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PhD Thesis

Investigating slopes of overconvergent modular forms

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Ad Elisa, per un futuro insieme.

Abstract

In this thesis we study the slopes of the Atkin's U_p operator acting on overconvergent modular forms. In the case of tame level 1 and for $p \in \{5, 7, 13\}$, we compute a quadratic lower bound for the Newton polygon of U_p. The methods of proof are explicit and rely on a certain deformation of the U_p operator and its characteristic power series. This gives us the possibility to compute the smallest possible slope for $p \in \{5, 7\}$ and to prove necessary and sufficient conditions on the weight such that the dimension of the cuspidal space is 1. This result allows us to exhibit some p-adic analytic families of modular forms in the framework of Coleman's theory. We then formulate a conjecture that would allow us to extend our analysis to all the congruence classes modulo $p - 1$. Finally, in the appendix, we present the results of some numerical experiments and we provide numerical evidence for the conjecture.

Resumé

I denne afhandling studerer vi hældninger af U_p-operatoren virkende på overkonvergente modulformer. I tilfældet af tamt niveau 1 og $p \in \{5, 7, 13\}$ bestemmer vi en kvadratisk, nedre grænse for Newton-polygonen for U_p. Bevismetoderne er eksplicitte og beror på en bestemt deformation af U_p-operatoren og dens karakteristiske potensrække. Dette giver os mulighed for at bestemme den mindst mulige hældning for $p \in \{5, 7\}$ og for at bevise nødvendige og tilstrækkelige betingelser for vægten således, at dimensionen af spidsrummet er 1. Dette resultat gør det muligt for os at fremvise nogle p-adisk analytiske familier af modulformer indenfor rammerne af Colemans teori. Derefter formulerer vi en formodning, der ville tillade en udvidelse af vores analyse til alle kongruensklasser modulo $p - 1$. Endelig præsenterer vi i et appendiks resultaterne af nogle numeriske eksperimenter, og vi giver numerisk evidens for formodningen.

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Contents

Chapter 1

Introduction

1.1 Overview

Fix $p > 2$ an odd prime number and let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$ with normalized valuation v_p such that $v_p(p) = 1$. Fix once and for all an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$.
A p-adic, or generalized, weight $\kappa = (s, [a])$ is a continuous $\mathbb{C}_p^*$-valued character of $\mathbb{Z}_p^*$ that can be represented as

$$\kappa(x) = \tau^a(x) \langle x \rangle^s$$

where $\langle x \rangle$ is the one-unit part of x, τ is the Teichmüller character, a is a representative of a congruence class in $\frac{\mathbb{Z}}{(p-1)\mathbb{Z}}$ and $s \in \mathbb{C}_p$ with $v_p(s) \geq \frac{1}{p-1} - 1$. The weight space $\mathcal{W}$ is then a disjoint union of $p-1$ connected components $\mathcal{W}_{[a]}$ obtained fixing the congruence class $[a]$. The ring $\mathbb{Z}$ embeds then diagonally in $\mathcal{W}$ and if $\kappa = (k, [k])$ for some $k \in \mathbb{Z}$, we have that $\kappa(x) = x^k$.
Let K be a finite extension of $\mathbb{Q}_p$ and let B be its ring of integers. For $k \in \mathbb{Z}$ denote by $M_k(N, B)$ the space of classical modular forms of weight k over $\Gamma_1(N) \cap \Gamma_0(p)$ and such that the coefficients of their q-expansion are in B.
For $r \in B$, Katz introduced in [Kat73] the space $M_k(N, B, r)$ of p-adic overconvergent modular forms of weight k and growth condition r and showed that $M_k(N, B, r)$ has the structure of a p-adic Banach space. We have that

$$M_k(N, B, r) \supseteq M_k(N, B).$$

The Atkin $U_p^{(k)}$ operator acts on $M_k(N, B, r)$ and its effect on q-expansions is

$$U_p^{(k)} \left(\sum_{n=0}^{\infty} a_n q^n \right) = \sum_{n=0}^{\infty} a_{np} q^n.$$

The operator $U_p^{(k)}$ is compact on $M_k(N, B, r)$ and stable on the subspace of cusp forms $S_k(N, B, r)$.
In this thesis we mostly deal with tame level $N = 1$.
We can then construct the characteristic power series $P(T)$ of the $U_p^{(k)}$ operator restricted to cusp forms. We then analytically extend this characteristic power series to $P_{[a]}(k, T)$ for all generalized weights $\kappa = (k, [a])$

in a given connected component $\mathcal{W}_{[a]}$ of the weight space. In Section 4.1.1 we describe the construction of $P(T)$ and $P_{[a]}(\kappa, T)$ with more details.
Let

$$P_{[a]}(k,T) = 1 + \sum_{m=1}^{\infty} C_{[a]}(m,k)T^m.$$

The Newton polygon of $P_{[a]}(k,T)$ is then defined as the lower convex hull of the set of points

$$\left\{ \left(m, v_p\left(C_{[a]}(m,k)\right)\right)\right\}.$$

This is also called the Newton polygon of $U_p^{(k)}$ and its slopes are called the slopes of $U_p^{(k)}$ or slopes of overconvergent cusp forms. We recall some facts about Newton polygons and their construction in Section 4.1.2.
Wan shows in [Wan98] that for $p \geq 5$ there is a parabola lying below the Newton polygon of $U_p^{(k)}$ for all $k \in \mathbb{Z}$ and all levels. He also shows that the quadratic term of the parabola is $\frac{6}{p+1}$ for tame level 1 and $p \in \{5, 7, 13\}$. Smithline in [Smi00] extends Wan's result for all primes and levels and gives a general formula to calculate the quadratic term of the parabola.
In the case of tame level 1, $p \in \{5, 7, 13\}$ and for the connected component $\mathcal{W}_{[0]}$ of the weight space we extend their results by computing explicitly the linear and constant terms. We are also able to compute the parabola for more general weights k.

Theorem 1.1. *For $p \in \{5, 7, 13\}$ and $k \in \mathbb{Z}_p$ we have the following lower bound on the valuation of the coefficients $C_{[0]}(m,k)$ of the characteristic power series $P_{[0]}(k,T)$:*

$$v_p\left(C_{[0]}(m,k)\right) \geq \left(\frac{6}{p+1}\right)m^2 - \left(1 - \frac{6}{p+1}\right)m - 1.$$

For $p \in \{5, 7\}$, but more generally $k \in \mathbb{C}_p$ such that $v_p(k) \geq 0$, the valuation of each coefficient $C_{[0]}(m,k)$ of the characteristic power series $P_{[0]}(k,T)$ satisfies

$$v_p\left(C_{[0]}(m,k)\right) \geq \left(\frac{6}{p+1} - \frac{1}{2}\right)m^2 - \left(\frac{3}{2} - \frac{6}{p+1}\right)m - 1.$$

If an eigenform f has q-expansion

$$f = \sum_{n=0}^{\infty} a_n q^n$$

and is normalized such that $a_1 = 1$, then its p-adic slope is $\alpha = v_p(a_p)$.
Coleman proves in [Col97, B5.7.1] the existence of p-adic analytic families of cuspidal eigenforms of fixed positive slope α, namely families of the form

$$F(\kappa) = \sum_{n=1}^{\infty} a_n(\kappa)q^n,$$

where each of the $a_n(\kappa)$ is an analytic function on the weight space $\mathcal{W}$ such that, for sufficiently large $k \in \mathbb{Z}$, the series $F(k)$ specializes to the q-expansion of a classical cusp form of fixed slope α. He also proves in [Col97, B3.5] that if the dimension of the space of cuspidal eigenforms of weight k_0 and slope α is 1 then there exists an integer $M \in \mathbb{Z}$ with the following property: for every sufficiently large $k_1 \in \mathbb{Z}$ such that

$$k_0 \equiv k_1 \mod (p-1)p^{n+M} \text{ for every } n \in \mathbb{N}$$

there exists a unique classical cusp form $F(k_1)$ over $\Gamma_0(p)$ of weight k_1 and slope α. Moreover the following congruence holds:

$$F(k_0) \equiv F(k_1) \mod p^{n+1},$$

where $F(k_0)$ is the unique normalized cuspidal form of slope α and weight k_0. When $p=2$ the problem is analyzed by Emerton in [Eme98] and Coleman in [Col97, Appendix II]. When $p=3$ the problem is studied by Coleman, Stevens and Teitelbaum in [CST98] and by Smithline in [Smi00].
In this context, as an application of Theorem 1.1, for $p \in \{5,7\}$ and for tame level 1 we are able to prove necessary and sufficient conditions on the weight k such that the cuspidal space has dimension 1 and to compute the slopes as well as the integers M in these cases.

Theorem 1.2. *Let $p=5$, and let k_0 and k_1 be integers in $4\mathbb{Z}$ such that $k_0 \geq 4$ and $k_1 \geq 4$. Assume furthermore that*

$$v_5(k_0-8) = v_5(k_1-8) \leq 1.$$

Then there exist unique classical normalized cuspidal eigenforms $F(k_0)$ and $F(k_1)$ over $\Gamma_0(5)$ of slope

$$\begin{cases} \alpha^{(5)}_{[0]}(k_0) = \alpha^{(5)}_{[0]}(k_1) = 1 & \text{if } v_5(k_0-8) = v_5(k_1-8) = 0 \\ \alpha^{(5)}_{[0]}(k_0) = \alpha^{(5)}_{[0]}(k_1) = 2 & \text{if } v_5(k_0-8) = v_5(k_1-8) = 1. \end{cases}$$

Moreover the following congruences hold:

$$\begin{cases} v_5\left(F(k_0)-F(k_1)\right) \geq v_5(k_0-k_1)+1 & \text{if } \alpha^{(5)}_{[0]}(k_0) = 1 \\ v_5\left(F(k_0)-F(k_1)\right) \geq v_5(k_0-k_1) & \text{if } \alpha^{(5)}_{[0]}(k_0) = 2. \end{cases}$$

Similarly, the following results holds for $p=7$.

Theorem 1.3. *Let $p=7$, and let k_0 and k_1 be integers in $6\mathbb{Z}$ such that $k_0 \geq 3$ and $k_1 \geq 3$. Assume furthermore that*

$$v_7(k_0-6) = v_5(k_1-6) = 0.$$

Then there exist unique classical normalized cuspidal eigenforms $F(k_0)$ and $F(k_1)$ over $\Gamma_0(7)$ of slope

$$\alpha^{(7)}_{[0]} = 1.$$

Moreover the following congruence holds:

$$v_7\left(F(k_0)-F(k_1)\right) \geq v_7(k_0-k_1)+1.$$

For a slightly more precise version of the statements we refer to Theorems 4.16 and 4.17.
This results can also be compared with the former Gouvêa-Mazur conjecture. The conjecture, stated in [GM92], can be formulated as follows:

Conjecture 1.4 (Gouvêa - Mazur). *Let $d(k,\alpha)$ be the dimension of the space spanned by the cuspidal eigenforms of weight k and slope $\alpha \in \mathbb{Q}$. Then if $k_0, k_1 \in \mathbb{Z}$ are both at least $2\alpha + 2$ and such that*

$$k_0 \equiv k_1 \mod (p-1)p^n$$

for some integer $n \geq \alpha$, then

$$d(k_0, \alpha) = d(k_1, \alpha).$$

Buzzard and Calegari provided remarkable counterexamples to the conjecture in [BC04]. Nevertheless, in many of the computed instances the conjecture seems to be even too weak.
Theorem 1.2 for example shows that, in the case of slope $\alpha = 2$, we can have that $d(k_0, 2) = d(k_1, 2) = 1$ under less restrictive assumptions. In fact this holds true if

$$k_0 \equiv k_1 \mod 4 \cdot 5$$

instead of congruence modulo $4 \cdot 5^2$ as the conjecture would have required.
The methods of proofs of our results are mostly explicit and rely on the fact that the modular curve $X_0(p)$ has genus zero.
In Chapter 2 we compute the action of the $U_p^{(0)}$ operator on classical modular functions and extend the computation to overconvergent modular functions using a specific basis for the space $S_k(1, B, r)$.
In Chapter 3, using a result of Coleman, we deform the weight zero operator to U_p^* to extend the computations to every weight k. To do so we use a result of Coleman about some specific Eisenstein series and their overconvergence rate; this restricts our computations to the congruence class of 0 modulo $p-1$.
In Chapter 4 we introduce the characteristic power series of the U_p^* operator and recall some basic facts about Newton polygons. Using a formula for the trace of the U_p operator due to Koike, we calculate the parabolas of Theorem 1.1. We apply the result in the framework of Coleman's theory of p-adic analytic families to prove a slightly more general version of Theorems 1.2 and 1.3. We then formulate a conjecture on a specific weight 1 Eisenstein series that allows us to extend our analysis to all the congruence classes $[a]$ modulo $p-1$ and the associated connected components $\mathcal{W}_{[a]}$ of the weight space.
In the appendices, we explain how to use Koike's formula to do numerical experiments. We then proceed to test our results by exhibiting lists of classical eigenforms of the predicted slopes and the congruences between then. We conclude by showing the numerical evidence for the conjecture formulated in Chapter 4.

