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Dynamical systems and algebras associated with separated graphs

PhD Thesis

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Abstract

In this thesis, we study partial dynamical systems and graph algebras arising from finitely separated graphs. The thesis consists of an introduction followed by three papers, the first of which is joint work with Pere Ara.

In Article [A], we introduce *convex subshifts*, an abstract generalisation of the partial dynamical systems associated with finite separated graphs. We define notions of a *finite* and *infinite type* convex subshift and show that all such dynamical systems arise from a finite bipartite separated graph up to a suitable type of equivalence. We then study various aspects of the ideal structure of the tame separated graph algebras for finite bipartite graphs: We represent the lattice of induced ideals by graph-theoretic data, compute all ideals of finite type in the reduced setting, and characterise both simplicity and primitivity.

In Article [B], we introduce a generalisation of Condition (K) to finitely separated graphs and show that it is equivalent to the partial action being essentially free as well as either of the tame algebras having the exchange property. We also demonstrate that Condition (K) is very restrictive, and as a consequence, the tame algebras are separative whenever they are exchange rings.

Finally, Article [C] completely characterises nuclearity of the tame graph C^*-algebras in terms of a graph-theoretic property. We also show that the full and reduced tame graph C^*-algebras coincide if and only if they are nuclear, and that otherwise the full algebra is in fact non-exact.

Resumé

Denne afhandling omhandler dynamiske systemer og grafalgebraer hørende til endeligt separerede grafer. Afhandlingen består af en introduktion og tre artikler, hvoraf den første er udarbejdet i samarbejde med Pere Ara.

I Artikel [A] introducerer vi *konvekse delskift*, som er en abstrakt generalisering af de partielle dynamiske systemer hørende til endelige, separerede grafer. Vi inddeler alle konvekse delskift i *endelig* og *uendelig type* og viser, at ethvert delskift af endelig type kan realiseres som virkningen hørende til en endelig, todelt, separeret graf op til en passende form for ækvivalens. Vi undersøger dernæst en række aspekter af idealstrukturen i de tamme grafalgebraer: Vi repræsenterer gitret af inducerede idealer ved grafteoretiske data, bestemmer alle idealer af endelig type i det reducerede tilfælde og karakteriserer såvel simplicitet som primitivitet.

I Artikel [B] introducerer vi en generalisering af Betingelse (K) for endeligt separerede grafer og viser, at den er ækvivalent til essentiel frihed for den partielle virkning såvel som exchange-egenskaben for de tamme algebraer. Vi viser også at Betingelse (K) er ganske restriktiv, og det følger heraf, at de tamme algebraer er separative, når de er exchange-ringe.

Endeligt gives der i Artikel [C] en komplet karakterisering af nuklearitet for de tamme graf-C^*-algebraer ved en grafteoretisk egenskab. Vi viser ligeledes, at den fulde og den reducerede tamme graf-C^*-algebra er sammenfaldende, hvis og kun hvis de er nukleære, og at den fulde algebra i modsat fald er ikke-eksakt.

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I would like to thank Pere for my visits to Barcelona and our extensive mail correspondence on separated graph algebras. I have picked up many ideas and tricks from you over the last years, and I hope you have learnt something from me as well.

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Contents

Part I

Thesis overview

In this first part of the thesis, we provide an overview of the results of the three articles that follow and put them into context. We first review the development of classical graph algebras and present the main motivation for studying the more general class of separated graph algebras. We then elaborate on the existing theory of separated graph algebras, before turning attention to our own results. Finally, we comment on the future of the subject.

1. Classical graph algebras

By a directed graph, we shall mean a quadruple $E = (E^0, E^1, r, s)$, where E^0 and E^1 are sets, and r and s are functions $E^1 \to E^0$. The elements of E^0 are called *vertices* while the elements of E^1 are called *edges*, and r and s determine the *range* and *source* of any edge. A vertex v is called a *sink* if $s^{-1}(v) = \emptyset$, a *source* if $r^{-1}(v) = \emptyset$, and *regular* if $0 < |r^{-1}(v)| < \infty$. The graph is referred to as being *column-finite* if $|r^{-1}(v)| < \infty$ for every $v \in E^0$, and it is plainly *finite* if both E^0 and E^1 are finite sets.

To any directed graph E, one may associate a *graph C^*-algebra $C^*(E)$* as the universal C^*-algebra for certain generators and relations represented by the graph. Since we will refer to this definition many times, we might as well just present it right away:

Definition. Let E denote any directed graph. The graph C^*-algebra $C^*(E)$ is the universal C^*-algebra for generators $E^0 \sqcup E^1$ with relations

(V) $uv = \delta_{u,v}v$ and $u = u^*$ for $u, v \in E^0$.
(E) $es(e) = r(e)e = e$ for $e \in E^1$.
(CK1) $e^*f = \delta_{e,f}s(e)$ for all $e, f \in E^1$.
(CK2) $v = \sum_{e \in r^{-1}(v)} ee^*$ for all regular vertices v.

The reader should note that we are applying the so-called *Raeburn convention*, so paths will have to be read from the right.

The basic idea when building complicated objects, such as C^*-algebras, from simple combinatorial objects, such as graphs, is of course to be able to understand properties of the complicated ones in terms of properties of the simpler ones. And, if the construction is to be truly meaningful, the complicated objects must also enjoy some interesting properties. The class of graph C^*-algebras certainly meet both these requirements. One can tell many things about $C^*(E)$ by merely looking at E in terms of ideal structure, (non-stable) K-theory, and the real and stable rank to just name some. On the other hand, it is a rather big class, containing for instance all AF-algebras and Kirchberg algebras with free K_1-group up to Morita equivalence. Having a unified theory for such diverse C^*-algebras, which are closed under many constructions, provides a natural playground for C^*-algebraists to build interesting examples and guide the general theory of C^*-algebras. For instance, graph C^*-algebras continue to inspire new ideas and provide a prominent test case within classification theory of non-simple separable, nuclear C^*-algebras.

While the general modern-day definition of $C^*(E)$ did not appear until around 2000, the field of graph C^*-algebras ought to be considered 20 years older, starting with the introduction of what became known as Cuntz-Krieger algebras in [16]. Specifically, to each $n \times n$-matrix A with $\{0, 1\}$-entries, no zero rows or columns, and satisfying a certain Condition (I), Cuntz and Krieger associated a unique C^*-algebra $\mathcal{O}_A$ generated by n

partial isometries under various relations on their initial and final projections. An Huef later discarded Condition (I), demanding instead that $\mathcal{O}_A$ is universal with respect to its defining generators and relations [2]. While many authors noted that the matrix A could be regarded as the adjacency matrix of a graph with n vertices, it was not until the work of Kumjian, Pask, Raeburn and Renault [25] in 1997 that graphs properly entered the picture, and in the follow-up paper [24] by the first three authors, the modern-day definition of the graph C^*-algebra of a column-finite graph finally appeared. The class of Cuntz-Krieger algebras could now be viewed as exactly the graph C^*-algebras of finite graphs with no sinks or sources – incidentally, Kumjian, Pask and Raeburn did not themselves use the term *graph C^*-algebras*, but referred instead to *Cuntz-Krieger algebras of directed graphs*. With this generalisation, an old friend such as the Toeplitz algebra now became an instance of this theory, and Coburn's Theorem became merely an application of the so-called Cuntz-Krieger Uniqueness Theorem. The definition of $C^*(E)$ for an arbitrary directed graph, E, would appear soon after in the paper [18] by Fowler, Laca and Raeburn, thereby giving the field its modern-day framework. While there are many subtleties in passing from column-finite to arbitrary graphs, we will mostly be concerned with the column-finite case throughout this introduction. We should also mention that while *graph algebras* always meant *graph C^*-algebras* until 2005, there has been a remarkable interest in purely algebraic counterparts called *Leavitt path algebras* since their introduction in [1] by Abrams and Pino. Given any field with involution K (some even work over an arbitrary commutative ring), one can define the Leavitt path algebra $L_K(E)$ to be the K-algebra with exactly the same set of generators and relations as $C^*(E)$. Many results from graph C^*-algebras have been reproved in this context, and today the two viewpoints constantly interact and inspire one another.

It was observed from the very beginning by Cuntz and Krieger that their algebras were intimately related to symbolic dynamics. Specifically, they showed that if the two-sided shift spaces $\overline{X}_A$ and $\overline{X}_B$, determined by matrices A and B, are flow equivalent, then the Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are stably isomorphic. In [15], Cuntz further deepened the relationship, showing that, under a certain Condition (II), all ideals of $\mathcal{O}_A$ corresponded to open and shift-invariant subspaces of $\overline{X}_A$. When Kumjian, Pask, Raeburn and Renault later initiated the use of graphs, they did so by representing $C^*(E)$ as a groupoid C^*-algebra $C^*(\mathcal{G}_E)$ for a certain *graph groupoid* $\mathcal{G}_E$: Any point in $\mathcal{G}_E$ is a triple (x, k, y), where $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1}$ are infinite paths in E, and k is an integer such that there exists $N \in \mathbb{N}$ for which $x_i = y_{i+k}$ whenever $i \geq N$. While they had to restrict to column-finite graphs without sources, this approach would later be generalised by Paterson to arbitrary graphs [28]. The most striking and direct relation between the graph C^*-algebras and symbolic dynamics, however, is the identification of $C^*(E)$ as a crossed product $C_0(\partial E) \rtimes \mathbb{F}(E^1)$ for a partial action $\theta^E : \mathbb{F}(E^1) \curvearrowright \partial E$ of the free group generated by the edge set on the so-called *boundary path space*. Such a description first emerged in work of Exel and Laca [17] for what has later been dubbed *Exel-Laca algebras*; these were an attempt at overcoming the assumption of column-finiteness, and so they in particular contain all graph C^*-algebras of column-finite graphs. In this case, ∂E is the set of all infinite paths in E along with all finite paths starting in a source (including the sources themselves), and the action is essentially just the shift: An edge e may act on a path $x = (x_n)_{n\geq 1}$ if and only if $s(e) = r(x_1)$, in which case $\theta^E_e(x) = ex$, while an inverse edge e^{-1} can act only on elements ex by $\theta^E_{e^{-1}}(ex) = x$. Moreover, an edge e can act on a source v by $\theta^E_e(v) = e$ if and only if $r(e) = v$. This description of $C^*(E)$ together with Exel's theory of crossed products for partial actions allows one to easily prove the classical results on ideal structure, the Cuntz-Krieger uniqueness theorem, and nuclearity of $C^*(E)$. Moreover, the Leavitt path algebra $L_K(E)$ may also be regarded as arising from this partial action as it is isomorphic to the algebraic crossed product $C_K(\partial E) \rtimes \mathbb{F}(E^1)$, where $C_K(\partial E)$ denotes the algebra of compactly supported, locally constant functions $\partial E \to K$. This common dynamical description of $L_K(E)$ and $C^*(E)$ explains many of their similarities.

The success of graph C^*-algebras has inspired various generalisations, including the C^*-algebras associated with topological graphs as defined and studied by Katsura in [19, 20, 21, 22], the higher-rank graphs of Kumjian and Pask [23], and – the main topic of this thesis – the separated graphs of Ara, Exel and Goodearl [5, 6, 7, 8]. Most of these generalisations seem to be motivated by the desire to include classes of C^*-algebras with more general behaviour while maintaining most of the results available for classical graph C^*-algebras. For instance, any Kirchberg algebra may be described as a topological graph C^*-algebra, and higher rank graphs give rise to both the irrational rotation algebras and the Bunce-Deddens algebras. The separated graph algebras differ in this regard: As we will see, they do not behave like ordinary graph algebras in many ways, and they are really invented to deal with a specific problem that we shall now describe.

2. The Fundamental Separativity Problem

A von Neumann algebra is simple (as a C^*-algebra) if and only if it is a factor of type I_n for $n < \infty$, II_1 or type III. In particular, either every non-zero projection is finite or they are all infinite. It was an open problem whether a similar dichotomy holds in the C^*-setting until Rørdam constructed a simple, separable, unital, nuclear C^*-algebra containing both a non-zero finite and an infinite projection in [29]. However, as Rørdam noted himself, his example is not of *real rank zero*: A C^*-algebra is said to have real rank zero if any self-adjoint element can be approximated by one with finite spectrum. C^*-algebraists often refer to this property as *having many projections*, and from this perspective, such algebras resemble von Neumann algebras – indeed every von Neumann algebra has real rank zero. He then went on to pose the questions if there exist simple, real rank zero C^*-algebras containing both a non-zero finite and an infinite projection – and if so, can one add nuclearity to the list of properties?

The first of Rørdam's question is naturally related to problems that have arisen in ring theory. A ring R is said to be (von Neumann) *regular* if for all $x \in R$, there exists $y \in R$ with $xyx = x$. This property is equivalent to being *absolutely flat*, i.e. that every module over A is flat. It was shown in [9] by Ara, Goodearl, O'Meara and Pardo that a number of open problems concerning regular rings have positive answers in the presence of *separativity*: A ring is called separative if its non-stable K-theory $\mathcal{V}(R)$ is a separative semigroup, that is if

$$2a = a + b = 2b \Rightarrow a = b$$

for $a, b \in \mathcal{V}(R)$. Consequently, they formulated what is now known as the Fundamental Separativity Problem (for regular rings): Is every regular ring separative? They focused on regular rings since the above mentioned open problems had originally been formulated in this context, but in fact their results concerned the larger class of *exchange rings.* A unital ring R is called an exchange ring if for every $x \in R$, there exists an idempotent $e \in R$ with $e \in Rx$ and $1 - e \in R(1 - x)$, and a C^*-algebra is an exchange ring if and only if it has real rank zero (see [9, Theorem 7.2] and [4, Theorem 3.8]). As such, exchange rings may be regarded as a general framework for studying both regular rings and real rank zero C^*-algebras. In the context of C^*-algebras, Ara, Goodearl, O'Meara and Pardo obtained (a)-(d) below for unital real rank zero C^*-algebras, while (e) was proved in [10, Theorem 3.1].

Theorem. *Let A be a unital C^*-algebra with real rank zero, and assume that A is separative.*

(a) *If A is finite, then A is stably finite.*
(b) *If A is simple and finite, then A has stable rank* 1*.*
(c) *The stable rank of A is either* 1*,* 2*, or* ∞*.*
(d) *The stable rank of A is finite if and only if the following cancellation property holds for projections $p, q \in M_\infty(A)$:*

$$1 \oplus 1 \oplus p \sim 1 \oplus q \Rightarrow 1 \oplus p \sim q.$$

(e) *The natural map $U(A)/U(A)^\circ \to K_1(A)$ is an isomorphism.*

We should note that (a) follows from a simple observation using only separativity, and that injectivity in (e) only makes use of real rank zero [26, Lemma 2.2]. Whether any of the statements hold universally within the class of real rank zero C^*-algebras is still unknown, but outside this class they all fail. Rørdam's example shows that (a) can fail even when A is simple and nuclear, and for any $n = 2, 3, 4, \ldots, \infty$, Villadsen constructed a finite, simple C^*-algebra of stable rank n [31].

The separativity problem is known to have a positive answer in a number of cases, of which we mention a few:

(1) If A has stable rank 1, then $\mathcal{V}(A)$ will even be cancellative by [12, Proposition 6.5.1] and hence separative.
(2) If $\mathcal{V}(A)$ enjoys n-cancellation, i.e. if $na = nb$ implies $a = b$, then A is certainly separative. This includes all Rickart C^*-algebras – and therefore all AW^*-algebras – by [3, Theorem 2.7],
(3) Any graph C^*-algebra is separative. This was established for finite graphs in [11] by showing that $\mathcal{V}(C^*(E))$ is a finitely generated refinement monoid, hence separative by results of Brookfield [13, Theorem 4.5 and Corollary 6.8]. Since separativity passes to limits and any graph C^*-algebra can be approximated by graph C^*-algebras of finite graphs, the result follows.
(4) Any extremally rich C^*-algebra of real rank zero is separative by [14, Comment 2.3 and Theorem 3.10].

Despite the above, it seems to be the common belief that the separativity problem has a negative answer for both regular rings and C^*-algebras. The main difficulty in constructing (candidates to) counterexamples is that we usually infer real rank zero of a given C^*-algebra by showing that it is somehow built from other C^*-algebras that are known to have real rank zero, and in this case, separativity will pass from the building blocks to the C^*-algebra in question. In order to properly attack the separativity problem, one therefore needs a class of non-separative building blocks with real rank at least one that can somehow be put together without gaining separativity, while at the same time obtaining real rank zero. Separated graph algebras is an attempt at obtaining such building blocks.

3. Separated graph algebras

A *separated graph* is a pair (E, C) consisting of a directed graph E and a *separation* C. Formally, a separation is a disjoint union $C = \bigsqcup_{v \in E^0} C_v$, where C_v is a partition of $r^{-1}(v)$ into non-empty subsets for all $v \in E^0$. Informally, one can think of C as a coloring of the edges, where one only cares about whether two edges have the same color if they have the same range. We shall say that (E, C) is *finitely separated* if every $X \in C$ is finite, that is if only finitely many edges of the same color have the same range, and we will focus exclusively on such graphs. Note that we can always regard a directed graph as a separated graph by giving it the *trivial separation* $\mathcal{T}$, where $\mathcal{T}_v = r^{-1}(v)$ whenever v is not a source. In this case, finitely separated simply means column-finite. The term "separated graph" was coined by Ara and Goodearl in [8], where they introduced and studied Leavitt paths algebras of such objects. Their basic idea was to generalise the definition of classical graph algebras in a completely different direction than the already existing generalisations to obtain algebras with more general non-stable K-theory. For any column-finite directed graph E, the non-stable K-theory $\mathcal{V}(L(E)) \cong \mathcal{V}(C^*(E))$ has precisely the form one would hope for: It is simply the abelian monoid $M(E)$ generated by the vertices E^0 under the relations $v = \sum_{e \in r^{-1}(v)} s(e)$ whenever v is not a source. As was noted above, this monoid is always separative, and so classical graph algebras cannot provide a counterexample to the Fundamental Separativity Problem. However, if instead one considers the abelian monoid $M(E, C)$ where, for every $v \in E^0 \setminus E^0_{\text{source}}$, the single relation $v = \sum_{e \in r^{-1}(v)} s(e)$ is replaced by the set of relations $v = \sum_{e \in X} s(e)$ for $X \in C_v$, then much more general abelian monoids – in fact all conical ones – can be obtained. This observation acts as a guideline for how one should alter the Cuntz-Krieger relations to the setting of separated graphs. Rather than summing over the entire set $r^{-1}(v)$ in (CK2), one should simply sum over each $X \in C_v$, and while it is not quite clear how (CK1) should be adapted, there is no need to require $e^* f = \delta_{e,f} s(e)$ if e and f do not have the same color. Ultimately, Ara and Goodearl defined $L_K(E, C)$ and $C^*(E, C)$ as follows:

Definition. Let (E, C) denote any finitely separated graph, and consider some field K with involution. The Leavitt path algebra $L_K(E, C)$ is the universal $*$-algebra (over K) and $C^*(E, C)$ is the universal C^*-algebra generated by $E^0 \sqcup E^1$ with relations

(V) $uv = \delta_{u,v} v$ and $u = u^*$ for $u, v \in E^0$.
(E) $es(e) = r(e)e = e$ for $e \in E^1$.
(SCK1) $e^* f = \delta_{e,f} s(e)$ for all $X \in C$ and $e, f \in X$.
(SCK2) $v = \sum_{e \in X} ee^*$ for all $v \in E^0$ and all $X \in C_v$.

Using results of Bergman, Ara and Goodearl were in fact able to compute $\mathcal{V}(L_K(E, C))$ and show that it is canonically isomorphic to $M(E, C)$ as described above [8, Theorem 4.3]. In particular, any conical abelian monoid can be realised as $\mathcal{V}(L_K(E, C))$ for an appropriate finitely separated graph (E, C), and so these algebras seem like excellent building blocks for attacking the separativity problem. Unfortunately, the relationship between the non-stable K-theories $\mathcal{V}(L_{\mathbb{C}}(E, C))$ and $\mathcal{V}(C^*(E, C))$ is still unknown. Ara and Goodearl have conjectured that the embedding $L_{\mathbb{C}}(E, C) \hookrightarrow C^*(E, C)$ induces an isomorphism, but so far no progress has been made on this important question. However, they were in fact able to compute the K-theory of $C^*(E, C)$ [7, Theorem 5.2], generalising the well known formulas for classical graph C^*-algebras, by applying a theorem of Thomsen on K-theory of universal amalgamated free product C^*-algebras. It is worth noting that $K_1(C^*(E,C))$, being a kernel, is always torsion-free.

The next major evolutionary step in the theory of separated graph algebras happened with the introduction of *tame* separated graph algebras by Ara and Exel in [5]. A set of partial isometries S is called tame if every element of the multiplicative semigroup $\langle S \cup S^* \rangle$ is also a partial isometry, and the fact that E^1 is generally not tame in $L_K(E,C)$ and $C^*(E,C)$ led them to the following definition.

Definition. Let (E,C) denote a finitely separated graph. The *abelianised* Leavitt path algebra $L_K^{\mathrm{ab}}(E,C)$ is the universal $*$-algebra and the *universal tame* graph C^*-algebra $\mathcal{O}(E,C)$ is the universal C^*-algebra generated by $E^0 \sqcup E^1$ with relations (V), (E), (SCK1), (SCK2) and E^1 being tame.

While the above definition makes perfect sense for all separated graphs, we have chosen to focus exclusively on finitely separated ones. In fact, Ara and Exel only considered finite and bipartite separated graphs in [5], since this is the realm to which their main construction naturally applies. We remark that the assumption of bipartiteness is a very minimal restriction. Indeed, given any separated graph algebra, the two-by-two matrices over this algebra can be realised as the corresponding separated graph algebra of an appropriate bipartite separated graph [5, Proposition 9.1].

The tame algebras are much more accessible than their non-tame extentions since they admit descriptions as crossed products for a partial action $\theta^{(E,C)} \colon \mathbb{F}(E^1) \curvearrowright \Omega(E,C)$ on a totally disconnected locally compact Hausdorff space. This was first proven by Ara and Exel in [5] whenever (E,C) is finite and bipartite, and we extend the result to finitely separated graphs in [B, Theorem 2.10]. For trivially separated graphs $(E,\mathcal{T})$, the partial action $\theta^{(E,\mathcal{T})}$ is easily seen to be conjugate to the canonical partial action of $\mathbb{F}(E^1)$ on the boundary path space ∂E, and so this result recovers one of the most useful descriptions of classical graph algebras as a very special case. As an important consequence, there is also a *reduced* tame graph C^*-algebra.

Definition. Let (E,C) denote a finitely separated graph. The reduced tame graph C^*-algebra $\mathcal{O}^r(E,C)$ is the reduced crossed product $\mathcal{O}^r(E,C) := C_0(\Omega(E,C)) \rtimes_r \mathbb{F}(E^1)$.

To any finite and bipartite separated graph (E,C), the main construction of [5] associates a sequence of such graphs (E_n,C^n) with $(E,C) = (E_0,C^0)$ and $s(E_n^1) = r(E_{n+1}^1)$ for which there are natural surjective $*$-homomorphisms

$$L_K(E_n,C^n) \to L_K(E_{n+1},C^{n+1}) \quad \text{and} \quad C^*(E_n,C^n) \to C^*(E_{n+1},C^{n+1}).$$

The kernel of the compositions $L_K(E,C) \to L_K(E_n,C^n)$ and $C^*(E,C) \to C^*(E_n,C^n)$ are proven to be exactly the ideal generated by all commutators $[\alpha\alpha^*, \beta\beta^*]$, where α and β are products of edges and adjoint edges of length at most n, and so it follows that

$$L_K^{\mathrm{ab}}(E,C) \cong \varinjlim L_K(E_n,C^n) \quad \text{and} \quad \mathcal{O}(E,C) \cong \varinjlim C^*(E_n,C^n).$$

Using the computation of non-stable K-theory by Ara and Goodearl, they were finally able to show that each monoid homomorphism $\mathcal{V}(L_K(E_n,C^n)) \to \mathcal{V}(L_K(E_{n+1},C^{n+1}))$ defines a unitary embedding, hence so does $\mathcal{V}(L_K(E,C)) \to \mathcal{V}(L_K^{\mathrm{ab}}(E,C))$, and that the limit $\varinjlim \mathcal{V}(L_K(E_n,C^n)) \cong \mathcal{V}(L_K^{\mathrm{ab}}(E,C))$ has the refinement property. In conclusion, passing to the tame quotient, one gets a much more well-behaved and easily accessible algebra while at the same time refining the non-stable K-theory. For this reason, the tame algebras should be regarded as better building blocks for attacking the separativity problem than their non-tame extensions.

The results of [5] allowed Ara and Exel to answer a question posed by Rørdam and Sierakowski on *relative type semigroups*. Given a (partial) action $\theta\colon G \curvearrowright \Omega$ of a discrete group G on a totally disconnected, locally compact Hausdorff space Ω, let $\mathbb{K} = \mathbb{K}(\Omega)$ denote the collection of compact open subsets $K \subset \Omega$. The relative type semigroup $S(\Omega, G, \mathbb{K})$ is the abelian monoid generated by the elements of $\mathbb{K}$ with relations $K_1 \sqcup K_2 = K_1 + K_2$ for all $K_1, K_2 \in \mathbb{K}$ and $\theta_g(K) = K$ whenever $g \in G$ and $K \in \mathbb{K}$ is in the domain of θ_g. In [30], Rørdam and Sierakowski studied purely infiniteness of the crossed product $C(\Omega) \rtimes_r G$ for essentially free actions of an exact group on the Cantor set. They proved that if the type semigroup is purely infinite, then so is the crossed product, and if, moreover, one assumes the type semigroup to be almost unperforated (meaning that $(n+1)a \leq n \cdot b$ implies $a \leq b$ for all $n \geq 2$), then these conditions are equivalent. Since no actions on the Cantor set had ever been shown to have a non-almost unperforated type semigroup at that time, they asked if such actions exist at all. In [5, Theorem 7.4], Ara and Exel were able to prove that the canonical monoid homomorphism

$$S(\Omega(E,C), \mathbb{F}(E^1), \mathbb{K}) \to \mathcal{V}(C_K(\Omega(E,C)) \rtimes \mathbb{F}(E^1)) = \mathcal{V}(L_K^{\mathrm{ab}}(E,C),$$

given by $1_K \mapsto [1_K \delta_1]$, is in fact an isomorphism. As a consequence, any conical abelian monoid may be embedded into a type semigroup, hence any cancellation property can fail for an appropriate action.

In the subsequent paper [6], Ara and Exel used the above approximation result together with Ara and Goodearl's computation of $K_*(C^*(E,C))$ to compute $K_*(\mathcal{O}(E,C))$ for any finitely separated graph (E,C). On the level of K_0, the quotient mapping induces a split monomorphism with a free abelian cokernel, and on K_1, it simply induces an isomorphism. Using a partial action version of the Pimsner-Voiculescu sequence proven by McClanahan in [27], they finally observed that $K_*(\mathcal{O}(E,C)) \cong K_*(\mathcal{O}^r(E,C))$. In particular, $K_1(\mathcal{O}(E,C)) \cong K_1(\mathcal{O}^r(E,C))$ is always torsion-free.

4. The approach and results of this thesis

The partial dynamical system associated to a finitely separated graph is a rather odd one at first sight, and its highly technical nature may easily lead one to think that its usefulness is limited. However, this is far from the truth, and we consistently embrace it as the natural approach to studying the tame algebras associated with finitely separated graphs. We will see how it allows for translation of many graph-theoretical properties into dynamical and algebraic phenomena, and, in fact, most of our results rely heavily on this approach.

We single out one part of the thesis as particularly important for properly understanding the essence and generality of these dynamical systems: our study of *convex subshifts* as defined in [A, Section 3]. A convex subshift, loosely speaking, is the shift action on a compact space of rooted trees whose edges are directed and labelled with a given, finite alphabet. *Shifting* simply means changing the root, and formally, a convex subshift is regarded as a partial action of the free group generated by the alphabet. Inspired by the classical study of subshifts, we introduce notions of *finite* and *infinite type*: A convex subshift is of finite type if it can be obtained from the space of *all* trees by forbidding finitely many balls, or equivalently, if there exists a natural number N such that one can determine whether a given tree belongs to the space by only inspecting the subtrees of radius N – in this case, we say that the convex subshift is *N-step*. If a convex subshift is not of finite type, naturally it is of infinite type. While the partial actions of separated graphs may seem somewhat arbitrary, the general notion of a convex subshift should appear fairly natural. However, by definition the former is a 1-step instance of the latter, and the main result of [A, Section 3] is an analogue of a classical result on graph representations of finite type subshifts: Any convex subshift of finite type can be represented (up to Kakutani equivalence) as the partial action of a finite bipartite separated graph. So rather than being somewhat special dynamical systems, the partial actions of separated graphs are in fact quite general. In addition, we can show a curious interplay between the viewpoint of convex subshifts and the main construction of [5]. Recall that for any finite bipartite separated graph (E, C), there is an associated sequence of such graphs (E_n, C^n) for which $(E_0, C^0) = (E, C)$. It follows from [5, Theorem 5.7] that the tame algebras of (E_n, C^n) do not depend on n, but the specific relationship between the corresponding dynamical systems was not understood. It turns out that $\theta^{(E_n,C^n)}$ is an instance of a natural construction in the realm of convex subshifts, specifically it is the *n-ball shift* of $\theta^{(E,C)}$, a generalisation of the higher block shifts from classical shift spaces. In particular, we recover [5, Theorem 8.3], which states that the vertices of E_n naturally correspond to n-balls of configurations in $\Omega(E, C)$.

Another major theme of this thesis is the study of open and invariant subspaces of the configuration space $\Omega(E, C)$ and the ideals they induce in the tame algebras. For a column-finite directed graph E, the lattice of open and invariant subspaces of the boundary path space ∂E is isomorphic to the lattice of hereditary and saturated vertex subsets, so one can read off all information from the graph. There is a similar notion of hereditary and *C-saturated* subsets, and to any such set H, one can can naturally associate an open invariant subspace $\Omega(E, C)^H \subset \Omega(E, C)$. There is also a notion of a quotient graph $(E/H, C/H)$, and the partial actions $\theta^{(E,C)}|_Z$ with $Z := \Omega(E, C) \setminus \Omega(E, C)^H$ and $\theta^{(E/H,C/H)}$ are essentially the same. On the level of tame algebras, we see that the quotient by the ideal $I(H)$ generated by H is exactly the corresponding tame algebra of $(E/H, C/H)$ [A, Theorem 5.5]. However, these subspaces $\Omega(E, C)^H$ are usually far from the only open and invariant ones. Even so, when (E, C) is finite and bipartite, we can in fact describe the lattice of open and invariant subspaces in terms of graph-theoretic data – we merely have to consider a much larger graph. Recall that $s(E^1_n) = r(E^1_{n+1})$ for every n, so the graphs (E_n, C^n) can naturally be glued together to give an infinite-layer graph (F_∞, D^∞), which we refer to as the *separated Bratteli diagram* of (E, C). In particular, there is a notion of hereditary and D^∞-saturated subsets of F^0_∞, and these exactly correspond to the open and invariant subspaces of $\Omega(E, C)$ by [A, Theorem 4.7]. Moreover, the complement $\Omega(E, C) \setminus \Omega(E, C)^H$ of such a subspace is of finite type if and only if H is of finite type, that is if H is the hereditary and D^∞-saturated closure of $H^{(n)} := H \cap E^0_n$ for some n.

While the lattice of open and invariant subspaces is too complicated to be determined directly from the graph itself, we show that many properties of the dynamical systems and algebras do in fact admit on-the-nose characterisations. This includes:

- (S) $\Omega(E, C)$ being a Cantor space (for finite graphs), [A, Proposition 9.6].
- (D1) Minimality of $\theta^{(E,C)}$, [B, Corollary 4.11].
- (D2) Topological freeness of $\theta^{(E,C)}$, [5, Theorem 10.5].

(D3) Essential freeness of $\theta^{(E,C)}$,[B, Theorem 6.13].
(A1) Simplicity of any of the separated graph algebras, [A, Theorem 8.1] and [B, Section 4].
(A2) The exchange property of any of the tame algebras, [B, Theorem 6.13].
(A3) Primeness of $L_K^{\mathrm{ab}}(E,C)$ and $\mathcal{O}^r(E,C)$ (for finite graphs satisfying (S)), [A, Theorem 9.10].
(A4) Nuclearity and exactness of $\mathcal{O}(E,C)$ and $\mathcal{O}^r(E,C)$, [C, Theorem 5.1].

Some of these properties, specifically (D1), (D3), (A1) and (A2), can only occur in rather special situations. As a result, we can show that tame separated graph algebras are only building blocks for attacking the separativity problem – they do not themselves provide a counterexample. Despite the fact that $\theta^{(E,C)}$ is usually not essentially free, when (E,C) is finite and bipartite, we can completely describe the *finite type ideals* $J \lhd \mathcal{O}^r(E,C)$ [A, Theorem 7.17]: We say that J is of finite type if $(J \cap C(\Omega(E,C))) \rtimes_r \mathbb{F} = I(H)$ for a hereditary and D^∞-saturated set H of finite type. This description relies on a weakening of topological freeness that we call *relative strong topological freeness*, and which is always enjoyed by $\theta^{(E,C)}$. Incidentally, this approach also allows one to reprove some of the fundamental results on classical graph C^*-algebras with very little work. The most subtle of the above characterisations is that of nuclearity and exactness, to which [C] is entirely dedicated. We establish a graph-theoretic *Condition* (N) characterising nuclearity of $\mathcal{O}(E,C)$ and $\mathcal{O}^r(E,C)$, and also prove that this is equivalent to the regular representation $\mathcal{O}(E,C) \to \mathcal{O}^r(E,C)$ being an isomorphism. Moreover, we show that $\mathcal{O}(E,C)$ is even non-exact when (E,C) does not satisfy Condition (N).

The earlier work of Ara, Exel and Goodearl, as sketched in the previous section, has required little inspection of the graph-theoretic properties of separated graphs. This changes dramatically when properly entering the dynamical domain, and so a key challenge has been the establishment of reasonable language and notation. Indeed, terminology can greatly affect ones ability to comprehend a theory, and, if well-chosen, will often allow one to obtain leaner and more elegant proofs. In addition, good terminology is all the more important when working with a naturally technical subject matter. The author would therefore like to express his hopes that the readers will find the thesis somewhat accessible in spite of its technical character.

5. A few problems for the future

We finally sketch some problems to be investigated further in the future.

The central question that motivated the study of separated graph algebras, whether or not one can construct a counterexample to the separativity problem, is of course still wide open. From [B, Theorem 6.13], it is apparent that one must consider quotients of the tame algebras corresponding to infinite type convex subshifts, which we have not studied at all. However, this is too vast a class to be investigated in full generality (for instance it contains all two-sided shift spaces as a tiny subclass), so one should rather study specific particularly well-behaved examples. Exploring whether some of the standard techniques for constructing infinite type subshifts can be adapted would be a first step in this direction. If one can understand the forbidden balls of an infinite type convex subshift $\mathbb{F} \curvearrowright \Omega$, then our theory will – in principle – allow one to compute the non-stable K-theory of the associated algebra $C_K(\Omega) \rtimes \mathbb{F}$ as a concrete direct limit. However, even for a minimal convex subshift, it is rather unclear how one would go about checking the exchange property/real rank zero of the associated algebras, and it is quite likely that new techniques will have to be developed for this.

Another problem is related to the latter of Rørdam's questions: Say that (E, C) is a finite bipartite separated graph, and H is an infinite type hereditary and D^{∞}-saturated subset of (F_{∞}, D^{∞}). Setting $H^{(n)} := H \cap E_n^0$, is it then possible for $\varinjlim_n \mathcal{O}^r(E_n/H^{(n)}, C^n/H^{(n)})$ to be nuclear even though every $(E_n/H^{(n)}, C^n/H^{(n)})$ does not satisfy Condition (N)? The author suspects that it is, although unaware of any examples. If it is not, then one should not expect to obtain nuclear C^*-algebras with exotic non-stable K-theory. Indeed, it seems very likely that the monoid $M(E, C)$ is both almost unperforated and separative whenever (E, C) satisfies Condition (N).

It remains unknown if the map $\mathcal{V}(L_{\mathbb{C}}(E, C)) \to \mathcal{V}(C^*(E))$ is an isomorphism, and this thesis does not offer any new insights into this important problem. Perhaps it is easier to decide whether or not the map $\mathcal{V}(L_{\mathbb{C}}^{\mathrm{ab}}(E, C)) \to \mathcal{V}(\mathcal{O}^r(E, C))$ is an isomorphism since the tame algebras are far more accessible, but even this question seems extremely difficult. A related problem is if all ideals of $\mathcal{O}^r(E, C)$ generated by projections are induced – while we prove that this is true for all ideals of finite type, our methods do not seem to cover the general case.

In another direction, it would be natural to study tame separated graph algebras of arbitrary separated graphs. The dynamical description ought to generalise by considering local configurations of the form

$$s^{-1}(v) \cup \{e_X^{-1} \colon X \in C_v, |X| < \infty\} \cup \{e_X^{-1} \colon X \in S\}$$

for $S \subset \{X \in C_v \colon |X| = \infty\}$, and the notion of *breaking vertices* should have a natural analogue in the separated setting. Quite a few of our results seem to generalise with minimal effort, while others might require new techniques, similar to those developed for non-column-finite graphs.

Part II

Articles

ARTICLE A

Convex subshifts, separated Bratteli diagrams, and ideal structure of tame separated graph algebras

This chapter contains the preprint version of the following article:

Pere Ara and Matias Lolk. Convex subshifts, separated Bratteli diagrams, and ideal structure of tame separated graph algebras. 2017.

A preprint version is publicly available at `http://arxiv.org/abs/1705.04495`.

CONVEX SUBSHIFTS, SEPARATED BRATTELI DIAGRAMS, AND IDEAL STRUCTURE OF TAME SEPARATED GRAPH ALGEBRAS

PERE ARA AND MATIAS LOLK

Abstract. We introduce a new class of partial actions of free groups on totally disconnected compact Hausdorff spaces, which we call *convex subshifts*. These serve as an abstract framework for the partial actions associated with finite separated graphs in much the same way as classical subshifts generalize the edge shift of a finite graph. We define the notion of a *finite type* convex subshift and show that any such subshift is Kakutani equivalent to the partial action associated with a finite bipartite separated graph. We then study the ideal structure of both the *full* and the *reduced tame graph C*-algebras*, $\mathcal{O}(E,C)$ and $\mathcal{O}^r(E,C)$, of a separated graph (E,C), and of the *abelianized Leavitt path algebra* $L_K^{\mathrm{ab}}(E,C)$ as well. These algebras are the (reduced) crossed products with respect to the above-mentioned partial actions, and we prove that there is a lattice isomorphism between the lattice of induced ideals and the lattice of hereditary D^∞-saturated subsets of a certain infinite separated graph (F_∞, D^∞) built from (E,C), called the *separated Bratteli diagram* of (E,C). We finally use these tools to study simplicity and primeness of the tame separated graph algebras.

Contents

1. Introduction

The study of C*-algebras associated to partial actions of groups on topological spaces is an important current line of investigation, see for instance [25] and the references therein.

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This of course includes the more traditional setting of globally defined group actions, and is in turn generalized by the often useful setting of C*-algebras associated to étale topological groupoids [45, 41, 27, 2, 35, 48].

In [26, Section 5], Exel describes any Cuntz-Krieger C*-algebra $\mathcal{O}_A$ as a crossed product of a commutative C*-algebra by a *partial action* of a non-abelian free group. This may be interpreted as follows. Given a $\{0,1\}$-matrix $A \in M_p(\{0,1\})$, one may consider the one-sided shift of finite type X_A consisting of the infinite sequences $(x_n) \in \{1,\dots,p\}^{\mathbb{N}}$ such that $A(x_n, x_{n+1}) = 1$ for all n. Then, a partial action of the free group $\mathbb{F}_p$ on the space X_A can be naturally defined so that $\mathcal{O}_A \cong C(X_A) \rtimes \mathbb{F}_p$. (This action is spelled out at the beginning of Section 3.) This description can be generalized to the more general context of graph C*-algebras, see e.g. [25, Theorem 37.8], [31].

One may also consider two-sided subshifts $\mathcal{X} \subseteq \{1,\dots,p\}^{\mathbb{Z}}$ (see e.g. [37]), where the shift map is a true homeomorphism, and thus the associated C*-algebra is simply the crossed product $C(\mathcal{X}) \rtimes \mathbb{Z}$, where the action is induced by the shift. The C*-algebra associated to a given two-sided shift is quite different from the C*-algebra associated to the corresponding one-sided shift. For instance, in the case of the full shift on $\{1,\dots,p\}$, we get the Cuntz algebra $\mathcal{O}_p$ from the one-sided shift, and we get the group C*-algebra $C^*(\mathbb{Z}_p \wr \mathbb{Z})$ of the lamplighter group $\mathbb{Z}_p \wr \mathbb{Z}$ from the two-sided shift.

In this paper, we propose a unified approach to one-sided and two-sided subshifts, under the general notion of a *convex subshift*. Given a finite alphabet A, define $\mathcal{C}(A)$ to be the space of all the (right-)convex subsets of the free group $\mathbb{F}(A)$ on A which contain 1. Then the *full convex shift* on A is defined using the natural action of $\mathbb{F}(A)$ on $\mathcal{C}(A)$ (see Definition 3.1 for the precise definition). A convex subshift is just the restriction of the full convex shift to a closed invariant subspace, and we say that a convex subshift is of *finite type* if it can be obtained by forbidding finitely many balls (or patterns).

We show that this notion is equivalent to the study of the dynamical systems associated to separated graphs, a concept recently coined by the first-named author and Ruy Exel [7]. Recall that a *separated graph* is a pair (E, C) consisting of a directed graph E and a set $C = \bigsqcup_{v \in E^0} C_v$, where each C_v is a partition of the set of edges whose terminal vertex is v. By [7, Theorem 9.1], the study of the dynamical systems associated to separated graphs can be reduced to the case of *bipartite separated graphs*, which are those separated graphs (E, C) such that $E^0 = E^{0,0} \sqcup E^{0,1}$ is the disjoint union of two layers $E^{0,0}$ and $E^{0,1}$, and $s(E^1) = E^{0,1}$, $r(E^1) = E^{0,0}$, where s and r are the source and range maps respectively. Given a finite bipartite separated graph (E, C), there exists a partial action $\theta^{(E,C)}$ of the free group $\mathbb{F} = \mathbb{F}(E^1)$ on a zero-dimensional metrizable compact space $\Omega(E, C)$ such that the tame C*-algebra $\mathcal{O}(E, C)$ is isomorphic to the partial crossed product $C(\Omega(E, C)) \rtimes \mathbb{F}$ ([7]) – this in turn allows for the definition of a *reduced* tame graph C^*-algebra $\mathcal{O}^r(E, C)$ as the corresponding reduced crossed product. A similar result holds for the abelianized Leavitt path algebra $L_K^{\mathrm{ab}}(E, C)$ over a field with involution K, allowing us to study both types of algebras at the same time.

One of our main results is Theorem 3.25, where we prove that any convex subshift of finite type is Kakutani equivalent to the dynamical system coming from a finite bipartite separated graph. This provides a far-reaching generalization of the well-known result that every shift of finite type is conjugate to an edge shift of a finite graph [37, Chapter 2].

The K-theory of the C*-algebras associated to separated graphs has been computed in [8]. For a finite bipartite separated graph (E, C), the K_0-group of both the reduced and the full tame graph C*-algebras of (E, C) can be computed in terms of an infinite separated graph (F_∞, D^∞), which is the union of a sequence of finite bipartite separated graphs (E_n, C^n). The infinite separated graph (F_∞, D^∞) has a structure which resembles very much the one of a usual Bratteli diagram. This leads us to define the new concept of a *separated Bratteli diagram* (Definition 2.8), and to refer to the graph (F_∞, D^∞) as the separated Bratteli diagram associated to (E, C).

Given a crossed product $\mathcal{O} = A \rtimes G$, we say that an ideal $J \lhd \mathcal{O}$ is *induced* if $J = (J \cap A) \rtimes G$, and we denote by $\mathrm{Ind}(\mathcal{O})$ the lattice of induced ideals. We show that the structure of induced ideals of the tame graph algebras of a finite separated graph (E, C) is completely determined by its associated separated Bratteli diagram (F_∞, D^∞). Concretely we obtain lattice isomorphisms

$$\mathrm{Ind}(L_K^{\mathrm{ab}}(E, C)) \cong \mathrm{Ind}(\mathcal{O}(E, C)) \cong \mathrm{Ind}(\mathcal{O}^r(E, C)) \cong \mathcal{H}(F_\infty, D^\infty),$$

where $\mathcal{H}(F_\infty, D^\infty)$ is the lattice of hereditary D^∞-saturated subsets of F_∞^0 (see Theorem 4.7). This generalizes the well-known result for ordinary graph C*-algebras [16, 17].

In sharp contrast with the situation for ordinary graphs, the quotient algebra $\mathcal{O}(E, C)/I(H)$ of an ideal generated by a hereditary D^∞-saturated subset H can be described in terms of the tame algebra of a separated graph only when H is of *finite type*, meaning that H can be generated by a hereditary C^n-saturated subset of vertices of the separated graph (E_n, C^n) for some $n \geq 0$. Otherwise, the quotient $\mathcal{O}(E, C)/I(H)$ can be described by means of a crossed product of a free group on a compact invariant subset of $\Omega(E, C)$, but it will in general not be a tame graph C*-algebra of a separated graph. Even so, its K-theory can be computed by using a corresponding separated Bratteli diagram $(F_\infty/H, D^\infty/H)$ (Theorem 6.4). This justifies the introduction of general separated Bratteli diagrams, and indicates that interesting examples (some of them of a pathological behaviour) can occur if we allow the consideration of hereditary D^∞-saturated of infinite type; see the comment after Example 6.9.

We present in some detail an example which is connected with the full two-sided shift. We believe this example is very useful to understand the various aspects of the theory that have been described above. It is also the example that allows to study the group algebra of the lamplighter group from the perspective of the theory of separated graphs. Using this, and generalizing [12], the first-named author and Joan Claramunt have obtained a concrete approximation of this group algebra by a sequence of finite-dimensional algebras [6].

We also study general (i.e. not necessarily induced) ideals of $\mathcal{O}^r(E, C)$. We say that an ideal J is of *finite type* if the corresponding induced ideal $I(H) = (J \cap C(\Omega(E, C))) \rtimes_r \mathbb{F}$ arises from a hereditary D^∞-saturated set H of finite type. By studying a weakening of topological freeness that we dub *relative strong topological freeness*, we are able to compute the lattice of finite type ideals. We also observe that our techniques, when applied to non-separated graphs, yield a complete characterization of the ideal structure.

We close the paper by using our tools to perform a study of simplicity and primeness in tame graph algebras of separated graphs.

In this paper, we consider three types of tame graph $*$-algebras associated to separated graphs, namely the *full tame graph C*-algebra* $\mathcal{O}(E,C)$, the *reduced tame graph C*-algebra* $\mathcal{O}^r(E,C)$, and the *abelianized Leavitt path algebra* $L_K^{\mathrm{ab}}(E,C)$ over a field with involution K. All three types of algebras have been introduced in [7]. The first two types generalize graph C*-algebras [44], in the sense that, denoting by $\mathcal{T}$ the trivial partition on each $r^{-1}(v)$, we have $\mathcal{O}(E,\mathcal{T}) = \mathcal{O}^r(E,\mathcal{T}) = C^*(E)$, where $C^*(E)$ is the graph C*-algebra of E. The third type generalizes Leavitt path algebras [4, 14], in the sense that $L_K^{\mathrm{ab}}(E,\mathcal{T}) = L_K(E)$, where $L_K(E)$ is the Leavitt path algebra of E.

Contents. We now describe the contents of the paper in more detail. In section 2, we recall relevant definitions and constructions from the existing theory of algebras associated with separated graphs, in particular the main results of [7]. We also introduce the notion of a *separated Bratteli diagram*. In section 3, we introduce a class of partial actions that we call *convex subshifts*. These serve as an abstract framework for the partial actions associated with finite separated graphs in much the same way as classical subshifts generalize the edge shift of a finite graph. We define the notion of a *finite type* convex subshift and show that any such subshift is Kakutani equivalent to the partial action associated with a finite bipartite separated graph (Theorem 3.25). Using the language of convex subshifts, we are also able to explain the precise relationship between the partial action of a finite bipartite separated graph (E,C) and its successors (E_n,C^n) for $n \geq 1$ (Theorem 3.22). In section 4, we begin our investigation of the lattice of induced ideals of the algebras $L^{\mathrm{ab}}(E,C)$, $\mathcal{O}(E,C)$ and $\mathcal{O}^r(E,C)$. The main result (Theorem 4.7) identifies this lattice with the lattice $\mathcal{L}(M(F_\infty,D^\infty))$ of order ideals of the monoid $M(F_\infty,D^\infty)$, as well as the lattice of hereditary and D^∞-saturated subsets of (F_∞,D^∞), providing both an algebraic and a graph theoretic perspective. In the following section, we study the specific ideals associated with hereditary and C-saturated subsets of (E,C), and we show that the corresponding quotients arise as separated graph algebras from the quotient graph (Theorem 5.5). In section 6, we combine the work of the previous sections to describe the quotient by an arbitrary induced ideal as a limit of separated graph algebras (Proposition 6.2), and we provide a dynamical description of this approximation. We also pay significant attention to a concrete example that illustrates the general theory (Examples 6.7, 6.9, and 6.10). In particular, we show that the crossed product of a two-sided finite type subshift can be realised as a full corner of the tame graph C*-algebra associated with a finite bipartite separated graph (Proposition 6.8). We proceed to study general ideals of finite type in section 7, and we provide a complete characterisation in terms of graph theoretic data (Theorem 7.12). This relies on an abstract study of *relatively strongly topologically free* partial actions that we initiate. We then study $\mathcal{V}$-simplicity of the separated graph algebras in section 8, showing that (up to Morita equivalence) the tame graph C*-algebras degenerate to either classical graph C*-algebras or group C*-algebras of free groups if $M(F_\infty, D^\infty) \cong \mathcal{V}(L^{\mathrm{ab}}(E,C))$ is order simple (Theorem 8.1). In section 9, we finally establish a criterion for $\Omega(E,C)$ to be a Cantor space (Proposition 9.6), and in this setting we characterize primeness of the algebras $L_K^{\mathrm{ab}}(E,C)$ and $\mathcal{O}^r(E,C)$ (Theorem 9.10).

2. Preliminary definitions

In this preliminary section, we will recall all the relevant definitions and constructions from the theory of algebras associated with separated graphs. The reader should note that we use the same conventions as in [7] and [8], but opposite to those of [11] and [10]. One important consequence is that all paths should be read from the right.

Definition 2.1. ([11]) A *separated graph* is a pair (E,C) where $E=(E^0,E^1,r,s)$ is a directed graph, $C=\bigsqcup_{v\in E^0} C_v$, and C_v is a partition of $r^{-1}(v)$ (into non-empty subsets) for every vertex v. In case v is a source, i.e. $r^{-1}(v)=\emptyset$, we take C_v to be the empty family of subsets of $r^{-1}(v)$. Given an edge e, we shall use the notation X_e for the element of C containing e.

If all the sets in C are finite, we say that (E,C) is a *finitely separated* graph. This necessarily holds if E is column-finite (that is, if $r^{-1}(v)$ is a finite set for every $v\in E^0$.)

The set C is a *trivial separation* of E in case $C_v=\{r^{-1}(v)\}$ for each $v\in E^0\setminus \mathrm{Source}(E)$. In that case, (E,C) is called a *trivially separated graph* or a *non-separated graph.*

Finally, (E,C) is called *bipartite* if the vertex set admits a partition $E^0=E^{0,0}\sqcup E^{0,1}$ with $s(E^1)=E^{0,1}$ and $r(E^1)=E^{0,0}$.

As the first of many different algebras assocated with separated graphs, we now define the Leavitt path algebra.

Definition 2.2. Let $(K,*)$ be a field with involution. The *Leavitt path algebra* of the separated graph (E,C) with coefficients in the field K is the $*$-algebra $L_K(E,C)$ with generators $\{v,e\mid v\in E^0, e\in E^1\}$, subject to the following relations:

(V) $vv'=\delta_{v,v'}v$ and $v=v^*$ for all $v,v'\in E^0$,

(E) $r(e)e=es(e)=e$ for all $e\in E^1$,

(SCK1) $e^*e'=\delta_{e,e'}s(e)$ for all $e,e'\in X$, $X\in C$, and

(SCK2) $v=\sum_{e\in X} ee^*$ for every finite set $X\in C_v$, $v\in E^0$.

The Leavitt path algebra $L_K(E)$ is just $L_K(E,C)$ where $C_v=\{r^{-1}(v)\}$ if $r^{-1}(v)\neq\emptyset$ and $C_v=\emptyset$ if $r^{-1}(v)=\emptyset$. An arbitrary field can be considered as a field with involution by taking the identity as the involution. However, our "default" involution over the complex numbers $\mathbb{C}$ will be the complex conjugation, and we will write $L(E,C):=L_{\mathbb{C}}(E,C)$.

We now recall the definition of the graph C*-algebra $C^*(E,C)$, introduced in [10].

Definition 2.3. The *graph C*-algebra* of a separated graph (E,C) is the C*-algebra $C^*(E,C)$ with generators $\{v,e\mid v\in E^0,\ e\in E^1\}$, subject to the relations (V), (E), (SCK1), (SCK2). In other words, $C^*(E,C)$ is the enveloping C*-algebra of $L(E,C)$.

In case (E,C) is trivially separated, $C^*(E,C)$ is just the classical graph C*-algebra $C^*(E)$. There is a unique *-homomorphism $L(E,C)\to C^*(E,C)$ sending the generators of $L(E,C)$ to their canonical images in $C^*(E,C)$. This map is injective by [10, Theorem 3.8(1)].

Since both $L_K(E,C)$ and $C^*(E,C)$ are universal objects with respect to the same sets of generators and relations, they can be studied in much the same way. A remarkable difference is that the non-stable K-theory $\mathcal{V}(L_K(E,C))$ *has* been computed for any separated graph (E,C), while the structure of $\mathcal{V}(C^*(E,C))$ is still unknown. However, it is conjectured in [10] that the natural map $L(E,C) \to C^*(E,C)$ induces an isomorphism $\mathcal{V}(L(E,C)) \to \mathcal{V}(C^*(E,C))$. See [5, Section 6] for a short discussion on this problem. We now describe $\mathcal{V}(L_K(E,C))$ as the *graph monoid* of (E,C).

Definition 2.4. Given a finitely separated graph (E,C), the *graph monoid* $M(E,C)$ is the abelian monoid with generators a_v for $v \in E^0$ and relations $a_v = \sum_{e\in X} a_{s(e)}$ for all $v \in E^0$ and $X \in C_v$.

Note that there is a natural map $M(E) \to \mathcal{V}(L_K(E,C))$ given by $a_v \mapsto [v]$, and this is in fact an isomorphism by [11, Theorem 4.3].

One issue with the algebras $L_K(E,C)$ and $C^*(E,C)$ is that the set of partial isometries represented by the edges E^1 is not *tame*, that is, products of edges and their adjoints are not in general partial isometries. This motivated Ruy Exel and the first named author to introduce certain quotients $L_K^{\mathrm{ab}}(E,C)$ and $\mathcal{O}(E,C)$ of $L_K(E,C)$ and $C^*(E,C)$, respectively, that we shall now describe.

Definition 2.5. For a finitely separated graph (E,C), let S denote the multiplicative subsemigroup of $L_K(E,C)$ generated by $E^1 \cup (E^1)^*$, and define an ideal (respectively a closed ideal)

$$J = \langle \alpha\alpha^*\alpha - \alpha \mid \alpha \in S\rangle = \langle [\alpha\alpha^*, \beta\beta^*] \colon \alpha, \beta \in S\rangle$$

of $L_K(E,C)$ (respectively of $C^*(E,C)$). Then we set

$$L_K^{\mathrm{ab}}(E,C) := L_K(E,C)/J \quad \text{and} \quad \mathcal{O}(E,C) := C^*(E,C)/J.$$

Observe that modding out J precisely forces E^1 to be tame in these quotients.

We now recall the main construction of [7] which will play an important role in what is to come.

Definition 2.6. Let (E,C) denote a finite bipartite separated graph, and write

$$C_u = \{X_1^u, \dots, X_{k_u}^u\}$$

for all $u \in E^{0,0}$. Then (E_1, C^1) is the finite bipartite separated graph defined by

- $E_1^{0,0} := E^{0,1}$ and $E_1^{0,1} := \{v(x_1, \dots, x_{k_u}) \mid u \in E^{0,0}, x_j \in X_j^u\}$,
- $E^1 := \{\alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u}) \mid u \in E^{0,0}, i = 1, \dots, k_u, x_j \in X_j^u\}$,
- $r_1(\alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u})) := s(x_i)$ and $s_1(\alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u})) := v(x_1, \dots, x_{k_u})$,
- $C_v^1 := \{X(x) \mid x \in s^{-1}(v)\}$, where

$$X(x_i) := \{\alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u}) \mid x_j \in X_j^u \text{ for } j \neq i\}.$$

A sequence of finite bipartite separated graphs $\{(E_n, C^n)\}_{n\geq 0}$ with $(E_0, C^0) := (E, C)$ is then defined inductively by letting (E_{n+1}, C^{n+1}) denote the 1-graph of (E_n, C^n). Finally, (F_n, D^n) denotes the union $\bigcup_{i=0}^n (E_n, C^n)$, and (F_∞, D^∞) is the infinite layer graph

$$(F_\infty, D^\infty) := \bigcup_{n=0}^\infty (F_n, D^n) = \bigcup_{n=0}^\infty (E_n, C^n).$$

Observe that (F_n, D^n) is a finite separated graph, while (F_∞, D^∞) is a finitely separated graph.

By [7, Theorem 5.1 and Theorem 5.7], there are canonical surjective $*$-homomorphisms $*$-homomorphisms

$$L_K(E_n, C^n) \to L_K(E_{n+1}, C^{n+1}) \quad \text{and} \quad C^*(E_n, C^n) \to C^*(E_{n+1}, C^{n+1})$$

such that

$$L_K^{ab}(E, C) \cong \varinjlim_n L_K(E_n, C^n) \quad \text{and} \quad \mathcal{O}(E, C) \cong \varinjlim_n C^*(E_n, C^n).$$

On the level of monoids, the induced monoid homomorphism

$$M(E_n, C^n) \cong \mathcal{V}(L_K(E_n, C^n)) \to \mathcal{V}(L_K(E_{n+1}, C^{n+1}) \cong M(E_{n+1}, C^{n+1})$$

is a unitary embedding, which refines the defining relations of $M(E_n, C^n)$. Consequently, the quotient map $L_K(E, C) \to L_K^{\mathrm{ab}}(E, C)$ induces a (universal) refinement

$$M(E, C) \cong \mathcal{V}(L_K(E, C)) \to \mathcal{V}(L_K^{\mathrm{ab}}(E, C)) \cong M(F_\infty, D^\infty).$$

One of the main advantages of dealing with the quotients $L_K^{\mathrm{ab}}(E, C)$ and $\mathcal{O}(E, C)$ is that, by [7, Corollary 6.12], they admit descriptions as crossed products

$$L_K^{\mathrm{ab}}(E, C) \cong C_K(E, C) \rtimes \mathbb{F} \quad \text{and} \quad \mathcal{O}(E, C) \cong C(\Omega(E, C)) \rtimes \mathbb{F}$$

of a topological partial action $\theta^{(E,C)} \colon \mathbb{F} \curvearrowright \Omega(E, C)$, referred to as the *canonical partial (E, C)-action.* Here, $\mathbb{F}$ is the free group generated by E^1, $\Omega(E, C)$ is a certain compact, zero-dimensional, metrisable space, and $C_K(\Omega(E, C))$ denotes the $*$-algebra of continuous functions $\Omega(E, C) \to K$ when K is given the discrete topology, that is, the $*$-algebra of locally constant functions $\Omega(E, C) \to K$. Every vertex $v \in F_\infty^0$ corresponds to a compact open subset $\Omega(E, C)_v \subset \Omega(E, C)$ such that $\Omega(E, C) = \bigsqcup_{v\in E_n^0} \Omega(E, C)_v$ for all n.

The crossed product description enables the definition of yet another tame graph C^*-algebra.

Definition 2.7. Let (E, C) denote a finite bipartite separated graph. Then $\mathcal{O}^r(E, C)$ is the reduced crossed product $C(\Omega(E, C)) \rtimes_r \mathbb{F}$ of the canonical partial (E, C)-action.

The separated graph (F_∞, D^∞) resembles very much the structure of a *Bratteli diagram* [18], but incorporating separations (and up to a change of the convention of the direction of the arrows). We formalize this with the following definition.

Definition 2.8. A separated (or colored) Bratteli diagram is an infinite separated graph (F, D). The vertex set F^0 is the union of finite, non-empty, pairwise disjoint sets $F^{0,j}$, $j \geq 0$. Similarly, the edge set F^1 is the union of a sequence of finite, non-empty, pairwise disjoint sets $F^{1,j}$, $j \geq 0$. The range and source maps satisfy $r(F^{1,j}) = F^{0,j}$ and $s(F^{1,j}) = F^{0,j+1}$ for all $j \geq 0$.

A Bratteli diagram is just a separated Bratteli diagram with the trivial separation. The graph (F_∞, D^∞) associated to a finite bipartite separated graph (E, C) is an example of a separated Bratteli diagram, and we will refer to it as the *separated Bratteli diagram* of the separated graph (E, C). We will find later other examples of separated Bratteli diagrams, related to quotients of tame graph C*-algebras, see Theorem 6.4.

The *graph monoid* of a Bratteli diagram (F, D) is the graph monoid $M(F, D)$ of the finitely separated graph (F, D), see Definition 2.4. The Grothendieck group $G(F, D)$ of the monoid $M(F, D)$ is an analogue of the dimension group associated to a Bratteli diagram, but it is not a dimension group in general. It is quite sensible to equip $G(F, D)$ with the structure of a pre-ordered abelian group, taking $G(F, D)^+ := \varphi(M(F, D))$, where $\varphi\colon M(F, D) \to G(F, D)$ is the canonical map.

The separated Bratteli diagram of a finite bipartite separated graph computes the K_0 of the tame graph algebras of the graph, as follows.

Theorem 2.9. *Let (E, C) be a finite bipartite separated graph, and let (F_∞, D^∞) be its separated Bratteli diagram.*

1. *There is a natural group isomorphism*
$$K_0(\mathcal{O}(E, C)) \cong K_0(\mathcal{O}^r(E, C)) \cong G(F_\infty, D^\infty).$$
2. *There is a natural isomorphism $\mathcal{V}(L_K^{\mathrm{ab}}(E, C)) \cong M(F_\infty, D^\infty)$, and so an isomorphism of pre-ordered abelian groups*
$$K_0(L_K^{\mathrm{ab}}(E, C)) \cong G(F_\infty, D^\infty)$$
for any field with involution K.

Proof. (2) follows from [7, Theorem 5.1], and (1) follows from [8, Theorem 4.4(c), Corollary 6.9]. ☐

In the following, we shall recall a very handy description of the canonical partial (E, C)-action introduced in [7], although we will make a slight change to the original definition, coherent with the conventions of [38] and [39]. We remark that while we only consider finite bipartite graphs, the dynamical picture can be extended to arbitrary finitely separated graphs [38, Definition 2.6 and Theorem 2.10]. For a comprehensive study of the general theory of partial actions and their crossed products, we refer the reader to [25].

Definition 2.10. Suppose that (E, C) is a separated graph, and let $\widehat{E}$ denote the *double* of E, i.e. the graph with

$$\widehat{E}^0 := E^0, \widehat{E}^1 := E^1 \sqcup (E^1)^{-1}, \hat{r}(e) := r(e) = \hat{s}(e^{-1}) \quad \text{and} \quad \hat{r}(e^{-1}) := s(e) =: \hat{s}(e).$$

A path in the double of E (with the convention that a path is read from the right to the left) is called *an admissible path* if

- $e \neq f$ for every subpath ef^{-1},
- $X_e \neq X_f$ for every subpath $e^{-1}f$.

We define the range and source of an admissible path to simply be the range and source in the double, and we view the vertices as the set of trivial admissible paths.

A *closed path* in (E, C) is a non-trivial admissible path α with $r(\alpha) = s(\alpha)$, and α is called a *cycle* if the concatenated word $\alpha\alpha$ is an admissible path as well. Either way, we shall say that α is *based* at $r(\alpha) = s(\alpha)$. A cycle is called *simple* if the only vertex repetition occurs at the end.

Definition 2.11. Suppose that (E, C) is finite bipartite separated graph, and let $\mathbb{F}$ denote the free group on E^1. Given $\xi \subset \mathbb{F}$ and $\alpha \in \xi$, the *local configuration* ξ_α of ξ at α is defined as

$$\xi_\alpha := \{\sigma \in E^1 \sqcup (E^1)^{-1} \mid \sigma \in \xi \cdot \alpha^{-1}\}.$$

Then $\Omega(E, C)$ is the set of $\xi \subset \mathbb{F}$ satisfying the following:

(a) $1 \in \xi$.
(b) ξ is *right-convex*: In view of (a), this exactly means that if $e_n^{\varepsilon_n} \cdots e_1^{\varepsilon_1} \in \xi$ for $e_i \in E^1$ and $\varepsilon_i \in \{\pm 1\}$, then $e_m^{\varepsilon_m} \cdots e_1^{\varepsilon_1} \in \xi$ as well for any $1 \leq m < n$.
(c) For every $\alpha \in \xi$, one of the following holds:
  (c1) $\xi_\alpha = s^{-1}(v)$ for some $v \in E^{0,1}$.
  (c2) $\xi_\alpha = \{e_X^{-1} \mid X \in C_v\}$ for some $v \in E^{0,0}$ and $e_X \in X$.

Observe that for $\xi \in \Omega(E, C)$, every $\alpha \in \xi$ is an admissible path. $\Omega(E, C)$ is made into a topological space by regarding it as a subspace of $\{0, 1\}^{\mathbb{F}}$. Thus it becomes a compact, zero-dimensional, metrisable space, and a topological partial action $\theta = \theta^{(E,C)} \colon \mathbb{F} \curvearrowright \Omega(E, C)$ with compact open domains is then defined by setting

- $\Omega(E, C)_\alpha := \{\xi \in \Omega(E, C) \mid \alpha^{-1} \in \xi\}$,
- $\theta_\alpha(\xi) := \xi \cdot \alpha^{-1}$ for $\xi \in \Omega(E, C)_{\alpha^{-1}}$.

This partial action is equivalent to the one defined in [7] under the map $\xi \mapsto \xi^{-1}$. We choose to invert the elements as we want a common terminology and notation for both the algebraic and the topological setting. The compact, open sets corresponding to the vertices of E are given by

$$\Omega(E, C)_v = \begin{cases} \Omega(E, C)_{e^{-1}} \text{ for some } e \in s^{-1}(v) & \text{if } v \in E^{0,1} \\ \bigsqcup_{e \in X} \Omega(E, C)_e \text{ for some } X \in C_v & \text{if } v \in E^{0,0} \end{cases},$$

and we note that this is independent of e and X by (c1) and (c2), respectively. The reader may think of $\Omega(E, C)_v$ as the set of configurations "starting" in v, and we shall regard $1 \in \xi$ as the trivial path v when $\xi \in \Omega(E, C)_v$. See Remark 3.23 for a description of $\Omega(E, C)_v$ when $v \in E_n^0$ for $n \geq 1$.

