\tau\in T_\omega(A)$ and positive element $b\in C^*(A,X)$.

Proof. It suffices to prove that, for each $m\geq 1$, there exists elements $d^{(l)}$, for $l=1,\dots,2^m$, satisfying the statement above. We prove the statement by induction on m.

Let $\beta_{k,2k,n}$ be given as in Lemma 6.7. Set $\gamma_{k,2,n}:=\beta_{k,2k,n}/(2k)$ and let $X\subseteq A_\omega$ be an arbitrary separable subset. By Lemma 6.7 there exist order zero maps $\psi_1,\dots,\psi_k\colon M_{2k}\to A_\omega\cap C^*(A,X)'$ satisfying

$$\tau\Big(\sum_{j=1}^k\psi_j(\mathbf{1}_{2k})b\Big)\geq\beta_{k,2k,n}\tau(b),$$

for all $\tau\in T_\omega(A)$ and positive elements $b\in C^*(A,X)$. Note also that we may arrange that $\sum_{j=1}^k\psi_j(\mathbf{1}_{2k})$ is a contraction. We first prove that, for every $\eta>0$, there exist positive, orthogonal contractions $d_\eta^{(1)},d_\eta^{(2)}\in A_\omega\cap C^*(A,X)'$ such that

$$\tau(d_\eta^{(i)}b)\geq\gamma_{k,2,n}\tau(b)-\eta,$$

for $i=1,2$, every $\tau\in T(A_\omega)$ and every positive element $b\in C^*(A,X)_+$.

Let $0<\eta<1/2$ be arbitrary. For $\delta_2>\delta_1\geq 0$, let $g_{\delta_1,\delta_2}\colon\mathbb{R}_+\to[0,1]$ be given by

$$g_{\delta_1,\delta_2}(t)=\begin{cases}0 & \text{for } 0\leq t\leq\delta_1\\ \frac{t-\delta_1}{\delta_2-\delta_1} & \text{for } \delta_1\leq t\leq\delta_2\\ 1 & \text{for } \delta_2\leq t.\end{cases}\tag{4}$$

Let

$$d_\eta^{(1)}=g_{\eta,2\eta}\Big(\sum_{j=1}^k\psi_j(e_{11})\Big),\quad d_\eta^{(2)}=\mathbf{1}-g_{0,\eta}\Big(\sum_{j=1}^k\psi_j(e_{11})\Big).$$

Clearly $d_\eta^{(0)}\perp d_\eta^{(1)}$. Fix arbitrary $\tau\in T_\omega(A)$ and positive $b\in C^*(A,X)$. Note that, since $d_\eta^{(1)}+\eta\mathbf{1}\geq\sum_{l=1}^k\psi_l(\mathbf{1})$, it follows that $\tau(d_\eta^{(1)}b)\geq\gamma_{k,2,n}\tau(b)-$

η. Furthermore,

$$\begin{aligned}\tau\big((\mathbf{1}-d_\eta^{(2)})b\big) &= \tau\Big(g_{0,\eta}\big(\sum_{j=1}^k \psi_j(e_{11})\big)b\Big)\\ &\le \lim_{p\to\infty}\tau\Big(\big(\sum_{j=1}^k \psi_j(e_{11})\big)^{1/p}b\Big)\\ &\le \sum_{j=1}^k \lim_{p\to\infty}\tau\big(\psi_j(e_{11})^{1/p}b\big)\\ &= \sum_{j=1}^k \lim_{p\to\infty}\tau\big((\psi_j)^{1/p}(e_{11})b\big)\\ &= \sum_{j=1}^k \lim_{p\to\infty}\frac{1}{2k}\tau\big((\psi_l)^{1/p}(\mathbf{1}_{2k})b\big)\\ &\le \frac{k}{2k}\tau(b)=\frac{1}{2}\tau(b).\end{aligned}$$

Here we have used that $\langle\sum_{j=1}^k \psi_j(e_{11})\rangle \le \sum_{j=1}^k\langle\psi_j(e_{11})\rangle$ in the Cuntz semigroup $\mathrm{Cu}\big(A_\omega\cap C^*(A,X)'\big)$, and that the map $a\mapsto \lim_{p\to\infty}\tau(a^{1/p}b)$ is a dimension function on $A_\omega\cap C^*(A,X)'$, for every $\tau\in T_\omega(A)$ and $b\in C^*(A,X)$. Additionally, we have used that the functional calculus for order zero maps $\varphi\colon B\to D$, see [28, Corollary 4.2], satisfies $f(\varphi)(p)=f(\varphi(p))$, whenever $p\in B$ is a projection. In particular, $\tau(d_\delta^{(2)}b)\ge\gamma_{k,2,n}\tau(b)$.

Since $\tau\in T_\omega(A)$, the positive element $b\in C^*(A,X)$ and $\eta>0$ was arbitrary, an easy application of the ε-test ([14, Lemma 3.1]), along with Lemma 6.8, yields orthogonal, positive elements $d^{(1)},d^{(2)}$ such that

$$\tau(d^{(i)}b)\ge\gamma_{k,2,n}\tau(b),$$

for every $i=1,2$, $\tau\in T_\omega(A)$ and positive element $b\in C^*(A,X)$. Hence, the induction start has been established

Now, suppose that $\gamma_{k,2^{m-1},n}>0$, and orthogonal positive elements $\overline{d}^{(l)}$, for $l=1,\dots,2^{m-1}$, have been found with the desired properties. Let $\gamma_{k,2^m,n}$ denote the positive number $\gamma_{k,2^{m-1},n}\cdot\gamma_{k,2,n}$ and

$$Y=X\cup\{\overline{d}^{(l)}\mid l=1,\dots,2^{m-1}\}.$$

By the induction start, there exists $d^{(1)},d^{(2)}\in A_\omega\cap C^*(A,Y)'$ such that $\tau(d^{(i)}x)\ge\gamma_{k,2,n}\tau(x)$, for $i=1,2$, every $\tau\in T_\omega(A)$, and every positive element $x\in C^*(A,Y)$. In particular, for every $l=1,\dots,2^{m-1}$, every $\tau\in T_\omega(A)$, and positive element $b\in C^*(A,X)$, we find

$$\tau(d^{(i)}\overline{d}^{(l)}b)\ge\gamma_{k,2,n}\tau(\overline{d}^{(l)}b)\ge\gamma_{k,2^m,n}\tau(b),$$

for $i=1,2$. Since $d^{(1)},d^{(2)}\in A_\omega\cap C^*(A,Y)'$, it follows that the elements $d^{(i)}\overline{d}^{(l)}$, for $i=1,2$ and $l=1,\dots,2^{m-1}$, are positive and pairwise orthogonal, whence the desired result follows. □

Remark 6.10. Let us briefly outline how to connect the results here, to the results of [24]. First, we recall the notation from this paper.

Given a C^*-algebra A, let

$$A_\infty := \ell^\infty(A)/\{(a_n)_n \in \ell^\infty(A) \mid \lim_{n\to\infty} \|a_n\| = 0\}.$$

Clearly, there is a natural embedding $A \to A_\infty$, defined exactly as the embedding $A \to A_\omega$. If, additionally, $T(A) \neq \emptyset$, we let $T_\infty(A)$ denote the set of tracial states on A_∞ on the form

$$\tau([(a_n)_n]) = \lim_{n\to\omega'} \tau_n(a_n), \quad (a_n)_n \in \ell^\infty(A),$$

where ω' is some ultrafilter on $\mathbb{N}$ and $(\tau_n)_n \subseteq T(A)$.