1.2 Overconvergent p-adic modular forms

In this section we shall recall some basic facts about overconvergent p-adic modular forms, mostly following the presentations given by Wan in [Wan98] and by Gouvêa in [Gou88]. The main references for the theory of p-adic modular forms are [Ser73], [Ser75], [Kat73], [Gou88] and [Col97].
Let $p \geq 5$ be a prime and fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. The construction of overconvergent p-adic modular forms is possible also when $p = 2, 3$, but it is

slightly different. The resulting theory is analogous, see [Kat73] for more details.
Let E_{p-1} be the classical Eisenstein series of weight $p-1$; for $p \geq 5$ it is a lift to characteristic zero of the Hasse invariant. Let K be a finite extension of the field $\mathbb{Q}_p$ and let B be its ring of integers. Let N be a positive integer such that $p \nmid N$. In this thesis, we mostly deal with with $N = 1$. Let $M_k(N, B)$ be the space of classical modular forms of weight k over $\Gamma_1(N)$ whose q-expansion have coefficients in B: it is a free B-module of finite rank. For every positive $j \in \mathbb{N}$, there exists a free B-submodule $A_j(N, B)$ of $M_{k+j(p-1)}(N, B)$ such that

$$M_{k+j(p-1)}(N, B) \cong E_{p-1} \cdot M_{k+(j-1)(p-1)}(N, B) \oplus A_j(N, B).$$

Note that the isomorphism is non-canonical, as it depends on a non-canonical section of the map given by multiplication by E_{p-1} between the classical spaces

$$M_{k+j(p-1)}(N, B) \xrightarrow{\cdot E_{p-1}} M_{k+(j+1)(p-1)}(N, B);$$

see [Kat73, Lema 2.6.1] for more details.
Set $A_0(N, B) = M_k(N, B)$. For an element $r \in B$ denote by $M_k(N, B, r)$ the space of r-overconvergent p-adic modular forms of classical weight k. Any form $f \in M_k(N, B, r)$ can be written uniquely in the form

$$f = \sum_{j \geq 0} \frac{\beta_j}{E_{p-1}^j},$$

where $\beta_j \in A_j(N, B)$ and $v_p\left(r^{-j}\beta_j\right) \to \infty$ in the sense that the valuation of the q-expansions coefficients of β_j grows rapidly enough as j goes to infinity. This is called the Katz expansion of f. Conversely, every element f with a Katz expansion is in $M_k(N, B, r)$. Moreover if $r = r_1 r_2$ we have the inclusion

$$M_k(N, B, r) \hookrightarrow M_k(N, B, r_1),$$

see [Gou88, Corollary I.2.7] for further details.
If r is invertible in B, the space $M_k(N, B, r)$ is the space of p-adic modular forms in the sense of Serre [Ser73]. It turns out that the space is too large to have a good spectral theory for the $U_p^{(k)}$ operator, hence we have to restrict to subspaces with $v_p(r) > 0$. In this case each of the spaces $M_k(N, B, r)$ is a finite $M_0(N, B, r)$-module.
The space $M_k(N, K, r) = M_k(N, B, r) \otimes K$ has the structure of a p-adic Banach space, obtained by taking $M_k(N, B, r)$ as the unit ball in $M_k(N, K, r)$.
One important operator acting on the space $M_k(N, B, 1)$ is the Frobenius operator V. Geometrically, the V operator can be seen as the pullback of a particular lifting of the usual Frobenius operator in characteristic p, hence it is a map of degree p. On q-expansions is given by

$$V\left(\sum_{n=0}^{\infty} a_n q^n\right) = \sum_{n=0}^{\infty} a_n q^{np}.$$

On classical forms, this maps on q-expansions maps classical forms of level N to modular forms of level Np. As classical modular forms of level Np are p-adically of level N, it makes sense as an operator on p-adic modular forms,

but one should note that in general V is not stable on $M_k(N, B, r)$, as it reduces the rate of overconvergence. See [Gou97, II.2] for the details of the geometric definition of the operator V and its properties.
Another important p-adic operator is $U_p^{(k)}$. The study of this operator and its eigenvalues will be the main focus of the following sections. It can be defined from the operator V in the following way:

$$p \cdot U_p^{(k)} = \mathrm{Tr}(V).$$

On q-expansions, $U_p^{(k)}$ acts as

$$U_p^{(k)}\left(\sum_{n=0}^{\infty} a_n q^n\right) = \sum_{n=0}^{\infty} a_{np} q^n.$$

Hence we have that, for $f \in M_k(N, B, r)$,

$$U_p^{(k)}(V(f)) = f.$$

The operator $U_p^{(k)}$ is a compact operator on the full space $M_k(N, B, 1)$, but, in general, is not stable on $M_k(N, B, r)$ if $v_p(r) > 0$.
Nevertheless, if $v_p(r) < \frac{p}{p+1}$ one has that

$$U_p^{(k)}\left(M_k(1, B, r)\right) \subseteq M_k(1, B, r^p) \otimes K = M_k(1, K, r^p).$$

On the other hand, we have the following inclusion:

$$\iota : M_k(N, B, r^p) \hookrightarrow M_k(N, B, r).$$

Then the $U_p^{(k)}$ operator can be seen acting on $M_k(N, B, r)$ via $\iota \circ U_p^{(k)}$ and we have the following result, cf. [Wan98, Lemma 2.2].

Proposition 1.5. *Let r such that $0 < v_p(r) < \frac{p}{p+1}$. Then the $U_p^{(k)}$ operator induces a compact endomorphism on the p-adic Banach space $M_k(N, K, r)$. Moreover $U_p^{(k)}$ is stable on the subspace of cusp forms $S_k(N, B, r)$.*

When the weight is clear from the context, we will often write the $U_p^{(k)}$ operator simply as U_p.

Chapter 2

The U_p operator in weight zero

2.1 The action of U_p on classical modular functions

Let p be one of the primes $2, 3, 5, 7, 13$. These are the primes p for which the modular curve $X_0(p)$ has genus 0. We first recall some notions about the Hauptmoduln and their link to the modular $j-$invariant. These are classical results due to Klein, cf. [Kle78], but a helpful and recent reference is the paper [Mai09] by Maier.

Proposition 2.1. *Let $p \in \{2, 3, 5, 7, 13\}$ and let η be the Dedekind η function. The function*

$$t_p(z) = \left(\frac{\eta(pz)}{\eta(z)}\right)^{\frac{24}{p-1}}$$

is a generator for the function field of $X_0(p)$.

If j is the classical j-invariant, we have

$$j = \frac{H_p(t_p)}{t_p} \tag{2.1}$$

for a certain polynomial H_p of degree $\deg H_p = p + 1$. These polynomials are as follows:

$$\begin{aligned}
H_2(x) &= (2^8x + 1)^3,\\
H_3(x) &= (3^5x + 1)^3(3^3x + 1),\\
H_5(x) &= (5^5x^2 + 2 \cdot 5^3x + 1)^3,\\
H_7(x) &= (7^4x^2 + 5 \cdot 7^2x + 1)^3(7^2x^2 + 13x + 1),\\
H_{13}(x) &= (13^4x^4 + 7 \cdot 13^3x^3 + 20 \cdot 13^2x^2 + 19 \cdot 13x + 1)^3(13x^2 + 5x + 1),
\end{aligned}$$

as given in [Mai09, Section 3].

Remark. Maier in [Mai09] uses a slightly different normalization for the Hauptmodul than we do, and hence his formulas are a bit different, but equivalent to the ones above.

In the rest of this section we obtain the modular equation and use it to recursively compute the action of the $U_p^{(0)}$ operator on powers of t_p. The importance of this technique will be explained in section 2.2. We follow the reasoning as in [Smi00, Section 3.3], but we shall provide a few more details and use a slightly different language.
Let V be the operator acting on q-expansion by $q \mapsto q^p$ as introduced in Section 1.2. Then $V(j)$ is a classical modular function on $\Gamma_0(p)$. Let W_p be the Atkin-Lehner involution on $X_0(p)$, given by the action of the matrix

$$\omega_p = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}.$$

Since j is invariant under the action of the full modular group $\mathrm{SL}_2(\mathbb{Z})$, in particular it is invariant under the action of the matrix

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have immediately that

$$V(j) = j \circ W_p.$$

As a consequence, using (2.1) we get the following relation:

$$\frac{H_p(V(t_p))}{V(t_p)} = V(j) = j \circ W_p = \frac{H_p\left(t_p \circ W_p\right)}{t_p \circ W_p}.$$

Recalling that $\Delta(z) = (2\pi)^{12}\eta(z)^{24}$, where Δ is the classical modular form of weight 12 and level 1, we obtain:

$$\begin{aligned} t_p \circ W_p &= \left(\frac{\Delta\left(-\frac{1}{z}\right)}{\Delta\left(-\frac{1}{pz}\right)}\right)^{\frac{1}{p-1}} = \left(\frac{\left(\frac{1}{z}\right)^{12}\Delta\left(z\right)}{\left(\frac{1}{pz}\right)^{12}\Delta\left(pz\right)}\right)^{\frac{1}{p-1}} \\ &= p^{-\frac{12}{p-1}}\left(\frac{\eta(z)}{\eta(pz)}\right)^{\frac{24}{p-1}} = \left(p^{\frac{12}{p-1}}t_p\right)^{-1} \\ &= G_p(t_p), \end{aligned}$$

where G_p is the rational function $G_p(x) = \left(p^{\frac{12}{p-1}}x\right)^{-1}$.
Thus we see that

$$(x, y) = (t_p, V(t_p))$$

is a solution to the equation

$$-\frac{H_p(G_p(x))}{G_p(x)} + \frac{H_p(y)}{y} = 0. \tag{2.2}$$

Clearly, this equation is identically satisfied by $(x, y) = (x, G_p(x))$. Note also that the polynomials H_p are of degree $p+1$, have constant term equal to 1 and leading coefficient $p^{\frac{12p}{p-1}}$. Hence, if we multiply the equation (2.2) by $x^p y$ and then divide by $p^{\frac{12}{p-1}}(G_p(x)-y) = 1 - p^{\frac{12}{p-1}}xy$, we obtain an equation of the form

$$F_p(x,y) = 0, \tag{2.3}$$

where $F_p(x,y) \in \mathbb{Z}[x,y]$ is a polynomial of degree p and monic with respect to x. We can express F_p as follows:

$$F_p(x,y) = x^p + \left(\sum_{s=1}^{p-1} a_s^{(p)}(y)x^{p-s}\right) - y$$

with certain polynomials $a_s^{(p)}(y)$ of degree at most p and without constant term. We refer to Appendix A for an explicit list of these polynomials for $p = 2,3,5,7,13$.
As $p^{\frac{12}{p-1}t_p}t_pV(t_p) \neq 1$, we conclude that

$$(x,y) = (t_p, V(t_p))$$

is also a solution to the equation (2.3).
As V is a map of degree p, cf. Section 1.2, it follows that there are precisely p solutions $\alpha_1,\ldots,\alpha_p$ that satisfy

$$V(\alpha_i) = t_p, \quad i = 1,\ldots,p.$$

Thus, for every $i = 1,\ldots,p$, we have that

$$F_p\left(V\left(\alpha_i\right), V\left(t_p\right)\right) = 0.$$

Whence, as $F_p \in \mathbb{Z}[x,y]$,

$$F_p\left(\alpha_i, t_p\right) = 0.$$

It follows that the α_i's are precisely the roots of the monic polynomial of degree p

$$F(x, t_p).$$

Recall from Section 1.2, that by definition the U_p operator is related to the trace of the Frobenius operator V by $pU_p = \operatorname{Tr} V$, hence

$$pU_p\left(t_p\right) = \sum_{i=1}^{p} \alpha_i = -a_1^{(p)}\left(t_p\right).$$

Note now that for any $n \in \mathbb{N}$ we have exactly p solutions $\beta_1,\ldots,\beta_p$ such that

$$V(\beta_i) = t_p^n, \quad i = 1,\ldots,p.$$

It is clear that, for $i = 1,\ldots,p$.

$$\beta_i = \alpha_i^n$$

So

$$pU_p\left(t_p^n\right) = \sum_{i=1}^{p} \alpha_i^n$$

can be computed recursively in terms of the $a_s^{(p)}(t_p)$ by Newton's formulas: let $u_j = pU_p\left(t_p^j\right)$, then we have

$$u_j + a_1^{(p)}(t_p)u_{j-1} + \cdots + a_{j-1}^{(p)}(t_p)u_1 + ja_j^{(p)}(t_p) = 0,$$

that is

$$u_j = -\left(a_1^{(p)}(t_p)u_{j-1} + \cdots + a_{j-1}^{(p)}(t_p)u_1 + ja_j^{(p)}(t_p)\right), \tag{2.4}$$

with the conventions $a_p^{(p)}(y) = -y$ and $a_j^{(p)}(y) = 0$ for $j > p$.
As a result, we find that

$$U_p\left(t_p^j\right) = \sum_i c_{ij}^{(p)} t_p^i$$

for certain coefficients $c_{ij}^{(p)}$ that we can compute with a recursive method.
A simple result on the coefficients $c_{ij}^{(p)}$ is the following:

Proposition 2.2. *Let the coefficients $c_{ij}^{(p)}$ be as above. Then $c_{ij}^{(p)} \neq 0$ only if*

$$\frac{j}{p} \leq i \leq pj.$$

Proof. This follows by induction on j using the recursion formula. Assume that the maximum nonzero power of t_p on the right hand side of the formula (2.4) is at most $i_{max}(j) = pj$.
Now consider

$$u_{j+1} = -\left(a_1^{(p)}(t_p)u_j + \cdots + a_j^{(p)}(t_p)u_1 + (j+1)a_{j+1}^{(p)}(t_p)\right).$$

Then, as the degree of the polynomials $a_s^p(y)$ is at most p, it is clear that the maximum nonzero power of t_p in the formula for u_{j+1} is at most
$jp + p = (j+1)p = i_{max}(j+1)$.
Similarly, assume that for any $j' \leq j$, the minimum nonzero power of t_p in the formula for $u_{j'}$ is at least

$$i_{min}(j') = \left\lceil \frac{j'}{p} \right\rceil.$$

As the polynomials $a_s^p(y)$ are without constant term, the minimum nonzero power of t_p that can appear in the formula for u_{j+1} is the same as the minimum possible nonzero power of t_p in the product $a_p^p(t_p)u_{j-p+1}$. Hence

$$i_{min}(j+1) = 1 + \left\lceil \frac{j-p+1}{p} \right\rceil = \left\lceil \frac{j+1}{p} \right\rceil.$$

□

The following result about the coefficients $c_{ij}^{(p)}$ is part of [Smi00, Proposition 3.3.3].

Proposition 2.3 (Smithline)**.** *Let $p \in \{2, 3, 5, 7, 13\}$. For the coefficients $c_{ij}^{(p)}$ above we have*

$$v_p\left(c_{ij}^{(p)}\right) \geq \gamma_p \cdot (pi - j) - 1,$$

where

$$\gamma_p = \frac{12}{p^2 - 1}.$$

In [Smi00] the proof of this statement is sketched for the case $p = 3$ whereas the other cases are left to the reader.
We have verified the proposition for each $p \in \{2, 3, 5, 7, 13\}$ using the direct method shown below in Proposition 2.5 for $p = 5$ (and using some automation, specifically Maple [Map15], to compute the recurrence formulas).