3. Convex subshifts

Given a finite alphabet A and any set $\mathcal{F}$ of finite words in A, one can consider the space $X_{\mathcal{F}} \subset A^{\mathbb{N}}$ of infinite one-sided sequences that do not contain any words from $\mathcal{F}$. In classical symbolic dynamics, $X_{\mathcal{F}}$ is then usually turned into a dynamical system by equipping it with the one-sided shift σ, but for the purpose of constructing interesting algebras, one would typically add additional dynamical structure. Specifically, a sequence may not only be shifted to the left, but new letters may also be introduced in the beginning, provided that we stay inside $X_{\mathcal{F}}$, of course.

Formally, this dynamical system can be regarded as a partial action with the free group $\mathbb{F}(A)$ acting on $X_{\mathcal{F}}$: If $a \in A$ and $x \in X_{\mathcal{F}}$, then a can act on x by $a.x = ax$ whenever $ax \in X_{\mathcal{F}}$, while the inverse a^{-1} can act on points $ax \in X_{\mathcal{F}}$ by $a^{-1}.ax = x = \sigma(ax)$. It is a standard fact that any subshift of finite type (meaning that $\mathcal{F}$ is finite) arises as the edge shift of some finite graph E with no sinks, and the graph C^*-algebra may then be recovered as the partial crossed product $C^*(E) \cong C(X_{\mathcal{F}}) \rtimes \mathbb{F}(A)$ (see for instance [25, Theorem 36.20] or [19, Theorem 3.1]). If E is a finite directed graph, one can also consider the *boundary path space* ∂E, where – in addition to the set of right infinite paths – one also includes the set of finite paths ending in a sink. Then there is natural partial action of $\mathbb{F}(E^1)$ on ∂E defined as above, and we still have $C^*(E) \cong C(\partial E) \rtimes \mathbb{F}(E^1)$. In fact, a similar description exists for arbitrary graphs.

In this section, we shall introduce a class of partial actions that one might consider as a generalisation of both the above partial actions on one-sided sequence spaces (including those of infinite type) and the boundary path spaces of finite graphs. We will see later (Example 6.7, Proposition 6.8) that also two-sided shifts can be recasted in this language. The basic idea is to give up the linear nature of a sequence and allow trees instead. The dynamics still arise from shifting, but there are usually many possible shifting directions with no one being canonical. We will refer to these dynamical systems as *convex subshifts* for reasons that should become apparent soon.

The motivation for introducing such systems is two-fold: On one hand, it provides a framework for the dynamical systems associated with finite separated graphs in which one has far more flexibility. On the other hand, the dynamical systems of finite bipartite separated graphs encompass a vast class of convex subshifts, in fact all *finite type* convex subshifts up to Kakutani-equivalence (see Theorem 3.25). As such, convex subshifts are to finite bipartite separated graphs as one-sided subshifts are to finite graphs. Finally, we shall see how the construction of (E_n, C^n) from (E, C) corresponds to a natural construction in the realm of convex subshifts.

The contents of this section are closely related to classical subshifts of free groups on the alphabet $\{0, 1\}$. Indeed, any convex subshift is the restriction of a free group subshift to a partial action, but the convexity requirement that we impose is *not* of finite type; hence a convex subshift would typically be regarded as an infinite type subshift. Moreover, it is most beneficial for us to define everything from scratch so that the formal framework of convex subshifts is similar to that of partial actions associated with separated graphs.

Throughout this section, A will be a finite alphabet and $\mathbb{F} = \mathbb{F}(A)$ will be the free group on A. Though some definitions still make sense for infinite alphabets, we choose to deal only with finite ones for the sake of simplicity.

Definition 3.1. Denote by $\mathcal{C} = \mathcal{C}(A)$ the set of right-convex subsets $\xi \subset \mathbb{F}$ for which $1 \in \xi$. As in Definition 2.11, right-convexity in this setting simply means that if $a_n^{\varepsilon_n} \cdots a_1^{\varepsilon_1} \in \xi$ for $a_i \in A$ and $\varepsilon_i \in \{\pm 1\}$, then $a_m^{\varepsilon_m} \cdots a_1^{\varepsilon_1} \in \xi$ as well for all $1 \leq m < n$. We topologize $\mathcal{C}$ as a subspace of $\mathcal{P}(\mathbb{F}) \cong \{0,1\}^{\mathbb{F}}$, making it into a compact, zero-dimensional, and metrizable space (with many isolated points). The *full convex shift on* A is then the partial action $\mathbb{F} \curvearrowright \mathcal{C}$ given by

- $\mathcal{C}_\alpha := \{\xi \in \mathcal{C} \mid \alpha^{-1} \in \xi\}$ for all $\alpha \in \mathbb{F}$.
- $\alpha.\xi := \xi \cdot \alpha^{-1}$ for $\xi \in \mathcal{C}_{\alpha^{-1}}$.

Viewing each $\xi \in \mathcal{C}$ as a tree, rooted in 1 and labelled relative to the root, the action of any $\alpha \in \xi$ on ξ is thus simply given by moving the root to α and relabelling accordingly.

Remark 3.2. We could also define $\mathcal{C}$ to be the set of *left-convex* subsets with

$$\mathcal{C}_\alpha = \{\xi \in \mathcal{C} \mid \alpha \in \xi\} \quad \text{and} \quad \alpha.\xi = \alpha \cdot \xi,$$

but we choose the above convention to match the one used for separated graphs. It is also very convenient that with our convention, α can act on ξ if and only if $\alpha \in \xi$ (as opposed to $\alpha^{-1} \in \xi$). By [29, Section 4] and [28, Proposition 4.5], the full convex shift is conjugate to the universal action for semi-saturated partial representations of $\mathbb{F}$.

Definition 3.3. A *convex subshift* is the restriction of the full convex shift $\mathbb{F} \curvearrowright \mathcal{C}$ to any closed invariant subspace $\Omega \subset \mathcal{C}$.

Definition 3.4. An *n-ball* is an element $B \in \mathcal{C}$ such that $|\alpha| \leq n$ for every $\alpha \in B$, together with the *radius* $r(B) = n$; a *ball* is then simply an n-ball for some n. Note that B as a set does not determine the radius $r(B)$, so one has to specify this. Now if $\xi \in \mathcal{C}$, we can always consider the n-ball

$$\xi^n := \{\alpha \in \xi \colon |\alpha| \leq n\},$$

and if B is a given n-ball, we shall write $\xi \not\ni B$ if $(\alpha.\xi)^n \neq B$ for all $\alpha \in \xi$. Finally, if Ω is any convex subshift, we will write

$$\mathcal{B}_n(\Omega) := \{\xi^n \mid \xi \in \Omega\} \quad \text{and} \quad \mathcal{B}(\Omega) := \bigsqcup_{n \geq 0} \mathcal{B}_n(\Omega)$$

for the set of *allowed* n-balls and *allowed* balls, respectively.

With all the relevant terminology in place, we can give an example of a convex subshift.

Definition 3.5. Let $\mathcal{F}$ denote any set of balls; we can then define a convex subshift by

$$\Omega^{\mathcal{F}} := \{\xi \in \mathcal{C} \mid \xi \not\ni B \text{ for all } B \in \mathcal{F}\},$$

and we shall refer to this as the convex subshift obtained from *forbidding* $\mathcal{F}$.

In fact, this is not *an* example, but rather *the* example.

Proposition 3.6. *If $\mathbb{F} \curvearrowright \Omega$ is a convex subshift, then $\Omega = \Omega^{\mathcal{F}}$ for some set of balls $\mathcal{F}$.*

Proof. Define $\mathcal{F}$ to be all the balls that do not occur in Ω, i.e. let

$$\mathcal{F} := \mathcal{B}(\mathcal{C}) \setminus \mathcal{B}(\Omega).$$

Clearly $\Omega \subset \Omega^{\mathcal{F}}$, so let us consider the reverse inclusion. Given any $\xi \in \Omega^{\mathcal{F}}$ and $n \geq 1$, it is enough to check that $\xi^n = \eta^n$ for some $\eta \in \Omega$ since Ω is closed. But since $\xi^n \notin \mathcal{F}$, we must have $\xi^n \in \mathcal{B}_n(\Omega)$, so $\xi^n = \eta^n$ for some $\eta \in \Omega$ as desired. □

Definition 3.7. A convex subshift $\mathbb{F} \curvearrowright \Omega$ is of *finite type* if $\Omega = \Omega^{\mathcal{F}}$ for some finite set of forbidden balls $\mathcal{F}$. Observe that for such a convex subshift, we can safely assume that all balls of $\mathcal{F}$ have the same radius R; we recognize this by saying that Ω is *R-step*. Observe that in this situation, Ω is *generated* by $\mathcal{B}_R(\Omega)$ in the following sense: An element $\xi \in \mathcal{C}$ belongs to Ω if and only if $(\alpha.\xi)^R \in \mathcal{B}_R(\Omega)$ for all $\alpha \in \xi$.

Before venturing on, we need to discuss how one might compare partial actions of different groups.

Definition 3.8. Consider partial actions $\theta\colon G \curvearrowright \Omega$ and $\theta'\colon H \curvearrowright \Omega'$ of discrete groups on topological spaces. Then θ and θ' are called *dynamically equivalent* and we shall write $\theta \approx \theta'$, if their transformation groupoids $\mathcal{G}_\theta$ and $\mathcal{G}_{\theta'}$ (see for instance [39, Example 2.3]) are isomorphic as topological groupoids. We now spell out exactly what this means: There is a homeomorphism $\varphi\colon \Omega \to \Omega'$ and continuous maps

$$a\colon \bigcup_{g\in G} \{g\} \times \Omega_{g^{-1}} \to H \quad \text{and} \quad b\colon \bigcup_{h\in H} \{h\} \times \Omega'_{h^{-1}} \to G$$

such that

(1) $\varphi(x) \in \Omega'_{a(g,x)^{-1}}$ and $\varphi(g.x) = a(g,x).\varphi(x)$,
(2) $\varphi^{-1}(y) \in \Omega_{b(h,y)^{-1}}$ and $\varphi^{-1}(h.y) = b(h,y).\varphi^{-1}(y)$,
(3) $b(a(g,x),\varphi(x)) = g$ and $a(b(h,y),\varphi^{-1}(y)) = h$,
(4) $a(g'g,x) = a(g',g.x)a(g,x)$ if $g.x \in \Omega_{g'^{-1}}$,
(5) $b(h'h,y) = b(h',h.y)b(h,y)$ if $h.y \in \Omega_{h'^{-1}}$

for all $g \in G$, $h \in H$, $x \in \Omega_{g^{-1}}$, $y \in \Omega'_{h^{-1}}$.

Remark 3.9. Given a locally compact Hausdorff étale groupoid $\mathcal{G}$, one can associate to it both a *universal* and a *reduced* groupoid C^*-algebra (see [45]), denoted $C^*(\mathcal{G})$ and $C^*_r(\mathcal{G})$, respectively. If $\mathcal{G} = \mathcal{G}_\theta$ for a partial action $\theta\colon G \curvearrowright \Omega$ of a discrete group on a locally compact Hausdorff space, then there are identifications $C^*(\mathcal{G}_\theta) \cong C_0(\Omega) \rtimes_\theta G$ and $C^*_r(\mathcal{G}_\theta) \cong C_0(\Omega) \rtimes_{\theta,r} G$ by [2, Theorem 3.3] and [35, Proposition 2.2]. For an *ample* groupoid $\mathcal{G}$ and any field K with involution, there is also a purely algebraic analogue $K\mathcal{G}$, known as the *Steinberg algebra* of $\mathcal{G}$ [47]. If a partial action θ as above acts on a totally disconnected space, then $\mathcal{G}_\theta$ is ample and $K\mathcal{G}_\theta \cong C_K(\Omega) \rtimes_\theta G$, where $C_K(\Omega)$ denotes the algebra of compactly supported continuous functions $\Omega \to K$, when K is endowed with the discrete topology. Hence if two partial actions θ and θ' as above are dynamically equivalent, i.e. $\mathcal{G}_\theta \cong \mathcal{G}_{\theta'}$, then they have isomorphic crossed products by base-preserving isomorphisms. In this light, groupoids provide a very flexible framework for identifying the crossed products of partial actions.

Definition 3.10. Consider partial actions $\theta\colon G \curvearrowright \Omega$ and $\theta'\colon H \curvearrowright \Omega'$ of discrete groups on topological spaces along with a group homomorphism $\Psi\colon G \to H$. A continuous map $\varphi\colon \Omega \to \Omega'$ is then called Ψ-*equivariant* if

(1) $\varphi(\Omega_g) \subset \Omega'_{\Psi(g)}$ for all $g \in G$,
(2) $\varphi(g.x) = \Psi(g).\varphi(x)$ for all $x \in \Omega_{g^{-1}}$.

The pair (φ, Ψ) is called a *conjugacy* if Ψ is an isomorphism and φ admits a Ψ^{-1}-equivariant inverse. However, conjugacy is often too rigid a notion and we therefore introduce another type of equivalence in between conjugacy and dynamical equivalence: The pair (φ, Ψ) is called a *direct dynamical equivalence* if

(a) φ is a homeomorphism,
(b) $\Omega_g \cap \Omega_{g'} = \emptyset$ for all $g \neq g'$ with $\Psi(g) = \Psi(g')$,
(c) $\Omega'_h = \bigcup_{\Psi(g)=h} \varphi(\Omega_g)$ for all $h \in H$,

and in this case we will write $\theta \xrightarrow{\approx} \theta'$.

Now let us see that our choice of name and notation is justified.

Proposition 3.11. *If $\theta \xrightarrow{\approx} \theta'$, then $\theta \approx \theta'$.*

Proof. Apply the notation from above. We then set $a(g,x) := \Psi(g)$ for all $g \in G$, $x \in \Omega_{g^{-1}}$, and given $h \in H$, $y \in \Omega'_{h^{-1}}$, we define $b(h,y)$ to be the unique element of G satisfying

$$\Psi(b(h,y)) = h \quad \text{and} \quad y \in \varphi(\Omega_{b(h,y)^{-1}}).$$

Then

$$\varphi^{-1}(h.y) = \varphi^{-1}(\Psi(b(h,y)).y) = b(h,y).\varphi^{-1}(y)$$

and

$$b(a(g,x),\varphi(x)) = b(\Psi(g),\varphi(x)) = g,$$

so (1)-(4) of Definition 3.8 surely hold. Finally, if $h' \in H$ and $h.y \in \Omega'_{h'^{-1}}$, then

$$\Psi(b(h',h.y)b(h,y)) = h'h$$

and

$$\begin{aligned} y \in h^{-1}.\big(\varphi(\Omega_{b(h',h.y)}) \cap \Omega'_h\big) &= \Psi(b(h,y))^{-1}.\big(\varphi(\Omega_{b(h',h.y)}) \cap \Omega'_{\Psi(b(h,y))}\big) \\ = \varphi\big(b(h,y)^{-1}.(\varphi(\Omega_{b(h',h.y)}) \cap \Omega'_{\Psi(b(h,y))})\big) &= \varphi\big(\Omega_{b(h,y)^{-1}b(h',h.y)^{-1}} \cap \Omega_{b(h,y)^{-1}}\big), \end{aligned}$$

hence

$$b(h'h,y) = b(h',h.y)b(h,y)$$

as desired. □

Remark 3.12. If (φ, Φ) is a direct dynamical equivalence from $\theta\colon G \curvearrowright \Omega$ to $\theta'\colon H \curvearrowright \Omega'$, then the induced isomorphisms $C_K(\Omega) \rtimes G \to C_K(\Omega') \rtimes H$ and $C_0(\Omega) \rtimes_{(r)} G \to C_0(\Omega') \rtimes_{(r)} H$ are simply given by $f\delta_g \mapsto f \circ \varphi^{-1}|_{\Omega'_{\Phi(g)}} \delta_{\Phi(g)}$ with inverses $f\delta_h \mapsto \sum_{\Phi(g)=h} f \circ \varphi|_{\Omega_g} \delta_g$.

In fact, any dynamical equivalence is a result of two direct dynamical equivalences:

Proposition 3.13. *If $\theta \approx \theta'$ for partial actions of G and H, respectively, then*

$$\theta \overset{\approx}{\leftarrow} \gamma \overset{\approx}{\rightarrow} \theta'$$

for some partial action γ of $G \times H$.

Proof. By otherwise replacing θ' with the conjugate partial action $\varphi \circ \theta' \circ \varphi^{-1}$, we may assume that θ and θ' act on the same space Ω and $\varphi = \mathrm{id}_\Omega$. Writing $a_g := a(g, -)$ and $b_h := b(h, -)$, we then first define domains by

$$\Omega_{(g,h)} := a_{g^{-1}}^{-1}(h^{-1}) = b_{h^{-1}}^{-1}(g^{-1})$$

for $(g, h) \in G \times H$. To see that the above equality holds, assume that $x \in a_{g^{-1}}^{-1}(h^{-1})$, i.e. $x \in \Omega_g$ with $a(g^{-1}, x) = h^{-1}$. Then $x \in \Omega_h$ and

$$b(h^{-1}, x) = b(a(g^{-1}, x), x) = g^{-1},$$

hence $x \in b_{h^{-1}}^{-1}(g^{-1})$, so $a_{g^{-1}}^{-1}(h^{-1}) = b_{h^{-1}}^{-1}(g^{-1})$ from symmetry. Then define the action of (g, h) by

$$(g, h).x := g.x = a(g, x).x = h.x$$

for all $x \in \Omega_{(g,h)^{-1}}$. It is straightforward to verify that this does indeed define a partial action $\gamma\colon G \times H \curvearrowright \Omega$. Then simply observe that the pairs $(\mathrm{id}_\Omega, \pi_G)$ and $(\mathrm{id}_\Omega, \pi_H)$, where π_G and π_H denote the projections onto G and H, respectively, are direct dynamical equivalences. □

Returning to the world of convex shifts, we recall that any n-step subshift X can be recoded into being 1-step using higher block shifts; one simply replaces the original alphabet with the n-blocks $\mathcal{B}_n(X)$ via the map

$$x \mapsto [x_1 \dots x_n][x_2 \dots x_{n+1}][x_3 \dots x_{n+2}] \dots .$$

In the following, we shall make a similar construction in the world of convex shifts – although with *blocks* replaced by *balls*.

Construction 3.14. Let $\theta\colon \mathbb{F} \curvearrowright \Omega$ denote any convex subshift, let $n \geq 1$ and consider the finite alphabet

$$A^{[n:\,\Omega]} := \Big\{ \big[(a.\xi)^n \overset{a}{\leftarrow} \xi^n\big] \mid \xi \in \Omega, a \in A \text{ such that } a \in \xi \Big\},$$

where each $\big[(a.\xi)^n \overset{a}{\leftarrow} \xi^n\big]$ is just a formal symbol. As every symbol is typically represented by many different configurations ξ, we will simply use the notation $[B \overset{a}{\leftarrow} B']$ where $B, B' \in \mathcal{B}_n(\Omega)$ in the future. We then consider the corresponding free group $\mathbb{F}^{[n:\,\Omega]} := \mathbb{F}(A^{[n:\,\Omega]})$ and introduce the notation

$$[B' \overset{a^{-1}}{\longleftarrow} B] := [B \overset{a}{\leftarrow} B']^{-1}.$$

Observe that if $\Omega \subset \Lambda$ is an inclusion of convex subshifts over A, then we obtain corresponding inclusions $A^{[n:\,\Omega]} \subset A^{[n:\,\Lambda]}$ and $\mathbb{F}^{[n:\,\Omega]} \subset \mathbb{F}^{[n:\,\Lambda]}$. When $\Omega = \mathcal{C}$ is the full convex shift, we will simply write $A^{[n]} := A^{[n:\,\mathcal{C}]}$ and $\mathbb{F}^{[n]} := \mathbb{F}^{[n:\,\mathcal{C}]}$, and define a group homomorphism

$$\Psi_n \colon \mathbb{F}^{[n]} \to \mathbb{F} \quad \text{given by} \quad \Psi_n([B \xleftarrow{a} B']) := a.$$

By a slight abuse of notation, we will also refer to Ψ_n when we really mean the restriction of Ψ_n to the subgroup $\mathbb{F}^{[n:\,\Omega]}$.

Our aim is to define a replacement for the above block-encoding, more specifically a map

$$\phi_n \colon \mathcal{C}(A) \to \mathcal{C}(A^{[n]}).$$

Given $\xi \in \Omega$, we first set $\phi_n(\xi, 1) := 1 \in \mathbb{F}^{[n]}$ and proceed to define $\phi_n(\xi, \alpha) \in \mathbb{F}^{[n]}$ for $1 \neq \alpha \in \xi$. Writing $\alpha = s_m \cdots s_1$ and $B_k = ((s_k \cdots s_1).\xi)^n$ for $k \leq m$ so that $B_0 = \xi^n$ and $B_m = (\alpha.\xi)^n$, we then set

$$\phi_n(\xi, \alpha) := [B_m \xleftarrow{s_m} B_{m-1}] \cdot [B_{m-1} \xleftarrow{s_{m-1}} B_{m-2}] \cdots [B_1 \xleftarrow{s_1} B_0],$$

allowing us to define ϕ_n by

$$\phi_n(\xi) := \{\phi_n(\xi, \alpha) \mid \alpha \in \xi\}.$$

It is clear from the construction that $\phi_n(\xi)$ is a right-convex subset of $\mathbb{F}^{[n]}$ containing 1, so that $\phi_n(\xi) \in \mathcal{C}(A^{[n]})$. Observe that if $\xi \in \Omega$, then $\phi_n(\xi, \alpha) \in \mathbb{F}^{[n:\,\Omega]}$ for all $\alpha \in \xi$, so that ϕ_n restricts to a map $\Omega \to \mathcal{C}(A^{[n:\,\Omega]})$. The *n-ball subshift* $\theta^{[n]}$ of θ is then simply the restricted action of $\mathbb{F}^{[n:\,\Omega]}$ on the image $\Omega^{[n]} := \phi_n(\Omega)$. We observe that ϕ_n has a Ψ_n-equivariant inverse $\psi_n \colon \Omega^{[n]} \to \Omega$ given by

$$\psi_n(\eta) := \{\Psi_n(\beta) \mid \beta \in \eta\},$$

and as both ϕ_n and ψ_n are obviously continuous, they are in fact homeomorphisms of Ω and $\Omega^{[n]}$. In particular, it follows that $\Omega^{[n]}$ is compact and invariant under the action of $\mathbb{F}^{[n:\,\Omega]}$, hence a convex subshift.

Using the same notation as above, we also define a map

$$\tilde{\Psi}_n \colon \{\beta \in \eta \mid \eta \in \mathcal{C}^{[n]}\} = \{\phi_n(\xi, \alpha) \mid \xi \in \mathcal{C}, \alpha \in \xi\} \to \mathcal{C}$$

by

$$\tilde{\Psi}_n\big([B_m \xleftarrow{s_m} B_{m-1}] \cdots [B_1 \xleftarrow{s_1} B_0]\big) := \bigcup_{k=0}^{m} (s_k \cdots s_1)^{-1}.B_k$$

and note that $\tilde{\Psi}_n(\phi_n(\xi, \alpha)) \subset \xi$ for all $\alpha \in \xi$. Consequently,

$$\psi_n(\eta) = \bigcup_{\beta \in \eta} \tilde{\Psi}_n(\beta)$$

for any $\eta \in \mathcal{C}^{[n]}$.

In the following we shall see that passing to higher ball shift does indeed allow one to recode any finite type convex shift into a 1-step convex shift. First though, we have to deal with the higher ball shifts of the full convex shift.

Lemma 3.15. *The n-ball subshift $\mathbb{F}^{[n]} \curvearrowright \mathcal{C}^{[n]}$ of the full convex shift is 1-step.*

Proof. It should be clear from the above construction that the 1-balls in $\mathcal{C}^{[n]}$ are exactly the sets of the form

$$\{[B(s) \xleftarrow{s} B]\}_{s\in B} \cup \{1\}$$

for some $B, B(s) \in \mathcal{B}_n(\mathcal{C})$ such that $[B(s) \xleftarrow{s} B] \in A^{[n]} \cup (A^{[n]})^{-1}$ for all $s \in B$, and we claim that these sets in fact generate $\mathcal{C}^{[n]}$ as a 1-step convex subshift of $\mathbb{F}^{[n]} \curvearrowright \mathcal{C}(A^{[n]})$. To this end, we let $\eta \in \mathcal{C}(A^{[n]})$ and assume that $(\beta.\eta)^1$ is of this form for any $\beta \in \eta$. Now set

$$\xi := \{\Psi_n(\beta) \mid \beta \in \eta\}$$

and observe that $\xi \in \mathcal{C}$: we ultimately wish to show that $\eta = \phi_n(\xi) \in \mathcal{C}^{[n]}$. Letting $B \in \mathcal{B}_n(\mathcal{C})$ denote the n-ball as above corresponding to η^1, we will first show that $\xi^n = B$. Observe that if

$$[B'_m \xleftarrow{s_m} B_{m-1}][B'_{m-1} \xleftarrow{s_{m-1}} B_{m-2}] \cdots [B'_2 \xleftarrow{s_2} B_1][B'_1 \xleftarrow{s_1} B] \in \eta,$$

then our assumption implies that $B_k = B'_k$ and $s_k \neq s_{k+1}^{-1}$ for all $1 \leq k \leq m-1$. In particular, $|\Psi_n(\beta)| = |\beta|$ for all $\beta \in \eta$, so that

$$\xi^n = \{\Psi_n(\beta) \mid \beta \in \eta^n\}.$$

Now if

$$\beta = [B_m \xleftarrow{s_m} B_{m-1}][B_{m-1} \xleftarrow{s_{m-1}} B_{m-2}] \cdots [B_2 \xleftarrow{s_2} B_1][B_1 \xleftarrow{s_1} B] \in \eta^n,$$

then of course $\Psi_n(\beta) = s_m \cdots s_1 \in B$, and if, conversely, $s_m \cdots s_1 \in B$ for $m \leq n$, then an inductive application of our standing assumption implies the existence of some β as above, hence $\xi^n = B$ as desired. Finally, applying our observation to $\beta.\eta$ for some arbitrary

$$\beta = [B_m \xleftarrow{s_m} B_{m-1}][B_{m-1} \xleftarrow{s_{m-1}} B_{m-2}] \cdots [B_2 \xleftarrow{s_2} B_1][B_1 \xleftarrow{s_1} B] \in \eta,$$

we see that $(\Psi_n(\beta).\xi)^n = B_m$, and so $\eta = \phi_n(\xi)$ from the way that ϕ_n is defined. □

The next lemma shows that if $n < R$, then we can recover the R-ball ξ^R from the $(R-n)$-ball $\phi_n(\xi)^{R-n}$.

Lemma 3.16. *If $\xi \in \mathcal{C}$ and $n < R$, then*

$$\xi^R = \bigcup_{\beta \in \phi_n(\xi)^{R-n}} \tilde{\Psi}_n(\beta).$$

Proof. We apply the notation of the above construction with $m \leq R-n$. For the inclusion $\supset$, it is enough to check that

$$(s_k \cdots s_1)^{-1}.B_k \subset \xi^R$$

for all $k \leq m$, and this holds simply because $B_k \subset (s_k \cdots s_1).\xi^R$. For the reverse inclusion, take $\gamma \in \xi^R$. If $|\gamma| \leq R-n$, then $\gamma \in \Psi_n(\phi_n(\xi)^{R-n}) \subset \bigcup_{\beta\in\phi_n(\xi)^{R-n}} \tilde{\Psi}_n(\beta)$, so we may assume that $|\gamma| > R-n$ and write $\gamma = \mu\alpha$ with $|\alpha| = R-n$. But then $\mu \in (\alpha.\xi)^n$, so

$$\gamma = \mu\alpha \in \alpha^{-1}.(\alpha.\xi)^n \subset \bigcup_{\beta \in \phi_n(\xi)^{R-n}} \tilde{\Psi}_n(\beta).$$

□

Definition 3.17. Let $n < R$ and $B \in \mathcal{B}_R(\mathcal{C})$; we then define

$$B^{[n]} := \phi_n(B)^{R-n} = \{\phi_n(B,\alpha) \mid \alpha \in B, |\alpha| \le R-n\} \in \mathcal{B}_{R-n}(\mathcal{C}^{[n]}).$$

Lemma 3.18. *If $n < R$, $B \in \mathcal{B}_R(\mathcal{C})$ and $\xi \in \mathcal{C}$, then*

$$\xi^R = B \text{ if and only if } \phi_n(\xi)^{R-n} = B^{[n]}.$$

Proof. Assuming $\xi^R = B$, we immediately see that

$$\begin{aligned}\phi_n(\xi)^{R-n} &= \{\phi_n(\xi,\alpha) \mid \alpha \in \xi\}^{R-n} = \{\phi_n(\xi,\alpha) \mid \alpha \in \xi, |\alpha| \le R-n\} \\ &= \{\phi_n(B,\alpha) \mid \alpha \in B, |\alpha| \le R-n\} = B^{[n]}.\end{aligned}$$

On the other hand, if $\phi_n(\xi)^{R-n} = B^{[n]}$, then

$$\xi^R = \bigcup_{\beta \in \phi_n(\xi)^{R-n}} \tilde{\Psi}_n(\beta) = \bigcup_{\beta \in \phi_n(B)^{R-n}} \tilde{\Psi}_n(\beta) = B^R = B$$

by Lemma 3.16. □

Corollary 3.19. *The allowed R-balls of the n-ball shift of a convex subshift $\theta\colon \mathbb{F} \curvearrowright \Omega$ is given by*

$$\mathcal{B}_R(\Omega^{[n]}) = \{B^{[n]} \mid B \in \mathcal{B}_{R+n}(\Omega)\}.$$

Moreover, if Ω is R-step and $n < R$, then $\Omega^{[n]}$ is $(R-n)$-step. In particular, $\Omega^{[R-1]}$ is 1-step.

Proof. The first claim follows immediately from Lemma 3.18. For the second one, we must also refer to Lemma 3.15. □

We can now prove that every finite type convex subshift can be recoded into a 1-step convex subshift.

Proposition 3.20. *If $\theta\colon \mathbb{F} \curvearrowright \Omega$ is a convex subshift and $n \ge 1$, then $\theta^{[n]} \xrightarrow{\approx} \theta$. In particular, any finite type convex subshift is directly dynamically equivalent to a 1-step convex subshift.*

Proof. We have already established that the map ψ_n is a Ψ_n-equivariant homeomorphism $\Omega^{[n]} \to \Omega$. Now assume that

$$\beta = [B_m \xleftarrow{s_m} B_{m-1}] \cdot [B_{m-1} \xleftarrow{s_{m-1}} B_{m-2}] \cdots [B_1 \xleftarrow{s_1} B_0]$$

and

$$\beta' = [B'_m \xleftarrow{s_m} B'_{m-1}] \cdot [B'_{m-1} \xleftarrow{s_{m-1}} B'_{m-2}] \cdots [B'_1 \xleftarrow{s_1} B'_0]$$

are distinct elements of $\mathbb{F}^{[n\colon \Omega]}$ satisfying $\Psi_n(\beta) = \Psi_n(\beta')$. Then $B_k \neq B'_k$ for some k, and

$$\big((s_k \cdots s_1).\psi_n(\eta)\big)^n = B_k \neq B'_k = \big((s_k \cdots s_1).\psi_n(\eta')\big)^n$$

for all $\eta \in \Omega^{[n]}_\beta$, $\eta' \in \Omega^{[n]}_{\beta'}$, hence $\Omega^{[n]}_\beta \cap \Omega^{[n]}_{\beta'} = \emptyset$ as desired. Finally, given any $\alpha \in \mathbb{F}$ and $\xi \in \Omega_\alpha$, we can consider $\phi_n(\xi) \in \Omega^{[n]}$. Then $\beta := \phi_n(\xi,\alpha) \in \phi_n(\xi)$ satisfies $\Psi_n(\beta) = \alpha$ and

$$\xi = \psi_n(\phi_n(\xi)) \in \psi_n(\Omega^{[n]}_\beta),$$

hence $\Omega_\alpha = \bigcup_{\Psi_n(\gamma)=\alpha} \psi_n(\Omega_\gamma^{[n]})$. We conclude that $\theta^{[n]} \overset{\approx}{\to} \theta$. The second part of the claim now follows directly from Proposition 3.11 and Corollary 3.19. ☐

One issue we have not yet dealt with is the identification of the n-ball shift and the n-fold 1-ball shift of a given convex subshift.

Proposition 3.21. *Consider any convex subshift $\theta \colon \mathbb{F} \curvearrowright \Omega$. Then the n-ball convex subshift $\theta^{[n]}$ is conjugate to the n-fold 1-ball convex subshift $\theta^{[1]\cdots[1]}$.*

Proof. We will show that $\theta^{[n+1]} \cong \theta^{[1][n]}$ for all $n \geq 1$ from which the claim follows inductively. We first define a map $\Phi \colon A^{[n+1\colon \Omega]} \to A^{[1\colon \Omega][n\colon \Omega^{[1]}]}$ by

$$\Phi([B_2 \overset{s}{\leftarrow} B_1]) := \big[B_2^{[1]} \xleftarrow{B_2^1 \overset{s}{\leftarrow} B_1^1} B_1^{[1]}\big] = \big[\phi_1(B_2)^n \xleftarrow{\phi_1(B_1,s)} \phi_1(B_1)^n\big],$$

which is easily seen to be a bijection by Lemma 3.19, so we obtain an induced isomorphism $\Phi \colon \mathbb{F}^{[n+1\colon \Omega]} \to \mathbb{F}^{[1\colon \Omega][n\colon \Omega^{[1]}]}$. The pair (φ, Φ), where $\varphi \colon \mathcal{C}(A^{[n+1\colon \Omega]}) \to \mathcal{C}(A^{[1\colon \Omega][n\colon \Omega^{[1]}]})$ is given by

$$\varphi(\eta) := \{\Phi(\beta) \mid \beta \in \eta\},$$

then defines a conjugacy. In order to see that it restricts to a conjugacy $\Omega^{[n+1]} \to \Omega^{[1][n]}$, it suffices to check that

$$\varphi(B^{[n+1]}) = B^{[1][n]}$$

for all $R \geq 1$ and $B \in \mathcal{B}_{R+n+1}(\Omega)$ due to Corollary 3.19. Recalling that

$$B^{[n+1]} = \phi_{n+1}(B)^R = \{\phi_{n+1}(B,\alpha) \mid \alpha \in B, |\alpha| \leq R\}$$

and

$$B^{[1][n]} = \phi_n(\phi_1(B)^{R+n})^R = \{\phi_n(\phi_1(B), \phi_1(B,\alpha)) \mid \alpha \in B, |\alpha| \leq R\},$$

in fact we only have to verify that

$$\Phi(\phi_{n+1}(B,\alpha)) = \phi_n(\phi_1(B), \phi_1(B,\alpha))$$

for all $\alpha \in B$ with $|\alpha| \leq R$. Writing $\alpha = s_m \cdots s_1$ and

$$t_k := \big[(s_k \cdots s_1.B)^1 \overset{s_k}{\leftarrow} (s_{k-1} \cdots s_1.B)^1\big] = \phi_1(s_{k-1} \cdots s_1.B, s_k)$$

so that

$$\phi_1(s_k \cdots s_1.B) = t_k \cdots t_1.\phi_1(B) \quad \text{and} \quad \phi_1(B,\alpha) = t_m \cdots t_1,$$

we then see that

$$\begin{aligned}
\Phi(\phi_{n+1}(B,\alpha)) &= \Phi\Big(\big[(s_m \cdots s_1.B)^{n+1} \overset{s_m}{\leftarrow} (s_{m-1} \cdots s_1.B)^{n+1}\big] \cdots \big[(s_1.B)^{n+1} \overset{s_1}{\leftarrow} B^{n+1}\big]\Big) \\
&= \big[\phi_1(s_m \cdots s_1.B)^n \xleftarrow{\phi_1(s_{m-1}\cdots s_1.B, s_m)} \phi_1(s_{m-1} \cdots s_1.B)^n\big] \cdots \big[\phi_1(s_1.B)^n \xleftarrow{\phi_1(B,s_1)} \phi_1(B)^n\big] \\
&= \big[(t_m \cdots t_1.\phi_1(B))^n \overset{t_m}{\leftarrow} (t_{m-1} \cdots t_1.\phi_1(B))^n\big] \cdots \big[(t_1.\phi_1(B))^n \overset{t_1}{\leftarrow} \phi_1(B)^n\big] \\
&= \phi_n(\phi_1(B), t_m \cdots t_1) = \phi_n(\phi_1(B), \phi_1(B,\alpha))
\end{aligned}$$

as desired. ☐

The following theorem explains the precise relationship between the partial actions $\theta^{(E,C)}$ and $\theta^{(E_n,C^n)}$, adding a dynamical dimension to [7, Theorem 8.3]. Note that while [7, Theorem 5.7] implies

$$L^{\mathrm{ab}}(E_n, C^n) \cong L^{\mathrm{ab}}(E, C) \quad \text{and} \quad \mathcal{O}(E_n, C^n) \cong \mathcal{O}(E, C)$$

for any finite bipartite graph and $n \geq 1$, the corresponding result for $\mathcal{O}^r$ is not immediate.

Theorem 3.22. *If (E, C) is a finite bipartite graph and $n \geq 1$, then $\theta^{(E_n,C^n)}$ is conjugate to the n-ball convex subshift $(\theta^{(E,C)})^{[n]}$. In particular, there are base-preserving isomorphisms*

$$L^{\mathrm{ab}}(E_n, C^n) \cong L^{\mathrm{ab}}(E, C), \quad \mathcal{O}(E_n, C^n) \cong \mathcal{O}(E, C) \quad \textit{and} \quad \mathcal{O}^r(E_n, C^n) \cong \mathcal{O}^r(E, C)$$

induced from the direct dynamical equivalence

$$\theta^{(E_n,C^n)} \cong (\theta^{(E,C)})^{[n]} \xrightarrow{\approx} \theta^{(E,C)}.$$

The isomorphisms on L^{ab}_K and $\mathcal{O}$ coincides with the ones induced by Φ_n as defined in [7]. *Moreover, there is a canonical bijective correspondence*

$$E_n^0 \ni v \mapsto B(v) \in \mathcal{B}_n(\Omega(E, C)),$$

and the homeomorphism $\Omega(E_n, C^n) \to \Omega(E, C)^{[n]} \to \Omega(E, C)$ restricts to a homeomorphism

$$\Omega(E_n, C^n)_v \to \{\xi \in \Omega(E, C) \mid \xi^n = B(v)\}$$

for all $v \in E_n^0$.

Proof. Set $A := E^1$ and $\Omega := \Omega(E, C)$ for notational simplicity; by Proposition 3.21, it suffices to verify our claims for $n = 1$. Observe first that $\mathcal{B}_1(\Omega) = \mathcal{B}_1^0(\Omega) \sqcup \mathcal{B}_1^1(\Omega)$, where

$$\mathcal{B}_1^0(\Omega) := \big\{\{1\} \sqcup \{x_1, \ldots, x_{k_u}\}^{-1} \mid u \in E^{0,0}, x_j \in X_j^u\big\} \quad \text{and} \quad \mathcal{B}_1^1(\Omega) := \{\{1\} \sqcup s^{-1}(v) \mid v \in E^{0,1}\},$$

using the standard notation. The alphabet $A^{[1\colon \Omega]}$ is therefore given by

$$A^{[1\colon \Omega]} = \big\{\big[\{x_1, \ldots, x_{k_u}\}^{-1} \xleftarrow{x_i} s^{-1}(s(x_i))\big] \mid u \in E^{0,0}, x_j \in X_j^u, i = 1, \ldots, k_u\big\}$$

and we can define an isomorphism $\Phi \colon \mathbb{F}^{[1\colon \Omega]} \to \mathbb{F}(E_1^1)$ by

$$\big[\{x_1, \ldots, x_{k_u}\}^{-1} \xleftarrow{x_i} s^{-1}(s(x_i))\big] \mapsto \alpha^{x_i}(x_1, \ldots, \widehat{x_i}, \ldots, x_{k_u})^{-1}.$$

Since both convex shifts are 1-step, we simply have to check that Φ induces a bijection

$$\mathcal{B}_1(\Omega^{[1]}) \to \mathcal{B}_1(\Omega(E_1, C^1)), \quad B \mapsto \Phi(B).$$

We first build some notation: Given any $u \in E^{0,0}$ and $(x_1, \ldots, x_{k_u}) \in \prod_{j=1}^{k_u} X_j^u$, we define

$$B_1^0(x_1, \ldots, x_{k_u}) := \Big\{\big[\{1\} \sqcup s^{-1}(s(x_i)) \xleftarrow{x_i^{-1}} \{1\} \sqcup \{x_1, \ldots, x_{k_u}\}^{-1}\big] : i = 1, \ldots, k_u\Big\}.$$

Also, for all $x_i \in X_i^u$, we set

$$Z(x_i) := \Big\{\{x_1, \ldots, \widehat{x_i}, \ldots, x_{k_u}\}^{-1} \mid x_j \in X_j^u, j \neq i\Big\}$$

and define

$$B_1^1(v, \{z(e)\}_{e \in s^{-1}(v)}) := \Big\{\big[\{1\} \sqcup z(e) \xleftarrow{e} \{1\} \sqcup s^{-1}(v)\big] : e \in s^{-1}(v)\Big\}$$

for any $v \in E^{0,1}$ and $z(e) \in Z(e)$. It is easily checked that $\mathcal{B}_1(\Omega^{[1]}) = \mathcal{B}_1^0(\Omega^{[1]}) \sqcup \mathcal{B}_1^1(\Omega^{[1]})$, where

$$\mathcal{B}_1^0(\Omega^{[1]}) := \Big\{ B_1^0(x_1, \dots, x_{k_u}) \mid u \in E^{0,0}, (x_1, \dots, x_{k_u}) \in \prod_{j=1}^{k_u} X_j^u \Big\}$$

and

$$\mathcal{B}_1^1(\Omega^{[1]}) := \Big\{ B_1^1(v, \{z(e)\}_{e \in s^{-1}(v)}) \mid v \in E^{0,1}, z(e) \in Z(e) \Big\}.$$

Now observe that

$$\begin{aligned}\Phi(B_1^0(x_1, \dots, x_{k_u})) &= \{1\} \sqcup \big\{ \alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u}) \mid i = 1, \dots, k_u \big\} \\ &= \{1\} \sqcup s_1^{-1}(v(x_1, \dots, x_{k_u}))\end{aligned}$$

hence Φ maps $B_1^0(\Omega^{[1]})$ onto the 1-balls of type (c1) (cf. Definition 2.11). Likewise, we see that

$$\Phi(B_1^1(v, \{z(e)\}_{e \in s^{-1}(v)})) = \{1\} \sqcup \big\{ \alpha^e(z(e))^{-1} \mid e \in s^{-1}(v) \big\},$$

so Φ maps $B_1^1(\Omega^{[1]})$ onto the 1-balls of type (c2). We conclude that the pair (φ, Φ) with $\varphi\colon \Omega^{[1]} \to \Omega(E_1, C^1)$ given by

$$\varphi(\xi) := \{\Phi(\alpha) \mid \alpha \in \xi\}$$

is a conjugacy of $\mathbb{F}^{[1:\Omega]} \curvearrowright \Omega^{[1]}$ and $\mathbb{F}(E_1^1) \curvearrowright \Omega(E_1, C^1)$. Consequently, there is a direct dynamical equivalence $\theta^{(E_n, C^n)} \cong (\theta^{(E,C)})^{[n]} \xrightarrow{\approx} \theta^{(E,C)}$, which induces base-preserving isomorphisms by Remark 3.9. It follows from Remark 3.12 that the isomorphisms on L^{ab} and $\mathcal{O}$ are exactly the ones of [7].

We now turn to the second part of the claim and set

$$B(v) := \begin{cases} \{1\} \sqcup s^{-1}(v) & \text{if } v \in E_1^{0,0} = E^{0,1} \\ \{1\} \sqcup \{x_1, \dots, x_{k_u}\}^{-1} & \text{if } v = v(x_1, \dots, x_{k_u}) \in E_1^{0,1} \end{cases}$$

for every $v \in E_1^0$; this clearly defines a bijective correspondence between E_1^0 and $\mathcal{B}_1(\Omega(E, C))$. Note that the homeomorphism $\Omega(E_1, C^1) \to \Omega(E, C)$ is induced by the group homomorphism $\mathbb{F}(E_1^1) \to \mathbb{F}(E^1)$ given by $\alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u}) \mapsto x_i^{-1}$, and it is easily checked that it maps $\Omega(E_1, C^1)_v$ onto $\{\xi \in \Omega(E, C) \mid \xi^1 = B(v)\}$. □

Remark 3.23. Theorem 3.22 shows that for $v \in E_n^0$ with $n \geq 1$, the subspace $\Omega(E, C)_v$ may be described as $\Omega(E, C)_v = \{\xi \in \Omega(E, C) \mid \xi^n = B(v)\}$.

Any partial action on a topological space may be viewed as the restriction of a global action [1, Theorem 2.5]. The globalisation is not Hausdorff in general [1, Proposition 2.10], but whenever it is, one may consider the relationship between the C^*-algebras of the partial action and its globalisation. However, it is also of natural interest to study restrictions of *partial actions*, in particular in cases where there is no Hausdorff globalisation, and they play a natural role in our main theorem about convex subshifts.

Definition 3.24 (See also [21, Definition 3.1], [36, Definition 2.17])**.** If $\theta\colon G \curvearrowright \Omega$ is a partial action on a topological space and $U \subset \Omega$ is an open subset, then we denote by $\theta|_U$ the restricted partial action $G \curvearrowright U$ with domains

$$U_g := \theta_g(U \cap \Omega_{g^{-1}}) \cap U,$$

and U is called *G-full* if

$$X = \{g.x \mid g \in G, x \in U \cap X_{g^{-1}}\}.$$

Finally, two partial actions $\theta\colon G \curvearrowright \Omega$ and $\gamma\colon H \curvearrowright \Omega'$ are called *Kakutani-equivalent* if there exist clopen subspaces $K \subset \Omega$ and $K' \subset \Omega'$, resp. G- and H-full, such that $\theta|_K \approx \gamma|_{K'}$.

If $\theta\colon G \curvearrowright \Omega$ and $\theta'\colon G \curvearrowright \Omega'$ are Kakutani equivalent partial actions on totally disconnected, locally compact spaces, then the groupoids $\mathcal{G}_\theta$ and $\mathcal{G}_{\theta'}$ are Kakutani equivalent in the sense of [21, Definition 3.1] and hence groupoid equivalent by [21, Theorem 3.2]. It follows that there are Morita-equivalences

$$C_K(\Omega) \rtimes_\theta G \sim_M C_K(\Omega) \rtimes_{\theta'} H \quad \text{and} \quad C_0(\Omega) \rtimes_{\theta,(r)} \sim_M C_0(\Omega') \rtimes_{\theta',(r)} H,$$

see for instance [40, Theorem 2.8], [48, Theorem 13] and [20, Theorm 5.1].

We can now state and prove the second main theorem about convex subshifts.

Theorem 3.25. *If $\theta\colon \mathbb{F} \curvearrowright \Omega$ is a convex subshift of finite type, then there is a finite bipartite separated graph (E, C) such that $\theta^{(E,C)}$ and θ are Kakutani equivalent.*

Proof. By Proposition 3.20, we may assume that Ω is 1-step. Now let $A_+^{[1:\,\Omega]}$ and $A_-^{[1:\,\Omega]}$ denote disjoint copies of the alphabet $A^{[1:\,\Omega]}$ with subscripts $+$ and $-$, and define a finite bipartite separated graph (E, C) by

- $E^{0,0} := \mathcal{B}_1(\Omega)$ and $E^{0,1} = A^{[1:\,\Omega]}$,
- $E^1 := A_+^{[1:\,\Omega]} \sqcup A_-^{[1:\,\Omega]}$,
- $r([B \overset{a}{\leftarrow} B']_+) := B$ and $r([B \overset{a}{\leftarrow} B']_-) := B'$,
- $s([B \overset{a}{\leftarrow} B']_+) := s([B \overset{a}{\leftarrow} B']_-) = [B \overset{a}{\leftarrow} B']$,
- $C_B := \{X_B(s) \mid 1 \neq s \in B\}$ for all $B \in E^{0,0}$, where for $a \in A$

$$X_B(a^{-1}) := \{[B \overset{a}{\leftarrow} B']_+ \mid B' \in E^{0,0} \text{ such that } B \overset{a}{\leftarrow} B'\},$$
$$X_B(a) := \{[B' \overset{a}{\leftarrow} B]_- \mid B' \in E^{0,0} \text{ such that } B' \overset{a}{\leftarrow} B\}.$$

Then consider the group homomorphism $\Phi\colon \mathbb{F}(E^1) \to \mathbb{F}^{[1:\,\Omega]}$ given by

$$\Phi([B \overset{a}{\leftarrow} B']_+) := [B \overset{a}{\leftarrow} B'] \quad \text{and} \quad \Phi([B \overset{a}{\leftarrow} B']_-) := 1$$

as well as the compact open $\mathbb{F}(E^1)$-full subspace

$$K := \bigsqcup_{u \in E^{0,0}} \Omega(E, C)_u.$$

Equipping K with the restricted partial action $\theta^{(E,C)}|_K\colon \mathbb{F}(E^1) \curvearrowright K$, we claim that $\varphi\colon K \to \Omega^{[1]}$ given by

$$\varphi(\xi) := \{\Phi(\alpha) \mid \alpha \in \xi\}$$

defines a Φ-equivariant homeomorphism, making the pair (φ, Φ) into a direct dynamical equivalence. First, let us check that φ even maps into $\Omega^{[1]}$, so take any $\xi \in K$. Observe that any length two admissible path $\alpha \in \xi$ is of the form

$$[B \xleftarrow{a} B']_+[B \xleftarrow{a} B']_-^{-1} \quad \text{or} \quad [B \xleftarrow{a} B']_-[B \xleftarrow{a} B']_+^{-1},$$

and these are mapped to $[B \xleftarrow{a} B']$ and $[B \xleftarrow{a} B']^{-1}$, respectively. It follows that any $\alpha \in \xi$ of length four is mapped to a length two word, hence any $\alpha \in \xi$ of length $2n$ is mapped to a word of length n. Note also that if $\alpha \in \xi$ has odd length, then

$$\alpha = [B \xleftarrow{a} B']_+^{-1}\beta \quad \text{or} \quad \alpha = [B \xleftarrow{a} B']_-^{-1}\beta$$

for some $[B \xleftarrow{a} B'] \in A^{[1:\Omega]}$, and in the latter case, $\Phi(\alpha) = \Phi(\beta)$. In the former case, there is a unique extension of length $|\alpha| + 1$ inside ξ, namely $[B \xleftarrow{a} B']_-[B \xleftarrow{a} B']_+^{-1}\beta \in \xi$, and $\Phi(\alpha) = \Phi([B \xleftarrow{a} B']_-\alpha)$, so in conclusion

$$\varphi(\xi) = \{\Phi(\alpha) \colon \alpha \in \xi, |\alpha| \text{ is even}\}.$$

In particular, φ is continuous and $\varphi(\xi)^1 = \Phi(\xi^2)$, so we only need to check that $\Phi(\xi^2) \in \mathcal{B}_1(\Omega^{[1]})$ for any $\xi \in K$. Assuming that $\xi \in \Omega(E, C)_B$, for every $1 \neq s \in B$ there is $B(s) \in \mathcal{B}_1(\Omega)$ such that

$$\begin{aligned}\Phi(\xi^2) &= \{1\} \sqcup \{[B \xleftarrow{a} B(a^{-1})]^{-1} \mid a \in A \cap B^{-1}\} \sqcup \{[B(a) \xleftarrow{a} B] \mid a \in A \cap B\} \\ &= \{1\} \sqcup \{[B(s) \xleftarrow{s} B] \mid 1 \neq s \in B\},\end{aligned}$$

and this is exactly an element of $\mathcal{B}_1(\Omega^{[1]})$. At this point we have verified that φ is a well-defined continuous Φ-equivariant map, and we now turn to the construction of an inverse. Define a group homomorphism $\Sigma \colon \mathbb{F}^{[1:\Omega]} \to \mathbb{F}(E^1)$ and a continuous Σ-equivariant map $\sigma \colon \Omega^{[1]} \to K$ by

$$\Sigma([B \xleftarrow{a} B']) := [B \xleftarrow{a} B']_+[B \xleftarrow{a} B']_-^{-1} \quad \text{and} \quad \sigma(\eta) := \operatorname{conv}\{\Sigma(\beta) \mid \beta \in \eta\},$$

where $\operatorname{conv}(H)$ for a set $H \subset \mathbb{F}(E^1)$ denotes the convex closure. Observing that Φ is a one-sided inverse of Σ, it follows that Σ is injective and hence that σ is a continuous Σ-equivariant map into $\mathcal{C}(E^1)$; but, we still have to verify that σ maps into K. Since $\sigma(\eta)^2 = \operatorname{conv}(\Sigma(\eta^1))$ for all $\eta \in \Omega^{[1]}$, it suffices to check that $\operatorname{conv}(\Sigma(\eta^1)) \in \mathcal{B}_2(\Omega(E, C))$. By construction, η^1 is of the form

$$\begin{aligned}\eta^1 &= \{1\} \sqcup \{[B(s) \xleftarrow{s} B] \mid 1 \neq s \in B\} \\ &= \{1\} \sqcup \{[B \xleftarrow{a} B(a^{-1})]^{-1} \mid a \in A \cap B^{-1}\} \sqcup \{[B(a) \xleftarrow{a} B] \mid a \in A \cap B\}\end{aligned}$$

for some $B, B(s) \in \mathcal{B}_1(\Omega)$, so

$$\begin{aligned}\Sigma(\eta^1) = \{1\} &\sqcup \{[B \xleftarrow{a} B(a^{-1})]_-[B \xleftarrow{a} B(a^{-1})]_+^{-1} \mid a \in A \cap B^{-1}\} \\ &\sqcup \{[B(a) \xleftarrow{a} B]_+[B(a) \xleftarrow{a} B]_-^{-1} \mid a \in A \cap B\}.\end{aligned}$$

Taking the convex closure of this, we clearly obtain a 2-ball of $\Omega(E, C)$, hence

$$\sigma(\eta) \in \Omega(E, C)_B \subset K$$

as desired. We now claim that φ and σ are in fact mutual inverses. Noting that $\Sigma(\Phi(\alpha)) = \alpha$ for $\alpha \in \xi$ of even length, we indeed have

$$\sigma(\varphi(\xi)) = \operatorname{conv}\{\Sigma(\Phi(\alpha)) \mid \alpha \in \xi, |\alpha| \text{ is even}\} = \operatorname{conv}\{\alpha \mid \alpha \in \xi, |\alpha| \text{ is even}\} = \xi$$

and

$$\varphi(\sigma(\eta)) = \varphi(\operatorname{conv}\{\Sigma(\beta) \mid \beta \in \eta\}) = \{\Phi(\Sigma(\beta)) \mid \beta \in \eta\} = \eta.$$

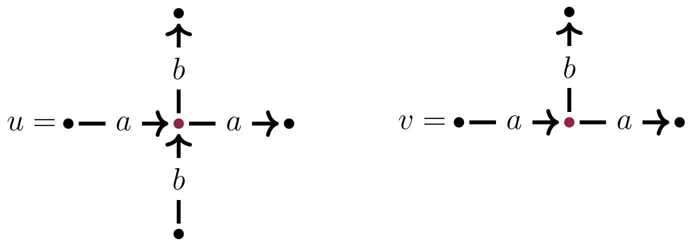
Letting $F := \operatorname{Im}(\Sigma) \leq \mathbb{F}(E^1)$ so that $\Sigma = \Phi|_F^{-1}$, we conclude that the partial actions $F \curvearrowright K$ and $\mathbb{F}^{[1:\,\Omega]} \curvearrowright \Omega^{[1]}$ are conjugate. Finally observing that, by the above observations, $K_\alpha = \emptyset$ for all $\alpha \in \mathbb{F}(E^1) \setminus F$, we conclude that (φ, Φ) is indeed a direct dynamical equivalence $\theta^{(E,C)}|_K \xrightarrow{\approx} \theta^{[1]}$, from which we obtain the desired direct dynamical equivalence as the composition $\theta^{(E,C)}|_K \xrightarrow{\approx} \theta^{[1]} \xrightarrow{\approx} \theta$. □

In view of the above theorem, the study of convex subshifts of finite type boils down to the study of dynamical systems associated with finite bipartite graphs, at least up to Katutani equivalence. In the following sections, we shall see how one can extract information about the open/closed invariant subspaces from the graph, illustrating the usefulness of having a graph representation.

We end this section with an application of Theorem 3.25 to a pair of concrete examples.

Example 3.26. Given a finite alphabet A, there is of course a finite bipartite separated graph (E, C) corresponding to the full convex shift on A as in Theorem 3.25. However, one can check that $|E^0| = 4(|A|^4 + |A|^2)$ and $|E^1| = 8|A|^4$, so even when $|A| = 2$ this is a fairly sizable graph. We shall therefore refrain from drawing it here.

Example 3.27. Consider the alphabet $A = \{a, b\}$ and the 1-step subshift $\mathbb{F}_2 \curvearrowright \Omega$ with $\mathcal{B}_1(\Omega) = \{u, v\}$, where $u = \{1, a^{\pm 1}, b^{\pm 1}\}$ and $v = \{1, a^{\pm 1}, b\}$ as illustrated just below. We will



then describe the separated graph (E, C) of Theorem 3.25. We have

$$E^{0,1} = A^{[1:\,\Omega]} = \{[u \xleftarrow{a} u], [u \xleftarrow{b} u], [v \xleftarrow{a} u], [u \xleftarrow{a} v], [u \xleftarrow{b} v], [v \xleftarrow{a} v]\}$$

and $E^1 = A_+^{[1:\,\Omega]} \sqcup A_-^{[1:\,\Omega]}$ with

$$r^{-1}(u) = \{[u \xleftarrow{a} u]_\pm, [u \xleftarrow{b} u]_\pm, [v \xleftarrow{a} u]_-, [u \xleftarrow{a} v]_+, [u \xleftarrow{b} v]_+\},$$

$$r^{-1}(v) = \{[v \xleftarrow{a} u]_+, [u \xleftarrow{a} v]_-, [u \xleftarrow{b} v]_-, [v \xleftarrow{a} v]_\pm\},$$

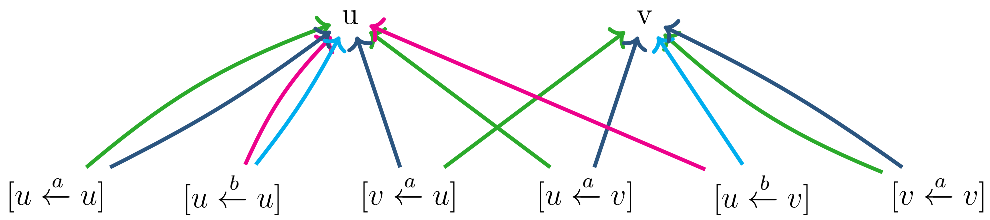
and the source map is simply the projection $E^1 = A_+^{[1:\,\Omega]} \sqcup A_-^{[1:\,\Omega]} \to A^{[1:\,\Omega]} = E^{0,1}$. The separation is given by

$$C_u = \{X_u(a), X_u(a^{-1}), X_u(b), X_u(b^{-1})\} \quad \text{and} \quad C_v = \{X_v(a), X_v(a^{-1}), X_v(b)\},$$

where

$$\begin{aligned} X_u(a) &= \{[u \overset{a}{\leftarrow} u]_-, [v \overset{a}{\leftarrow} u]_-\}, & X_u(a^{-1}) &= \{[u \overset{a}{\leftarrow} u]_+, [u \overset{a}{\leftarrow} v]_+\}, \\ X_u(b) &= \{[u \overset{b}{\leftarrow} u]_-\}, & X_u(b^{-1}) &= \{[u \overset{b}{\leftarrow} u]_+, [u \overset{b}{\leftarrow} v]_+\}, \\ X_v(a) &= \{[u \overset{a}{\leftarrow} v]_-, [v \overset{a}{\leftarrow} v]_-\}, & X_v(a^{-1}) &= \{[v \overset{a}{\leftarrow} u]_+, [v \overset{a}{\leftarrow} v]_+\}, \\ X_v(b) &= \{[u \overset{b}{\leftarrow} v]_-\}\ . \end{aligned}$$

We can therefore picture (E, C) as follows:



4. The lattice of induced ideals

In this section, we describe the lattice of induced ideals of the algebras $L^{\mathrm{ab}}(E, C)$, $\mathcal{O}(E, C)$ and $\mathcal{O}^r(E, C)$ for (E, C) finite and bipartite in terms of graph-theoretic data, specifically certain sets of vertices in the infinite layer graph (F_∞, D^∞) (see Definition 2.6).

We first settle the meaning of the various types of ideals that we shall encounter. When dealing with C*-algebras, we will consider only closed ideals, so that the word *ideal* will mean *closed ideal* in this case. For a general ring R, an ideal $I \lhd R$ is called a *trace ideal* if it is generated by the entries of some set of idempotent matrices over R, and we denote the lattice of trace ideals by $\mathrm{Tr}(R)$. The lattice of idempotent-generated ideals $\mathrm{Idem}(R)$ then sits as a sublattice of $\mathrm{Tr}(R)$. Given a crossed product (algebraic or C^*-algebraic, reduced or universal) $\mathcal{O} = A \rtimes_{(r)} G$, we say that an ideal $J \lhd \mathcal{O}$ is *induced* if $J = (J \cap A) \rtimes_{(r)} G$, and we denote by $\mathrm{Ind}(\mathcal{O})$ the lattice of induced ideals. Finally, if M is an abelian monoid, then a submonoid $I \subset M$ is called an *order-ideal* if $x + y \in I$ implies $x, y \in I$. The lattice of order ideals of M will be denoted by $\mathcal{L}(M)$.

The basic tools in our analysis are the following results.

Theorem 4.1. [11, Proposition 10.10] *For any ring R, there is a lattice isomorphism*

$$\mathcal{L}(\mathcal{V}(R))) \cong \mathrm{Tr}(R).$$

Moreover, if $\mathcal{V}(R)$ is generated by the classes $[e]$ of idempotents of R, then $\mathrm{Tr}(R) = \mathrm{Idem}(R)$.

Theorem 4.2 ([25])**.** *Let $G \curvearrowright A$ be a partial action of a discrete group G on a C^*-algebra. Then the map $J \mapsto J \cap A$ defines a bijective correspondence between* $\mathrm{Ind}(A \rtimes_{(r)} G)$ *and the lattice of invariant ideals of A. Moreover, if I is an invariant ideal of A, then*

$$(A \rtimes G)/(I \rtimes G) \cong (A/I) \rtimes G,$$

and if G is exact, then $(A \rtimes_r G)/(I \rtimes_r G) \cong (A/I) \rtimes_r G$ as well.

Proof. If J is an ideal of $A \rtimes_{(r)} G$, then $J \cap A$ is a G-invariant ideal of A by [25, Proposition 23.11], and if I is an invariant ideal of A, then $I \rtimes_{(r)} G$ is an ideal of $A \rtimes_{(r)} G$ by [25, Proposition 21.12 and 21.15]. In particular, we have the above mentioned bijective correspondence. Moreover, $(A \rtimes G)/(I \rtimes G) \cong (A/I) \rtimes G$ by [25, Proposition 21.15], and in case G is exact, then $(A \rtimes_r G)/(I \rtimes_r G) \cong (A/I) \rtimes_r G$ as well by [25, Theorem 21.18]. □

Remark 4.3. We note that it is straightforward to prove a result completely analogous to Theorem 4.2 for partial actions on $*$-algebras.

We now recall some definitions from [11], adapted to our choice of conventions. These notions generalize the classical notions of hereditary and saturated subsets of vertices, cf. [44, Chapter 4], to the separated setting.

Definition 4.4. [11, Definition 6.3] Let (E, C) be a finitely separated graph. A subset H of E^0 is said to be *hereditary* if for any $e \in E^1$, we have $r(e) \in H$ implies $s(e) \in H$, and H is said to be *C-saturated* if for any $v \in E^0$ and $X \in C_v$, $s(X) \subset H$ implies $v \in H$. We denote by $\mathcal{H}(E, C)$ the lattice of hereditary C-saturated subsets of E^0.

Theorem 4.5. *Let (E, C) be a finite bipartite separated graph. Then there are lattice isomorphisms*

$$\mathrm{Idem}(L_K^{\mathrm{ab}}(E, C)) \cong \mathcal{L}(M(F_\infty, D^\infty)) \cong \mathcal{H}(F_\infty, D^\infty).$$

If $H \in \mathcal{H}(F_\infty, D^\infty)$, the ideal $I(H)$ of $L_K^{\mathrm{ab}}(E, C)$ associated to H through this isomorphism is the ideal generated by all the projections $\pi_{n,\infty}(v)$, where $v \in H \cap E_n^0$, and $\pi_{n,\infty} \colon L(E_n, C^n) \to L_K^{\mathrm{ab}}(E, C)$ is the natural quotient map.

Proof. By [7, Corollary 5.9], we have an isomorphism $\mathcal{V}(L^{\mathrm{ab}}(E, C)) \cong M(F_\infty, D^\infty)$. In particular, $\mathcal{V}(L^{\mathrm{ab}}(E, C))$ is generated by the classes of the idempotents in $L^{\mathrm{ab}}(E, C)$ corresponding to the vertices in F_∞. By Theorem 4.1, we obtain

$$\mathcal{L}(M(F_\infty, D^\infty)) \cong \mathcal{L}(\mathcal{V}(L^{\mathrm{ab}}(E, C))) \cong \mathrm{Tr}(L^{\mathrm{ab}}(E, C)) = \mathrm{Idem}(L^{\mathrm{ab}}(E, C)).$$

On the other hand, by [11, Corollary 6.10], we have $\mathcal{L}(M(F_\infty, D^\infty)) \cong \mathcal{H}(F_\infty, D^\infty)$, so that we finally obtain a lattice isomorphism $\mathrm{Idem}(L^{\mathrm{ab}}(E, C)) \cong \mathcal{H}(F_\infty, D^\infty)$. □

Let Ω be a zero-dimensional metrizable locally compact Hausdorff space, and let $\mathbb{K} = \mathbb{K}(\Omega)$ be the subalgebra of $\mathcal{P}(\Omega)$ consisting of all the compact open subsets of Ω. Let $\theta \colon G \curvearrowright \Omega$ be a partial action of a discrete group G by continuous transformations on Ω such that $\Omega_g \in \mathbb{K}$ for all $g \in G$. Observe that $\mathbb{K}$ is then automatically G-invariant. The (relative) type semigroup $S(\Omega, G, \mathbb{K})$ has been defined in [7, Definition 7.1], see also [34] and [46]. The semigroup

$S(\Omega, G, \mathbb{K})$ is indeed a conical refinement monoid, and we obtain the following description of its lattice $\mathcal{L}(S(\Omega, G, \mathbb{K}))$ of order ideals.