Now, let A be a unital and separable C^*-algebra with $T(A) \neq \emptyset$, such that $\text{L-Div}_k(F(A), 2) \leq \text{L-Div}_k(F(A), 2k) = n < \infty$. Let $\gamma := \gamma_{k,k,n} > 0$ be given as in Lemma 6.9 and $\beta_{k,2,n} > 0$ be given as in Lemma 6.7. Fix an arbitrary separable subspace $X \subseteq A_\omega$. Now, by Lemma 6.7, there exist ccp. order zero maps $\varphi_1, \ldots, \varphi_k \colon M_2 \to A_\omega \cap C^*(A, X)'$ such that

$$\tau\Big(\sum_{j=1}^{k} \varphi_j(1_m) b\Big) \geq \beta\tau(b),$$

for all $\tau \in T(A_\omega)$ and all positive elements $b \in C^*(A, X)$. Let $Y := X \cup \text{span}\{\varphi_j(M_2) \mid j = 1, \ldots, k\}$ and, using Lemma 6.9, choose pairwise orthogonal, positive contractions $d^{(1)}, \ldots, d^{(k)}$, such that

$$\tau(d^{(i)} c) \geq \gamma\tau(c),$$

for every $i = 1, \ldots, k$, $\tau \in T_\omega(A)$ and positive element $c \in C^*(A, Y)$. Let $\Phi \colon M_2 \to A_\omega \cap C^*(A, X)'$ be given by

$$\Phi(x) = \sum_{j=1}^{k} d^{(j)} \varphi_j(x), \quad x \in M_2.$$

Straightforward computations show that Φ is a ccp. order zero map satisfying $\tau(\Phi(\mathbf{1}_2)b) \geq (\gamma\beta)\tau(b)$, for every $\tau \in T_\omega(A)$ and every positive element $b \in C^*(A, X)$. Using Lemma 6.8 and a diagonal argument, it follows that, for every separable subspace $Z \subseteq A_\infty$, there exists a ccp. order zero map $\Psi \colon M_2 \to A_\infty \cap A' \cap Z'$, such that

$$\tau(\Psi(\mathbf{1}_2) z) \geq (\beta\gamma)\tau(z),$$

for every $\tau \in T_\infty(A)$ and positive element $z \in C^*(A, X)$. Hence, it follows from [24, Lemma 4.5], that there exists a ccp. order zero map $\tilde{\Psi} \colon M_2 \to A_\infty \cap A' \cap Z'$ such that $\tau(\tilde{\Psi}(\mathbf{1}_2)) = 1$, for all $\tau \in T_\infty(A)$.

Theorem 6.11. *Suppose A is a unital, simple, separable, nuclear and stably finite C^*-algebra with property (SI). Then the following are equivalent.*

(i) *There exists $k \geq 1$ such that* $\text{L-Div}_k(F(A), m) < \infty$*, for all $m \geq 1$.*
(ii) *There exists $k \geq 1$ such that* $\text{L-Div}_k(F(A), 2k) < \infty$.
(iii) $A \otimes \mathcal{Z} \cong A$.

Proof. (i)⇒(ii) is trivial.

(ii)⇒(iii) By Remark 6.10, there exists a ccp. order zero map $\Phi\colon M_2 \to A_\infty \cap A'$ such that $\tau(\Phi(\mathbf{1}_2)) = 1$ for all $\tau \in T_\infty(A)$. Hence, by the proof of the main result in [16], see also [24, Theorem 2.6], the desired result follows.

(iii)⇒(i): This follows from the fact that, if $A \otimes \mathcal{Z} \cong A$, then there is a unital embedding $\mathcal{Z} \to F(A)$, whence $\text{L-Div}_1(F(A), m) \leq m+1$, for every $m \geq 2$, by [23, Lemma 4.2]. □

The remainder of this section will be spent proving that the assumption that A has property (SI), in the above theorem, is redundant if we assume that $F(A)$ is k-locally almost divisible, rather than $\text{L-Div}_k(F(A), m) < \infty$, for all $m \geq 2$.

Recall the following definition from [27]. Given a C^*-algebra A and a positive element $a \in M_\infty(A) = \bigcup_{n\geq 1} M_n(A)$, we let $\text{her}(a) \subseteq M_\infty$ denote the hereditary subalgebra generated by a in $M_m(A) \subseteq M_\infty(A)$, where $m \geq 1$ is chosen such that $a \in M_m(A)$. If A is unital and τ is a tracial state on A, then we also let τ denote the extension of τ to $M_\infty(A)$.

Definition 6.12. Let A be a unital and exact C^*-algebra. Then we say that A is *tracially m-almost divisible* if, for any positive contraction $a \in M_\infty(A)$, $\varepsilon > 0$ and $0 \neq l \in \mathbb{N}$, there exists a ccp. order zero map $\psi\colon M_l \to \text{her}(a) \subseteq M_\infty(A)$, such that
$$\tau(\psi(\mathbf{1}_l)) \geq \frac{1}{m+1}\tau(a) - \varepsilon,$$
for all $\tau \in T(A)$.

We use Lemma 6.9 to establish tracial m-almost divisibility of A, when $F(A)$ is k-locally almost divisible, for some $k \geq 1$.

Lemma 6.13. *Let A be a unital, separable and exact C^*-algebra such that $T(A) \neq \emptyset$. If there is an integer $k \geq 1$, such that $F(A)$ is k-locally almost divisible, then there exists $\eta_k > 0$ such that, for any positive contraction $a \in M_\infty$, $\varepsilon > 0$ and integer $l \geq 1$, there exists a ccp. order zero map $\psi\colon M_l \to her(a) \subseteq M_\infty(A)$ satisfying*
$$\tau(\psi(\mathbf{1}_l)) \geq \eta_k \tau(a) - \varepsilon,$$
for all $\tau \in T(A)$.

Proof. Note that, for any $n \in \mathbb{N}$, the map $\kappa\colon A \to M_n(A)$, given by $\kappa(a) = \text{diag}(a, a, \ldots, a)$, induces an isomorphism $F(A) \to F(M_n(A))$. Hence, it suffices to prove the statement for a positive contraction $a \in A$.

It follows from the proof of Lemma 6.7 that, with $\beta_k = 1/(2k^2)$, there exist, for any integer $l \geq 1$, ccp. order zero maps $\varphi_1, \ldots, \varphi_k\colon M_l \to F(A)$ satisfying
$$\tau(\sum_{j=1}^{k} \varphi_j(\mathbf{1}_l)b) \geq \beta_k \tau(b),$$
for all positive elements $b \in A$ and any $\tau \in T_\omega(A)$. Let $B \subseteq F(A)$ denote the C^*-algebra generated by the images of the φ_j's. Choose $\gamma_k := \gamma_{k,k,2k+1} > 0$ as in Lemma 6.9 and pairwise orthogonal, positive elements $d^{(1)}, \ldots, d^{(k)} \in F(B, A)$ such that
$$\tau(d^{(i)}x) \geq \gamma_k \tau(x),$$

for all positive elements $x \in C^*(A, B)$ and all $\tau \in T_\omega(A)$. Let $\eta_k := \gamma_k \cdot \beta_k$.

Fix a positive contraction $a \in A$, $\varepsilon > 0$ and $l \geq 1$. Let $\varphi_1, \dots, \varphi_k \colon M_l \to F(A)$ be given as above, and let $\overline{\psi} \colon M_l \to D := \overline{aA_\omega a} \subseteq A_\omega$ denote the ccp. order zero map given by

$$\overline{\psi}(x) = a \sum_{j=1}^{k} d^{(j)} \varphi_j(x), \quad x \in M_l.$$

It is straightforward to check that $\overline{\psi}$ is a ccp. order zero map, and, by the choices above, for any $\tau \in T_\omega(A)$ we have

$$\tau(\overline{\psi}(\mathbf{1}_l)) = \sum_{j=1}^{k} \tau(d^{(j)} a \varphi_j(\mathbf{1}_l)) \geq \gamma_k \cdot \tau\big(\sum_{j=1}^{k} \varphi_j(\mathbf{1}_l) a\big) \geq \eta_k \cdot \tau(a).$$

In fact, going through the proofs, it follows that we may choose $\eta_k \geq 1/(4k^3)$.

Now, note that $\mathrm{her}(a)_\omega \subseteq A_\omega$ is a hereditary subalgebra containing a, whence $D \subseteq \mathrm{her}(a)_\omega$. Thus we may lift $\overline{\psi}$ to a sequence of ccp. order zero maps $\psi_n \colon M_l \to \mathrm{her}(a)$, since ccp. order zero maps with finite-dimensional domain are liftable. An application of Lemma 6.8 therefore completes the proof. □

Proposition 6.14. *Let A be a unital, simple, separable and exact C^*-algebra with $T(A) \neq \emptyset$. If there exists an integer $k \geq 1$ such that $F(A)$ is k-locally almost divisible, then there exists an integer $\tilde{m} \geq 1$, such that A has strong tracial $\tilde{m}$-comparison.*

Proof. It follows from Proposition 5.7 (and the comments above), that A has $(k-1)$-comparison in the sense of [27, Definition 3.1(i)]. Let η_k be given as in Lemma 6.13 and $\overline{m}$ denote the least integer such that $1/(\overline{m}+1) \leq \eta_k$. By the proof, we may choose $\overline{m} \leq 4k^3 - 1$. Since A has $(k-1)$-comparison and is tracially $\overline{m}$-almost divisible, it follows from [27, Proposition 3.9], that there exists $\tilde{m}$, depending on k and $\overline{m}$, such that A has strong tracial $\tilde{m}$-comparison. □

Nota that the following proposition relies heavily on [14], which in turn relies on [16].