Remark. We found that the bound of Proposition 2.3 is sharp in many instances, for each of the primes $p = 2, 3, 5, 7, 13$.

Let now $p = 5$, $t = t_5$, $U_5^{(0)} = U$, $a_s(t) = a_s^{(5)}(t_5)$ and $c_{ij}^{(5)} = c_{ij}$. Then, using Newtons's formula (2.4) we have the following relations:

$$\begin{aligned} U(1) =& 1, \\ 5 \cdot U(t) =& -a_1(t), \\ 5 \cdot U(t^2) =& -a_1(t)U(t) - 2a_2(t), \\ 5 \cdot U(t^3) =& -a_1(t)U(t^2) - a_2(t)U(t) - 3a_3(t), \\ 5 \cdot U(t^4) =& -a_1(t)U(t^3) - a_2(t)U(t^2) - a_3(t)U(t) - 4a_4(t), \end{aligned}$$

as well as the following recurrence formula for $j \geq 5$:

$$U(t^j) = -a_1(t)U(t^{j-1}) - a_2(t)U(t^{j-2}) - a_3(t)U(t^{j-3}) - a_4(t)U(t^{j-4}) + tU(t^{j-5}).$$

Once again, cf. Appendix A for an explicit display of the polynomials $a_s(y)$. As a result we obtain the recurrence formula for the coefficients c_{ij}:

Proposition 2.4. *The coefficients c_{ij} satisfy the following recursion formula:*

$$\begin{aligned} c_{i,j} =& 3^2 5^2 7 c_{i-1,j-1} + 2^2 5^5 13 c_{i-2,j-1} + 3^2 5^7 7 c_{i-3,j-1} + 2 \cdot 3 \cdot 5^{10} c_{i-4,j-1} \\ &+ 5^{12} c_{i-5,j-1} + 2^2 5^2 13 c_{i-1,j-2} + 3^2 5^4 7 c_{i-2,j-2} + 2 \cdot 3 \cdot 5^7 c_{i-3,j-2} \\ &+ 5^9 c_{i-4,j-2} + 3^2 5 \cdot 7 c_{i-1,j-3} + 2 \cdot 3 \cdot 5^4 c_{i-2,j-3} + 5^6 c_{i-3,j-3} \\ &+ 2 \cdot 3 \cdot 5 c_{i-1,j-4} + 5^3 c_{i-2,j-4} + c_{i-1,j-5} \end{aligned}$$

for $j \geq 5$ with the convention that $c_{ij} = 0$ for $i < 0$.

Proposition 2.5. *For the coefficients $c_{ij} = c_{ij}^{(5)}$ we have the following lower bound on the 5-adic valuation:*

$$v_5(c_{ij}) \geq \frac{1}{2} \cdot (5i - j) - 1.$$

Proof. We can compute the coefficients c_{ij} for $j = 0, 1, 2, 3, 4$ and verify the lower bound directly (recall that $c_{ij} = 0$ when $i > 5j$ by Proposition 2.2). After that, the bound can be established by induction on i as follows. Put

$$h(i, j) = \frac{1}{2} \cdot (5i - j) - 1$$

and suppose that the we have $v_5(c_{ij}) \geq h(i, j)$ whenever $j < m$. To establish the lower bound on $v_5(c_{i,m})$ for all i, we use the recurrence formula from Proposition 2.4, applying the induction hypothesis to every term on the right hand side. For example we can compute

$$v_5\left(3^2 5^2 7 c_{i-1,m-1}\right) \geq 2 + h(i - 1, m - 1) = \frac{1}{2} \cdot (5i - m) - 1 = h(i, m).$$

We find that this holds for every term of the recurrence formula and the result follows.

□

2.2 The action of U_p on overconvergent modular functions

We want to express the action of the $U_p^{(0)}$ operator on overconvergent modular forms of weight zero as explicitly as possible. Take now $p \in \{2, 3, 5, 7, 13\}$. Let B be the ring of integers of a finite extension of $\mathbb{Q}_p$. Let t_p be a generator of the function field of $X_0(p)$ as in section 2.1 and recall now the following result due to Loeffler [Loe07, Corollary 2.2],

Proposition 2.6 (Loeffler)**.** *For any r such that $0 \leq v_p(r) < \frac{p}{p+1}$, the space $S_0(1, B, r)$ of p-adic cusp functions of weight 0, tame level 1 and growth condition r has an orthonormal basis*

$$\left\{ \left(\frac{t_p}{d} \right)^i \right\}_{i=1}^{\infty}$$

for any $d \in \mathbb{C}_p$ such that $v_p(d) = -\frac{12}{p-1} v_p(r)$.

Remark. Following [Col97, A1] if E is a p-adic Banach space over B, a subset $\{e_i : i \in \mathbb{N}\} \subset E$ is said to be an orthonormal basis if every element x of E can be expressed uniquely as a sum

$$x = \sum_{i \in \mathbb{N}} a_i e_i$$

such that $\lim_{i \to \infty} v_p(a_i) = \infty$ and such that for every i we have $\|e_i\| = 1$ with respect to the norm of E. It is then clear that Proposition 2.6 implies that $\{(t_p)^i\}_{i \geq 1}$ is an orthonormal basis. By abusing the language, we will also call the

$$\left\{ \left(\frac{t_p}{d} \right)^i \right\}_{i=1}^{\infty}$$

an orthonormal basis of $S_0(1, B, r)$.

Now choose r such that $0 \leq v_p(r) < \frac{1}{p+1}$, hence we have that $-\frac{12}{p^2-1} < v_p(d) \leq 0$. Recall from Section 1.2 that the image of $r-$overconvergent modular functions via the U_p operator are r^p-overconvergent modular functions. As $0 \leq v_p(r^p) < \frac{p}{p+1}$, we can express the action of the U_p operator on $S_0(1, B, r)$ in term of this basis:

$$U_p \left(\left(\frac{t_p}{d} \right)^j \right) = \sum_{i=0}^{\infty} c_{ij}^{(p)}(d) \left(\frac{t_p}{d} \right)^i.$$

For some coefficients $c_{ij}^{(p)}(d)$ depending on d.
As we have

$$U_p \left(\left(\frac{1}{d} \right)^i \right) = \left(\frac{1}{d} \right)^i,$$

it is then clear that we have

$$c_{ij}^{(p)}(d) = d^{i-j} c_{ij}^{(p)} \tag{2.5}$$

where the coefficients $c_{ij}^{(p)}$ are as in section 2.1.

Crucial Remark. In general, it can be quite hard to compute the overconvergence rate of a given p-adic modular form. On the other hand, if $p \in \{2, 3, 5, 7, 13\}$ and the weight is zero, we can check its rate of overconvergence thanks to Proposition 2.6. A p-adic modular function f is r-overconvergent if it has a unique expression in terms of the orthonormal basis

$$f = \sum_{i=0}^{\infty} d^i b_i \left(\frac{t_p}{d}\right)^i$$

for some coefficients b_i and d taken as above such that $v_p(d^i b_i) \geq 0$ for every i. Hence, in this case, we can check the overconvergence rate by looking at growth rate of the valuation of the coefficients b_i.

Remark. In the following sections, we will often ask to make a suitable choice of d instead of r. It is clear that these choices are equivalent in the context of Proposition 2.6.

Proposition 2.7. *Let $p \in \{2, 3, 5, 7, 13\}$ and let $d \in \mathbb{C}_p$ be such that*

$$-\gamma_p = -\frac{12}{p^2 - 1} < v_p(d) < 0.$$

Then the following lower bound on the valuation of the coefficients $c_{ij}^{(p)}(d)$ holds:

$$v_p\left(c_{ij}^{(p)}(d)\right) \geq \gamma_p \cdot (pi - j) + 1 + v_p(d) \cdot (i - j).$$

Proof. It is an immediate consequence of Proposition 2.3 and the relation (2.5). $\square$

Chapter 3

Deformation of the U_p operator to any weight k

3.1 Coleman's trick and deformation

We want to construct a deformed version of the U_p operator, denoted by U_p^*, that in a sense interpolates p-adically the action of the usual U_p operator on spaces of overconvergent p-adic modular forms of different weights.
The key observation is due to Coleman and it is used in [Col97, Section B3]. One can also see [Smi00, Theorem 2.3.2] as a reference.

Proposition 3.1 (Coleman's trick)**.** *Let k_1 and k_2 be in $\mathbb{Z}$. Let B be a finite extension of $\mathbb{Q}_p$ and let $r \in B$ such that*

$$0 \leq v_p(r) < \frac{1}{p+1}.$$

Let $U_p^{(k_1)}$ and $U_p^{(k_2)}$ be the U_p operator acting on p-adic modular modular forms of weight respectively k_1 and k_2. Let G be a modular form of weight $k_1 - k_2$ such that the modular function

$$g = \frac{G}{V\left(G\right)}$$

is a $1-$unit in the ring $M_0(N, B, r)$.
Then the operators $U_p^{(k_1)}(\cdot)$ and $U_p^{(k_2)}(g\cdot)$ are conjugate.

Crucial Remark. In order to gain explicitness from the use of Proposition 3.1, we have to make suitable choices of the form G and we need to explicitly compute the rate of overconvergence of the modular function $g = \frac{G}{V(G)}$. The latter, in general, is a very difficult task.

Definition 3.2. Let G and g be as in Proposition 3.1 and let $k' \in \mathbb{Z}$ be the weight of G. The U_p^* operator acts on modular forms f_k of varying weight $k \in \mathbb{Z}$ as a deformation of the $U_p^{(0)}$ operator as follows:

$$U_p^*(f_k) = U_p^{(0)}\left(g^{\frac{k}{k'}} \cdot f_k\right).$$

Note that the definition of U_p^* depends on k, even if it is omitted in the notation.
It is then clear by the definition that U_p^* only makes sense if their weight is a multiple of k'. For this reason, suitable choices of the form G have to made.

3.2 The U_p^* operator for tame level 1, $p = 5, 7, 13$ and $[a] = [0]$

The goal of this section is to construct explicitly the U_p^* operator, as defined in section 3.1, in the case of tame level 1 and $p = 5, 7, 13$. The specific choice of the series

$$e_{p-1} = \frac{E_{p-1}}{V\left(E_{p-1}\right)}$$

that we will make will limit this construction to the connected component $\mathcal{W}_{[0]}$, the one associated to the congruence class [0] modulo $p - 1$.

Remark. For $p = 2, 3$ one can not take the series E_{p-1} as a lift of the Hasse invariant, but a similar construction of the deformed U_p^* operator can be done. See [Eme98] for $p = 2$; the case $p = 3$ has been analyzed in [CST98] and [Smi00].

In this case, a suitable choice of the modular form G as introduced in Proposition 3.1 is given by the following result, due to Coleman, cf. [Col97, Section B]. An explicit proof was provided by Wan [Wan98, Lemma 2.1].

Proposition 3.3 (Coleman). *Let $p \geq 5$. For any r such that*

$$0 \leq v_p(r) < \frac{1}{p+1}$$

the function

$$e_{p-1} = \frac{E_{p-1}}{V\left(E_{p-1}\right)}$$

is a 1-unit in the ring $M_0\left(1, B, r\right)$.

An immediate consequence is the following statement.

Corollary 3.4. *Let $p \in \{5, 7, 13\}$. For any r such that*

$$0 \leq v_p(r) < \frac{1}{p+1}$$

the function

$$e_{p-1} = \frac{E_{p-1}}{V\left(E_{p-1}\right)}$$

is an r-overconvergent modular function of level 1 and expansion

$$e_{p-1} = 1 + \sum_{i=1}^{\infty} d^i b_i \left(\frac{t_p}{d}\right)^i$$

with t_p as in Chapter 2 and d such that

$$v_p(d) = -\frac{12}{p-1} v_p(r).$$

Furthermore, for every $i \in \mathbb{N}$

$$v_p(b_i) \geq \frac{12}{p^2-1} i = \gamma_p \cdot i.$$

Proof. As $p \in \{5, 7, 13\}$ and as e_{p-1} is a modular function, it must have an expansion in term of the Hauptmopdul t_p. As e_{p-1} is r-overconvergent, the condition on d follows from Proposition 2.6.
Finally, as e_{p-1} is overconvergent we must have that $v_p\left(d^i b_i\right) \geq 0$ by Proposition 2.6. Hence it follows that

$$v_p(b_i) > -v_p(d) \cdot i \geq \frac{12}{p^2-1} i.$$

□

For the remaining of the section, unless specified, let $p \in \{5, 7, 13\}$.
Let $k \in \mathbb{C}_p$ and assume. Consider, purely formally at first, the expansion of $e_{p-1}^{\frac{k}{p-1}}$ via the binomial series:

$$e_{p-1}^{\frac{k}{p-1}} = \sum_{n=0}^{\infty} \binom{\frac{k}{p-1}}{n} x^n$$

where

$$x = e_{p-1} - 1 = \sum_{i=1}^{\infty} d^i b_i \left(\frac{t_p}{d}\right)^i.$$

Still purely formally we then can express

$$e_{p-1}^{\frac{k}{p-1}} = 1 + \sum_{n=1}^{\infty} \alpha_n(d,k) \left(\frac{t_p}{d}\right)^n \tag{3.1}$$

with

$$\alpha_n(d,k) = d^n \sum_{i=1}^{n} \binom{\frac{k}{p-1}}{i} \sum b_{\sigma_1}^{\tau_1} \cdots b_{\sigma_i}^{\tau_i}$$

where the inner sum is taken over all $\sigma_s, \tau_s \in \mathbb{N}$ such that

$$\sigma_1\tau_1 + \sigma_2\tau_2 + \cdots + \sigma_s\tau_s = n.$$

We want now to estimate the p-adic valuation of $\alpha_n(d,k)$. The expression (3.1) makes sense when $k \in \mathbb{Z}$ and $k \equiv 0 \mod p-1$. On the other hand, we can extend the coefficients $\alpha_n(d,k)$ formally to all $k \in \mathbb{C}_p$ by continuity. This extension will be helpful in the following chapter when we will use geometric techniques to study the deformed operator U_p^*.