Lemma 4.6. *Let $\theta\colon G \curvearrowright \Omega$ be a partial action of a discrete group G by continuous transformations on Ω such that $\Omega_g \in \mathbb{K}$ for all $g \in G$. Then there are mutually inverse, order-preserving maps*

$$\varphi\colon \mathcal{L}(S(\Omega, G, \mathbb{K})) \to \mathbb{O}^G(\Omega), \qquad \psi\colon \mathbb{O}^G(\Omega) \to \mathcal{L}(S(\Omega, G, \mathbb{K})),$$

$$\varphi(I) = \bigcup\{K \in \mathbb{K} \mid [K] \in I\}, \qquad \psi(U) = \langle [K] \mid K \in \mathbb{K}, K \subseteq U\rangle,$$

where $\mathbb{O}^G(\Omega)$ is the lattice of G-invariant open subsets of Ω, and, for $T \subseteq S(\Omega, G, \mathbb{K})$, $\langle T\rangle$ stands for the order ideal of $S(\Omega, G, \mathbb{K})$ generated by T.

Proof. Write $S := S(\Omega, G, \mathbb{K})$ and take $I \in \mathcal{L}(S)$. Clearly $U := \varphi(I)$ is an open subset of Ω. If $x \in \Omega_{g^{-1}} \cap U$ for some $g \in G$, then there is $K \in \mathbb{K}$ with $[K] \in I$ such that $x \in X_{g^{-1}} \cap K$. But now we have $\theta_g(x) \in \theta_g(\Omega_{g^{-1}} \cap K)$ with

$$[\theta_g(\Omega_{g^{-1}} \cap K)] = [\Omega_{g^{-1}} \cap K] \leq [K] \in I,$$

and so $[\theta_g(\Omega_{g^{-1}} \cap K)] \in I$ because I is an order ideal of S. It follows that U is G-invariant.

Let U be an invariant open subset of Ω. Then $\psi(U) \in \mathcal{L}(S)$ by definition of ψ. It is clear that φ and ψ are order-preserving maps. We have to show that $(\varphi \circ \psi)(U) = U$ and $(\psi \circ \varphi)(I) = I$ for $U \in \mathbb{O}^G(X)$ and $I \in \mathcal{L}(S)$. For $U \in \mathbb{O}^G(\Omega)$, let $K \in \mathbb{K}$ be such that $[K] \in \psi(U)$. Then there are $K_1, \dots, K_r \in \mathbb{K}$ such that $K_j \subseteq U$ for $j = 1, \dots, r$ and

$$[K] \leq [K_1] + [K_2] + \cdots + [K_r].$$

Using the refinement property of S and the definition of the type semigroup, one obtains a decomposition $K = \sqcup_{i=1}^n K_i'$ such that $K_i' \in \mathbb{K}$ for each i, and $g_1, \dots, g_n \in G$ such that, for each i, $K_i' \subseteq \Omega_{g_i^{-1}}$ and $\theta_{g_i}(K_i') \subseteq K_j$ for some $j = 1, \dots, r$. It follows that

$$K_i' \subseteq \theta_{g_i^{-1}}(K_j \cap \Omega_{g_i}) \subseteq \theta_{g_i^{-1}}(U \cap \Omega_{g_i}) \subseteq U,$$

where the last containment follows from the fact that U is G-invariant. We deduce that $K \subseteq U$, and so $\varphi(\psi(U)) \subseteq U$. The other containment $U \subseteq \varphi(\psi(U))$ follows from the fact that Ω has a basis of compact open subsets.

Finally, let $I \in \mathcal{L}(S)$. It is clear that $I \subseteq \psi(\varphi(I))$. To show the reverse inclusion, it is enough to check that, if $K \in \mathbb{K}$ and $K \subseteq \varphi(I)$, then $[K] \in I$. By compactness of K, there are $K_1, \dots, K_r \in \mathbb{K}$ such that $[K_i] \in I$ and $K \subseteq \cup_{i=1}^r K_i$, and thus

$$[K] \leq [K_1] + \cdots + [K_r] \in I.$$

Since I is an order ideal of S, we see that $[K] \in I$, as desired. □

We can now obtain a description of the lattice of induced ideals of tame graph algebras.

Theorem 4.7. *Let (E, C) be a finite bipartite separated graph. Then there is a lattice isomorphism*

$$\mathrm{Ind}(L^{\mathrm{ab}}(E, C)) \cong \mathrm{Ind}(\mathcal{O}^{(r)}(E, C)) \cong \mathcal{L}(M(F_\infty, D^\infty)) \cong \mathcal{H}(F_\infty, D^\infty).$$

Moreover for $H \in \mathcal{H}(F_\infty, D^\infty)$, we have

$$L^{\mathrm{ab}}(E,C)/I(H) \cong C_K(Z) \rtimes_{\theta|^*_Z} \mathbb{F} \quad \text{and} \quad \mathcal{O}^{(r)}(E,C)/I(H) \cong C(Z) \rtimes_{(r),\theta|^*_Z} \mathbb{F},$$

where $Z := \Omega(E,C) \setminus U$ with $U := \bigcup_{v \in H} \Omega(E,C)_v$.

Proof. It follow from [7, Theorem 7.4] that there is a natural isomorphism

$$S := S(\Omega(E,C), \mathbb{F}, \mathbb{K}) \cong M(F_\infty, D^\infty).$$

Combining this with Theorem 4.2 (or Remark 4.3), [11, Corollary 6.10] and Lemma 4.6, we obtain

$$\mathrm{Ind}(L^{\mathrm{ab}}(E,C)) \cong \mathrm{Ind}(\mathcal{O}^{(r)}(E,C)) \cong \mathbb{O}^{\mathbb{F}}(\Omega(E,C)) \cong \mathcal{L}(S) \cong \mathcal{L}(M(F_\infty, D^\infty)) \cong \mathcal{H}(F_\infty, D^\infty).$$

The last part follows from Theorem 4.2 and the definitions of the lattice isomorphisms. □

Remark 4.8. We believe it is likely that Theorem 4.5 generalizes to the setting of tame graph C*-algebras, at least for the reduced ones. This would mean that we have a lattice isomorphism

$$\mathrm{Proj}(\mathcal{O}^r(E,C)) \cong \mathcal{L}(M(F_\infty, D^\infty)) \cong \mathcal{H}(F_\infty, D^\infty),$$

where $\mathrm{Proj}(\mathcal{O}^r(E,C))$ denotes the lattice of ideals of $\mathcal{O}^r(E,C)$ which are generated by their projections. By Theorem 4.7, this is equivalent to saying that every ideal generated by projections is induced. In Section 7, we will prove this for ideals $I \lhd \mathcal{O}^r(E,C)$ of *finite type*.

5. The ideals associated to hereditary C-saturated subsets of (E,C)

In this section, we will analyze the induced ideals of $L^{\mathrm{ab}}(E,C)$, $\mathcal{O}(E,C)$ and $\mathcal{O}^r(E,C)$ arising from hereditary C-saturated subsets of E^0, as opposed to the general study of ideals corresponding to hereditary D^∞-saturated subsets of (F_∞, D^∞). By Theorem 3.22, (E,C) and (E_n, C^n) give rise to the same algebras for all $n \geq 0$, so we can apply the corresponding results to any hereditary C^n-saturated subset of the separated graph (E_n, C^n).

First, we shall give a concrete description of the hereditary and D^∞-saturated closure of a subset $H \in \mathcal{H}(E,C)$ inside (F_∞, D^∞).

Lemma 5.1. *Let (E,C) be a finite bipartite separated graph and take $H \in \mathcal{H}(E,C)$. Let*

$$H_1 := \{s_1(X(x)) \mid x \in E^1 \text{ and } s(x) \in H\} \cup (H \cap E_1^{0,0}).$$

Then $H_1 \in \mathcal{H}_{(E_1,C^1)}$ and $H \cup H_1 \in \mathcal{H}_{(F_1,D^1)}$ is the hereditary closure of H inside (F_1, D^1).

Proof. H_1 is clearly hereditary; if $r_1(e) \in H \cap E_1^{0,0} = H \cap E^{0,1}$, then $e \in X(x)$ for some $x \in E^1$ with $s(x) = r_1(e)$, hence $s_1(e) \in s_1(X(x)) \subseteq H_1$. Next, we show C^1-saturation. Suppose that $s(X(x)) \subseteq H_1$ for some $x \in E^1$, and write $w := s(x)$. Also, set $x = x_i \in X_i^u$ with $u := r(x)$ and $C_u = \{X_1^u, \dots, X_i^u, \dots, X_{k_u}^u\}$. If $k_u = 1$, then necessarily $w \in H$ by the definition of H_1, so suppose that $k_u > 1$. If for some $j \neq i$, we have $s(x_j) \in H$ for all $x_j \in X_j^u$, then $u \in H$ by C-saturation, and so $s(x) = s(x_i) \in H$ because H is hereditary. Thus, we may assume that, for all $j \neq i$, there exists $x_j \in X_j^u$ such that $s(x_j) \notin H$. Now, consider the vertex

$$v := v(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{k_u}) \in E_1^{0,1}.$$

Then $v \in H_1$ because $v = s(\alpha^{x_i}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{k_u})) \in s(X(x_i)) = s(X(x)) \subseteq H_1$. But by the definition of H_1, there must be $k \in \{1, \dots, k_u\}$ such that $s(x_k) \in H$. Hence $k = i$ and $w = s(x_i) \in H$, as desired. It is now clear that $H \cup H_1 \in \mathcal{H}(F_1, D^1)$, and $H \cup H_1$ is obviously nothing but the hereditary closure of H inside (F_1, D^1). □

Notation 5.2. Given $H \in \mathcal{H}(E, C)$, we define a sequence $H_n \in \mathcal{H}(E_n, C^n)$ in an inductive way, so that

$$H_n := \{s(X(x)) \mid x \in E^1_{n-1} \text{ and } s(x) \in H_{n-1}\} \cup (H_{n-1} \cap E_n^{0,0}).$$

Then, by Lemma 5.1, $H_\infty := \bigcup_{n=0}^\infty H_n \in \mathcal{H}(F_\infty, D^\infty)$ is the hereditary closure of H inside (F_∞, D^∞).

We have thus showed the following lemma:

Lemma 5.3. *Let (E, C) be a finite bipartite separated graph. Then there is an injective order-preserving map*

$$\mathcal{H}(E, C) \to \mathcal{H}(F_\infty, D^\infty)$$

sending $H \in \mathcal{H}(E, C)$ to $H_\infty \in \mathcal{H}(F_\infty, D^\infty)$. Moreover, the ideal $I(H)$ of $\mathcal{O}^r(E, C)$ generated by H is precisely the ideal $I(H_\infty)$, and similar statements hold for $\mathcal{O}(E, C)$ and for $L_K^{\mathrm{ab}}(E, C)$.

We also mention the following easy description of the open, invariant subspace associated with $H \in \mathcal{H}(E, C)$ in terms of configuration spaces. Recall that if $\xi \in \Omega(E, C)_v$, then $1 \in \xi$ is regarded as the trivial path v, so that $r(1) = v$ in this situation.

Lemma 5.4. *Let (E, C) be a finite bipartite graph and let $H \in \mathcal{H}(E, C)$. Then*

$$\bigcup_{v \in H_\infty} \Omega(E, C)_v = \{\xi \in \Omega(E, C) \mid r(\alpha) \in H \text{ for some } \alpha \in \xi\}.$$

Proof. Since $I(H) = I(H_\infty)$, we have

$$U = \theta_{\mathbb{F}}\Big(\bigcup_{v \in H} \Omega(E, C)_v\Big) = \{\xi \in \Omega(E, C) \mid r(\alpha) \in H \text{ for some } \alpha \in \xi\}.$$

□

Given a finitely separated graph (E, C) and a hereditary C-saturated subset H of E^0, we denote by E/H the subgraph of E with $(E/H)^0 := E^0 \setminus H$ and $(E/H)^1 := \{e \in E^1 \mid s(e) \notin H\}$. Similarly, for any subset $X \subseteq E^1$, define

$$X/H := X \cap s^{-1}(E^0 \setminus H).$$

For $v \in (E/H)^0$, we set

$$(C/H)_v := \{X/H \mid X \in C_v\},$$

which is a partition of $r_{E/H}^{-1}(v)$, and $C/H := \bigsqcup_{v \in E^0 \setminus H} (C/H)_v$. Observe that $X/H \neq \emptyset$ for all $X \in C_v$ with $v \in E^0 \setminus H$, because H is C-saturated.

Theorem 5.5. *Let (E,C) be a finite bipartite separated graph and let $H \in \mathcal{H}(E,C)$. Then there is a natural $*$-algebra isomorphism*

$$L_K^{\mathrm{ab}}(E,C)/I(H) \cong L_K^{\mathrm{ab}}(E/H,C/H).$$

Likewise, there are natural C^*-algebra isomorphisms

$$\mathcal{O}(E,C)/I(H) \cong \mathcal{O}(E/H,C/H) \quad \text{and} \quad \mathcal{O}^r(E,C)/I(H) \cong \mathcal{O}^r(E/H,C/H).$$

Proof. As observed in the proof of [11, Corollary 3.12], it is easy to show using universal properties that the map sending $v + I(H) \mapsto v$ for $v \in (E/H)^0$ and $e + I(H) \mapsto e$ for $e \in (E/H)^1$ extends to a $*$-isomorphism

$$C^*(E,C)/I(H) \longrightarrow C^*(E/H,C/H).$$

Likewise, we obtain a $*$-isomorphism $L_K(E,C)/I(H) \cong L_K(E/H,C/H)$ for any field with involution K. It is straightforward to check that

$$\big(E_1/H_1, C^1/H_1\big) = \big((E/H)_1, (C/H)^1\big).$$

Indeed we have that $(E_1/H_1)^{0,0} = (E/H)^{0,1} = ((E/H)_1)^{0,0}$ and that $(E_1/H_1)^{0,1}$ is the set of all the vertices $v(x_1,x_2,\dots,x_k)$ such that $x_i \in X_i/H$ for all i, where $C_v = \{X_1,X_2,\dots,X_k\}$ for some $v \in E^{0,0} \setminus H = (E/H)^{0,0}$. This shows that $(E_1/H_1)^0 = ((E/H)_1)^0$, and similarly $(E_1/H_1)^1 = ((E/H)_1)^1$ and $C^1/H_1 = (C/H)^1$. We thus obtain the following commutative diagram

$$\begin{array}{ccccc}
L(E,C)/I(H) & & \xrightarrow{\quad\quad\cong\quad\quad} & & L(E/H,C/H) \\
\big\downarrow & & & & \big\downarrow \\
L(E_1,C^1)/I(H_1) & \xrightarrow{\cong} & L(E_1/H_1,C^1/H_1) & \xrightarrow{\cong} & L((E/H)_1,(C/H)^1)
\end{array},$$

where all the maps are the canonical ones. Applying this observation inductively gives identifications $L(E_n,C^n)/I(H_n) \cong L((E/H)_n,(C/H)_n)$ commuting with the connecting homomorphisms, and hence we obtain an isomorphism

$$L^{\mathrm{ab}}(E,C)/I(H) = L^{\mathrm{ab}}(E,C)/I(H_\infty) \cong L^{\mathrm{ab}}(E/H,C/H)$$

of the limits. The same proof applies to $\mathcal{O}$.

We now give a proof for $\mathcal{O}^r$, which uses the dynamical interpretation of these algebras. Let $\mathbb{F}'$ denote the free group on $(E/H)^1$, which we can regard as a subgroup of $\mathbb{F}$. Let $U := \bigcup_{v\in H_\infty} \Omega(E,C)_v$ be the open invariant subset of $\Omega(E,C)$ associated to H_∞, and set $Z := \Omega(E,C) \setminus U$. By Theorem 4.7, we have

$$\mathcal{O}^r(E,C)/I(H) = \mathcal{O}^r(E,C)/I(H_\infty) \cong C(Z) \rtimes_{r,\theta|_Z^*} \mathbb{F}.$$

Let us denote by θ' the restriction of θ to Z. If $x \in E^1$ and $s(x) \in H$, then the domain and codomain of θ'_x is empty, so we have that

$$\mathcal{O}^r(E,C)/I(H) \cong C(Z) \rtimes_{r,(\theta')^*} \mathbb{F}',$$

and we only have to show that the action θ' of $\mathbb{F}'$ on Z is conjugate to $\theta^{(E/H,C/H)}$. Observe that by Lemma 5.4,

$$Z = \{\xi \in \Omega(E,C) \mid r(\alpha) \notin H \text{ for all } \alpha \in \xi\}.$$

We now claim that in fact $Z = \Omega(E/H, C/H)$, so let $\xi \in \Omega(E/H, C/H)$. Since every $\alpha \in \xi$ satisfies $r(\alpha) \in (E/H)^0 = E^0 \setminus H$, we simply need to verify that the local configurations ξ_α are either of type (c1) or (c2) with respect to (E,C). Assume first that ξ is of type (c1) with respect to $(E/H, C/H)$, i.e. that $\xi_\alpha = s_{E/H}^{-1}(r(\alpha))$. Now since H is hereditary, we must have

$$\xi_\alpha = s_{E/H}^{-1}(r(\alpha)) = s^{-1}(r(\alpha)),$$

so ξ_α is indeed of type (c1) with respect to (E,C). Next, assume that ξ_α is of type (c2) with respect to $(E/H, C/H)$, i.e. that $\xi_\alpha = \{e_{X/H}^{-1} \mid X/H \in (C/H)_{r(\alpha)}\}$ for some $e_{X/H} \in X/H$. From H being C-saturated, we have $(C/H)_{r(\alpha)} = \{X/H \mid X \in C_{r(\alpha)}\}$, so setting $e_X := e_{X/H}$ for all $X \in C_{r(\alpha)}$,

$$\xi_\alpha = \{e_{X/H}^{-1} \mid X/H \in (C/H)_{r(\alpha)}\} = \{e_X^{-1} \mid X \in C_{r(\alpha)}\}$$

is of type (c2) with respect to (E,C). The converse inclusion is completely straightforward and does not use (explicitly) the assumptions about H being hereditary and C-saturated. Finally, since the dynamics is completely determined by the configurations, and the configuration spaces agree, we conclude that the two partial action are in fact conjugate.

Observe that, by Theorem 4.7 and Remark 4.8(1), the same proof applies to $\mathcal{O}$ and L^{ab} respectively, so we obtain a second proof for those. □

6. The ideals associated to hereditary D^∞-saturated subsets of (F_∞, D^∞)

Recall from Section 4 that every induced ideal of a tame graph algebra corresponds to a set $H \in \mathcal{H}(F_\infty, D^\infty)$. In this section, we shall describe the induced ideal, or rather its quotient, in terms of the intersections $H \cap E_n^0$. We shall also consider a number of examples.

The proof of the following lemma is straightforward.

Lemma 6.1. *Let $H \in \mathcal{H}(F_\infty, D^\infty)$, and, for each $n \geq 0$, set $H^{(n)} := H \cap E_n^0$. Then $H^{(n)}$ is a hereditary C^n-saturated subset of E_n^0, and $H = \cup_{n=0}^\infty H^{(n)}$.*

We now describe the hereditary and D^∞-saturated closure of $H^{(n)}$ inside (F_∞, D^∞): Following Notation 5.2, we will denote by $H_\infty^{(n)}$ the hereditary closure of $H^{(n)}$ in F_∞^0, that is, $H_\infty^{(n)} = \cup_{m\geq n} H_m^{(n)}$, where

$$H_m^{(n)} = \{s(X(x)) \mid x \in E_{m-1}^1 \text{ and } s(x) \in H_{m-1}^{(n)}\} \cup (H_{m-1}^{(n)} \cap E_{m-1}^{0,1}).$$

Observe that $H_\infty^{(n)}$ is just the hereditary closure of $H^{(n)}$ in F_∞^0, but we proved in Lemma 5.1 that it is $(D^{\geq n})$-saturated, that is, if $v \in F^{0,m}$ with $m \geq n$ and $X \in C_v^m$ are such that

$s(X) \subseteq H_\infty^{(n)}$, then $v \in H_\infty^{(n)}$. Now define

$$H^n := (H \cap F_{n-1}^0) \cup H_\infty^{(n)} = (\bigcup_{i=0}^{n-1} H^{(i)}) \cup H_\infty^{(n)}$$

and observe that H^n is exactly the hereditary and D^∞-saturated closure of $H^{(n)}$ inside (F_∞, D^∞). Note also that $F_\infty \setminus H$ has the structure of a separated Bratteli diagram, which we denote by $(F_\infty/H, D^\infty/H)$. Here

$$(F_\infty/H)^{0,n} = F^{0,n} \setminus H, \qquad D^\infty/H = \bigsqcup_{n=0}^{\infty} C^n/H^{(n)}.$$

Proposition 6.2. *Let (E, C) denote a finite bipartite separated graph, let $H \in \mathcal{H}(F_\infty, D^\infty)$ and apply the above notation. Also, let U_n denote the open invariant subspace of $\Omega(E, C)$ corresponding to H^n, and set $Z_n := \Omega(E, C) \setminus U_n$ as well as $Z := \bigcap_{n=0}^\infty Z_n$. Then*

$$\mathcal{O}^r(E, C)/I(H) \cong C(Z) \rtimes_r \mathbb{F} \cong \varinjlim_n C(Z_n) \rtimes_r \mathbb{F} \cong \varinjlim_n \mathcal{O}^r(E_n/H^{(n)}, C^n/H^{(n)}),$$

where the connecting homomorphisms are simply induced from restriction of functions. The same statement holds with $\mathcal{O}$ or L^{ab} in place of $\mathcal{O}^r$.

Proof. Since H^n is the hereditary and D^∞-saturated closure of $H^{(n)}$, we have $I(H^n) = I(H^{(n)})$ as ideals of $\mathcal{O}^r(E, C)$. It follows that

$$\mathcal{O}^r(E_n/H^{(n)}, C^n/H^{(n)}) \cong \mathcal{O}^r(E_n, C^n)/I(H^{(n)}) \cong \mathcal{O}^r(E, C)/I(H^n) \cong C(Z_n) \rtimes_r \mathbb{F}$$

from Theorem 3.22 and Theorem 5.5. Now let U denote the open invariant set corresponding to H. We have $H = \bigcup_{n=0}^\infty H^n$ by construction, so $U = \bigcup_{n=0}^\infty U_n$ and $Z = \Omega(E, C) \setminus U$. Thus, we obtain the identification

$$\mathcal{O}^r(E, C)/I(H) \cong C(Z) \rtimes_r \mathbb{F},$$

and clearly $C(Z) \rtimes_r \mathbb{F} \cong \varinjlim_n C(Z_n) \rtimes_r \mathbb{F}$. The same proof applies to $\mathcal{O}$ and L^{ab}. □

Remark 6.3. Recall from Theorem 3.22 that every $v \in E_n^0$ for $n \geq 1$ corresponds to an n-ball $B(v) \in \mathcal{B}_n(\Omega(E, C))$. The open set U_n of Proposition 6.2 may therefore be described as the set of configurations $\xi \in \Omega(E, C)$ satisfying the following: There is some $v \in H^{(n)}$ and $\alpha \in \xi$ for which $\theta_\alpha(\xi)^n = B(v)$. In other words, we can describe the set Z_n as

$$Z_n = \{\xi \in \Omega(E, C) \mid \xi \not\equiv B(v) \text{ for all } v \in H^{(n)}\}.$$

Thus, the descending filtration $(Z_n)_n$ exactly removes the configurations with forbidden n-balls at the n'th step. In particular, we see that the restricted action $\theta|_Z \colon \mathbb{F} \curvearrowright Z$ is of finite type in the sense of Definition 3.7 if and only if $H = H^n$ for some $n \geq 0$.

As another consequence, we see that *every* convex shift $\theta \colon \mathbb{F} \curvearrowright \Omega$ can be represented, up to Kakutani equivalence, by a separated Bratteli diagram: If (E, C) is the graph representing the full convex shift (see Example 3.26), then θ is Kakutani equivalent to the restriction of $\theta^{(E,C)}$ to some closed invariant subspace Z. It follows that θ may be described as above by the separated Bratteli diagram $(F_\infty/H, D^\infty/H)$, where $H \in \mathcal{H}(F_\infty, D^\infty)$ corresponds to $U := \Omega(E,C) \setminus Z$.

We now describe K_0 of these quotient algebras in terms of their associated separated Bratteli diagrams.

Theorem 6.4. *Let (E,C) be a finite bipartite separated graph, let H be a proper hereditary D^∞-saturated subset of F_∞, and let $(F_\infty/H, D^\infty/H)$ be its associated separated Bratteli diagram.*

(1) *There is a natural group isomorphism*

$$K_0(\mathcal{O}(E,C)/I(H)) \cong K_0(\mathcal{O}^r(E,C)/I(H)) \cong G(F_\infty/H, D^\infty/H).$$

(2) *There is a natural isomorphism $\mathcal{V}(L^{\mathrm{ab}}_K(E,C)/I(H)) \cong M(F_\infty/H, D^\infty/H)$, and so an isomorphism of pre-ordered abelian groups*

$$K_0(L^{\mathrm{ab}}_K(E,C)/I(H)) \cong G(F_\infty/H, D^\infty/H)$$

for any field with involution K.

Proof. We adopt the above notation. It is straightforward to check that, for $n \geq 1$ we have

$$M(F_{\geq n}/H^{(n)}_\infty, D^{\geq n}/H^{(n)}_\infty) \cong M(F_\infty/H^n, D^\infty/H^n)$$

and

$$G(F_{\geq n}/H^{(n)}_\infty, D^{\geq n}/H^{(n)}_\infty) \cong G(F_\infty/H^n, D^\infty/H^n).$$

Therefore, it follows from Theorem 2.9 and an easy calculation that there are commutative diagrams

$$\begin{array}{ccc} K_0(\mathcal{O}(E_n/H^{(n)}, C^n/H^{(n)})) & \xrightarrow{\cong} & G(F_\infty/H^n, D^\infty/H^n) \\ \downarrow & & \downarrow \\ K_0(\mathcal{O}(E_{n+1}/H^{(n+1)}, C^{n+1}/H^{(n+1)})) & \xrightarrow{\cong} & G(F_\infty/H^{n+1}, D^\infty/H^{n+1}) \end{array}$$

for all $n \geq 1$. Using Proposition 6.2, we obtain

$$\begin{aligned} K_0(\mathcal{O}(E,C)/I(H)) &\cong \varinjlim_n K_0(\mathcal{O}(E_n/H^{(n)}, C^n/H^{(n)})) \\ &\cong \varinjlim_n G(F_\infty/H^n, D^\infty/H^n) \\ &= G(F_\infty/H, D^\infty/H). \end{aligned}$$

This gives (1) for $\mathcal{O}$. The same arguments give, using Theorem 2.9, the result for $\mathcal{O}^r$ and for L^{ab}. ☐

We record some useful properties of the monoid $M(F_\infty/H, D^\infty/H)$.

Proposition 6.5. *Let (E, C) be a finite bipartite separated graph, let H be a proper hereditary D^∞-saturated subset of F_∞, and let $(F_\infty/H, D^\infty/H)$ be its associated separated Bratteli diagram. Then we have a natural monoid isomorphism*

$$M(F_\infty/H, D^\infty/H) \cong M(F_\infty, D^\infty)/M(H),$$

where $M(H)$ is the order-ideal of $M(F_\infty, D^\infty)$ generated by H. In particular, $M(F_\infty/H, D^\infty/H)$ is a refinement monoid.

Proof. The isomorphism follows from [10, Construction 6.8]. Since $M(F_\infty, D^\infty)$ is a refinement monoid, the quotient monoid $M(F_\infty, D^\infty)/M(H)$ is also a refinement monoid, by [13, Lemma 4.3]. □

We now consider a number of examples.

Example 6.6. For integers $1 \le m \le n$, define the separated graph $(E(m, n), C(m, n))$, where

(1) $E(m, n)^0 := \{v, w\}$ (with $v \neq w$).
(2) $E(m, n)^1 := \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\}$ (with $n + m$ distinct edges).
(3) $s(\alpha_i) = s(\beta_j) = v$ and $r(\alpha_i) = r(\beta_j) = w$ for all i, j.
(4) $C(m, n) = C(m, n)_v := \{X, Y\}$, where $X := \{\alpha_1, \dots, \alpha_n\}$ and $Y := \{\beta_1, \dots, \beta_m\}$.

See Figure 1 just below for a picture in the case $m = 2$, $n = 3$. We refer the reader to [9] and [7, Example 9.3] for more information on this example.

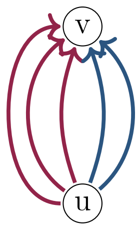


FIGURE 1. The separated graph $(E(2, 3), C(2, 3))$

The C*-algebras $\mathcal{O}_{m,n}$ and $\mathcal{O}^r_{m,n}$ studied in [9] are precisely the C*-algebras $\mathcal{O}(E(m, n), C(m, n))$ and $\mathcal{O}^r(E(m, n), C(m, n))$, respectively, in the notation of the present paper. It was shown in [9] that these two C*-algebras are not isomorphic, and we will now show that the C*-algebra $\mathcal{O}^r_{m,n}$ is not simple if $m \ge 2$. Let $(E, C) := (E(m, n), C(m, n))$. It is clear that $\mathcal{H}(E, C)$ only contains the trivial sets $\emptyset, E^0$.

Adopting the notation of [9], we write

$$\Omega^u := \Omega(E, C) = X^u \sqcup Y^u,$$

where $X^u = \sqcup_{i=1}^n H_i^u = \sqcup_{j=1}^m V_j^u$, and Y^u is homeomorphic to each of the sets H_i^u, V_j^u by homeomorphisms $h_i^u \colon Y^u \to H_i^u$ and $v_j^h \colon Y^u \to V_j^u$. The maps h_i^u, v_j^u are the universal maps defining the universal (m, n)-dynamical system. Indeed, we have $h_i^u = \theta_{\alpha_i}$ and $v_j^u = \theta_{\beta_j}$ for all i, j.

Let us now describe the separated graph (E_1, C^1). We have $E_1^{0,0} = \{w\}$ and

$$E_1^{0,1} = \{w_{ij} : 1 \le i \le n, 1 \le j \le m\}.$$

Now there are $n+m$ elements in C^1_w, namely $X_i := X(\alpha_i)$ and $Y_j := X(\beta_j)$ for $i = 1, \dots, n$ and $j = 1, \dots, m$. Note that $|X_i| = m$ and $|Y_j| = n$. Moreover, $s(X_i) = \{w_{ij} : j = 1, \dots, m\}$ and $s(Y_j) = \{w_{ij} : i = 1, \dots, n\}$.

If $m = n \ge 2$, then set $H := \{w_{ij} : i \ne j\}$. Then H is a maximal hereditary and C^1-saturated subset of E_1^0. Moreover, $(E_1/H, C^1/H)$ consists of n cycles based at the vertex w, so that

$$\mathcal{O}^r_{n,n}/I(H) \cong \mathcal{O}^r(E_1/H, C^1/H) \cong M_{n+1}(C^*_{\text{red}}(\mathbb{F}_n)),$$

which is a simple C*-algebra. However $\mathcal{O}_{n,n}/I(H) \cong C^*(\mathbb{F}_n)$ is not simple. (Incidentally, note that this gives another proof that $\mathcal{O}^r_{n,n} \ne \mathcal{O}_{n,n}$ for $n \ge 2$.)

If $2 \le m < n$, then set

$$H := \{w_{ij} : i \ne j \text{ and } 1 \le i \le n, 1 \le j \le m-1\} \cup \{w_{im} : 1 \le i \le m-1\}.$$

Then H is again a maximal hereditary C^1-saturated subset of E_1^0. However, in this case the quotient C*-algebra $\mathcal{O}^r_{m,n}/I(H)$ is not even $\mathcal{V}$-simple, as we will show in Section 8. It is therefore clear that the universal $(E_1/H, C^1/H)$-system is not equivalent to the (m,n)-system (X, Y) considered in the proof of [9, Proposition 3.9]. Indeed, observe that $v_i^{-1} \circ h_i = \mathrm{id}_Y$ for $i = 1, \dots, m-1$ in that example, and this is not necessarily true in a $(E_1/H, C^1/H)$-system. The (m,n)-dynamical system just mentioned shows that the algebra $M_{m+1}(\mathcal{O}_{n-m+1}) \cong M_{n+1}(\mathcal{O}_{n-m+1})$ is a simple quotient of $\mathcal{O}_{m,n}$, where $\mathcal{O}_k$ denotes the usual Cuntz algebra. Indeed, we can define a surjective $*$-homomorphism $\mathcal{O}_{m,n} \to M_{m+1}(\mathcal{O}_{n-m+1})$ by

$$\alpha_i \mapsto \begin{cases} 1 \otimes e_{i+1,1} & \text{if } i = 1, \dots, m-1 \\ s_{i-m+1} \otimes e_{m+1,1} & \text{if } i = m, \dots, n \end{cases} \quad , \quad \beta_j \mapsto 1 \otimes e_{j+1,1} \text{ for } j = 1, \dots, m,$$

$w \mapsto 1 \otimes e_{1,1}$ and $v \mapsto \sum_{j=2}^{m+1} 1 \otimes e_{j,j}$. However, it is not clear to the authors whether the same algebra $M_{m+1}(\mathcal{O}_{n-m+1})$ appears as a simple quotient of the *reduced* tame C*-algebra $\mathcal{O}^r_{m,n}$.

We now present an example relating our theory with classical symbolic dynamics; hopefully, it can also serve as an exemplification of the general theory of the previous sections. We only consider the case where the alphabet is $\{0, 1\}$, but similar statements can be made for an arbitrary finite alphabet, considering a corresponding variation of the separated graph considered below.

Example 6.7. Let (E, C) be the separated graph described in Figure 2, with $C_v = \{X, Y\}$ and $X = \{\alpha_0, \alpha_1\}$ and $Y = \{\beta_0, \beta_1\}$.

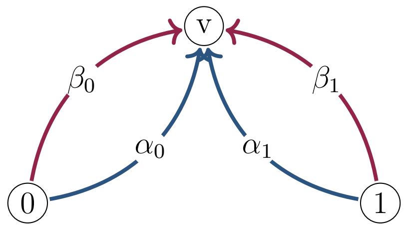


FIGURE 2. The separated graph underlying the lamplighter group

This example has been considered in [7, Example 9.7], where it is observed that

$$v\mathcal{O}(F,D)v \cong C(\mathcal{X}) \rtimes_\sigma \mathbb{Z}, \qquad vL_K^{\mathrm{ab}}(F,D)v \cong C_K(\mathcal{X}) \rtimes_\sigma \mathbb{Z},$$

where $\mathcal{X} = \{0,1\}^{\mathbb{Z}}$ and σ is the usual shift homeomorphism on $\mathcal{X}$. Note that $\mathcal{O}(E,C) = \mathcal{O}^r(E,C)$ in this case. Indeed it can be easily seen that (E,C)-dynamical systems are in one-to-one correspondence with usual dynamical systems $(\mathcal{Y},\varphi)$, where φ is a homeomorphism of the compact Hausdorff space $\mathcal{Y}$, with the additional information of a partition $\mathcal{Y} = \mathcal{Y}_0 \sqcup \mathcal{Y}_1$ of $\mathcal{Y}$ into clopen subsets $\mathcal{Y}_0, \mathcal{Y}_1$. To obtain the corresponding (E,C)-system, take $\Omega := \mathcal{Y}' \sqcup \mathcal{Y}$, where $\mathcal{Y}' = \mathcal{Y}_0' \sqcup \mathcal{Y}_1'$ is a disjoint copy of $\mathcal{Y}$, and where $\Omega_v := \mathcal{Y}$, $\mathcal{Y}_i'$ correspond to the vertices labeled by i, the maps α_i correspond to the identification of elements of $\mathcal{Y}_i'$ with elements of $\mathcal{Y}_i$, and the maps β_i are induced by the homeomorphism φ. It is easily checked that $v(C(\Omega) \rtimes \mathbb{F})v \cong C(\mathcal{Y}) \rtimes_\varphi \mathbb{Z}$ in this situation. The unique equivariant continuous map $\Omega \to \Omega(E,C)$ predicted by [7, Corollary 6.11], restricted to $\mathcal{Y}$, is the fundamental map in symbolic dynamics (see e.g. [37, §6.5]) sending each element x in $\mathcal{Y}$ to the sequence $(a_n)_{n\in\mathbb{Z}}$ recording to which of the sets $\mathcal{Y}_0$ or $\mathcal{Y}_1$ belongs $\varphi^n(x)$, that is, $a_n = i \iff \varphi^n(x) \in \mathcal{Y}_i$, for $n \in \mathbb{Z}$.

We here give a dynamical interpretation in terms of the associated canonical sequence $\{(E_n, C^n)\}$ of bipartite separated graphs. Let $\mathcal{X} = \{0,1\}^{\mathbb{Z}}$ as above. For a finite word $a_1a_2\cdots a_n \in \{0,1\}^n$, we will write

$$[a_1a_2\cdots a_{i-1}\underline{a_i}a_{i+1}\cdots a_n] := \{x \in \mathcal{X} : x_{j-i} = a_j \text{ for } j = 1,2,\dots,n\}.$$

The space $\Omega(E,C)$ is of the form $\Omega(E,C) = \mathcal{X}' \sqcup \mathcal{X}$, where $\mathcal{X}' \cong \mathcal{X} = \{0,1\}^{\mathbb{Z}}$. By a slight abuse of language, we will identify $\mathcal{X}'$ with $\mathcal{X}$ notationally, so that we will obtain two partitions of the space $\{0,1\}^{\mathbb{Z}}$ into clopen subsets for each separated graph (E_n, C^n), corresponding to the sets of vertices $E_n^{0,0}$ and $E_n^{0,1}$ respectively, and maps between these clopen subsets corresponding to the edges E_n^1.

The first layer of the sequence $\{(E_n,C^n)\}_{n\geq 0}$ corresponds to a trivial decomposition $\mathcal{X}^0 = \mathcal{X}$ and to the decomposition

$$\mathcal{X}_0^1 = [\underline{0}], \qquad \mathcal{X}_1^1 = [\underline{1}].$$

The maps corresponding to the edges are the maps $\alpha_i \colon \mathcal{X}_i^1 \to \mathcal{X}$ and $\beta_i \colon \mathcal{X}_i^1 \to \mathcal{X}$ defined by $\alpha_i = \mathrm{id}|_{\mathcal{X}_i^1}$ and $\beta_i = \sigma|_{\mathcal{X}_i^1}$. We describe now the clopen sets corresponding to the separated graph (E_n, C^n) for any $n \geq 1$. Let $w = a_1a_2\cdots a_n \in \{0,1\}^n$ be a word of length n. If $n = 2m$

is even, define

$$\mathcal{X}_w^n := [a_1 \cdots a_m \underline{a_{m+1}} a_{m+2} \cdots a_{2m}].$$

If $n = 2m+1$ is odd, define

$$\mathcal{X}_w^n := [a_1 \cdots a_m \underline{a_{m+1}} a_{m+2} \cdots a_{2m+1}].$$

The set $F_n^0 = E_n^{0,0}$ has exactly 2^n elements, and thus $F_{n+1}^0 = E_n^{0,1}$ has exactly 2^{n+1} elements. The vertices in $E_n^{0,0}$ correspond to a decomposition

$$\mathcal{X} = \bigsqcup_{w \in \{0,1\}^n} \mathcal{X}_w^n.$$

Let $n = 2m$ and take $w \in \{0,1\}^n$. Then $C_w^n = \{X_1^w, X_2^w\}$, where $X_1^w = \{\alpha_{w0}^{n+1}, \alpha_{w1}^{n+1}\}$ and $X_2^w = \{\beta_{0w}^{n+1}, \beta_{1w}^{n+1}\}$. The edges α_{wi}^{n+1} correspond to the maps, denoted in the same way, $\alpha_{wi}^{n+1} \colon \mathcal{X}_{wi}^{n+1} \to \mathcal{X}_w^n$, where each is simply the identity on the respective domain. Similarly, the edges β_{iw}^{n+1} correspond to the maps $\beta_{iw}^{n+1} \colon \mathcal{X}_{iw}^{n+1} \to \mathcal{X}_w^n$ given by the restriction of σ to the respective domains.

Now let $n = 2m+1$ and take $w \in \{0,1\}^n$. In this case, we have $C_w^n = \{X_1^w, X_2^w\}$, where $X_1^w = \{\alpha_{0w}^{n+1}, \alpha_{1w}^{n+1}\}$ and $X_2^w = \{\beta_{w0}^{n+1}, \beta_{w1}^{n+1}\}$. The edges α_{iw}^{n+1} again correspond to the maps, denoted in the same way, $\alpha_{iw}^{n+1} \colon \mathcal{X}_{iw}^{n+1} \to \mathcal{X}_w^n$, acting as the identity on the respective domains, while the edges β_{wi}^{n+1} correspond to the maps $\beta_{wi}^{n+1} \colon \mathcal{X}_{wi}^{n+1} \to \mathcal{X}_w^n$ given by the restriction of σ^{-1} to the respective domains.

Now it is quite easy to observe that $\Omega(E,C)_v \cong \mathcal{X}$ canonically. Indeed, by [8, p. 3008], we have $\Omega(E,C)_v \cong \varprojlim (F_\infty^{0,2j}, r_{2j})$, and in this case, the maps $r_{2j} \colon F_\infty^{0,2j} \to F_\infty^{0,2j-2}$ are the maps sending awb to w, where $w \in \{0,1\}^{2j-2}$ and $a, b \in \{0,1\}$.

We can now relate the ideals of $\mathcal{O}(E,C)$ with some sets of words, and the quotients of $\mathcal{O}(E,C)$ with the subshifts of the shift $(\mathcal{X}, \sigma)$. Recall from [37, Chapter 1] that a *subshift* of $\mathcal{X} = \{0,1\}^{\mathbb{Z}}$ is a subspace $\mathcal{X}_{\mathcal{F}}$ which can be described as the set of all elements x in $\mathcal{X}$ not containing any block from a fixed family $\mathcal{F}$ of finite words in the alphabet $\{0,1\}$. (A block of x is a finite subsequence of consecutive terms in x.) The family $\mathcal{F}$ is called the family of *forbidden words* of the subshift. By [37, Theorem 6.1.21], the subshifts of $\mathcal{X}$ are exactly the invariant closed subsets of $\mathcal{X}$. A subshift Z is said to be *of finite type* if there exists a finite set $\mathcal{F}$ such that $Z = \mathcal{X}_{\mathcal{F}}$.

Proposition 6.8. *Let (E,C) be the separated graph described in Example 6.7, and adopt the notation used there.*

1. *For each subset $\mathcal{F}$ of words, there is a unique hereditary D^∞-saturated subset $H_{\mathcal{F}}$ of the separated graph (F_∞, D^∞) such that $\mathcal{F}$ and $H_{\mathcal{F}}$ generate the same subshift, that is $\mathcal{X}_{\mathcal{F}} = \mathcal{X}_{H_{\mathcal{F}}}$.*
2. *$\mathcal{X}_{\mathcal{F}}$ is a shift of finite type if and only if there is some $n < \infty$ such that $H_{\mathcal{F}}$ is generated by $H := H_{\mathcal{F}} \cap E_n^0$. In this case, $\theta^{(E_n/H, C^n/H)}$ and σ are Kakutani equivalent, so in particular there are Morita-equivalences*

$$C_K(X_{\mathcal{F}}) \rtimes_\sigma \mathbb{Z} \sim L_K^{\mathrm{ab}}(E_n/H, C^n/H) \quad \textit{and} \quad C(\mathcal{X}_{\mathcal{F}}) \times_\sigma \mathbb{Z} \sim \mathcal{O}(E_n/H, C^n/H).$$

Proof. (1): By Example 6.7, the set of vertices of the graph F_∞ can be identified with the set of all finite words in the alphabet $\{0,1\}$.

Given a set $\mathcal{F}$ of words, the space $\mathcal{X}_{\mathcal{F}}$ is a closed invariant subset of $\mathcal{X}$. One can easily show that

$$\mathcal{X} \setminus \mathcal{X}_{\mathcal{F}} = \bigcup_{n=0}^{\infty} \bigcup_{w \in W_n} \bigcup_{j \in \mathbb{Z}} \sigma^j(\mathcal{X}_w^n),$$

where for each $n \geq 0$, the set W_n is the set of words of length n containing a block coming from $\mathcal{F}$, and $\mathcal{X}_w^n$ are the subsets of $\mathcal{X}$ defined in Example 6.7. Observe that the set $W := \bigcup_{n=0}^{\infty} W_n$ is precisely the hereditary closure of $\mathcal{F}$ in the graph F_∞. The set $H_{\mathcal{F}}$ is the D^∞-saturation of W, and it generates the same open invariant subset as $\mathcal{F}$ and as W. We therefore have $\mathcal{X}_{\mathcal{F}} = \mathcal{X}_{H_{\mathcal{F}}}$. The uniqueness of $H_{\mathcal{F}}$ comes from Theorem 4.7.

(2): The first statement follows immediately from (1). Using the above identification of $\theta^{(E_n,C^n)}$, we see that $\bigsqcup_{v \in E^{0,0} \cap H} \Omega(E_n, C^n)_v = X_{\mathcal{F}}$, so that the restriction of $\theta^{(E_n/H, C^n/H)}$ to the full, clopen subspace

$$\bigsqcup_{v \in E_n^{0,0} \setminus H} \Omega(E_n/H, C^n/H)_v$$

is directly dynamically equivalent to σ via the group homomorphism $\mathbb{F}((E_n/H)^1) \to \mathbb{Z}$ given by

- $\alpha_{wi}^{n+1} \mapsto 0$ and $\beta_{iw}^{n+1} \mapsto 1$ if n is even,
- $\alpha_{iw}^{n+1} \mapsto 0$ and $\beta_{wi}^{n+1} \mapsto -1$ if n is odd.

In particular, $\theta^{(E_n/H, C^n/H)}$ and σ are Kakutani equivalent, so we obtain Morita equivalences as above. This concludes the proof. □

Example 6.9. The cycles in the different graphs (E_n, C^n) are determined by the complements of some hereditary C^n-saturated subsets of E_n^0. They correspond to periodic orbits in the shift $(\mathcal{X}, \sigma)$. For instance, consider the word $w = 0110$, which is primitive, that is, it is not a square. The successive rotations of w are the words

$$w_1 = w, \quad w_2 = 1100, \quad w_3 = 1001, \quad w_4 = 0011.$$

Consider the separated graph (E_3, C^3), and the C^3-saturation H of the set

$$E_3^{0,1} \setminus \{w_1, w_2, w_3, w_4\}.$$

It is easily seen that $E_3^{0,1} \setminus H = \{w_1, w_2, w_3, w_4\}$, and $E_3^{0,0} \setminus H = \{v_1, v_2, v_3, v_4\}$, where

$$v_1 = 110, \quad v_2 = 100, \quad v_3 = 001, \quad v_4 = 011.$$

The separated graph $(E_3/H, C^3/H)$ is essentially given by a cycle of length 4. We have

$$\mathcal{O}(E,C)/I(H_\infty) \cong \mathcal{O}(E_3/H, C^3/H) \cong M_8(C(\mathbb{T})).$$

Note that if we start with a word which is not primitive, for instance $w = 0101$, then there is a non-trivial C^2-saturation H_2 induced by H in (E_2, C^2), and there is a non-trivial C^1-saturation H_1 induced by H_2 in (E_1, C^1), which is such that $E_1^{0,1} \setminus H_1 = \{10, 01\}$ (that is, $H_1 = \{00, 11\}$), and everything is reduced to the 2-cycle generated by 01.

The periodic orbits give trivial examples of minimal subshifts. More interesting examples are provided by the minimal Cantor subshifts, whose underlying space is a Cantor set. By [49], every strong orbit equivalence class of minimal Cantor systems contains a minimal Cantor subshift. By combining this with [30, Theorem 2.1], we see that, for every $*$-isomorphism class of C^*-algebra crossed products $C(\Omega) \rtimes \mathbb{Z}$ of minimal actions on the Cantor set Ω, there is a representative coming from a minimal subshift. These C^*-algebras are classified by their ordered Grothendieck groups with distinguished order-units, which may be any simple dimension group with order-unit (see [30, Theorems 1.12 and 1.14]). See also [23] and [33] for further information about the conjugacy classes of minimal subshifts. We remark that these examples imply that, in spite of the results in [38], the theory of separated graph C*-algebras leads to non-trivial examples (i.e. not coming from ordinary directed graphs) of simple nuclear C*-algebras with stable rank one and real rank zero, since it is well-known that the C*-algebras associated to minimal Cantor systems enjoy these properties (see [42] and [24, p. 184]). By the results of the second-named author ([38]) this is not possible for the tame C*-algebra of a separated graph, but we see now that, factoring out a suitable maximal ideal generated by projections of the algebra $\mathcal{O}(E,C)$ appearing in Example 6.8, we may obtain such examples.

The following example appears for instance in [22].

Example 6.10 (The even shift)**.** Consider the subshift $\mathcal{Y}$ of $(\mathcal{X},\sigma)$ defined by taking as a set of forbidden words $\mathcal{F} = \{01^{2n+1}0 \mid n \geq 0\}$. That is, in a word of $\mathcal{Y}$ there is always an even number of consecutive 1's between two 0's. In this case, we have $H^{(2)} = \{010\}$, $H^{(4)} = \{01110\} \cup H'_4$, where H'_4 is the family of words of length 4 or 5 containing as subwords the word 010. In general

$$H^{(2i)} = \{01^{2i-1}0\} \cup H'_{2i},$$

where H'_{2i} is the set of words of length $2i$ or $2i+1$ containing some subword of the form $01^{2j-1}0$ with $j < i$. This gives rise to a subshift which is not finite.

7. A complete description of the ideals of finite type

In this section, we completely determine the structure of the *finite type* ideals of $\mathcal{O}^r(E,C)$:

Definition 7.1. Let (E,C) denote a finite bipartite separated graph and let (F_∞, D^∞) be the separated Bratteli diagram of (E,C). Given an arbitrary ideal $J \lhd \mathcal{O}^r(E,C)$, there is some $H_J \in \mathcal{H}(F_\infty, D^\infty)$ for which $I(H_J) = (J \cap C(\Omega(E,C))) \rtimes_r \mathbb{F}$. We will say that any $H \in \mathcal{H}(F_\infty, D^\infty)$ is of finite type if $H = H^n$ for some $n \geq 0$, and an ideal J is of finite type if H_J is so. Finally, the lattices of finite type vertex sets and ideals will be denoted by $\mathcal{H}_{\text{fin}}(F_\infty, D^\infty)$ and $\mathcal{I}_{\text{fin}}(\mathcal{O}^r(E,C))$, respectively.

Given any partial action $\theta\colon G \curvearrowright \Omega$ and a point $x \in \Omega$, the stabiliser of x is the subgroup

$$\text{Stab}(x) := \{g \in G \mid x \in \Omega_{g^{-1}} \text{ and } \theta_g(x) = x\}.$$

Recall that θ is *called topologically free* if, given any open subspace U and any $1 \neq g \in G$, there exists $x \in U$ with $g \notin \text{Stab}(x)$. If θ is topologically free, then $C_0(\Omega) \rtimes_r G$ has the *intersection property* by [25, Theorem 29.5], i.e. any non-zero ideal $J \triangleleft C_0(\Omega) \rtimes_r G$ has non-trivial intersection $C_0(U) = J \cap C_0(\Omega) \neq \{0\}$. It follows that J contains the non-zero induced ideal $I := C_0(U) \rtimes_r G$, but it does not tell us anything about the quotient J/I. The problem arises when the restriction of θ to $Z := \Omega \setminus U$ is not topologically free, and a partial action is said to be *essentially free* if all such restrictions are topologically free. However, while topological freeness of $\theta^{(E,C)}$ is quite frequent (we recall the characterisation given in [7] just below), essential freeness is extremely rare as shown in [38]. In this section, we will introduce a weakening of topological freeness that still allows one to obtain information about the ideals, and show that it is always enjoyed by $\theta^{(E,C)}$. In particular, the restriction $\theta^{(E,C)}|_Z$ to any closed invariant subspace Z of finite type will also have this property. From these observations, we can completely characterize the structure of finite type ideals of $\mathcal{O}^r(E,C)$. It is also worth noting that our methods, when applied to non-separated graphs, yield a complete description of the ideals of the graph C^*-algebra.

We first recall *Condition* (*L*) of [7], using a slightly different terminology:

Definition 7.2. Consider any finite bipartite separated graph (E,C). A vertex $v \in E^0$ is said to *admit a choice* if there exists an admissible path $\beta = e\alpha$ with $s(\beta) = v$, and an element $X_e \neq X \in C_{r(e)}$ with $|X| \geq 2$. The graph is said to satisfy Condition (L) if for every simple cycle σ, the base vertex $s(\sigma)$ admits a choice.

Theorem 7.3 ([7, Theorem 10.5])**.** *Let* (E,C) *denote a finite, bipartite separated graph. Then* $\theta^{(E,C)}$ *is topologically free if and only if* (E,C) *satisfies Condition* (*L*)*.*

We now introduce the appropriate weakening of topological freeness.

Definition 7.4. Let $\theta\colon G \curvearrowright \Omega$ be any partial action of a discrete group on a locally compact Hausdorff space. For any $x \in \Omega$, we shall say that θ is

- *topologically free in* x if, given any $1 \neq g \in \mathrm{Stab}(x)$ and an open neighbourhood U of x, there exists $y \in U$ with $g \notin \mathrm{Stab}(y)$.
- *strongly topologically free in* x if, given any $1 \neq g_1, \dots, g_n \in \mathrm{Stab}(x)$ and any open neighbourhood U of x, there exists $y \in U$ with $g_1, \dots, g_n \notin \mathrm{Stab}(y)$.

We will denote by Ω^{TF} the set of points $x \in \Omega$ in which θ is topologically free. If θ is strongly topologically free in every $x \in \Omega^{\mathrm{TF}}$, it is said to be *relatively strongly topologically free*.

Observe that if x is an interior point of Ω^{TF}, then θ is automatically strongly topologically free in x. In particular, topologically free partial actions are strongly topologically free in all points.

Remark 7.5. We will now expand a bit on the situation for $\theta^{(E,C)}$, and we first borrow a bit of graph theory from [38, Section 3]. If $v \in E^0$ does not admit a choice, every closed path α based at v has a unique word decomposition $\alpha = \gamma^{-1}\beta\gamma$ for a (possibly trivial) admissible path γ and a cycle β, and we will say that α is a *simple closed path* if γ does not repeat a vertex, and β is a simple cycle. The set

$$\mathbb{F}_v := \{\text{closed paths based at } v\} \cup \{1\}$$

defines a subgroup $\mathbb{F}_v \leq \mathbb{F}$, and every closed path based at v is a reduced product of simple closed paths. It follows that such v admits a unique simple closed path (up to inversion) if and only if $\mathbb{F}_v \cong \mathbb{Z}$. Moreover, $\Omega(E,C)_v = \{\xi\}$ is a one-point set and $\mathrm{Stab}(\xi) = \mathbb{F}_v$, so $\xi \notin \Omega(E,C)^{\mathrm{TF}}$ if and only if v admits a closed path. It follows from the proof of [7, Theorem 10.5] that every $\eta \in \Omega(E,C) \setminus \Omega(E,C)^{\mathrm{TF}}$ is of the form $\eta = \theta_\gamma(\xi)$ for some $\gamma \in \xi$ with ξ as above – in particular, the complement $\Omega(E,C) \setminus \Omega(E,C)^{\mathrm{TF}}$ is discrete.

Before we progress any further, let us consider a somewhat trivial, but not uninteresting, example.

Example 7.6. If $\theta\colon G \curvearrowright \Omega$ is any partial action, and $x \in \Omega^{\mathrm{TF}}$ satisfies $\mathrm{Stab}(x) \cong \mathbb{Z}$, then θ is automatically strongly topologically free in x. Indeed, given $1 \neq g_1, \ldots, g_n \in \mathrm{Stab}(x)$, we may write $g_i = g^{k_i}$ with g a generator of $\mathrm{Stab}(x)$, and we can safely assume that $k_i > 0$ for all i. For any open neighbourhood U of x, we consider $h := g^{k_1 \cdots k_n} \in G$ and pick $y \in U$ with $h \notin \mathrm{Stab}(y)$ using topological freeness in x. Then we obviously have $g_i \notin \mathrm{Stab}(y)$ as well, so θ is indeed strongly topologically free in x.

The main result of [39] is a characterisation of nuclearity and exactness of $\mathcal{O}^{(r)}(E,C)$ in terms of a graph theoretic *Condition* (N). Another equivalent condition is that every stabiliser of $\theta^{(E,C)}$ is either trivial or isomorphic to $\mathbb{Z}$, so from the above example we obtain the following proposition:

Proposition 7.7. *If (E,C) satisfies Condition (N), then the restriction of $\theta^{(E,C)}$ to any closed invariant subspace is relatively strongly topologically free.*

For general separated graphs, we can only handle restrictions to closed invariant subspaces of finite type.

Proposition 7.8. *If (E,C) is any finite bipartite separated graph, then $\theta^{(E,C)}$ is relatively strongly topologically free.*

Proof. Whenever we write $\beta\alpha$ for admissible paths α and β in the following, we shall mean the concatenated product, that is, we do not allow for cancellation. Now consider any $\xi \in \Omega(E,C)^{\mathrm{TF}}$ and $g_1, \ldots, g_n \in \mathrm{Stab}(\xi)$, assuming, without loss of generality, that no pair g_i, g_j of distinct elements generate a rank one subgroup. For every i, we may write $g_i = \mu_i^{-1}\sigma_i\mu_i$ for cycles σ_i. Since the action is topologically free in ξ, there exist admissible paths β_i and edges x_i with $|X_{x_i}| \geq 2$, such that $x_i^{-1}\beta_i\mu_i \in \xi$. Either $x_i^{-1}\beta_i\sigma_i$ or $x_i^{-1}\beta_i\sigma_i^{-1}$ must be admissible, and without loss of generality, we can assume the former. Now given any open neighbourhood U of ξ, we have

$$\xi \in \Omega(E,C)_B := \{\eta \in \Omega(E,C) \mid B \subset \eta\} \subset U$$

for a sufficiently big ball $B := \xi^N$. We may of course assume that $N > |x_i^{-1}\beta_i\mu_i|$ for all i. Picking $l_i \geq 1$ such that $|\sigma_i^{l_i}\mu_i| \geq N$ and $x_i \neq y_i \in X_{x_i}$, we then consider the set

$$\omega := B \cup \{y_i^{-1}\beta_i\sigma_i^{l_i}\mu_i \mid i = 1, \ldots, n\}.$$

It should be clear that there exists $\eta \in \Omega(E, C)$ with $\omega \subset \eta$, so in particular $\eta \in U$. Finally, observe that $g_i \notin \operatorname{Stab}(\eta)$ by construction since $x_i^{-1}\beta_i\mu_i \in B \subset \eta$ and

$$x_i^{-1}\beta_i\mu_i \notin \eta \cdot \mu_i^{-1}\sigma_i^{-l_i}\mu_i = \theta_{g_i^{l_i}}(\eta),$$

so θ is indeed strongly topologically free in ξ. □

Now let us instead consider a non-example.

Example 7.9. For $n \in \mathbb{Z}$, we define $f_n^1, f_n^2 \in \{0,1\}^{\mathbb{Z}^2}$ by

$$f_n^1(a,b) = \begin{cases} 0 & \text{if } b > n \\ 1 & \text{if } b \leq n \end{cases} \quad \text{and} \quad f_n^2(a,b) = \begin{cases} 0 & \text{if } a > n \\ 1 & \text{if } a \leq n \end{cases}.$$

Then the $\mathbb{Z}^2$-shift on $\{0,1\}$ restricts to an action $\theta\colon \mathbb{Z}^2 \curvearrowright \Omega$ on the compact Hausdorff space $\Omega := \{f_n^1, f_n^2, 0, 1 \mid n \in \mathbb{Z}\}$. Every f_n^i is isolated with $f_n^i \to 0$ for $n \to -\infty$ and $f_n^i \to 1$ for $n \to \infty$. By construction, $\operatorname{Stab}(f_n^1) = \mathbb{Z} \oplus 0$ and $\operatorname{Stab}(f_n^2) = 0 \oplus \mathbb{Z}$ for all n, so $\Omega^{\mathrm{TF}} = \{0,1\}$. However, θ is not strongly topologically free in 0 or 1 since any f_n^i is either fixed by a or b. In conclusion, relative strong topological freeness is not automatic.

Remark 7.10. We have no examples of partial actions of free groups that are not relatively strongly topologically free, but we suspect that such examples exist. However, it is notable that whenever a partial action $\theta\colon \mathbb{F} \curvearrowright \Omega$ of a free group is topologically free in x, and we consider only two elements $g_1, g_2 \in \operatorname{Stab}(x)$, then given any open neighbourhood U of x we *can* find $y \in U$ with $g_1, g_2 \notin \operatorname{Stab}(y)$. Indeed, we may assume that g_1 and g_2 do not generate a free subgroup of rank one, so that the commutator $[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2$ is non-trivial. Let V_i be an open neighbourhood of x so that $\theta_{g_i}(V_i) \subset U$ and consider $V := U \cap V_1 \cap V_2$. Applying topological freeness with respect to $[g_1, g_2]$, we obtain $y \in V$ with $[g_1, g_2] \notin \operatorname{Stab}(y)$. Obviously, we cannot have $g_1, g_2 \in \operatorname{Stab}(y)$, and if $g_1, g_2 \notin \operatorname{Stab}(y)$, then we are done. We may therefore assume that exactly one of g_1 and g_2 belongs to $\operatorname{Stab}(y)$. If $g_1 \in \operatorname{Stab}(y)$, we consider $y \neq y' := \theta_{g_2}(y) \in U$ instead. Observe that if $g_1 \in \operatorname{Stab}(y')$, then

$$\theta_{[g_1,g_2]}(y) = \theta_{g_1^{-1}g_2^{-1}g_1}(y') = \theta_{g_1^{-1}g_2^{-1}}(y') = \theta_{g_1^{-1}}(y) = y,$$

which contradicts our choice of y. We conclude that $g_1, g_2 \notin \operatorname{Stab}(y')$ as desired. In case $g_2 \in \operatorname{Stab}(y)$, we apply the exact same argument with g_1 and g_2 reversed: this is possible since $[g_2, g_1] = [g_1, g_2]^{-1} \notin \operatorname{Stab}(y)$.

It is evident from the definition that $\Omega \setminus \Omega^{\mathrm{TF}}$ is an open invariant subspace, so there is a corresponding “obstruction ideal”:

Definition 7.11. Given a partial action $\theta\colon G \curvearrowright \Omega$ of a discrete group on a locally compact Hausdorff space, we define an ideal $J^o = J^o_\theta := C_0(\Omega \setminus \Omega^{\mathrm{TF}}) \rtimes_r G$ of $C_0(\Omega) \rtimes_r G$.

We now put relative strong topological freeness to work. The proof of the following theorem is modelled over that of [25, Theorem 29.5], but the statement is somewhat more general.

Theorem 7.12. *Let $\theta\colon G \curvearrowright \Omega$ denote a partial action of a discrete group G on a locally compact Hausdorff space. Suppose that G is exact and that θ is relatively strongly topologically free. If $J \cap C_0(\Omega) = \{0\}$ for an ideal $J \lhd C_0(\Omega) \rtimes_r G$, then $J \subset J^o$.*

Proof. Assuming that $J \not\subset J^o$, we take any $a \in J \setminus J^o$ and consider $w := a^*a \in J \setminus J^o$. Now consider the commutative diagram

$$\begin{array}{ccc} C_0(\Omega) \rtimes_r G & \xrightarrow{\;p_*\;} & C_0(\Omega^{\mathrm{TF}}) \rtimes_r G \\ E \big\downarrow & & \big\downarrow F \\ C_0(\Omega) & \xrightarrow[\;p\;]{} & C_0(\Omega^{\mathrm{TF}}) \end{array},$$

where E and F are the canonical conditional expectations. By exactness of G, we have $\ker(p_*) = J^o$, hence $p_*(w) \neq 0$. Faithfulness of F then implies

$$p(E(w)) = F(p_*(w)) \neq 0,$$

so $f := E(w)$ attains a non-zero value on Ω^{TF}. Let $x_0 \in \Omega^{\mathrm{TF}}$ be such that $|f(x_0)| = \sup_{x\in\Omega^{\mathrm{TF}}} |f(x)|$ and take any $0 < \varepsilon < \frac{|f(x_0)|}{2}$. By Urysohn's Lemma (applied to the one point compactification of Ω), there exists $u \in C_0(\Omega)$ with $0 \leq u \leq 1$, $u(x_0) = 1$ and $u(x) = 0$ whenever $|f(x)| \geq |f(x_0)| + \varepsilon/4$. Consider $z := uw \in J \setminus J^o$ and note that $E(z) = uE(w) = uf$, hence $|f(x_0)| \leq \|E(z)\| \leq |f(x_0)| + \varepsilon/4$. We claim that there exists a function $h \in C_0(\Omega)$ such that

(1) $0 \leq h \leq 1$,
(2) $\|hE(z)h\| > \|E(z)\| - \varepsilon$,
(3) $\|hzh - hE(z)h\| < \varepsilon$.

To see this, we first pick $b \in C_0(\Omega) \rtimes_{\mathrm{alg}} G$ with $\|z - b\| < \varepsilon/4$ and write $b = b_1 + \sum_{g\in T} b_g\delta_g$ for a finite set $T \subset G\setminus\{1\}$. If $T = \emptyset$, then the claim is easily verified, so we may assume that $T \neq \emptyset$. Since θ is strongly topologically free in x_0, we can find $x_1 \in \Omega$ with $|b_1(x_1) - b_1(x_0)| < \varepsilon/4$ and $T \cap \mathrm{Stab}(x_1) = \emptyset$. We may apply [25, Lemma 29.4] for every $g \in T$ to obtain $h_g \in C_0(\Omega)$ with $0 \leq h_g \leq 1$, $h_g(x_1) = 1$ and $\|h_g(b_g\delta_g)h_g\| < \frac{\varepsilon}{2|T|}$. Setting $h := \prod_{g\in T} h_g$, (2) then follows from the calculation

$$\begin{aligned} \|hE(z)h\| &> \|hE(b)h\| - \varepsilon/4 = \|hb_1h\| - \varepsilon/4 \\ &\geq |h(x_1)b_1(x_1)h(x_1)| - \varepsilon/4 = |b_1(x_1)| - \varepsilon/4 \\ &> |b_1(x_0)| - \varepsilon/2 > |E(z)(x_0)| - 3\varepsilon/4 = |f(x_0)| - 3\varepsilon/4 \\ &\geq \|E(z)\| - \varepsilon. \end{aligned}$$

In order to check (3), we first observe that

$$\|hbh - hb_1h\| = \|\sum_{g\in T} h(b_g\delta_g)h\| \leq \sum_{g\in T} \|h_g(b_g\delta_g)h_g\| < \varepsilon/2,$$

hence

$$\|hzh - hE(z)h\| \le \|hzh - hbh\| + \|hbh - hb_1h\| + \|hb_1h - hE(z)h\| < \varepsilon.$$

Having verified the claim, we let π denote the quotient map $C_0(\Omega) \rtimes_r G \to (C_0(\Omega) \rtimes_r G)/J$. Since $z \in J$ and $J \cap C_0(\Omega) = 0$, we have

$$\begin{aligned}\|E(z)\| &< \|hE(z)h\| + \varepsilon = \|\pi(hE(z)h - hzh)\| + \varepsilon \\ &\le \|hE(z)h - hzh\| + \varepsilon < 2\varepsilon.\end{aligned}$$

But at the same time, $\|E(z)\| \ge |f(x_0)| > 2\varepsilon$, a contradiction. □

We need to specialize the above theorem a bit before we can apply it to our setting. The following is an ever useful observation.