Proposition 6.15. *Let A be a unital, simple, separable, nuclear and stably finite C^*-algebra. If $F(A)$ is k-locally almost divisible, for some $k \geq 1$, then A has local weak comparison. In particular, A has property (SI).*

Proof. The first statement follows from Proposition 6.14 and [14, Lemma 2.4]. The last statement then follows from [14, Corollary 5.10]. □

Theorem 6.16. *Let A be a unital, simple, separable, nuclear and stably finite C^*-algebra. If there exists an integer $k \geq 1$ such that $F(A)$ is k-locally almost divisible, then $A \otimes \mathcal{Z} \cong A$.*

Proof. It follows from Proposition 6.15, that A has property (SI), whence Theorem 6.11 implies that $A \otimes \mathcal{Z} \cong A$. □

Finally, we recover Winter's seminal result, see [27], that any unital, simple, separable and infinite-dimensional C^*-algebra with $\dim_{\mathrm{nuc}}(A) < \infty$, satisfies $A \otimes \mathcal{Z} \cong A$. Although the proof avoids the more technical parts of [27], it should be noted that many results of this section depend, both explicitly and implicitly, on results and techniques from [27]. Furthermore, we do require the full force of Kirchberg and Rørdam's proof that local weak comparison imply property (SI), for unital, simple, separable, nuclear, stably finite C^*-algebras, and [29, Proposition 4.3]

Corollary 6.17. *Let A be a unital, simple, separable C^*-algebra such that $\dim_{\mathrm{nuc}}(A) < \infty$. Then $A \otimes \mathcal{Z} \cong A$.*

Proof. Suppose that $\dim_{\mathrm{nuc}}(A) = n < \infty$. Since A is non-elementary, it follows from [29, Proposition 4.3] that $\mathrm{cov}(F(A)) \leq (n+1)^2$. It therefore follows from Proposition 5.2 and (the proof of) Lemma 5.5, that $F(A)$ is k-locally almost divisible, with $k := 2(n+1)^2$. Hence, the desired result follows from Theorem 6.16. □

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A NOTE ON SIGMA IDEALS

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ABSTRACT. We show that for any unital, separable and $\mathcal{Z}$-stable C^*-algebra such that the set of extremal tracial states contains a compact and connected subset with at least 2 points, there exists a closed 2-sided ideal in the central sequence algebra which is not a σ-ideal.

1. INTRODUCTION

The notion of σ-ideals was introduced by Kirchberg in [4], and shown to be a particularly useful notion for the study of central sequence algebras. In particular, if D is a C^*-algebra, $I \subseteq D$ is a σ-ideal and $B \subseteq D$ is a separable C^*-subalgebra, it follows that the sequence

$$0 \to I \cap B' \to D \cap B' \to D/I \cap \big(B/(B \cap I)\big)'$$

is short exact, and strongly locally semi-split (see [4, Proposition 1.6]). This result was used in [5, Remark 4.7] to give an elegant proof of [5, Theorem 3.3].

Between [4, Corollary 1.7] and [5, Remark 4.7], it is shown that many naturally occurring closed 2-sided ideals of the ultrapower A_ω of a C^*-algebra A are, in fact, σ-ideals, but the question of whether *every* closed 2-sided ideal of A_ω is a σ-ideal was left unanswered. Although the general feeling seems to be that the answer to this question should be no, there are, to the best of the authors knowledge, no counter-examples in the literature.

In this note, we show that for a substantial class of unital, separable and $\mathcal{Z}$-stable C^*-algebras A, there exists an ideal $I \subseteq A_\omega$, which is not a σ-ideal. The proof relies on the fact any σ-ideal is stable under application of the ε-test (see Definition 2.1 for the precise definition). In fact, this property characterizes the σ-ideals of A_ω, by Proposition 2.3. We also make extensive use of the properties of central sequence algebras established in [5] and [2].

2. σ-IDEALS

Fix a free ultrafilter ω on $\mathbb{N}$. Given a C^*-algebra A, let $\ell^\infty(A)$ denote the set of bounded sequences in A, and the *ultrapower* A_ω of A be given by

$$A_\omega = \ell^\infty(A)/\{(a_n)_n \in \ell^\infty(A) \mid \lim_{n\to\omega} \|a_n\| = 0\}.$$

Given $(a_n)_n \in \ell^\infty(A)$, let $[(a_n)] \in A_\omega$ denote the image of $(a_n)_n$ under the quotient map and identify A with the image under the embedding $A \to A_\omega$, given by $a \mapsto [(a, a, a, \dots)]$. We use $F(A)$ to denote the *central sequence algebra* $A_\omega \cap A'$. The notation $F(A)$ was introduced by Kirchberg in [4], wherein the definition of $F(A)$ was extended, in a useful way, to all separable C^*-algebras. Although we only consider unital C^*-algebras in this note, we retain the notation to emphasize the connection with Kirchberg's work.

First we show that being a σ-ideal in $F(A)$ is equivalent to being stable under application of the ε-test. To be more precise, consider the following definition.

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Definition 2.1. Let A be a separable C^*-algebra, let ω be a free ultrafilter on $\mathbb{N}$ and let $I \subseteq A_\omega \cap A'$ be an ideal. We say that I *satisfies the ε-test property for ideals* if the following statement holds.

Suppose that, for some $r > 0$, and for each $n \in \mathbb{N}$ we are given a subset $X_n \subseteq (A)_r$ and, for each $k \in \mathbb{N}$, we are given a sequence $(f_n^{(k)})_{n\geq 1}$ of functions $f_n^{(k)} \colon X_n \to [0, \infty)$. For each $k \in \mathbb{N}$ define a new function $f_\omega^{(k)} \colon \prod_{n=1}^\infty X_n \to [0, \infty]$ by

$$f_\omega^{(k)}((s_1, s_2, \dots)) = \lim_{n\to\omega} f_n^{(k)}(s_n), \quad (s_n)_{n\geq 1} \in \prod_{n=1}^\infty X_n.$$

If, for each $m \in \mathbb{N}$ and $\varepsilon > 0$, there exists $s^\varepsilon = (s_n^\varepsilon)_{n\geq 1} \in \prod_{n\geq 1} X_n$ such that

$$f_\omega^{(k)}(s^\varepsilon) < \varepsilon, \quad \text{for } k = 1, \dots, m,$$

and $\pi_\omega((s_1^\varepsilon, s_2^\varepsilon, \dots)) \in I$, then there exists $t = (t_n)_{n\geq 1} \in \prod_{n\geq 1} X_n$, such that

$$f_\omega^{(k)}(t) = 0, \quad \text{for all } k \in \mathbb{N},$$

and $\pi_\omega((t_1, t_2, \dots)) \in I \subseteq A_\omega \cap A'$.

The following definition can be found in [4].

Definition 2.2. Let D be a C^*-algebra, and $J \subseteq D$ be an ideal. Then J is a *σ-ideal* if, for every separable sub-C^*-algebra $C \subseteq D$, there exists a positive contraction $e \in J$, such that $e \in C' \cap J$ and $ec = c$, for all $c \in J \cap C$.

Note that if $J \subseteq D$ is a σ-ideal and $B \subseteq D$ is a separable C^*-subalgebra, then $I \cap B' \subseteq D \cap B'$ is a σ-ideal.

Proposition 2.3. *Let A be a unital, separable C^*-algebra and $I \subseteq F(A)$ be an ideal. Then the following are equivalent:*

(i) *I is a σ-ideal.*
(ii) *For any $d \in I$, there exists $e \in I$ such that $ed = d$.*
(iii) *I satisfies the ε-test property for ideals.*

Proof. Obviously (i) implies (ii). We show (ii)$\Rightarrow$(iii)$\Rightarrow$(i).

Assume that I satisfies condition (ii). Let $\{a_k\}_{k\in\mathbb{N}} \subseteq (A)_1$ be a dense countable set. Suppose that, for some $r > 0$ and each $n \in \mathbb{N}$, we are given a subset $X_n \subseteq (A)_r$ and, for each $k \in \mathbb{N}$, let $f_n^{(k)} \colon X_n \to [0, \infty)$ be a sequence of functions. Let $f_\omega^{(k)}$ be given as in Definition 2.1. Assume further that, for each $m \in \mathbb{N}$, there exists $s^{(m)} := (s_1^{(m)}, s_2^{(m)}, \dots) \in \prod_{n=1}^\infty X_n$ such that $f_\omega^{(k)}(s^{(m)}) < \frac{1}{m}$, for $k = 1, \dots, m$, and $\pi_\omega(s^{(m)}) \in I \subseteq A_\omega \cap A'$. Let $c_0 \in C := C^*(\{\pi_\omega(s^{(m)}) \mid m \in \mathbb{N}\}) \subseteq I$ be a strictly positive element. By assumption, we may choose an element $e \in I$ such that $ec_0 = c_0$, whence $ec = c$, for all $c \in C^*(\{\pi_\omega(s^{(m)}) \mid m \in \mathbb{N}\})$.