Proposition 3.5. *For every $n \geq 1$, $k \in \mathbb{C}_p$ and $v_p(k) \geq 0$ the following inequality holds:*

$$v_p(\alpha_n(d,k)) \geq \left(v_p(d) + \gamma_p - \frac{1}{p-1}\right) n,$$

where $\gamma_p = \frac{12}{p^2-1}$ as before.

Proof. From the definition of $\alpha_n(d,k)$ above, it follows that

$$v_p(\alpha_n(d,k)) \geq v_p(d) \cdot n + \min_{1 \leq i \leq n} \left\{ v_p\left(\binom{\frac{k}{p-1}}{i}\right) + \gamma_p \cdot n \right\},$$

as the valuation of each of the products $b_{\sigma_1}^{\tau_1} \cdots b_{\sigma_i}^{\tau_i}$ is greater or equal than $\gamma_p \cdot n$ by Corollary 3.4. Note now the trivial lower bound $v_p(i!) \leq \frac{i}{p-1}$, hence

$$v_p\left(\binom{\frac{k}{p-1}}{i}\right) \geq v_p\left(\frac{1}{i!}\right) \geq -\frac{i}{p-1},$$

therefore the minimum occurs when $i = n$ and we have

$$v_p(\alpha_n(d,k)) \geq \left(v_p(d) + \gamma_p - \frac{1}{p-1}\right) n.$$

□

Corollary 3.6. *When $p \in \{5, 7\}$, the expression (3.1) defines an r-overconvergent modular function whenever r is chosen such that*

$$0 \leq v_p(r) < \frac{1}{p+1} - \frac{1}{12}$$

and k is specialized to an integer in $\mathbb{Z}$.

Proof. Such a choice of r implies that

$$-\frac{12}{p-1}\left(\frac{1}{p+1} - \frac{1}{12}\right) < v_p(d) \leq 0$$

using Proposition 2.6. This implies, together with the inequality given in Proposition 3.5, that the valuation of the coefficients $\alpha_n(d,k)$ is positive, as

$$\gamma_p - \frac{1}{p-1} = \frac{12}{p^2-1} - \frac{1}{p-1} = \frac{12}{p-1}\left(\frac{1}{p+1} - \frac{1}{12}\right).$$

□

Corollary 3.7. *If $k \in \mathbb{Z}_p$, then for $p \in \{5, 7, 13\}$, the expression (3.1) defines an r-overconvergent modular function whenever r is chosen such that*

$$0 \leq v_p(r) < \frac{1}{p+1}$$

and k is specialized to an integer in $\mathbb{Z}$.

Proof. If $k \in \mathbb{Z}_p$, we have that

$$v_p\left(\binom{\frac{k}{p-1}}{i}\right) \geq 0,$$

hence we can refine the lower bound on the valuation of the coefficients $\alpha_n(d,k)$ of Proposition 3.5:

$$v_p\left(\alpha_n(d,k)\right) \geq \left(v_p(d) + \gamma_p\right) n$$

for every $n \geq 1$. Then the expression

$$e_{p-1}^{\frac{k}{p-1}} = 1 + \sum_{n=1}^{\infty} \alpha_n(d,k) \left(\frac{t_p}{d}\right)^n$$

defines an r-overconvergent modular function for the chosen r. □

Remark. When $p = 13$, the number

$$\frac{1}{p+1} - \frac{1}{12}$$

is negative, hence the result of Corollary 3.6 does not have an analogous. Nevertheless, if we restrict our assumption of $k \in \mathbb{C}_p$ to p-adic integers we can still obtain some bounds on the Newton polygon of U_{13}^*, using Corollary 3.7. On the other hand, we will need to consider weights k outside of some p-adic discs, i.e. we will need to impose conditions like $v_p(k - b) < 1$ for some b. With $k \in \mathbb{Z}_p$, the corresponding regions of the weight space would be trivial. This is one of the obstruction to using the methods here presented to get accurate information about the 13-adic family. See Lemma 4.15 and the proof of Theorem 4.16 for details. In the case of $p = 13$ one could possibly describe the 13-adic family by choosing a suitable (but not too large) extension of $\mathbb{Z}_p$ as a domain for the weights.

We summarize these results in the next proposition.

Proposition 3.8. *Suppose that $p \in \{5, 7\}$ and $k \in \mathbb{C}_p$ with $v_p(k) \geq 0$. Consider the expression*

$$e_{p-1}^{\frac{k}{p-1}} = \sum_{n=0}^{\infty} \alpha_n(d,k) \left(\frac{t}{d}\right)^n.$$

Then, whenever $0 \leq v_p(r) < \frac{1}{p+1} - \frac{1}{12}$ and d is chosen such that $v_p(d) = -\frac{12}{p-1} v_p(r)$ we have for every $n \in \mathbb{N}$:

$$v_p(\alpha_n(d,k)) \geq \left(\gamma_p - \frac{1}{p-1} + v_p(d)\right) n.$$

Moreover the expression $e_{p-1}^{\frac{k}{p-1}}$ defines an r-overconvergent modular function if k is specialized to an integer in $\mathbb{Z}$.

If $p \in \{5, 7, 13\}$ *and* $k \in \mathbb{Z}_p$, $0 \le v_p(r) < \frac{1}{p+1}$ *and* d *is chosen such that* $v_p(d) = -\frac{12}{p-1} v_p(r)$ *we have instead the following bound on the valuation of the coefficients* $\alpha_n(d, k)$

$$v_p(\alpha_n(d, k)) \ge (\gamma_p + v_p(d))\, n.$$

Moreover the expression $e_{p-1}^{\frac{k}{p-1}}$ *defines an* r*-overconvergent modular function if* k *is specialized to an integer in* $\mathbb{Z}$*.*

Corollary 3.9. *Let* p, r, d, e_{p-1} *be as in Proposition 3.8. Let* $k \in \mathbb{Z}$ *such that* $k \equiv 0 \mod p - 1$*. Then the deformed operator* U_p^* *introduced in Definition 3.2 is defined by the formula*

$$U_p^*(f) = U_p^{(0)} \left(e_{p-1}^{\frac{k}{p-1}} f \right)$$

for r*-overconvergent functions* f*.*

Note again that the definition of U_p^* depends on k even though k is suppressed from the notation.

3.3 Computing the action

For $p = 5, 7, 13$, let U_p^* be the deformed operator as defined in the previous section. Consider now the basis of $S_k(1, B, r)$

$$\left\{ \left(\frac{t_p}{d} \right)^j \right\}_{j=1}^{\infty}$$

as defined by Proposition 2.6 and let $t_p = t$ and $e_{p-1} = \frac{E_{p-1}}{V(E_{p-1})}$ as before. The action of the U_p^* operator on this basis is then given by the formula

$$\begin{aligned} U_p^* \left(\left(\frac{t}{d} \right)^j \right) &= U_p^{(0)} \left(e_{p-1}^{\frac{k}{p-1}} \left(\frac{t}{d} \right)^j \right) = \sum_{n=0}^{\infty} \left(\alpha_n(d, k) \cdot U_p^{(0)} \left(\left(\frac{t}{d} \right)^{j+n} \right) \right) \\ &= \sum_{n=0}^{\infty} \left(\alpha_n(d, k) \sum_{i=0}^{\infty} c_{i,j+n}(d) \left(\frac{t}{d} \right)^i \right), \end{aligned}$$

where the coefficients $c_{i,j}(d)$ are as introduced in Section 2.1, namely

$$c_{i,j}(d) = d^{i-j} c_{i,j}^{(p)}.$$

In the following we will suppress the prime p from the notation when it is clear from the context.
Note now that the inner sum in the formula above is in fact finite, as we have $c_{i,j+n}(d) = 0$ for $i > p(j + n)$, cf. Proposition 2.2. Hence we can interchange the two sums and we obtain

$$U_p^* \left(\left(\frac{t}{d} \right)^j \right) = \sum_{i=0}^{\infty} c_{i,j}(d, k) \left(\frac{t}{d} \right)^i \tag{3.2}$$

where
$$c_{i,j}(d,k) = \sum_{n=0}^{\infty} \alpha_n(d,k) c_{i,j+n}(d).$$
Again, the latter sum is in fact a finite sum since $c_{i,j+n}(d) = 0$ when $j+n > pi$ by Proposition 2.2.
Note that we denote by ξ_p the row vector
$$\xi_p = \left(1, \frac{t_p}{d}, \left(\frac{t_p}{d}\right)^2, \dots\right),$$
we can express the formula (3.2) more compactly as
$$U_p^* \left(\xi_p^T\right) = M_p^* \left(\xi_p\right)$$
where $M_p^* = (c_{i,j}(d,k))$ is the infinite matrix representing the action of U_p^*.
The following Proposition describes some lower bounds on the valuation of the coefficients $c_{i,j}(d,k)$ that will be helpful in Chapter 4.

Proposition 3.10. *Suppose that* $p \in \{5,7\}$*,* $k \in \mathbb{C}_p$ *with* $v_p(k) \geq 0$ *and* r *such that* $0 \leq r < \frac{1}{p+1} - \frac{1}{12}$*. Let* $\gamma_p = \frac{12}{p^2-1}$*. Then:*
$$v_p\left(c_{i,j}(d,k)\right) \geq \left(\gamma_p - \frac{1}{p-1}\right)(pi-j) - 1 + v_p(d)(i-j)$$
uniformly in k*.*
Suppose that $p \in \{5,7,13\}$*,* $k \in \mathbb{Z}_p$ *and* r *such that* $0 \leq r < \frac{1}{p+1}$*. Let* $\gamma_p = \frac{12}{p^2-1}$*. Then:*
$$v_p\left(c_{i,j}(d,k)\right) \geq \gamma_p(pi-j) - 1 + v_p(d)(i-j).$$
uniformly in k*.*

Proof. Combining the definition of $c_{i,j}(d,k)$ with Propositions 3.8 and 2.7 with the fact that $c_{i,j+n}(d) = 0$ when $j > pi - n$, we can obtain the first inequality:
$$\begin{aligned}
&v_p(c_{i,j}) \geq \min_{0 \leq n \leq pi-j} \{\alpha_n(d,k) + v_p\left(c_{i,j+n}(d)\right)\} \\
&\geq \min_{0 \leq n \leq pi-j} \left\{ \left(\gamma_p - \frac{1}{p-1} + v_p(d)\right) n + (pi-j-n)\gamma_p - 1 + v_p(d)(i-j-n) \right\} \\
&= \min_{0 \leq n \leq pi-j} \left\{ \gamma_p(pi-j) - \frac{n}{p-1} - 1 + v_p(d)(i-j) \right\} \\
&= \left(\gamma_p - \frac{1}{p-1}\right)(pi-j) - 1 + v_p(d) \cdot (i-j),
\end{aligned}$$
as the minimum occurs when $n = pi - j$.
The second inequality follows easily using the second part of Proposition 3.10:
$$\begin{aligned}
v_p(c_{i,j}) &\geq \min_{0 \leq n \leq pi-j} \{\alpha_n(d,k) + v_p\left(c_{i,j+n}(d)\right)\} \\
&\geq \min_{0 \leq n \leq pi-j} \left\{ \left(\gamma_p + v_p(d)\right) n + (pi-j-n)\gamma_p - 1 + v_p(d)(i-j-n) \right\} \\
&= \gamma_p(pi-j) - 1 + v_p(d) \cdot (i-j).
\end{aligned}$$
□

Chapter 4

Main results and conjectures

4.1 Characteristic power series, Newton polygons

4.1.1 The characteristic power series of the U_p operator

To study the U_p operator and its spectral theory, it is very helpful to consider its characteristic power series.
For $p \geq 5$, let U_p be the operator acting on p-adic modular forms of weight k. As U_p is a compact operator acting on the p-adic Banach space $M_k(1, B, r)$, following the general theory of [Ser62, Section 5] we can define the characteristic power series of $U_p^{(k)} = U_p$ restricted to the space of cusp forms $S_k(1, B, r)$:

$$P(T) = \det \left(1 - TU_p \Big|_{S_k(1,B,r)} \right),$$

that is the p-adic analogue of the Fredholm determinant for p-adic Banach spaces.
We can summarize the important properties about the power series $P(T)$ with the following proposition, cf. [Gou88, Section II.3.4]

Proposition 4.1. *Let $P(T)$ be the characteristic power series of U_p restricted to $S_k(1, B, r)$. Then $\lambda \neq 0$ is an eigenvalue for U_p if and only if $P\left(\lambda^{-1}\right) = 0$ and the dimension of the eigenspace of cusp forms corresponding to λ is precisely the multiplicity of λ^{-1} as a root of $P(T)$.*
Moreover, the power series $P(T)$ defines a p-adic entire function and can be expressed in terms of the trace of the powers of the U_p operator as

$$P(T) = \exp \left(- \sum_{n=1}^{\infty} \left(\frac{\operatorname{Tr}\left((U_p)^n \right) T^n}{n} \right) \right).$$

If we now consider the weight k as a variable in a given connected component $\mathcal{W}_{[a]}$ of the weight space, we can study the spectral theory of the deformed U_p^*

operator introduced in Definition 3.2. Let $M_p^*(k)$ be the matrix of its action on $S_k(1, B, r)$ and consider its two-variable power series

$$P_{[a]}(k, T) = \det(1 - TM_p^*(k)) = 1 + \sum_{m=1}^{\infty} C_{[a]}(m, k)T^m,$$

Where each of the coefficient $C_{[a]}(m, k)$ is now a power series in k:

$$C_{[a]}(m, k) = \sum_{i=0}^{\infty} \beta_i k^i,$$

with $v_p(\beta_i) \to \infty$ as $i \to \infty$.

Remark. The U_p^* operator and its matrix $M_p^*(k)$ are defined only for $k \in \mathbb{Z}$. On the other hand, formally we can extend by continuity the coefficients of the matrix $M_p^*(k)$ to any $k \in \mathbb{C}_p$. This allows us to consider the series $P_{[a]}(\kappa, T)$ formally for any $\kappa \in \mathcal{W}$. In the next sections, we will make use of some geometric techniques later in this chapter, before specializing the results back to $k \in \mathbb{Z}$.