Lemma 7.13. *Let $\theta\colon G \curvearrowright \Omega$ denote a partial action of a discrete group on a locally compact Hausdorff space. If $x \in \Omega$ is isolated, then*

$$1_x(C_0(\Omega) \rtimes_{(r)} G)1_x \cong C^*_{(r)}(\mathrm{Stab}(x)),$$

where 1_x denotes the indicator function in x.

Proof. By [9, Proposition 6.1 and Corollary 6.3], we have embeddings

$$C^*_{(r)}(\mathrm{Stab}(x)) \cong C(\{x\}) \rtimes_{(r)} \mathrm{Stab}(x) \hookrightarrow C_0(\Omega) \rtimes_{(r)} \mathrm{Stab}(x) \hookrightarrow C_0(\Omega) \rtimes_{(r)} G,$$

and the composition clearly maps onto the corner $1_x(C_0(\Omega) \rtimes_{(r)} G)1_x$. □

Definition 7.14. Let $\theta\colon G \curvearrowright \Omega$ denote a partial action on a locally compact Hausdorff space, and suppose that $\mathcal{U} \subset \mathbb{O}^G(\Omega)$ is a collection of open invariant subsets $U \subset \Omega$ for which the restriction $\theta|_{Z_U}$ to $Z_U := \Omega \setminus U$ satisfies the following two conditions:

(1) $\theta|_{Z_U}$ is relatively strongly topologically free,
(2) the space $W_U := Z_U \setminus Z_U^{\mathrm{TF}}$ is discrete.

Observe that if $U \subset V$ for $U, V \in \mathcal{U}$, then $W_U \setminus V \subset W_V$. For any U, we may therefore choose a set of representatives Λ_U for the orbit space W_U/G such that $\Lambda_U \setminus V \subset \Lambda_V$ whenever $U \subset V$. We then introduce a set

$$\mathcal{I}_{\mathcal{U}}(\theta) := \Big\{ \big(U, (I_U^x)_{x\in\Lambda_U}\big) \mid U \in \mathcal{U}, I_U^x \text{ is a proper ideal of } C_r^*(\mathrm{Stab}(x)) \Big\}$$

and equip it with the partial order

$$\big(U, (I_U^x)_{x\in\Lambda_U}\big) \le \big(V, (I_V^x)_{x\in\Lambda_V}\big) \Leftrightarrow U \subset V \text{ and } I_U^x \subset I_V^x \text{ for all } x \in \Lambda_U \setminus V.$$

For notational simplicity, we will usually write $I_U^\bullet = (I_U^x)_{x\in\Lambda_U}$, and we finally write $\mathcal{I}_{\mathcal{U}}(C_0(\Omega) \rtimes_r G)$ for the collection of ideals $J \lhd C_0(\Omega) \rtimes_r G$ satisfying $J \cap C_0(\Omega) = C_0(U)$ for some $U \in \mathcal{U}$.

Corollary 7.15. *Let $\theta\colon G \curvearrowright \Omega$ denote a partial action of an exact group on a locally compact Hausdorff space with the setup from Definition 7.14. Then there is a canonical order isomorphism*

$$\mathcal{I}_{\mathcal{U}}(\theta) \to \mathcal{I}_{\mathcal{U}}(C_0(\Omega) \rtimes_r G), \quad (U, I_U^\bullet) \mapsto J(U, I_U^\bullet),$$

with the following properties:

(1) $J(U, I_U^\bullet) \cap C_0(\Omega) = C_0(U)$.
(2) *The quotient* $J(U, I_U^\bullet)/(C_0(U) \rtimes_r G)$ *is canonically Morita equivalent to* $\bigoplus_{x\in\Lambda_U} I_U^x$.

Proof. Let us write $J_U^o := J^o_{\theta|_{Z_U}}$. By Definition 7.14(2), every orbit in W_U is clopen, so there is a canonical identification

$$J_U^o \cong \bigoplus_{x\in\Lambda_U} C_0(G.x) \rtimes_r G.$$

From Lemma 7.13, $C_r^*(\mathrm{Stab}(x))$ sits as a full corner of $C_0(G.x) \rtimes_r G$ for all $x \in \Lambda_U$, and we let $\tilde{I}_U^x$ denote the ideal generated by I_U^x inside $C_0(G.x) \rtimes_r G$. We may then define an ideal $\tilde{I}_U := \bigoplus_{x\in\Lambda_U} \tilde{I}_U^x \lhd J_U^o$, and letting π_U denote the quotient map $C_0(\Omega) \rtimes_r G \to C_0(Z_U) \rtimes_r G$, we finally set $J(U, I_U^\bullet) := \pi_U^{-1}(\tilde{I}_U)$. We proceed to verify the properties of the map $(U, I_U^\bullet) \mapsto J(U, I_U^\bullet)$, assuming that $U \subset V$. Observe that the quotient map

$$\pi_{U,V} \colon C_0(Z_U) \rtimes_r G \to C_0(Z_V) \rtimes_r G$$

maps $C_r^*(G.x) \rtimes_r G$ to $\{0\}$ if $x \in V$, and that it maps $C_0(G.x) \rtimes_r G \subset J_U^o$ identically to $C_0(G.x) \rtimes_r G \subset J_V^o$ otherwise. It follows that $(U, I_U^\bullet) \leq (V, I_V^\bullet)$ if and only if $J(U, I_U^\bullet) \subset J(V, I_V^\bullet)$. Both (1) and (2) should be obvious from the above, and injectivity is immediate from these. To see that our map is surjective, take any ideal $J \lhd C_0(\Omega) \rtimes_r G$ with $J \cap C_0(\Omega) = C_0(U)$ for $U \in \mathcal{U}$, and consider $J_U := J/(C_0(U) \rtimes_r G) \lhd C_0(Z_U) \rtimes_r G$. Since $\theta|_{Z_U}$ is assumed relatively strongly topologically free, we see that $J_U \subset J_U^o$ from Theorem 7.12. In particular, we may write $J_U \cong \bigoplus_{x\in\Lambda_U} \tilde{I}_U^x$ for ideals $\tilde{I}_U^x \lhd C_0(G.x) \rtimes_r G$. Observe that $\tilde{I}_U^x$ must be proper, for otherwise we would have $x \in U$. Setting $I_x^U := 1_x(\tilde{I}_U^x)1_x$, we finally have $J(U, I_U^\bullet) = J$. $\square$

We are almost ready to put our results to use at this point, but we still need to introduce some bookkeeping.

Construction 7.16. Consider any finite bipartite separated graph (E, C), and let (F_∞, D^∞) be its Bratteli diagram. We denote by $\mathfrak{C}(E, C)$ the set of vertices $v \in E^0$ which admit no choices and a simple cycle α such that every closed path based at v is a power of α, modulo the relation

$$u \sim v \Leftrightarrow u \text{ and } v \text{ belong to the same cycle.}$$

It follows from Remark 7.5 that $\mathfrak{C}(E, C)$ is in canonical bijective correspondence with the orbit space $W/\mathbb{F}$ for

$$W := \{\xi \in \Omega(E, C) \setminus \Omega(E, C)^{\mathrm{TF}} \mid \mathrm{Stab}(\xi) \cong \mathbb{Z}\}.$$

Observe that there is a natural map $\mathfrak{C}(E, C) \to \mathfrak{C}(E_1, C^1)$ given by $[v] \mapsto [v]$ whenever $v \in E^{0,1}$, and that it is in fact a bijection. This can either be seen by direct arguments, by Theorem 3.22, or most easily by [38, Lemma 5.2]. Consequently, if $H \in \mathcal{H}_{\mathrm{fin}}(F_\infty, D^\infty)$ satisfies $H = H^n$, we may identify $\mathfrak{C}(E_n/H^{(n)}, C^n/H^{(n)})$ with $\mathfrak{C}(E_m/H^{(m)}, C^m/H^{(m)})$ whenever $m \geq n$; formally, we do this by setting

$$\mathfrak{C}(H) := \varinjlim_{m\geq n} \mathfrak{C}(E_m/H^{(m)}, C^m/H^{(m)}).$$

Observe that whenever $H_1 \subset H_2$ for $H_1, H_2 \in \mathcal{H}_{\text{fin}}(F_\infty, D^\infty)$, we have an inclusion

$$\{\mathfrak{c} \in \mathfrak{C}(H_1) \mid \mathfrak{c} \not\subset H_2\} \subset \mathfrak{C}(H_2).$$

Indeed, whenever $m \geq n$ for sufficiently large n, we have representations

$$\mathfrak{C}(H_1) = \mathfrak{C}(E_m/H_1^{(m)}, C^m/H_1^{(m)}) \quad \text{and} \quad \mathfrak{C}(H_1) = \mathfrak{C}(E_m/H_2^{(m)}, C^m/H_2^{(m)}),$$

and $H_1^{(m)} \subset H_2^{(m)}$. Consequently, there in an inclusion

$$\{\mathfrak{c} \in \mathfrak{C}(E_m/H_1^{(m)}, C^m/H_1^{(m)}) \mid \mathfrak{c} \not\subset H_2^{(m)}\} \subset \mathfrak{C}(E_m/H_2^{(m)}, C^m/H_2^{(m)}),$$

and this does not depend on m. Letting $\mathbb{O}_p(\mathbb{T})$ denote the collection of proper open subsets of $\mathbb{T}$, we finally define a set

$$\mathcal{I}_{\text{fin}}(E, C) = \big\{(H, T) \mid H \in \mathcal{H}_{\text{fin}}(F_\infty, D^\infty), T \in \mathbb{O}_p(\mathbb{T})^{\mathfrak{C}(H)}\big\}$$

and equip it with the partial ordering

$$(H_1, T_1) \leq (H_2, T_2) \Leftrightarrow H_1 \subset H_2 \text{ and } T_1(\mathfrak{c}) \subset T_2(\mathfrak{c}) \text{ for all } \mathfrak{c} \in \mathfrak{C}(H_1) \text{ with } \mathfrak{c} \not\subset H_2.$$

Theorem 7.17. *For any finite bipartite separated graph (E, C), there is a canonical lattice isomorphism*

$$\mathcal{I}_{\text{fin}}(E, C) \to \mathcal{I}_{\text{fin}}(\mathcal{O}^r(E, C)) \quad , \quad (H, T) \mapsto I(H, T),$$

with the following properties:

1. $H_{I(H,T)} = H$.
2. *The quotient $I(H, T)/I(H)$ is Morita equivalent to $\bigoplus_{\mathfrak{c} \in \mathfrak{C}(H)} C_0(T(\mathfrak{c}))$.*

In particular, a finite type ideal of $\mathcal{O}^r(E, C)$ is generated by its projections if and only if it is induced.

Proof. Let $\mathcal{U} := \{\Omega(E, C)^H \mid H \in \mathcal{H}_{\text{fin}}(F_\infty, D^\infty)\}$ and observe that Corollary 7.15 may be applied thanks to Remark 7.5, Proposition 7.8 and Theorem 7.12. Since $\mathbb{F}_n$ is C^*-simple for all $n \geq 2$ [43], we see that $I_U^x = 0$ whenever $\text{Stab}(x) \not\cong \mathbb{Z}$, and proper ideals of $C_r^*(\mathbb{Z}) \cong C(\mathbb{T})$ correspond to proper open subsets $T \subset \mathbb{T}$. Finally, if $H = H^n$ and $U = \Omega(E, C)^H$, then the orbits of points $\xi \in W_U$ with stabiliser $\mathbb{Z}$ correspond canonically to the elements of $\mathfrak{C}(H)$. The above statement is therefore exactly the conclusion that can be drawn from Corollary 7.15. □

Remark 7.18. We have no hope of achieving a similar result for arbitrary ideals of $\mathcal{O}^r(E, C)$ except in special cases. Indeed, we suspect that for an infinite type subspace Z, the restriction $\theta^{(E,C)}|_Z \colon \mathbb{F} \curvearrowright Z$ need not be relatively strongly topologically free, and one can easily find examples where the space $Z \setminus Z^{\text{TF}}$ is not discrete.

We finally apply our results to classical graph C^*-algebras to provide a new proof for the description of the ideal lattice first obtained by Hong and Szymański in [32]. We first recall a bit of terminology and a few results.

Definition 7.19 ([16])**.** Let E denote any directed graph and denote the set of hereditary and saturated subsets by $\mathcal{H}(E)$. For any $H \in \mathcal{H}(E)$, there is a set of *breaking vertices* for H given by

$$H_\infty^{\text{fin}} := \{v \in E^0 \setminus H \colon |r^{-1}(v)| = \infty \text{ and } 0 < |r^{-1}(v) \cap s^{-1}(E^0 \setminus H)| < \infty\},$$

and pairs (H, B) with $B \subset H_\infty^{\text{fin}}$ are called *admissible*. For any such pair, one defines a quotient graph $E/(H, B)$ as

$$(E/(H,B))^0 := (E^0 \setminus H) \cup \{\beta(v) \mid v \in H_\infty^{\text{fin}} \setminus B\},$$
$$(E/(H,B))^1 := s^{-1}(E^0 \setminus H) \cup \{\beta(e) \mid e \in E^1, s(e) \in H_\infty^{\text{fin}} \setminus B\}$$

with r, s extended by $r(\beta(e)) := r(e)$ and $s(\beta(e)) := \beta(s(e))$. We finally order the admissible pairs (H, B) by

$$(H_1, B_1) \le (H_2, B_2) \Leftrightarrow H_1 \subset H_2 \text{ and } B_1 \setminus B_2 \subset H_2.$$

Theorem 7.20. *Let E denote any directed graph. There exists a partial action $\theta^E \colon \mathbb{F} \curvearrowright \partial E$ of $\mathbb{F} := \mathbb{F}(E^1)$ on a totally disconnected, locally compact Hausdorff space ∂E with the following properties:*

(1) $C^*(E) \cong C_0(\partial E) \rtimes \mathbb{F}(E^1)$ *canonically.*
(2) *There is a canonical lattice isomorphism*

$$\{(H, B) \mid H \in \mathcal{H}(E) \text{ and } B \subset H_\infty^{\text{fin}}\} \to \mathbb{O}^{\mathbb{F}}(\partial E), \quad (H, B) \mapsto U(H, B),$$

and $\theta^E|_{\partial E \setminus U(H,B)} \approx \theta^{E/(H,B)}$ *for every admissible pair (H, B). In particular,*

$$C^*(E)/I(H, B) \cong C^*(E/(H, B)),$$

where $I(H, B)$ is the ideal induced from $U(H, B)$.

Proof. (1) is proven in [25, Chapter 37], and (2) follows from the description of gauge-invariant ideals in [16] (one can also prove (2) directly with fairly little work). □

As for the separated graphs, we have to introduce some bookkeeping:

Definition 7.21. For any directed graph E, we let $\mathfrak{C}(E)$ denote the set of vertices $v \in E^0$ which admit a cycle without any entries, modulo the relation

$$u \sim v \Leftrightarrow u \text{ and } v \text{ belong to the same cycle.}$$

Observe that if (H, B) is an admissible pair, then $\mathfrak{C}(E/(H, B)) = \mathfrak{C}(E/H)$ since the additional vertices in E/H are all sources. Given any $H \in \mathcal{H}(E)$, we set $\mathfrak{C}(H) := \mathfrak{C}(E/H)$ and observe that if $H_1 \subset H_2$, then we have an inclusion

$$\{\mathfrak{c} \in \mathfrak{C}(H_1) \mid \mathfrak{c} \not\subset H_2\} \subset \mathfrak{C}(H_2).$$

We may in turn define a lattice

$$\mathcal{I}(E) := \{(H, B, T) \mid H \in \mathcal{H}, B \subset H_\infty^{\text{fin}}, T \in \mathbb{O}_p(\mathbb{T})^{\mathfrak{C}(H)}\}$$

ordered by

$$(H_1, B_1, T_1) \le (H_2, B_2, T_2) \Leftrightarrow (H_1, B_1) \le (H_2, B_2) \text{ and } T_1(\mathfrak{c}) \subset T_2(\mathfrak{c}) \quad \text{for all } \mathfrak{c} \in \mathfrak{C}(H_1) \text{ with } \mathfrak{c} \not\subset H_2.$$

Finally, we denote the ideal lattice of $C^*(E)$ by $\mathcal{I}(C^*(E))$, and given any $J \in \mathcal{I}(C^*(E))$, we write (H_J, B_J) for the admissible pair satisfying $J \cap C_0(\partial E) = C_0(U(H_J, B_J))$.

Theorem 7.22. *For any directed graph E, there is a canonical lattice isomorphism*

$$\mathcal{I}(E) \to \mathcal{I}(C^*(E)), \quad (H, B, T) \mapsto I(H, B, T),$$

with the following properties:

(1) *$H_{I(H,B,T)} = H$ and $B_{I(H,B,T)} = B$.*
(2) *The quotient $I(H, B, T)/I(H, B)$ is Morita equivalent to $\bigoplus_{\mathfrak{c} \in \mathfrak{C}(H,B)} C_0(T(\mathfrak{c}))$.*

Proof. Let $\mathcal{U} := \mathbb{O}^{\mathbb{F}}(\partial E)$ and observe that Corollary 7.15 applies due to Theorem 7.20 and Example 7.6, since any stabiliser is either trivial or isomorphic to $\mathbb{Z}$. For any $U = U(H, B) \in \mathcal{U}$, the orbits of points $x \in W_U$ with $\text{Stab}(x) \cong \mathbb{Z}$ correspond to the elements of $\mathfrak{C}(H)$, so Corollary 7.15 reduces to the above statement. □

Remark 7.23. Observe that Theorem 7.12 also provides a new proof of Szymański's general Cuntz-Krieger Uniqueness Theorem [50] when applied to the boundary path space action θ^E.

Remark 7.24. We finally remark that Theorem 7.12 has an analogue for algebraic crossed products $C_K(\Omega) \rtimes G$, where $C_K(\Omega)$ denotes the algebra of compactly supported locally constant function $\Omega \to K$ when K is given the discrete topology. The assumptions will have to be slightly different: on one hand, there is no need for exactness of the group, but on the other hand, one needs the space Ω to be totally disconnected to have sufficiently many continuous functions. The proof should be a bit simpler, although one will have to avoid C^*-techniques. The description of the lattice of ideals of the Leavitt path algebra $L_K(E)$ of an arbitrary graph E, obtained in [3, Theorem 2.8.10], can also be obtained using this approach.

8. $\mathcal{V}$-SIMPLICITY

We continue our investigation of the ideal structure with the study of $\mathcal{V}$-simplicity, that is, we want to compute the algebras of graphs (E, C) for which the monoid $M(F_\infty, D^\infty)$ is order simple. By Theorem 4.7, this is equivalent to saying that $I \cap C(\Omega(E, C)) = 0$ for every proper ideal I of $\mathcal{O}^r(E, C)$, or $\mathcal{H}(F_\infty, D^\infty) = \{\emptyset, F_\infty^0\}$. We will simply say that (E, C) is *simple* in this case, and any of the algebras $\mathcal{O}(E, C)$, $\mathcal{O}^r(E, C)$ and $L^{\text{ab}}(E, C)$ will be called *$\mathcal{V}$-simple.* The second named author proves similar results in [38, Section 4] by studying minimality of the partial action $\theta^{(E,C)}$; our study here, on the other hand, is purely graph-theoretical.

We state the main result of this section right away:

Theorem 8.1. *Let (E, C) be a simple finite bipartite separated graph. Then $L_K(E, C) = L_K^{\text{ab}}(E, C)$ and $C^*(E, C) = \mathcal{O}(E, C)$, and one of the following holds:*

(1) $L_K(E,C)$ *is isomorphic to a simple Leavitt path algebra, and* $C^*(E,C)$ *is isomorphic to a simple graph* C^**-algebra of a non-separated graph.*
(2) $L_K(E,C)$ *is Morita-equivalent to* $K[\mathbb{F}_n]$ *and* $C^*(E,C)$ *is Morita-equivalent to* $C^*(\mathbb{F}_n)$*, where* $\mathbb{F}_n$ *is a free group of rank* n *with* $1 \le n < \infty$*.*

In the latter case, $\mathcal{O}^r(E,C)$ *is Morita equivalent to* $C^*_r(\mathbb{F}_n)$*.*

From Theorem 8.1 and the fact that the reduced group C*-algebra $C^*_r(\mathbb{F}_n)$ of a free group $\mathbb{F}_n$ of rank $n > 1$ is simple [43], we obtain:

Corollary 8.2. *Let* (E,C) *be a finite bipartite separated graph. If* $\mathcal{O}^r(E,C)$ *is* $\mathcal{V}$*-simple, then either* $\mathcal{O}^r(E,C)$ *is isomorphic to a simple* C^**-algebra, or it is Morita equivalent to* $C(\mathbb{T})$*.*

We develop the proof in various steps. We begin with a simple observation.

Lemma 8.3. *Let* (E,C) *be a finite bipartite separated graph. Then* (E,C) *is simple if and only if* $\mathcal{H}(E_n,C^n) = \{\emptyset, E_n^0\}$ *for all* $n \ge 0$*.*

Proof. Let $n \ge 0$ be given. It follows from Lemma 5.3 that there is an injective order-preserving map $\mathcal{H}(E_n,C^n) \to \mathcal{H}(F_\infty,D^\infty)$, so $\mathcal{H}(E_n,C^n)$ is trivial if $\mathcal{H}(F_\infty,D^\infty)$ is trivial.

Conversely assume that $\mathcal{H}(E_n,C^n)$ is trivial for all $n \ge 0$. If $H \in \mathcal{H}(F_\infty,D^\infty)$, then $H = \cup_{n=0} H^{(n)}$, where $H^{(n)} := H \cap E_n^0 \in \mathcal{H}(E_n,C^n)$ for all n. If $H^{(n)} = E_n^0$ for some n, then $H = F_\infty^0$. Otherwise $H^{(n)} = \emptyset$ for all n, so $H = \emptyset$. □

A useful property of the separated graphs (E_n,C^n) associated to a finite bipartite separated graph (E,C) is the following: if $X \in C_v^n$ and $n \ge 1$, then $s(x) \ne s(y)$ whenever x,y are different elements of X. This follows immediately from the definition of these graphs.

We now obtain some necessary conditions for $\mathcal{V}$-simplicity.

Lemma 8.4. *Let* (E,C) *be a simple finite bipartite separated graph. Then for all* $v \in F_\infty^0$*, there is at most one* $X \in D_v^\infty$ *such that* $|X| > 1$*.*

Proof. It suffices to show the result for all $v \in E^{0,0}$. Suppose there exist distinct X,Y in C_v such that $|X| > 1$ and $|Y| > 1$. Take a vertex

$$w := v(x_1,y_1,z_1,\dots,z_t) \in E_1^{0,1},$$

where $x_1 \in X$, $y_1 \in Y$ and $z_i \in Z_i$, where $C_v = \{X,Y,Z_1,\dots,Z_t\}$. The singleton $\{w\}$ is then hereditary and C^1-saturated in E_1, because the elements X of C^1 having edges that start at w are of the form $X = X(x)$, where $x \in \{x_1,y_1,z_1,\dots,z_t\}$, and so they all have more than one element, and moreover the sources of the vertices of X are all different. So $\mathcal{H}(E_1,C^1)$ is non-trivial, contradicting Lemma 8.3. □

At this point we can already show the coincidence between the universal graph algebras and their tame quotients.

Corollary 8.5. *If* (E,C) *is a simple finite bipartite separated graph, then the natural maps* $L_K(E,C) \to L_K^{\mathrm{ab}}(E,C)$ *and* $C^*(E,C) \to \mathcal{O}(E,C)$ *are isomorphisms.*

Proof. Let $n \geq 0$. By Lemma 8.4, we have $[ee^*, ff^*] = 0$ for $e, f \in E^1_n$. Therefore it follows from [7, Theorem 5.1(a)] that the maps $L_K(E_n, C^n) \to L_K(E_{n+1}, C^{n+1})$ and $C^*(E_n, C^n) \to C^*(E_{n+1}, C^{n+1})$ are isomorphisms. Since this holds for each $n \geq 0$, we get $L_K(E,C) \cong L^{\rm ab}_K(E,C)$ and $C^*(E,C) \cong \mathcal{O}(E,C)$. □

Lemma 8.6. *Let (E,C) be a simple finite bipartite separated graph. Then, for all $w \in E^{0,1}_n$ with $n \geq 1$, there exists $X \in C^n$ such that $|X| = 1$ and $s(X) = \{w\}$.*

Proof. The only point where we use that $n \geq 1$ is the property that all the source vertices of edges coming from the same set $X \in C$ are distinct. Thus, it suffices to prove the result for an arbitrary finite bipartite separated graph (E,C) such that, for every $X \in C$, we have $s(x) \neq s(y)$ whenever x, y are distinct elements of X. Using this condition, we get that if $w \in E^{0,1}$ and $|X| > 1$ for all $X \in C$ such that $w \in s(X)$, then $\{w\}$ is a non-trivial hereditary and C-saturated subset of (E,C), which contradicts our hypothesis. □

Definition 8.7. Let (E,C) be a simple finite bipartite separated graph. A vertex $v \in E^{0,0}$ is said to be of *type A* in case there is a (unique) $X \in C_v$ such that $|X| > 1$, and v is said to be of *type B* in case $|X| = 1$ for all $X \in C_v$. Note that, by Lemma 8.4, every vertex $v \in E^{0,0}$ is either of type A or of type B.

If $v \in E^{0,0}$ is of type A, we denote by X^v the unique element in C_v having more than one element.

Lemma 8.8. *Let (E,C) be a simple finite bipartite separated graph. Then for each $w \in E^{0,1}$ there exists at most one $X \in C$ such that $|X| = 1$, $s(X) = \{w\}$, and $X \in C_v$ for a vertex $v \in E^{0,0}$ of type A.*

Proof. Suppose that X, Y are distinct, $X \in C_v$, $Y \in C_{v'}$, for v, v' vertices of type A, that $|X| = |Y| = 1$, and that $s(X) = s(Y) = \{w\}$. Let $X = \{x\}$ and $Y = \{y\}$. Then $X(x)$ and $X(y)$ are two distinct elements of C^1_w, and $|X(x)| > 1$, $|X(y)| > 1$ because there are $X' \in C_v$ and $Y' \in C_{v'}$ with $|X'| > 1$ and $|Y'| > 1$. This contradicts Lemma 8.4. □

We say that a type B vertex v is *of type* B_1 if, given any $w \in E^{0,1}$, there is at most one $X \in C_v$ such that $|X| = 1$ and $s(X) = \{w\}$. Say that v is type B_2 in case v is not type B_1.

The following definitions apply to a general finite bipartite separated graph (E,C). Recall that, for $e \in E^1$, we denote by X_e the unique element of C such that $e \in X_e$.

Definition 8.9. Let (E,C) be a finite bipartite separated graph. An admissible path γ is said to a 1-*path* in case all the edges $e \in E^1$ appearing in the path (with exponent ± 1) satisfy that $|X_e| = 1$. Length zero paths are also considered 1-paths. Two vertices v, w of E are said to be 1-connected, denoted by $v \sim w$, if there is a 1-path from v to w. A 1-cycle is a 1-path which is also a cycle.

Remark 8.10. Observe that the relation $\sim$ on E^0 is an equivalence relation. In fact one can easily show that if γ_1, γ_2 are 1-paths with $r(\gamma_1) = s(\gamma_2)$, then the reduced product $\gamma_2 \cdot \gamma_1$ (i.e. the path obtained after cancellation of terms ee^{-1} or $e^{-1}e$ in the concatenation of γ_2 and γ_1) is also a 1-path, which gives the transitivity of the relation $\sim$. It is obvious that $\sim$ is symmetric and reflexive.

Lemma 8.11. *Let (E,C) be a simple finite bipartite separated graph. The following hold:*

(a) *Two distinct vertices of type A are not 1-connected.*
(b) *The vertices of type A and the vertices of type B_2 are not 1-connected.*
(c) *Let v be a vertex of type A and let w a vertex of type B such that there is a 1-cycle based at w. Then v is not 1-connected to w.*

Proof. (a) Let v, v' be two distinct vertices of type A. We show that v is not 1-connected to v' by induction on the length of a minimal 1-path between v and v'. Suppose that there is a path $e_2e_1^{-1}$ from v to v' such that $|X_{e_i}| = 1$ for $i = 1, 2$. Let w be the vertex in $E^{0,1}$ such that $\{w\} = s(X_{e_1}) = s(X_{e_2})$. Then $X(e_1)$ and $X(e_2)$ are two different elements of C^1_w having more than one element each, contradicting Lemma 8.4.

Assume that there are no 1-paths of length $\leq 2(n-1)$ between two vertices of type A, and let γ be a path of length $2n$ between two vertices v, v' of type A. Write $\gamma = e_{2n}e_{2n-1}^{-1}\cdots e_2e_1^{-1}$, where each X_{e_i} has only one element (namely e_i). Then the vertices w, w' of E_1 such that $\{w\} = s(X_{e_1})$ and $\{w'\} = s(X_{e_{2n}})$ are of type A in E_1, because v and v' are of type A (in E), and moreover there is a 1-path in E_1 from w to w' of length $2(n-1)$, leading to a contradiction. This shows that there is no 1-path of length $\leq 2n$ between distinct vertices of type A.

(b) Let v be a vertex of type A and let v' be a vertex of type B_2. We show that v is not 1-connected to v' by induction on the length of a minimal 1-path between v and v'. Suppose that there is a path $e_2e_1^{-1}$ from v to v' such that $|X_{e_i}| = 1$ for $i = 1, 2$. Then in the separated graph (E_1, C^1), we have a vertex w with $\{w\} = s(X_{e_1}) = s(X_{e_2})$ and $X, Y \in C^1_w$, where $X = X(e_1)$ and $Y = X(e_2)$. If there is $Z \in C_{v'}$ such that $Z \neq X_{e_2}$, $|Z| = 1$ and $s(Z) = s(X_{e_2})$, then setting $Z = \{z\}$, we get $X(z) \neq Y$ and $s(X(z)) = s(Y)$. Now let $Y = \{y'\}$ and $X(z) = \{z'\}$. Note that, since v is of type A, we have $|X| = |X(e_1)| > 1$. In conclusion, we have that C^1_w has at least three sets $X, Y, X(z)$, with $|X| > 1$, $|Y| = |X(z)| = 1$ and $s(Y) = s(X(z))$. Thus, in (E_2, C^2) we have a vertex $\{w'\} = s(Y) = s(X(z))$ with $C_{w'}$ containing two sets $X(y')$ and $X(z')$ with more than one element. This contradicts Lemma 8.4.

Assume now that there is no $Z \in C_{v'}$ such that $Z \neq X_{e_2}$, $|Z| = 1$ and $s(Z) = s(X_{e_2})$. Since v' is of type B_2 by hypothesis, there are two distinct sets $Z, T \in C_{v'}$ such that $|Z| = |T| = 1$ and $s(Z) = s(T)$. Now the vertex w is type A in (E_1, C^1), and the vertex $w' = s(Z) = s(T)$ is type B_2 in (E_1, C^1). Indeed, if w' is of type A, then there is an edge f from w' to a vertex $v'' \in E^{0,0}$ of type A such that $|X_f| = 1$, and then we could apply the argument in the above paragraph to the 1-path $f^{-1}z$, where $Z = \{z\}$, to arrive at a contradiction. Moreover there is a 1-path $e'_2(e'_1)^{-1}$ from w to w' such that $X(e_2) = \{e'_1\}$ and $X(z) = \{e'_2\}$. Set $T = \{t\}$. Then $|X(z)| = |X(t)| = 1$ and $s(X(z)) = s(X(t))$ in (E_1, C^1), so that we are in the same situation as before, but replacing (E, C) with (E_1, C^1). Consequently, we arrive at a contradiction with the fact that $\mathcal{H}(E_3, C^3)$ only contains the trivial subsets.

This shows the case where the length of the path is 2. If we have a path of length $2n$, then it gives rise to a path of length $2n-2$ between a vertex of type A and a vertex of type B_2 in the graph (E_2, C^2), leading to a contradiction.

(c) The proof is similar to the proof of (b), we leave the details to the reader. □

In the following lemma, we describe the $\mathcal{V}$-simple algebras corresponding to graphs with only vertices of type A.

Lemma 8.12. *Let (E,C) be a simple finite bipartite separated graph, and suppose that the source vertices of edges coming from the same set $X \in C$ are all distinct. If all the vertices in $E^{0,0}$ are of type A, then $L_K(E,C) = L_K^{\mathrm{ab}}(E,C)$ is isomorphic to a Leavitt path algebra $L_K(\overline{E})$, and $C^*(E,C) = \mathcal{O}(E,C)$ is isomorphic to a graph C^*-algebra $C^*(\overline{E})$ of a non-separated graph $\overline{E}$ with $\mathcal{H}(\overline{E}) = \{\emptyset, \overline{E}^0\}$.*

Proof. By Lemma 8.6 and Lemma 8.8, for each $w \in E^{0,1}$ there exists a unique $Y \in C$ such that $|Y| = 1$ and $s(Y) = \{w\}$. We denote by y_w the unique edge in such a group Y. It is clear that $E^1_- := E^1 \setminus \{y_w \mid w \in E^{0,1}\}$ and $E^1_+ := \{y_w \mid w \in E^{0,1}\}$ defines a non-separated orientation in the sense of [38, Definition 3.8], so $L_K(E,C) \cong L_K(\overline{E})$ and $C^*(E,C) \cong C^*(\overline{E})$ by [38, Proposition 3.11], where $\overline{E}$ is the graph obtained from (E,C) by inverting all the edges of E^1_+. Moreover, $\overline{E}$ contains no hereditary and saturated subsets since (E,C) contains no hereditary and C-saturated subsets. □

Lemma 8.13. *Let (E,C) be a simple finite bipartite separated graph. Assume that $v \in E^0$ is of type B, and that v is not 1-connected to a vertex of type A. Then v is a full projection and there are identifications*

$$vL^{\mathrm{ab}}(E,C)v \cong K[\mathbb{F}_v], \quad v\mathcal{O}(E,C)v \cong C^*(\mathbb{F}_v) \quad \textit{and} \quad v\mathcal{O}^r(E,C)v \cong C^*_r(\mathbb{F}_v),$$

where $\mathbb{F}_v$ is the group of closed paths based at v.

Proof. From (E,C) being simple, we see that v is full. We claim that $\Omega(E,C)_v$ is in fact a one-point space. If it were not, then there would exist some admissible path $e\gamma$ with $s(\gamma) = v$ along with $X \in C_{r(e)}$ satisfying $|X| > 1$ and $X \neq X_e$. We can of course assume that $e\gamma$ is minimal with these properties. Assuming that γ passes through a type A vertex, we can write $\gamma = \gamma_2\gamma_1$, where γ_1 is the maximal initial subpath for which $r(\gamma_1)$ is of type A. But then $e\gamma_2$ is a 1-path between vertices of type A, contradicting Lemma 8.11. We deduce that $e\gamma$ is itself a 1-path, so that v is 1-connected to a type A vertex, in conflict with our assumption. It follows that the partial action $\theta^{(E,C)}$ restricts to the trivial global action $\mathbb{F}_v \curvearrowright \Omega(E,C)_v$, hence

$$vL^{\mathrm{ab}}_K(E,C)v \cong C_K(\Omega(E,C)_v) \rtimes \mathbb{F}_v \cong K \rtimes \mathbb{F}_v \cong K[\mathbb{F}_v]$$

and

$$v\mathcal{O}^{(r)}(E,C)v \cong C(\Omega(E,C)_v) \rtimes_{(r)} \mathbb{F}_v \cong \mathbb{C} \rtimes_{(r)} \mathbb{F}_v \cong C^*_{(r)}(\mathbb{F}_v)$$

by Lemma 7.13. □

We are now ready to prove Theorem 8.1.

Proof of Theorem 8.1. Passing, if necessary, to (E_1, C^1), we can assume that for each $X \in C$ we have that $s(x) \neq s(x')$ whenever x, x' are distinct elements of X. If $E^{0,0}$ consists entirely of vertices of type A, then we can reach (1) by Lemma 8.12. If $E^{0,0}$ contains both vertices of type A and of type B, and a vertex of type A is 1-connected to a vertex of type B, then its 1-connected component cannot contain any 1-cycle, by Lemma 8.11. In that case, the vertices of type B in it will vanish in the graph (E_n, C^n) for some n.

Indeed, let v be a vertex of type A and let γ be a non-trivial 1-path starting at v. By Lemma 8.11(a), all vertices in $E^{0,0}$ visited by γ are of type B. If the 1-connected component of v does not contain 1-cycles, then all the vertices visited by γ must be distinct. So we may assume that γ is of maximal length. Suppose for instance that

$$\gamma = e_{2r}e_{2r-1}^{-1}\cdots e_2e_1^{-1}$$

is of even length. Then the vertex $s(e_1)$ is of type A in (E_1, C^1), and there is a 1-path $\gamma' = (e'_{2r})^{-1}e'_{2r-1}\cdots(e'_2)^{-1}$ of length $2r-1$ in (E_1, C^1), where each e'_i belongs to $X(e_i)$, for $i = 2, \dots, 2r$. Moreover all the 1-paths of (E_1, C^1) starting at $s(e_1)$ are of this form, and we conclude that the 1-connected component of the vertex $s(e_1)$ only contains 1-paths of length $\leq 2r-1$. The case where γ is of odd length is treated in the same way. Now let n be the maximum of the lengths of all the maximal 1-paths starting at vertices of type A in (E, C). The above argument shows that the graph (E_n, C^n) has the property that no vertex of type A is 1-connected to a vertex of type B.

So we can assume, in addition, that no vertex of type A is 1-connected to a vertex of type B. But then Lemma 8.13 applies. □

9. Primeness

Recall that a ring is called *prime* if the product of any two non-zero ideals is non-zero. In this final section, we characterize primeness of the algebras $L_K^{\mathrm{ab}}(E, C)$ and $\mathcal{O}^r(E, C)$ in purely graph theoretical terms when the configuration space $\Omega(E, C)$ is a Cantor space. In order to check this hypothesis, we also develop a test for the existence of isolated points.

We first introduce a partial version of a well known concept.

Definition 9.1. A partial action $\theta\colon G \curvearrowright \Omega$ is called *topologically transitive* if for any two non-empty open subsets $U, U' \subset G$, there exists $g \in G$ such that $\theta_g(U \cap \Omega_{g^{-1}}) \cap U' \neq \emptyset$.

The main technical tool for our analysis is the fact that for a topologically free partial action $\theta\colon G \curvearrowright \Omega$ on a totally disconnected compact Hausdorff space, the algebras $C_K(\Omega) \rtimes G$ and $C(\Omega) \rtimes_r G$ are prime if and only if θ is topologically transitive. This should be clear from the fact that both these crossed products enjoy the intersection property (see [15, Lemma 4.2] and [25, Theorem 29.5]).

Throughout this section, we will write $\mathfrak{i}_d(\alpha)$ and $\mathfrak{t}_d(\alpha)$ for the initial (i.e right most) and terminal (i.e. left most) edge, respectively, of a non-trivial admissible α when viewed as a path in the double $\widehat{E}$.

Definition 9.2. Let (E, C) denote a finite bipartite separated graph. If $B \in \mathcal{B}_n(\Omega(E, C))$, then we define a clopen subspace by

$$\Omega(E, C)_B := \{\xi \in \Omega(E, C) \mid \xi^n = B\},$$

and we will say that B is an n-ball *at* v, where $v \in E^0$ is such that $\Omega(E,C)_B \subset \Omega(E,C)_v$. By the *boundary of* B, we shall mean the set

$$\partial B := \{\mathfrak{t}_d(\alpha) \mid \alpha \in B \text{ is maximal}\}.$$

Definition 9.3. Let (E,C) denote a finite bipartite graph, and consider a *path closed* subset $A \subset E^1 \cup (E^1)^{-1}$, that is, a subset satisfying

$$\mathfrak{i}_d(\alpha) \in A \Rightarrow \mathfrak{t}_d(\alpha) \in A$$

for any admissible path α. From (E,C) being bipartite, this may also be phrased as follows:

- If ef^{-1} is admissible and $f^{-1} \in A$, then $e \in A$.
- If $e^{-1}f$ is admissible and $f \in A$, then $e^{-1} \in A$.

We will say that such A is *∂-closed* if the following holds:

(1) Assume $|s^{-1}(v)| \geq 2$ and let $e \in s^{-1}(v)$. If $s^{-1}(v) \setminus \{e\} \subset A$, then $e^{-1} \in A$ as well.
(2) Assume $|C_v| \geq 2$ and let $X \in C_v$. If $Y^{-1} \cap A \neq \emptyset$ for any $Y \in C_v \setminus \{X\}$, then $X \subset A$ as well.

It is clear that an intersection of ∂-closed sets is again ∂-closed, so every $A \subset E^1 \cup (E^1)^{-1}$ is contained in a minimal ∂-closed subset $\overline{A}$. We then associate a set of vertices to A by

$$V(A) := \{v \in E^{0,0} \mid X^{-1} \cap \overline{A} \neq \emptyset \text{ for all } X \in C_v\} \cup \{v \in E^{0,1} \mid s^{-1}(v) \subset \overline{A}\}.$$

Lemma 9.4. *Let (E,C) denote a finite bipartite separated graph, and consider a path closed subset $A \subset E^1 \cup (E^1)^{-1}$. Then $v \in V(A)$ if and only if there exists some $n \geq 1$ and an n-ball $B \in \mathcal{B}_n(\Omega(E,C))$ at v such that $\partial B \subset A$.*

Proof. First suppose that $v \notin V(A)$. We will show that $\partial B \not\subset \overline{A}$ for any $n \geq 1$ and any n-ball B at v, and we shall argue by induction on n. If $n = 1$, then this is clear by definition of $V(A)$. Assuming that it holds for some $n \geq 1$, consider any $(n+1)$-ball B and write

$$B^n := \{\alpha \in B \colon |\alpha| \leq n\}.$$

From our inductive assumption, there exists a maximal admissible path $\alpha \in B^n$ such that $\mathfrak{t}_d(\alpha) \notin \overline{A}$. If α is also maximal in B, then surely $\partial B \not\subset \overline{A}$, so we may assume that it is not. We then divide into the two cases $r(\alpha) \in E^{0,0}$ and $r(\alpha) \in E^{0,1}$. In the former, we can write $\alpha = e\beta$, and there exists some $X_e \neq Y \in C_{r(\alpha)}$ with $Y^{-1} \cap \overline{A} = \emptyset$. By definition of $\Omega(E,C)$, we then have $y^{-1}\alpha \in B$ for some $y \in Y$. In the other case, we may instead write $\alpha = e^{-1}\beta$. Since $e^{-1} \notin \overline{A}$, we see that $f \notin \overline{A}$ for some $e \neq f \in s^{-1}(r(\alpha))$, and we consider the path $f\alpha \in B$. Either way, we have found an element of the boundary ∂B which is not contained in $\overline{A}$, so in particular not in A.

For the converse implication, we first introduce a bit of handy notation, specifically we define a partition $\overline{A} = \bigsqcup_{m=0}^{\infty} A_m$. First set $A_0 := A$, let $m \geq 0$ and assume that A_k has been defined for all $k \leq m$. Then for any $e \in E^1$, we declare that $e^{-1} \in A_{m+1}$ if

$$e^{-1} \notin \bigsqcup_{k=0}^{m} A_k, \quad |s^{-1}(s(e))| \geq 2 \quad \text{and} \quad s^{-1}(s(e)) \setminus \{e\} \subset \bigsqcup_{k=0}^{m} A_k.$$

Similarly, $e \in A_{m+1}$ if

$$e \notin \bigsqcup_{k=0}^{m} A_k, \quad |C_{r(e)}| \geq 2 \quad \text{and} \quad Y^{-1} \cap \bigsqcup_{k=0}^{m} A_k \neq \emptyset \text{ for any } Y \in C_{r(e)} \setminus \{X_e\}.$$

Now set

$$T := \{e \in E^1 : |C_{r(e)}| = 1\} \cup \{e^{-1} \in (E^1)^{-1} : |s^{-1}(s(e))| = 1\}$$

and observe that $A \cap T = \overline{A} \cap T$. Whenever $\sigma \in \overline{A}$, we write m_σ for the number satisfying $\sigma \in A_{m_\sigma}$, and if $X^{-1} \cap \overline{A} \neq \emptyset$ for some $X \in C$, we set $m_X := \min\{m_{x^{-1}} \mid x^{-1} \in X^{-1} \cap A\}$. Given $v \in V(A)$, we then define

$$n_v := \begin{cases} \max\{m_e \mid e \in s^{-1}(v)\} + 1 & \text{if } v \in E^{0,1} \\ \max\{m_X \mid X \in C_v\} + 1 & \text{if } v \in E^{0,0} \end{cases},$$

claiming that $\partial B \subset A$ for an n_v-ball B at v. Specifically, we will show that whenever $n \leq n_v$, there is an n-ball B_n at v satisfying $\partial B_n \subset \bigsqcup_{k=0}^{n_v-n} A_k$, and we proceed by induction over n. For $n = 1$, this is clear. Assuming that the claim has been verified for some $1 \leq n < n_v$, and letting i be such that $v \in E^{0,i}$, we consider the cases of $i + n$ being odd and even, separately. If it is odd, then there is a unique $(n+1)$-ball B_{n+1} containing B_n given by

$$B_{n+1} = B_n \cup \big\{e\alpha : \alpha \in B_n, |\alpha| = n \text{ and } e \in s^{-1}(r(\alpha)) \setminus \{\mathsf{t}_d(\alpha)^{-1}\}\big\}$$

with boundary

$$\partial B_{n+1} = (\partial B_n \cap T) \cup \bigcup_{f \in \partial B_n \setminus T} s^{-1}(s(f)) \setminus \{f\},$$

so it follows immediately from the above definition and A being path closed that $\partial B_{n+1} \subset \bigsqcup_{k=0}^{n_v-n-1} A_k$. If $i + n$ is even, then $\partial B_n \setminus T \subset E^1$, and for any $f \in \partial B_n \setminus T$, $Y \in C_{r(f)} \setminus \{X_f\}$, we can choose an edge $e_{f,Y} \in Y$ such that $e_{f,Y}^{-1} \in \bigsqcup_{k=0}^{n_v-n-1} A_k$. Now define an $(n+1)$-ball B_{n+1} by

$$B_{n+1} := B_n \cup \big\{e_{\mathsf{t}_d(\alpha),Y}^{-1}\alpha : \alpha \in B_n, |\alpha| = n \text{ and } Y \in C_{r(\alpha)} \setminus \{X_{\mathsf{t}_d(\alpha)}\}\big\}$$

and observe that

$$\partial B_{n+1} = (\partial B_n \cap T) \cup \big\{e_{f,Y}^{-1} \mid f \in \partial B_n \setminus T \text{ and } Y \in C_{r(f)} \setminus \{X_f\}\big\},$$

hence $\partial B_{n+1} \subset \bigsqcup_{k=0}^{n_v-n-1} A_k$ as desired. Considering the particular case $n = n_v$ and $B := B_{n_v}$, we finally see that $\partial B \subset A$. □

Definition 9.5. Let (E, C) denote a bipartite separated graph. A *choice path* is an admissible path $\alpha = e\beta$ such that there exists $X_e \neq X \in C_{r(\alpha)}$ with $|X| \geq 2$. Given any edge $f \in E^1$, we will say that f is a *dead end* if there is no choice path starting with f, and likewise f^{-1} is a dead end if no choice path starts with f^{-1}.

Proposition 9.6. *Let (E, C) denote a finite bipartite separated graph, and define*

$$A_{\mathrm{DE}} := \{\textit{dead ends of } E^1 \cup (E^1)^{-1}\}.$$

Then $\Omega(E,C)_v$ contains an isolated point if and only if $v \in V(A_{\mathrm{DE}})$. Consequently, $\Omega(E,C)$ is a Cantor space if and only if $V(A_{\mathrm{DE}}) = \emptyset$.

Proof. Observe first that $A := A_{\mathrm{DE}}$ is path closed. It follows from Lemma 9.4 that $v \in V(A)$ if and only if there exists a ball B at v such that $\partial B \subset A$. If this is the case, then $\Omega(E,C)_B$ is a one-point space, so $\Omega(E,C)_v$ does indeed contain an isolated point. If $v \notin V(A)$, then given any ball B at v, there exists $\sigma \in \partial B$ which is not a dead end. Consequently, $\Omega(E,C)_B$ contains at least two configurations, so $\Omega(E,C)_v$ does not contain any isolated points. □

Remark 9.7. It is worth mentioning that if $\Omega(E,C)$ is a Cantor space, then $\theta^{(E,C)}$ is automatically topologically free. Indeed, by [7, Theorem 10.5], $\theta^{(E,C)}$ is topologically free if and only if for every vertex $v \in E^{0,1}$ on a cycle, there exists some $e \in s^{-1}(v)$ which is not a dead end. And if no such e existed, then $\Omega(E,C)_v$ would be a one-point space.

Definition 9.8. Let (E,C) denote a finite bipartite separated graph. We will say that two sets $A, A' \subset E^1 \cup (E^1)^{-1}$ *can be linked* if there exist $\sigma \in A$, $\sigma' \in A'$ and an admissible path α such that the concatenation $\sigma^{-1}\alpha\sigma'$ is admissible. If this is not the case, then the pair A, A' is *unlinkable*. Morover, it is *maximal unlinkable* if for any larger pair $A \subset D$, $A' \subset D'$ we have

$$D \text{ and } D' \text{ are unlinkable} \Rightarrow A = D, A' = D'.$$

Finally, (E,C) is said to have *the Linking Property* if ∂B and $\partial B'$ can be linked for any two balls $B, B' \in \mathcal{B}(\Omega(E,C))$.

Lemma 9.9. *If (E,C) has the Linking Property, then $\theta^{(E,C)}$ is topologically transitive, and if $\Omega(E,C)$ is a Cantor space, then the reverse implication holds as well.*

Proof. Assume first that (E,C) has the Linking Property. Then given any two open sets $U, U' \subset \Omega(E,C)$, there are balls B, B' such that $\Omega(E,C)_B \subset U$ and $\Omega(E,C)_{B'} \subset U'$. Since B and B' can be linked, there exist $\sigma \in \partial B$, $\sigma' \in \partial B'$ and an admissible path α for which $\sigma^{-1}\alpha\sigma'$ is admissible. Let $\gamma \in B$ and $\gamma' \in B'$ be such that $\mathsf{t}_d(\gamma) = \sigma$ and $\mathsf{t}_d(\gamma') = \sigma'$, and set $\beta := \gamma^{-1}\alpha\gamma'$. The situation is depicted below in Figure 3.

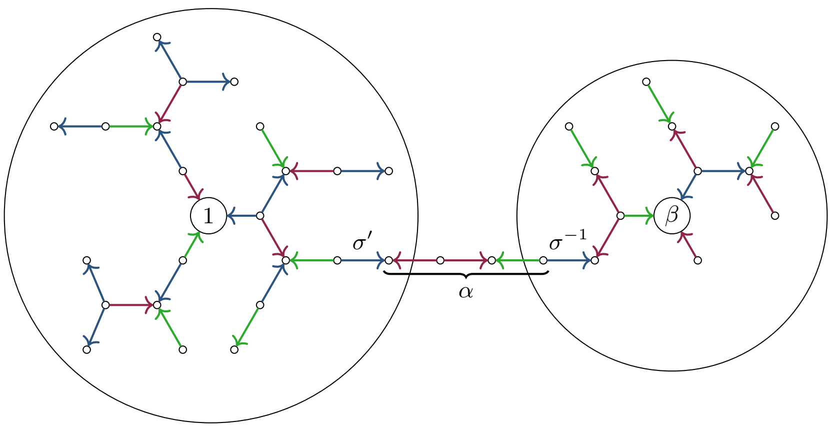


FIGURE 3. B' (to the left) and B (to the right) linked by the admissible path α.

It follows that

$$\emptyset \neq \theta_\beta(\Omega(E,C)_B \cap \Omega(E,C)_{\beta^{-1}}) \cap \Omega(E,C)_{B'} \subset \theta_\beta(U \cap \Omega(E,C)_{\beta^{-1}}) \cap U',$$

so $\theta^{(E,C)}$ is indeed topologically transitive. Conversely, assume now that $\theta^{(E,C)}$ is topologically transitive and $\Omega(E,C)$ is a Cantor space. Consider any two balls B, B' and assume without loss of generality that both have radius n. If $\beta \in \mathbb{F}$ with $|\beta| < 2n$ is such that

$$\theta_\beta\big(\Omega(E,C)_B \cap \Omega(E,C)_{\beta^{-1}}\big) \cap \Omega(E,C)_{B'} \neq \emptyset,$$

then take any two distinct points ξ, ξ' in this open set, using the assumption about isolated points. By the Hausdorff property, these can be separated by open neighbourhoods U_1, U_1' of ξ and ξ', respectively, contained in the above intersection. We have $\theta_{\beta^{-1}}(U_1) \subseteq \Omega(E,C)_B$, $U_1' \subseteq \Omega(E,C)_{B'}$, and

$$\theta_\beta\big(\theta_{\beta^{-1}}(U_1) \cap \Omega(E,C)_{\beta^{-1}}\big) \cap U_1' = \emptyset.$$

By applying this procedure sufficiently many times, we see that there are non-empty open subsets $V \subseteq \Omega(E,C)_B$ and $V' \subseteq \Omega(E,C)_{B'}$, such that

$$\theta_\beta\big(V \cap \Omega(E,C)_{\beta^{-1}}\big) \cap V' = \emptyset$$

for all $\beta \in \mathbb{F}$ with length $|\beta| < 2n$. However, by topological transitivity, there is some $\beta \in \mathbb{F}$ for which this intersection is non-empty, so $|\beta| \geq 2n$. It follows that B and B' can be linked. □

We are now in a position to characterize primeness.

Theorem 9.10. *Assume that $\Omega(E,C)$ is a Cantor space. Then either algebra $L_K^{\mathrm{ab}}(E,C)$ or $\mathcal{O}^r(E,C)$ is prime if and only if $V(A) = \emptyset$ or $V(A') = \emptyset$ for all maximal unlinkable pairs $A, A' \subset E^1 \cup (E^1)^{-1}$.*

Proof. Observe that, by maximality, A and A' as above are path closed. By Remark 9.7, $\theta^{(E,C)}$ is topologically free, so $L_K^{\mathrm{ab}}(E,C)$ and $\mathcal{O}^r(E,C)$ are both prime if and only if $\theta^{(E,C)}$ is topologically transitive, which is again equivalent to the Linking Property by Lemma 9.9. So we really have to check that the Linking Property is equivalent to the above condition. But this is clear from Lemma 9.4. □

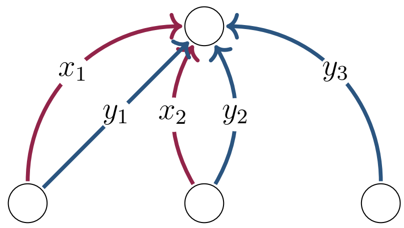
Remark 9.11. It is of course also natural to ask if interesting prime algebras can be constructed from a finite bipartite separated graph (E,C), where the configuration space $\Omega(E,C)$ does contain isolated points. However, this is not the case: For sufficiently big n, there must exist a vertex $v \in E_n^{0,1}$ such that $\Omega(E_n, C^n)_v$ is a one-point space, i.e. every $e \in s^{-1}(v)$ is a dead end. By topological transitivity, the orbit of this single point is dense, so in particular v can be connected to any other vertex by an admissible path. But then (E_n, C^n) must satisfy Condition (C) of [38, Definition 3.5]. Note that v cannot admit exactly one simple closed path (up to inversion), for then it would generate an ideal Morita equivalent to $K[\mathbb{Z}]$ or $C(\mathbb{T})$, depending on the situation, by 7.13. If v does not admit a closed path, then both algebras degenerate to graph algebras of a non-separated graph by [38, Theorem 5.7].

Finally, if v admits at least two simple closed paths (up to inversion), then it generates an ideal Morita equivalent with the group algebra $K[\mathbb{F}_v]$ in the algebraic setting and $C^*_r(\mathbb{F}_v)$ in the C^*-algebraic (here $\mathbb{F}_v$ denotes the group of all closed paths based at v), and the quotient is a classical graph algebra.

Example 9.12. We now apply our work to a few examples:

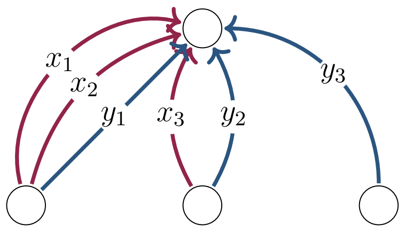
(1) If $(E, C) = (E(m, n), C(m, n))$ as in Example 6.6, then $A_{\mathrm{DE}} = \emptyset$ and every pair of edges can be linked, so $\Omega(E, C)$ is a Cantor space and the algebras $L^{\mathrm{ab}}_{m,n}$ and $\mathcal{O}^r_{m,n}$ are prime.

(2) Consider the graph (E, C) as pictured just below:



Note that $A_{\mathrm{DE}} = \{y_3^{-1}\}$ has closure $\overline{A_{\mathrm{DE}}} = \{x_1, x_2, y_1^{-1}, y_2^{-1}, y_3^{-1}\}$, so $V(A_{\mathrm{DE}}) = \emptyset$. It follows that $\Omega(E, C)$ is a Cantor space. However, the set $A := \{x_1^{-1}, x_2^{-1}, y_1, y_2, y_3\}$ is not linked to itself and $V(A) = \{s(y_3)\}$, so the algebras are not prime.

(3) Now consider the following variation of the above graph:



Once again we have $V(A_{\mathrm{DE}}) = \emptyset$, so $\Omega(E, C)$ is indeed a Cantor space. Observe that there is a unique pair of maximal unlinkable subsets, namely

$$A = A' := \{x_1, x_2, x_3, y_1^{-1}, y_2^{-1}, y_3^{-1}\},$$

and $V(A) = \emptyset$. It follows that $L^{\mathrm{ab}}_K(E, C)$ and $\mathcal{O}^r(E, C)$ *are* prime in this case.

Remark 9.13. We finally remark that topological transitivity of $\theta^{(E,C)}$ can be phrased very simply in terms of the separated Bratteli diagram (F_∞, D^∞). Since the vertices of F_∞ correspond to the balls of $\Omega(E, C)$, and there is a direct dynamical equivalence $\theta^{(E_n,C^n)} \xrightarrow{\approx} \theta^{(E,C)}$ for any n, the partial action $\theta^{(E,C)}$ is topologically transitive if and only if for all n, any two vertices $u, v \in E_n^0$ can be connected by an admissible path in (E_n, C^n), or, alternatively, that any two vertices in F_∞ lay over a common vertex $w \in F_\infty^0$. Consequently, one can give another proof of Theorem 9.10 by checking that the graph theoretical condition passes from (E, C) to (E_1, C^1).

References

[1] F. Abadie, *Enveloping actions and Takai duality for partial actions*, J. Funct. Anal. **197** (2003), 14–67.

[2] F. Abadie, *On partial actions and groupoids*, Proc. Amer. Math. Soc. **132** (2004), 1037-1047.

[3] G. Abrams, P. Ara, M. Siles Molina, *Leavitt Path Algebras*, to appear in Lecture Notes in Mathematics (Springer).

[4] G. Abrams, G. Aranda Pino, *The Leavitt path algebra of a graph*, J. Algebra **293** (2005), 319–334.

[5] P. Ara, *Purely infinite simple reduced C*-algebras of one-relator separated graphs*, J. Math. Anal. Appl. **393** (2012), 493–508.

[6] P. Ara, J. Claramunt, *Approximating the group algebra of the lamplighter by finite-dimensional algebras.* In preparation.

[7] P. Ara, R. Exel, *Dynamical systems associated to separated graphs, graph algebras, and paradoxical decompositions*, Adv. Math. **252** (2014), 748–804.

[8] P. Ara, R. Exel, *K-theory for the tame C*-algebra of a separated graph.* Journal of Functional Analysis **269** (2015), 2995-3041.

[9] P. Ara, R. Exel, T. Katsura, *Dynamical systems of type (m, n) and their C*-algebras*, Ergodic Theory Dynam. Systems **33** (2013), 1291–1325.

[10] P. Ara, K. R. Goodearl, *C*-algebras of separated graphs*, J. Funct. Anal. **261** (2011), 2540–2568.

[11] P. Ara, K. R. Goodearl, *Leavitt path algebras of separated graphs.* J. reine angew. Math. **669** (2012), 165–224.

[12] P. Ara, K. R. Goodearl, *The realization problem for some wild monoids and the Atiyah problem*, Trans. Amer. Math. Soc. http:dx.doi.org/10.1090/tran/6889. Article published electronically on December 7, 2016. ArXiv:1410.8838v2 [math.RA].

[13] P. Ara, K. R. Goodearl, K.C. O'Meara, E. Pardo, *Separative cancellation for projective modules over exchange rings*, Israel J. Math. **105** (1998), 105–137.

[14] P. Ara, M. A. Moreno, E. Pardo, *Nonstable K-theory for graph algebras*, Algebr. Represent. Theory **10** (2007), 157–178.

[15] J. Brown, L. O. Clark, C. Farthing, A. Sims, *Simplicity of algebras associated to étale groupoids*, Semigroup Forum **88** (2014), 433–452.

[16] T. Bates, J. Hong, I. Raeburn and W. Szymański, *The ideal structure of the C*-algebras of infinite graphs*, Illinois J. Math. **46** (2002), 1159–1176.

[17] T. Bates, D. Pask, I. Raeburn and W. Szymański, *The C*-algebras of row-finite graphs*, New York J. Math. **6** (2000), 307–324 (electronic).

[18] O. Bratteli, *Inductive limits of finite dimensional C*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234.

[19] Toke M. Carlsen and Nadia S. Larsen, *Partial actions and KMS states on relative graph C-algebras*, J. Funct. Anal. **271** (2016) 2090-2132.

[20] L. O. Clark and A. Sims, *Equivalent groupoids have Morita equivalent Steinberg algebras*, J. Pure Appl. Algebra **219** (2015), 2062–2075.

[21] Toke M. Carlsen, Efren Ruiz, and Aidan Sims, *Equivalence and stable isomorphism of groupoids, and diagonal-preserving stable isomorphisms of graph C*-algebras and Leavitt path algebras*, Proc. Amer. Math. Soc. **145** (2017), 1581–1592.

[22] M. Dokuchaev, R. Exel, *Partial actions and subshifts*, preprint 2015, arXiv:1511.00939v1 [math.OA].

[23] T. Downarowicz, A. Maass, *Finite-rank Bratteli-Vershik diagrams are expansive*, Ergodic Theory Dynam. Systems **28** (2008), 739–747.

[24] G. A. Elliott, *On the classification of C*-algebras of real rank zero*, J. Reine Angew. Math. **443** (1993), 179–219.

[25] R. Exel, *Partial Dynamical Systems, Fell Bundles and Applications.* `www.mtm.ufsc.br/~exel/papers/pdynsysfellbun.pdf`.

[26] R. Exel, *Amenability for Fell bundles*, J. Reine Angew. Math. **492** (1997), 41–73.

[27] Ruy Exel, *Inverse semigroups and combinatorial C*-algebras*, Bull. Braz. Math. Soc. (N.S.) **39** (2008), 191–313.

[28] R. Exel and M. Laca, *Cuntz–Krieger algebras for infinite matrices*, J. reine angew. Math. **512** (1999), 119–172.

[29] R. Exel, M. Laca, J. Quigg, *Partial dynamical systems and C*-algebras generated by partial isometries*, J. Operator Theory **47** (2002), 169–186.

[30] T. Giordano, I. F. Putnam, G. F. Skau, *Topological orbit equivalence and C-crossed products*, J. Reine Angew. Math. **469** (1995), 51–111.

[31] D. Gonçalves, D. Royer, *Leavitt path algebras as partial skew group rings*, Comm. Algebra **42** (2014), 3578–3592.

[32] J.H. Hong and W. Szymański, *The primitive ideal space of the C -algebras of infinite graphs*, J. Math. Soc. Japan **56** (2004), 4564.

[33] S-M. Høynes, *F*inite-rank Bratteli-Vershik diagrams are expansive - -a new proof, arXiv:1411.3371v1[math.DS].

[34] D. Kerr, P. W. Nowak, *Residually finite actions and crossed products*, Ergodic Theory and Dynamical Systems **32** (2012), 1585–1614.

[35] X. Li, *Partial transformation groupoids attached to graphs and semigroups*, arXiv:1603.09165 [math.OA].

[36] X. Li, *Dynamic characterizations of quasi-isometry, and applications to cohomology*, arXiv:1604.07375v3 [math.GR].

[37] D. Lind, B. Marcus, An introduction to symbolic dynamics and coding. Cambridge University Press, Cambridge, 1995.

[38] M. Lolk, *Exchange rings and real rank zero C^*-algebras associated with finitely separated graphs*, preprint 2017.

[39] M. Lolk, *On nuclearity and exactness of the tame C^*-algebras associated with finitely separated graphs*, preprint 2017.

[40] P. S. Muhly, J. N. Renault and D. P. Williams, *Equivalence and isomorphism for groupoid C^*-algebras*, J. Operator Theory **17** (1987), 3–22.

[41] A. L. T. Paterson, *Graph inverse semigroups, groupoids and their C*-algebras*, J. Operator Theory **48** (2002), 645–662.

[42] I. F. Putnam, *On the topological stable rank of certain transformation group C*-algebras*, Ergodic Theory Dynam. Systems **10** (1990), 197–207.

[43] Robert T. Powers, *Simplicity of the C^*-algebra associated with the free group on two generators*, Duke Math. J. 42 (1975), no. 1, 151–156.

[44] I. Raeburn, Graph algebras. CBMS Regional Conference Series in Mathematics, 103. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2005.

[45] J. Renault, *A groupoid approach to C-algebras*, Lecture Notes in Mathematics, 793. Springer, Berlin, 1980.

[46] M. Rørdam, A. Sierakowski, *Purely infinite C*-algebras arising from crossed products*, Ergodic Theory and Dynamical Systems **32** (2012), 273–293.

[47] B. Steinberg, *A groupoid approach to discrete inverse semigroup algebras*, Adv. Math. **223** (2010), 689727.

[48] A. Sims and D. P. Williams, *Renault's equivalence theorem for reduced groupoid C^*-algebras*, J. Operator Theory **68** (2012), 223–239.
[49] F. Sugisaki, *On the subshift within a strong orbit equivalence class for minimal homeomorphisms*, Ergodic Theory Dynam. Systems **27** (2007), 971–990.
[50] W. Szymański, *General Cuntz-Krieger uniqueness theorem*, Internat. J. Math. **13** (2002) 549555.

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ARTICLE B

Exchange rings and real rank zero C^*-algebras associated with finitely separated graphs

This chapter contains the preprint version of the following article:

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A preprint version is publicly available at `http://arxiv.org/abs/1705.04494`.