Let $(e_1, e_2, \dots) \in \ell^\infty(A)$ be a lift of $e \in I$, such that $\sup_n \|e_n\| \leq \|e\|$. Define functions $g_n^{(k)} \colon (A)_r \to [0, \infty)$ by

$$\begin{aligned} g_n^{(1)}(z) &= \|e_n z - z\|, \\ g_n^{(2k)}(z) &= \|a_k z - z a_k\|, && k \geq 1 \\ g_n^{(2k+1)}(z) &= f_n^{(k)}(z), && k \geq 1 \end{aligned}$$

Since $\pi_\omega(s^{(m)}) \in A_\omega \cap A'$ and $e\pi_\omega(s^{(m)}) = \pi_\omega(s^{(m)})$, for each $m \in \mathbb{N}$, it follows that $g_\omega^{(l)}(s^{(m)}) < \frac{1}{m}$, for $l = 1, \dots, 2m+1$, whence [4, Lemma A.1] implies the existence of $t' \in \prod_{n\in\mathbb{N}} X_n \subseteq \ell^\infty(A)$ such that $g_\omega^{(k)}(t') = 0$, for all $k \in \mathbb{N}$. In particular $f_\omega^{(k)}(t') = g_\omega^{(2k+1)}(t') = 0$, for all $k \in \mathbb{N}$. Furthermore, $t := \pi_\omega(t') \in A_\omega \cap A'$, and since $et = t$ and $e \in I$, it follows that $t \in I$. Hence I satisfies the ε-test property for ideals.

Assume that I satisfies (iii). Let $C \subseteq F(A)$ be a separable sub-C^*-algebra, $(c_k)_k \subseteq C^*(A \cup C)$ be a dense, countable subset, and $c_0 \in C \cap I$ be a strictly positive contraction. For each $k \geq 0$, let $(c_n^{(k)})_n \in \ell^\infty(A)$ be a lift of c_k, and functions $f_n^{(k)} \colon (A)_1 \to [0, \infty)$ be given by

$$f_n^{(1)}(x) = \|c_n^{(0)} x^* x - c_n^{(0)}\|,$$
$$f_n^{(k)}(x) = \|c_n^{(k-1)} x^* x - x^* x c_n^{(k-1)}\|, \quad k \geq 2.$$

Let $(e_\alpha)_\alpha \subseteq I \subseteq F(A)$ be a quasi-central approximate unit. For each $m \in \mathbb{N}$, choose α_m such that $e_m := e_{\alpha_m}$ satisfies $\|c_0 e_m - c_0\| < \frac{1}{m}$ and $\|c_i e_m - e_m c_i\| < \frac{1}{m}$, for $i = 1, \ldots, m$. Let $s^{(m)} := (s_1^{(m)}, s_2^{(m)}, \ldots) \in \ell^\infty(A)$ be a positive and contractive lift of $e_m^{1/2}$.

Let $\varepsilon > 0$ be arbitrary, and choose $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$. Then

$$f_\omega^{(1)}(s^{(m)}) = \lim_{n \to \omega} \|c_n^{(0)}(s_n^{(m)})^2 - c_n^{(0)}\| = \|c_0 e_m - c_0\| < \frac{1}{m} < \varepsilon,$$

and, for $l = 1, \ldots, m$,

$$f_\omega^{(l+1)}(s^{(m)}) = \lim_{n \to \omega} \|c_n^{(l)}(s_n^{(m)})^2 - (s_n^{(m)})^2 c_n^{(l)}\| = \|c_l e_m - e_m c_l\| < \frac{1}{m} < \varepsilon.$$

Hence, since I satisfies the ε-test property for ideals, there exists $e^{(0)} \in \ell^\infty(A)$ such that $f_\omega^{(k)}(e) = 0$, for all $k \in \mathbb{N}$, and $\pi_\omega(e^{(0)}) \in I$. Setting $e := \pi_\omega(e^{(0)*} e^{(0)})$ it follows that $e \in I \cap C'$ and $c_0 e = e c_0 = c_0$. Since $c_0 \in I \cap C$ is strictly positive, $ec = c$, for all $c \in C \cap I$, and hence I is a σ-ideal. □

We now proceed to an example of an ideal in $F(A)$ which is *not* a σ-ideal. Given a non-empty subset $S \subseteq T(A)$, let $\| \cdot \|_{\omega,S}$ denote the semi-norm on A_ω given by

$$\big\| [(a_n)_n] \big\|_{\omega,S} = \lim_{n \to \omega} \sup_{\tau \in S} \tau(a_n^* a_n)^{1/2},$$

and $J_S \subseteq A_\omega$ denote the ideal $J_S = \{x \in F(A) \mid \|x\|_{\omega,S} = 0\}$. Note that $J_S \subseteq A_\omega$ is a σ-ideal, for every non-empty subset $S \subseteq T(A)$, by [5, Remark 4.7].

In the following results we use the theory of W^*-bundles, as introduced by Ozawa in [6] (see [6, Section 5] for the definition of a W^*-bundle), and further developed by other authors, in [2] and [3]. Most importantly, we use that for a given unital, separable C^*-algebra, and a compact, non-empty subset $K \subseteq \partial_e T(A)$, there exists an associated W^*-bundle M_K over K ([6, Theorem 3]), such that the natural map $A \to M_K$ induces an isomorphism $F(A)/(F(A) \cap J_K) \cong M_K^\omega \cap M_K'$ (this is a straightforward modification of [2, Lemma 3.10]). We let K^ω denote the compact Hausdorff space such that $C(K^\omega) \cong C(K)_\omega$. Furthermore, let $E_K \colon M_K^\omega \cap M_K' \to C(K^\omega)$ denote the conditional expectation coming from the W^*-bundle structure on $M_K^\omega \cap M_K'$ (see [2, Proposition 3.9]), and $\| \cdot \|_{2,u}$ denote the norm on $M_K^\omega \cap M_K'$, given by $\|x\|_{2,u} = \|E_K(x^*x)^{1/2}\|$. Finally, for each $\lambda \in K^\omega$, let π_λ denote the GNS representation of $M_K^\omega \cap M_K'$ associated with the tracial state $\tau_\lambda := \mathrm{ev}_\lambda \circ E_K$, and let $I_\lambda \subseteq M_k^\omega \cap M_K'$ denote the ideal $C_0(K^\omega \backslash \{\lambda\})(M_K^\omega \cap M_K')$.

Lemma 2.4. *Let A be a unital and separable C^*-algebra and $K \subseteq \partial_e T(A)$ be a non-empty compact subset. With M_K given as above, we have*

$$\ker(\pi_\lambda) = \overline{I_\lambda}^{\|\cdot\|_{2,u}},$$

for every $\lambda \in K^\omega$.

Proof. It follows, from basic properties of the GNS-representation, that

$$\begin{aligned}\ker(\pi_\lambda) &= \{x \in M_K^\omega \cap M_K' \mid \tau_\lambda(x^*x) = 0\} \\ &= \{x \in M_K^\omega \cap M_K' \mid E_K(x^*x) \in C_0(K^\omega \backslash \{\lambda\})\}.\end{aligned}$$

For arbitrary elements $f \in C_0(K_\omega \backslash \{\lambda\})$ and $y \in M_K^\omega \cap M_K'$ we have

$$E_K((fy)^*(fy)) = E_K\big((f^*f)(y^*y)\big) = (f^*f)E_K(y^*y) \in C_0(K^\omega \backslash \{\lambda\}),$$

since E is a conditional expectation. In particular, it follows that $I_\lambda \subseteq \ker(\pi_\lambda)$. Clearly, $\ker(\pi_\lambda)$ is closed in $\|\cdot\|_{2,u}$, whence one inclusion follows. For the other inclusion, let $(f_\alpha)_\alpha \subseteq C(K^\omega \backslash \{\lambda\})$ denote an approximate unit, and suppose $x \in \ker(\pi_\lambda)$. Then $E_K(x^*x) \in C_0(K^\omega \backslash \{\lambda\})$. Note that

$$\|x - f_\alpha x\|_{2,u} = \|E_K\big((1-f_\alpha)x^*x(1-f_\alpha)\big)^{1/2}\| = \|(1-f_\alpha)E_K(x^*x)^{1/2}\|,$$

whence $\lim_\alpha \|x - f_\alpha x\|_{2,u} = 0$. Since $f_\alpha x \in I_\lambda$, for each α, the desired result follows. ☐

We let $\rho_K \colon F(A) \to M_K^\omega \cap M_K'$ denote the quotient map. Note that, for arbitrary $x \in F(A)$, we have $\|x\|_{\omega,K} = \|\rho_K(x)\|_{2,u}$.