4.1.2 Newton polygons

Newton Polygons are a classical tool that is extremely helpful in p-adic analysis. For a classical reference see [Gou97, Section 6.4].

Definition 4.2. Let K be a finite extension of $\mathbb{Q}_p$ and let $F(T) = \sum a_i T^i$ be a power series with coefficients in K such that $a_0 \neq 0$ and $v_p(a_i) \to \infty$ as $i \to \infty$. The Newton polygon of F is the lower convex hull of the set of points

$$\{(i, v_p(a_i))\}.$$

Given a line segment of extremes $(i_0, v_p(a_{i_0}))$ and $(i_1, v_p(a_{i_1}))$, its length l is the length of the projection on the first axis: $l = i_1 - i_0$ and its slope α, as expected, is given by

$$\alpha = \frac{v_p(a_{i_1}) - v_p(a_{i_0})}{i_1 - i_0}.$$

The most important results about Newton Polygons is described in the following proposition, cf. [Gou97, Corollary 6.4.11].

Proposition 4.3. *Let $F(T)$ be a power series as above, and let $\alpha_1, \alpha_2, \ldots$ be the slopes of the line segments of its Newton polygon. Let $l_1, l_2, \ldots$ be the the length of the line segments. Then for every $j \geq 1$, the series has exactly l_j roots of valuation $-\alpha_j$.*

4.2 Results for $p = 5, 7, 13$

The goal of this section is to study the U_p^* operator for $p = 5, 7, 13$ via the Newton polygons of its characteristic power series $P_{[0]}(k, T)$, as defined in Section 4.1.
The idea is to find under which conditions on the weight k, the U_p^*-eigenspace associated to the smallest possible slope has dimension 1. Combining

Propositions 4.1 and 4.3, this amounts to prove under what assumptions on the weight k the length of the first segment of the Newton polygon of $P_{[0]}(k,T)$ is 1. Finally, using Coleman's theory about p-adic analytic families of fixed finite slope, we will be able to prove the Theorems of Section 1.1.

4.2.1 Lower bounds for the Newton polygon

One more helpful tool that we will need in the following are Serre's estimates on p-adic compact operator. The original result is in [Ser62, Section 6, Lemme 3], but a presentation in a similar setting can be found in [Kil08, Theorem 13].

Proposition 4.4 (Serre)**.** *Let $P_{[a]}(k,T) = 1 + \sum C_{[a]}(m,k)$ be the characteristic power series of the U_p^* operator restricted to cusp forms as defined in Section 4.1.1 and let $c_{i,j}(d,k)$ be the coefficients of the matrix M_p^* describing its action, as defined in Section 3.3. Define*

$$s_i(d,k) = \inf_j \{v_p(c_{i,j}(d,k))\}.$$

Then we have, for every $m \geq 1$

$$v_p\left(C_{[a]}(m,k)\right) \geq \sum_{i=0}^{m} s_i(d,k).$$

Proposition 4.5. *Let $p \in \{5,7\}$, $P_{[0]}(k,T)$, $c_{i,j}(d,k)$ and $s_i(d,k)$ as in Proposition 4.4. Assume $k \in \mathbb{C}_p$ and that $v_p(k) \geq 0$. Furthermore assume that k is in the component $\mathcal{W}_{[0]}$ of the weight space corresponding to the class $[0]$ modulo $p-1$. Then we can choose d such that*

$$s_i(d,k) \geq i\left(\frac{12}{p+1} - 1\right) - 1.$$

Proof. Using the bounds for $c_{i,j}(d,k)$ provided by Proposition 3.10, it is clear that, for fixed i, the minimum occurs when j is as large as possible. By Proposition 2.2 this happens when $j = pi$, as for $j > pi$ we have instead that $c_{i,j}(d,k) = 0$. Hence:

$$s_i(d,k) = \inf_j \{v_p(c_{i,j}(d,k))\} \geq i(1-p)\cdot v_p(d) - 1.$$

Recalling that d is such that $v_p(d) = -\frac{12 v_p(r)}{p-1}$ by Proposition 2.6; we can then choose r such that $0 \leq v_p(r) < \frac{1}{p+1} - \frac{1}{12}$ thanks to Proposition 3.10. This implies that we can choose $d \in \mathbb{C}_p$ such that

$$v_p(d) = -\frac{12}{p^2-1} + \frac{1}{p-1} + \varepsilon$$

for some arbitrarily small positive real number ε. It follows that:

$$\begin{aligned}
s_i(d,k) &\geq -i(p-1)\cdot v_p(d) - 1 \\
&\geq -i(p-1)\cdot\left(-\frac{12}{p^2-1} + \frac{1}{p-1} + \varepsilon\right) - 1 \\
&\geq -i\left(-\frac{12}{p+1} + 1 + \varepsilon(p-1)\right) - 1.
\end{aligned}$$

As we can take ε to be arbitrarily small, we obtain the inequality we wanted. $\square$

Proposition 4.6. *Let $p \in \{5, 7, 13\}$, $P_{[0]}(k, T)$, $c_{i,j}(d, k)$ and $s_i(d, k)$ as in Proposition 4.4. Assume $k \in \mathbb{Z}_p$ and that k is in the component $\mathcal{W}_{[0]}$ of the weight space corresponding to the class $[0]$ modulo $p - 1$. Then we can choose d such that*

$$s_i(d, k) \geq i \left(\frac{12}{p+1} \right) - 1.$$

Proof. The proof is the same as for Proposition 4.5, but using the second part of Proposition 3.10. $\square$

We can now prove the bounds on the valuation of the characteristic power series of the U_p operator as stated in Section 1.1.

Theorem 4.7. *For $p \in \{5, 7\}$, $k \in \mathbb{C}_p$ and $v_p(k) \geq 0$ the valuation of each coefficient $C_{[0]}(m, k)$ of the characteristic power series $P_{[0]}(k, T)$ is bounded by*

$$v_p\left(C_{[0]}(m, k)\right) \geq \left(\frac{6}{p+1} - \frac{1}{2} \right) m^2 - \left(\frac{3}{2} - \frac{6}{p+1} \right) m - 1.$$

For $p \in \{5, 7, 13\}$, and $k \in \mathbb{Z}_p$ the valuation of each coefficient $C_{[0]}(m, k)$ of the characteristic power series $P_{[0]}(k, T)$ is bounded by

$$v_p\left(C_{[0]}(m, k)\right) \geq \left(\frac{6}{p+1} \right) m^2 - \left(1 - \frac{6}{p+1} \right) m - 1.$$

Proof. From Proposition 4.5 one has immediately

$$\begin{aligned} v_p\left(C_{[0]}(m, k)\right) &\geq \sum_{i=0}^{m} i \left(\frac{12}{p+1} - 1 \right) - 1 \\ &\geq \frac{1}{2} \left(\frac{12}{p+1} - 1 \right) \cdot m(m+1) - m - 1 \\ &= \left(\frac{6}{p+1} - \frac{1}{2} \right) m^2 - \left(\frac{3}{2} - \frac{6}{p+1} \right) m - 1. \end{aligned}$$

The second part is proven in the same way from Proposition 4.6. $\square$

In order to compute explicitly the coefficients $C_{[a]}(m, k)$, by Proposition 4.1 it is enough to compute the trace of powers of the operator $U_p^{(k)}$. This can be done with Koike's formula, cf. [Koi75]:

$$\operatorname{Tr}\left(\left(U_p^{(k)} \right)^n \right) = - \sum_{\substack{0 \leq u < \sqrt{p^n} \\ (u,p)=1}} \mathrm{H}(u^2 - 4p^n) \cdot \frac{(\gamma(u))^k}{\gamma(u)^2 - p^n} - 1, \tag{4.1}$$

where $\mathrm{H}(D)$ denotes the Hurwitz class number of D, and $\gamma(u)$ is the unique p-adic unic root of the equation

$$x^2 - ux + p^n = 0.$$

We refer to Appendix B for the details about the formula. For the purpose of this section, it is enough to know that a PARI/GP [PG17b] implementation of the formula allows us to compute the coefficients $C_{[a]}(m,k)$ up to a desired p-adic precision for specific class $[a]$ modulo $p-1$; again refer to Appendix B for the details about the computational part.

Remark. In this chapter, if $\lambda_1, \lambda_2 \in \mathbb{C}_p$ with the notation

$$\lambda_1 \equiv \lambda_2 \mod p^\delta$$

for some integer δ, we mean

$$v_p(\lambda_1 - \lambda_2) \geq \delta.$$

for the normalized valuation v_p.

4.2.2 The smallest slope when $p = 5$ and $[a] = [0]$

Let $p = 5$. Due to the fact that we used $e_4 = \frac{E_4}{V(E_4)}$ in the construction of the U_5^* operator in Section 3.2, we can do the computations for weights $k \in \mathbb{C}_5$ such that $v_5(k) \geq 0$ and such that k is in the component $\mathcal{W}_{[0]}$ of the weight space associated to the class [0] modulo 4: for the remaining of this section, we always assume that k belongs to $\mathcal{W}_{[0]}$.
The following statements describe the smallest possible slope of the Newton polygon of $P_{[0]}(k,T)$ in this setting.

Proposition 4.8. *Let $k \in \mathbb{C}_5$ with $v_5(k) \geq 0$ and furthermore assume that $v_5(k-8) < 2$. Let $C_{[0]}(1,k)$ be the first coefficient of the characteristic power series $P_{[0]}(k,T)$ of U_5^* restricted to cusp forms. Then*

$$v_5\left(C_{[0]}(1,k)\right) = 1 + v_5(k-8) < 3.$$

Proof. Using Koike's formula (4.1), we get the following polynomial:

$$\begin{aligned} C_{[0]}(1,k) &\equiv 60 - 20k + 25k^2 \mod 5^3 \\ &\equiv 20(3-k) + 25k^2 \mod 5^3. \end{aligned}$$

Hence

$$v_5\left(C_{[0]}(1,k)\right) \geq \min\{1 + v_5(3-k), 2 + 2v_5(k)\}.$$

And surely the inequality above is an equality if $v_5(3-k) = 0$ or $v_5(3-k) > 1$, because in this cases $v_5(k) = 0$).
If we assume instead $0 < v_5(k-3) \leq 1$, to be able to compute $v_5\left(C_{[0]}(1,k)\right)$ mod 5^3 we need to impose $v_5\left(C_{[0]}(1,k)\right) < 3$, that is

$$\begin{aligned} 20(3-k) + 25k^2 &\not\equiv 0 \mod 5^3 \\ 4(3-k) + 5k^2 &\not\equiv 0 \mod 5^2 \end{aligned}$$

Since we are assuming $0 < v_5(k-3) \leq 1$, it is enough to check the cases $k \equiv 8, 13, 18, 23 \mod 25$. By checking all this possibilities, we see that $C_{[0]}(1,k) \equiv 0 \mod 5^3$ only if $k = 8 \mod 5^2$. It follows that, as long as $v_5(k-8) < 2$,

$$v_5\left(C_{[0]}(1,k)\right) = 1 + v_5(k-8).$$

$\square$

Remark. Clearly, we could get more refined version of Proposition 4.8 by computing $C_{[0]}(1,k)$ modulo higher power of 5 and finding sharper conditions on k. On the other hand, the result here obtained is enough to compute the length of the first segment of the Newton polygon, as shown in the following.

If we specialize the bound of Theorem 4.7 to $p = 5$, we get that for every $m \geq 1$.

$$v_5\left(C_{[0]}(m,k)\right) \geq \frac{1}{2}m^2 - \frac{1}{2}m - 1.$$

To prove that the length of the first segment of the Newton polygon of $P_{[0]}(k,T)$ is 1, is equivalent to prove that for every $m > 1$

$$\frac{v_5\left(C_{[0]}(m,k)\right)}{m} > v_5\left(C_{[0]}(1,k)\right).$$

By using Proposition 4.8, we can assume that $v_5\left(C_{[0]}(1,k)\right) < 3$ whenever $v_5(k-8) < 2$, the problem then amounts to solve the following inequality:

$$v_5\left(C_{[0]}(m,k)\right) \geq \frac{1}{2}m^2 - \frac{1}{2}m - 1 \geq 3m,$$

that is $m > 7$.
The missing cases are $m = 2, 3, 4, 5, 6, 7$, and we can compute them with Koike's formula.

The case $m = 2$

Using Koike's formula we can compute $C_{[0]}(2,k)$:

$$C_{[0]}(2,k) \equiv 5^4(4 \cdot 5 + 9k + 6k^2 + 24k^3 + 21k^5 + 3 \cdot 5k^5) \mod 5^6.$$

By inspection, we see that

$$\begin{cases} v_5\left(C_{[0]}(2,k)\right) \geq 5, & \text{if } k \equiv 0,1,2,3 \mod 5 \\ v_5\left(C_{[0]}(2,k)\right) = 4, & \text{if } k \equiv 4 \mod 5 \end{cases}$$

It follows that the inequality $v_5\left(C_{[0]}(2,k)\right) > 2 \cdot v_5\left(C_{[0]}(1,k)\right)$ holds true if $k \equiv 4 \mod 5$, as in this case we have that $v_5(k-8) = 0$, hence $v_5\left(C_{[0]}(1,k)\right) = 1$ by Proposition 4.8.
By further inspection, assuming $k \not\equiv 4 \mod 5$, one can see that

$$\begin{cases} v_5\left(C_{[0]}(2,k)\right) \geq 6, & \text{if } k \equiv 8, 12, 16, 20 \mod 5^2 \\ v_5\left(C_{[0]}(2,k)\right) = 5, & \text{otherwise.} \end{cases}$$

It follows that the inequality $v_5\left(C_{[0]}(2,k)\right) \geq 6 > 2 \cdot v_5\left(C_{[0]}(1,k)\right)$ holds true if $k \equiv 8, 12, 16, 20 \mod 5^2$.
On the other hand, if $k \equiv 0, 1, 2, 5, 6, 7, 10, 11, 15, 17, 21, 22 \mod 5^2$ we have that $v_5(k-8) = 0$ and the required inequality also holds by Proposition 4.8.
Finally, if $k \equiv 3, 13, 18, 23 \mod 5^2$ we have that $v_5(k-8) = 1$ and the inequality is still true.