EXCHANGE RINGS AND REAL RANK ZERO C^*-ALGEBRAS ASSOCIATED WITH FINITELY SEPARATED GRAPHS

MATIAS LOLK

Abstract. We introduce a generalisation of Condition (K) to finitely separated graphs and show that it is equivalent to essential freeness of the associated partial action as well as the exchange property of any of the associated tame algebras. As a consequence, we can show that any tame separated graph algebra with the exchange property is separative.

Introduction

A *finitely separated graph* is a directed graph with a partition of the edges into finite subsets, which might be thought of as an edge-colouring, so that edges with distinct ranges have different colours. Ara and Goodearl first introduced *Leavitt path algebras* $L_K(E,C)$ and *graph C^*-algebras* $C^*(E,C)$ associated with a separated graph (E,C) in [6] and [5], respectively, and showed that any conical abelian monoid may be realised as the non-stable K-theory $\mathcal{V}(L_K(E,C))$ of a Leavitt path algebra of a finitely separated graph. They also conjectured that the inclusion $L_{\mathbb{C}}(E,C) \hookrightarrow C^*(E,C)$ induces an isomorphism on non-stable K-theory, but this important problem remains open. While the edges and their adjoints define partial isometries in these algebras, their products need not be partial isometries – we say that E^1 is not a *tame* set of partial isometries in $L_K(E,C)$ and $C^*(E,C)$. This led Ara and Exel to define quotients $L_K(E,C) \twoheadrightarrow L_K^{\mathrm{ab}}(E,C)$ and $C^*(E,C) \twoheadrightarrow \mathcal{O}(E,C)$ in which E^1 is exactly forced to be tame, as well as a further *reduced quotient* $\mathcal{O}^r(E,C)$ in the C^*-setting. Amazingly, passing to these much more well behaved quotients only enriches the non-stable K-theory (at least on the level of Leavitt path algebras) in the sense that the induced monoid homomorphism $\mathcal{V}(L_K(E,C)) \to \mathcal{V}(L_K^{\mathrm{ab}}(E,C))$ is a refinement. Moreover, if the above conjecture holds, then the canonical embedding $L_{\mathbb{C}}^{\mathrm{ab}}(E,C) \hookrightarrow \mathcal{O}(E,C)$ will induce an isomorphism on non-stable K-theory as well.

Following [1], a ring R (possibly non-unital) is called an *exchange ring* if for any $x \in R$, there exists an idempotent $e \in R$ and elements $r, s \in R$ such that $e = xr$ and $e = s + x - xs$. For the class of C^*-algebras, this property coincides with the (to C^*-algebraists more familiar) notion of *real rank zero* [1, Theorem 3.8]. The Fundamental Separativity Problem for exchange rings (see [20], [7])) asks whether every exchange ring R is *separative*, that is whether the cancellation property

$$2a = a + b = 2b \Rightarrow a = b$$

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holds in the non-stable K-theory $\mathcal{V}(R)$. A positive answer to this problem would provide positive answers to a number of open problems in both ring-theory and operator algebras [7, Sections 6 and 7]. In light of the highly general non-stable K-theory of separated graph algebras, it is therefore natural to ask when the tame algebras $L_K^{\mathrm{ab}}(E,C)$, $\mathcal{O}(E,C)$ and $\mathcal{O}^r(E,C)$ are exchange rings. The main result of the present paper provides a somewhat discouraging, but not unexpected, answer to this question: If any one of these is an exchange ring, then it will be a classical graph algebra in case (E,C) is finite and approximately a classical graph algebra in case (E,C) is only finitely separated. In particular, it will be separative. On our way to proving this result, we also take a minor detour (Section 4) to obtain a characterisation of simplicity for these algebras, and the conclusion is similar to the one above: Any of the algebras $L_K(E,C)$, $L_K^{\mathrm{ab}}(E,C)$, $C^*(E,C)$ and $\mathcal{O}(E,C)$ can only be simple if it is in fact a classical graph algebra, while $\mathcal{O}^r(E,C)$ may also be Morita equivalent to the reduced group C^*-algebra $C_r^*(\mathbb{F}_n)$ for $n \geq 2$. This result (although only in the setting of finite separated graphs) is also obtained by Ara and the author in [8], but with quite different arguments.

Our proofs of the above mentioned results rest heavily on a dynamical description of the tame algebras $L_K^{\mathrm{ab}}(E,C)$, $\mathcal{O}(E,C)$ and $\mathcal{O}^r(E,C)$. This was first obtained for finite bipartite graphs in [2] by Ara and Exel, and we generalise this description to finitely separated graphs in Section 2. In Section 3, we investigate when $L_K(E,C)$ and $C^*(E,C)$ degenerate to graph algebras of non-separated graphs, and we combine the results of Section 2 and 3 to characterise simplicity of the various algebras in Section 4. We then study degeneracy of the tame algebras $L_K^{\mathrm{ab}}(E,C)$, $\mathcal{O}(E,C)$ and $\mathcal{O}^r(E,C)$ in Section 5, before characterising the exchange property of the various algebras as well as essential freeness of the associated partial action in terms of a graph-theoretic Condition (K) in Section 6.

1. Preliminary definitions

In this section, we recall the necessary definitions and results from the existing theory on algebras associated with separated graphs.

Definition 1.1. A *separated graph* (E,C) is a graph $E = (E^0, E^1, r, s)$ together with a *separation* $C = \bigsqcup_{v \in E^0} C_v$, where each C_v is a partition of $r^{-1}(v)$ into non-empty subsets. If v is a source, i.e. if $r^{-1}(v) = \emptyset$, then we simply take C_v to be the empty partition, and for any $e \in E^1$, the set in $C_{r(e)}$ containing e will be denoted $[e]$. As soon as we start building various objects out of separated graphs, we will only consider *finitely separated* ones, meaning that every $X \in C$ is finite. Finally, any directed graph E may be regarded as a separated graph by giving it the *trivial separation* $\mathcal{T}_v = \{r^{-1}(v)\}$ for all $v \in E^0 \setminus E^0_{\mathrm{source}}$. Note that in this case, finitely separated simply means column-finite.

A *complete* subgraph (F,D) of (E,C) is a subgraph such that $D_v = \{X \in C_v \mid X \cap F^1 \neq \emptyset\}$ for every $v \in F^0$. The inclusion of a complete subgraph defines a morphism in the category of finitely separated graphs (see [6, Definition 8.4] for details on the appropriate notion of morphism), and every finitely separated graph is the direct limit of its finite complete subgraphs [6, Proposition 3.5 and Definition 8.4]. Moreover, as all the constructions from separated graphs are functorial and preserve direct limits, it is often sufficient to consider only finite graphs.

Definition 1.2. Let K denote any field. The *Leavitt path algebra* $L_K(E,C)$ associated with a finitely separated graph (E,C) is the $*$-algebra (over K) generated by $E^0 \sqcup E^1$ with relations

(V) $uv = \delta_{u,v}v$ and $u = u^*$ for $u, v \in E^0$,

(E) $es(e) = r(e)e = e$ for $e \in E^1$,

(SCK1) $e^*f = \delta_{e,f}s(e)$ if $[e] = [f]$,

(SCK2) $v = \sum_{e\in X} ee^*$ for all $v \in E^0$ and $X \in C_v$,

and the graph C^*-algebra $C^*(E,C)$ is the universal C^*-algebra with respect to these generators and relations. In other words, $C^*(E,C)$ is the enveloping C^*-algebra of $L_{\mathbb{C}}(E,C)$. The reader should be aware that we use the convention of [2], [3], [8] and [23], often referred to as the *Raeburn-convention*, opposite to that of [5] and [6] – consequently, paths will have to be read from the right. ◀

Note that if $C = \mathcal{T}$, then (SCK1) and (SCK2) are simply the ordinary (CK1) and (CK2) axioms for classical graph algebras, so $L_K(E,C) = L_K(E)$ and $C^*(E,\mathcal{T}) = C^*(E)$. It was observed in [6, Proposition 3.6] and [5, Proposition 1.6] that the assignments $(E,C) \mapsto L_K(E,C)$ and $(E,C) \mapsto C^*(E,C)$ extend to continuous functors from the category of finitely separated graphs.

Recall that a set of partial isometries S is set to be *tame* if all products of elements from $S \cup S^*$ are also partial isometries.

Definition 1.3 ([3, Definition 2.4])**.** Let (E,C) denote a finitely separated graph. Then $L_K^{\mathrm{ab}}(E,C)$ is the $*$-algebra (over K) and $\mathcal{O}(E,C)$ is the universal C^*-algebra generated by $E^0 \sqcup E^1$ with relations (V), (E), (SCK1), (SCK2), and E^1 being tame. We refer to $L_K^{\mathrm{ab}}(E,C)$ as the *abelianised Leavitt path algebra* of (E,C) and to $\mathcal{O}(E,C)$ as *universal tame graph C^*-algebra* of (E,C). Since we invoke more relations than above, there are canonical quotient maps

$$L_K(E,C) \to L_K^{\mathrm{ab}}(E,C) \quad \text{and} \quad C^*(E,C) \to \mathcal{O}(E,C).$$

It was proven in [3, Proposition 7.2] that the assignment $(E,C) \mapsto \mathcal{O}(E,C)$ extends to a continuous functor, and the same proof applies to $(E,C) \mapsto L_K^{\mathrm{ab}}(E,C)$.

Definition 1.4. A separated graph (E,C) is called *bipartite* if there exists a partition of the vertex set $E^0 = E^{0,0} \sqcup E^{0,1}$ with $s(E^1) = E^{0,1}$ and $r(E^1) = E^{0,0}$. ◀

Whenever (E,C) is a separated graph, there is a canonical bipartite replacement $\mathbf{B}(E,C)$ as defined in [2, Proposition 9.1] and [3, Definition 7.4]. By [3, Proposition 7.5], the assignment $(E,C) \mapsto \mathbf{B}(E,C)$ is functorial and there are natural isomorphisms of functors

$$\mathbf{M}_2 \circ C^* \cong C^* \circ \mathbf{B} \quad \text{and} \quad \mathbf{M}_2 \circ \mathcal{O} \cong \mathcal{O} \circ \mathbf{B},$$

where $\mathbf{M}_2$ is the functor taking A to $M_2(A)$, and the same proof applies to L_K and L_K^{ab}. Therefore, one can often restrict to finite bipartite graphs, which was exactly the setup of [2].

Moreover, if (E,C) is finite and bipartite, then there is a sequence of finite, bipartite separated graphs (E_n, C^n) as defined in [2, Construction 4.4], such that $L_K^{\mathrm{ab}}(E,C) \cong \varinjlim_n L_K(E_n, C^n)$ and $\mathcal{O}(E,C) \cong \varinjlim_n C^*(E_n, C^n)$ for appropriate connecting $*$-homomorphisms. We have $(E,C) = (E_0, C^0)$, and every (E_{n+1}, C^{n+1}) is constructed in the same way from (E_n, C^n), so all the graphs (E_n, C^n) give rise to the same tame algebras.

While only finite bipartite separated graphs were considered in [2], one can associate a partial action $\theta^{(E,C)} \colon \mathbb{F} \curvearrowright \Omega(E,C)$ to any finitely separated graph (E,C). Here $\mathbb{F}$ is the free group generated by the edge set E^1, and $\Omega(E,C)$ is a zero-dimensional, locally compact and metrisable space (see Definition 2.6). We will verify in Theorem 2.10 that [2, Corollary 6.12] generalises, i.e. that

$$L_K^{\mathrm{ab}}(E,C) \cong C_K(\Omega(E,C)) \rtimes \mathbb{F} \quad \text{and} \quad \mathcal{O}(E,C) \cong C_0(\Omega(E,C)) \rtimes \mathbb{F}$$

for all finitely separated graphs (E,C), where $C_K(\Omega(E,C))$ is the $*$-algebra of locally constant, compactly supported functions $\Omega(E,C) \to K$. This allows for the definition of a *reduced* tame C^*-algebra as defined in [3, Definition 6.8].

Definition 1.5. If (E,C) is a finitely separated graph, then *the reduced tame C^*-algebra* associated with (E,C) is the reduced crossed product $\mathcal{O}^r(E,C) := C_0(\Omega(E,C)) \rtimes_r \mathbb{F}$.

Definition 1.6. Let E denote a graph. A non-trivial path in E is a finite, non-empty sequence of edges $\alpha = e_n e_{n-1} \cdots e_2 e_1$ satisfying $r(e_i) = s(e_{i+1})$ for all $i = 1, \ldots, n-1$. A *subpath* of α is simply a subsequence $e_m e_{m-1} \cdots e_{j+1} e_j$, and the range and source of α is defined by $r(\alpha) := r(e_n)$ and $s(\alpha) := s(e_1)$. We will sometimes use the notation $\alpha \colon u \to v$ for a path with source u and range v, and we shall regard the vertices E^0 as the set of *trivial paths* with $r(v) := v =: s(v)$.

The *double* $\hat{E}$ of E is the graph obtained from E by adding an edge e^{-1} going in the reverse direction for any $e \in E^1$. Namely, $\hat{E}$ is the graph with vertices $\hat{E}^0 := E^0$ and edges

$$\hat{E}^1 := E^1 \sqcup \{e^{-1} \mid e \in E^1\},$$

where r and s are extended from E^1 by $r(e^{-1}) := s(e)$ and $s(e^{-1}) := r(e)$. The map

$$E^0 \sqcup E^1 \to E^0 \sqcup \hat{E}^1 \quad \text{given by} \quad v \mapsto v \quad \text{and} \quad e \mapsto e^{-1}$$

then extends canonically to an order-reversing, order two bijection of the paths of $\widehat{E}$, denoted by $\alpha \mapsto \alpha^{-1}$.

Now if (E,C) is a separated graph, an *admissible path* α in (E,C) is a path in the double $\hat{E}$, such that

(1) any subpath $e^{-1} f$ with $e, f \in E^1$ satisfies $[e] \neq [f]$,
(2) any subpath $e f^{-1}$ with $e, f \in E^1$ satisfies $e \neq f$,

and the set of admissible paths of (E,C) is denoted $\mathcal{P}(E,C)$. If α is a non-trivial admissible path, we shall use the notation $\mathfrak{i}_{\mathrm{d}}(\alpha)$ and $\mathfrak{t}_{\mathrm{d}}(\alpha)$ for the initial (i.e. rightmost) and terminal (i.e. leftmost) symbol of α, respectively; for instance

$$\mathfrak{i}_{\mathrm{d}}(ef^{-1}) = f^{-1} \quad \text{and} \quad \mathfrak{t}_{\mathrm{d}}(ef^{-1}) = e.$$

Letting $\pi\colon \hat{E}^1 \to E^1$ denote the projection given by $\pi(e) := e =: \pi(e^{-1})$, we then set

$$\mathfrak{i}(\alpha) := \pi(\mathfrak{i}_{\mathrm{d}}(\alpha)) \quad \text{as well as} \quad \mathfrak{t}(\alpha) := \pi(\mathfrak{t}_{\mathrm{d}}(\alpha)).$$

If α is an admissible path and $X, Y \in C$, we shall say that $X^{-1}\alpha$, respectively, $X^{-1}\alpha Y$ is an *admissible composition* if $x^{-1}\alpha$, respectively, $x^{-1}\alpha y$ is admissible for some (hence any) $x \in X$ and $y \in Y$. We finally introduce a partial order $\leq$ on $\mathcal{P}(E, C)$ given by

$$\beta \leq \alpha \Leftrightarrow \beta \text{ is an initial subpath of } \alpha.$$

In particular, whenever $s(\alpha) = s(\beta)$, there is a maximal initial subpath $\alpha \wedge \beta \leq \alpha, \beta$. ◀

We shall use the exact same terminology as above when inverse edges e^{-1} are replaced by adjoint edges e^*.

Notation 1.7. Given admissible paths α and β in (E, C), we will write $\beta\alpha$ for the concatenated product, which may or may not be an admissible path. However, one may also view α and β as elements of the free group $\mathbb{F}$ on E^1, in which a product can also be formed, allowing for cancellation of edges and their inverses. To distinguish notationally between these two products, we will always write $\beta \cdot \alpha$ when the product is formed in $\mathbb{F}$.

Remark 1.8. Note that in light of the defining relations of $C^*(E, C)$, any non-zero product $\alpha \in C^*(E, C)$ of elements from $E^1 \sqcup (E^1)^*$ can be written as

$$\alpha = \alpha_n \alpha_{n-1} \cdots \alpha_2 \alpha_1,$$

where each α_i is a non-trivial admissible path, and

(1) $\mathfrak{i}_{\mathrm{d}}(\alpha_{i+1}) \in E^1$ and $\mathfrak{t}_{\mathrm{d}}(\alpha_i) = \mathfrak{i}_{\mathrm{d}}(\alpha_{i+1})^*$,
(2) $|[\mathfrak{i}_{\mathrm{d}}(\alpha_{i+1})]| \geq 2$

for all $i = 1, \dots, n-1$.

Definition 1.9. A non trivial admissible path α in a separated graph is called a *closed path* if $r(\alpha) = s(\alpha)$, and it is called a *cycle* if $\alpha\alpha$ is an admissible path as well. In either case, we shall say that α is *based* at $r(\alpha) = s(\alpha)$. An admissible path α will be referred to as *simple* if it does not meet the same vertex twice, i.e. if $r(\alpha_1) = r(\alpha_2)$ for $\alpha_1, \alpha_2 \leq \alpha$ implies $\alpha_1 = \alpha_2$, while a cycle is called a *simple cycle* if the only vertex repetition occurs at the end, that is if $\alpha_1 < \alpha_2 \leq \alpha$ and $r(\alpha_1) = r(\alpha_2)$ implies $\alpha_1 = s(\alpha)$ and $\alpha_2 = \alpha$. Any closed path α is called *base-simple* if $s(\alpha)$ is only repeated at the end. ◀

As for non-separated graphs, there is a notion of *hereditary* and *saturated* sets, giving rise to ideals in our algebras.

Definition 1.10 ([6, Definition 6.3 and Definition 6.5])**.** Let (E, C) denote a finitely separated graph. A set of vertices $H \subset E^0$ is called *hereditary* if $r(e) \in H$ implies $s(e) \in H$ for all $e \in E^1$, and it is called *C-saturated* if for all $v \in E^0$ and $X \in C_v$, $s(X) \subset H$ implies $v \in H$.

We will write $\mathcal{H}(E,C)$ for the lattice of hereditary and C-saturated sets. Finally, for any such $H \in \mathcal{H}(E,C)$, we define a quotient graph $(E/H, C/H)$ by

- $E/H)^0 := E^0 \setminus H$ and $(E/H)^1 := r^{-1}(H)$,
- $(C/H)_v := \{X/H \mid X \in C_v\}$ where $X/H := X \cap r^{-1}(H)$,

with the range and source restricted from E.

2. Dynamical systems associated with finitely separated graphs

In this section, we generalise Ara and Exel's dynamical description of the tame algebras $L_K^{\mathrm{ab}}(E,C)$ and $\mathcal{O}(E,C)$ in [2] to finitely separated graphs, allowing us to define $\mathcal{O}^r(E,C)$ for such graphs. We also address the relationship between the partial actions $\theta^{(E,C)}$ and $\theta^{\mathbf{B}(E,C)}$ associated to a finitely separated graph and its bipartite sibling, respectively. First, however, we will recall the basics of partial actions.

Definition 2.1. A *partial action* $\theta\colon G \curvearrowright \Omega$ of a discrete group G on a topological space Ω is a family of homeomorphisms of open subspaces $\{\theta_g\colon \Omega_{g^{-1}} \to \Omega_g\}_{g\in G}$, such that

- $\theta_g(\Omega_{g^{-1}} \cap \Omega_h) \subset \Omega_{gh}$ for all $g, h \in G$,
- $\theta_g(\theta_h(x)) = \theta_{gh}(x)$ for all $g, h \in G$ and $x \in \Omega_{h^{-1}} \cap \Omega_{h^{-1}g^{-1}}$,

and we will always assume Ω to be locally compact Hausdorff. Completely similarly, one can define the concept of a partial action on a $(C)^*$-algebra, demanding that the domains should be (closed) ideals. Hence, θ as above translates into a partial C^*-action $\theta^*\colon G \curvearrowright C_0(\Omega)$ given by $C_0(\Omega)_g := C_0(\Omega_g)$ and $\theta_g^*(f) := f \circ \theta_g^{-1}$ for all $g \in G$. As for global actions, one can associated both a *full* and a *reduced* crossed product, and there is a canonical surjective $*$-homomorphism

$$C_0(\Omega) \rtimes G \to C_0(\Omega) \rtimes_r G,$$

called the *regular representation.* We will often write $\rtimes_{(r)}$ to indicate that a given statement concerns both crossed products. If the space Ω is totally disconnected and K is any field with involution, there is also a meaningful, purely algebraic partial action $\theta^*\colon G \curvearrowright C_K(\Omega)$ on the $*$-algebra $C_K(\Omega)$ of compactly supported, locally constant functions, and this gives rise to a single algebraic crossed product $C_K(\Omega) \rtimes G$. We refer the reader to [18] for a comprehensive treatment of crossed products associated with partial actions.

Returning to the topological setting, a subspace $U \subset \Omega$ is called *invariant* if $\theta_g(x) \in U$ for all $g \in G$ and $x \in U \cap \Omega_{g^{-1}}$. Observe that whenever U is open and invariant, then $Z := \Omega \setminus U$ is closed and invariant, so θ naturally restricts to partial actions of both U and Z, giving rise to sequences

$$0 \to C_K(U) \rtimes G \to C_K(\Omega) \rtimes G \to C_K(Z) \rtimes G \to 0$$

and

$$0 \to C_0(U) \rtimes_{(r)} G \to C_0(\Omega) \rtimes_{(r)} G \to C_0(Z) \rtimes_{(r)} G \to 0.$$

On the level of full crossed products and in the purely algebraic context, this sequence is always exact, but for reduced crossed products, one must also require the group to be exact

[18, Theorem 22.9]; as we are really only interested in free groups, this is not a problem. The *orbit* through any $x \in \Omega$ is the set

$$\theta_G(x) := \{\theta_g(x) \mid g \in G \text{ such that } x \in \Omega_{g^{-1}}\},$$

and the action is called *minimal* if every orbit is dense in Ω, or equivalently if the only open invariant subspaces are the trivial ones. The action is called *topologically free* if for every $1 \neq g \in G$, the set of fixed points

$$\Omega^g := \{x \in \Omega_{g^{-1}} \mid \theta_g(x) = x\} \subset \Omega$$

has empty interior. Minimality is of course a necessary condition for simplicity of any of the above crossed products, and if the action is topologically free, then it is also sufficient in both the algebraic and reduced context (see [8, Remark 3.9], [12, Lemma 3.1 and Theorem 4.1] and [18, Corollary 29.8]). A partial action is called *essentially free* if the restriction to every closed invariant subset is topologically free. Essential freeness (together with exactness of the group in the reduced setting) guarantees that all ideals of the algebraic and reduced crossed product are induced from open invariant subspaces (see [16, Corollary 3.7] and [18, Theorem 29.9]).

Now if $U \subset \Omega$ is any open subspace (not necessarily invariant), we may still define a restricted partial action $\theta|_U \colon G \curvearrowright U$ with domains $U_g := \theta_g(U \cap \Omega_{g^{-1}}) \cap U$. Following [15] and [22], such a restriction will be called *full* (or *G-full*) if $\Omega = \{\theta_g(x) \mid g \in G, x \in U \cap \Omega_{g^{-1}}\}$. ◀

We now recall a number of different types of equivalences between partial actions.

Definition 2.2. Suppose that $\theta \colon G \curvearrowright \Omega$ and $\theta' \colon H \curvearrowright \Omega'$ are partial actions and $\Phi \colon G \to H$ is a group homomorphism. A continuous map $\varphi \colon \Omega \to \Omega'$ is called Φ-*equivariant* if

- $\varphi(\Omega_g) \subset \Omega'_{\Phi(g)}$ for all $g \in G$,
- $\theta'_{\Phi(g)}(\varphi(x)) = \varphi(\theta_g(x))$ for all $g \in G$ and $x \in \Omega_{g^{-1}}$.

If $G = H$ and $\Phi = \mathrm{id}_G$, then φ is simply called equivariant (or possibly G-equivariant). The pair (φ, Φ) is called a *conjugacy* if φ is a homeomorphism, Φ is an isomorphism, and φ^{-1} is Φ^{-1}-equivariant. However, conjugacy is often too rigid a notion and we therefore consider a few other types of equivalences: Following [8], the pair (φ, Φ) is called a *direct dynamical equivalence* if

(a) φ is a homeomorphism,
(b) $\Omega_g \cap \Omega_{g'} = \emptyset$ for all $g \neq g'$ with $\Phi(g) = \Phi(g')$,
(c) $\Omega'_h = \bigcup_{\Phi(g)=h} \varphi(\Omega_g)$ for all $h \in H$.

If, moreover, Φ is injective, the pair (φ, Φ) will be called a *direct quasi-conjugacy. Dynamical equivalence* and *quasi-conjugacy* are then simply the equivalence relations on partial actions generated by these two non-symmetric relations. It should be obvious that any dynamical property is preserved by dynamical equivalence. In fact, by [8, Proposition 3.11 and Proposition 3.13], dynamical equivalence is exactly the same as isomorphism of the transformation groupoids $\mathcal{G}_\theta$ and $\mathcal{G}_{\theta'}$. Finally, borrowing from [22] and [15], θ and θ' are called *Kakutani* *equivalent* if there exist full, clopen subspaces $K \subset \Omega$ and $K' \subset \Omega'$ such that the restrictions $\theta|_K$ and $\theta'|_{K'}$ are dynamically equivalent. As was noted just below [8, Definition 3.24], Kakutani equivalence implies Morita equivalence of the associated crossed products. ◀

The following results about full subspaces and Kakutani equivalence will be useful later.

Lemma 2.3. *Suppose that $\theta\colon G \curvearrowright \Omega$ is a partial action on a locally compact Hausdorff space, and let $U \subset \Omega$ denote an open, full subspace. Then there is a bijective correspondence*

$$\{\textit{open } \theta\textit{-invariant subsets of } \Omega\} \to \{\textit{open } \theta|_U\textit{-invariant subsets of } U\} \textit{ given by } V \mapsto U \cap V.$$

Moreover, for any open and θ-invariant $V \subset \Omega$, the following hold:

(1) *$U \cap V$ is full in V.*
(2) *$U \cap (\Omega \setminus V)$ is full in $\Omega \setminus V$.*

Proof. Any intersection $U \cap V$ clearly defines an open $\theta|_U$-invariant subset of U, so we simply have to build an inverse. Suppose that $W \subset U$ is open and $\theta|_U$-invariant, and define an open and θ-invariant subset of Ω by

$$V := \bigcup_{g \in G} \theta_g(W \cap \Omega_{g^{-1}}).$$

Observe that if $x \in W \cap \Omega_{g^{-1}}$, then either $\theta_g(x) \notin U$ or $\theta_g(x) \in W$, hence

$$U \cap V = \bigcup_{g \in G} U \cap \theta_g(W \cap \Omega_{g^{-1}}) = W.$$

Now whenever $y \in V$, then by fullness of U, we have $y = \theta_h(x)$ for some $h \in G$ and $x \in U \cap \Omega_{h^{-1}}$. From invariance, we see that $x \in V$, so $y \in \theta_h(U \cap V \cap \Omega_{h^{-1}})$. Consequently

$$\bigcup_{g \in G} \theta_g(U \cap V \cap \Omega_{g^{-1}}) = V,$$

so the above map is indeed a bijective correspondence, and (1) holds. (2) then follows immediately from (1) and fullness of U. □

Lemma 2.4. *Topological freeness is preserved under Kakutani equivalence.*

Proof. Obviously, topological freeness is preserved under direct dynamical equivalence, so we simply have to show that a partial action θ is topologically free if and only if a G-full restriction $\theta|_K$ to a clopen subspace $K \subset \Omega$ is topologically free. Assume the latter and take some $x \in \Omega^g$ along with an open neighbourhood $U \subset \Omega_{g^{-1}}$ of x. By fullness, there exists some $h \in G$ and $y \in K \cap \Omega_{h^{-1}}$ such that $x = \theta_h(y)$. We then define an open neighbourhood of y in K by

$$V := \theta_{h^{-1}}(U \cap \theta_h(K \cap \Omega_{h^{-1}}))$$

and observe that $y \in K^{h^{-1}gh}$. Now since $\theta|_K$ is topologically free, we can find $y' \in V \cap K_{h^{-1}g^{-1}h}$ such that $\theta_{h^{-1}gh}(y') \neq y'$. Setting $x' := \theta_h(y') \in U$, we then have

$$x' = \theta_h(y') \neq \theta_h(\theta_{h^{-1}gh}(y')) = \theta_{gh}(y') = \theta_g(x')$$

as desired. The other implication is trivial. □

Corollary 2.5. *Essential freeness is preserved under Kakutani equivalence.*

Proof. Once again, it suffices to verify the claim for a partial action $\theta\colon G \curvearrowright \Omega$ and the restriction $\theta|_K$ to a clopen, full subset $K \subset \Omega$. But then the claim follows immediately from Lemma 2.3 and Lemma 2.4. □

We now describe the partial action giving rise to a crossed product description of $\mathcal{O}(E, C)$.

Definition 2.6. Suppose that (E, C) is a finitely separated graph, and let $\mathbb{F}$ denote the free group on E^1. Also, denote by E^0_{iso} the set of isolated vertices with the discrete topology. Given $\xi \subset \mathbb{F}$ and $\alpha \in \xi$, the *local configuration* ξ_α of ξ at α is the set

$$\xi_\alpha := \{\sigma \in E^1 \sqcup (E^1)^{-1} \mid \sigma \in \xi \cdot \alpha^{-1}\}.$$

Then $\Omega(E, C)$ is the disjoint union of E^0_{iso} and the set of $\xi \subset \mathbb{F}$ satisfying the following:

(a) $1 \in \xi$.
(b) ξ is *right-convex*. In view of (a), this exactly means that if $e_n^{\varepsilon_n} \cdots e_1^{\varepsilon_1} \in \xi$ for $e_i \in E^1$ and $\varepsilon_i \in \{\pm 1\}$, then $e_m^{\varepsilon_m} \cdots e_1^{\varepsilon_1} \in \xi$ as well for any $1 \le m < n$.
(c) For every $\alpha \in \xi$, there is some $v \in E^0$ and distinguished $e_X \in X$ for each $X \in C_v$, such that

$$\xi_\alpha = s^{-1}(v) \sqcup \{e_X^{-1} \mid X \in C_v\}.$$

$\Omega(E, C)$ is made into a topological space by regarding it as a subspace of $\{0, 1\}^{\mathbb{F}} \sqcup E^0_{\mathrm{iso}}$. Thus it becomes a zero-dimensional, locally compact Hausdorff space, which is compact if and only if E^0 is a finite set. A topological partial action $\theta = \theta^{(E,C)}\colon \mathbb{F} \curvearrowright \Omega(E, C)$ with compact-open domains is then defined by setting

- $\Omega(E, C)_\alpha := \{\xi \in \Omega(E, C) \setminus E^0_{\mathrm{iso}} \mid \alpha^{-1} \in \xi\}$ for $\alpha \neq 1$,
- $\theta_\alpha(\xi) := \xi \cdot \alpha^{-1}$ for $\xi \in \Omega(E, C)_{\alpha^{-1}}$.

In case (E, C) is a finite bipartite graph, this partial action is conjugate to the one defined in [2] under the map $\xi \mapsto \xi^{-1}$. We choose to invert the configurations so that the terminologies related to the algebras and the dynamical systems agree. We set $\Omega(E, C)_{s(e)} := \Omega(E, C)_{e^{-1}}$ for every $e \in E^1$ and $\Omega(E, C)_u := \bigsqcup_{e \in X} \Omega(E, C)_e$ for every $X \in C_u$. Note that this is well-defined due to the above condition (c). If u is an isolated vertex, we simply set $\Omega(E, C)_u := \{u\}$. Finally, in the case of a trivial separation, we will write $\Omega(E) := \Omega(E, \mathcal{T})$ and $\theta^E := \theta^{(E,\mathcal{T})}$.

Remark 2.7. The partial action θ^E is easily seen to be conjugate to the canonical partial action of $\mathbb{F}$ on the boundary path space ∂E (see [14] and adjust the definition to the Raeburn-convention). However, there are also graphs with non-trivial separations that give rise to boundary path space actions. Indeed, Proposition 3.11 and Proposition 3.20 together identifies a class of such graphs, where the identification is made by an actual conjugacy. Relaxing conjugacy to dynamical equivalence and considering only finite bipartite graphs, we further strengthen this result with Theorem 5.7.

Remark 2.8. If (F, D) is a complete subgraph of (E, C), then there is a natural $\mathbb{F}(F^1)$-equivariant surjection $p\colon \bigsqcup_{v\in F^0} \Omega(E,C)_v \to \Omega(F,D)$ given by

$$p(\xi) = \begin{cases} \xi \cap \mathbb{F}(F^1) & \text{if } \xi \in \Omega(E,C)_v \text{ for } v \notin F^0_{\text{iso}} \\ v & \text{if } \xi \in \Omega(E,C)_v \text{ for } v \in F^0_{\text{iso}} \end{cases}.$$

Consequently, it induces $\mathbb{F}(F^1)$-equivariant embeddings

$$C_K(\Omega(F,D)) \xrightarrow{p^*} C_K\big(\bigsqcup_{v\in F^0} \Omega(E,C)_v\big) \hookrightarrow C_K(\Omega(E,C))$$

and

$$C_0(\Omega(F,D)) \xrightarrow{p^*} C_0\big(\bigsqcup_{v\in F^0} \Omega(E,C)_v\big) \hookrightarrow C_0(\Omega(E,C))$$

from which we obtain $*$-homomorphisms

$$C_K(\Omega(F,D)) \rtimes \mathbb{F}(F^1) \to C_K(\Omega(E,C)) \rtimes \mathbb{F}(F^1) \to C_K(\Omega(E,C)) \rtimes \mathbb{F}(E^1)$$

and

$$C_0(\Omega(F,D)) \rtimes_{(r)} \mathbb{F}(F^1) \to C_0(\Omega(E,C)) \rtimes_{(r)} \mathbb{F}(F^1) \to C_0(\Omega(E,C)) \rtimes_{(r)} \mathbb{F}(E^1).$$

Finally, observe that taking the limits over the finite complete subgraphs with inclusions, we have

$$C_K(\Omega(E,C)) \cong \varinjlim_{(F,D)} C_K(\Omega(F,D)) \quad \text{and} \quad C_0(\Omega(E,C)) \cong \varinjlim_{(F,D)} C(\Omega(F,D))$$

for any finitely separated graph (E, C), and if $E^0_{\text{iso}} = \emptyset$, we have the same approximations when considering only finite complete subgraphs (F, D) with $F^0_{\text{iso}} = \emptyset$. ◀

We introduce a bit of terminology related to the closed subspaces of $\Omega(E,C)$, while the definition of $\Omega(E,C)$ is still fresh in mind.

Definition 2.9. An (E, C)-*animal* is a right-convex subset $\omega \subset \xi$ of a configuration $\xi \in \Omega(E,C) \setminus E^0_{\text{iso}}$ such that $\{1\} \subsetneq \omega$. It is called finite if it has finite cardinality, and for any animal ω, we can define a compact subset of $\Omega(E,C)$ by

$$\Omega(E,C)_\omega := \{\xi \in \Omega(E,C) \mid \omega \subset \xi\},$$

which is open if ω is finite. Given any non-empty subset $\{1\} \neq S \subset \mathbb{F}$ such that $\alpha \cdot \beta^{-1}$ is an admissible path for any pair of distinct $\alpha, \beta \in S \cup \{1\}$, observe that the right-convex closure $\langle S\rangle := \operatorname{conv}(S \cup \{1\})$ of $S \cup \{1\}$ inside $\mathbb{F}$ defines an (E, C)-animal. In order to avoid confusion, the reader should also note that we have the slightly annoying identity $\Omega(E,C)_\alpha = \Omega(E,C)_{\{\alpha^{-1}\}}$.

The *balls* are a particularly important type of animals: An n-ball is simply a set of the form $\xi^n := \{\alpha \in \xi \colon |\alpha| \le n\}$ together with the radius n (we sometimes want to distinguish balls with the same underlying set and different radii). If (E, C) is finite, then any finite animal is contained in a ball, and the compact-open subsets $\Omega(E,C)_B$ corresponding to the balls B form a basis for the topology. We will denote the set of n-balls by $\mathcal{B}_n(\Omega(E,C))$. ◀

We now prove that $\theta^{(E,C)}$ does in fact provide a dynamical description of $L_K^{\mathrm{ab}}(E,C)$ and $\mathcal{O}(E,C)$. While the original proof from [2] proceeds in a constructive manner, applying the machinery of [19] to translate the defining relations into restrictions on the local configurations ξ_α for $\xi \in \mathcal{P}(\mathbb{F})$ with $1 \in \xi$, we will aim for a more direct and conceptually easier, but also somewhat unmotivated proof.

Theorem 2.10. *For any finitely separated graph (E,C), there are canonical isomorphisms*

$$L_K^{\mathrm{ab}}(E,C) \cong C_K(\Omega(E,C)) \rtimes \mathbb{F} \quad \text{and} \quad \mathcal{O}(E,C) \cong C_0(\Omega(E,C)) \rtimes \mathbb{F}.$$

Proof. We may assume without loss of generality that $E^0_{\mathrm{iso}} = \emptyset$. Denote by 1_α and 1_v the indicator function on $\Omega(E,C)_\alpha = \{\xi \in \Omega(E,C) \mid \alpha^{-1} \in \xi\}$ (remember the slightly confusing inversion) and $\Omega(E,C)_v$, respectively, and write $u_\alpha := 1_\alpha \delta_\alpha$ for all $\alpha \in \mathbb{F}$. We then consider the elements u_e and $p_v := 1_v$ in $C_K(\Omega(E,C)) \rtimes \mathbb{F}$ for $e \in E^1$ and $v \in E^0$, claiming that they form a tame (E,C)-family. (V) and (E) are both clear, while (SCK1) follows from the calculation

$$u_e^* u_f = u_{e^{-1}} u_f = \theta^*_{e^{-1}}(1_e 1_f) u_{e^{-1}f} = \delta_{e,f} 1_{e^{-1}} = \delta_{e,f} p_{s(e)}$$

whenever $[e] = [f]$. Noting that $u_e u_e^* = u_e u_{e^{-1}} = 1_e$ for $e \in E^1$, we also see that

$$\sum_{e \in X} u_e u_e^* = \sum_{e \in X} 1_e = p_v$$

for any $v \in E^0$ and $X \in C_v$, so that (SCK2) is satisfied. Now let α denote any reduced product of edges and inverse edges $\alpha = \alpha_n \cdots \alpha_1$. We then introduce the notation $\underline{e} := e$, $\underline{e^{-1}} := e^*$ and $\underline{\alpha} := \underline{\alpha_n} \cdots \underline{\alpha_1}$, and claim that

$$u_{\underline{\alpha}} := u_{\underline{\alpha_n}} \cdots u_{\underline{\alpha_1}} = u_\alpha.$$

Assuming the claim holds for products of length $n-1$ and writing $\beta = \alpha_{n-1} \cdots \alpha_1$, we see that

$$u_{\underline{\alpha}} = u_{\underline{\alpha_n}} u_{\underline{\beta}} = 1_{\alpha_n} 1_\alpha u_\alpha = u_\alpha,$$

where we used right-convexity to conclude that $1_{\alpha_n} 1_\alpha = 1_\alpha$. It follows that

$$u_{\underline{\alpha}} u_{\underline{\alpha}}^* = u_\alpha u_\alpha^* = 1_\alpha,$$

so in particular the set $\{u_e \mid e \in E^1\}$ is tame. From universality, we therefore obtain a $*$-homomorphism $\varphi\colon L_K^{\mathrm{ab}}(E,C) \to C_K(\Omega(E,C)) \rtimes \mathbb{F}$ given by $e \mapsto u_e$ and $v \mapsto p_v$.

We now begin the construction of an inverse by first building a $*$-homomorphism

$$\rho\colon C_K(\Omega(E,C)) \to L_K^{\mathrm{ab}}(E,C).$$

To this end, let (F,D) denote any finite complete subgraph with $F^0_{\mathrm{iso}} = \emptyset$, let $n \geq 1$ and write

$$1_B := \prod_{\alpha \in B} 1_{\alpha^{-1}} \in C_K(\Omega(F,D))$$

for any $B \in \mathcal{B}_n(\Omega(F,D))$; 1_B is merely the indicator function on the subspace $\Omega(F,D)_B$. We then define finite-dimensional subalgebras

$$\mathfrak{B}_n^{(F,D)} := \mathrm{span}\{1_B \mid B \in \mathcal{B}_n(\Omega(F,D))\} \subset C_K(\Omega(F,D))$$

along with inclusions $\phi_n^{(F,D)}\colon \mathfrak{B}_n^{(F,D)} \to \mathfrak{B}_{n+1}^{(F,D)}$ given by

$$\phi_n^{(F,D)}(1_B) = \sum_{B \subset B' \in \mathcal{B}_{n+1}(\Omega(F,D))} 1_{B'},$$

and observe that $C_K(\Omega(F,D)) = \varinjlim_n \mathfrak{B}_n^{(F,D)}$. Now if $(F,D) \subset (G,L)$, then the inclusion of Remark 2.8 restricts to an inclusion $\mathfrak{B}_n^{(F,D)} \hookrightarrow \mathfrak{B}_n^{(G,L)}$, which makes the diagram

$$\begin{array}{ccc} \mathfrak{B}_n^{(F,D)} & \xrightarrow{\phi_n^{(F,D)}} & \mathfrak{B}_{n+1}^{(F,D)} \\ \big\downarrow & & \big\downarrow \\ \mathfrak{B}_n^{(G,L)} & \xrightarrow[\phi_n^{(G,L)}]{} & \mathfrak{B}_{n+1}^{(G,L)} \end{array}$$

commute. In conclusion, defining a $*$-homomorphism out of $C_K(\Omega(E,C))$ is the same as defining a family of $*$-homomorphisms out of the algebras $\mathfrak{B}_n^{(F,D)}$ that respects both the horizontal and vertical maps above. Now consider the self-adjoint linear map

$$\rho_n^{(F,D)}\colon \mathfrak{B}_n^{(F,D)} \to L_K^{\mathrm{ab}}(E,C) \quad \text{given by} \quad \rho_n^{(F,D)}(1_B) = \prod_{\alpha \in B} \underline{\alpha}^* \underline{\alpha},$$

which is well-defined since $E^1 \subset L_K^{\mathrm{ab}}(E,C)$ is tame (see for instance [18, Proposition 12.8]). Checking that $\rho_n^{(F,D)}$ is also multiplicative exactly amounts to showing

$$\rho_n^{(F,D)}(1_{B_1})\rho_n^{(F,D)}(1_{B_2}) = 0$$

for all $B_1 \neq B_2$. Since

$$\rho_n^{(F,D)}(1_B) \leq \rho_m^{(F,D)}\big(1_{B^m}\big),$$

where $B^m := \{\alpha \in B\colon |\alpha| \leq m\}$, for all $m \leq n$, we may assume that $B_1^{n-1} = B_2^{n-1}$. Consequently there is some $\beta \in B_1, B_2$ of length $|\beta| = n-1$ and $X \in C_{r(\beta)}$ with distinct $x_1, x_2 \in X$ such that $\alpha_1 := x_1^{-1}\beta \in B_1$ and $\alpha_2 := x_2^{-1}\beta \in B_2$. We see that

$$\underline{\alpha_1}^* \underline{\alpha_1} \underline{\alpha_2}^* \underline{\alpha_2} = x_1 \underline{\beta}^* \underline{\beta} x_1^* x_2 \underline{\beta}^* \underline{\beta} x_2 = 0,$$

so $\rho_n^{(F,D)}(1_{B_1})\rho_n^{(F,D)}(1_{B_2}) = 0$ as well. In order to see that these $*$-homomorphisms respect both the vertical and horizontal inclusions above, simply note that both follow from an inductive application of the following observations. Given any finite (E,C)-animal ω, the following hold:

(1) If $1 \neq \beta \in \omega$ and $\omega_\beta \cap X^{-1} = \emptyset$ for some $X \in C_{r(\beta)}$, then

$$\prod_{\alpha \in \omega} \underline{\alpha}^* \underline{\alpha} = \sum_{x \in X} \prod_{\alpha \in \omega \cup \{x^{-1}\beta\}} \underline{\alpha}^* \underline{\alpha}.$$

(2) If $1 \neq \beta \in \omega$ and $e \notin \omega_\beta$ for some $e \in s^{-1}(r(\beta))$, then

$$\prod_{\alpha \in \omega} \underline{\alpha}^* \underline{\alpha} = \prod_{\alpha \in \omega \cup \{e\beta\}} \underline{\alpha}^* \underline{\alpha}.$$

We thereby obtain a unique $*$-homomorphism $\rho\colon C_K(\Omega(E,C)) \to L_K^{\mathrm{ab}}(E,C)$ characterised by $\rho(1_\alpha) = \underline{\alpha}\underline{\alpha}^*$ for any $\alpha \in \mathbb{F}$. Now observe that

$$\varphi \circ \rho(1_\alpha) = \varphi(\underline{\alpha}\underline{\alpha}^*) = u_{\underline{\alpha}} u_{\underline{\alpha}}^* = 1_\alpha$$

for any α, so the composition $\varphi \circ \rho$ is nothing but the inclusion $C_K(\Omega(E,C)) \hookrightarrow C_K(\Omega(E,C)) \rtimes \mathbb{F}$. Since E^1 is tame in $L_K^{\mathrm{ab}}(E,C)$, by the implication (iii)⇒(i) of [18, Proposition 12.13] which holds in an arbitrary unital $*$-algebra, there is a semi-saturated partial representation σ of $\mathbb{F}$ on the unitalisation of $L_K^{\mathrm{ab}}(E,C)$ given by $\sigma(\alpha) := \underline{\alpha}$ for all $\alpha \neq 1$, so that $\rho(1_\alpha) = p(\alpha) := \sigma(\alpha)\sigma(\alpha)^*$. We claim that the pair (ρ, σ) is a covariant representation. It suffices to check that

$$\sigma(\alpha)\rho(1_{\alpha^{-1}}1_\beta)\sigma(\alpha)^* = \rho(\theta_\alpha^*(1_{\alpha^{-1}}1_\beta))$$

for all $\alpha, \beta \in \mathbb{F}$, and from [18, Proposition 9.8(iii)], we have $p(\beta)\sigma(\alpha)^* = \sigma(\alpha)^* p(\alpha\beta)$. We now see that

$$\begin{aligned} \sigma(\alpha)\rho(1_{\alpha^{-1}}1_\beta)\sigma(\alpha)^* &= \sigma(\alpha)p(\alpha^{-1})p(\beta)\sigma(\alpha)^* = \sigma(\alpha)p(\beta)\sigma(\alpha)^* = \sigma(\alpha)\sigma(\alpha)^* p(\alpha \cdot \beta) \\ &= p(\alpha)p(\alpha \cdot \beta) = \rho(1_\alpha 1_{\alpha \cdot \beta}) = \rho(\theta_\alpha(1_{\alpha^{-1}}1_\beta)) \end{aligned}$$

as desired, so there is an induced $*$-homomorphism $\rho \times \sigma\colon C_K(\Omega(E,C)) \rtimes \mathbb{F} \to L_K^{\mathrm{ab}}(E,C)$. Since

$$(\rho \times \sigma) \circ \varphi(e) = \rho \times \sigma(1_e \delta_e) = ee^*e = e$$

for all $e \in E^1$, we have $(\rho \times \sigma) \circ \varphi = \mathrm{id}$. Moreover, the fact that $\varphi \circ \rho$ is simply the inclusion $C_K(\Omega(E,C)) \hookrightarrow C_K(\Omega(E,C)) \rtimes \mathbb{F}$ together with the observation

$$\varphi \circ (\rho \times \sigma)(u_\alpha) = \varphi(\underline{\alpha}\underline{\alpha}^*\underline{\alpha}) = \varphi(\underline{\alpha}) = u_{\underline{\alpha}} = u_\alpha$$

for all $\alpha \in \mathbb{F}$ implies that $\varphi \circ (\rho \times \sigma) = \mathrm{id}$ as well. It follows that $L_K^{\mathrm{ab}}(E,C) \cong C_K(\Omega(E,C)) \rtimes \mathbb{F}$ as desired, and the C^*-case is completely similar. □

We now see that the $*$-homomorphisms coming from inclusions of complete subgraphs are simply those of Remark 2.8.

Lemma 2.11. *Let (E,C) denote a finitely separated graph, and consider an embedding of a complete subgraph $(F,D) \overset{\iota}{\hookrightarrow} (E,C)$. Then the $*$-homomorphisms*

$$L_K^{\mathrm{ab}}(\iota)\colon L_K^{\mathrm{ab}}(F,D) \to L_K^{\mathrm{ab}}(E,C) \quad \text{and} \quad \mathcal{O}(\iota)\colon \mathcal{O}(F,D) \to \mathcal{O}(E,C)$$

are exactly the ones of Remark 2.8. Consequently, there is a unique $*$-homomorphism

$$\mathcal{O}^r(\iota)\colon \mathcal{O}^r(F,D) \to \mathcal{O}^r(E,C)$$

making the diagram

$$\begin{array}{ccc} \mathcal{O}(F,D) & \xrightarrow{\mathcal{O}(\iota)} & \mathcal{O}(E,C) \\ \downarrow & & \downarrow \\ \mathcal{O}^r(F,D) & \xrightarrow[\mathcal{O}^r(\iota)]{} & \mathcal{O}^r(E,C) \end{array}$$

commute.

Proof. Simply observe that $L_K^{\mathrm{ab}}(\iota)$ and $\mathcal{O}(\iota)$ agree with the homomorphisms of Remark 2.8 on the generators. ☐

We can also characterise $\theta^{(E,C)}$ by a very useful universal property, corresponding to the universal property of $\mathcal{O}(E,C)$. Recall that whenever $\{\theta_a \mid a \in A\}$ is a family of homeomorphisms of open subspaces of a space Ω, there is a canonical partial action of $\mathbb{F}(A)$ on Ω: If $\alpha = a_n^{\varepsilon_n} \cdots a_1^{\varepsilon_1}$ is a reduced word, then $\theta_\alpha := \theta_{a_n}^{\varepsilon_n} \cdots \theta_{a_1}^{\varepsilon_1}$, where $\cdot$ denotes the maximal composition of two functions. It is verified in [18, Proposition 4.7] that this does indeed define a partial action.

Definition 2.12. Suppose that Ω is a locally compact Hausdorff space and (E,C) is a finitely separated graph. An (E,C)-*action* on Ω is the canonical action of $\mathbb{F} = \mathbb{F}(E^1)$ induced by a family $\{\theta_e \colon \Omega_{e^{-1}} \to \Omega_e \mid e \in E^1\}$ of partial homeomorphisms of compact open subspaces with the following properties:

1. There is a decomposition $\Omega = \bigsqcup_{v \in E^0} \Omega_v$ for compact open subspaces $\Omega_v \subset \Omega$.
2. If $e \in E^1$, then $\Omega_{s(e)} = \Omega_{e^{-1}}$.
3. If $v \in E^0$ and $X \in C_v$, then $\Omega_v = \bigsqcup_{e \in X} \Omega_e$.

An (E,C)-action $\theta \colon \mathbb{F} \curvearrowright \Omega$ is called *universal* if, given any other (E,C)-action $\theta' \colon \mathbb{F} \curvearrowright \Omega'$, there exists a unique $\mathbb{F}$-equivariant continuous map $f \colon \Omega' \to \Omega$ such that $f(\Omega'_v) \subset f(\Omega_v)$ for any isolated vertex v (this will automatically hold for any other vertex due to equivariance). Observe that a universal (E,C)-action is unique up to canonical conjugacy. Finally, if $C = \mathcal{T}$, we will simply suppress the separation, referring instead to an E-action. ◀

The following can be obtained from Theorem 2.10 by applying duality, but we choose to a give a concrete proof for clarity.

Proposition 2.13. *The partial action $\theta^{(E,C)}$ is the universal (E,C)-action for any finitely separated graph (E,C). If $\theta \colon \mathbb{F} \curvearrowright \Omega$ is any other (E,C)-action, then the unique equivariant map $f \colon \Omega \to \Omega(E,C)$ satisfying $p(\Omega_v) \subset \Omega(E,C)_v$ for $v \in E^0_{\mathrm{iso}}$ is given by*

$$p(x) = \begin{cases} \mathbb{F}^x := \{\alpha \in \mathbb{F} \mid x \in \Omega_{\alpha^{-1}}\} & \text{if } x \in \bigsqcup_{v \in E^0 \setminus E^0_{\mathrm{iso}}} \Omega_v \\ v & \text{if } x \in \Omega_v \text{ for } v \in E^0_{\mathrm{iso}} \end{cases}.$$

Proof. It is clear that $\theta^{(E,C)}$ is itself an (E,C)-action, and that the restriction of p to $\bigsqcup_{v \in E^0_{\mathrm{iso}}} \Omega_v$ is the unique map satisfying $p(\Omega_v) \subset \Omega(E,C)_v$. Considering any $x \in \bigsqcup_{v \in E^0 \setminus E^0_{\mathrm{iso}}} \Omega_v$, it is also clear that $\mathbb{F}^x$ is a right-convex set containing 1. We have $x \in \Omega_v$ for a unique $v \in E^0$, so $x \in \Omega_{e^{-1}}$ if and only if $e \in s^{-1}(v)$, and $x \in \Omega_{e_X}$ for a unique $e_X \in X$ for all $X \in C_v$. Hence the local configuration of $\mathbb{F}^x$ at 1 is given by

$$s^{-1}(v) \sqcup \{e_X^{-1} \mid X \in C_v\}$$

as in Definition 2.6(c). Now if $\alpha \in \mathbb{F}^x$, then we may apply this observation to $\theta_\alpha(x)$ to see that $\mathbb{F}^x_\alpha = \mathbb{F}^{\theta_\alpha(x)}_1$ is of the same type, thus $\mathbb{F}^x \in \Omega(E, C)$. Equivariance and continuity of $x \mapsto \mathbb{F}^x$ is obvious, and if $\varphi\colon \Omega \to \Omega(E, C)$ is any equivariant map, then necessarily $\varphi(x) = \mathbb{F}^{\varphi(x)} \supset \mathbb{F}^x$. Now since $\mathbb{F}^x, \mathbb{F}^{\varphi(x)} \in \Omega(E, C)$, we must have $\mathbb{F}^{\varphi(x)} = \mathbb{F}^x$, and so $\varphi(x) = \mathbb{F}^x = p(x)$. □

Remark 2.14. Note that if (F, D) is a complete subgraph of (E, C), and θ is the restriction of $\theta^{(E,C)}$ to $\bigsqcup_{v\in F^0} \Omega(E, C)_v$, then the map p from Remark 2.8 is exactly the map p of Proposition 2.13. ◀

Now that we have a dynamical system associated to every finitely separated graph, we will shortly consider the relationship between the dynamics of (E, C) and its bipartite sibling $\mathbf{B}(E, C)$ as defined in [3, Definition 7.4]. First though, we have to introduce a bit of terminology.

Definition 2.15. For any topological Ω, we will write $\overrightarrow{\Omega} = \Omega^1 \sqcup \Omega^0$, where each

$$\Omega^i = \{\xi^i \mid \xi \in \Omega\}$$

is simply a copy of Ω. Given a partial homeomorphism φ of Ω, we define a partial homeomorphism $\overrightarrow{\varphi}\colon \mathrm{dom}(\varphi)^1 \to \mathrm{im}(\varphi)^0$ by $\overrightarrow{\varphi}(\xi^1) = \varphi(\xi)^0$. Now if $\theta\colon \mathbb{F}(A) \curvearrowright \Omega$ is a partial action induced from a family of partial homeomorphisms $\{\theta_a\}_{a\in A}$, we define the *double action* of θ to be the partial action $\overrightarrow{\theta}\colon \mathbb{F}(A) * \mathbb{Z} \curvearrowright \overrightarrow{\Omega}$ induced by the family $\{\overrightarrow{\theta_a}\}_{a\in A}$ and $\sigma := \overrightarrow{\mathrm{id}_\Omega}$.

Proposition 2.16. *Consider a partial action θ as in Definition 2.15. For any $i = 0, 1$, there is a direct quasi-conjugacy $\theta \to \overrightarrow{\theta}|_{\Omega^i}$ and a direct dynamical equivalence $\overrightarrow{\theta}|_{\Omega_i} \to \theta$.*

Proof. We only consider the case $i = 1$; the other one is completely analogous. Denote the generator of $\mathbb{Z}$ by s so that $\mathbb{F}(A) * \mathbb{Z} = \mathbb{F}(A \cup \{s\})$, and consider the injective homomorphism $\Phi\colon \mathbb{F}(A) \to \mathbb{F}(A \cup \{s\})$ given by $\Phi(a) = s^{-1}a$ for all $a \in A$ as well as the embedding $\varphi\colon \Omega \to \overrightarrow{\Omega}$ onto Ω^1. Then

$$\big(\overrightarrow{\theta}\big)_{\Phi(a)} = \big(\overrightarrow{\theta}\big)_{s^{-1}a} = \sigma^{-1} \circ \overrightarrow{\theta_a} = \varphi \circ \theta_a$$

for all $a \in A$, so (φ, Φ) is a conjugacy of θ and the restricted partial action $\mathrm{im}(\Phi) \curvearrowright \Omega^1$. It simply remains to check that $\Omega^1_\beta = \emptyset$ for all $\beta \in \mathbb{F}(A \cup \{s\}) \setminus \mathrm{im}(\Phi)$. Observe that such β, as a reduced word, must contain a subword either of one of the forms sa, as, aa' for $a, a' \in A$ or an inverse of one of these. In every case, we see that $\Omega^1_\beta = \emptyset$, as desired. Since Φ^{-1} can be extended to a group homomorphism $\Psi\colon \mathbb{F}(A \cup \{s\}) \to \mathbb{F}(A)$, namely $\Psi(a) = a$ for $a \in A$ and $\Psi(s) = 1$, we see that (φ^{-1}, Ψ) defines a direct dynamical equivalence $\overrightarrow{\theta}|_{\Omega^1} \to \theta$. □

We now relate the partial actions of (E, C) and $\mathbf{B}(E, C)$.

Proposition 2.17. *Let (E,C) denote a finitely separated graph and write $(\tilde{E},\tilde{C}) := \mathbf{B}(E,C)$ as well as $\theta := \theta^{(E,C)}$. Then there is a direct dynamical equivalence $\theta^{(\tilde{E},\tilde{C})} \to \overrightarrow{\theta}$. In particular, $\theta^{(\tilde{E},\tilde{C})}$ and θ are Kakutani equivalent.*

Proof. Recall that $(\tilde{E},\tilde{C})$ is the finitely separated bipartite graph given by

- $\tilde{E}^{0,i} := \{v_i \mid v \in E^0\}$ for $i = 0,1$,
- $\tilde{E}^1 := \{\tilde{e} \mid e \in E^1\} \cup \{h_v \mid v \in E^0\}$,
- $\tilde{r}(\tilde{e}) := r(e)_0$ and $\tilde{s}(\tilde{e}) := s(e)_1$ for all $e \in E^1$,
- $\tilde{r}(h_v) := v_0$ and $\tilde{s}(h_v) := v_1$ for all $v \in E^0$,
- $\tilde{C}_{v_0} := \{\{h_v\}, \tilde{X} \mid X \in C_v\}$ where $\tilde{X} := \{\tilde{e} \mid e \in X\}$ for all $v \in E^0$.

As above, we denote the generator of the factor $\mathbb{Z}$ by s, and we will write $\Omega := \Omega(E,C)$. We first define an $(\tilde{E},\tilde{C})$-action γ on $\overrightarrow{\Omega}$ by

- $\big(\overrightarrow{\Omega}\big)_{v_i} := \Omega_v^i$ for all $v \in E^0$ and $i = 0,1$,
- $\big(\overrightarrow{\Omega}\big)_{\tilde{e}^{-1}} := \Omega^1_{e^{-1}}$, $\big(\overrightarrow{\Omega}\big)_{\tilde{e}} := \Omega^0_e$ and $\gamma_{\tilde{e}} := \overrightarrow{\theta_e}$ for all $e \in E^1$,
- $\big(\overrightarrow{\Omega}\big)_{h_v^{-1}} := \Omega^1_v$, $\big(\overrightarrow{\Omega}\big)_{h_v} := \Omega^0_v$ and $\gamma_{h_v} := \overrightarrow{\mathrm{id}_{\Omega_v}} = \sigma|_{\Omega_v^1}$ for all $v \in E^0$.

Observe that there is a direct dynamical equivalence $(\mathrm{id},\Phi)\colon \gamma \to \overrightarrow{\theta}$, where $\Phi(e) = e$ and $\Phi(h_v) = s$ for all $e \in E^1$ and $v \in E^0$. Now, by the universal property of $\theta^{(\tilde{E},\tilde{C})}$, there is a unique $\mathbb{F}(\tilde{E}^1)$-equivariant continuous map $\varphi\colon \overrightarrow{\Omega} \to \Omega(\tilde{E},\tilde{C})$, and we claim that this is in fact a conjugacy. To see this, we first define injective group homomorphisms $\Psi^1,\Psi^0\colon \mathbb{F}(E^1) \to \mathbb{F}(\tilde{E}^1)$ by

$$\Psi^1(e) = h_{r(e)}^{-1}\tilde{e} \quad \text{and} \quad \Psi^0(e) = \tilde{e}h_{s(e)}^{-1},$$

and observe (just as in Proposition 2.16) that the identification $\Omega \cong \Omega^i$ together with Ψ^i defines a direct quasi-conjugacy $\theta \to \gamma|_{\Omega^i}$. Next, define an (E,C)-action γ^i on

$$\Omega^i(\tilde{E},\tilde{C}) := \bigsqcup_{v\in E^0} \Omega(\tilde{E},\tilde{C})_{v_i}$$

for $i = 0,1$ by

- $\Omega^i(\tilde{E},\tilde{C})_v := \Omega(\tilde{E},\tilde{C})_{v_i}$ for all $v \in E^0$,
- $\Omega^i(\tilde{E},\tilde{C})_{e^{\pm 1}} := \Omega(\tilde{E},\tilde{C})_{\Psi^i(e^{\pm 1})}$ and $\gamma^i_e := \theta^{(\tilde{E},\tilde{C})}_{\Psi^i(e)}$ for all $e \in E^1$.

From the universal property of (E,C) and the observations just above, there is a unique $\mathbb{F}(E^1)$-equivariant continuous map $\psi^i\colon \Omega^i(\tilde{E},\tilde{C}) \to \Omega^i$, and $\psi^i \circ \varphi|_{\Omega^i} = \mathrm{id}_{\Omega^i}$ by uniqueness. Setting $\psi := \psi^1 \sqcup \psi^0\colon \Omega(\tilde{E},\tilde{C}) \to \Omega$ so that $\psi \circ \varphi = \mathrm{id}_{\overrightarrow{\Omega}}$, we claim that ψ is in fact $\mathbb{F}(\tilde{E}^1)$-equivariant. By construction, it is equivariant under both $\mathrm{im}(\Psi^1)$ and $\mathrm{im}(\Psi^0)$, so we simply have to check that it is also equivariant under the action of every h_v, i.e. that the diagram

$$\begin{array}{ccc} \Omega^1(\tilde{E},\tilde{C}) & \xrightarrow{\bigsqcup_{v\in E^0}\theta_{h_v}^{(\tilde{E},\tilde{C})}} & \Omega^0(\tilde{E},\tilde{C}) \\ \psi_1 \downarrow & & \downarrow \psi_0 \\ \Omega^1 & \xrightarrow[\sigma]{} & \Omega^0 \end{array}$$

commutes. Note that all four entries carry partial actions of $\mathbb{F}(E^1)$, and that the maps are all equivariant with respect to these actions. Since the action of $\mathbb{F}(E^1)$ on Ω^0 is the universal (E, C)-action, uniqueness of $\mathbb{F}(E^1)$-equivariant maps $\Omega^1(\tilde{E},\tilde{C}) \to \Omega^0$ guarantees that the diagram actually commutes. We conclude that $\varphi \circ \psi$ is $\mathbb{F}(\tilde{E}^1)$-equivariant, hence $\varphi \circ \psi = \mathrm{id}_{\Omega(\tilde{E},\tilde{C})}$ as desired. □

We finally observe that hereditary and C-saturated subsets, just as for finite bipartite separated graphs, give rise to ideals in the tame algebras. When $\xi \in \Omega(E, C)_v$ is a configuration, we regard $1 \in \xi$ as the trivial path v and so $r(1) := v$ by convention.

Definition 2.18. Given a hereditary and C-saturated subset $H \subset E^0$, we define

$$\Omega(E, C)^H := \{\xi \in \Omega(E, C) \mid r(\alpha) \in H \text{ for some } \alpha \in \xi\}.$$

Theorem 2.19. *Let (E, C) denote a finitely separated graph, and consider a hereditary and C-saturated set of vertices $H \subset E^0$. Then $\Omega(E, C)^H$ is an open and invariant subspace, and there is a direct quasi-conjugacy $\theta^{(E/H,C/H)} \to \theta^{(E,C)}|_Z$ where $Z := \Omega(E, C) \setminus \Omega(E, C)^H$. Letting $I(H)$ denote the induced ideal in the various algebras, which is exactly the ideal generated by H, we therefore have isomorphisms*

$$L_K^{\mathrm{ab}}(E, C)/I(H) \cong L_K^{\mathrm{ab}}(E/H, C/H) \quad \text{and} \quad \mathcal{O}^{(r)}(E, C)/I(H) \cong \mathcal{O}^{(r)}(E/H, C/H).$$

Proof. Simply observe that the proof of [8, Theorem 5.5] (or rather the second part of it) generalises with minimal effort. □

3. Degeneracy of $L_K(E, C)$ and $C^*(E, C)$

In this section, we give a sufficient condition for $L_K(E, C)$ and $C^*(E, C)$ to be isomorphic to a graph algebra of a non-separated graph; we regard this as a degenerating situation since these algebras are well studied. This isomorphism is always implemented by reversing certain edges of the separated graph, a technique also used by Duncan in [17]. The concepts and theorems of this section will be used heavily in subsequent sections on simplicity and the exchange property.

First we will need to introduce the following essential definitions.

Definition 3.1 ([8, Definition 9.5])**.** Let (E, C) denote a finitely separated graph. An admissible path α is called a *choice path* if there is an admissible composition $X^{-1}\alpha$ for some $X \in C$ with $|X| \geq 2$.

Definition 3.2. Let (E, C) denote a finitely separated graph. We shall say that $e \in E^1$ *admits a choice* if there is a choice path α satisfying $\mathfrak{i}_{\mathrm{d}}(\alpha) = e$, while an inverse edge e^{-1} admits a choice if $|[e]| \geq 2$, or if there is a choice path α with $\mathfrak{i}_{\mathrm{d}}(\alpha) = e^{-1}$. A set $X \in C$ then admits a choice if e^{-1} admits a choice for some $e \in X$, and finally a vertex $v \in E^0$ is said to admit exactly

$$|\{e \in s^{-1}(v) \mid e \text{ admits a choice}\}| + |\{X \in C_v \mid X \text{ admits a choice}\}|$$

choices. ◀

The important distinction – as we will see – is between those vertices that admit no, those that admit exactly one, and those that admit at least two choices. The following easy lemma guarantees that the equivalence relation of being on the same cycle respects this distinction.