Proposition 2.5. *Suppose A is a unital, separable C^*-algebra, and $K \subseteq \partial_e T(A)$ is a non-empty, compact subset, such that $\rho_K^{-1}(I_\lambda) \subseteq F(A)$ is a σ-ideal, for each $\lambda \in K^\omega$. Then, for every $x \in M_K^\omega \cap M_K'$, the map $K_\omega \ni \lambda \mapsto \|\pi_\lambda(x)\|$ is continuous.*

Proof. We first show that $\ker(\pi_\lambda) = I_\lambda$, for all $\lambda \in K^\omega$. Note that, it is sufficient to show that $(I_\lambda)_1 \subseteq M_K^\omega \cap M_K'$ is closed in $\|\cdot\|_{2,u}$. Indeed, suppose this is the case, and that a contraction $x \in M_K^\omega \cap M_K'$ belongs to the closure of I_λ, with respect to $\|\cdot\|_{2,u}$. Let $\varepsilon > 0$ be given, and let $(e_\alpha)_\alpha \subseteq I_\lambda$ be an approximate unit. If $y \in I_\lambda$ is given such that $\|x-y\|_{2,u} < \varepsilon/3$, and β is given such that $\|y - e_\beta y\| \leq \|y - e_\beta y\|_{2,u} < \varepsilon/3$, then

$$\|x - e_\beta x\|_{2,u} \leq \|x - y\|_{2,u} + \|y - e_\beta y\|_{2,u} + \|e_\beta(y-x)\|_{2,u} < \varepsilon.$$

Hence $\lim_\alpha \|x - e_\alpha x\|_{2,u} = 0$, and since each $e_\alpha x$ is a contraction, the claim follows.

Let $J_\lambda := \rho_K^{-1}(I_\lambda)$, and $(x_n)_n \subseteq (I_\lambda)_1$ and $x \in (M_K^\omega \cap M_K')_1$ be given such that $\lim_{n\to\infty} \|x - x_n\|_{2,u} = 0$. Choosing $y_i \in (J_\lambda)_1$ and $y \in F(A)_1$ such that $\rho(y) = x$ and $\rho(y_i) = x_i$, for $i \geq 1$, it follows that

$$\lim_{n\to\infty} \|y - y_n\|_{\omega,K} = \lim_{n\to\infty} \|x - x_n\|_{2,u} = 0.$$

By Proposition 2.3, J_λ satisfies the ε-test property for ideals, whence there exists $z \in (J_\lambda)_1$ such that $\|y - z\|_{\omega,K} = 0$, and therefore $x = \rho_K(y) = \rho_K(z) \in I_\lambda$.

Now, fix $x \in M_K^\omega \cap M_K'$ and let $g_x \colon K^\omega \to \mathbb{R}_+$ denote the map $g_x(\lambda) = \|\pi_\lambda(x)\|$. We aim to prove that g_x is continuous. Since $M_K^\omega \cap M_K'$ is a W^*-bundle, it follows that g_x is lower semi-continuous. Indeed, by basic properties of the GNS-representation, for each $\lambda \in K^\omega$ we have that

$$g_x(\lambda) = \sup\{\tau_\lambda(c^*x^*xc) \mid \tau_\lambda(c^*c) \leq 1\} = \sup\{E(c^*x^*xc)(\lambda) \mid E(c^*c) \leq 1\}.$$

Thus g_x is lower semi-continuous, being the point-wise supremum of a family of continuous functions. Furthermore, since $I_\lambda = \ker(\pi_\lambda)$, it follows that, for each $\lambda \in K^\omega$, we have

$$g_x(\lambda) = \inf\{\|(1 - f(\lambda))x + fx\| \mid f \in C(K^\omega)\}.$$

Thus, being the pointwise infimum of a family of continuous functions, it follows that g_x is also upper semi-continuous, and therefore continuous. ☐

A stronger version of the following lemma was proven in [1, Proposition 1.5]. We supply a different proof.

Lemma 2.6. *Suppose K is a compact and connected Hausdorff space. Then K^ω is also connected.*

Proof. We prove the contrapositive statement, i.e., if K^ω is not connected, then neither is K. Hence, assume there exists non-zero positive contractions $f, g \in C(K)_\omega$, such that $fg = 0$ and $f + g = \mathbf{1}$. Let $(f_n)_n, (g_n)_n \in C(K)_\omega$ be positive, contractive lifts of f and g such that $f_n g_n = 0$, for all $n \in \mathbb{N}$. Since

$$\lim_{n\to\omega} \|\mathbf{1} - (f_n + g_n)\| = \|\mathbf{1} - (f + g)\| = 0,$$

it follows that, there exists $k \in \mathbb{N}$, such that both f_k and g_k are non-zero, $f_k g_k = 0$ and $\inf_{\lambda\in K} \big(f_k(\lambda) + g_k(\lambda)\big) > 0$, i.e., for all $\lambda \in K$ either $f_k(\lambda) > 0$ or $g_k(\lambda) > 0$. Let $U = \{\lambda \in K \mid f_k(\lambda) > 0\}$ and $V = \{\lambda \in K \mid g_k(\lambda) > 0\}$. Then $U, V \subseteq K$ are both open and non-empty, $U \cap V = \emptyset$ and $U \cup V = K$. Hence K cannot be connected. □

Proposition 2.7. *Let A be a unital, separable and $\mathcal{Z}$-stable C^*-algebra such that $\partial_e T(A)$ contains a compact and connected set K with $|K| \geq 2$. Then there is some point $\lambda \in K^\omega$ such that $\rho_K^{-1}(I_\lambda) \subseteq F(A)$ is not a σ-ideal.*

Proof. Since $A \otimes \mathcal{Z} \cong A$, it follows from [2, Proposition 3.11], [6, Theorem 15] and (the proof of) '(i)⇒(ii)' in [5, Proposition 5.12] (note that the assumptions listed in the proposition are only needed for the implication '(iii)⇒(iv)') that $M_K \cong C_\sigma(K, \mathcal{R})$, where $\mathcal{R}$ denotes the hyperfinite II_1-factor. Since K contains at least two points, there exists a non-zero, positive contraction $f \in C(K_\omega) \cong C(K)_\omega$, and points $\lambda_1, \lambda_2 \in K_\omega$ such that $f(\lambda_1) = 1$ and $f(\lambda_2) = 0$. It follows from [2, Lemma 3.17], that there exists a projection $p \in M_K^\omega \cap M_K'$, such that $\tau_\lambda(p) = f(\lambda)$, for all $\lambda \in K^\omega$. In particular, $p \in M_K^\omega \cap M_K'$ is a non-zero projection, since $\tau_{\lambda_1}(p) = 1$, such that $\|p\|_{2,\tau_{\lambda_2}} = \tau_{\lambda_2}(p) = 0$, and therefore $\pi_{\lambda_2}(p) = 0$. Since K^ω is connected, by Lemma 2.6, this implies that the map $K^\omega \ni \lambda \mapsto \|\pi_\lambda(p)\|$ cannot be continuous, whence it follows from Proposition 2.5 that, there exists some $\lambda \in K^\omega$, such that $\rho^{-1}(I_\lambda)$ is not a σ-ideal. □

Remark 2.8. We remark that the above proposition also yields an ideal in A_ω which is not a σ-ideal. Indeed, note that M_K^ω is also a W^*-bundle over K_ω and, since $C(K_\omega) \subseteq M_K^\omega$ is central, for each $\lambda \in K_\omega$ we have

$$C_0(K^\omega\backslash\{\lambda\})(M_K^\omega \cap M_K') = (C_0(K^\omega\backslash\{\lambda\})M_K^\omega) \cap M_K'.$$

Let $J_\lambda \subseteq A_\omega$ denote the ideal $\pi_K^{-1}(C_0(K_\omega\backslash\{\lambda\})M_K^\omega)$, where $\pi_K \colon A_\omega \to A_\omega/J_K$ denotes the quotient map. By the above, $\rho_K^{-1}(I_\lambda) = J_\lambda \cap A'$. Since $\rho_K^{-1}(I_\lambda)$ is not a σ-ideal, neither is J_λ, thus proving the claim.