The case $m = 3$

Using Koike's formula we can compute:

$$C_{[0]}(3, k) \equiv 5^8 k^2 (3 + 3k + 4k^2 + 2k^4 + 2k^5 + k^6) \mod 5^9.$$

By inspection we note that

$$v_5(3 + 3k + 4k^2 + 2k^4 + 2k^5 + k^6) \geq 1 \text{ if } k \equiv 1, 2, 3, 4 \mod 5.$$

On the other hand, if $k \equiv 0 \mod 5$ we have that $v_5(k^2) \geq 2$, hence for all k such that $v_5(k) \geq 0$ we have that

$$v_5 \left(C_{[0]}(3, k) \right) \geq 9 > 3 \cdot C_{[0]}(1, k)$$

as required.

The cases $m = 4, 5, 6, 7$

Using Koike's formula as above, we get that for each k such that $v_5(k) \geq 0$:

$$\begin{aligned} C_{[0]}(4, k) &\equiv 0 \mod 5^{12}, \\ C_{[0]}(5, k) &\equiv 0 \mod 5^{15}, \\ C_{[0]}(6, k) &\equiv 0 \mod 5^{18}, \\ C_{[0]}(7, k) &\equiv 0 \mod 5^{21}. \end{aligned}$$

This implies that, if $m \in \{4, 5, 6, 7\}$, then $v_5 \left(C_{[0]}(m, k) \right) \geq 3m$ and the inequality is verified for all m.
We summarize this discussion with the following statement.

Proposition 4.9. *Let $k \in \mathbb{C}_5$, $v_5(k) \geq 0$ and assume that $k \in \mathcal{W}_{[0]}$. Furthermore assume that $v_5(k - 8) < 2$. Then the first segment of the Newton polygon of the characteristic power series of the U_5^* operator restricted to cusp forms has length one. Furthermore, the slope of this segment is*

$$\alpha_{[0]}^{(5)}(k) = 1 + v_5(k - 8).$$

Remark. The result is optimal, in the sense that the computations above imply that if $v_5(k - 8) \geq 2$ then the minimal slope is at least 3 and the dimension of the cuspidal space is strictly bigger then one.

4.2.3 The smallest slope when $p = 7$ and $[a] = [0]$

We want to obtain statements for $p = 7$ similar to those found in Section 4.2.2 for $p = 5$. The method is essentially the same. Once again, as we used the series $e_6 = \frac{E_6}{V(E_6)}$ in the construction of the U_7^* operator, we can do the computations for weights $k \in \mathbb{C}_7$ such that $v_7(k) \geq 0$ and such that k is in the component $\mathcal{W}_{[0]}$ of the weight space associated to the class [0] modulo 6.

Proposition 4.10. *Let $k \in \mathbb{C}_7$, $v_7(k) \geq 0$ and furthermore, assume that $v_7(k - 6) < 1$. Let $C_{[0]}(1, k)$ be the first coefficient of the characteristic power series $P_{[0]}(k, T)$ of U_7^* restricted to cusp forms. Then*

$$v_7 \left(C_{[0]}(1, k) \right) = 1 + v_5(k - 6) < 2.$$

Proof. Using Koike's formula (4.1), we get the following polynomial:

$$\begin{aligned} C_{[0]}(1,k) &\equiv 14 - 14k \mod 7^2 \\ &\equiv 2 \cdot 7(1+k) \mod 7^2. \end{aligned}$$

Hence

$$v_7\left(C_{[0]}(1,k)\right) = 1 + v_7(k-6),$$

provided that $v_7(k-6) < 1$. ☐

If we specialize the bound of Theorem 4.7 to $p = 7$, we get that for every $m \geq 1$.

$$v_7\left(C_{[0]}(m,k)\right) \geq \frac{1}{4}m^2 - \frac{3}{4}m - 1.$$

To prove that the length of the first segment of the Newton polygon of $P_{[0]}(k,T)$ is 1, is equivalent to prove that for every $m > 1$

$$\frac{v_7\left(C_{[0]}(m,k)\right)}{m} > v_7\left(C_{[0]}(1,k)\right).$$

By using Proposition 4.10, we can assume that $v_7\left(C_{[0]}(1,k)\right) < 2$ whenever $v_7(k-6) < 1$; the problem then amounts to solve the following inequality:

$$v_7\left(C_{[0]}(m,k)\right) \geq \frac{1}{4}m^2 - \frac{3}{4}m - 1 \geq 2m,$$

that is $m > 11$.
The missing cases are $m = 2, 3, \ldots, 11$, and we can compute them with Koike's formula.

The case $m = 2$

Using Koike's formula we can compute $C_{[0]}(2,k)$:

$$C_{[0]}(2,k) \equiv 7^3(1 + 3k + k^2 + 6k^3) \mod 7^4.$$

We can then find that

$$\begin{cases} v_7\left(C_{[0]}(2,k)\right) \geq 4, & \text{if } k \equiv 0,1,2,3 \mod 7 \\ v_7\left(C_{[0]}(2,k)\right) = 3, & \text{if } k \equiv 4,5,6 \mod 7. \end{cases}$$

It follows that the inequality $v_7\left(C_{[0]}(2,k)\right) \geq 4 > 2 \cdot v_7\left(C_{[0]}(1,k)\right)$ holds true if $k \equiv 4,5,6 \mod 7$.
On the other hand, if $k \equiv 0,1,2,3 \mod 7$, then we have that $v_7(k-6) = 0$, hence $v_7\left(C_{[0]}(1,k)\right) = 1$ by Proposition 4.10 and the inequality holds also in this case.

The cases $m = 3, 4, \ldots, 11$

Using Koike's formula as above, we get that for every k such that $v_7(k) \geq 0$:

$$\begin{aligned} C_{[0]}(3,k) &\equiv 0 \mod 7^6, \\ C_{[0]}(4,k) &\equiv 0 \mod 7^8, \end{aligned}$$

$$\begin{aligned}
C_{[0]}(5,k) &\equiv 0 \mod 7^{10},\\
C_{[0]}(6,k) &\equiv 0 \mod 7^{12},\\
C_{[0]}(7,k) &\equiv 0 \mod 7^{14},\\
C_{[0]}(8,k) &\equiv 0 \mod 7^{16},\\
C_{[0]}(9,k) &\equiv 0 \mod 7^{18},\\
C_{[0]}(10,k) &\equiv 0 \mod 7^{20},\\
C_{[0]}(11,k) &\equiv 0 \mod 7^{22}.
\end{aligned}$$

This implies that, if $m \in \{3, 4, 5, 6, 7, 8, 9, 10, 11\}$, then $v_7\left(C_{[0]}(m,k)\right) \geq 2m$ and the inequality is verified for all m.
We summarize this discussion with the following statement.

Proposition 4.11. *Let $k \in \mathbb{C}_7$, $v_7(k) \geq 0$ and assume that $k \in \mathcal{W}_{[0]}$. Furthermore assume that $v_7(k-6) < 1$. Then the first segment of the Newton polygon of the characteristic power series of the U_7^* operator restricted to cusp forms has length one. Furthermore, the slope of this segment is*

$$\alpha_{[0]}^{(7)}(k) = 1 + v_7(k-6).$$

Remark. The result is optimal, in the sense that the computations above imply that if $v_7(k-6) \geq 1$ then the minimal slope is at least 2 and the dimension of the cuspidal space is strictly bigger then one.

4.2.4 p-adic analytic families and congruences

In Sections 4.2.2 and 4.2.3 we proved under what conditions on the weight k, for the primes $p = 5, 7$, the first segment of the Newton polygon of the characteristic power series $P_{[0]}(k,T)$ has length one. Using Propositions 4.3 and 4.1, we can rephrase the results with the following statements.

Proposition 4.12. *Let $p = 5$. Let $k \in \mathbb{C}_5$, $v_5(k) \geq 0$ and assume that k lies in the the connected component $\mathcal{W}_{[0]}$ of the weight space associated to the class $[0]$ modulo 4. Furthermore assume that $v_5(k-8) < 2$. Then there is a unique root λ^{-1} of $P_{[0]}(k,T)$ and such that*

$$v_5(\lambda) = \alpha_{[0]}^{(5)}(k) = 1 + v_5(k-8).$$

Furthermore λ is an eigenvalue of the operator U_5^* and the dimension of the eigenspace of cusp forms corresponding to λ is 1.

Proposition 4.13. *Let $p = 7$. Let $k \in \mathbb{C}_7$, $v_5(k) \geq 0$ and assume that k lies in the the connected component $\mathcal{W}_{[0]}$ of the weight space associated to the class $[0]$ modulo 6. Furthermore assume that $v_7(k-6) < 1$. Then there is a unique root λ^{-1} of $P_{[0]}(k,T)$ and such that*

$$v_7(\lambda) = \alpha_{[0]}^{(7)}(k) = 1 + v_7(k-6).$$

Furthermore λ is an eigenvalue of the operator U_7^* and the dimension of the eigenspace of cusp forms corresponding to λ is 1.

Definition 4.14. Let $B(c, p^{-\delta}) \subset \mathbb{C}_p$ be the open ball in $\mathbb{C}_p$ of centre c and such that $x \in B(c, p^{-\delta})$ if $v_p(x - c) > \delta$.
Similarly, let $B[c, p^{-\delta}]$ be the closed ball of centre c and such that $x \in B[c, p^{-\delta}]$ if $v_p(x - c) \geq \delta$.

To obtain explicit congruences about different forms in the same family, we will need the following general result, due to Coleman, Stevens and Teitelbaum.

Lemma 4.15 ([CST98, p. 147])**.** *Let S be a finite subset of points of $B(0, p^{-H}) \subset \mathbb{C}_p$, for some $H \in \mathbb{R}$ and let h be a function from S into $\mathbb{R}$. Let $f(k)$ be a p-adic analytic function with values in $\mathbb{C}_p$ and such that $v_p(f(k)) \geq 0$ for all k in the region*

$$V = B(0, p^{-H}) \setminus \bigcup_{a \in S} B[a, p^{-h(a)}],$$

Then, for every $x_1, x_2 \in V$,

$$v_p(f(x_1) - f(x_2)) \geq v_p(x_1 - x_2) + \inf_{a \in S}\{-H, h(a) - v_p((x_1 - a)(x_2 - a))\}.$$

We can now state our results in term of p-adic analytic families and prove the theorems of Section 1.1.

Theorem 4.16. *Let $p = 5$. Let $X^5_{[0]}$ be a subset of the connected component $\mathcal{W}_{[0]}$ of the weight space associated to the class $[0]$ modulo 4; furthermore assume that $X^5_{[0]}$ is a strict neighborhood of*

$$\bar{X}^5_{[0]} = B[0, 1] \setminus B\left(8, 5^{-2+\varepsilon}\right) \subset \mathcal{W}_{[0]},$$

for some $\varepsilon \in \mathbb{R}_{>0}$.
Then, for every $n \geq 2$, there is an analytic function $a_n(k)$ on $X^5_{[0]}$ such that the formal power series

$$F^{(k)}_{[0]}(q) = q + \sum_{n=2}^{\infty} a_n(k) q^n$$

specializes, for $k_0 \in \mathbb{Z}$, to the q-expansion of an overconvergent normalized 5-adic cuspidal eigenform form of tame level 1, generalized weight $(k_0, [k_0])$ and slope

$$\alpha^{(5)}_{[0]}(k_0) = 1 + v_5(k_0 - 8).$$

Furthermore $\alpha^{(5)}_{[0]}(k_0)$ is the minimal possible slope among overconvergent 5-adic cuspidal eigenforms of generalized weight $(k_0, [0])$ and $F^{(k_0)}_{[0]}(q)$ is the unique normalized overconvergent form with this slope.
Moreover, if k_0 and k_1 are in $4\mathbb{Z}$, $k_0 \geq 4$ and $k_1 \geq 4$, the forms

$$F^{(k_0)}_{[0]}(q) \quad \text{and} \quad F^{(k_1)}_{[0]}(q)$$

are two classical cuspidal eigenforms of weight k_0 and k_1 over $\Gamma_0(5)$ and we have that uniformly for $n \geq 2$

$$v_5\left(a_n(k_0) - a_n(k_1)\right) \geq v_5(k_0 - k_1) + 1 + \min\{0, 1 - v_5\left((k_0 - 8)(k_1 - 8)\right)\}.$$

Remark. The introduction of $X^5_{[0]}$ in Theorem 4.16 is simply the translation in a more geometric language of the same conditions of Proposition 4.12: we in fact have that

$$k \in B[0,1] \iff v_5(k) \geq 0$$
$$k \notin B\left(8, 5^{-2+\varepsilon}\right) \iff v_5(k-8) \leq 2-\varepsilon.$$

Proof of Theorem 4.16. The existence in general a p-adic analytic family of fixed finite slope $\alpha > 0$ passing through a given eigenform of slope α has been proved by Coleman in [Col97, B5.7.1] for any dimension of the eigenspace.
By Proposition 4.12, with our assumption on k_0, the dimension of the eigenspace is one and there is a unique overconvergent 5-adic cusp form of slope $\alpha^{(5)}_{[0]}(k_0)$. Note that in this case the existence of the family is easier to prove as a special case of [Col97, B5.7]; this procedure has been shown with more details in [CST98, Section II].
Denote by

$$F^{(k)}_{[0]}(q) = q + \sum_{n=2}^{\infty} a_n(k) q^n$$

the 5-adic analytic family that specializes to this unique eigenform when $k = k_0$. Note that we actually have one family for every possible slope $\alpha^{(5)}_{[0]}(k)$. Furthermore, this is the smallest possible slope among overconvergent 5-adic cuspidal eigenforms of weight k_0 by being the first (hence the smallest) slope of the Newton polygon of the series $P_{[0]}(k, T)$ by Proposition 4.9. As we have that $\alpha^{(5)}_{[0]}(k) < 3$ for every k, it follows that $k_i > \alpha^{(5)}_{[0]}(k_i) + 1$ for $i = 0, 1$.
Hence, by [Col97, B3.5], the forms obtained specializing the family to k_0 and k_1 must be classical forms of generalized weight $(k_i, [k_i]) = (k_i, [0])$, i.e. classical forms of weight k_i over $\Gamma_0(5)$ as the classical weights embed diagonally in the generalized weights. Moreover from [Col97, B3.5] also follows that there exist $M \in \mathbb{Z}$ such that

$$v_5\left(F^{(k_0)}_{[0]}(q) - F^{(k_1)}_{[0]}(q)\right) \geq v_5\left(k_0 - k_1\right) + 1 - M$$

uniformly for every coefficient $a_n(k_i)$.
Finally we can explicitly bound M by using Lemma 4.15 on the analytic functions $a_n(k)$ and taking $H < 0$, $S = \{8\}$ and $h(8) = 2$. If $-H$ is sufficiently close to zero, the region V of the lemma is contained in $X^5_{[0]}$. Hence for every $n \geq 2$ we have that

$$\begin{aligned} v_5(a_n(k_0) - a_n(k_1)) &\geq v_5(k_0 - k_1) + \min\{-H, 2 - v_p((k_0-8)(k_1-8)\} \\ &= v_5(k_0 - k_1) + 1 + \min\{0, 1 - v_p((k_0-8)(k_1-8)\}, \end{aligned}$$

where the last equality holds because k_0 and k_1 are integers. □

Remark. Following the notation of [Col97, B3.5], we hence proved that $M \leq 0$ when the 5-adic slope is 1, i.e when $v_5(k_0 - 8) = 0$. The result is optimal, in the sense that the we are able to find integers k_0 and k_1 in $4\mathbb{Z}$ such that the inequality of Theorem 4.16 is actually an equality, hence $M = 0$ in this case.
The theorem also implies that $M \leq 1$ when the 5-adic slope is 2, i.e. when $v_5(k_0 - 8) = 1$. On the other hand, numerical experiments suggest that we could also have $M = 0$ when the slope is 2. See Appendix C for the details.