Lemma 3.3. *Let (E, C) denote a finitely separated graph. If u and v are on the same cycle, then*

1. *u admits no choices if and only v admits no choices,*
2. *u admits exactly one choice if and only v admits exactly one choice,*
3. *u admits at least two choices if and only if v admits at least two choices.*

Proof. Say that $\beta\alpha$ is a cycle with $s(\alpha) = u$ and $r(\alpha) = v$. Observe that if $\sigma \in r^{-1}(u)^{-1} \cup s^{-1}(u)$ admits a choice at u, then so does either $\mathfrak{i}_{\mathrm{d}}(\beta)$ or $\mathfrak{t}_{\mathrm{d}}(\alpha)^{-1}$, as either $\sigma\beta$ or $\sigma\alpha^{-1}$ is admissible. Now assume that $\sigma, \tau \in r^{-1}(u)^{-1} \cup s^{-1}(u)$ give rise to two different choices at u. If $\sigma\beta$ is not admissible, then both $\tau\beta$ and $\sigma\alpha^{-1}$ must be admissible, hence v admits at least two choices as well. Likewise we may assume, without loss of generality, that $\tau\beta$ is admissible, thereby verifying (3). (2) now follows automatically. □

Definition 3.4. If α is a cycle passing through v, then we will say that α admits

1. no choices, if v admits no choices,
2. exactly one choice, if v admits exactly one choice,
3. at least two choices, if v admits at least two choices.

By Lemma 3.3, this is independent of the choice of v on α.

Definition 3.5. Let (E, C) denote a finitely separated graph. We will say that (E, C) satisfies Condition (C) if every $v \in E^0$ admits at most one choice. ◀

Recall that a non-separated graph E is said to satisfy Condition (L) if for every cycle $\alpha = e_n \cdots e_1$, there is some $1 \leq k \leq n$ and $f \neq e_k$ with $r(e_k) = r(f)$. The edge f is usually referred to as an *entry* of α. It is well known that $C^*(E)$ is simple if and only if E satisfies Condition (L) and has only trivial hereditary and saturated subsets. Ara and Exel defined Condition (L) for finite bipartite separated graphs in [2], and as it will play an important role in the next few sections, we now redefine it in the language of this paper for arbitrary finitely separated graphs.

Definition 3.6 ([2, Definition 10.2])**.** A finitely separated graph (E, C) is said to satisfy Condition (L) if any simple cycle admits a choice.

Theorem 3.7 ([2, Theorem 10.5])**.** *Let (E,C) denote a finitely separated graph. Then $\theta^{(E,C)}$ is topologically free if and only if (E,C) satisfies Condition (L).*

Proof. The strategy from [2, Theorem 10.5] easily generalises to arbitrary finitely separated graphs. □

Definition 3.8. A *non-separated orientation* of a finitely separated graph (E,C) is a decomposition $E^1 = E^1_- \sqcup E^1_+$ such that $[e]=\{e\}$ for every $e \in E^1_+$ and one of the following holds for any $v \in E^0$:

(1) $E^1_- \cap r^{-1}(v) \in C_v$ and $E^1_+ \cap s^{-1}(v) = \emptyset$.
(2) $E^1_- \cap r^{-1}(v) = \emptyset$ and $|E^1_+ \cap s^{-1}(v)| \leq 1$.

This is a special case of an *orientation*, which is defined in [23, Definition 3.11]. We shall often regard the partition $E = E^1_- \sqcup E^1_+$ as a map $\mathfrak{o}\colon E^1 \to \{-1,1\}$ with

$$\mathfrak{o}(e) := \begin{cases} 1 & \text{if } e \in E^1_+ \\ -1 & \text{if } e \in E^1_- \end{cases}.$$

An admissible path of the form

$$e_n^{\mathfrak{o}(e_n)} e_{n-1}^{\mathfrak{o}(e_{n-1})} \cdots e_2^{\mathfrak{o}(e_2)} e_1^{\mathfrak{o}(e_1)}$$

will then be called *positively oriented*, while a path of the form

$$e_n^{-\mathfrak{o}(e_n)} e_{n-1}^{-\mathfrak{o}(e_{n-1})} \cdots e_2^{-\mathfrak{o}(e_2)} e_1^{-\mathfrak{o}(e_1)}$$

will be called *negatively oriented*. By [23, Lemma 3.12], every admissible path α decomposes as $\alpha = \alpha_- \alpha_+$, where α_+ and α_- are positively and negatively oriented, respectively. ◀

The point of a non-separated orientation is that it allows us to turn a separated graph into a non-separated one.

Definition 3.9. Assume that (E,C) is a finitely separated graph. If (E,C) admits a non-separated orientation $E^1 = E^1_- \sqcup E^1_+$, then we can define a corresponding column-finite directed graph $\overline{E} = (\overline{E}^0, \overline{E}^1, \overline{r}, \overline{s})$ by $\overline{E}^0 = \{\overline{v} \mid v \in E^0\}$, $\overline{E}^1 = \{\overline{e} \mid e \in E^1\}$,

$$\overline{r}(\overline{e}) = \begin{cases} r(e) & \text{if } e \in E^1_- \\ s(e) & \text{if } e \in E^1_+ \end{cases} \quad \text{and} \quad \overline{s}(\overline{e}) = \begin{cases} s(e) & \text{if } e \in E^1_- \\ r(e) & \text{if } e \in E^1_+ \end{cases}.$$

◀

Before considering the relationship between the dynamics and algebras of (E,C) and $\overline{E}$, we first record a graph-theoretical lemma for later use.

Lemma 3.10. *Assume that $\mathfrak{o}$ is a non-separated orientation of (E,C), and let $\overline{E}$ denote the resulting non-separated graph. The map*

$$\overline{e_n}\,\overline{e_{n-1}} \cdots \overline{e_2}\,\overline{e_1} \mapsto e_n^{-\mathfrak{o}(e_n)} e_{n-1}^{-\mathfrak{o}(e_{n-1})} \cdots e_2^{-\mathfrak{o}(e_2)} e_1^{-\mathfrak{o}(e_1)}$$

is a bijection between the paths of $\overline{E}$ and the negatively oriented paths of (E,C).

Proof. First observe that $\overline{r}(\overline{e}) = r(e^{-\mathfrak{o}(e)})$ and $\overline{s}(\overline{e}) = s(e^{-\mathfrak{o}(e)})$ for all $e \in E^1$, so the above correspondence takes inverse paths of $\overline{E}$ to paths in the double $\hat{E}$, and vice versa. We simply have to check that the paths in the double are also admissible, i.e. that if $e_i \in E^1_-$ and $e_{i+1} \in E^1_+$, then $[e_i] \neq [e_{i+1}]$. But this is clear since either $X \subset E^1_-$ or $X \subset E^1_+$ for all $X \in C$. □

Proposition 3.11. *Assume that (E, C) is a finitely separated graph with a non-separated orientation, and let $\overline{E}$ denote the resulting non-separated graph. Then $\theta^{(E,C)}$ is conjugate to $\theta^{\overline{E}} \colon \mathbb{F} \curvearrowright \Omega(\overline{E})$,*

$$L_K(E, C) = L^{\mathrm{ab}}_K(E, C) \cong L(\overline{E}) \quad \textit{and} \quad C^*(E, C) = \mathcal{O}(E, C) \cong C^*(\overline{E}).$$

Moreover, a subset $H \subset E^0$ is hereditary and C-saturated if and only if $\overline{H} = \{\overline{v} \mid v \in H\}$ is hereditary and saturated in $\overline{E}$.

Proof. First, observe that we can define an (E, C)-action γ on $\Omega(\overline{E})$ by

- $\Omega(\overline{E})_v := \Omega(\overline{E})_{\overline{v}}$ for $v \in E^0$,
- $\Omega(\overline{E})_{e^{\pm 1}} := \Omega(\overline{E})_{\overline{e}^{\mp 1}}$ and $\gamma_e := \theta^{\overline{E}}_{\overline{e}^{-1}}$ for $e \in E^1_+$,
- $\Omega(\overline{E})_{e^{\pm 1}} := \Omega(\overline{E})_{\overline{e}^{\pm 1}}$ and $\gamma_e := \theta^{\overline{E}}_{\overline{e}}$ for $e \in E^1_-$,

as well as an $\overline{E}$-action σ on $\Omega(E, C)$ by

- $\Omega(E, C)_{\overline{v}} := \Omega(E, C)_v$ for $v \in E^0$,
- $\Omega(E, C)_{\overline{e}^{\pm 1}} := \Omega(E, C)_{e^{\mp 1}}$ and $\sigma_{\overline{e}} := \theta^{(E,C)}_{e^{-1}}$ for $e \in E^1_+$,
- $\Omega(E, C)_{\overline{e}^{\pm 1}} := \Omega(E, C)_{e^{\pm 1}}$ and $\sigma_{\overline{e}} := \theta^{(E,C)}_{e}$ for $e \in E^1_-$.

We then obtain equivariant maps $\Omega(\overline{E}) \to \Omega(E, C)$ and $\Omega(E, C) \to \Omega(\overline{E})$ from the universal properties, and by uniqueness, these must be mutual inverses. Consequently,

$$\mathcal{O}(E, C) \cong C_0(\Omega(E, C)) \rtimes \mathbb{F} \cong C_0(\Omega(\overline{E})) \rtimes \mathbb{F} \cong C^*(\overline{E}).$$

Likewise, one can check that $p_{\overline{v}} = v$ for $v \in E^0$ and

$$t_{\overline{e}} = \begin{cases} e & \text{if } e \in E^1_- \\ e^* & \text{if } e \in E^1_+ \end{cases}$$

for $e \in E^1$ defines an $\overline{E}$-family inside $C^*(E, C)$, so that the isomorphism $C^*(\overline{E}) \to \mathcal{O}(E, C)$ factors through $C^*(E, C)$ as a surjection. It follows that $C^*(\overline{E}) \cong C^*(E, C)$ as well. The same argument applies to the Leavitt path algebras.

For the last part of the proposition, suppose that $H \subset E^0$ is hereditary and C-saturated with respect to (E, C). In order to check that $\overline{H}$ is hereditary in $\overline{E}$, we assume that $\overline{r}(\overline{e}) \in \overline{H}$. If $e \in E^1_-$, then $\overline{r(e)} = \overline{r}(\overline{e}) \in \overline{H}$ and so $r(e) \in H$. It follows that $\overline{s}(\overline{e}) = \overline{s(e)} \in \overline{H}$, so let us instead assume that $e \in E^1_+$. Then $\overline{s(e)} = \overline{r}(\overline{e}) \in \overline{H}$, so $s(e) \in H$. C-saturation and $[e] = \{e\}$ imply $r(e) \in H$, hence $\overline{s}(\overline{e}) = \overline{r(e)} \in \overline{H}$ as well. To see that $\overline{H}$ is saturated, suppose that $\overline{s}(\overline{r}^{-1}(\overline{v})) \subset \overline{H}$ for some $v \in E^0$ with $\overline{r}^{-1}(\overline{v}) \neq \emptyset$. First assume that v satisfies Definition 3.8(1). Then $\overline{r}^{-1}(\overline{v}) = \{\overline{e} \mid e \in E^1_- \cap r^{-1}(v)\}$ and

$$\overline{s}(\overline{r}^{-1}(\overline{v})) = \{\overline{s(e)} \mid e \in E^1_- \cap r^{-1}(v)\} \subset \overline{H},$$

so $s(E^1_- \cap r^{-1}(v)) \subset H$. C-saturation now implies $v \in H$ as well. Assuming instead that v satisfies (2) of Definition 3.8, we must have $E^1_+ \cap s^{-1}(v) = \{e\}$ and $\overline{r}^{-1}(\overline{v}) = \{\overline{e}\}$ for some e, hence $\overline{s}(\overline{r}^{-1}(v)) = \{\overline{r(e)}\} \subset \overline{H}$, i.e. $r(e) \in H$. We deduce that $v = s(e) \in H$ from H being hereditary, so $v \in \overline{H}$ as desired. The other implication can easily be proven by analogous arguments, and we therefore leave it to the reader. □

While the concept of a non-separated orientation is quite handy for technical purposes, it is certainly not a very natural one. Instead we shall give a sufficient graph-theoretic condition for the existence of such an orientation below in Proposition 3.20. First though, we need a number of minor technical results to introduce and apply the notion of a *simple closed path.*

Lemma 3.12. *If α is a closed path based at a vertex v, which admits no choices, then neither does any vertex on α.*

Proof. This is obvious. □

Lemma 3.13. *Assume that v admits no choices and that α, β are admissible paths with $r(\alpha) = s(\beta) = v$. Then the reduced product $\beta \cdot \alpha$ is an admissible path.*

Proof. Set $\gamma := \alpha^{-1} \wedge \beta$ and write $\alpha = \gamma^{-1}\alpha'$, $\beta = \beta'\gamma$. We then need to verify that the reduced product $\beta \cdot \alpha = \beta'\alpha'$ is admissible. By construction, we must have $r(\alpha') = s(\beta')$ and $\mathfrak{t}_{\mathrm{d}}(\alpha')^{-1} \neq \mathfrak{i}_{\mathrm{d}}(\beta')$, so assuming that $\mathfrak{t}_{\mathrm{d}}(\alpha') \in E^1$ and $\mathfrak{i}_{\mathrm{d}}(\beta') \in (E^1)^{-1}$, we must simply check that $[\mathfrak{i}_{\mathrm{d}}(\beta')^{-1}] \neq [\mathfrak{t}_{\mathrm{d}}(\alpha')]$. But this is evident as v would otherwise admit a choice. □

Proposition 3.14. *Assume that a vertex $v \in E^0$ admits no choices. Then every closed path α based at v decomposes uniquely as $\alpha = \gamma^{-1}\beta\gamma$ for a cycle β, and*

$$\mathbb{F}_v := \{\textit{closed paths based at } v\} \cup \{1\}$$

forms a free subgroup $\mathbb{F}_v \leq \mathbb{F}$.

Proof. For the first part, set $\gamma := \alpha \wedge \alpha^{-1}$ and write $\alpha = \gamma^{-1}\beta\gamma$ – we then claim that β is a cycle. If it were not, then we would have $\mathfrak{i}_{\mathrm{d}}(\beta) \in (E^1)^{-1}$, $\mathfrak{t}_{\mathrm{d}}(\beta) \in E^1$, $[\mathfrak{t}_{\mathrm{d}}(\beta)] = [\mathfrak{i}_{\mathrm{d}}(\beta)^{-1}]$ and $\mathfrak{t}_{\mathrm{d}}(\beta) \neq \mathfrak{i}_{\mathrm{d}}(\beta)^{-1}$, hence $r(\beta)$ would admit a choice, contradicting Lemma 3.12. The second part of the claim is immediate from Lemma 3.13 and the Nielsen-Schreier Theorem. □

Definition 3.15. Assume that $v \in E^0$ admits no choices. Then a non-trivial closed path α based at v is called a *simple closed path* if $\alpha = \gamma^{-1}\beta\gamma$ for a simple admissible path γ and a simple cycle β. Obviously, if α is a simple closed path, then so is α^{-1}, and so we say that v admits

$$\frac{1}{2} \cdot \big|\{\text{simple closed paths } \alpha \text{ based at } v\}\big| = \left|\frac{\{\text{simple closed paths } \alpha \text{ based at } v\}}{\alpha \sim \alpha^{-1}}\right|$$

simple closed paths *up to inversion.*

Proposition 3.16. *Assume that $v \in E^0$ admits no choices. Then every closed path based at v is a reduced product of simple closed paths based at v.*

Proof. We claim that any closed path α based at v admits a decomposition

$$\alpha = \alpha' \cdot \gamma^{-1}\beta\gamma,$$

where γ is a simple admissible path, β is a simple cycle, and α' is a closed path based at v with $|\alpha'| < |\alpha|$. An inductive application of this claim surely proves the lemma.

To prove the claim, take $\beta' \leq \alpha$ to be minimal with the property that there exists some $\gamma < \beta'$ with $r(\gamma) = r(\beta')$, and write $\beta' = \beta\gamma$. Then γ is a simple admissible path by construction, and β is a closed path such that the base vertex admits no choices, as seen from Lemma 3.12. Since the only vertex repetition on β happens at the endpoints, it follows from the first part of Proposition 3.14 that β is a cycle, hence a simple cycle. It follows immediately from minimality that $\gamma^{-1} \cdot \beta = \gamma^{-1}\beta$, and the concatenation is admissible due to Lemma 3.13. Now write $\alpha = \sigma\beta\gamma$ and define $\alpha' := \sigma \cdot \gamma$. Lemma 3.13 guarantees that α' is admissible, hence a closed path, and we clearly have $\alpha = \alpha' \cdot \gamma^{-1}\beta\gamma$. Finally observing that

$$|\alpha'| \leq |\sigma| + |\gamma| < |\alpha|,$$

the proof is complete. □

Remark 3.17. If v admits no choices, and if Λ is a set of representatives for the set of simple closed paths based at v modulo inversion, then $\mathbb{F}_v$ is generated by Λ due to Proposition 3.16. However, $\mathbb{F}_v$ need not be freely generated by Λ. For instance, both vertices in the graph

admit three simple closed paths up to inversion, yet $\mathbb{F}_v \cong \mathbb{F}_2$ for either vertex v. But this is not a problem since we only need to distinguish between the three cases

- $|\Lambda| = 0$ in which $\mathbb{F}_v = \{1\}$,
- $|\Lambda| = 1$ in which $\mathbb{F}_v \cong \mathbb{Z}$,
- $|\Lambda| \geq 2$ in which $\mathbb{F}_v \cong \mathbb{F}_n$ for some $2 \leq n \leq \infty$.

◀

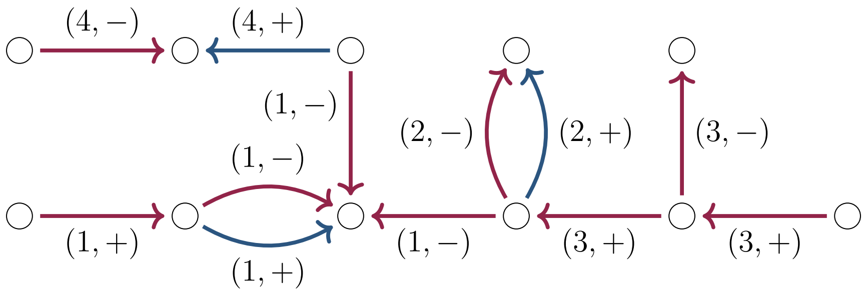
We will also need the following somewhat odd corollary.

Corollary 3.18. *Assume that $v \in E^0$ admits no choices and at most one simple closed path up to inversion. If v admits a cycle α, then it admits a unique simple cycle, β, up to inversion, and $\alpha = \beta^n$ for some $n \in \mathbb{Z}$.*

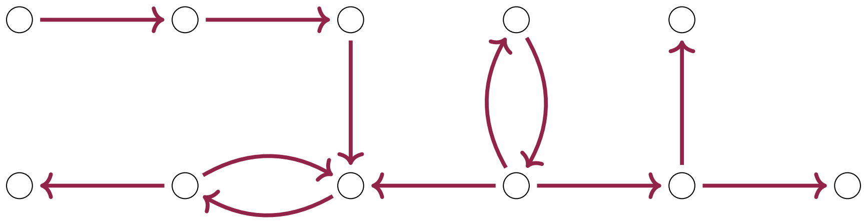
Proof. If α is a cycle based at v, then $\alpha = \beta^n$ for a simple closed path β due to Proposition 3.16. But then β must be a cycle as well, hence a simple cycle. □

The proof of Proposition 3.20 is fairly technical and requires the treatment of four different types of edges. We recommend having the following example in mind when reading through the proof of the proposition.

Example 3.19. In the following graph, all edges have been labelled with both the type and a choice of non-separated orientation as defined in the proof of Proposition 3.20:



The resulting non-separated graph is:



◀

In the following, we will say that an edge e is *on* a path α, if either of the letters e or e^{-1} are present in the symbol expansion of α.

Proposition 3.20. *Let (E, C) denote a finitely separated Condition (C) graph, and assume that every vertex admitting no choices admits at most one simple closed path up to inversion. Then (E, C) can be equipped with a non-separated orientation.*

Proof. The construction of $\mathfrak{o}(e)$ for $e \in E^1$ will proceed in the following four steps:

(1) Either e or e^{-1} admits a choice (equivalently, $r(e)$ admits a choice).
(2) e is not of type (1), but e is on a (simple) cycle.
(3) e is not of type (1) or (2), but e is on a (simple) closed path.
(4) e is not of type (1), (2), or (3), i.e. neither e nor e^{-1} admits a choice, and e is not on a closed path.

Type (1): We simply set

$$\mathfrak{o}(e) = \begin{cases} 1 & \text{if } e \text{ admits a choice} \\ -1 & \text{if } e^{-1} \text{ admits a choice} \end{cases}.$$

Observe that if $v \in E^0$ admits a choice, then exactly one of e and e^{-1} admits a choice for every $e \in r^{-1}(v) \cup s^{-1}(v)$, so before defining $\mathfrak{o}$ on the remaining edges, we might as well check that it satisfies Definition 3.8 at such v. If $\mathfrak{o}(e) = 1$, i.e. if e admits a choice, then either some $f \in s^{-1}(r(e))$ with $f \neq e$ or some $X \in C_{r(e)}$ with $e \notin X$ admits a choice. And since every vertex admits at most one choice, we must have $[e] = \{e\}$. Likewise, Definition 3.8(1) and (2) hold simply because every vertex admits at most one choice.

Type (2): Define an equivalence relation on the set of type (2) edges by

$$e \approx f \Leftrightarrow e \text{ and } f \text{ are on the same (simple) cycle.}$$

Observe that $\approx$ is transitive due to Corollary 3.18. Now choose a representative e for each equivalence class modulo $\approx$ as well as a simple cycle

$$\alpha = e_n^{\varepsilon_n} e_{n-1}^{\varepsilon_{n-1}} \cdots e_2^{\varepsilon_2} e_1^{\varepsilon_1}$$

that e is on. We then set $\mathfrak{o}(e_i) := \varepsilon_i$ for all $i = 1, \dots, n$; this is well-defined, because α is simple.

Type (3): If e is a type (3) edge, then $e = \mathfrak{i}(\alpha)$ for a (up to inversion) unique simple closed path α. This allows us to define $\mathfrak{o}(e)$ so that $e^{\mathfrak{o}(e)} = \mathfrak{i}_{\mathrm{d}}(\alpha)$; note that this does not depend on the choice of α over α^{-1} by Proposition 3.14.

Observe that if $u \in E^0$ admits a closed path but no choices, then we have defined $\mathfrak{o}$ on all edges $e \in r^{-1}(u) \cup s^{-1}(u)$. Before defining the orientation of a type (4) edge, we will therefore check that Definition 3.8 is satisfied at such u. We should distinguish between two cases; when u admits and does not admit a cycle.

First assume that u admits a simple cycle $\alpha = e_n^{\varepsilon_n} e_{n-1}^{\varepsilon_{n-1}} \cdots e_2^{\varepsilon_2} e_1^{\varepsilon_1}$, and that the orientation is defined as above. Note that no $e \in r^{-1}(u)$ admits a choice for then u would it self admit a choice, but if $e \in s^{-1}(u)$, then e^{-1} admits a choice if and only if $|[e]| \geq 2$. Now by construction

$$\mathfrak{o}(e) = \begin{cases} -1 & \text{if } e \in s^{-1}(u) \text{ and } |[e]| \geq 2 \\ \varepsilon_1 & \text{if } e = e_1 \\ \varepsilon_n & \text{if } e = e_n \\ -1 & \text{if } e \in s^{-1}(u) \text{ is not on a cycle} \\ 1 & \text{if } e \in r^{-1}(u) \text{ is not on a cycle} \end{cases}.$$

Simply observing that

$$\mathfrak{o}^{-1}(-1) \cap r^{-1}(u) = \begin{cases} \emptyset & \text{if } \varepsilon_1 = 1 \\ \{e_1\} & \text{if } \varepsilon_1 = -1 \end{cases}$$

and

$$\mathfrak{o}^{-1}(1) \cap s^{-1}(u) = \begin{cases} \{e_1\} & \text{if } \varepsilon_1 = 1 \\ \emptyset & \text{if } \varepsilon_1 = -1 \end{cases},$$

we then see that Definition 3.8 is satisfied at u.

Next, assume that u is not on a cycle, but that $\alpha = \gamma^{-1}\beta\gamma$ is a closed path based at u. Observe again that all $e \in r^{-1}(u)$ are of type (3), but that $e \in s^{-1}(u)$ is of type (1) if $|[e]| \geq 2$, and otherwise it is of type (3). If $\mathfrak{i}_{\mathrm{d}}(\alpha) \in (E^1)^{-1}$, then $e^{-1}\alpha e$ defines a closed path for all $e \in r^{-1}(u)$, $e \neq \mathfrak{i}(\alpha)$, hence $\mathfrak{o}(\mathfrak{i}(\alpha)) = -1$ and $\mathfrak{o}(e) = 1$. Moreover, if $e \in s^{-1}(u)$, then either $|[e]| \geq 2$ or e is of type (3) and $e\alpha e^{-1}$ defines a closed path, hence $\mathfrak{o}(e) = -1$ either way. We conclude that Definition 3.8(1) is satisfied in this case, and similarly one can check that Definition 3.8(2) is satisfied when $\mathfrak{i}_{\mathrm{d}}(\alpha) \in E^1$.

Type (4): Finally, let $U \subset E^0$ denote the set of $u \in E^0$ admitting no choices and no closed paths, and define an equivalence relation on U by

$$u \sim v \Leftrightarrow \text{ there is an admissible path of type (4) edges } u \to v.$$

For every equivalence class, we then pick a unique representative. If e is any edge of type (4), and u is the representative of the equivalence class of $r(e)$, then there is a unique admissible path α with $r(\alpha) = u$ and $\mathfrak{i}(\alpha) = e$, and we define $\mathfrak{o}(e)$ so that $e^{\mathfrak{o}(e)} = \mathfrak{i}_{\rm d}(\alpha)$. Verifying that $\mathfrak{o}$ satisfies Definition 3.8 is completely analogous to what we did just above. ☐

Corollary 3.21. *Let (E, C) denote a finitely separated graph satisfying both Condition (C) and Condition (L). Then (E, C) has a non-separated orientation for which the resulting graph $\overline{E}$ satisfies Condition (L).*

Proof. Assume that $v \in E^0$ does not admit a choice, and assume in order to reach a contradiction that v admits a closed path $\alpha = \gamma^{-1}\beta\gamma$ with β a cycle. Then $s(\beta)$ admits a choice by assumption, hence so does v by Lemma 3.12. The claim then follows by invoking Proposition 3.20. ☐

4. A characterisation of simplicity

In this section, we compute all graph algebras of finitely separated graphs giving rise to minimal partial actions, and as a result, we are able to characterise simplicity of these C^*-algebras. A similar result is obtained in [8, Theorem 8.1] for finite bipartite graphs, but the two proofs are quite different. Indeed, the one in [8] proceeds via a graph-theoretic investigation of the separated Bratteli diagram (F_∞, D^∞), while the below proof combines the contents of Section 3 with a simple dynamical observation (Lemma 4.3).

Definition 4.1. Let (E, C) denote a finitely separated graph. If $X, Y \in C$ satisfy $|X|, |Y| \geq 2$, then a *choice connector between X and Y* is an admissible path α for which $Y^{-1}\alpha X$ an admissible composition. If (E, C) does not satisfy Condition (C), we define the *maximal choice distance* to be

$$m_{\rm CD}(E, C) := \sup\{n \mid \text{there exists a choice connector of length } n\}.$$

◀

We have the following trivial, but quite handy, observation.

Lemma 4.2. *Given any set $A \subset E^1 \sqcup (E^1)^{-1}$, the function $s_A \colon \Omega(E, C) \to \mathbb{Z}_+$ given by*

$$s_A(\xi) := |\{\mathfrak{i}_d(\alpha) \colon \alpha \in \xi, \mathfrak{t}_d(\alpha) \in A\}|$$

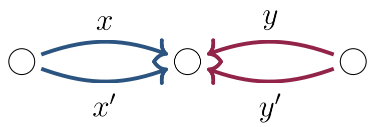
is lower semi-continuous, i.e. $\liminf_{\eta \to \xi} s_A(\eta) \geq s_A(\xi)$ *for any* $\xi \in \Omega(E, C)$.

Proof. This is obvious. ☐

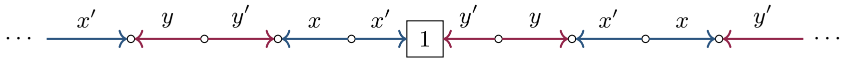
Lemma 4.3. *Let (E, C) denote a finitely separated graph. If $\theta^{(E,C)}$ is minimal, then (E, C) satisfies Condition (C).*

Proof. We argue by contraposition, assuming that (E, C) does not satisfy Condition (C). Consider any admissible composition $Y^{-1}\alpha X$ with $|X|, |Y| \geq 2$, fix some $x \in X$, $y \in Y$ and set $A := \{x^{-1}, y^{-1}\}$. Also, pick any configuration ξ with $\{x^{-1}, y^{-1}\alpha\} \subset \xi$, so that $s_A(\xi) \geq 2$. We may then construct a configuration η with the property $\mathfrak{t}_d(\beta) \neq x^{-1}, y^{-1}$ for all $\beta \in \eta$. Indeed, starting from any vertex and constructing η inductively, one may simply refrain from choosing x^{-1} when reaching $r(x)$, and similarly for y (see Example 4.4 for what such a configuration might look like). It follows that $s_A(\theta_\alpha(\eta)) \leq 1$ for any $\alpha \in \eta$, so in particular $\xi \notin \overline{\theta_{\mathbb{F}}(\eta)}$ by Lemma 4.2. □

Example 4.4. Consider the separated graph



which does not satisfy Condition (C). In this case, there is only the following choice of a configuration η as in the proof of Lemma 4.3:



A configuration η as in the proof of Lemma 4.3.

◀

The rest of this section essentially just exploits Condition (C) in order to apply the results of Section 3. However, before we can give the first application of this property, we will need to introduce yet another graph-theoretic notion that will come in handy in the proof of Proposition 4.6.

Definition 4.5. An admissible path α is called *forced* if $[e] = 1$ for all edges e, such that e^{-1} (or e^* when regarding α as an element of a graph algebra) is in the symbol expansion of α. Observe that $\alpha^*\alpha = s(\alpha)$ whenever α is forced.

Proposition 4.6. *If* (E, C) *satisfies Condition* (*C*), *then*

$$L_K(E, C) = L_K^{\mathrm{ab}}(E, C) \quad \text{and} \quad C^*(E, C) = \mathcal{O}(E, C).$$

Proof. We simply have to verify that $\alpha = \alpha\alpha^*\alpha$ in $L_K(E, C)$ for all products of elements from the set $E^1 \cup (E^1)^* \subset L_K(E, C)$. Recall from Remark 1.8 that any such non-zero α is of the form $\alpha = \alpha_n \cdots \alpha_1$, where each α_i is a non-trivial admissible path satisfying

- $\mathfrak{i}_{\mathrm{d}}(\alpha_{i+1}) \in E^1$ and $\mathfrak{t}_{\mathrm{d}}(\alpha_i) = \mathfrak{i}_{\mathrm{d}}(\alpha_{i+1})^*$,
- $|[\mathfrak{i}_{\mathrm{d}}(\alpha_{i+1})]| \geq 2$

for all $i = 1, \ldots, n-1$. We first claim that Condition (C) implies $n \leq 2$. Indeed if $n \geq 3$, then $\mathfrak{i}_{\mathrm{d}}(\alpha_2) \in E^1$ and $\mathfrak{t}_{\mathrm{d}}(\alpha_2) \in (E^1)^*$, so $|\alpha_2| \geq 2$ and we can consider the admissible path α_2' obtained from removing the initial and terminal symbol (if $|\alpha_2| = 2$ so that $\alpha_2 = e^* f$ for $e, f \in E^1$, we set $\alpha_2' := r(f)$). Then α_2' will be a choice connector, contradicting Condition

(C). If $n=1$, then $\alpha=\alpha_1$ must be of the form $\alpha=\sigma_2\sigma_1^*$, where both σ_1 and σ_2 are forced, and consequently

$$\alpha\alpha^*\alpha=\sigma_2\sigma_1^*\sigma_1\sigma_2^*\sigma_2\sigma_1^*=\sigma_2\sigma_1^*=\alpha.$$

Assuming $n=2$ instead, both α_1^* and α_2 must be forced, and so the situation is the same as in the case $n=1$. ☐

We include a proof of following observation, which is also used in various forms in [8], for clarity.

Proposition 4.7. *Let (E,C) denote a finitely separated graph, assume that $v\in E^0$ admits no choices. Then there are identifications*

$$vL_K^{\mathrm{ab}}(E,C)v\cong K[\mathbb{F}_v] \quad \textit{and} \quad v\mathcal{O}^{(r)}(E,C)v\cong C^*_{(r)}(\mathbb{F}_v),$$

where $K[\mathbb{F}_v]$ is the group ring of $\mathbb{F}_v$ with coefficient in K.

Proof. Observing that $\Omega(E,C)_v$ is a one-point space and that the partial action of $\mathbb{F}$ restricts to the trivial global action $\mathbb{F}_v\curvearrowright\Omega(E,C)_v$, we deduce that

$$vL_K^{\mathrm{ab}}(E,C)v\cong C_K(\Omega(E,C)_v)\rtimes_{\mathrm{alg}}\mathbb{F}_v\cong K\rtimes_{\mathrm{alg}}\mathbb{F}\cong K[\mathbb{F}_v]$$

and

$$v\mathcal{O}^{(r)}(E,C)v\cong C(\Omega(E,C)_v)\rtimes_{(r)}\mathbb{F}_v\cong\mathbb{C}\rtimes_{(r)}\mathbb{F}_v\cong C^*_{(r)}(\mathbb{F}_v),$$

by invoking [8, Lemma 7.13]. ☐

Having made all the preparations, we are now able to describe all algebras associated with finitely separated graphs for which the partial action is minimal.

Theorem 4.8. *Let (E,C) is a finitely separated graph. If (E,C) satisfies Condition (C) and $\mathcal{H}(E,C)=\{\emptyset,E^0\}$, then exactly one of the following holds:*

(1) *Every cycle admits exactly one choice. In that case*

$$L_K(E,C)=L_K^{\mathrm{ab}}(E,C)$$

is isomorphic to a simple Leavitt path algebra $L(\overline{E})$, and

$$C^*(E,C)=\mathcal{O}(E,C)\cong\mathcal{O}^r(E,C)$$

is isomorphic to a simple graph C^*-algebra $C^*(\overline{E})$.

(2) *There is a vertex, which admits no choices and exactly one simple closed path up to inversion. Then $L_K(E,C)=L_K^{\mathrm{ab}}(E,C)$ is isomorphic to a Leavitt path algebra $L_K(\overline{E})$ and Morita equivalent to the algebra of Laurent polynomials $K[\mathbb{Z}]=K[x,x^{-1}]$, while $C^*(E,C)=\mathcal{O}(E,C)\cong\mathcal{O}^r(E,C)$ is isomorphic to a graph C^*-algebra $C^*(\overline{E})$ and Morita equivalent to $C(\mathbb{T})$.*

(3) *There is a vertex $v\in E^0$, which admits no choices and at least two simple closed paths up to inversion. In that case, there are Morita equivalences*

$$L_K(E,C)=L_K^{\mathrm{ab}}(E,C)\sim K[\mathbb{F}_v],\quad C^*(E,C)=\mathcal{O}(E,C)\sim C^*(\mathbb{F}_v)$$

and $\mathcal{O}^r(E,C) \sim C_r^*(\mathbb{F}_v)$, where $\mathbb{F}_v$ denotes the free subgroup of rank at least two consisting of all the closed paths based at v as well as the empty word.

Proof. First, we recall that the quotient maps $L_K(E,C) \to L_K^{\mathrm{ab}}(E,C)$ and $C^*(E,C) \to \mathcal{O}(E,C)$ are isomorphisms in any case by Proposition 4.6 (strictly speaking, we only need to invoke the result for case 3). Now, if every cycle admits exactly one choice, then we obtain (1) immediately by Corollary 3.21. If this is not the case, then some $v \in E^0$ admits a closed path but no choices, so Proposition 4.7 applies. Moreover, as v generates E^0 as a hereditary and C-saturated set, it defines a full projection in $L_K(E,C)$, $C^*(E,C)$ and the quotients, hence they are all Morita-equivalent to their respective corners obtained by cutting down with v. If v admits at least two simple closed paths up to inversion, then neither can be a multiple of the other, hence $\mathbb{F}_v$ will be a free group of rank at least two.

Finally, we observe that if there is only one simple closed path based at v, then no $u \in E^0$ can admit at least two simple closed paths and no choices, for then $C(\mathbb{T})$ and $C^*(\mathbb{F}_n)$ would be Morita-equivalent for some $n \geq 2$. Now Proposition 3.20 applies to give (2). □

As a consequence, we can completely characterise the simple C^*-algebras associated with finitely separated graphs.

Corollary 4.9. *Let (E,C) denote a finitely separated graph. Then the algebras $L_K(E,C)$ and $L_K^{\mathrm{ab}}(E,C)$ as well as the C^*-algebras $C^*(E,C)$ and $\mathcal{O}(E,C)$ are simple if and only if the following holds:*

1. *(E,C) satisfies Condition (C),*
2. *$\mathcal{H}(E,C) = \{\emptyset, E^0\}$,*
3. *every cycle admits exactly one choice.*

In that case, $L_K(E,C) = L_K^{\mathrm{ab}}(E,C)$ is isomorphic to the Leavitt path algebra and

$$C^*(E,C) = \mathcal{O}(E,C) \cong \mathcal{O}^r(E,C)$$

is isomorphic to the graph C^*-algebra of a non-separated graph.

Proof. If either algebra is simple, then $\theta^{(E,C)}$ is minimal. By Lemma 4.7 and Theorem 2.19, this implies that (E,C) satisfies Condition (C) and contains only trivial hereditary and C-saturated subsets. Now the result is immediate from Theorem 4.8, since (full) group algebras of free groups are not simple. □

Corollary 4.10. *Let (E,C) denote a finitely separated graph. Then the C^*-algebra $\mathcal{O}^r(E,C)$ is simple if and only if*

1. *(E,C) satisfies Condition (C),*
2. *$\mathcal{H}(E,C) = \{\emptyset, E^0\}$,*

and one of the following holds:

(3a) *Every cycle admits exactly one choice. In that case, $\mathcal{O}^r(E,C)$ is isomorphic to a classical graph C^*-algebra.*

(3b) *There is a vertex $v \in E^0$, which admits no choices and at least two simple closed paths up to inversion. In that case, $\mathcal{O}^r(E,C)$ is Morita-equivalent to $C^*_r(\mathbb{F}_v)$, where $\mathbb{F}_v$ denotes the free subgroup of rank at least two consisting of all the closed paths based at v as well as the empty word.*

Proof. The proof is completely similar to that of Corollary 4.9, except that $C^*_r(\mathbb{F}_n)$ is in fact simple for every $2 \leq n \leq \infty$ [25]. □

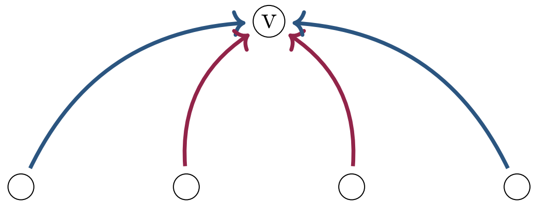
Finally, we can also characterise minimality of $\theta^{(E,C)}$:

Corollary 4.11. *Let (E,C) denote a finitely separated graph. Then $\theta^{(E,C)}$ is minimal if and only if (E,C) satisfies Condition (C) and $\mathcal{H}(E,C) = \{\emptyset, E^0\}$.*

Proof. One implication follows immediately from 2.19 and Lemma 4.3. For the other one, note that Theorem 4.8 applies, and that if (1) or (3) of the Theorem 4.8 holds, then $\theta^{(E,C)}$ must be minimal due to simplicity of the graph algebras. Assuming (2) instead, there is a vertex v which admits no choices. Since v generates E^0 as a hereditary and C-saturated set, we see that $\Omega(E,C)$ is nothing but the orbit of the one-point set $\Omega(E,C)_v$, hence minimal. □

5. Degeneracy of the tame algebras

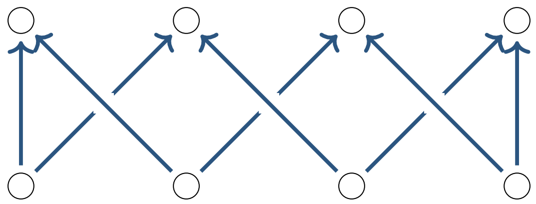
In Section 3, we saw that the Leavitt path algebra and graph C^*-algebra degenerate under certain conditions, including Condition (C). On the other hand, even very simple separated graphs without Condition (C) can produce quite complicated algebras. For instance, if (E,C) denotes the graph



of [2, Example 9.4], then $C^*(E,C)$ is Morita equivalent to the universal unital C^*-algebra generated by two projections , namely

$$vC^*(E,C)v \cong \mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2 \cong \{f \in C([0,1], M_2(\mathbb{C})) \mid f(0), f(1) \text{ diagonal}\},$$

while $\mathcal{O}(E,C) \cong \bigoplus_{i=1}^4 M_3(\mathbb{C})$. Indeed, (E_1, C^1) is the trivially separated graph



to which we can apply the standard formula for finite non-separated graphs without cycles. In this short section, we shall explore when the tame algebras degenerate to graph algebras of non-separated graphs by combining our work in Section 3 with the fact that (E_n, C^n) and (E, C) produce the same tame algebras. We briefly recall the definition of (E_1, C^1).

Definition 5.1 ([2, Construction 4.4])**.** Let (E, C) denote a finite bipartite separated graph, and write

$$C_u = \{X_1^u, \dots, X_{k_u}^u\}$$

for all $u \in E^{0,0}$. Then (E_1, C^1) is the finite bipartite separated defined by

- $E_1^{0,0} := E^{0,1}$ and $E_1^{0,1} := \{v(x_1, \dots, x_{k_u}) \mid u \in E^{0,0}, x_j \in X_j^u\}$,
- $E^1 := \{\alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u}) \mid u \in E^{0,0}, i = 1, \dots, k_u, x_j \in X_j^u\}$,
- $r_1(\alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u})) := s(x_i)$ and $s_1(\alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u})) := v(x_1, \dots, x_{k_u})$,
- $C_v^1 := \{X(x) \mid x \in s^{-1}(v)\}$, where

$$X(x_i) := \{\alpha^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u}) \mid x_j \in X_j^u \text{ for } j \neq i\}.$$

We also define a map $\mathfrak{r}\colon E_1^0 \to E^0$ by $\mathfrak{r}(v) := v$ for $v \in E_1^{0,0} = E^{0,1}$ and

$$\mathfrak{r}(v(x_1, \dots, x_{k_u})) := u$$

for all $u \in E^{0,0}$ and $(x_1, \dots, x_{k_u}) \in \prod_{i=1}^{k_u} X_i^u$. ◀

The following technical lemma will prove most useful.

Lemma 5.2. *Assume that (E, C) is a finite bipartite graph. The assignments $v \mapsto \mathfrak{r}(v)$ and $\alpha^e(*) \mapsto e^{-1}$ extend to a length-preserving surjective map $\Psi\colon \mathcal{P}(E_1, C^1) \to \mathcal{P}(E, C)$ with the following properties*

1. *If $\alpha, \beta \in \mathcal{P}(E_1, C^1)$ satisfy $r(\alpha) = s(\beta)$, then $\Psi(\beta)\Psi(\alpha)$ is admissible if and only if $\beta\alpha$ is admissible.*
2. *If $\alpha \in \mathcal{P}(E, C)$ with $r(\alpha), s(\alpha) \in E^{0,1}$, then*
$$r_1(\Psi^{-1}(\alpha)) = \{r(\alpha)\} \quad \text{and} \quad s_1(\Psi^{-1}(\alpha)) = \{s(\alpha)\}.$$
3. *If $\alpha \in \mathcal{P}(E, C)$ with $r(\alpha) \in E^{0,0}$ and $s(\alpha) \in E^{0,1}$, so that we may write $\alpha = x\beta$ for $x \in E^1$, then*
$$r_1(\Psi^{-1}(\alpha)) = s_1(X(x)) \quad \text{and} \quad s_1(\Psi^{-1}(\alpha)) = \{s(\alpha)\}.$$
4. *If $\alpha \in \mathcal{P}(E, C)$ is non-trivial with $r(\alpha), s(\alpha) \in E^{0,0}$, so that we may write $\alpha = x\beta y^{-1}$ for $x, y \in E^1$, then*
$$(r_1, s_1)(\Psi^{-1}(\alpha)) = s_1(X(x)) \times s_1(X(y)).$$
5. *Let $x\alpha \in \mathcal{P}(E, C)$ and consider a lift $\beta \in \Psi^{-1}(\alpha)$. Then $x\alpha$ is a choice path if and only if $|X(x)| \geq 2$ and $X(x)^{-1}\beta$ is an admissible composition in (E_1, C^1). Consequently, any $v \in E^{0,1} = E_1^{0,0}$ admits the same number of choices in (E, C) and (E_1, C^1).*
6. *The restriction of Ψ to the set of closed paths based at vertices admitting no choices is injective.*

Proof. We extend the assignment $\alpha^e(*) \mapsto e^{-1}$ to a group homomorphism $\Psi\colon \mathbb{F}(E_1^1) \to \mathbb{F}(E^1)$, and claim that for $e, f \in E_1^1$, the following hold:

(a) If $r(e) = r(f)$, then $e^{-1}f$ is admissible if and only if $\Psi(e^{-1})\Psi(f)$ is admissible,
(b) If $s(e) = s(f)$, then ef^{-1} is admissible if and only if $\Psi(e)\Psi(f^{-1})$ is admissible.

To this end, write $e = \alpha^{x_i}(x_1, \dots, \hat{x}_i, \dots, x_k)$ and $f = \alpha^{y_j}(y_1, \dots, \hat{y}_j, \dots, y_l)$. In situation (a), we have

$$s(x_i) = r\big(\alpha^{x_i}(x_1, \dots, \hat{x}_i, \dots, x_k)\big) = r\big(\alpha^{y_j}(y_1, \dots, \hat{y}_j, \dots, y_l)\big) = s(y_j),$$

hence $\Psi(e^{-1})\Psi(f) = x_i y_j^{-1}$ is admissible if and only if $x_i \neq y_j$. And since $r(e) = r(f)$, we note that $e^{-1}f$ is admissible if and only if $X(x_i) = [e] \neq [f] = X(y_j)$, which is certainly equivalent to $x_i \neq y_j$. Moving on to (b), we have

$$v(x_1, \dots, x_k) = s\big(\alpha^{x_i}(x_1, \dots, \hat{x}_i, \dots, x_k)\big) = s\big(\alpha^{y_j}(y_1, \dots, \hat{y}_j, \dots, y_l)\big) = v(y_1, \dots, y_l),$$

so $\Psi(e)\Psi(f^{-1}) = x_i^{-1}x_j$ is admissible if and only if $i \neq j$, which is equivalent to $e \neq f$, or ef^{-1} being admissible. It follows that the restriction of Ψ to $\mathcal{P}(E_1, C^1)$ along with the assignment of the vertices defines a length-preserving map $\mathcal{P}(E_1, C^1) \to \mathcal{P}(E, C)$ satisfying (1). Observe, in view of (1), that it is enough to check (2) for admissible paths $\alpha = x_i^{-1}x_j$ of length two, and such α lifts to a path in ef^{-1} with

$$e := \alpha^{x_i}(x_1, \dots, \hat{x}_i, \dots, x_k) \quad \text{and} \quad f := \alpha^{x_j}(x_1, \dots, \hat{x}_j, \dots, x_k),$$

where $x_l \in X_l^u$ is arbitrary for $l \neq i, j$. (3) and (4) then follow immediate by applying (2) to β and invoking (1). In particular, Ψ is surjective. Now consider claim (5) and assume that $X(x)^{-1}\beta$ is an admissible composition with $|X(x)| \geq 2$. Then $x\alpha$ is in the image of Ψ, hence admissible. Moreover, $|X(x)| \geq 2$ implies that there is some $[x] \neq X \in C_{r(x)}$ with $|X| \geq 2$, so $x\alpha$ is in fact a choice path. The reverse implication uses the exact same arguments. Finally, consider claim (6) and recall that if α is a closed path and $s(\alpha)$ does not admit any choices, then neither does any vertex on α. Consequently, Ψ is injective on the set of edges and vertices that such α may pass through. But then Ψ is surely injective on the set of all such closed paths. □

Corollary 5.3. *Let (E, C) denote a finite bipartite separated graph. If $2 \leq m_{\mathrm{CD}}(E, C) < \infty$, then*

$$m_{\mathrm{CD}}(E_1, C^1) = m_{\mathrm{CD}}(E, C) - 2,$$

and if $m_{\mathrm{CD}}(E, C) = 0$, then (E_1, C^1) satisfies Condition (C).

Proof. Simply observe from Lemma 5.2(5) that if $\beta \in \mathcal{P}(E_1, C^1)$ and $\alpha := \Psi(\beta)$, then β is a choice connector in (E_1, C^1) between $X(x)$ and $X(y)$ if and only if $x\alpha y^{-1}$ is a choice connector in (E, C). □

In the following lemma, finiteness of E is crucial.

Lemma 5.4. *Let (E, C) denote a finite separated graph, and assume that every cycle admits at most one choice. Then $m_{\mathrm{CD}}(E, C) < \infty$.*

Proof. Assume in order to reach a contradiction that there is a choice connector α of length $|\alpha| \geq 3 \cdot |E^0|$. Then α must pass some vertex $v \in E^0$ three times, i.e. there are closed paths β and γ based at v such that $\gamma\beta$ is admissible. Since v is on a choice connector, it cannot admit any cycles, hence neither β nor γ are cycles. But then $\gamma\beta$ must itself be a cycle, giving us our desired contradiction. □

Corollary 5.5. *Let (E, C) denote a finite bipartite separated graph. If every cycle admits at most one choice, then (E_n, C^n) will satisfy Condition (C) for sufficiently large n.*

Proof. This is immediate from Lemma 5.4 and Corollary 5.3. □

We now make the final preparations before obtaining the main theorem of this section.

Lemma 5.6. *Let (E, C) denote a finite bipartite graph, and assume that every vertex without a choice admits at most one simple closed path up to inversion. Then (E_1, C^1) satisfies the same property.*

Proof. Assume that $v \in E_1^0$ does not admit a choice. Without loss of generality, we may assume that $v \in E_1^{0,0} = E^{0,1}$ since every non-trivial path must pass a vertex in this layer. By Lemma 5.2(5), v does not admit a choice in (E, C) either, hence it admits at most one simple closed path up to inversion in (E, C). It follows from Lemma 5.2(6) that v admits at most one simple closed path up to inversion in (E_1, C^1) as well. □

Theorem 5.7. *Let (E, C) denote a finite bipartite separated graph. If every cycle admits at most one choice, and every vertex without a choice admits at most simple closed path up to inversion, then (E_n, C^n) admits a non-separated orientation for sufficiently large n. Consequently, there exists a finite non-separated graph $F := \overline{E_n}$ and a direct dynamical equivalence $\theta^F \to \theta^{(E,C)}$. In particular,*

$$L_K^{\mathrm{ab}}(E, C) \cong L_K(F) \quad \text{and} \quad \mathcal{O}(E, C) \cong \mathcal{O}^r(E, C) \cong C^*(F).$$

Proof. By Corollary 5.5 and Lemma 5.6, (E_n, C^n) will satisfy the requirements of Proposition 3.20 for sufficiently large n, so the result follows by combining Proposition 3.11 and [8, Theorem 3.22]. □

6. The exchange property, real rank zero and essentially free actions

Recall that a non-separated graph E is said to satisfy *Condition* (K) if every vertex on a cycle admits at least two simple cycles. The main point of Condition (K) is that it implies Condition (L) and is preserved when passing to any quotient graph E/H. It is well known that it is equivalent to $L_K(E)$ being an exchange ring [10, Theorem 4.5], $C^*(E)$ having real rank zero [21, Theorem 3.5], and the graph groupoid being essentially principal [24, Proposition 8]. In this section, we introduce the appropriate generalisation of Condition (K) to finitely separated graphs and prove an analogous result: Condition (K) is equivalent to $L_K^{\mathrm{ab}}(E, C)$ being an exchange ring, real rank zero of both $\mathcal{O}(E, C)$ and $\mathcal{O}^r(E, C)$, and essential freeness of $\theta^{(E,C)}$.

We refer the reader to [1, Theorem 1.2 and Definition 1.3] and [13, Theorem 2.6] for various equivalent definitions of exchange rings and real rank zero C^*-algebras, respectively. By [1, Theorem 3.8], these two concepts agree for C^*-algebras. The class of exchange rings is closed under ideals, quotients, extensions where idempotents can be lifted modulo the ideal [1, Theorem 2.3], corners [4, Corollary 1.5], direct limits, and Morita equivalence between idempotent rings (C^*-algebras for instance) [4, Theorem 2.3].

Definition 6.1. A finitely separated graph (E, C) is said to satisfy *Condition* (K) if every vertex $v \in E^0$ on a cycle satisfies the following:

(1) v admits exactly one choice.
(2) v admits at least two base-simple cycles up to inversion.

It is apparent that any finite bipartite Condition (K) graph (E, C) satisfies the assumptions of Theorem 5.7, so that $\mathcal{O}(E, C)$ will degenerate to a graph C^*-algebra $C^*(F)$ with $F := \overline{E_n}$ for some n. However, in order to conclude that F satisfies the usual Condition (K), we first have to check that it is preserved when passing from (E, C) to (E_n, C^n). ◀

Dealing with base-simple cycles is somewhat complicated in the realm of separated graphs since cycles need not decompose into a product of base-simple cycles. However, when we add Definition 6.1(1) to the equation, this problem disappears.

Lemma 6.2. *Let (E, C) denote a finitely separated graph. If $v \in E^0$ admits exactly one choice, then any cycle based at v is a concatenated product of base-simple cycles*

Proof. First observe that whenever γ is a cycle based at v, exactly one of $\mathfrak{i}_{\mathrm{d}}(\gamma)$ and $\mathfrak{t}_{\mathrm{d}}(\gamma)^{-1}$ admits a choice. Now take any cycle α based at v and let $\beta \le \alpha$ denote the minimal closed initial subpath: It suffices to check that β must be a cycle. Assume in order to reach a contradiction that it is not, and take a minimal cycle $\beta_n \cdots \beta_1 \le \alpha$ written as a concatenated product of base-simple closed paths with $\beta_1 = \beta$. Observe that, by minimality, both $\beta_n\beta_1$ and $\beta_n^{-1}\beta_1$ are cycles. Now if $\mathfrak{i}_{\mathrm{d}}(\beta)$ admits a choice, then so does $\mathfrak{i}_{\mathrm{d}}(\beta_n^{-1}) = \mathfrak{t}_{\mathrm{d}}(\beta_n)^{-1}$ and vice versa, contradicting the above observation applied to $\gamma = \beta_n\beta_1$. □

Remark 6.3. It is easy to check that a finitely separated graph (E, C) satisfies Condition (K) if and only if its bipartite sibling $\mathbf{B}(E, C)$ satisfies Condition (K). We leave this to the reader.

Lemma 6.4. *Let (E, C) denote a finite bipartite graph. If (E, C) satisfies Condition (K), then so does (E_1, C^1).*

Proof. Suppose that $v \in E_1^0$ admits a cycle α in (E_1, C^1); by otherwise replacing v with another vertex on α, we may assume that $v \in E_1^{0,0}$. Then v admits the cycle $\Psi(\alpha)$ in (E, C), hence it admits exactly one choice and at least two distinct base-simple cycles in (E, C) by assumption. Using Lemma 5.2(5), we conclude that it admits exactly one choice in (E_1, C^1) as well, and lifting these cycles arbitrarily to (E_1, C^1) using Lemma 5.2(2), we obtain two distinct base-simple cycles based at v in (E_1, C^1) as desired. □

Lemma 6.5. *Let (E, C) denote a finitely separated graph satisfying Condition (K). If H is a hereditary and C-saturated set, then the quotient graph $(E/H, C/H)$ satisfies Condition (K) as well.*

Proof. Assume that $v \in (E/H)^0 = E^0 \setminus H$ admits a cycle in $(E/H, C/H)$. Then v admits at least two distinct base-simple cycles in (E, C), and noting that for every cycle α, either α or α^{-1} is forced, we see that these cycles are contained in $(E/H, C/H)$ as well. Now if β is a minimal choice path with $s(\beta) = v$, then β too must be contained in $(E/H, C/H)$, so v admits exactly one choice in $(E/H, C/H)$ as well. □

We will now apply the main result of Section 5.

Corollary 6.6. *Let (E, C) denote a finite bipartite separated graph satisfying Condition (K). Then there exists a finite non-separated graph F with Condition (K) and a direct dynamical equivalence $\theta^F \to \theta^{(E,C)}$. Consequently, $\theta^{(E,C)}$ is essentially free,*

$$L_K^{\mathrm{ab}}(E, C) \cong L_K(F) \quad \text{and} \quad \mathcal{O}(E, C) \cong \mathcal{O}^r(E, C) \cong C^*(F).$$

In particular, $L_K^{\mathrm{ab}}(E, C)$ is an exchange ring and $\mathcal{O}(E, C) \cong \mathcal{O}^r(E, C)$ has real rank zero.

Proof. The first part follows immediately from Theorem 5.7 with $F = \overline{E_n}$. Moreover, E_n satisfies Condition (K) by Lemma 6.4, hence so does F by Lemma 3.10. It follows from [10, Theorem 4.5] and [21, Theorem 3.5] that $L_K(F)$ is an exchange ring and $C^*(F)$ has real rank zero, respectively. Moreover, the ideals of $C^*(F)$ are exactly those generated by hereditary and saturated subsets of F^0, which correspond to the hereditary and C^n-saturated subsets of E_n^0 by Proposition 3.11. It follows that the closed and invariant subsets of $\Omega(E_n, C^n)$ exactly correspond to the hereditary and C^n-saturated subsets of E_n^0. Now since $(E_n/H, C^/H)$ satisfies Condition (K) for any such H by Lemma 4.3, we see that $\theta^{(E_n,C^n)}$ is essentially free using Theorem 3.7. Finally, $\theta^{(E,C)}$ must then be essentially free since there is a direct dynamical equivalence $\theta^{(E_n,C^n)} \to \theta^{(E,C)}$ by [8, Theorem 3.22]. □

Corollary 6.7. *Let (E, C) denote a finite separated graph satisfying Condition (K). Then $\theta^{(E,C)}$ is essentially free,*

$$L_K^{\mathrm{ab}}(E, C) \cong L_K(F) \quad \text{and} \quad \mathcal{O}(E, C) \cong \mathcal{O}^r(E, C) \cong C^*(F)$$

for a finite graph F with Condition (K). In particular, $L_K^{\mathrm{ab}}(E, C)$ is an exchange ring and $\mathcal{O}(E, C) \cong \mathcal{O}^r(E, C)$ has real rank zero.

Proof. Applying Corollary 6.6 to $\mathbf{B}(E, C)$, it follows from Corollary 2.5 and Proposition 2.17 that $\theta^{(E,C)}$ is essentially free as well. Moreover, there are isomorphisms

$$M_2(L_K^{\mathrm{ab}}(E, C)) \cong L_K^{\mathrm{ab}}(\mathbf{B}(E, C)) \cong L_K(F) \quad \text{and} \quad M_2(\mathcal{O}(E, C)) \cong \mathcal{O}(\mathbf{B}(E, C)) \cong C^*(F),$$

where F is a graph satisfying Condition (K), by [2, Proposition 9.1]. We may then apply [11, Theorem 6.1] (along with the final comment in the introduction of [11]) to obtain a graph G for which $L_K^{\mathrm{ab}}(E, C) \cong L_K(G)$ and $\mathcal{O}(E, C) \cong C^*(G)$. □

In order to extend Corollary 6.7 to arbitrary finitely separated graphs, we need to be able to approximate any such Condition (K) graph by its finite complete Condition (K) subgraphs.

Lemma 6.8. *Every finitely separated Condition (K) graph is a direct limit of its finite complete Condition (K) subgraphs.*

Proof. Let (E,C) denote a finitely separated Condition (K) graph with a finite complete subgraph (F,D). We then claim that there is an intermediate finite complete subgraph $(F,D) \subset (G,L) \subset (E,C)$ satisfying Condition (K), and we first observe that if $v \in F^0$ admits a cycle α in (F,D), then it automatically admits a choice in (F,D) as well: By assumption, it admits exactly one choice in (E,C), so if β is a minimal path with $s(\beta) = v$ leading to a choice X, then either $x^{-1}\beta \leq \alpha$ or $x^{-1}\beta \leq \alpha^{-1}$ for some $x \in X$. From (F,D) being a complete subgraph, it follows that $X \in D$, so v admits exactly one choice in (F,D) as well.

Now assume that v admits only one base-simple cycle in (F,D) up to inversion, and consider some other base-simple cycle $e_n^{\varepsilon_n} \cdots e_1^{\varepsilon_1}$ based at v in (E,C). We then extend the subgraph by the set of edges $\bigcup_{i=1}^n [e_i]$ as well as the ranges and sources of these edges to a finite complete subgraph. Observe that all the added vertices either admit no or at least two base-simple cycles up to inversion, so applying this procedure sufficiently many times leaves us with a finite complete subgraph (G,L) satisfying Condition (K). ☐

Corollary 6.9. *Let (E,C) denote a finitely separated graph satisfying Condition (K). Then the partial action $\theta^{(E,C)}$ is essentially free, and there are finite non-separated Condition (K) graphs $(F_n)_{n\geq 1}$ such that*

$$L_K^{\mathrm{ab}}(E,C) \cong \varinjlim_n L_K(F_n) \quad \textit{and} \quad \mathcal{O}(E,C) \cong \mathcal{O}^r(E,C) \cong \varinjlim_n C^*(F_n)$$

for appropriate connecting homomorphisms. In particular, $L_K^{\mathrm{ab}}(E,C)$ is an exchange ring and $\mathcal{O}(E,C) \cong \mathcal{O}^r(E,C)$ has real rank zero.

Proof. By Lemma 6.8, we can find an increasing union of finite complete Condition (K) subgraphs (G_n, L^n) of (E,C) such that $(E,C) = \varinjlim_n (G_n, L^n)$. Then $L_K^{\mathrm{ab}}(G_n, L^n) \cong L_K(F_n)$ and $\mathcal{O}^{(r)}(G_n, L^n) \cong C^*(F_n)$ for some non-separated graph F_n satisfying Condition (K) by Corollary 6.7. Recalling from [3, Proposition 7.2] that $\mathcal{O}$ is a continuous functor and that the same proof applies to L_K^{ab}, we see that

$$L_K^{\mathrm{ab}}(E,C) \cong \varinjlim_n L_K(F_n) \quad \text{and} \quad \mathcal{O}(E,C) \cong \mathcal{O}^r(E,C) \cong \varinjlim_n C^*(F_n),$$

and as the exchange property passes to limits, it follows from Lemma 6.8 and Corollary 6.7 that $L_K^{\mathrm{ab}}(E,C)$ is an exchange ring, and $\mathcal{O}(E,C) \cong \mathcal{O}^r(E,C)$ has real rank zero.

We move on to checking essential freeness. Assume that $\Omega \subset \Omega(E,C)$ is a closed invariant subspace and $\xi \in \Omega$ is fixed by $1 \neq \alpha \in \mathbb{F}$. Taking any finite animal $\omega \subset \xi$ with $\alpha \in \omega$, we must verify that $\theta_\alpha^{(E,C)}(\eta) \neq \eta$ for some $\eta \in \Omega \cap \Omega(E,C)_\omega$. By Lemma 6.8, there is a finite complete Condition (K) subgraph (F, D) of (E, C) such that $\omega \subset \mathbb{F}(F^1)$, and we consider the canonical surjective $\mathbb{F}(F^1)$-equivariant continuous map

$$p\colon \Omega(E,C)_{F^0} = \bigsqcup_{v\in F^0} \Omega(E,C)_v \to \Omega(F,D)$$

given by $p(\eta) = \eta \cap \mathbb{F}(F^1)$. Then $\Omega' := p(\Omega(E,C)_{F^0} \cap \Omega)$ is a closed invariant subspace of $\Omega(F, D)$, and $p(\xi) \in \Omega'$ is fixed by α. Moreover, $\Omega(F, D)_\omega \cap \Omega'$ is an open neighbourhood of $p(\xi)$ in Ω', so by essential freeness there is some $\zeta \in \Omega(F, D)_\omega \cap \Omega'$ with $\theta_\alpha^{(F,D)}(\zeta) \neq \zeta$. Now any lift $\eta \in \Omega$ of ζ will do the job. □

Having proved the positive part of the main result of this section, we now begin an investigation of finitely separated graphs not satisfying Condition (K). The lemma just below takes care of the situation in which a cycle admits at least two choices.

Lemma 6.10. *Let (E, C) denote a finitely separated graph. If some cycle admits at least two choices, then there is a configuration $\xi \in \Omega(E, C)$ with stabiliser* $\mathrm{Stab}(\xi) \cong \mathbb{Z}$*, such that ξ is isolated in $\overline{\theta_{\mathbb{F}}(\xi)}$.*

Proof. Observe that one of the following holds:

(1) There is a cycle α and an admissible path β with the following properties:
   (a) Both compositions $\beta\alpha$ and $\beta\alpha^{-1}$ are admissible.
   (b) $\mathsf{t}_d(\beta) = x^{-1}$ for some $x \in E^1$ with $|[x]| \geq 2$.
(2) There is a cycle α with subpaths $x^{-1} \leq \alpha$ and $y^{-1}\beta \leq \alpha^{-1}$ such that $|[x]|, |[y]| \geq 2$.