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A NOTE ON ASYMPTOTIC REGULARITY

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Abstract. We give a characterization of asymptotic regularity in terms of the Cuntz semigroup, for simple, separable C^*-algebras, and show that any simple, separable C^*-algebra which is neither stably finite nor purely infinite is not asymptotically regular either.

1. Introduction

In this note, we consider asymptotic regularity, as defined by Ng in [9]. The original motivation was to answer the question of whether the corona factorization property is equivalent to ω-comparison, for simple and separable C^*-algebras, and the study carried out here was meant to facilitate an answer to this question. Unfortunately, not much progress was made in this direction. However, we *do* obtain a characterization of asymptotic regularity, in terms of the Cuntz semigroup, for such C^*-algebras. As an application, we show that asymptotic regularity implies dichotomy, i.e., if a simple and separable C^*-algebra A is asymptotically regular, then either A is stably finite, in the sense that $\mathrm{Cu}(A)$ admits a functional which is finite on $\mathrm{Cu}(A)_{\ll\infty}$, or A is purely infinite.

2. Cuntz semigroups

We consider general Cuntz semigroups, as introduced in [5]. Throughout, $(S, \leq)$ will denote a positively ordered abelian monoid, that is, an ordered abelian semigroup, with neutral element 0, satisfying that $0 \leq x$, for all $x \in S$. If S is a positively ordered, abelian semigroup, which admits suprema of increasing sequences, then, given $x, y \in S$, we write $x \ll y$ if, whenever $(y_n)_n$ is an increasing sequence in S with $y \leq \sup_n y_n$, we have $x \leq y_k$, for some $k \geq 1$.

Definition 2.1 (Cu-semigroups)**.** Let S be a positively ordered, abelian monoid. Then we say that S is a Cu-*semigroup*, if the following axioms are satisfied.

(i) Every increasing sequence $(x_n)_n$ in S admits a supremum.
(ii) Every element $x \in S$ is the supremum of a sequence $(x_n)_n$ in S satisfying $x_n \ll x_{n+1}$, for all $n \geq 1$.
(iii) If $x, x', y, y' \in S$ satisfy $x' \ll x$ and $y' \ll y$, then $x' + y' \ll x + y$.
(iv) If $(x_n)_n$ and $(y_n)_n$ are increasing sequences in S, then $\sup_n(x_n+y_n) = \sup_n x_n + \sup_n y_n$.

It was shown in [5] that the Cuntz semigroup $\mathrm{Cu}(A)$ of a C^*-algebra A is a Cu-semigroup. We refer the reader to [1] for an introduction to the Cuntz semigroup $\mathrm{Cu}(A)$ of a C^*-algebra A. Here, we will use the picture of $\mathrm{Cu}(A)$ as equivalence classes $\langle a\rangle$, of positive elements in $a \in A \otimes \mathbb{K}$. In this picture,

$a, b \in (A \otimes \mathbb{K})_+$ satisfy $\langle a \rangle \leq \langle b \rangle$ if, and only if, $a \precsim b$ in $A \otimes \mathbb{K}$, i.e., there exists a sequence $(x_n)_n \subseteq A \otimes \mathbb{K}$ such that $\lim_{n\to\infty} \|a - x_n^* b x_n\| = 0$, and addition is given by $\langle a \rangle + \langle b \rangle = \langle a' + b' \rangle$, where $a', b' \in (A \otimes \mathbb{K})_+$ are orthogonal and satisfy $\langle a \rangle = \langle a' \rangle$ and $\langle b \rangle = \langle b' \rangle$. In particular, if $a, b \in (A \otimes \mathbb{K})_+$ are orthogonal elements, then $\langle a \rangle + \langle b \rangle = \langle a + b \rangle$. Furthermore, $\langle a \rangle \ll \langle b \rangle$ if, and only if, there exists $\varepsilon > 0$ such that $\langle a \rangle \leq \langle (b - \varepsilon)_+ \rangle$. Here $(b - \varepsilon)_+$ denotes $h_\varepsilon(b)$, where $h_\varepsilon \colon \mathbb{R}_+ \to \mathbb{R}_+$ is the continuous function given by $h_\varepsilon(t) = \max\{0, t - \varepsilon\}$.

In the following, we shall exclusive be interested in Cu-semigroups S with a maximal element $\infty \in S$. Whenever $S = \mathrm{Cu}(A)$ for a separable (or, more generally, σ-unital) C^*-algebra, such an element always exist. We use $S_{\ll\infty}$ to denote the set $S_\infty = \{x \in S \mid x \ll \infty\}$.

Definition 2.2. Let S be a Cu-semigroup. A *functional* λ on S is a semigroup map $\lambda \colon S \to \mathbb{R}_+$, satisfying $\lambda(x) \leq \lambda(y)$, whenever $x \leq y$, and $\lambda(\sup_n x_n) = \sup \lambda(x_n)$, whenever $(x_n)_n$ is an increasing sequence in S.

Finally, given elements $x, y \in S$, we write $x <_s y$, if there exists $n \geq 1$ such that $(n+1)x \leq ny$. By [10, Proposition 2.1], the relation $<_s$ is transitive.

3. Asymptotic regularity and the Cuntz semigroup

We seek to characterize asymptotic regularity (as defined by Ng) for simple, separable C^*-algebras in terms of the Cuntz semigroup. We start with Ng's definition. Recall that a C^*-algebra D is said to have *property (S)* if it has no unital quotients and admits no bounded 2-quasitraces. We refer the reader to [2] for the definition and properties of 2-quasitraces.

Definition 3.1. Let A be a separable C^*-algebra. Then A is said to be asymptotically regular if, for any full hereditary C^*-subalgebra $D \subseteq A \otimes \mathbb{K}$ with property (S), there exists an integer $n \in \mathbb{N}$, such that $M_n(D)$ is stable.

Consider the following property of a Cu-semigroup.

Definition 3.2. Let S be a simple Cu-semigroup. We say that S has *asymptotic ω-comparison* if the following holds: whenever $y_1, y_2, \ldots$ is a sequence of non-zero elements in $S_{\ll\infty}$, such that $y_i <_s y_{i+1}$, for all $i \geq 1$, there exists $n \in \mathbb{N}$, such that $n \cdot \sum_{i=m}^{\infty} y_i = \infty$, for all $m \geq 1$.

Inspired by [3, Theorem 4.2.3], we aim to show that a simple, separable C^*-algebra A is asymptotically regular if and only if $\mathrm{Cu}(A)$ has asymptotic ω-comparison. We need the following results from [4].

Lemma 3.3. *Let S be a simple* Cu*-semigroup. Then, either there exists a non-zero functional which is finite on $S_{\ll\infty}$, or, for every non-zero $z \in S$, there exists $n \in \mathbb{N}$, such that $nz = \infty$.*

Lemma 3.4. *Let S be a simple* Cu*-semigroup. Then, for every $x \in S_{\ll\infty}$, there exists $m \in \mathbb{N}$ with the property that, whenever $y \in S$ satisfies $m\lambda(x) < \lambda(y)$, for all functionals λ finite on $S_{\ll\infty}$, we have $x <_s y$.*

The following lemma is an elaboration of results appearing in [4].

Lemma 3.5. *Let S be a simple* Cu*-semigroup. Then the following are equivalent:*

(i) *S has asymptotic ω-comparison.*
(ii) *Whenever $y_1, y_2, \dots$ is a sequence of non-zero elements in $S_{\ll\infty}$ satisfying $\lambda(\sum_{i=1}^{\infty} y_i) = \infty$, for all non-zero functionals λ on S, there exists $n \in \mathbb{N}$, such that $n \cdot \sum_{i=m}^{\infty} y_i = \infty$, for all $m \in \mathbb{N}$.*

Proof. The implication (ii)⇒(i) is clear. Indeed, suppose $y_1, y_2, \dots$ is a sequence of non-zero elements in $S_{\ll\infty}$, satisfying $y_i <_s y_{i+1}$, for all $i \geq 1$. Then $\lambda(y_i) < \lambda(y_{i+1})$, for all $i \geq 1$ and all functionals λ on S, and since S is simple, it follows that $\lambda(y_1) \neq 0$, for all non-zero functionals λ. In particular, $\lambda(\sum_{i=1}^{\infty} y_i) = \infty$, for all non-zero functionals λ on S.