Theorem 4.17. *Let $p = 7$. Let $X^7_{[0]}$ be a subset of the connected component $\mathcal{W}_{[0]}$ of the weight space associated to the class $[0]$ modulo 6; furthermore assume that $X^7_{[0]}$ is a strict neighborhood of*

$$\bar{X}^7_{[0]} = B[0,1] \setminus B\left(6, 7^{-1+\varepsilon}\right) \subset \mathcal{W}_{[0]},$$

for some $\varepsilon \in \mathbb{R}_{>0}$.
Then, for every $n \geq 2$, there is an analytic function $a_n(k)$ on $X^7_{[0]}$ such that the formal power series

$$F^{(k)}_{[0]}(q) = q + \sum_{n=2}^{\infty} a_n(k) q^n$$

specializes, for $k_0 \in \mathbb{Z}$, to the q-expansion of an overconvergent normalized 7-adic cusp form of tame level 1, generalized weight $(k_0, [k_0])$ and slope

$$\alpha^{(7)}_{[0]} = 1.$$

Furthermore $\alpha^{(7)}_{[0]}$ is the minimal possible slope among overconvergent 7-adic cusp forms of generalized weight $(k_0, [k_0])$ and $F^{(k_0)}_{[0]}(q)$ is the unique normalized overconvergent form with this slope.
Moreover, if k_0 and k_1 are in $6\mathbb{Z}$, $k_0 \geq 3$ and $k_1 \geq 3$, the forms

$$F^{(k_0)}_{[0]}(q) \quad \text{and} \quad F^{(k_1)}_{[0]}(q)$$

are two classical cuspidal eigenforms of weight k_0 and k_1 over $\Gamma_0(7)$ and we have that uniformly for $n \geq 2$

$$v_7\left(a_n(k_0) - a_n(k_1)\right) \geq v_7(k_0 - k_1) + 1.$$

Proof. The proof is essentially the same as for Theorem 4.16, but using the results of Proposition 4.13. When we specialize to $k_0 \in \mathbb{Z}$, the condition $k_0 \in X^7_{[0]}$ simply means $v_7(k-6) = 0$, hence there is only the case of slope 1 and the application of Lemma 4.15 is easier. □

Remark. Following the notation of [Col97, B3.5], we hence proved that $M \leq 0$ when the 7-adic slope is 1, i.e when $v_7(k_0 - 6) = 0$. The result is optimal, in the sense that the we are able to find integers k_0 and k_1 in $6\mathbb{Z}$ such that the inequality of Theorem 4.17 is actually an equality, hence $M = 0$. See Appendix C for the details.

4.3 Conjectural results

The results of Section 4.2 relies on the explicit construction for the U_p^* in the cases $p = 5, 7$ given in Corollary 3.9. We deformed the operator $U_p^{(0)}$ to weight k by using powers of a specific modular function e_{p-1}:

$$e_{p-1} = \frac{E_{p-1}}{V\left(E_{p-1}\right)}.$$

As we used an Eisenstein series of weight $p-1$, we can only to deform to weights lying in the connected component $\mathcal{W}_{[0]}$ associated to the class $[0]$ modulo $p-1$. To study the geometry in the other connected components, a different Eisenstein series have to be chosen.
For $p>2$, following [Col97, B1] let $E_1 = E_{(1,0)}$ be the weight 1 Eisenstein series on $X_1(p)$ defined by

$$E_1(q) = 1 + \frac{2}{L_p(0,\mathbb{1})} \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n \\ p\nmid d}} \tau^{-1}(d) \right) q^n,$$

where τ is the Teichmüller character:

$$\tau : \mathbb{Z} \longrightarrow \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\times} \longrightarrow \boldsymbol{\mu}_{p-1} \subseteq \mathbb{Z}_p.$$

Following [DS05, Section 4.8], there exist two primitive Dirichlet characters modulo p, denoted by ψ and φ, such that we can write E_1 as

$$E_1^{\psi,\varphi} = \delta(\varphi)L(0,\psi) + \delta(\psi)L(0,\phi) + 2\sum_{n=1}^{\infty} \sigma_0^{\psi,\varphi}(n) q^n$$

where

$$\begin{cases} \delta(\psi) = 1, \text{ if } \psi = \mathbb{1} \\ \delta(\psi) = 0, \text{ else} \end{cases}$$

and

$$\sigma_0^{\psi,\varphi} = \sum_{d|n} \psi\left(\frac{n}{d}\right)\varphi(d).$$

We can then take $\psi = \mathbb{1}$ and $\varphi = \tau^{-1}$ with the convention that $\varphi(d) = 0$ if $d|p$. Furthermore we have that

$$L(0,\varphi) = -B_{1,\varphi} = -\frac{1}{p}\sum_{a=0}^{p-1} \varphi(a)\cdot a$$

by [DS05, Section 4.7]. Hence we can write E_1 as

$$E_1 = 1 - \frac{2}{B_{1,\varphi}} \sum_{n=1}^{\infty} \left(\sum_{d|n}^{\infty} \varphi(d) \right) q^n.$$

We would like to deform the $U_p^{(0)}$ operator to any weight $k \in \mathbb{Z}$ by using powers of the p-adic modular function

$$e_1 = \frac{E_1}{V\left(E_1\right)}.$$

The issue is that we do not know explicitly the overconverge rate of e_1, in other words we do not have a result analogous to Proposition 3.3 valid for e_1. Nevertheless, some numerical evidence suggest that we can formulate the following conjecture.

Conjecture 4.18. *Let $p \in \{5, 7, 13\}$. For any r such that*

$$0 \leq v_p(r) < \frac{1}{p+1}$$

the function

$$e_1 = \frac{E_1}{V(E_1)}$$

is an r-overconvergent 5-adic modular function of level 1.

In, if $p \in \{5, 7, 13\}$ we can test the overconvergence rate of e_1 by looking at its unique t_p-expansion. Moreover we can also check numerically the results on classical modular forms implied by the conjecture. See Appendix C for details about this numerical evidence.
Assuming Conjecture 4.18, all the results of Chapter 3 extend with the same proofs to e_1 and its t_p-expansion, hence we can define a more general version of the operator U_p^*.

Proposition 4.19. *Assume Conjecture 4.18. Let $p \in \{5, 7, 13\}$ and let $k \in \mathbb{Z}$. Then the deformed U_p^* operator introduced in Definition 3.2 is given by the formula*

$$U_p^*(f) = U_p^{(0)}\left(e_1^k f\right)$$

for r-overconvergent functions f.

We can then study the characteristic power series $P_{[a]}(k, T)$ introduced in Section 4.1.1 for any class $[a]$ modulo $p-1$, as the bounds provided by Theorem 4.7 can now be applied to any class $[a]$.

Theorem 4.20. *Assume Conjecture 4.18. Let $p \in \{5, 7\}$, $k \in \mathbb{C}_p$ with $v_p(k) \geq 0$ and let $[a]$ be an even congruence class modulo $p-1$. Then the valuation of each coefficient $C_{[a]}(m, k)$ of the characteristic power series $P_{[a]}(k, T)$ is bounded by*

$$v_p\left(C_{[a]}(m,k)\right) \geq \left(\frac{6}{p+1} - \frac{1}{2}\right) m^2 - \left(\frac{3}{2} - \frac{6}{p+1}\right) m - 1.$$

Let $p \in \{5, 7, 13\}$, $k \in \mathbb{Z}_p$ and let $[a]$ be an even congruence class modulo $p-1$. Then the valuation of each coefficient $C_{[a]}(m, k)$ of the characteristic power series $P_{[a]}(k, T)$ is bounded by

$$v_p\left(C_{[a]}(m,k)\right) \geq \left(\frac{6}{p+1}\right) m^2 - \left(1 - \frac{6}{p+1}\right) m - 1.$$

The techniques and proofs are analogous as the one used in Sections 4.2.2 and 4.2.3 for $[a] = [0]$.

4.3.1 The smallest slope when $p = 5$ and $[a] = [2]$

In this section, assume that $k \in \mathbb{C}_5$ is in the connected component $\mathcal{W}_{[2]}$ of the weight space associated to the class [2] modulo 4. The following statements describe the smallest possible slope of the Newton polygon of $P_{[2]}(k, T)$.

Proposition 4.21. *Assume Conjecture 4.18. Let $k \in \mathbb{C}_5$ with $v_5(k) \geq 0$ and furthermore assume that $v_5((k-10)(k-14)) < 2$. Let $C_{[2]}(1,k)$ be the first coefficient of the characteristic power series $P_{[2]}(k,T)$ of U_5^* restricted to cusp forms. Then*

$$v_5\left(C_{[2]}(1,k)\right) = 2 + v_5\left((k-10)(k-14)\right) < 4.$$

Proof. Using Koike's formula (4.1), we obtain the following congruence:

$$C_{[2]}(1,k) \equiv 2 \cdot 5^2 k(1+k) \mod 5^3.$$

This implies that $v_5\left(C_{[2]}(1,k)\right) = 2$ if $v_5(k(k-4)) < 1$.
To gather more information in the cases $k \equiv 0$ and $k \equiv 4 \mod 5$ we need to compute $C_{[2]}(1,k)$ at higher 5-adic precision:

$$v_5\left(C_{[2]}(1,k)\right) \equiv 5^2(17k + 7k^2 + 3 \cdot 5k^3) \mod 5^4.$$

By inspection, assuming that $k \equiv 0, 4 \mod 5$ we see that

$$\begin{cases} v_5\left(C_{[2]}(1,k)\right) \geq 4, & \text{if } k \equiv 10, 14 \mod 5^2 \\ v_5\left(C_{[2]}(1,k)\right) = 3, & \text{if } k \equiv 0,4,5,9,15,19,20,24 \mod 5^2. \end{cases}$$

Note now that if $v_5(k-10) > 0$ then $v_5(k-14) = 0$ and vice versa. Hence if we assume that

$$v_5(k-10) + v_5(k-14) = v_5((k-10)(k-14)) < 2,$$

we can write

$$v_5\left(C_{[2]}(1,k) v_5\left(C_{[2]}(1,k)\right) = 2 + v_5\left((k-10)(k-14)\right)\right).$$

□

If we assume Conjecture 4.18, we can specialize the bound of Theorem 4.20 to $p = 5$, we get that for every $m \geq 1$.

$$v_5\left(C_{[2]}(m,k)\right) \geq \frac{1}{2}m^2 - \frac{1}{2}m - 1.$$

To prove that the length of the first segment of the Newton polygon of $P_{[2]}(k,T)$ is 1, is equivalent to prove that for every $m > 1$

$$\frac{v_5\left(C_{[2]}(m,k)\right)}{m} > v_5\left(C_{[2]}(1,k)\right).$$

By using Proposition 4.21, we can assume that $v_5\left(C_{[2]}(1,k)\right) < 4$ whenever $v_5((k-10)(k-14)) < 2$; the problem then amounts to solve the following inequality:

$$v_5\left(C_{[2]}(m,k)\right) \geq \frac{1}{2}m^2 - \frac{1}{2}m - 1 \geq 4m,$$

that is $m > 9$.
The missing cases are $m = 2,3,4,5,6,7,8,9$, and we can compute them with Koike's formula.

The case $m = 2$

Using Koike's formula we can compute $C_{[2]}(2,k)$:

$$C_{[2]}(2,k) \equiv 5^6(4\cdot 5 + 3k + 23k^2 + 3\cdot 5k^3 + 3\cdot 5k^4 + 2k^5 + 17k^6 + 5k^7) \mod 5^8.$$

By inspection, we see that

$$\begin{cases} v_5\left(C_{[2]}(2,k)\right) \geq 8, & \text{if } k \equiv 0,4 \mod 5 \\ v_5\left(C_{[2]}(2,k)\right) = 7, & \text{otherwise.} \end{cases}$$

It follows that the inequality $v_5\left(C_{[2]}(2,k)\right) \geq 8 > 2\cdot v_5\left(C_{[2]}(1,k)\right)$ holds true if $k \equiv 0,4 \mod 5$.
On the other hand, if $k \equiv 1,2,3 \mod 5$ we have that $v_5((k-10)(k-14)) = 0$ hence $v_5\left(C_{[2]}(1,k)\right) = 2$ by Proposition 4.21 and the required inequality holds also in this case.