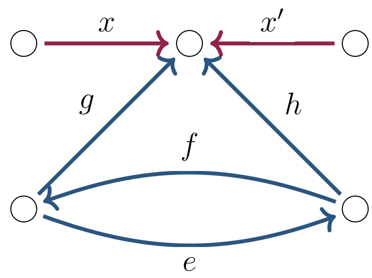
In case of (1), we consider the animal $\omega := \langle \beta\alpha^n \mid n \in \mathbb{Z}\rangle$. Being α-periodic, we may extend it to an α-periodic configuration ξ such that $\mathsf{t}_d(\gamma) = x^{-1}$ entails $\gamma \in \omega$ (see also Example 6.11 for what such ξ might look like for a particular graph). For the sake of completeness, let us carry out the actual construction of such ξ: First consider the finite animal $\langle \mathsf{t}_d(\alpha)^{-1}, \alpha, \beta\rangle$ and extend it arbitrarily to a configuration η such that $\mathsf{t}_d(\gamma) = x^{-1}$ for $\gamma \in \eta$ entails $\gamma \in \langle \mathsf{t}_d(\alpha)^{-1}, \alpha, \beta\rangle$: This can be done as in the proof of Lemma 4.3 by never choosing to go down x^{-1} when extending. Then consider the animal

$$\chi := \{\gamma \in \eta \mid \gamma \ngeq \mathsf{t}_d(\alpha)^{-1}, \alpha\}$$

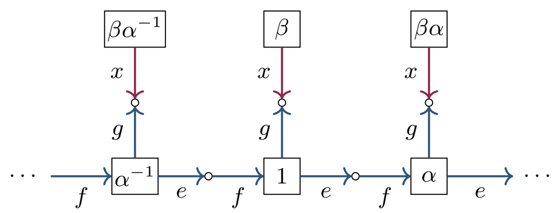
and define $\xi := \bigsqcup_{n\in\mathbb{Z}} \chi \cdot \alpha^n$. It should be clear that $\xi \in \Omega(E, C)$, and by construction it is fixed by α. Let $\gamma \in \xi$ and assume that $\mathsf{t}_d(\gamma) = x^{-1}$; we may then write $\gamma = \gamma' \cdot \alpha^n$ uniquely with $\gamma' \in \chi$ and $n \in \mathbb{Z}$. Now if $\mathsf{t}_d(\gamma') = \mathsf{t}_d(\gamma) = x^{-1}$, we have $\gamma' < \alpha$ or $\gamma' \leq \beta$ by construction of η, and if $\mathsf{t}_d(\gamma') \neq \mathsf{t}_d(\gamma)$, then γ' must be cancelled out completely by α^n, hence $\gamma' < \alpha$. In either case, we see that $\gamma \in \omega$ as required. It follows that $s_{\{x^{-1}\}}(\theta_\gamma(\xi)) \leq 2$ whenever $\gamma \in \xi$ is not a power of α, while $s_{\{x^{-1}\}}(\xi) = 3$. We conclude from Lemma 4.2 that ξ is isolated in the closure of its own orbit and that $\mathrm{Stab}(\xi) \cong \mathbb{Z}$.

Now consider (2), and assume without loss of generality that (1) does not hold. In this case, the animal $\omega = \langle \alpha^n \mid n \in \mathbb{Z}\rangle$ can be extended uniquely to a configuration ξ, which is necessarily α-periodic. Setting $A := \{x^{-1}, y^{-1}\}$, we see that $s_A(\theta_\gamma(\xi)) = 1$ for all $\gamma \notin \omega$ while $s_A(\xi) = 2$, so ξ is once again isolated in $\overline{\theta_{\mathbb{F}}(\xi)}$ and has stabiliser $\mathrm{Stab}(\xi) \cong \mathbb{Z}$. □

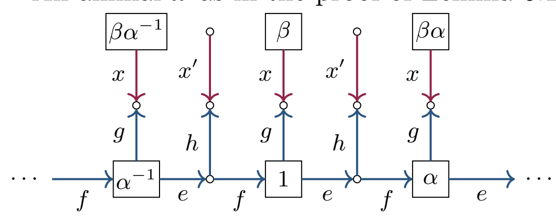
Example 6.11. We will now consider the separated graph



satisfying (1) in the proof of Lemma 6.10, and see what the configuration ξ might look like. We take α to be the cycle $\alpha = fe$ and β to be the path $\beta = x^{-1}g$, so the animal ω and the configuration ξ may be pictured as below:



An animal ω as in the proof of Lemma 6.10.



A configuration ξ as in the proof of Lemma 6.10.

◀

Next, we consider the situation in which a vertex admits exactly one choice and one base-simple cycle up to inversion.

Lemma 6.12. *Let (E, C) denote a finitely separated graph, and assume that $v \in E^0$ admits exactly one choice and exactly one base-simple cycle up to inversion. Then there is $H \in \mathcal{H}(E, C)$ with $v \notin H$, such that v admits a cycle but no choices in $(E/H, C/H)$.*

Proof. Let α denote the unique base-simple cycle based at v such that α^{-1} is forced. By possibly translating the cycle, we may assume that there is $X \in C_v$ with $|X| \geq 2$ such that $\mathfrak{t}_{\mathrm{d}}(\alpha) = x^{-1}$ for some $x \in X$. Now define

$$H := \{u \in E^0 \mid \text{there is no forced path } v \to u\}$$

and observe that H is hereditary: If $e \in E^1$ and $s(e) \notin H$, i.e. if there is a forced path $\beta\colon v \to s(e)$, then $e \cdot \beta$ is a forced path $v \to r(e)$ as well, hence $r(e) \notin H$. In order to check that H is also C-saturated, we assume that $u \notin H$ with $\beta\colon v \to u$ forced, and take any $Y \in C_u$. If $Y = \{y\}$ is a singleton or $y := \mathfrak{t}_d(\beta)^{-1} \in Y$, then $y^{-1} \cdot \beta$ is forced as well, so $s(y) \notin H$. If $|Y| \geq 2$, then we must have $\beta = v$ and $X = Y$ so that $s(x) \notin H$. We conclude that H is indeed a hereditary and C-saturated subset. We proceed to check that v does not admit any choices in the quotient graph $(E/H, C/H)$, and it suffices to verify that $X/H = \{x\}$ as the same argument may be applied to any other vertex on α. If there were some other $x' \in X/H$, then, by definition of H, there would exist a forced path $\beta\colon v \to s(x')$. But then $x'\beta$ would a cycle, which is clearly not a power of α, so we have reached a contradiction. $\square$

Finally, we are ready to patch everything together and obtain our main theorem.

Theorem 6.13. *Let (E, C) denote a finitely separated graph. The following are equivalent:*

(1) *(E, C) satisfies Condition (K).*
(2) *$\theta^{(E,C)}$ is essentially free.*
(3) *$\mathcal{O}(E, C)$ has real rank zero.*
(4) *$\mathcal{O}^r(E, C)$ has real rank zero.*
(5) *$L_K^{\mathrm{ab}}(E, C)$ is an exchange ring.*
(6) *$\mathcal{O}(E, C)$ is the direct limit of real rank zero graph C^*-algebras of finite non-separated graphs.*
(7) *$L_K^{\mathrm{ab}}(E, C)$ is the direct limit of Leavitt path algebras of finite non-separated graphs with the exchange property.*

If (E, C) is finite, then we may replace (6) *and* (7) *with the conditions*

(6') *$\mathcal{O}(E, C)$ is isomorphic to a real rank zero graph C^*-algebra of a finite non-separated graph.*
(7') *$L_K^{\mathrm{ab}}(E, C)$ is isomorphic to a Leavitt path algebra of a finite non-separated graph with the exchange property.*

Proof. In any case, (1) implies (2)-(7) due to Corollary 6.9, and if (E, C) is finite, then (6') and (7') follow from Corollary 6.7. Now suppose that (E, C) does not satisfy Condition (K). Then there is a vertex v on a cycle such that one of the following holds:

(i) v admits no choices,
(ii) v admits exactly one choice and one base-simple cycle up to inversion,
(iii) v admits at least two choices.

In the case of (i), the compact-open subspace $\Omega(E, C)_v$ is nothing but an isolated point with stabiliser $\mathbb{F}_v$, so the partial action is not even topologically free. Moreover, as we observed in Proposition 4.7,

$$vL_K^{\mathrm{ab}}(E, C)v \cong K[\mathbb{F}_v], \quad v\mathcal{O}(E, C)v \cong C^*(\mathbb{F}_v) \quad \text{and} \quad v\mathcal{O}^r(E, C)v \cong C_r^*(\mathbb{F}_v),$$

so neither is an exchange ring. If (ii) holds, then there is hereditary and C-saturated subset $H \subset E^0$ as in Lemma 6.12, giving rise to an invariant closed subspace on which the restricted action is directly quasi-conjugate to the partial action $\theta^{(E/H, C/H)}$ by Theorem 2.19. The quotient graph $(E/H, C/H)$ satisfies (i), so the first case applies. Finally, Lemma 6.10 applies to (iii) to give a point ξ with non-trivial stabiliser, such that ξ is isolated in $\Omega := \overline{\theta_{\mathbb{F}}(\xi)}$. We immediately see that the restricted partial action is not topologically free,

$$1_\xi\big(C_K(\Omega) \rtimes \mathbb{F}\big)1_\xi \cong K[\mathbb{Z}] \quad \text{and} \quad 1_\xi\big(C(\Omega) \rtimes_{(r)} \mathbb{F}\big)1_\xi \cong C(\mathbb{T}).$$

It follows that none of the above crossed product are exchange rings, so neither are $L_K^{\mathrm{ab}}(E, C)$, $\mathcal{O}(E, C)$ and $\mathcal{O}^r(E, C)$. □

Corollary 6.14. *Let (E, C) denote a finitely separated graph. If either of the algebras $L_K^{\mathrm{ab}}(E, C)$, $\mathcal{O}(E, C)$ and $\mathcal{O}^r(E, C)$ is an exchange ring, then it is also separative.*

Proof. This is immediate from Theorem 6.13 and [9, Theorem 3.5, Proposition 4.4 and Theorem 7.1]. □

The above corollary shows that the tame algebras of finitely separated graphs do not provide a solution to the Fundamental Separativity Problem for exchange rings. However, as was noted in [8], the crossed product $C(\mathcal{X}) \rtimes_\sigma \mathbb{Z}$ of any two-sided subshift is Morita equivalent to a quotient of a separated graph C^*-algebra, corresponding to the restriction of the partial action to a closed invariant subspace. In particular, interesting real rank zero C^*-algebras, which are not graph C^*-algebras, may arise from separated Bratteli diagrams (see [8, Definition 2.8]). The question therefore remains if one can find a finite bipartite separated graph (E, C) and a hereditary D^∞-saturated subset of the associated separated Bratteli diagram, corresponding to a closed invariant subspace $\Omega \subset \Omega(E, C)$ of infinite type (in the sense of [8, Section 3]), such that the monoid

$$\mathcal{V}(L_K^{\mathrm{ab}}(E, C)) \cong M(F_\infty/H, D^\infty/H) \cong \varinjlim_n M(E_n/H^{(n)}, C^n/H^{(n)}),$$

where $H^{(n)} := H \cap E_n^0$, is non-separative and the limit algebras

$$\varinjlim_n L^{\mathrm{ab}}(E_n/H^{(n)}, C^n/H^{(n)}) \cong C_K(\Omega) \rtimes \mathbb{F} \quad \text{and} \quad \varinjlim_n \mathcal{O}^r(E_n/H^{(n)}, C^n/H^{(n)}) \cong C(\Omega) \rtimes_r \mathbb{F}$$

are exchange rings. One strategy would be to start out with a suitable graph (E, C), for which the monoid $M(E, C)$ is non-separative, and then try to remove more and more obstructions to the exchange property the bigger n gets, while still maintaining injectivity of the composition

$$M(E, C) \to M(E_n, C^n) \to M(E_n, /H^{(n)}, C^n/H^{(n)}).$$

One could then hope to remove *all* obstructions to the exchange property in the limit algebras.

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References

[1] P. Ara. Extensions of exchange rings. *J. Algebra*, 197(2):409–423, 1997.
[2] P. Ara and R. Exel. Dynamical systems associated to separated graphs, graph algebras, and paradoxical decompositions. *Adv. Math.*, 252:748–804, 2014.
[3] P. Ara and R. Exel. K-theory for the tame C*-algebra of a separated graph. *J. Funct. Anal.*, 269(9):2995–3041, 2015.
[4] P. Ara, M. Gómez Lozano, and M. Siles Molina. Local rings of exchange rings. *Comm. Algebra*, 26(12):4191–4205, 1998.
[5] P. Ara and K. R. Goodearl. C^*-algebras of separated graphs. *J. Funct. Anal.*, 261(9):2540–2568, 2011.
[6] P. Ara and K. R. Goodearl. Leavitt path algebras of separated graphs. *J. Reine Angew. Math.*, 669:165–224, 2012.
[7] P. Ara, K. R. Goodearl, K. C. O'Meara, and E. Pardo. Separative cancellation for projective modules over exchange rings. *Israel J. Math.*, 105:105–137, 1998.
[8] P. Ara and M. Lolk. Convex subshifts, separated bratteli diagrams, and ideal structure of tame separated graph algebras. *ArXiv e-prints*, 2017.
[9] P. Ara, M. A. Moreno, and E. Pardo. Nonstable K-theory for graph algebras. *Algebr. Represent. Theory*, 10(2):157–178, 2007.
[10] G. Aranda Pino, E. Pardo, and M. Siles Molina. Exchange Leavitt path algebras and stable rank. *J. Algebra*, 305(2):912–936, 2006.
[11] S. E. Arklint, J. Gabe, and E. Ruiz. Hereditary C^*-subalgebras of graph C^*-algebras. arXiv:1604.03085v2 [math.OA].
[12] J. Brown, L. O. Clark, C. Farthing, and A. Sims. Simplicity of algebras associated to étale groupoids. *Semigroup Forum*, 88(2):433–452, 2014.
[13] L. G. Brown and G. K. Pedersen. C^*-algebras of real rank zero. *J. Funct. Anal.*, 99(1):131–149, 1991.
[14] T. M. Carlsen and N. S. Larsen. Partial actions and KMS states on relative graph C^*-algebras. *J. Funct. Anal.*, 271(8):2090–2132, 2016.
[15] T. M. Carlsen, E. Ruiz, and A. Sims. Equivalence and stable isomorphism of groupoids, and diagonal-preserving stable isomorphisms of graph C^*-algebras and Leavitt path algebras. *Proc. Amer. Math. Soc.*, 145(4):1581–1592, 2017.
[16] L. O. Clark, C. Edie-Michell, A. an Huef, and A. Sims. Ideals of steinberg algebras of strongly effective groupoids, with applications to leavitt path algebras. *ArXiv e-prints.*
[17] B. L. Duncan. When universal edge-colored directed graph c*-algebras are exact. *ArXiv e-prints.*
[18] R. Exel. *Partial Dynamical Systems, Fell Bundles and Applications.* 2014.
[19] R. Exel, M. Laca, and J. Quigg. Partial dynamical systems and C^*-algebras generated by partial isometries. *J. Operator Theory*, 47(1):169–186, 2002.
[20] K. R. Goodearl. von Neumann regular rings and direct sum decomposition problems. In *Abelian groups and modules (Padova, 1994)*, volume 343 of *Math. Appl.*, pages 249–255. Kluwer Acad. Publ., Dordrecht, 1995.
[21] J. A. Jeong. Real rank of C^*-algebras associated with graphs. *J. Aust. Math. Soc.*, 77(1):141–147, 2004.
[22] X. Li. Dynamic characterizations of quasi-isometry, and applications to cohomology. *ArXiv e-prints.*
[23] M. Lolk. On nuclearity and exactness of the tame C^*-algebras associated with finitely separated graphs. *ArXiv e-prints*, 2017.
[24] A. L. T. Paterson. Graph inverse semigroups, groupoids and their C^*-algebras. *J. Operator Theory*, 48(3, suppl.):645–662, 2002.
[25] R. T. Powers. Simplicity of the C^*-algebra associated with the free group on two generators. *Duke Math. J.*, 42:151–156, 1975.

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ARTICLE C

On nuclearity and exactness of the tame C^*-algebras associated with finitely separated graphs

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ON NUCLEARITY AND EXACTNESS OF THE TAME C^*-ALGEBRAS ASSOCIATED WITH FINITELY SEPARATED GRAPHS

MATIAS LOLK

Abstract. We introduce a graph theoretic property called Condition (N) for finitely separated graphs and prove that it is equivalent to both nuclearity and exactness of the associated universal tame graph C^*-algebra.

Introduction

A *finitely separated graph* is a directed graph with a partition of the edges into finite subsets, which might be thought of as an edge colouring, so that edges with distinct ranges have different colours. To any such graph (E, C), Ara and Exel introduced a C^*-algebra $\mathcal{O}(E, C)$ in [4, 5], referred to as the universal *tame* C^*-algebra of (E, C). It is generated by the vertices and the edges of the graph with relations similar to the ordinary Cuntz-Krieger relations, taking into account the colouring, so that the edge set E^1 defines a tame set of partial isometries. Among many other results, they provided a very useful dynamical description of $\mathcal{O}(E, C)$ when (E, C) is finite and bipartite, and this was generalised to finitely separated graphs by the author in [17]. Specifically, $\mathcal{O}(E, C)$ may be identified with a universal crossed product $C_0(\Omega(E, C)) \rtimes \mathbb{F}$ for a partial action $\theta^{(E,C)}$ of a free group $\mathbb{F} = \mathbb{F}(E^1)$ on a locally compact, zero-dimensional Hausdorff space $\Omega(E, C)$. The potency of these partial actions was immediately demonstrated when they were used to answer a question of Rørdam and Sierakowski [20] about relative type semigroups in the negative [4, Section 7].

Recently, Ara and the author have introduced the more general notion of a *convex subshift* [9, Section 3] and shown that all convex subshifts of *finite type* arise, up to Kakutani equivalence, as the partial action associated with a finite bipartite graph. As such, finitely separated graphs may be viewed as combinatorial models for a wide class of partial actions, which includes both one-sided and two-sided shift spaces, but which also includes many new and previously unstudied dynamical systems.

Nuclearity and exactness of $\mathcal{O}(E, C)$ have not been systematically studied before, but both nuclearity and non-exactness have been observed in a number of examples. If E is any column-finite graph, it may be regarded as a finitely separated graph by equipping it with the *trivial separation* $\mathcal{T}$, and in this case, the tame C^*-algebra $\mathcal{O}(E, \mathcal{T})$ is simply the classical graph C^*-algebra $C^*(E)$, hence nuclear. Likewise, if $\mathcal{X}$ is any two-sided subshift of finite type, then the crossed product $C(\mathcal{X}) \rtimes \mathbb{Z}$ is Morita equivalent to $\mathcal{O}(E, C)$ for an appropriate finite bipartite separated graph by [9, Proposition 6.8], so in this case $\mathcal{O}(E, C)$ is nuclear as well. In the other direction, one can easily identify $C^*(\mathbb{F}_n)$ with a tame separated graph

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C^*-algebra: the graph is nothing but a single vertex with n loops of different colours. A much more interesting class of non-exact examples were considered by Ara, Exel and Katsura in [6], where they studied C^*-algebras $\mathcal{O}_{m,n}$ for $2 \leq m < n < \infty$ that might be considered as two-parameter versions of the Cuntz algebras.

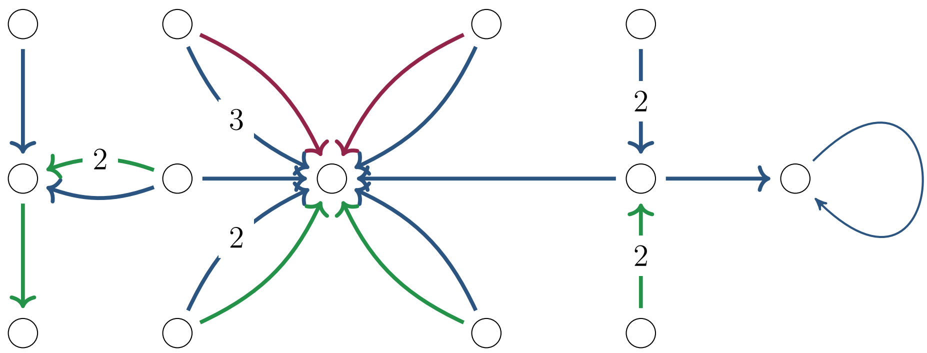
In this paper, we characterise nuclearity and exactness in terms a graph-theoretic property that we call Condition (N). To this end, we first introduce a notion of topological amenability for partial actions, and prove, using the theory of groupoid C^*-algebras, that a partial action of a discrete group on a locally compact Hausdorff space is topologically amenable if and only if the corresponding crossed products are nuclear (Theorem 2.8). Given any finitely separated graph (E, C), we then identify two complementary subgraphs $(E_{\mathrm{Br}}, C^{\mathrm{Br}})$ and $(E_{\mathrm{BF}}, C^{\mathrm{BF}})$, called the *branching subgraph* and the *branch free subgraph*, respectively. The C^*-algebra $\mathcal{O}(E_{\mathrm{Br}}, C^{\mathrm{Br}})$ naturally appears as a quotient of $\mathcal{O}(E, C)$, and in Section 3, we prove that both nuclearity and exactness of $\mathcal{O}(E_{\mathrm{Br}}, C^{\mathrm{Br}})$ is equivalent to Condition (N). We can use the approach of [6] to establish necessity (Proposition 3.7), but sufficiency takes more work: The main step is the construction of a *proper orientation* of the branching subgraph (Theorem 3.17) in the presence of Condition (N). In Section 4, we prove that $\mathcal{O}(E_{\mathrm{BF}}, C^{\mathrm{BF}})$ is *always* nuclear, and that nuclearity of $\mathcal{O}(E_{\mathrm{Br}}, C^{\mathrm{Br}})$ and $\mathcal{O}(E_{\mathrm{BF}}, C^{\mathrm{BF}})$ implies nuclearity of $\mathcal{O}(E, C)$, before putting everything together in section 5 to obtain our main theorem (Theorem 5.1). We finally study a number of examples.

1. Preliminary definitions

In this section, we recall the necessary definitions and results from the existing theory on algebras associated with separated graphs in a slightly condensed version – the reader may consult [9, Section 2] and [17, Section 1] for more details. Most importantly, we describe the C^*-algebra $\mathcal{O}(E, C)$ as a universal crossed product for a partial action.

Definition 1.1. A *finitely separated graph* (E, C) is a graph $E = (E^0, E^1, r, s)$ together with a *separation* $C = \bigsqcup_{v \in E^0} C_v$, where each C_v is a partition of $r^{-1}(v)$ into non-empty finite subsets. In case $r^{-1}(v) = \emptyset$, we simply take C_v to be the empty partition, and for any edge $e \in E^1$, we will denote the element of C containing e by $[e]$.

Example 1.2. Below is an example of a finite separated graph:



We gladly use the same colour for edges with different ranges when depicting separated graphs – otherwise one would need nine colours here – so the colouring should only be understood as a partition of the edges going into a given vertex. The numbering indicates the number of edges, so that we may simply write a number, say 42, instead of visually representing each of the 42 edges. This particular graph will serve as our main example throughout the paper. ◀

Recall that a set of partial isometries S is called *tame* if every product formed from $S \cup S^*$ is again a partial isometry.

Definition 1.3 ([5, Definition 2.4])**.** Let (E, C) denote a finitely separated graph. The *universal tame graph C^*-algebra $\mathcal{O}(E, C)$* is the universal C^*-algebra generated by $E^0 \sqcup E^1$ with relations

(V) $uv = \delta_{u,v} v$ and $u = u^*$ for $u, v \in E^0$,
(E) $es(e) = r(e)e = e$ for $e \in E^1$,
(SCK1) $e^* f = \delta_{e,f} s(e)$ if $[e] = [f]$,
(SCK2) $v = \sum_{e \in X} ee^*$ for all $v \in E^0$ and $X \in C_v$,
(T) $E^1 \subset \mathcal{O}(E, C)$ is tame.

The reader should note that we use the convention of [4], [5], [9] and [17], often referred to as the *Raeburn-convention*, opposite to the one used in [7] and [8].

Remark 1.4. A subgraph (F, D) of (E, C) is called *complete* if

$$D_v = \{X \in C_v \mid X \cap F^1 \neq \emptyset\}$$

for every $v \in F^0$, and by universality there is an induced $*$-homomorphism $\mathcal{O}(F, D) \to \mathcal{O}(E, C)$. Since every finitely separated graph is the direct limit of its finite complete subgraphs (see [8, Section 3] for the precise meaning of this), and $\mathcal{O}$ is a continuous functor from the category of finitely separated graphs [5, Proposition 7.2], we see that $\mathcal{O}(E, C)$ may always be approximated by C^*-algebras $\mathcal{O}(F, D)$ for (F, D) a finite separated graph. ◀

We now recall a bit of terminology related to separated graphs from [17].

Definition 1.5. Let (E, C) denote a (finitely) separated graph. The *double* $\widehat{E}$ of E is the graph given by $\widehat{E}^0 := E^0$ and $\widehat{E}^1 = E^1 \sqcup \{e^{-1} \mid e \in E^1\}$ with range and source maps extended by $r(e^{-1}) := s(e)$ and $s(e^{-1}) = r(e)$. An *admissible path* α in (E, C) is a path (read from the right) in $\widehat{E}$ such that

(1) any subpath ef^{-1} with $e, f \in E^1$ satisfies $e \neq f$,
(2) any subpath $e^{-1} f$ with $e, f \in E^1$ satisfies $[e] \neq [f]$.

We regard the vertices $v \in E^0$ as the *trivial* admissible paths with $r(v) := v =: s(v)$, and if α is a non-trivial admissible path, we will write $\mathfrak{i}_d(\alpha)$ and $\mathfrak{t}_d(\alpha)$ for the initial and terminal symbol of α, respectively; for instance

$$\mathfrak{i}_d(ef^{-1}) = f^{-1} \quad \text{and} \quad \mathfrak{t}_d(ef^{-1}) = e.$$

We then extend the range and source functions to admissible paths by the formulas $r(\alpha) := r(\mathfrak{t}_d(\alpha))$ and $s(\alpha) := s(\mathfrak{i}_d(\alpha))$. For admissible paths α and β, we will denote the concatenation by $\beta\alpha$, and we will always write $\beta \cdot \alpha$ if we allow for cancellation of mutual inverses.

A *closed path* in (E, C) is a non-trivial admissible path α with $r(\alpha) = s(\alpha)$, and α is called a *cycle* if the concatenation $\alpha\alpha$ is an admissible path as well. Either way, we shall say that α is *based* at $r(\alpha) = s(\alpha)$, and α is called *base-simple* if $r(\beta) \neq s(\alpha)$ for all proper subpaths $s(\alpha) < \beta < \alpha$. We finally define a natural partial order $\leq$ on the set of admissible paths by

$$\beta \leq \alpha \Leftrightarrow \beta \text{ is an initial subpath of } \alpha,$$

and whenever $s(\alpha) = s(\beta)$, we write $\alpha \wedge \beta \leq \alpha, \beta$ for the maximal initial subpath. ◀

The notion of a *partial action* is fundamental to this work, and so we briefly recall the essentials.

Definition 1.6. A partial action $\theta \colon G \curvearrowright \Omega$ of a discrete group G on a topological space Ω is a family of homeomorphisms of open subspaces $\{\theta_g \colon \Omega_{g^{-1}} \to \Omega_g\}_{g \in G}$, such that

- $\theta_g(\Omega_{g^{-1}} \cap \Omega_h) \subset \Omega_{gh}$ for all $g, h \in G$,
- $\theta_g(\theta_h(x)) = \theta_{gh}(x)$ for all $g, h \in G$ and $x \in \Omega_{h^{-1}} \cap \Omega_{h^{-1}g^{-1}}$.

For any point $x \in \Omega$, we will write $G^x := \{g \in G \mid x \in \Omega_{g^{-1}}\}$ for the group elements that can act on x. Now if $\theta' \colon G \curvearrowright \Omega'$ is another partial action, then a map $\varphi \colon \Omega \to \Omega'$ is called *G-equivariant* if

- $\varphi(\Omega_g) \subset \Omega'_g$ for all $g \in G$,
- $\theta'_g(\varphi(x)) = \varphi(\theta_g(x))$ for all $g \in G$ and $x \in \Omega_{g^{-1}}$.

Similarly to the above, one can define the concept of a partial action on a C^*-algebra, demanding that the domains should be closed two-sided ideals. Hence, if Ω is a locally compact Hausdorff space, then θ translates into a partial C^*-action $\theta^* \colon G \curvearrowright C_0(\Omega)$. As is the case for global actions, one can associate both a *full* and a *reduced* crossed product to a partial action, and there is a canonical surjective $*$-homomorphism

$$C_0(\Omega) \rtimes G \to C_0(\Omega) \rtimes_r G,$$

called the *regular representation.* We refer the reader to [13] for a comprehensive treatment of the theory of partial actions and their crossed products. ◀

We now proceed to describe the partial dynamical system associated with a finitely separated graph, introduced in [4] for finite bipartite graphs and extended to the more general setting in [17].

Definition 1.7 ([17, Definition 2.6])**.** Suppose that (E, C) is a finitely separated graph, and let $\mathbb{F}$ denote the free group on E^1. Given $\xi \subset \mathbb{F}$ and $\alpha \in \xi$, the *local configuration* ξ_α of ξ at α is the set

$$\xi_\alpha := \{s \in E^1 \sqcup (E^1)^{-1} \mid s \in \xi \cdot \alpha^{-1}\}.$$

Then $\Omega(E, C)$ is the disjoint union of the discrete space E^0_{iso} and the set of $\xi \subset \mathbb{F}$ satisfying the following:

(a) $1 \in \xi$.

(b) ξ is *right-convex*: In view of (a), this exactly means that if $e_n^{\varepsilon_n} \cdots e_1^{\varepsilon_1} \in \xi$ for $e_i \in E^1$ and $\varepsilon_i \in \{\pm 1\}$, then $e_m^{\varepsilon_m} \cdots e_1^{\varepsilon_1} \in \xi$ as well for any $1 \leq m < n$.

(c) For every $\alpha \in \xi$, there is some $v \in E^0$ and $e_X \in X$ for each $X \in C_v$, such that

$$\xi_\alpha = s^{-1}(v) \sqcup \{e_X^{-1} \mid X \in C_v\}.$$

$\Omega(E, C)$ is made into a topological space by regarding it as a subspace $\{0,1\}^{\mathbb{F}} \sqcup E^0_{\text{iso}}$. One can easily check that it becomes a zero-dimensional, locally compact Hausdorff space, which is compact if and only if E^0 is a finite set. A topological partial action $\theta = \theta^{(E,C)} \colon \mathbb{F} \curvearrowright \Omega(E, C)$ with compact-open domains is then defined by setting

$$\Omega(E, C)_\alpha := \{\xi \in \Omega(E, C) \mid \alpha^{-1} \in \xi\} \quad \text{and} \quad \theta_\alpha(\xi) := \xi \cdot \alpha^{-1}$$

for $\alpha \in \mathbb{F}$ and $\xi \in \Omega(E, C)_{\alpha^{-1}}$. It follows from (a), (b) and (c) above that $\Omega(E, C)_\alpha$ is non-empty if and only if α is an admissible path. We finally set $\Omega(E, C)_{s(e)} := \Omega(E, C)_{e^{-1}}$ for every $e \in E^1$ and

$$\Omega(E, C)_v := \bigsqcup_{e \in X} \Omega(E, C)_e$$

for every $X \in C_v$. Note that in case $X \in C_v$ and $v = s(e)$, these two definitions coincide. If v is isolated, we simply set $\Omega(E, C)_v := \{v\}$. The reader may think of $\Omega(E, C)_v$ as the set of configurations “starting” in v, and we have $\Omega(E, C) = \bigsqcup_{v \in E^0} \Omega(E, C)_v$. ◀

We will also need the notion of an *animal*.

Definition 1.8 ([17, Definition 2.9])**.** An (E, C)-*animal* is a right-convex subset $\omega \subset \xi$ of a configuration $\xi \in \Omega(E, C) \setminus E^0_{\text{iso}}$ such that $\{1\} \subsetneq \omega$. It is called finite if it is a finite set, and we can define a compact subset of $\Omega(E, C)$ by

$$\Omega(E, C)_\omega := \{\xi \in \Omega(E, C) \mid \omega \subset \xi\},$$

which is open if ω is finite. It is easy to check that if $\{1\} \neq S \subset \mathbb{F}$ is any non-empty subset such that $\alpha \cdot \beta^{-1}$ is admissible for distinct $\alpha, \beta \in S \cup \{1\}$, then the right-convex closure $\langle S \rangle := \operatorname{conv}(S \cup \{1\})$ of $S \cup \{1\}$ inside $\mathbb{F}$ defines an (E, C)-animal. We warn the reader that we have the somewhat confusing identity $\Omega(E, C)_\alpha = \Omega(E, C)_{\{\alpha^{-1}\}}$.

Our main tool for studying the tame universal C^*-algebra of a separated graph is the following result, which was first obtained for finite bipartite separated graphs in [4].

Theorem 1.9 ([17, Theorem 2.10])**.** *Let (E, C) denote a finitely separated graph. Then there is a canonical isomorphism of C^*-algebras $\mathcal{O}(E, C) \cong C_0(\Omega(E, C)) \rtimes \mathbb{F}$.* ◀

As a consequence of this theorem, there is also a natural *reduced* tame C^*-algebra.

Definition 1.10 ([5, Definition 6.8])**.** *The reduced tame graph C^*-algebra* of a finitely separated graph (E, C) is the reduced crossed product $\mathcal{O}^r(E, C) := C_0(\Omega(E, C)) \rtimes_r \mathbb{F}$. ◀

Just as for non-separated graphs, there are certain sets of vertices that naturally correspond to ideals.

Definition 1.11 ([8, Definition 6.3 and Definition 6.5])**.** Let (E, C) denote a finitely separated graph. Given any subset of vertices $H \subset E^0$, the *full subgraph* (E_H, C^H) of H is given by $E_H^0 := H$,

$$E_H^1 := \{e \in E^1 \mid r(e), s(e) \in H\}$$

with restricted range and source maps, and separation $C^H := \{X \cap E_H^1 \mid X \cap E_H^1 \neq \emptyset\}$. The set H is called *hereditary* if $r(e) \in H$ implies $s(e) \in H$ for any $e \in E^1$, and it is called *C-saturated* if $s(X) \subset H$ for $X \in C_v$ implies $v \in H$. The set of hereditary and C-saturated subsets $H \subset E^0$ is denoted $\mathcal{H}(E, C)$, and for all $H \in \mathcal{H}(E, C)$, we also have a *quotient graph* $(E/H, C/H)$; this is simply the full subgraph of $E^0 \setminus H$. ◀

Modding out the ideal generated by a hereditary and C-saturated subset, one obtains the tame graph C^*-algebra of the corresponding quotient graph.

Theorem 1.12 ([9, Theorem 5.5], [17, Theorem 2.19])**.** *Let (E, C) denote a finitely separated graph and consider any $H \in \mathcal{H}(E, C)$. The ideal $I(H)$ in $\mathcal{O}(E, C)$ generated by H is the ideal induced from the open and invariant subspace $U(H) := \theta_{\mathbb{F}}\big(\bigsqcup_{v \in H} \Omega(E, C)_v\big)$, and there is a canonical isomorphism $\mathcal{O}(E, C)/I(H) \cong \mathcal{O}(E/H, C/H)$.* ◀

Recall that in the non-separated setting, the ideal $I(H)$ corresponding to a hereditary and saturated set canonically contains $C^*(E_H)$ as a full corner. This fails in the separated setting for two reasons:

(1) The canonical map

$$p \colon \bigsqcup_{v \in H} \Omega(E, C)_v \to \Omega(E_H, C^H)$$

of [17, Remark 2.8] is usually not injective.

(2) There are usually admissible paths α with $r(\alpha), s(\alpha) \in H$, which are not entirely contained in (E_H, C^H).

In section 4, we will consider certain hereditary and C-saturated subsets H, where situation (2) does not occur. We will then show that nuclearity of $\mathcal{O}(E, C)$ can be inferred from nuclearity of $\mathcal{O}(E_H, C^H)$ and $\mathcal{O}(E/H, C/H)$ in spite of (1).

2. Topologically amenable partial actions

In this section, we define the notion of *topological amenability* for partial actions of discrete groups on locally compact Hausdorff spaces. Using the machinery of groupoids, we then check that our definition is equivalent to nuclearity of the crossed product C^*-algebras. The author would like to thank Claire Anantharaman-Delaroche for suggesting a groupoid approach to this problem.

We first present the necessary definitions from groupoid theory. The reader is referred to [18] for a comprehensive treatment of topological groupoids and their C^*-algebras.

Definition 2.1. A groupoid $\mathcal{G}$ is a set with a distinguished subset $\mathcal{G}^{(0)} \subset \mathcal{G}$, *range* and *source* maps $r, s\colon \mathcal{G} \to \mathcal{G}^{(0)}$ and a partial composition

$$\{(\alpha,\beta) \in \mathcal{G} \times \mathcal{G} \mid s(\alpha) = r(\beta)\} = \mathcal{G}^{(2)} \ni (\alpha,\beta) \mapsto \alpha \cdot \beta,$$

such that

(1) $r(\alpha\cdot\beta) = r(\alpha)$ and $s(\alpha\cdot\beta) = s(\beta)$ for $(\alpha,\beta) \in \mathcal{G}^{(2)}$,
(2) $r(\alpha) = \alpha = s(\alpha)$ for all $\alpha \in \mathcal{G}^{(0)}$,
(3) $r(\alpha)\cdot\alpha = \alpha = \alpha\cdot s(\alpha)$ for all $\alpha \in \mathcal{G}$,
(4) $(\alpha\cdot\beta)\cdot\gamma = \alpha\cdot(\beta\cdot\gamma)$ for $(\alpha,\beta),(\beta,\gamma) \in \mathcal{G}^{(2)}$,
(5) for all $\alpha \in \mathcal{G}$, there is $\alpha^{-1} \in \mathcal{G}$ such that

$$\alpha\cdot\alpha^{-1} = r(\alpha) \quad \text{and} \quad \alpha^{-1}\cdot\alpha = s(\alpha).$$

We shall use the fiber notation $\mathcal{G}^x := r^{-1}(x)$ for $x \in \mathcal{G}^{(0)}$. ◀

Alternatively, a groupoid can be defined as a small category in which all morphisms are invertible: Letting $\mathcal{G}$ denote the collection of morphisms and identifying the objects $\mathcal{G}^{(0)}$ of the category with the collection of identity morphisms, we have natural range and source maps as above, an associative partial composition and inverses. However, for our purposes it is easier to stress all the axioms explicitly as we shall immediately impose extra structure.

Definition 2.2. A *topological groupoid* $\mathcal{G}$ is a groupoid equipped with a topology such that all the operations (i.e. range, source, composition and inversion) are continuous when $\mathcal{G}^{(0)} \subset \mathcal{G}$ is given the subspace topology. Moreover, $\mathcal{G}$ is called an étale groupoid, if r and s are local homeomorphisms. ◀

The reason for passing to groupoids is that we can encode a partial action into a groupoid.

Example 2.3 (Transformation groupoid)**.** Consider a partial topological action $\theta\colon G \curvearrowright \Omega$ of a discrete group G on a locally compact Hausdorff space Ω. To any such partial action, we can associate a groupoid $\mathcal{G}_\theta$ as follows: Set

$$\mathcal{G}_\theta := \{(g,x) \in G \times \Omega \mid x \in \Omega_{g^{-1}}\} \quad , \quad \mathcal{G}_\theta^{(0)} := \{1\} \times \Omega,$$

and define range and source maps $r, s\colon \mathcal{G}_\theta \to \mathcal{G}_\theta^{(0)}$ by

$$r(g,x) := (1, \theta_g(x)) \quad \text{and} \quad s(g,x) := (1,x).$$

Then $\alpha = (g,x) \in \mathcal{G}_\theta$ and $\beta = (h,y) \in \mathcal{G}_\theta$ are composable if and only if $x = \theta_h(y)$, in which case we set

$$\alpha\cdot\beta = (g,x)\cdot(h,y) := (gh,y).$$

We define an inversion by $(g,x)^{-1} = (g^{-1}, \theta_g(x))$, and giving $\mathcal{G}_\theta$ the subspace topology of $G \times \Omega$, it is easily checked that it becomes an étale locally compact Hausdorff groupoid, called the *transformation groupoid* of θ. In the future we shall always identify $\mathcal{G}^{(0)} = \{1\} \times \Omega$ and Ω. ◀

In the following, whenever μ is a measure supported on $\mathcal{G}^{s(\alpha)}$, $\alpha\cdot\mu$ is the measure supported on $\mathcal{G}^{r(\alpha)}$ defined by $\alpha\cdot\mu(A) := \mu(\alpha^{-1}\cdot A)$.

Definition 2.4 ([19, Definition 2.6]). Let $\mathcal{G}$ denote a locally compact Hausdorff groupoid. A *topological approximate invariant mean* on $\mathcal{G}$ is a net $(m_i)_{i\in I}$, where each m_i is a family $(m_i^x)_{x\in\mathcal{G}^{(0)}}$, m_i^x being a probability measure on $\mathcal{G}^x = r^{-1}(x)$, such that

(1) for all $i \in I$ and $f \in C_c(\mathcal{G})$, the map $\mathcal{G}^{(0)} \ni x \mapsto \int_{\mathcal{G}^x} f \, dm_i^x$ is continuous,

(2) $\sup_{\alpha\in\mathcal{K}} \|\alpha \cdot m_i^{s(\alpha)} - m_i^{r(\alpha)}\| \to 0$ for all compact subsets $\mathcal{K} \subset \mathcal{G}$,

where the norm expression of (2) denotes the total variation of the measures. The groupoid $\mathcal{G}$ is said to be *topologically amenable* if it admits a topological approximate invariant mean.

Remark 2.5. Topological amenability has a number of different definitions in the literature, one being the existence of a *topological invariant density* (see [19, Definition 2.7]). This is also the approach in [12, Section 5.6], but by [3, Proposition 2.2.13] these two definitions are equivalent in the presence of a *continuous Haar system*, which $\mathcal{G}_\theta$ always possesses [2, Proposition 2.2]. ◀

We now specialise to partial actions of a discrete group G on a locally compact Hausdorff space. To this end, define

$$\mathrm{Prob}(G) := \{\mu \in \ell^1(G) \mid \mu \geq 0 \text{ and } \|\mu\|_1 = 1\}.$$

Then G acts on $\mathrm{Prob}(G)$ by $g.\mu(h) := \mu(hg)$.

Definition 2.6. Consider a partial action $\theta\colon G \curvearrowright \Omega$ of a discrete group G on a locally compact Hausdorff space Ω. A *topological approximate invariant mean* for θ is a net $(m_i)_{i\in I}$, where each m_i is a family $(m_i^x)_{x\in\Omega}$ with $m_i^x \in \mathrm{Prob}(G)$ supported on $G^x = \{g \in G \mid x \in \Omega_{g^{-1}}\}$, such that

(1) for all $i \in I$ and $g \in G$, the map $\Omega_{g^{-1}} \ni x \mapsto m_i^x(g)$ is continuous,

(2) $\sup_{x\in K} \|g.m_i^x - m_i^{\theta_g(x)}\|_1 \to 0$ for any $g \in G$ and all compact subsets $K \subset \Omega_{g^{-1}}$,

where the norm expression in (2) is the usual norm on $\ell^1(G)$. The partial action is said to be *topologically amenable* if it admits a topological approximate invariant mean. ◀

Note that for global actions, the above definition of topological amenability is equivalent to the classical one, see for instance [12, Definition 4.3.5]. In order to verify that it is the appropriate generalisation to partial actions, we simply check that it is indeed a special case of Definition 2.4.

Proposition 2.7. *A topological partial action $\theta\colon G \curvearrowright \Omega$ of a discrete group on a locally compact Hausdorff space is topologically amenable in the sense of Definition 2.6 if and only if the transformation groupoid $\mathcal{G}_\theta$ is topologically amenable in the sense of Definition 2.4.*

Proof. Given a topological approximate invariant mean $(m_i)_{i\in I}$ on $\mathcal{G}_\theta$, define $\mu_i^x \in \mathrm{Prob}(G)$ by $\mu_i^x(g) := m_i^x(g^{-1}, \theta_g(x))$ for all $g \in G^x$, and set $\mu_i^x(g) := 0$ for $g \notin G^x$. By assumption, m_i^x is supported on the discrete set

$$\mathcal{G}_\theta^x = r^{-1}(x) = \{(g^{-1}, \theta_g(x)) \in G \times \Omega \mid g \in G^x\},$$

which is mapped bijectively to G^x under the map $(g^{-1}, \theta_g(x)) \mapsto g$, so μ_i^x is well-defined. To check (1) for some given $i \in I$ and $g \in G$, fix $x_0 \in \Omega_{g^{-1}}$. Then we may take a neighbourhood

$x_0 \in U \subset \Omega_{g^{-1}}$ and a compactly supported continuous function $f \in C_c(\{g^{-1}\} \times \Omega_g) \subset C_c(\mathcal{G}_\theta)$ satisfying $0 \notin f(\{g^{-1}\} \times \theta_g(U))$. Now by assumption, the function

$$x \mapsto \int_{\mathcal{G}_\theta^x} f dm_i^x = \sum_{h \in G^x} f(h^{-1}, \theta_h(x)) \cdot m_i^x(h^{-1}, \theta_h(x)) = f(g^{-1}, \theta_g(x)) \cdot \mu_i^x(g)$$

is continuous, hence so is $U \ni x \mapsto \mu_i^x(g)$. For (2), observe first that

$$g.\mu_i^x(h) = \mu_i^x(hg) = m_i^x(g^{-1}h^{-1}, \theta_{hg}(x)) = (g, x) \cdot m_i^x\big(h^{-1}, \theta_h(\theta_g(x))\big)$$

for all $g \in G$, $x \in \Omega_{g^{-1}}$ and $h \in G^{\theta_g(x)} = G^x \cdot g^{-1}$. Given any compact set $K \subset \Omega_{g^{-1}}$, we therefore have

$$\sup_{x \in K} \left\| g.\mu_i^x - \mu_i^{\theta_g(x)} \right\|_1 = 2 \cdot \sup_{x \in K} \left\| g.\mu_i^x - \mu_i^{\theta_g(x)} \right\| = 2 \cdot \sup_{\gamma \in \{g\} \times K} \left\| \gamma \cdot m_i^{s(\gamma)} - m_i^{r(\gamma)} \right\| \to 0$$

as desired. We conclude that $(\mu_i)_{i \in I}$ is indeed a topological approximate invariant mean for θ. The reverse implication is almost identical: Given a topological approximate invariant mean $(\mu_i)_{i \in I}$ for θ, we set

$$m_i^x(g, y) := \begin{cases} \mu_i^x(g^{-1}) & \text{if } x = \theta_g(y) \\ 0 & \text{otherwise} \end{cases}$$

and observe that m_i^x is indeed a probability measure supported on $\mathcal{G}_\theta^x$. Let $f \in C_c(\mathcal{G}_\theta)$ and observe that $F := \big\{ g \in G \mid f(\{g^{-1}\} \times \Omega_g) \neq \{0\} \big\}$ is a finite subset; we then have

$$\int_{\mathcal{G}_\theta^x} f dm_i^x = \sum_{g \in F \cap G^x} f(g^{-1}, \theta_g(x)) \mu_i^x(g)$$

for all $x \in \Omega = \mathcal{G}_\theta^{(0)}$. While every summand $x \mapsto f(g^{-1}, \theta_g(x))\mu_i^x(g)$ is continuous on $\Omega_{g^{-1}}$, the set $F \cap G^x$ need not vary continuously. To get around this, fix some $x_0 \in \Omega$ and set $F_{x_0} := F \cap G^{x_0}$; we may of course assume that $F_{x_0} \subsetneq F$. From f being compactly supported, there exist open neighbourhoods U_g of x_0 for all $g \in F \setminus F_{x_0}$ such that either

- $U_g \cap \Omega_{g^{-1}} = \emptyset$, or
- $U_g \cap \Omega_{g^{-1}} \neq \emptyset$ and $f\big(\{g^{-1}\} \times \theta_g(U_g \cap \Omega_{g^{-1}})\big) = \{0\}$.

Defining an open neighbourhood of x_0 by $U := \bigcap_{g \in F \setminus F_{x_0}} U_g$, we then have

$$\int_{\mathcal{G}_\theta^x} f dm_i^x = \sum_{g \in F_{x_0}} f(g^{-1}, \theta_g(x)) \mu_i^x(g)$$

for all $x \in U$. Since $x \mapsto \mu_i^x(g^{-1})$ is continuous in x_0 for every $g \in F_{x_0}$, we conclude that $x \mapsto \int_{\mathcal{G}_\theta^x} f dm_i^x$ is indeed continuous. (2) follows just as above by noting that every compact set $\mathcal{K} \subset \mathcal{G}_\theta$ is of the form $\bigsqcup_{i=1}^n \{g_i\} \times K_i$ for group elements $g_1, \ldots, g_n$ and compact subsets $K_i \subset \Omega_{g_i^{-1}}$. □

The real reason for our digression to groupoids is that one can associate a *full* and a *reduced* groupoid C^*-algebra $C^*(\mathcal{G})$ and $C^*_r(\mathcal{G})$ to any étale locally compact Hausdorff groupoid, and in the case $\mathcal{G} = \mathcal{G}_\theta$, there are canonical isomorphisms

$$C^*(\mathcal{G}_\theta) \cong C_0(\Omega) \rtimes_\theta G \quad \text{and} \quad C^*_r(\mathcal{G}_\theta) \cong C_0(\Omega) \rtimes_{\theta,r} G,$$

proven in [2, Theorem 3.3 (2)] and [16, Proposition 2.2]. We thus obtain the following for free.

Theorem 2.8. *Consider a partial action $\theta\colon G \curvearrowright \Omega$ of a discrete group on a locally compact Hausdorff space. Then $C_0(\Omega) \rtimes_r G$ is nuclear if and only if θ is topologically amenable in the sense of Definition 2.6, and in that case, the regular representation*

$$C_0(\Omega) \rtimes G \to C_0(\Omega) \rtimes_r G$$

is an isomorphism. In particular, $C_0(\Omega) \rtimes G$ is nuclear if and only if $C_0(\Omega) \rtimes_r G$ is nuclear.

Proof. In view of Remark 2.5 and Proposition 2.7, the first part follows immediately from [12, Theorem 5.6.18 and Corollary 5.6.17]. Finally, if the full crossed product is nuclear, then so is the reduced, since nuclearity passes to quotients. □

In the setting of global actions, topological amenability can always be pulled back by continuous equivariant maps. In the partial setting, however, one has to be a little more careful.

Definition 2.9 ([14, Definition 2.2])**.** Suppose that G acts partially on Ω and Ω' and that $f\colon \Omega \to \Omega'$ is equivariant, so that $G^x \subset G^{f(x)}$ for all $x \in \Omega$. Then f is called *d-bijective* if $G^{f(x)} = G^x$ for all $x \in \Omega$.

Proposition 2.10 ([14, Proposition 2.4])**.** *Assume that $\theta\colon G \curvearrowright \Omega$ and $\theta'\colon G \curvearrowright \Omega'$ are partial actions of a discrete group on locally compact Hausdorff spaces, and that $f\colon \Omega \to \Omega'$ is a continuous, equivariant and d-bijective map. If θ' is topologically amenable, then so is θ.*

Proof. Let $(m_i)_{i\in I}$ denote a topological approximate invariant mean for θ' and define $\mu_i^x := m_i^{f(x)}$ for all $x \in \Omega$ and $i \in I$. First observe that each μ_i^x is a probability measure on G^x since f is d-bijective, and that $\Omega_{g^{-1}} \ni x \mapsto \mu_i^x(g)$ is continuous, being the composition of f and $\Omega'_{g^{-1}} \ni y \mapsto m_i^y(g)$. Finally, if $K \subset \Omega_{g^{-1}}$ is compact, then

$$\sup_{x\in K} \|g.\mu_i^x - \mu_i^{\theta_g(x)}\|_1 = \sup_{x\in K} \|g.m_i^{f(x)} - m_i^{f(\theta_g(x))}\|_1 = \sup_{y\in f(K)} \|g.m_i^y - m_i^{\theta'_g(y)}\|_1 \to 0,$$

hence θ is indeed topologically amenable. □

One particular simple situation giving rise to a topological approximate invariant mean is the existence of a *topological Følner net* for the action.

Definition 2.11. Let $\theta\colon G \curvearrowright \Omega$ denote a partial action of a discrete group on a locally compact Hausdorff space, and denote by $\mathcal{F}(G)$ the set of non-empty finite subsets of G endowed with the discrete topology. A *topological Følner net for θ* is a net $(F_i)_{i\in I}$ of continuous (i.e. locally constant) functions $F_i\colon \Omega \to \mathcal{F}(G)$, $x \mapsto F_i^x$, such that $F_i^x \subset G^x$ for all $x \in \Omega$, and for every $g \in G$ and all compact subsets $K \subset \Omega_{g^{-1}}$,

$$\sup_{x\in K} \frac{|F_i^x \cdot g^{-1} \setminus F_i^{\theta_g(x)}|}{|F_i^x|} \to 0.$$

◀

Here is the observation that justifies Definition 2.11.

Proposition 2.12. *Let $\theta\colon G \curvearrowright \Omega$ denote a partial action of a discrete group on a locally compact Hausdorff space. If θ has a topological Følner net, then it is topologically amenable.*

Proof. Assume that $(F_i)_{i\in I}$ is a topological Følner net for θ, and define

$$m_i\colon \Omega \to \mathrm{Prob}(G) \quad \text{by} \quad m_i^x := \frac{1}{|F_i^x|} \cdot 1_{F_i^x},$$

where 1_F is the characteristic function on a set F. Then each m_i^x is certainly a probability measure with support $F_i^x \subset G^x$, satisfying (1) of Definition 2.6. In order to check (2), let $g \in G$ and compact $K \subset \Omega_{g^{-1}}$ be given. We then have

$$\begin{aligned}
\|g.m_i^x - m_i^{\theta_g(x)}\|_1 &= \frac{|F_i^x \cdot g^{-1} \setminus F_i^{\theta_g(x)}|}{|F_i^x|} + \frac{|F_i^{\theta_g(x)} \setminus F_i^x \cdot g^{-1}|}{|F_i^{\theta_g(x)}|} \\
&\quad + |F_i^x \cdot g^{-1} \cap F_i^{\theta_g(x)}| \cdot \Big|\frac{1}{|F_i^x \cdot g^{-1}|} - \frac{1}{|F_i^{\theta_g(x)}|}\Big| \\
&= \frac{|F_i^x \cdot g^{-1} \setminus F_i^{\theta_g(x)}|}{|F_i^x|} + \frac{|F_i^{\theta_g(x)} \cdot g \setminus F_i^{\theta_{g^{-1}}(\theta_g(x))}|}{|F_i^{\theta_g(x)}|} \\
&\quad + |F_i^x \cdot g^{-1} \cap F_i^{\theta_g(x)}| \cdot \frac{\big||F_i^{\theta_g(x)}| - |F_i^x \cdot g^{-1}|\big|}{|F_i^x| \cdot |F_i^{\theta_g(x)}|} \\
&\le 2 \cdot \frac{|F_i^x \cdot g^{-1} \setminus F_i^{\theta_g(x)}|}{|F_i^x|} + 2 \cdot \frac{|F_i^{\theta_g(x)} \cdot g \setminus F_i^{\theta_{g^{-1}}(\theta_g(x))}|}{|F_i^{\theta_g(x)}|}
\end{aligned}$$

for all $x \in K$. Setting $K' := \theta_g(K) \subset \Omega_g$, we deduce that

$$\sup_{x\in K} \|g.m_i^x - m_i^{\theta_g(x)}\|_1 \le 2 \cdot \sup_{x\in K} \frac{|F_i^x \cdot g^{-1} \setminus F_i^{\theta_g(x)}|}{|F_i^x|} + 2 \cdot \sup_{y \in K'} \frac{|F_i^y \cdot g \setminus F_i^{\theta_{g^{-1}}(y)}|}{|F_i^y|} \to 0.$$

□

Given a partial action $\theta\colon G \curvearrowright \Omega$ with certain properties, one might desire a global action with the same properties. In terms of purely dynamical properties, this can always be accomplished by considering the *minimal globalisation* $\widetilde{\theta}\colon G \curvearrowright \widetilde{\Omega}$ of [1, Theorem 2.5]. While the space $\widetilde{\Omega}$ resembles Ω locally, it might have very different global properties. The Hausdorff property might not even pass from Ω to $\widetilde{\Omega}$ [1, Example 2.9], but if the domains are clopen, such pathological examples do not exist [1, Proposition 2.10]. However, compactness will *usually* not be preserved, and so it is natural to ask if there exist "good" compactifications of $\widetilde{\theta}$. One particular case of interest is that of a topologically amenable partial action $\theta\colon G \curvearrowright \Omega$ on a compact Hausdorff space with clopen domains. By the above, the minimal globalisation is topologically amenable and acts on a locally compact Hausdorff space. A good compactification of $\widetilde{\theta}$ in this context would be a topologically amenable one, and so the one-point compactification is not desirable if G is non-amenable. Below, we will provide a good compactification in this setup for *right-convex* partial actions of a free group.

Definition 2.13. A partial action $\mathbb{F} \curvearrowright \Omega$ of a free group is called *convex* if

$$\mathbb{F}^x = \{\alpha \in \mathbb{F} \mid x \in \Omega_{\alpha^{-1}}\}$$

is a right-convex subset of $\mathbb{F}$ for all $x \in \Omega$. ◀

We will need the following technical lemma.

Lemma 2.14. *Consider a continuous, surjective map of Hausdorff spaces $p\colon \Omega \to \Upsilon$, and assume that for any $y \in \Upsilon$, there exists an open neighbourhood U of y for which $\overline{p^{-1}(U)}$ is compact. If C is any compactification of Υ, then $\Omega \sqcup \partial\Upsilon$ is a Hausdorff compactification of Ω when equipped with the smallest topology making $\Omega \hookrightarrow \Omega \sqcup \partial\Upsilon$ open and $p \sqcup \mathrm{id}\colon \Omega \sqcup \partial\Upsilon \to C$ continuous.*

Proof. We first show that $\Omega \sqcup \partial\Upsilon$ is Hausdorff, so consider any pair of distinct points $x_1, x_2 \in \Omega \sqcup \partial\Upsilon$. Since Ω is open in $\Omega \sqcup \partial\Upsilon$, we may assume that at least one of the points belongs to $\partial\Upsilon$. Then $p \sqcup \mathrm{id}(x_1) \neq p \sqcup \mathrm{id}(x_2)$, so they can be separated by $\Upsilon \sqcup \partial\Upsilon$ being Hausdorff and continuity of $p \sqcup \mathrm{id}$. In order to demonstrate compactness, take any net $(x_i)_{i\in I}$ in $\Omega \sqcup \partial\Upsilon$ and consider the net $(y_i)_{i\in I}$ with $y_i = p \sqcup \mathrm{id}(x_i)$. By compactness of C, it has a convergent subnet $(y_i)_{i\in J}$ and we denote the limit point by y. If $y \in \Upsilon$, then there exist an open neighbourhood U of y for which $\overline{p^{-1}(U)}$ is compact, and since $\overline{p^{-1}(U)}$ contains a subnet of $(x_i)_{i\in I}$, it has a convergent subnet. Assume instead that $y \in \partial\Upsilon$. Then, since every open neighbourhood of y inside $\Omega \sqcup \partial\Upsilon$ is of the form $(p \sqcup \mathrm{id})^{-1}(U)$ for an open neighbourhood $y \in U \subset C$, we see that $(x_i)_{i\in J}$ converges towards y. □

Theorem 2.15. *Suppose that $\theta\colon \mathbb{F} \curvearrowright \Omega$ is a convex partial action of a free group of rank at least two on a compact Hausdorff space with clopen domains. Then there exists a Hausdorff compactification $\widehat{\Omega}$ of $\widetilde{\Omega}$ and an extension $\widehat{\theta}\colon \mathbb{F} \curvearrowright \widehat{\Omega}$ of $\widetilde{\theta}$, such that the restriction $\widehat{\theta}\colon \mathbb{F} \curvearrowright \widehat{\Omega}\backslash\widetilde{\Omega}$ is conjugate to the canonical boundary action $\mathbb{F} \curvearrowright \partial\mathbb{F}$. In particular, if θ is topologically amenable, then so is $\widehat{\theta}$.*

Proof. Recall that

$$\widetilde{\Omega} = \frac{\mathbb{F} \times \Omega}{\sim}, \quad \text{where } (\alpha, x) \sim (\beta, y) \Leftrightarrow x \in \Omega_{\alpha^{-1}\beta} \text{ and } y = \theta_{\beta^{-1}\alpha}(x).$$

The action is simply induced from the group, $\widetilde{\theta}_\beta([\alpha, x]) = [\beta\alpha, x]$, and Ω embeds as a clopen subspace by the map $\iota\colon \Omega \to \widetilde{\Omega}$, $\iota(x) = [1, x]$. Now since θ is assumed convex, to each pair

$(\alpha, x) \in \mathbb{F} \times \Omega$ there is unique minimal word $\sigma(\alpha, x) \in \mathbb{F}$ such that $\sigma(\alpha, x)^{-1} \cdot \alpha \in \mathbb{F}^x$, and it is straightforward to check that σ respects the relation $\sim$. We claim that σ is also continuous, so consider any pair (α, x). Assuming first that $\sigma(\alpha, x) = 1$, we note that $\{\alpha\} \times \Omega_{\alpha^{-1}}$ is an open neighbourhood of (α, x) on which σ attains the value 1. If $\sigma(\alpha, x) \neq 1$, we write α_x for the maximal subword (read from the right) of α satisfying $\alpha_x \in \mathbb{F}^x$ and denote the following letter by s. Then σ is constant on the open neighbourhood

$$\{\alpha\} \times \big(\Omega_{\alpha_x^{-1}} \setminus \Omega_{(s\alpha_x)^{-1}}\big),$$

where we invoke the assumption of clopen domains, hence it is indeed continuous. In conclusion, σ drops to a continuous map $\sigma\colon \widetilde{\Omega} \to \mathbb{F}$. Also observe that if $\beta \in \mathbb{F}$ does not contain $\sigma([\alpha, x])^{-1}$ as a subword (read from the right), then

$$\sigma(\widetilde{\theta}_\beta([\alpha, x])) = \sigma([\beta \cdot \alpha, x]) = \beta \cdot \sigma([\alpha, x]).$$

Now define $\widehat{\Omega} := \widetilde{\Omega} \cup \partial\mathbb{F}$ as a set and equip it with the smallest topology that makes the inclusion $\widetilde{\Omega} \hookrightarrow \widehat{\Omega}$ open and the map $\sigma \cup \mathrm{id}\colon \widehat{\Omega} \to \mathbb{F} \cup \partial\mathbb{F}$ continuous. By Lemma 2.14, this makes $\widehat{\Omega}$ into a compact Hausdorff space, and we extend the action by the ordinary action of $\mathbb{F}$ on its boundary. It follows immediately from the above observation that the action on $\widetilde{\Omega}$ is compatible with that on the boundary, and so we obtain a short exact sequence

$$0 \to C_0(\widetilde{\Omega}) \rtimes \mathbb{F} \to C(\widehat{\Omega}) \rtimes \mathbb{F} \to C(\partial\mathbb{F}) \rtimes \mathbb{F} \to 0.$$

Since both the ideal and the quotient are nuclear, we conclude that the extension is nuclear as well, hence $\widehat{\theta}$ is topologically amenable. □

3. Condition (N) and proper orientability of the branching subgraph

In this section, we introduce Condition (N) for finitely separated graphs and prove that it is equivalent to both exactness and nuclearity of the C^*-algebra associated to a certain subgraph. But first, we introduce quite a bit of terminology.

Definition 3.1. A non-trivial admissible path α in a separated graph (E, C) is said to *allow a return* if there is an admissible path β making $\beta\alpha$ a closed path, and a set $X \in C_v$ then allows a return if e^{-1} allows a return for some $e \in X$. A vertex $v \in E^0$ is called a *branching vertex* if

$$|\{e \in s^{-1}(v) \mid e \text{ allows a return}\}| + |\{X \in C_v \mid X \text{ allows a return}\}| \geq 3,$$

and a branching vertex v is said to *admit a local orientation* if one of the following holds:

(1) There exists $X_v \in C_v$ such that for every base-simple closed path α at v, either $\mathfrak{i}_d(\alpha) \in X_v^{-1}$ or $\mathfrak{t}_d(\alpha) \in X_v$.
(2) There is an edge $e_v \in s^{-1}(v)$ such that for every base-simple closed path α at v, either $\mathfrak{i}_d(\alpha) = e_v$ or $\mathfrak{t}_d(\alpha) = e_v^{-1}$.

Observe that if v is a branching vertex satisfying (1), then it does not satisfy (2) and X_v is unique. Likewise, if v is a branching vertex satisfying (2), then it does not satisfy (1) and

e_v is unique. We will therefore refer to the branching vertices admitting local orientations as either *type* (1) or (2).

Lemma 3.2. *If v is a branching vertex admitting a local orientation, then it satisfies either Definition 3.1(1) or Definition 3.1(2) for arbitrary closed paths α based at v.*

Proof. We verify the claim for Definition 3.1(1); the proof in the second case is exactly the same. If α is an arbitrary closed path, we can decompose into base-simple closed paths $\alpha = \alpha_n \cdots \alpha_1$. Assuming that the claim holds for all products of $n-1$ base-simple closed paths, we either have $\mathfrak{i}_d(\alpha) = \mathfrak{i}_d(\alpha_{n-1} \cdots \alpha_1) \in X_v^{-1}$, in which case we are done, or $\mathfrak{t}_d(\alpha_{n-1}) \in X_v$. In the latter case, we must have $\mathfrak{i}_d(\alpha_n) \notin X_v^{-1}$ so $\mathfrak{t}_d(\alpha) = \mathfrak{t}_d(\alpha_n) \in X_v$. □

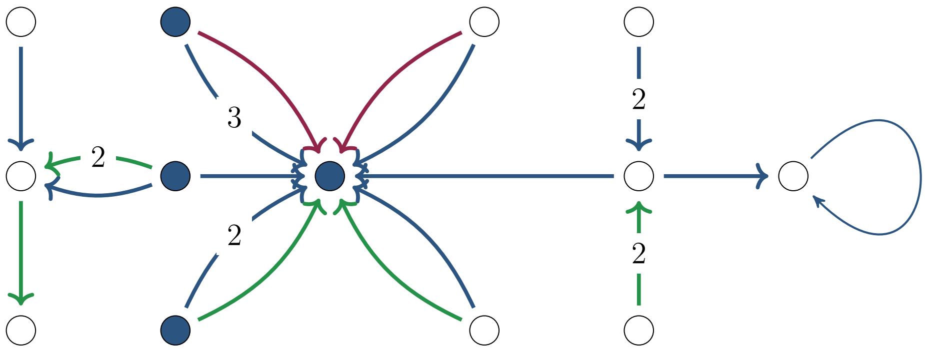
Remark 3.3. If v admits a local orientation, one should regard X_v^{-1} or e_v (depending on the type of v) as the proper exits of v, while the rest of $s^{-1}(v) \sqcup r^{-1}(v)^{-1}$ should be regarded as entries. Then any cycle based at v will depart from v using one and arrive at v using the other, so there is a canonical orientation of the cycle, i.e. a canonical choice between itself and its inverse. This is why we call it a local orientation. ◀

We now have the language to define Condition (N).

Definition 3.4. A finitely separated graph (E, C) is said to satisfy *Condition (N)* if any branching vertex $v \in E^0$ admits a local orientation. ◀

It is worth noting that Condition (N) trivially passes to subgraphs.

Example 1.2 (continued). Below, we have marked the branching vertices in blue:



The reader may check that this graph actually satisfies Condition (N) as the right branching vertex is of type (1), and the left ones are of type (2). ◀

Rather than simply negating the above definition, we would also like to have a constructive understanding of what it means for a graph not to satisfy Condition (N). For clarity, we first introduce the following technical lemma.