Assume that S has asymptotic ω-comparison, and let $y_1, y_2, \dots$ be a sequence as in (ii). We show that we can find a sequence $y_1', y_2', \dots$ in $S_{\ll\infty}$ such that $y_1' <_s y_2' <_s y_3' <_s \cdots$ and, for each $m \geq 1$, there exists $l(= l(m)) \geq 1$ such that $\sum_{i=m}^{\infty} y_i' = \sum_{i=l}^{\infty} y_i$, from which the desired result will follow.

If there are no functionals finite on $S_{\ll\infty}$ then $x <_s z$ for every x, z, with z non-zero, by Lemma 3.3, whence we can simply choose $y_i' = y_i$, for all $i \geq 1$. Hence, suppose there exist functionals on S finite on $S_{\ll\infty}$. We show, by induction, that there exists a strictly increasing sequence of integers $(n_i)_{i\geq 0}$, such that $y_i' := \sum_{j=n_{i-1}+1}^{n_i} y_j$ satisfies $y_i' <_s y_{i+1}'$, for all $i \geq 1$. Set $n_0 := 0$, $n_1 := 1$, and $y_1' := y_1$. Suppose that we have found $n_0, \dots, n_m$ for some $m \geq 1$. Using Lemma 3.4, let $N \in \mathbb{N}$ be given such that, whenever $z \in S$ satisfies $N\lambda(y_j') < \lambda(z)$, for all functionals λ finite on $S_{\ll\infty}$, then $y_j' <_s z$. Note that, for all non-zero functionals on S, we have

$$\lambda\Big(\sum_{j=n_m+1}^{\infty} y_j\Big) = \lim_{M\to\infty} \sum_{j=n_m+1}^{M} \lambda(y_j) = \infty,$$

and since the set of functionals is compact (see [6] for a proof and definition of the topology), it follows that there exist $n_{m+1} \in \mathbb{N}$, such that $N\lambda(y_j') < \lambda(\sum_{j=n_m+1}^{n_{m+1}} y_j)$, whenever λ is non-zero and finite on $S_{\ll\infty}$. In particular, by choice of N, $y_{m+1}' := \sum_{j=n_m+1}^{n_{m+1}} y_j$ satisfies $y_m' <_s y_{m+1}'$. It is obvious from the construction of the y_i''s, that the sequence $y_1', y_2', \dots$ has the desired properties. □

We briefly consider the condition $n \cdot \sum_{i=m}^{\infty} y_i = \infty$, for some n and all m.

Lemma 3.6. *Let S be a* Cu*-semigroup and $y_1, y_2, \dots$ be any sequence of elements in S. If there exists $n \in \mathbb{N}$ such that $n \cdot \sum_{i=m}^{\infty} y_i = \infty$, for all $m \in \mathbb{N}$, then, for every $j \geq 1$, it holds that $\sum_{k=1}^{j} y_k \leq n \cdot \sum_{i=j+1}^{\infty} y_i$. Conversely, if there exists $n \in \mathbb{N}$, such that $n \cdot \sum_{i=1}^{\infty} y_i = \infty$ and $n \cdot \sum_{k=1}^{j} y_k \leq n \cdot \sum_{l=j+1}^{\infty} y_l$, for all $j \geq 1$, then $2n \cdot \sum_{l=m}^{\infty} y_l = \infty$, for all $m \geq 1$.*

Proof. The first statement is obvious. For the second statement, suppose that $y_1, y_2, \dots$ is a sequence in S, and $n \in \mathbb{N}$ is given such that $n \cdot \sum_{i=1}^{\infty} y_i = \infty$ and, $n \cdot \sum_{k=1}^{j} y_k \leq n \cdot \sum_{i=j+1}^{\infty} y_i$, for all $j \geq 1$. Let $m \geq 1$ be arbitrary. Then $n \cdot \sum_{k=1}^{m-1} y_k \leq n \cdot \sum_{l=m}^{\infty} y_l$, by assumption, whence

$$2n \cdot \sum_{l=m}^{\infty} y_l = n \cdot \sum_{l=m}^{\infty} y_l + n \cdot \sum_{l=m}^{\infty} y_l \geq n \cdot \sum_{i=1}^{\infty} y_i = \infty. \qquad \square$$

We need one final lemma, and some notation, before we get to the main result. Given a C^*-algebra D, and a strictly positive contraction $c \in D$, let $L_c(D) := \{a \in D_+ \mid f_\varepsilon(c)a = a \text{ for some } \varepsilon > 0\}$, where $f_\varepsilon \colon [0,1] \to [0,1]$ is given by

$$f_\varepsilon(t) = \begin{cases} 0 & \text{if } 0 \le t \le \varepsilon/2 \\ 2\varepsilon^{-1}t - 1 & \text{if } \varepsilon/2 \le t \le \varepsilon \\ 1 & \text{if } t \ge \varepsilon. \end{cases} \tag{1}$$

Note that, if $a, b \in L_c(D)$, then $a + b \in L_c(D)$ and $f_\delta(c)df_\delta(c) \in L_c(D)$, for every $\delta > 0$ and $d \in D_+$. The following lemma is a mild elaboration of [10, Lemma 4.3].

Lemma 3.7. *Let D be a σ-unital C^*-algebra with property (S), and let $c \in D$ be a strictly positive contraction. Then, for every $a \in L_c(D)$, there exists $b \in D_+$ such that $ab = 0$, $\langle a \rangle <_s \langle b \rangle$, and $b \in L_c(D)$.*

Proof. Choose $\varepsilon' > 0$ such that $f_{\varepsilon'}(c)a = a$. Let $e := f_{\varepsilon'}(c)$, and note that $a \precsim (e - 1/2)_+$. Since $e \in L_c(D)$, and D has property (S), there exists $b_0 \in D_+$ such that $eb_0 = 0$ and $\langle e \rangle <_s \langle b_0 \rangle$, by [10, Proposition 4.5]. Furthermore, by [10, Lemma 2.6], there exists some $\delta > 0$, such that $\langle (e - 1/2)_+ \rangle <_s \langle (b_0 - \delta)_+ \rangle$. Note that $(f_{1/n}(c))_{n \ge 1}$ is an approximate unit for D, since $c \in D$ is strictly positive, whence $\|b_0 - f_{1/m}(c)b_0 f_{1/m}(c)\| < \delta$, for some $m \ge 1$, and we may assume that $\varepsilon := 1/m < \varepsilon'/2$. Set $b := f_\varepsilon(c)b_0 f_\varepsilon(c)$, and note that

$$ab = aef_\varepsilon(c)b_0 f_\varepsilon(c) = aeb_0 f_\varepsilon(c) = 0,$$

and $b \in L_c(D)$. Furthermore, since $\|b_0 - b\| < \delta$, it follows from [8, Lemma 2.5], that

$$\langle a \rangle \le \langle (e - 1/2)_+ \rangle <_s \langle (b_0 - \delta)_+ \rangle \le \langle b \rangle. \qquad \square$$

In the proof of the following proposition we write $a \perp b$, if $a, b \in D$ are positive elements, in some C^*-algebra D, satisfying $ab = 0$.

Proposition 3.8. *Let A be a simple and separable C^*-algebra. Then the following are equivalent:*

(i) $\mathrm{Cu}(A)$ *has asymptotic ω-comparison.*
(ii) A *is asymptotically regular.*

Proof. Assume $\mathrm{Cu}(A)$ has asymptotic ω-comparison, and let $D \subseteq A \otimes \mathbb{K}$ be a non-zero hereditary sub-C^*-algebra, with property (S). Let $c \in D_+$ be a strictly positive element. Then, for every $m \ge 1$, $c \otimes \mathbf{1}_m \in D \otimes M_m \cong M_m(D)$ is a strictly positive element, whence it follows from [7, Theorem 2.1 and Proposition 2.2], that it suffices to prove the existence of $n \in \mathbb{N}$ such that, for all $\varepsilon > 0$, there exists $b \in (D \otimes M_n)_+$ satisfying $(c \otimes \mathbf{1}_n - \varepsilon)_+ \perp b$, and $(c \otimes \mathbf{1}_n - \varepsilon)_+ \precsim b$.

Let $(\varepsilon_n)_{n \ge 1}$ be a decreasing sequence of positive real numbers such that $\varepsilon_n \to 0$. We prove, by induction, that there exists a sequence $(b_n)_n$ of pairwise orthogonal, positive elements in D, such that $\langle b_i \rangle <_s \langle b_{i+1} \rangle$, $b_i \in L_c(D)$, $\langle (c - \varepsilon_i)_+ \rangle <_s \langle b_i \rangle$, and $(c - \varepsilon_i)_+ \perp b_i$, for all $i \ge 1$. The induction start follows from lemma 3.7. Hence, suppose that we have found $b_1, \dots, b_n$ with the desired properties, for some $n \ge 1$. Note that, since $b_1, \dots, b_n \in$

$L_c(D)$, it follows that $(c-\varepsilon_{n+1})_+ + b_1 + \cdots + b_n \in L_c(D)$, whence, applying Lemma 3.7 again, it follows that there exists $b_{n+1} \in L_c(D)$ such that $\langle (c-\varepsilon_{n+1})_+\rangle + \langle b_1\rangle + \cdots + \langle b_n\rangle <_s \langle b_{n+1}\rangle$, and $(c-\varepsilon_{n+1})_+ + b_1 + \cdots + b_n \perp b_{n+1}$. In particular, it follows that b_{n+1} has the desired properties. For $i \geq 1$, let $y_i := \langle b_i\rangle \in \mathrm{Cu}(A)$. By asymptotic ω-comparison it follows that there exists $n \in \mathbb{N}$ such that $n \cdot \sum_{i=m}^{\infty} y_i = \infty$ for all $m \in \mathbb{N}$. Let $\varepsilon > 0$ be arbitrary and choose $m \in \mathbb{N}$ such that $\varepsilon_m < \varepsilon$. Then there exists $N \in \mathbb{N}$, such that

$$\langle (c \otimes \mathbf{1}_n - \varepsilon_m)_+\rangle \leq n \cdot \sum_{i=m}^{N} \langle b_i\rangle = \Big\langle \sum_{i=m}^{N} b_i \otimes \mathbf{1}_n \Big\rangle,$$

and since $(c \otimes \mathbf{1}_n - \varepsilon)_+ \leq (c \otimes \mathbf{1}_n - \varepsilon_m)_+ \perp \sum_{i=m}^{N} b_i \otimes \mathbf{1}_n$, the desired result follows.

Now, suppose A is asymptotically regular, and that $y_1, y_2, \ldots$ is a sequence of non-zero elements in $\mathrm{Cu}(A)$, such that $y_i <_s y_{i+1}$, for all $i \geq 1$. Choose pairwise orthogonal positive elements $b_i \in A \otimes \mathbb{K}$ such that $\|b_i\| \leq 2^{-i}$ and $y_i = \langle b_i\rangle$. Set $b := \sum_{i=1}^{\infty} b_i$, and let $D \subseteq (A \otimes \mathbb{K})_+$ denote the hereditary sub-C^*-algebra generated by b. We show that D has property (S). It is easy to see that $\lambda(\langle b\rangle) = \infty$ for all functionals λ on $\mathrm{Cu}(A)$, hence D does not admit any bounded 2-quasitrace; indeed, any such trace would extend to $D \otimes \mathbb{K} \cong A \otimes \mathbb{K}$, and thus give rise to a functional on $\mathrm{Cu}(A)$ finite on $\langle b\rangle$. Similarly, assuming for a contradiction, that D is unital, it follows that b is invertible, being a strictly positive element in D, and therefore $\sum_{i=1}^{m} b_i$ is invertible, for some $m \in \mathbb{N}$. Therefore $b_k = 0$, for all $k > m$, since these elements are orthogonal to an invertible element, which implies $b_i = 0$, for all $i \geq 1$ (since $\langle b_i\rangle <_s \langle b_j\rangle$ whenever $i < j$); a contradiction. Hence, D has property (S), and therefore $D \otimes M_n$ is stable, for some $n \in \mathbb{N}$, by asymptotic regularity of A. Therefore, $\infty = \langle b \otimes \mathbf{1}_n\rangle = n \cdot \langle b\rangle = n \cdot \sum_{i=1}^{\infty} \langle b_i\rangle$, since $b \otimes \mathbf{1}_n$ is a strictly positive element in $D \otimes M_n \cong (D \otimes M_n) \otimes \mathbb{K} \cong A \otimes \mathbb{K}$. By Lemma 3.6, it suffices to prove that $\sum_{k=1}^{j} \langle (b_k \otimes \mathbf{1}_n - \varepsilon)_+\rangle \leq \sum_{i=j+1}^{\infty} \langle b_i \otimes \mathbf{1}_n\rangle$, for every $j \geq 1$ and $\varepsilon > 0$.

Fix $\varepsilon > 0$ and, for ease of notation, let $d_i := b_i \otimes \mathbf{1}_n$, for all $i \geq 1$. For every $j \geq 1$ and $m \geq 1$, let $e_m^{(j)} := \sum_{i=j}^{m} f_{1/m}(d_i)$, and $e_m := e_m^{(1)}$, with f as in (1). Note that $(e_m)_{m\geq 1}$ is an approximate unit for $D \otimes M_n$, and that $e_m^{(j)} \precsim \sum_{i=j}^{m} d_i$, for all $j, m \in \mathbb{N}$, since $f_{1/m}(d_i)$ belongs to the hereditary subalgebra of D generated by d_i, for all $i \geq 1$. Now, since $D \otimes M_n$ is stable, it follows that there exists $a \in D \otimes M_n$, such that $\sum_{k=1}^{j} d_k \perp a$ and $\sum_{k=1}^{j} d_k \precsim a$. We may therefore choose $\delta > 0$, such that $\sum_{k=1}^{j} (d_k - \varepsilon)_+ \precsim (a - \delta)_+$. We may also choose $m \in \mathbb{N}$, such that $\|a - e_m a e_m\| < \delta$, whence $(a-\delta)_+ \precsim e_m a e_m$. Now, since $\sum_{k=1}^{k} d_i \perp a$, it follows that

$$e_m a^{1/2} = \Big(\sum_{i=1}^{m} f_{1/m}(d_i)\Big) a^{1/2} = \Big(\sum_{i=j+1}^{m} f_{1/m}(d_i)\Big) a^{1/2} = e_m^{(j+1)} a^{1/2}$$

In particular, $e_m a e_m \leq \|a\| \big(e_m^{(j+1)}\big)^2$, whence

$$\sum_{k=1}^{j} (d_k - \varepsilon)_+ \precsim (a-\delta)_+ \precsim e_m a e_m \precsim \big(e_m^{(j+1)}\big)^2 \precsim \sum_{i=j+1}^{m} d_i. \qquad \square$$

Proposition 3.9. *Let A be a simple and separable C^*-algebra, which is neither stably finite or purely infinite. Then A is* not *asymptotically regular.*

Proof. Since A is not stably finite, it follows from Lemma 3.3, that every non-zero $z \in \mathrm{Cu}(A)$ satisfies $nz = \infty$, for some $n \in \mathbb{N}$. As A is not purely infinite, there exists $x \in \mathrm{Cu}(A)_{\ll\infty}$, such that $x \neq \infty$. Note that, since A is a simple and non-elementary C^*-algebra, we may, for each non-zero $y \in \mathrm{Cu}(A)$, choose non-zero $z \in \mathrm{Cu}(A)$, such that $2z \leq y$ (this follows from Glimm's halving lemma, see for instance [11, Proposition 3.10]). Hence, with $x_0 := x$, we may find a sequence $x_1, x_2, \ldots \in \mathrm{Cu}(A)$, of non-zero elements, such that $2^{i+1}x_i \leq x_{i-1}$, for all $i \geq 1$. We show that $\sum_{i=j}^{\infty} x_i \leq 2x_j$, for all $j \geq 1$.

Fix arbitrary $j \geq 1$. We show, by induction, that $2x_n + \sum_{i=j}^{n-1} x_i \leq 2x_j$, for all $n \geq j+1$. The induction start is clear, since $2x_{j+1} + x_j \leq x_j + x_j = 2x_j$. Hence, assume the statement is true for some $n \geq j+1$. Then we have

$$2x_{n+1} + \sum_{i=j}^{n} x_i \leq x_n + \sum_{i=j}^{n} x_i = 2x_n + \sum_{i=j}^{n-1} x_i \leq 2x_j.$$

Hence $\sum_{i=j}^{n} x_i \leq 2x_n + \sum_{i=j}^{n-1} x_i \leq 2x_j$, for all n, and, taking the supremum over n of the left hand side, we therefore obtain $\sum_{i=j}^{\infty} x_i \leq 2x_j$. In particular we find that $2^j \sum_{i=j}^{\infty} x_i \leq 2^{j+1}x_j \leq x_0$ is a finite element. Since every element in $\mathrm{Cu}(A)$ is eventually infinite, it follows that $x_i <_s x_{i+1}$, for all i, but there exists no $n \in \mathbb{N}$ such that $n \cdot \sum_{i=m}^{\infty} x_i = \infty$ for all $m \in \mathbb{N}$, and therefore, by the above proposition, A cannot be asymptotically regular. □

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