Lemma 3.5. *Let P and S denote sets with the following structure: There are functions $\iota, \tau\colon P \to S$, an associative partial composition*

$$\{(\alpha_2, \alpha_1) \in P^2 \mid \iota(\alpha_2) \neq \tau(\alpha_1)\} \to P \quad , \quad (\alpha_2, \alpha_1) \mapsto \alpha_2\alpha_1,$$

and a function $P \ni \alpha \mapsto \alpha^{-1} \in P$*, such that*

$$\iota(\alpha_2\alpha_1) = \iota(\alpha_1), \quad \tau(\alpha_2\alpha_1) = \tau(\alpha_2), \quad \iota(\alpha^{-1}) = \tau(\alpha), \quad \text{and} \quad \tau(\alpha^{-1}) = \iota(\alpha).$$

Moreover, assume that for all $\alpha_1, \alpha_2, \alpha_3 \in P$*,*

$$|\iota(\{\alpha_1, \alpha_2, \alpha_3\})| \leq 2 \quad \text{or} \quad |\tau(\{\alpha_1, \alpha_2, \alpha_3\})| \leq 2.$$

If $|\iota(P)| = |\tau(P)| \geq 3$*, then there is a unique* $s \in S$ *satisfying* $s \in \{\iota(\alpha), \tau(\alpha)\}$ *for all* $\alpha \in P$*.*

Proof. Take $\alpha_1, \alpha_2, \alpha_3 \in P$ with distinct $\iota(\alpha_i)$'s and define $\mathcal{E}(\alpha) := \{\iota(\alpha), \tau(\alpha)\}$ for all $\alpha \in P$. We first claim that $\mathcal{E}(\alpha_i) \cap \mathcal{E}(\alpha_j) \neq \emptyset$ for all i, j, and without loss of generality we may take $i = 1$ and $j = 2$. Assume in order to reach a contradiction that $\mathcal{E}(\alpha_1) \cap \mathcal{E}(\alpha_2) = \emptyset$. If $\iota(\alpha_1) \neq \tau(\alpha_1)$, then

$$|\iota(\{\alpha_1, \alpha_1^{-1}, \alpha_2\})| = |\tau(\{\alpha_1, \alpha_1^{-1}, \alpha_2\})| = 3,$$

hence $\iota(\alpha_1) = \tau(\alpha_1)$, and similarly we must have $\iota(\alpha_2) = \tau(\alpha_2)$. We deduce that either

$$\mathcal{E}(\alpha_1) \cap \mathcal{E}(\alpha_3) = \emptyset \quad \text{or} \quad \mathcal{E}(\alpha_2) \cap \mathcal{E}(\alpha_3) = \emptyset,$$

and without loss of generality we may assume the former. But then

$$|\iota(\{\alpha_1, \alpha_2, \alpha_3^{-1}\alpha_1\alpha_3\})| = |\tau(\{\alpha_1, \alpha_2, \alpha_3^{-1}\alpha_1\alpha_3\})| = 3,$$

giving us our desired contradiction. We now even claim that

$$\mathcal{E}(\alpha_1) \cap \mathcal{E}(\alpha_2) \cap \mathcal{E}(\alpha_3) \neq \emptyset.$$

Assuming the contrary, we can arrange that

$$\tau(\alpha_1) = \iota(\alpha_2), \quad \tau(\alpha_2) = \iota(\alpha_3), \quad \text{and} \quad \tau(\alpha_3) = \iota(\alpha_1)$$

by applying the first part and possibly interchanging the indices and taking inverses, hence

$$|\iota(\{\alpha_1, \alpha_2, \alpha_3\})| = |\tau(\{\alpha_1, \alpha_2, \alpha_3\})| = 3.$$

We conclude that

$$\mathcal{E}(\alpha_1) \cap \mathcal{E}(\alpha_2) \cap \mathcal{E}(\alpha_3) = \{s\}$$

for some $s \in S$. This implies that there are distinct $s_1, s_2 \neq s$ and $i \neq j$ such that $\mathcal{E}(\alpha_i) = \{s, s_1\}$ and $\mathcal{E}(\alpha_j) = \{s, s_2\}$. Now if $\alpha \in P$ is arbitrary, then by taking suitable inverses $\beta = \alpha^{\varepsilon}$, $\beta_i = \alpha_i^{\varepsilon_i}$, and $\beta_j = \alpha_j^{\varepsilon_j}$ (i.e. $\varepsilon, \varepsilon_i, \varepsilon_j \in \{-1, 1\}$), we can arrange that

$$|\iota(\{\beta, \beta_i, \beta_j\})| = 3.$$

This allows us to apply the above conclusions, hence

$$\emptyset \neq \mathcal{E}(\beta) \cap \mathcal{E}(\beta_i) \cap \mathcal{E}(\beta_j) \subset \mathcal{E}(\alpha_i) \cap \mathcal{E}(\alpha_j) = \{s\}.$$

We deduce that $\mathcal{E}(\beta) \cap \mathcal{E}(\beta_i) \cap \mathcal{E}(\beta_j) = \{s\}$, so in particular $s \in \mathcal{E}(\beta) = \mathcal{E}(\alpha)$. Uniqueness of s is clear from $\mathcal{E}(\alpha_i) \cap \mathcal{E}(\alpha_j) = \{s\}$. □

Proposition 3.6. *A finitely separated graph* (E, C) *does not satisfy Condition* (N) *if and only if there is a branching vertex* $v \in E^0$ *and cycles* $\alpha = \delta\gamma$ *and* $\beta = \varepsilon\gamma$ *based at* v *with* $\gamma = \alpha \wedge \beta < \alpha, \beta$*, such that* $\beta \cdot \alpha^{-1} = \varepsilon\delta^{-1}$ *and* $\beta\alpha$ *are cycles.*

Proof. It is clear that if such α and β exist, then v does not admit a local orientation. Now let v denote any branching vertex in (E, C) and assume instead that such α and β do not exist. Letting π denote the map $s^{-1}(v) \sqcup r^{-1}(v)^{-1} \to s^{-1}(v) \sqcup C_v$ given by $\pi(e) := e$ and $\pi(e^{-1}) := [e]$, we define sets $P := \{\text{closed paths based at } v\}$ and $S := s^{-1}(v) \sqcup C_v$ along with maps $\iota, \tau \colon P \to S$ given by $\iota(\alpha) := \pi(\mathfrak{i}_d(\alpha))$ and $\tau(\alpha) := \pi(\mathfrak{t}_d(\alpha)^{-1})$. Obviously, given two closed paths $\alpha_1, \alpha_2 \in P$, the concatenated product $\alpha_2\alpha_1$ is in P if and only if $\iota(\alpha_2) \neq \tau(\alpha_1)$. In fact, the only assumption of Lemma 3.5 which is not obviously satisfied is that either $|\iota(\{\alpha_1, \alpha_2, \alpha_3\})| \leq 2$ or $|\tau(\{\alpha_1, \alpha_2, \alpha_3\})| \leq 2$ for all triples $\alpha_1, \alpha_2, \alpha_3 \in P$. Assume in order to reach a contradiction that

$$|\iota(\{\alpha_1, \alpha_2, \alpha_3\})| = |\tau(\{\alpha_1, \alpha_2, \alpha_3\})| = 3$$

for one such triple. Then $\alpha := \alpha_2^{-1}\alpha_1$ and $\beta := \alpha_3^{-1}\alpha_1$ are cycles with $\alpha \wedge \beta = \alpha_1 < \alpha, \beta$ such that both $\beta \cdot \alpha^{-1} = \alpha_3^{-1}\alpha_2$ and $\beta\alpha = \alpha_3^{-1}\alpha_1\alpha_2^{-1}\alpha_1$ are cycles as well, contradicting our assumption. It now follows immediately from Lemma 3.5 that v admits a local orientation. $\square$

With the above characterisation at hand, we can already prove that Condition (N) is a necessary condition for exactness of $\mathcal{O}(E, C)$.

Proposition 3.7. *Let (E, C) denote a finitely separated graph and consider the statements*

1. *The C^*-algebra $\mathcal{O}(E, C)$ is exact.*
2. *Every stabiliser of the partial action $\theta^{(E,C)}$ is amenable (hence trivial or cyclic).*
3. *(E, C) satisfies Condition (N).*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. Assuming that (E, C) does not satisfy Condition (N), there are cycles α and β as in Proposition 3.6. Observe that any reduced product of α's, β's and their inverses is admissible, and denote by F the free subgroup of $\mathbb{F}$ generated by α and β. If F were of rank 1, then we would have $\alpha = \gamma^m$ and $\beta = \gamma^n$ for some cycle γ and non-zero integers m, n. As $\beta\alpha$ is admissible, we see that m and n have the same sign, contradicting $\alpha \wedge \beta < \alpha, \beta$. We conclude that $F \cong \mathbb{F}_2$. Now $\omega := \langle F \rangle$ defines an (E, C)-animal, and we can find a configuration $\xi \in \Omega(E, C)_\omega$ with $F \leq \text{Stab}(\xi)$. Formally, this construction can be carried out as follows: Take any $\eta \in \Omega(E, C)$ with $\{\alpha, \beta, \mathfrak{t}_d(\alpha)^{-1}, \mathfrak{t}_d(\beta)^{-1}\} \subset \eta$ and consider the animal

$$\chi := \{\gamma \in \eta \mid \gamma \not\geq \alpha, \beta, \mathfrak{t}_d(\alpha)^{-1}, \mathfrak{t}_d(\beta)^{-1}\}.$$

Then one may verify that $\xi := \bigsqcup_{\sigma \in F} \chi \cdot \sigma$ defines a configuration, and by construction $F \leq \text{Stab}(\xi)$, so (2) does not hold. It finally follows that $\mathcal{O}(E, C)$ is non-exact by [6, Proposition 7.1(i)]. $\square$

The aim of the rest of this paper is to prove that Condition (N) in fact implies nuclearity of $\mathcal{O}(E, C)$. Roughly speaking, the idea is to decompose the graph into one part 'spanned' by the branching vertices, and a complementary part containing no branching vertices, and then deal with these two subgraphs separately. In the remainder of this section, we will treat the former graph.

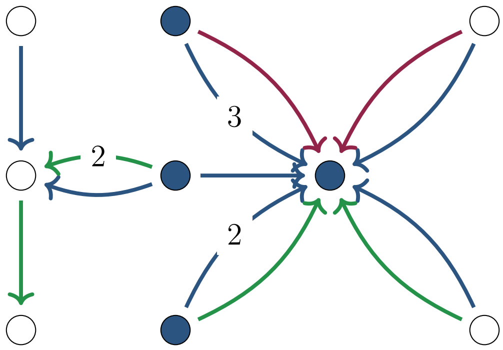
Definition 3.8. Given any finitely separated graph (E, C), define a relation on E^0 by $u \multimap v$ if there is an admissible path $\alpha\colon u \to v$ and a cycle β based at v, such that $\alpha^{-1}\beta\alpha$ is admissible. Note that $\multimap$ is transitive, but in general it is not reflexive, symmetric or antisymmetric. In fact, $u \multimap u$ if and only if u admits a cycle as we may take $\alpha = u$.

Definition 3.9. Let (E, C) denote a finitely separated graph. The *branching subgraph* $(E_{\mathrm{Br}}, C^{\mathrm{Br}})$ is the full subgraph with vertex set

$$E^0_{\mathrm{Br}} := \{u \in E^0 \mid u \multimap v \text{ for a branching vertex } v\}.$$

Remark 3.10. Note that if there is a closed path based at u passing through a branching vertex v, then automatically $u \multimap v$.

Example 1.2 (continued). The branching subgraph of our example is as indicated below:



◀

Next, we introduce the notion of a (*proper*) *orientation*.

Definition 3.11. Let (E, C) denote a finitely separated graph. A *proper orientation* of (E, C) is a decomposition $E^1 = E^1_+ \sqcup E^1_-$ such that, for every $v \in E^0$, one of the following holds:

(1) $E^1_- \cap r^{-1}(v) \in C_v$ and $E^1_+ \cap s^{-1}(v) = \emptyset$.
(2) $E^1_- \cap r^{-1}(v) = \emptyset$ and $|E^1_+ \cap s^{-1}(v)| = 1$.

If (2) is replaced by the weaker assumption

(2') $E^1_- \cap r^{-1}(v) = \emptyset$ and $|E^1_+ \cap s^{-1}(v)| \leq 1$,

then it will simply be called an *orientation*. We shall often regard an orientation as a map $\mathfrak{o}\colon E^1 \to \{-1, 1\}$, where

$$\mathfrak{o}(e) = \begin{cases} 1 & \text{if } e \in E^1_+ \\ -1 & \text{if } e \in E^1_- \end{cases},$$

and as in [9], an admissible path of the form

$$e_n^{\mathfrak{o}(e_n)} e_{n-1}^{\mathfrak{o}(e_{n-1})} \cdots e_2^{\mathfrak{o}(e_2)} e_1^{\mathfrak{o}(e_1)}$$

will be referred to as *positively oriented*, while an admissible path of the form

$$e_n^{-\mathfrak{o}(e_n)} e_{n-1}^{-\mathfrak{o}(e_{n-1})} \cdots e_2^{-\mathfrak{o}(e_2)} e_1^{-\mathfrak{o}(e_1)}$$

is called *negatively oriented.* ◀

We first mention a pair of trivial, yet important, observations.

Lemma 3.12. *Assume that (E, C) is oriented. Then every path α is of the form $\alpha = \alpha_-\alpha_+$, where α_+ and α_- are (possibly trivial) positively and negatively oriented paths, respectively.*

Proof. We simply have to check that if $f^\varepsilon e^{-\mathfrak{o}(e)}$ is admissible, then $\varepsilon = -\mathfrak{o}(f)$. Assume first that $\mathfrak{o}(e) = -1$; then $r(e)$ must satisfy Definition 3.11(1) since $E^1_- \cap r^{-1}(v) \neq \emptyset$. Now if $\varepsilon = 1$ so that $s(f) = r(e)$, then necessarily $\mathfrak{o}(f) = -1 = -\varepsilon$. Conversely if $\varepsilon = -1$, then $r(f) = r(e)$ with $[e] \neq [f]$, so $\mathfrak{o}(f) = 1 = -\varepsilon$. The case $\mathfrak{o}(e) = 1$ is completely similar. □

Lemma 3.13. *Assume that (E, C) is properly oriented and let $\xi \in \Omega(E, C)$. Then for every $n \geq 1$, there is a unique positively oriented admissible path $\xi_n \in \xi$ of length n.*

Proof. This is clear from Definition 3.11. □

We can now easily prove that properly oriented graphs give rise to topologically amenable actions.

Proposition 3.14. *If (E, C) is properly oriented, then the partial action $\theta^{(E,C)}$ admits a topological Følner sequence.*

Proof. Define $F^\xi_n := \{\xi_k \mid k \leq n\}$ for all $\xi \in \Omega(E, C)$ and $n \geq 1$, using the notation of Lemma 3.13. Given any $\alpha \in \mathbb{F}$ and $\xi \in \Omega(E, C)_{\alpha^{-1}}$, we write $\alpha = \alpha_-\alpha_+$ and, for the sake of notational simplicity, set $\eta := \theta_{\alpha_+}(\xi)$. Then α_-^{-1} is positively oriented, so

$$F^\xi_{|\alpha_+|+n} = \{\beta \mid 1 < \beta \leq \alpha_+\} \sqcup F^\eta_n \cdot \alpha_+ \quad \text{and} \quad F^{\theta_\alpha(\xi)}_{|\alpha_-|+n} = \{\beta \mid 1 < \beta \leq \alpha_-^{-1}\} \sqcup F^\eta_n \cdot \alpha_-^{-1}$$

for all $n \geq 1$. We see that

$$F^\xi_{|\alpha_+|+n} \cdot \alpha^{-1} = \{\beta \mid \alpha_-^{-1} \leq \beta < \alpha^{-1}\} \sqcup F^\eta_n \cdot \alpha_-^{-1}$$

hence

$$\frac{\left|F_{|\alpha|+n} \cdot \alpha^{-1} \setminus F^{\theta_\alpha(\xi)}_{|\alpha|+n}\right|}{\left|F^\xi_{|\alpha|+n}\right|} \leq \frac{|\alpha|}{|\alpha| + n}$$

with the upper bound converging to 0 as $n \to \infty$ uniformly on $\Omega(E, C)_{\alpha^{-1}}$. □

In view of the above result, our goal is to extend the local orientations of a Condition (N) graph into a proper orientation of the entire branching subgraph. This requires some preparation, which we provide just below. First though, we need a bit more terminology.

Definition 3.15. A vertex v is called *weakly branching* if there is a cycle passing through both v and a branching vertex, and an edge e is called *critical* if e does not allow a return and $r(e)$ is weakly branching.

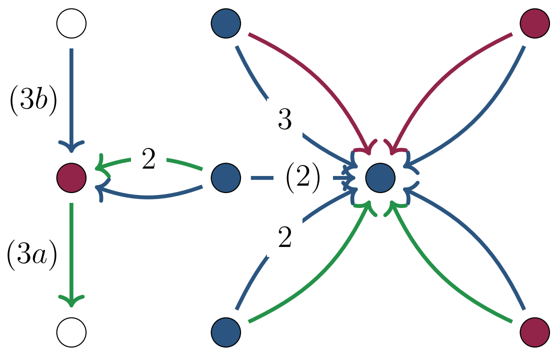
Lemma 3.16. *Let (E, C) denote a finitely separated graph. Any edge $e \in E^1_{\mathrm{Br}}$ satisfies exactly one of the following:*

(1) *e is on a cycle passing through a branching vertex.*

(2) *e is critical.*
(3a) *If α is a closed path in $(E_{\mathrm{Br}}, C^{\mathrm{Br}})$, based at $r(e)$ and passing through a branching vertex, then $\mathfrak{i}_d(\alpha) \in [e]^{-1}$.*
(3b) *If α is a closed path in $(E_{\mathrm{Br}}, C^{\mathrm{Br}})$, based at $s(e)$ and passing through a branching vertex, then $\mathfrak{i}_d(\alpha) = e$.*

Proof. It is clear that the above statements are mutually exclusive. In order to see that they cover all edges, assume that $e \in E^1_{\mathrm{Br}}$ does not satisfy (2), (3b) or (3a). Then we can take closed paths α_1, α_2 in $(E_{\mathrm{Br}}, C^{\mathrm{Br}})$ passing through branching vertices, such that $s(\alpha_1) = s(e)$, $s(\alpha_2) = r(e)$, $\mathfrak{i}_d(\alpha_1) \neq e$ and $\mathfrak{i}_d(\alpha_2) \notin [e]^{-1}$. We may assume that $\mathfrak{t}_d(\alpha_1) \neq e^{-1}$, so $e\alpha_1 e^{-1}$ is admissible as well as $\mathfrak{t}_d(\alpha_2) \neq e$. Observe that if $e^{-1}\alpha_2$ is admissible, then $\alpha_1\alpha_1 e^{-1}\alpha_2 e$ is a cycle passing through a branching vertex, so let us assume it is not. Then β_2 is a cycle, so $r(e)$ is weakly branching. It follows from e being non-critical that e admits a return, and we let β denote an admissible path for which βe is a closed path. Finally, if $\mathfrak{t}_d(\beta) \neq e^{-1}$, then $\beta\alpha_2 e$ is a cycle, and if $\mathfrak{t}_d(\beta) = e^{-1}$, then $\alpha_1\beta e$ is a cycle, so we obtain (1) either way. □

Example 1.2 (continued). Just below, we have marked the weakly branching (but non-branching) vertices as red and marked every non-type (1) edge with its type in parentheses:



◀

Now we are ready to build a proper orientation of the branching subgraph.

Theorem 3.17. *If (E, C) is a finitely separated graph satisfying Condition (N), then the branching subgraph $(E_{\mathrm{Br}}, C^{\mathrm{Br}})$ admits a proper orientation. In particular, the tame C^*-algebra $\mathcal{O}(E_{\mathrm{Br}}, C^{\mathrm{Br}})$ is nuclear.*

Proof. To simplify the notation, let us assume that any vertex $u \in E^0$ satisfies $u \multimap v$ for some branching vertex v, i.e. that (E, C) is its own branching subgraph. Now let v denote a branching vertex and let $e \in r^{-1}(v) \cup s^{-1}(v)$ be of type (1) as stated in Lemma 3.16. We then first define an orientation $\mathfrak{o}_v(e)$ of e relative to v by

$$\mathfrak{o}_v(e) := \begin{cases} -1 & \text{if } v \text{ is of type (1) and either } e \in X_v \text{ or } s(e) = v \\ 1 & \text{if } v \text{ is of type (1), } r(e) = v \text{ and } e \notin X_v \\ 1 & \text{if } v \text{ is of type (2) and either } e = e_v \text{ or } r(e) = v \\ -1 & \text{if } v \text{ is of type (2), } s(e) = v \text{ and } e \neq e_v \end{cases}$$

Note that if e is a loop, i.e. if $r(e) = s(e)$, then $r(e)$ is necessarily of type (1) and $e \in X_{r(e)}$, so the above is well-defined. Next, we define the orientation of arbitrary type (1) edges. Given any cycle $\alpha = e_n^{\varepsilon_n} \cdots e_1^{\varepsilon_1}$ based at v, observe that $\mathfrak{o}_v(e_1) = \varepsilon_1$ if and only if $\mathfrak{o}_v(e_n) = \varepsilon_n$ since $e_1^{\varepsilon_1} e_n^{\varepsilon_n}$ is admissible. We can therefore extend the orientation at v by declaring

$$\mathfrak{o}_\alpha(e_i) := \begin{cases} \varepsilon_i & \text{if } \mathfrak{o}_v(e_1) = \varepsilon_1 \\ -\varepsilon_i & \text{if } \mathfrak{o}_v(e_1) = -\varepsilon_1 \end{cases},$$

and consequently $\mathfrak{o}_\alpha(e_i) = \mathfrak{o}_{\alpha^{-1}}(e_i)$. We claim that the orientation is in fact independent of the cycle in question, but in order to see this, we need the following claim:

Claim: If $e_2^{\varepsilon_2} \alpha e_1^{\varepsilon_1} \colon v \to u$ is an admissible path between branching vertices which may be extended to a cycle, then $\mathfrak{o}_v(e_1) = \varepsilon_1$ if and only if $\mathfrak{o}_u(e_2) = \varepsilon_2$.

Proof of claim. Observe that in all situations, if $\mathfrak{o}_v(e_1) \neq \varepsilon_1$, then there is a closed path γ based at v making $e_2^{\varepsilon_2} \alpha e_1^{\varepsilon_1} \gamma e_1^{-\varepsilon_1} \alpha^{-1} e_2^{-\varepsilon_2}$ admissible. If u is type (1), this implies $e_2^{\varepsilon_2} \in X_u$ so $\varepsilon_2 = 1 \neq -1 = \mathfrak{o}_u(e_2)$, and if u is type (2), then $e_2^{-\varepsilon_2} = e_u$ so $\varepsilon_2 = -1 \neq 1 = \mathfrak{o}_u(e_2)$. This proves one implication. To obtain the other, consider any extension $f_2^{\delta_2} \beta f_1^{\delta_1} e_2^{\varepsilon_2} \alpha e_1^{\varepsilon_1}$ to a cycle. If $\varepsilon_1 = \mathfrak{o}_v(e_1)$, then $\delta_2 = \mathfrak{o}_v(f_2)$ as well by the above, so $\mathfrak{o}_u(f_1) = \delta_1$ by the first implication. Using the above observation once more, we finally arrive at $\mathfrak{o}_u(e_2) = \varepsilon_2$ as well. ☐

Now if $\alpha = \alpha_2 e \alpha_1$ and $\beta = \beta_2 e \beta_1$ are cycles based at branching vertices v and u, respectively, then $\alpha_2 e \beta_1 \beta_2 e \alpha_1$ is a cycle based at v passing through u, so evidently $\mathfrak{o}_\alpha(e) = \mathfrak{o}_\beta(e)$ by the above claim. We conclude that setting $\mathfrak{o}(e) := \mathfrak{o}_\alpha(e)$ for any choice of cycle α is well-defined. Finally, for edges of other types we simply set

$$\mathfrak{o}(e) := \begin{cases} -1 & \text{if } e \text{ is of type (2) or (3a)} \\ 1 & \text{if } e \text{ is of type (3b)} \end{cases},$$

so all that remains is to check the axioms.

First take $u \in E^0$ to be weakly branching; we then divide into the following two situations:

(i) There is some type (1) edge $e \in r^{-1}(u)$ with $\mathfrak{o}(e) = -1$.
(ii) There is some type (1) edge $e \in s^{-1}(u)$ with $\mathfrak{o}(e) = 1$.

As any weakly branching vertex admits a cycle passing through a branching vertex, and every such cycle is either positively or negatively oriented by construction, clearly one of these will always hold. Observe also that both cannot hold at u: If there were type (1) edges $e \in r^{-1}(u)$ and $f \in s^{-1}(u)$ with $\mathfrak{o}(e) = -1$ and $\mathfrak{o}(f) = 1$, then taking any positively oriented cycle α with $\mathfrak{i}_d(\alpha) = e^{-1}$, we would have $\mathfrak{t}_d(\alpha) \neq f^{-1}$. But then u is branching, and the assumption contradicts the definition of $\mathfrak{o}$ at u. Hence (i) and (ii) are mutually exclusive.

Assuming (i), we claim that u actually satisfies Definition 3.11(1). If there were some $f \in s^{-1}(v)$ with $\mathfrak{o}(f) = 1$, then by the above f would be of type (3b). But as $u = s(f)$ admits a cycle, this is surely not the case, hence $\mathfrak{o}^{-1}(1) \cap s^{-1}(u) = \emptyset$. Next, assume $[f] = [e]$; we must show that $\mathfrak{o}(f) = -1$. If f is of type (1), there is a cycle α with $\mathfrak{i}_d(\alpha) = f^{-1}$. Taking any positively oriented cycle β with $\mathfrak{i}_d(\beta) = e^{-1}$, we see that the cycle $\beta\alpha$ must be positively oriented, hence $\mathfrak{o}(f) = -1$. Now as the orientation of any type (2) or type (3a) edge is -1, it simply remains to check that f is not of type (3b). Assuming that it is, there exists a closed path β based at u making $f^{-1}\beta f$ admissible. Taking some negatively oriented admissible path $\alpha\colon v \to u$ with v branching and $\mathfrak{t}_d(\alpha) = e$, we then see that $\alpha^{-1}\beta\alpha$ is a closed path. However, this is impossible since α is negatively oriented. We conclude that $\mathfrak{o}(f) = -1$ as desired. Finally, we must show that $\mathfrak{o}(f) = 1$ if $r(f) = u$ and $[f] \neq [e]$. Supposing first that f is of type (1), we can take a cycle α with $\mathfrak{i}_d(\alpha) = f^{-1}$. Now if $\mathfrak{t}_d(\alpha) \in [e]$, then necessarily $\mathfrak{o}(f) = 1$, and if $\mathfrak{t}_d(\alpha) \notin [e]$, then u is branching, in which case the claim is clear. Next, we show that f cannot be of type (2). If u is branching, this is obvious, so let us assume it is not. Then we may take an admissible path $\alpha\colon u \to v$ with v branching and $\mathfrak{i}_d(\alpha) = e^{-1}$ as well as a closed path β based at v, such that $\alpha^{-1}\beta\alpha$ is admissible. Now observe that $f^{-1}\alpha^{-1}\beta\alpha f$ is a closed path, so f is not of type (2). It is also clear that f cannot be of type (3a), hence $\mathfrak{o}(f) = 1$. We conclude that $\mathfrak{o}^{-1}(-1) \cap r^{-1}(u) = [e]$, so $\mathfrak{o}$ does indeed satisfy Definition 3.11(1) at u. Having done the harder case in details, we leave it to the reader to check that if (ii) holds for some weakly branching vertex u, then Definition 3.11(2) is satisfied at u.

Finally, assume that $u \in E^0$ is not weakly branching, and let $\alpha\colon u \to v$ and $\beta\colon v \to v$ be implementing the relation $u \multimap v$ for some branching vertex v. We once more divide into two different scenarios:

(a) The initial symbol satisfies $\mathfrak{i}_d(\alpha) \in (E^1)^{-1}$.
(b) The initial symbol satisfies $\mathfrak{i}_d(\alpha) \in E^1$.

We only consider (a) and leave (b) to the reader. First observe that if $\alpha'\colon u \to v'$ and $\beta'\colon v' \to v'$ implement the relation $u \multimap v'$ for a branching vertex v', then necessarily $\mathfrak{i}_d(\alpha') \in (E^1)^{-1}$ and $[\mathfrak{i}_d(\alpha)^{-1}] = [\mathfrak{i}_d(\alpha')^{-1}]$; otherwise $\alpha'^{-1}\beta'\alpha'\alpha^{-1}\beta\alpha$ would be a cycle passing through a branching vertex. Now whenever $[e] = [\mathfrak{i}_d(\alpha)^{-1}]$, we claim that such α' and β' with $\mathfrak{i}_d(\alpha') = e^{-1}$ exist. Assuming this is not the case for one such edge e, there is an admissible path $\alpha' e\colon u \to v'$ and a cycle $\beta'\colon v' \to v'$ with v' branching, such that $e^{-1}\alpha'^{-1}\beta\alpha' e$ is admissible. But then α' and β' contradict the observation we have just made. We conclude that Definition 3.11(1) is indeed satisfied at u, thereby concluding the proof. □

4. The Branch Free Subgraph

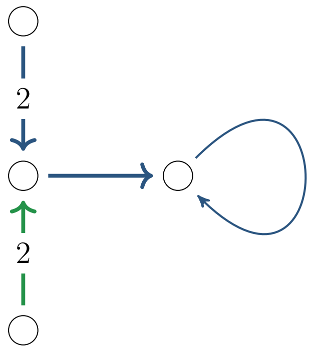
In this section, we study the subgraph obtained as the complement of the branching subgraph, and our ultimate goal is to prove that the associated C^*-algebra is nuclear.

Definition 4.1. Let (E, C) denote a finitely separated graph. The *branch free subgraph* $(E_{\mathrm{BF}}, C^{\mathrm{BF}})$ is the full subgraph with vertex set $E^0_{\mathrm{BF}} := E^0 \setminus E^0_{\mathrm{Br}}$, and the *acyclic* subgraph $(E_{\mathrm{Ac}}, C^{\mathrm{Ac}})$ is the full subgraph with vertex set

$$E^0_{\mathrm{Ac}} := E^0 \setminus \{u \in E^0 \mid u \multimap v \text{ for some } v\}$$

Note that by definition, $E^0_{\mathrm{Ac}} \subset E^0_{\mathrm{BF}}$.

Example 1.2 (continued). This is the branch free subgraph of our example:



Note that it cannot be given an orientation as one can 'change direction' using the doubled blue or green edges. ◀

Before investigating these two subgraphs more closely, we introduce a very useful notion in the study of nuclearity.

Definition 4.2. Let (E,C) denote a finitely separated graph. A hereditary and C-saturated subset $H \subset E^0$ will be called *return free* if every admissible path α with $r(\alpha), s(\alpha) \in H$ is entirely contained in (E_H, C^H).

Proposition 4.3. *Let (E,C) denote a finitely separated graph. If $H \in \mathcal{H}(E,C)$ is return free and both $\mathcal{O}(E_H, C^H)$ and $\mathcal{O}(E/H, C/H)$ are nuclear, then so is $\mathcal{O}(E,C)$.*

Proof. By Theorem 1.12, there is a short exact sequence

$$0 \to I(H) \to \mathcal{O}(E,C) \to \mathcal{O}(E/H, C/H) \to 0,$$

and since nuclearity passes to extensions, we simply have to verify that $I(H)$ is nuclear. Note that $\Omega(E,C)_H := \bigsqcup_{v\in H} \Omega(E,C)_v$ is an $\mathbb{F}$-full clopen subspace of $U(H)$, so $I(H)$ is Morita equivalent to the crossed product $C(\Omega(E,C)_H) \rtimes \mathbb{F}$ for the restricted action $\theta^{(E,C)}|_{\Omega(E,C)_H}$ (see [9, Definition 3.24] for the appropriate definitions and references). Since H is assumed return free, the domain of any path $\alpha \in \mathbb{F} \setminus \mathbb{F}(E^1_H)$ is empty, hence

$$C(\Omega(E,C)_H) \rtimes \mathbb{F} \cong C(\Omega(E,C)_H) \rtimes \mathbb{F}(E^1_H).$$

Now observe that as in [17, Remark 2.8], there is a canonical d-bijective, $\mathbb{F}(E^1_H)$-equivariant surjective map $\Omega(E,C)_H \to \Omega(E_H, C^H)$ given by

$$\xi \mapsto \begin{cases} \xi \cap \mathbb{F}(E^1_H) & \text{if } \xi \in \Omega(E,C)_v \text{ for } v \notin (E_H)^0_{\text{iso}} \\ v & \text{if } \xi \in \Omega(E,C)_v \text{ for } v \in (E_H)^0_{\text{iso}} \end{cases}.$$

Since the partial action $\theta^{(E_H,C^H)}$ is topologically amenable by assumption, so is the partial action $\mathbb{F}(E^1_H) \curvearrowright \Omega(E,C)_H$ by Proposition 2.10. It follows that $C(\Omega(E,C)_H) \rtimes \mathbb{F}$ and, in turn, $I(H)$ is nuclear. □

With the above proposition in mind, the following lemma simplifies our task tremendously.

Lemma 4.4. *E^0_{BF} and E^0_{Ac} are return free hereditary and C-saturated sets for any finitely separated graph (E,C).*

Proof. We only consider the case of E^0_{BF} as the proofs are virtually identical, and we first verify that it is hereditary. Assuming that $s(e) \in E^0_{\mathrm{Br}}$ for some $e \in E^1$, there is an admissible path $\alpha\colon s(e) \to v$ and a cycle β based at v such that $\alpha^{-1}\beta\alpha$ is admissible. Setting $\gamma := \alpha \cdot e^{-1}$, we see that $\gamma^{-1}\beta\gamma$ is admissible as well, hence $r(e) \in E^0_{\mathrm{Br}}$.

We move on to checking C-saturation, so let $u \in E^0_{\mathrm{Br}}$ and consider any $X \in C_u$. Take α and β as above implementing the relation $u \multimap v$ for some branching vertex v. If $\mathfrak{i}_d(\beta\alpha) \in X^{-1}$, then we certainly have $s(X) \cap E^0_{\mathrm{Br}} \neq \emptyset$, so we may assume that $\beta\alpha x$ is admissible for any $x \in X$. But then $(\alpha x)^{-1}\beta(\alpha x)$ implements the relation $s(x) \multimap v$, so in fact $s(X) \subset E^0_{\mathrm{Br}}$. We conclude that E^0_{BF} is indeed C-saturated.

Finally, we claim that E^0_{BF} is return free. Assume in order to reach a contradiction that there actually is an admissible path α with $r(\alpha), s(\alpha) \in E^0_{\mathrm{BF}}$, which is not contained in the branch free subgraph. Taking α to be minimal with these properties, we can write $\alpha = e_2^{-1}\beta e_1$ for $e_1, e_2 \in E^1 \setminus (E^1_{\mathrm{Br}} \cup E^1_{\mathrm{BF}})$ and β an admissible path in the branching subgraph. Then we can take admissible paths $\alpha_i\colon r(e_i) \to v_i$, with v_i a branching vertex, and cycles β_i based at v_i for $i = 1, 2$ making $\alpha_i^{-1}\beta_i\alpha_i$ admissible. As $s(e_i) \in E^0_{\mathrm{BF}}$, we must have $\mathfrak{i}_d(\alpha_i) \in [e_i]^{-1}$ for $i = 1, 2$. But then $(\alpha_2\beta e_1)^{-1}\beta_2(\alpha_2\beta e_1)$ is admissible, contradicting $v_1 \in E^0_{\mathrm{BF}}$. $\square$

We easily obtain nuclearity in the acyclic case.

Proposition 4.5. *The tame C^*-algebra $\mathcal{O}(E_{\mathrm{Ac}}, C^{\mathrm{Ac}})$ is locally AF for any finitely separated graph (E, C). In particular, it is nuclear.*

Proof. We claim that $\mathcal{O}(E, C)$ is locally AF whenever (E, C) admits no cycles, and by continuity of $\mathcal{O}$, we may assume (E, C) to be finite. Now simply observe that any admissible path has length at most $3 \cdot |E^0|$; otherwise it would contain a cycle. It follows that $\Omega(E, C)$ is a finite space and $\Omega(E, C)_\alpha \neq \emptyset$ for only finitely many $\alpha \in \mathbb{F}$, hence $\mathcal{O}(E, C) = C(\Omega(E, C)) \rtimes \mathbb{F}$ is finite dimensional. $\square$

Now we are ready to deal with the final obstacle before the main theorem.

Theorem 4.6. *$\mathcal{O}(E_{\mathrm{BF}}, C^{\mathrm{BF}})$ is nuclear for any finitely separated graph (E, C).*

Proof. We claim that $\mathcal{O}(E, C)$ is nuclear whenever (E, C) is a finitely separated graph without branching vertices, and by continuity of $\mathcal{O}$, we might as well assume (E, C) to be finite. Moreover, as $\mathcal{O}(E_{\mathrm{Ac}}, E^{\mathrm{Ac}})$ is nuclear by Proposition 4.5, we may further reduce to the quotient graph $(E/H, C/H)$ for $H := E^0_{\mathrm{Ac}}$ due to Proposition 4.3 and Lemma 4.4. In conclusion, we may take (E, C) to be a finite graph such that for all $u \in E^0$, $u \multimap v$ for some $v \in E^0$.

Let $V \subset E^0$ denote the set of vertices admitting a cycle and write $v_1 \sim v_2$ for $v_1, v_2 \in V$ if there is a cycle passing through both v_1 and v_2. This is clearly an equivalence relation, and we will prove the theorem by induction over $|V/{\sim}|$, the number of equivalence classes. For the induction start, assume $|V/{\sim}| = 1$. We claim that for any $\xi \in \Omega(E, C)$, viewing it as a tree, there is a bi-infinite linear subset $F^\xi \subset \xi$ with the following properties

(1) $F^{\theta_\alpha(\xi)} = F^\xi \cdot \alpha^{-1}$ whenever $\alpha \in \xi$,
(2) $\operatorname{dist}(F^\xi, 1) \leq 3|E^0|$ in the ordinary tree metric of ξ,

(3) $\xi \mapsto F_n^\xi := F^\xi \cap \xi^{n+3|E^0|}$ is locally constant for any n,

where $\xi^k := \{\beta \in \xi \colon |\beta| \leq k\}$. It should be clear from these properties that (F_n) will be a topological Følner sequence, hence $\mathcal{O}(E,C)$ will be nuclear by Theorem 2.8 and Proposition 2.12. Specifically, define

$$F^\xi := \{\alpha \in \xi \mid r(\alpha) \in V\},$$

where $r(1) := v$ for v such that $\xi \in \Omega(E,C)_v$ by convention. In order to see that F^ξ enjoys the above properties, we first need to verify the following minor claim.

Claim: Let $X \in C$. If for some $e \in X$ there exists an admissible path γ with $\mathfrak{i}_d(\gamma) = e^{-1}$ and $r(\gamma) \in V$, then any $f \in X$ has this property.

Proof of claim. Let $e \in X$ and γ be as above, and assume in order to reach a contradiction that some $f \in X$ does not have the above property. Then, since $s(f) \multimap v$ for some $v \in V$, there is an admissible path $\alpha f \colon s(f) \to v$ and a cycle $\beta \colon v \to v$ such that $\alpha^{-1}\beta\alpha$ is admissible. Now by $v \sim r(\gamma)$, the admissible path $\gamma\alpha^{-1}$ can be extended to a closed path based at v. But from $\alpha^{-1}\beta\alpha$ being admissible, we may then conclude that v is branching, a contradiction. □

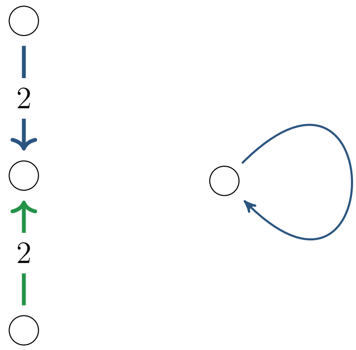
One immediate consequence of the claim (and the assumption that any u satisfies $u \multimap v$ for some v) is that any $\xi \in \Omega(E,C)$ is necessarily infinite. Now observe that if an admissible path passes a vertex three times (including the source and range), then that vertex must admit a cycle. In particular, there is some $\alpha \in F^\xi$ with $|\alpha| \leq 3|E^0|$. Another consequence of the claim is that for all $\alpha \in F^\xi$, at least two neighbours of α in ξ are contained in F^ξ. It therefore only remains to check that F^ξ is contained a linear subset. Assume in order to reach a contradiction that this is not the case for some $\xi \in \Omega(E,C)$. Then, by possibly replacing ξ with a translate, we can find non-trivial admissible paths $\alpha_1, \alpha_2, \alpha_3 \in \xi$ with $r(\alpha_i) \in V$ such that the concatenation $\alpha_i^{-1}\alpha_j$ is admissible for $i \neq j$. We claim that $u := s(\alpha_i) \in V$. If this were not the case, then we could take a non-trivial admissible path $\alpha \colon u \to v$ and a cycle β at v implementing the relation $u \multimap v$ for some $v \in V$. We may then apply the same argument as above in the proof of the claim to show that v is branching, a contradiction. Now since $u \in V$ and $u \sim r(\alpha_i)$ for $i = 1,2,3$, each α_i may be extended to a closed path. But then u is branching – we conclude that F^ξ is indeed a bi-infinite linear subset. It is obvious that it satisfies (1) and (3), and we have already seen that (2) holds as well, thereby concluding the proof of the induction start.

For the inductive step, let $|V/{\sim}| \geq 2$ and suppose that the claim holds whenever the number of equivalence classes is at most $|V/{\sim}| - 1$. We let the relation $\multimap$ descend to the equivalence classes $\mathfrak{u}, \mathfrak{v} \in V/{\sim}$ by

$$\mathfrak{u} \multimap \mathfrak{v} \Leftrightarrow u \multimap v \text{ for some } u \in \mathfrak{u} \text{ and } v \in \mathfrak{v},$$

and note that if $u \multimap v$, then $u' \multimap v$ for any $u' \sim u$. Now observe that $\multimap$ becomes antisymmetric on the set of equivalence classes, so there exist $\mathfrak{u}, \mathfrak{v} \in V/{\sim}$ with $\mathfrak{u} \not\multimap \mathfrak{v}$. Defining $H := \{u \in E^0 \mid u \not\multimap v \text{ for any } v \in \mathfrak{v}\}$, it is then straightforward to check that H is a return free hereditary and C-saturated set. Being return free, any cycle in (E,C) is properly contained in one of the graphs (E_H, C^H) and $(E/H, C/H)$, and since $\mathfrak{u} \subset H$ and $\mathfrak{v} \subset (E/H)^0$, the two graphs satisfy the inductive assumption. We conclude that $\mathcal{O}(E_H, C^H)$ and $\mathcal{O}(E/H, C/H)$ are both nuclear, hence so is $\mathcal{O}(E, C)$ by Proposition 4.3. This finishes the inductive step and, in turn, the proof. □

Example 1.2 (continued). Decomposing our branch free subgraph as in the above proof will leave us with the two subgraphs



each of which produces nuclear C^*-algebras.

Remark 4.7. In the above proof, we relied on an inductive argument to prove that for a finite graph without branching vertices (E, C), the quotient graph $(E/H, C/H)$ for $H := E^0_{\mathrm{Ac}}$ gives rise to a nuclear C^*-algebra. It is also possible to prove this claim directly by construction a very natural topological Følner sequence (F_n), where F_n^ξ is simply the n-ball $\xi^n := \{\alpha \in \xi \colon |\alpha| \leq 1\}$. However, we prefer the above proof due to the noteworthy technical difficulties that arise when verifying the Følner property of this sequence.

5. The main theorem and examples

In this final section, we give a short proof of the main theorem and finally consider a few examples of Condition (N) graphs and their graph monoids.

Note that quite a few of the implications in the main theorem follow from general theory. Indeed, the implications (3) ⇔ (4) ⇒ (2) are immediate from Theorem 2.8, which is an application of the groupoid C^*-algebra theory of Renault and Anantharaman-Delaroche, while (2) ⇒ (5) follows from Exel's theory of partial crossed products, specifically [13, Proposition 25.12], since free groups are exact [12, Proposition 5.1.8].

Theorem 5.1. *For any finitely separated graph (E, C), the following are equivalent:*

1. *(E, C) satisfies Condition (N).*
2. *The regular representation $\mathcal{O}(E, C) \to \mathcal{O}^r(E, C)$ is an isomorphism.*
3. *The C^*-algebra $\mathcal{O}^r(E, C)$ is nuclear.*
4. *The C^*-algebra $\mathcal{O}(E, C)$ is nuclear.*
5. *The C^*-algebra $\mathcal{O}(E, C)$ is exact.*
6. *Every stabiliser of the partial action $\theta^{(E,C)}$ is amenable (hence trivial or cyclic).*

Proof. We have already proven (5) $\Rightarrow$ (6) $\Rightarrow$ (1) in Proposition 3.7, so in light of the above comments, it remains only to verify (1) $\Rightarrow$ (4). But by Proposition 4.3 and Lemma 4.4, it suffices to show that $\mathcal{O}(E_{\mathrm{Br}}, C^{\mathrm{Br}})$ and $\mathcal{O}(E_{\mathrm{BF}}, C^{\mathrm{BF}})$ are nuclear, and this was done in Theorem 3.17 and Theorem 4.6, respectively. □

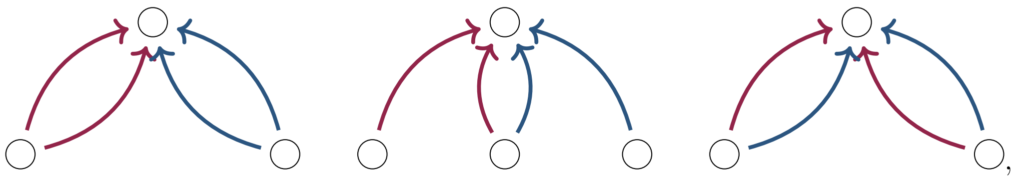
We obtain the following purely graph-theoretic consequence of the above theorem. Recall from [4] that if (E, C) is finite and bipartite, then one can associate a sequence (E_n, C^n) of finite bipartite graphs to $(E, C) = (E_0, C^0)$, such that $\mathcal{O}(E, C) \cong \varinjlim_n C^*(E_n, C^n)$ for certain connecting homomorphisms.

Corollary 5.2. *Let (E, C) denote a finite bipartite graph. Then (E_n, C^n) satisfies Condition (N) for some n if and only if it does so for all n.*

Proof. This is immediate from Theorem 5.1 since $\mathcal{O}(E_m, C^m) \cong \mathcal{O}(E_n, C^n)$ for all m, n. □

We finally consider some examples.

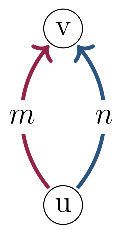
Example 5.3. If $|s^{-1}(v)| + |C_v| \leq 2$ for any $v \in E^0$, then (E, C) contains no branching vertices and therefore satisfies Condition (N) trivially. This covers examples such as



which were considered in [4, Example 9.5, 9.6, 9.7] as well as [9, Example 6.7].

Example 5.4. If E is any column-finite graph, then E clearly satisfies Condition (N) when regarded as a trivially separated graph. In fact, $\mathfrak{o}(e) := -1$ for all $e \in E^1$ defines an orientation, but unless E contains no sources, it is not a proper orientation. This can be circumvented in two different ways: Either one mods out by the acyclic subgraph as we have done above, or one adds heads to all sources. Either way, the new graph will admit a proper orientation, giving another proof of nuclearity of classical graph C^*-algebras of column-finite graphs.

Example 5.5. If $(E, C) := (E(m, n), C(m, n))$ for some $2 \leq m \leq n < \infty$ is the graph



as in [4, Example 9.3], so that (E, C) gives rise to the (m, n)-dynamical systems of [6], then u is a branching vertex without a local orientation. We therefore recover the fact from [6, Theorem 7.2] that $\mathcal{O}_{m,n} := \mathcal{O}(E, C)$ is non-exact. ◀

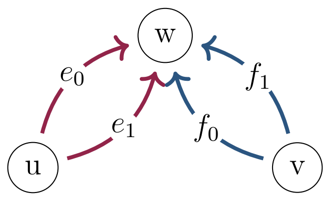
For every finitely separated graph (E, C), there is a natural *graph monoid* $M(E, C)$ as defined in [8]: It is the universal abelian monoid with generators E^0 and relations $v = \sum_{e \in X} s(e)$ for all $v \in E^0$ and $X \in C_v$. By [8, Proposition 4.4], every conical abelian monoid can be represented as $M(E, C)$ for an appropriate finitely separated bipartite graph (E, C). Conversely, the graph monoid $M(E)$ of a non-separated column-finite graph is always quite well-behaved: Whenever E is finite, $M(E)$ is a refinement monoid by [10, Proposition 4.4], hence primely generated by [11, Corollary 6.8]. It follows that $M(E)$ satisfies the extensive list of properties given by [11, Theorem 5.19], of which many pass to direct limits. Since any column-finite graph E is a direct limit of its finite complete subgraphs and the assignment $E \mapsto M(E)$ is continuous by [10, Lemma 3.4], the monoid $M(E)$ satisfies all such properties. Among these are

- *Unperforation*: If $n \cdot a \leq n \cdot b$, then $a \leq b$ for all integers $n \geq 2$.
- *Pseudo-cancellation*: If $a + c \leq b + c$, then there is some a_1 with $a_1 + c \leq c$ for which $a \leq b + a_1$.

Below we shall see two basic examples of finite bipartite separated graphs satisfying Condition (N) for which the graph monoids do not enjoy unperforation and pseudo-cancellation, respectively.

Recall that $M(E) \cong \mathcal{V}(L_K(E)) \cong \mathcal{V}(C^*(E))$ by [10, Theorem 3.5 and Theorem 7.1]. In the separated setting, we have $M(E, C) \cong \mathcal{V}(L_K(E, C))$ due to [8, Theorem 4.3], and the quotient map $L_K(E, C) \to L_K^{\mathrm{ab}}(E, C)$ induces a refinement $\mathcal{V}(L_K(E, C)) \to \mathcal{V}(L_K^{\mathrm{ab}}(E, C))$ by [4, Corollary 5.9]. However, it is still an open problem whether the inclusion $L_{\mathbb{C}}(E, C) \hookrightarrow C^*(E, C)$ induces an isomorphism $\mathcal{V}(L_{\mathbb{C}}(E, C)) \to \mathcal{V}(C^*(E, C))$. If this happens to be the case, then $\mathcal{V}(L_{\mathbb{C}}^{\mathrm{ab}}(E, C)) \cong \mathcal{V}(\mathcal{O}(E, C))$ as well by [4, Theorem 5.7], in which case there will be a natural refinement $M(E, C) \to \mathcal{V}(\mathcal{O}(E, C))$.

Example 5.6. Consider the Condition (N) graph



of [4, Example 9.5] and denote it by (E, C). Then

$$M(E, C) = \langle u, v, w \mid 2u = w = 2v \rangle \cong \langle u, v \mid 2u = 2v \rangle,$$

but $u \not\leq v$ and $v \not\leq u$, so $M(E, C)$ is not unperforated. It follows from [4, Theorem 7.4] that the relative type semigroup $S(\Omega(E, C), \mathbb{F}, \mathbb{K})$ of the partial action $\theta^{(E,C)}$ is isomorphic to $\mathcal{V}(L_K^{\mathrm{ab}}(E, C))$, so in particular, $M(E, C)$ unitarily embeds into the type semigroup, and this will be unperforated as well. We will now identify the partial action $\theta^{(E,C)}$ up to Kakutani equivalence (see [15, Definition 2.14]) with a concrete partial $\mathbb{F}_2$-action. Consider the sequence space $\mathcal{X} := \{0, 1\}^{\mathbb{Z}}$ along with the clopen subspaces

$$\mathcal{X}_{i\bullet} := \{x \in \mathcal{X} \mid x_{-1} = i\} \quad \text{and} \quad \mathcal{X}_{\bullet i} := \{x \in \mathcal{X} \mid x_0 = i\}$$

for $i = 0, 1$. We then define partial homeomorphisms $\varphi_L : X_{0\bullet} \to X_{1\bullet}$ and $\varphi_R : X_{\bullet 0} \to X_{\bullet 1}$ given by

$$\varphi_L(\dots x_{-3}x_{-2}0\bullet x_0x_1x_2\dots) := (\dots x_1x_0 1\bullet x_{-2}x_{-3}x_{-4}\dots)$$
$$\varphi_R(\dots x_{-3}x_{-2}x_{-1}\bullet 0x_1x_2\dots) := (\dots x_3x_2x_1\bullet 1x_{-1}x_{-2}x_{-3}\dots)$$

and consider the semi-saturated partial action $\theta : \mathbb{F}[L, R] \curvearrowright \mathcal{X}$ induced by φ_L and φ_R. Then θ is quasi-conjugate (see [17, Definition 2.2]) to the restriction $\theta^{(E,C)}|_\Omega$ of $\theta^{(E,C)}$ to the full (see [9, Definition 3.24]) clopen subspace $\Omega := \Omega(E, C)_w$ as follows. Observe first that every configuration $\xi \in \Omega$ may be represented by an ordered pair (ξ^{red}, ξ^{blue}) of infinite admissible paths

$$\xi^{red} := \dots f_{\underline{i_{-4}}} f^{-1}_{i_{-4}} e_{\underline{i_{-3}}} e^{-1}_{i_{-3}} f_{\underline{i_{-2}}} f^{-1}_{i_{-2}} e_{\underline{i_{-1}}} e^{-1}_{i_{-1}} \quad \text{and} \quad \xi^{blue} := \dots e_{\underline{i_3}} e^{-1}_{i_3} f_{\underline{i_2}} f^{-1}_{i_2} e_{\underline{i_1}} e^{-1}_{i_1} f_{\underline{i_0}} f^{-1}_{i_0} ,$$

for $i_k \in \{0, 1\}$, where $\underline{0} := 1$ and $\underline{1} := 0$. ξ^{red} represents the part of the configuration that initially travels by the inverse of a red edge, while ξ^{blue} initially travels by the inverse of a blue edge. An element $\alpha \in \mathbb{F}$ may act on ξ if and only if $\alpha \leq \xi^{red}$ or $\alpha \leq \xi^{blue}$, and the result $\theta^{(E,C)}_\alpha(\xi)$ lies in Ω if and only if α is of even length. If α denotes the simple cycle $\alpha = e_{\underline{i_{-1}}} e^{-1}_{-1}$, we have

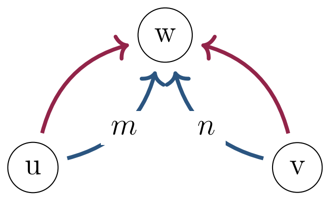
$$\theta^{(E,C)}_\alpha(\xi)^{red} = \dots f_{\underline{i_2}} f^{-1}_{i_2} e_{\underline{i_1}} e^{-1}_{i_1} f_{\underline{i_0}} f^{-1}_{i_0} e_{i_{-1}} e^{-1}_{\underline{i_{-1}}} \quad \text{and} \quad \theta^{(E,C)}_\alpha(\xi)^{blue} = \dots f_{\underline{i_{-4}}} f^{-1}_{i_{-4}} e_{\underline{i_{-3}}} e^{-1}_{i_{-3}} f_{\underline{i_{-2}}} f^{-1}_{i_{-2}},$$

and a similar situation occurs for $\alpha = f_{\underline{i_0}} f^{-1}_{i_0}$. Now consider the group homomorphism $\Psi : \mathbb{F}[L, R] \to \mathbb{F}$ given by $\Psi(L) = e^{-1}_1 e_0$ and $\Psi(R) = f^{-1}_1 f_0$, as well as the homeomorphism $\psi : \mathcal{X} \to \Omega$ defined by $\psi(x) = (\xi^{red}_x, \xi^{blue}_x)$, where

$$\xi^{red}_x := \dots f_{\underline{x_{-4}}} f^{-1}_{x_{-4}} e_{\underline{x_{-3}}} e^{-1}_{x_{-3}} f_{\underline{x_{-2}}} f^{-1}_{x_{-2}} e_{\underline{x_{-1}}} e^{-1}_{x_{-1}} \quad \text{and} \quad \xi^{blue}_x := \dots e_{\underline{x_3}} e^{-1}_{x_3} f_{\underline{x_2}} f^{-1}_{x_2} e_{\underline{x_1}} e^{-1}_{x_1} f_{\underline{x_0}} f^{-1}_{x_0} .$$

It is clear from our above observation that the pair (ψ, Ψ) defines a quasi-conjugacy $\theta \to \theta^{(E,C)}|_\Omega$, so θ and $\theta^{(E,C)}$ are indeed Kakutani equivalent.

Example 5.7. Finally, consider the graph



with $m \geq 2$, $n \geq 1$, and denote it by (E, C). Then

$$M(E, C) = \langle u, v, w \mid w = u + v = m \cdot u + n \cdot v \rangle \cong \langle u, v \mid u + v = m \cdot u + n \cdot v \rangle,$$

and we claim that $M(E, C)$ does not enjoy pseudo-cancellation as above with

$$a := (m - 1) \cdot u, \quad b := v \quad \text{and} \quad c := u.$$

These elements surely satisfy the assumption, namely that

$$a + c = m \cdot u \leq m \cdot u + n \cdot v = u + v = b + c,$$

so if pseudo-cancellation were present, then there should exist some $a_1 \in M(E,C)$ with $a_1 + c \leq c$ and $a \leq b + a_1$. However, the first inequality entails $a_1 = 0$, hence

$$(m-1) \cdot u = a \leq b = v,$$

which is absurd. It follows that the type semigroup $S(\Omega(E,C), \mathbb{F}, \mathbb{K})$ is not pseudo-cancellative either. Contrary to Example 5.6, we have no simple description of the partial action $\theta^{(E,C)}$ as the configurations are much more complicated in this case. ◀

Example 5.6 and Example 5.7 show that one should expect to find nuclear tame separated graph C^*-algebras of a rather different nature than that of classical graph C^*-algebras, in particular with a more general projection structure. However, while both unperforation and pseudo-cancellation may fail in the graph monoid $M(E,C)$ of a Condition (N) graph, the author expects that it will always enjoy the following important cancellation properties:

- *Almost unperforation*: If $(n+1) \cdot a \leq n \cdot b$ for some $n \geq 2$, then $a \leq b$.
- *Separation*: If $2a = a + b = 2b$, then $a = b$.

Observe that if any monoid theoretic property, which passes to limits, holds for all finite bipartite Condition (N) graphs (E,C), then it will automatically hold for

$$S(\Omega(E,C), \mathbb{F}, \mathbb{K}) \cong \mathcal{V}(L^{ab}(E,C)) \cong \varinjlim_n M(E_n, C^n)$$

as well by Corollary 5.2 and [4, Corollary 5.9]. As a consequence, the author expects $\theta^{(E,C)}$ to not be topologically amenable whenever it presents a counterexample to the topological version of Tarski's theorem considered in [4, Section 7].

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References

[1] F. Abadie. Enveloping actions and Takai duality for partial actions. *J. Funct. Anal.*, 197(1):14–67, 2003.
[2] F. Abadie. On partial actions and groupoids. *Proc. Amer. Math. Soc.*, 132(4):1037–1047 (electronic), 2004.
[3] C. Anantharaman-Delaroche and J. Renault. *Amenable groupoids*, volume 36 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*. L'Enseignement Mathématique, Geneva, 2000. With a foreword by G. Skandalis and Appendix B by E. Germain.
[4] P. Ara and R. Exel. Dynamical systems associated to separated graphs, graph algebras, and paradoxical decompositions. *Adv. Math.*, 252:748–804, 2014.
[5] P. Ara and R. Exel. K-theory for the tame C*-algebra of a separated graph. *J. Funct. Anal.*, 269(9):2995–3041, 2015.
[6] P. Ara, R. Exel, and T. Katsura. Dynamical systems of type (m,n) and their C*-algebras. *Ergodic Theory Dynam. Systems*, 33(5):1291–1325, 2013.
[7] P. Ara and K. R. Goodearl. C^*-algebras of separated graphs. *J. Funct. Anal.*, 261(9):2540–2568, 2011.
[8] P. Ara and K. R. Goodearl. Leavitt path algebras of separated graphs. *J. Reine Angew. Math.*, 669:165–224, 2012.

[9] P. Ara and M. Lolk. Convex subshifts, separated bratteli diagrams, and ideal structure of tame separated graph algebras. *ArXiv e-prints*, 2017.

[10] P. Ara, M. A. Moreno, and E. Pardo. Nonstable K-theory for graph algebras. *Algebr. Represent. Theory*, 10(2):157–178, 2007.

[11] G. Brookfield. Cancellation in primely generated refinement monoids. *Algebra Universalis*, 46(3):343–371, 2001.

[12] N. P. Brown and N. Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[13] R. Exel. *Partial Dynamical Systems, Fell Bundles and Applications*. 2014.

[14] R. Exel and C. Starling. Amenable actions of inverse semigroups. *Ergodic Theory Dynam. Systems*, 37(2):481–489, 2017.

[15] X. Li. Dynamic characterizations of quasi-isometry, and applications to cohomology. *ArXiv e-prints*.

[16] X. Li. Partial transformation groupoids attached to graphs and semigroups. *ArXiv e-prints*.

[17] M. Lolk. Exchange rings and real rank zero C^*-algebras associated with finitely separated graphs. *ArXiv e-prints*, 2017.

[18] J. Renault. *A groupoid approach to C^*-algebras*, volume 793 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[19] J. Renault. Topological amenability is a Borel property. *Math. Scand.*, 117(1):5–30, 2015.

[20] M. Rørdam and A. Sierakowski. Purely infinite C^*-algebras arising from crossed products. *Ergodic Theory Dynam. Systems*, 32(1):273–293, 2012.

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Bibliography

[A] Pere Ara and Matias Lolk. Convex subshifts, separated Bratteli diagrams, and ideal structure of tame separated graph algebras. Preprint, arXiv:1705.04495, 2017.

[B] Matias Lolk. Exchange rings and real rank zero C^*-algebras associated with finitely separated graphs. Preprint, arXiv:1705.04494, 2017.

[C] Matias Lolk. On nuclearity and exactness of the tame C^*-algebras associated with finitely separated graphs. Preprint, arXiv:1705.04500, 2017.

[1] G. Abrams and G. Aranda Pino. The Leavitt path algebra of a graph. *J. Algebra*, 293(2):319–334, 2005.

[2] Astrid an Huef. *The Cuntz-Krieger algebras,*. PhD thesis, University of Newcastle, 1993. Honours Thesis.

[3] P. Ara. Left and right projections are equivalent in Rickart C^*-algebras. *J. Algebra*, 120(2):433–448, 1989.

[4] P. Ara. Extensions of exchange rings. *J. Algebra*, 197(2):409–423, 1997.

[5] P. Ara and R. Exel. Dynamical systems associated to separated graphs, graph algebras, and paradoxical decompositions. *Adv. Math.*, 252:748–804, 2014.

[6] P. Ara and R. Exel. K-theory for the tame C*-algebra of a separated graph. *J. Funct. Anal.*, 269(9):2995–3041, 2015.

[7] P. Ara and K. R. Goodearl. C^*-algebras of separated graphs. *J. Funct. Anal.*, 261(9):2540–2568, 2011.

[8] P. Ara and K. R. Goodearl. Leavitt path algebras of separated graphs. *J. Reine Angew. Math.*, 669:165–224, 2012.

[9] P. Ara, K. R. Goodearl, K. C. O'Meara, and E. Pardo. Separative cancellation for projective modules over exchange rings. *Israel J. Math.*, 105:105–137, 1998.

[10] P. Ara, K. R. Goodearl, K. C. O'Meara, and R. Raphael. K_1 of separative exchange rings and C^*-algebras with real rank zero. *Pacific J. Math.*, 195(2):261–275, 2000.

[11] P. Ara, M. A. Moreno, and E. Pardo. Nonstable K-theory for graph algebras. *Algebr. Represent. Theory*, 10(2):157–178, 2007.

[12] B. Blackadar. *K-theory for operator algebras*, volume 5 of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, second edition, 1998.

[13] G. Brookfield. Cancellation in primely generated refinement monoids. *Algebra Universalis*, 46(3):343–371, 2001.

[14] Lawrence G. Brown and Gert K. Pedersen. Non-stable K-theory and extremally rich C^*-algebras. *J. Funct. Anal.*, 267(1):262–298, 2014.

[15] J. Cuntz. A class of C^*-algebras and topological Markov chains. II. Reducible chains and the Ext-functor for C^*-algebras. *Invent. Math.*, 63(1):25–40, 1981.

[16] Joachim Cuntz and Wolfgang Krieger. A class of C^*-algebras and topological Markov chains. *Invent. Math.*, 56(3):251–268, 1980.

[17] R. Exel and M. Laca. Cuntz-Krieger algebras for infinite matrices. *J. Reine Angew. Math.*, 512:119–172, 1999.

[18] N. J. Fowler, M. Laca, and I. Raeburn. The C^*-algebras of infinite graphs. *Proc. Amer. Math. Soc.*, 128(8):2319–2327, 2000.

[19] T. Katsura. A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras. I. Fundamental results. *Trans. Amer. Math. Soc.*, 356(11):4287–4322, 2004.

[20] T. Katsura. A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras. II. Examples. *Internat. J. Math.*, 17(7):791–833, 2006.

[21] T. Katsura. A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras. III. Ideal structures. *Ergodic Theory Dynam. Systems*, 26(6):1805–1854, 2006.

[22] T. Katsura. A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras. IV. Pure infiniteness. *J. Funct. Anal.*, 254(5):1161–1187, 2008.

[23] A. Kumjian and D. Pask. Higher rank graph C^*-algebras. *New York J. Math.*, 6:1–20, 2000.

[24] A. Kumjian, D. Pask, and I. Raeburn. Cuntz-Krieger algebras of directed graphs. *Pacific J. Math.*, 184(1):161–174, 1998.

[25] A. Kumjian, D. Pask, I. Raeburn, and J. Renault. Graphs, groupoids, and Cuntz-Krieger algebras. *J. Funct. Anal.*, 144(2):505–541, 1997.

[26] H. Lin. Approximation by normal elements with finite spectra in C^*-algebras of real rank zero. *Pacific J. Math.*, 173(2):443–489, 1996.

[27] K. McClanahan. K-theory for partial crossed products by discrete groups. *J. Funct. Anal.*, 130(1):77–117, 1995.

[28] A. L. T. Paterson. Graph inverse semigroups, groupoids and their C^*-algebras. *J. Operator Theory*, 48(3, suppl.):645–662, 2002.

[29] M. Rørdam. A simple C^*-algebra with a finite and an infinite projection. *Acta Math.*, 191(1):109–142, 2003.

[30] M. Rørdam and A. Sierakowski. Purely infinite C^*-algebras arising from crossed products. *Ergodic Theory Dynam. Systems*, 32(1):273–293, 2012.

[31] Jesper Villadsen. On the stable rank of simple C^*-algebras. *J. Amer. Math. Soc.*, 12(4):1091–1102, 1999.