The case $m = 3$

Using Koike's formula we can compute:

$$v_5\left(C_{[2]}(3,k)\right) \equiv 5^{11}k(3k^{10}+4k^9+k^8+4k^6+2k^5+3k^4+3k^2+4k+1) \mod 5^{12}.$$

By inspection we note that

$$v_5(3k^{10}+4k^9+k^8+4k^6+2k^5+3k^4+3k^2+4k+1) \geq 1 \text{ if } k \equiv 1,2,3,4 \mod 5.$$

On the other hand, if $k \equiv 0 \mod 5$ we have that $v_5(k) \geq 1$, hence for all k such that $v_5(k) \geq 0$ we have that

$$v_5\left(C_{[2]}(3,k)\right) \geq 12 > 3\cdot C_{[2]}(1,k)$$

as required.

The cases $m = 4,5,\ldots,9$

Using Koike's formula as above, we get that for each k such that $v_5(k) \geq 0$:

$$\begin{aligned} C_{[2]}(4,k) &\equiv 0 \mod 5^{16}, \\ C_{[2]}(5,k) &\equiv 0 \mod 5^{20}, \\ C_{[2]}(6,k) &\equiv 0 \mod 5^{24}, \\ C_{[2]}(7,k) &\equiv 0 \mod 5^{28}, \\ C_{[2]}(8,k) &\equiv 0 \mod 5^{32}, \\ C_{[2]}(9,k) &\equiv 0 \mod 5^{36}. \end{aligned}$$

This implies that, if $m \in \{4,5,6,7,8,9\}$, then $v_5\left(C_{[2]}(m,k)\right) \geq 4m$ and the inequality is verified for all m.
We summarize this discussion with the following statement.

Proposition 4.22. *Assume Conjecture 4.18. Let $k \in \mathbb{C}_5$, $v_5(k) \geq 0$ and assume that $k \in \mathcal{W}_{[2]}$. Furthermore assume that $v_5((k-10)(k-14)) < 2$. Then the first segment of the Newton polygon of the characteristic power series of the U_5^* operator restricted to cusp forms has length one. Furthermore, the slope of this segment is*

$$\alpha_{[2]}^{(5)}(k) = 2 + v_5((k-10)(k-14)).$$

Remark. The result is optimal, in the sense that the computations above imply that if

$$v_5(k-10) \geq 2 \quad \text{or} \quad v_5(k-14) \geq 2$$

then the minimal slope is at least 4 and the dimension of the cuspidal space is strictly bigger then one.

4.3.2 The smallest slope when $p = 7$ and $[a] = [2]$

In this section, assume that $k \in \mathbb{C}_7$ is in the connected component $\mathcal{W}_{[2]}$ of the weight space associated to the class [2] modulo 6. The following statements describe the smallest possible slope of the Newton polygon of $P_{[2]}(k, T)$.

Proposition 4.23. *Assume Conjecture 4.18. Let $k \in \mathbb{C}_7$ with $v_7(k) \geq 0$; assume that $v_7(k-1) < 1$ and that $v_7(k-14) < 2$. Let $C_{[2]}(1,k)$ be the first coefficient of the characteristic power series $P_{[2]}(k,T)$ of U_7^* restricted to cusp forms. Then*

$$v_7\left(C_{[2]}(1,k)\right) = 2 + v_7\left((k-8)(k-14)\right) < 4.$$

Proof. Using Koike's formula (4.1) we obtain the following:

$$C_{[2]} = 7^2 k(k-1) \mod 7^3.$$

Hence

$$v_7\left(C_{[2]}(1,k) = 2 + v_7(k(k-8))\right)$$

Provided that $v_7(k) < 1$ and $v_7(k-8) < 1$. To refine the result when $v_7(k) \geq 1$ we now compute

$$C_{[2]} = 7^2(2 \cdot 7 + 13k + 43k^2 + 28k^3) \mod 7^4.$$

Assuming that $v_7(k) \geq 1$, by inspection we observe that

$$\begin{cases} v_7\left(C_{[2]}(1,k)\right) \geq 4, & \text{if } k \equiv 14 \mod 7^2 \\ v_7\left(C_{[2]}(1,k)\right) = 3, & \text{if } k \equiv 0, 7, 21, 28, 35, 42 \mod 7^2. \end{cases}$$

This implies that

$$v_7\left(C_{[2]}\right) = 2 + v_7\left((k-8)(k-14)\right),$$

provided that $v_7(k-8) < 1$ and $v_7(k-14) < 2$. □

If we assume Conjecture 4.18, we can specialize the bound of Theorem 4.20 to $p = 7$, we get that for every $m \geq 1$.

$$v_7\left(C_{[2]}(m,k)\right) \geq \frac{1}{4}m^2 - \frac{3}{4}m - 1.$$

To prove that the length of the first segment of the Newton polygon of $P_{[2]}(k,T)$ is 1, is equivalent to prove that for every $m > 1$

$$\frac{v_7\left(C_{[2]}(m,k)\right)}{m} > v_7\left(C_{[2]}(1,k)\right).$$

By using Proposition 4.23, we can assume that $v_7\left(C_{[2]}(1,k)\right) < 4$ whenever $v_7(k-8) < 1$ and $v_7(k-14) < 2$; the problem then amounts to solve the following inequality:

$$v_7\left(C_{[2]}(m,k)\right) \geq \frac{1}{4}m^2 - \frac{3}{4}m - 1 \geq 4m,$$

that is $m > 19$.
The missing cases are $m = 2, 3, \ldots, 19$, and we can compute them with Koike's formula.

The case $m = 2$

Using Koike's formula we can compute $C_{[2]}(2,k)$:

$$C_{[2]}(2,k) \equiv 7^5(7^2+3^3\cdot 7k+58k^2+309k^3+222k^4+6k^5+33\cdot 7k^6+7^2k^7) \mod 7^8.$$

By inspection we see that

$$\begin{cases} v_7\left(C_{[2]}(2,k)\right) \geq 8, & \text{if } k \equiv 14 \mod 7^2 \\ v_7\left(C_{[2]}(2,k)\right) = 7, & \text{if } k \equiv 0,7,21,28,35,42 \mod 7^2 \\ v_7\left(C_{[2]}(2,k)\right) \geq 6, & \text{if } k \equiv 1 \mod 7 \\ v_7\left(C_{[2]}(2,k)\right) \geq 5, & \text{otherwise.} \end{cases}$$

Note that all the possible k modulo 7^3 have been checked, and the results are grouped in the formula above.
It follows from Proposition 4.23 that

$$v_7\left(C_{[2]}(2,k) > 2 \cdot v_7\left(C_{[2]}(1,k)\right)\right)$$

for every k, provided that $v_7(k-14) < 2$ and $v_7(k-8) < 1$.

The case $m = 3$

Using Koike's formula we can compute $C_{[2]}(3,k)$:

$$\begin{gathered} C_{[2]}(3,k) \equiv 7^9k(2\cdot 7^2 + 13\cdot 7k + 251k^2 + 186k^3 + 237k^4 + 19\cdot 7k^5 + 102k^6 \\ +202k^7 + 260k^8 + 34\cdot 7k^9 + 3\cdot 7^2k^{10}) \mod 7^{12}. \end{gathered}$$

By inspection we see that

$$\begin{cases} v_7\left(C_{[2]}(2,k)\right) \geq 12, & \text{if } k \equiv 0 \mod 7 \\ v_7\left(C_{[2]}(2,k)\right) \geq 9, & \text{otherwise.} \end{cases}$$

Note that all the possible k modulo 7^3 have been checked, and the results are grouped in the formula above. It follows from Proposition 4.23 that

$$v_7\left(C_{[2]}(3,k)\right) > 3 \cdot v_7\left(C_{[2]}(1,k)\right)$$

for every k, provided that $v_7(k-14) < 2$ and $v_7(k-8) < 1$.

The case $m = 4$

Using Koike's formula we can compute $C_{[2]}(4,k)$:

$$\begin{aligned} C_{[2]}(4,k) \equiv 7^{15}k^3(1{+}3k + k^2 + 3k^4 + k^5 + 5k^6 + 4k^7 \\ +6k^8 + 4k^{10} + 6k^{11} + k^{12}) \mod 7^{16}. \end{aligned}$$

We straightforward inspection, we note that

$$v_7\left(C_{[2]}(4k)\right) \geq 16$$

for every k such that $v_7(k) \geq 1$, hence by Proposition 4.23 we have that

$$v_7\left(C_{[2]}(4,k) > 4 \cdot v_7\left(C_{[2]}(1,k)\right)\right)$$

for every k, provided that $v_7(k-14) < 2$ and $v_7(k-8) < 1$.

The cases $m = 5, 6, \ldots, 19$

Using Koike's formula as above, we get that for each k such that $v_7(k) \geq 0$:

$$\begin{aligned}
C_{[2]}(5,k) &\equiv 0 \mod 7^{20},\\
C_{[2]}(6,k) &\equiv 0 \mod 7^{24},\\
C_{[2]}(7,k) &\equiv 0 \mod 7^{28},\\
C_{[2]}(8,k) &\equiv 0 \mod 7^{32},\\
C_{[2]}(9,k) &\equiv 0 \mod 7^{36},\\
C_{[2]}(10,k) &\equiv 0 \mod 7^{40},\\
C_{[2]}(11,k) &\equiv 0 \mod 7^{44},\\
C_{[2]}(12,k) &\equiv 0 \mod 7^{48},\\
C_{[2]}(13,k) &\equiv 0 \mod 7^{52},\\
C_{[2]}(14,k) &\equiv 0 \mod 7^{56},\\
C_{[2]}(15,k) &\equiv 0 \mod 7^{60},\\
C_{[2]}(16,k) &\equiv 0 \mod 7^{64},\\
C_{[2]}(17,k) &\equiv 0 \mod 7^{68},\\
C_{[2]}(18,k) &\equiv 0 \mod 7^{72},\\
C_{[2]}(19,k) &\equiv 0 \mod 7^{76}.
\end{aligned}$$

Note that the calculation of $C_{[2]}(m,k)$ for the last few m is fairly intensive from a computational point of view and took some days on a robust desktop machine.

This implies that, if $m \in \{5, 6, \ldots, 19\}$, then

$$v_7\left(C_{[2]}(m,k)\right) \geq 4m > m \cdot v_7\left(C_{[2]}(1,k)\right)$$

and the inequality is verified for all m.
We summarize this discussion with the following statement.

Proposition 4.24. *Assume Conjecture 4.18. Let $k \in \mathbb{C}_7$, $v_7(k) \geq 0$ and assume that $k \in \mathcal{W}_{[2]}$. Furthermore assume that $v_7(k-8) < 1$ and that $v_7(k-14) < 2$. Then the first segment of the Newton polygon of the characteristic power series of the U_7^* restricted to cusp forms operator has length one. Furthermore, the slope of this segment is*

$$\alpha_{[2]}^{(7)}(k) = 2 + v_7((k-8)(k-14)).$$

Remark. The result is optimal, in the sense that the computations above imply that if $v_7(k-8) \geq 1$ or $v_7(k-14) \geq 2$ then the minimal slope is respectively at least 3 or at least 4 and the dimension of the cuspidal space is strictly bigger then one.

4.3.3 The smallest slope when $p = 7$ and $[a] = [4]$

In this section, assume that $k \in \mathbb{C}_7$ is in the connected component $\mathcal{W}_{[4]}$ of the weight space associated to the class [24 modulo 6. The following statements describe the smallest possible slope of the Newton polygon of $P_{[4]}(k,T)$.

Proposition 4.25. *Assume Conjecture 4.18. Let $k \in \mathbb{C}_7$ with $v_7(k) \geq 0$; assume that $v_7(k-10) < 2$. Let $C_{[4]}(1,k)$ be the first coefficient of the characteristic power series $P_{[4]}(k,T)$ of U_7^* restricted to cusp forms. Then*

$$v_7\left(C_{[4]}(1,k)\right) = 1 + v_7(k-10) < 3.$$

Proof. Using Koike's formula (4.1) we obtain the following:

$$C_{[4]} = 7(k-3) \mod 7^2.$$

Hence

$$v_7\left(C_{[4]}(1,k) = 1 + v_7(k-3)\right)$$

Provided that $v_7(k-3) < 1$. To refine the result when $v_7(k-3) \geq 1$ we now compute

$$C_{[4]} = 7(4 + 8k + 7k^2) \mod 7^3.$$

Assuming that $v_7(k-3) \geq 1$, by inspection we observe that

$$\begin{cases} v_7\left(C_{[4]}(1,k)\right) \geq 3, & \text{if } k \equiv 10 \mod 7^2 \\ v_7\left(C_{[4]}(1,k)\right) = 3, & \text{if } k \equiv 0, 7, 21, 28, 35, 42 \mod 7^2. \end{cases}$$

This implies that

$$v_7\left(C_{[2]}\right) = 1 + v_7(k-10),$$

provided that $v_7(k-10) < 2$. ∎

If we assume Conjecture 4.18, we can specialize the bound of Theorem 4.20 to $p = 7$, we get that for every $m \geq 1$.

$$v_7\left(C_{[4]}(m,k)\right) \geq \frac{1}{4}m^2 - \frac{3}{4}m - 1.$$

To prove that the length of the first segment of the Newton polygon of $P_{[4]}(k,T)$ is 1, is equivalent to prove that for every $m > 1$

$$\frac{v_7\left(C_{[4]}(m,k)\right)}{m} > v_7\left(C_{[4]}(1,k)\right).$$

By using Proposition 4.25, we can assume that $v_7\left(C_{[2]}(1,k)\right) < 3$ whenever $v_7(k-10) < 2$; the problem then amounts to solve the following inequality:

$$v_7\left(C_{[4]}(m,k)\right) \geq \frac{1}{4}m^2 - \frac{3}{4}m - 1 \geq 3m,$$

that is $m > 15$.
The missing cases are $m = 2, 3, \ldots, 15$, and we can compute them with Koike's formula.

The case $m = 2$

Using Koike's formula we can compute $C_{[4]}(2,k)$:

$$C_{[4]}(2,k) \equiv 7^4(39 + 24k + 8k^2 + 12k^3 + k^4) \mod 7^6.$$

By inspection we see that

$$\begin{cases} v_7\left(C_{[4]}(2,k)\right) \geq 6, & \text{if } k \equiv 3 \mod 7 \\ v_7\left(C_{[4]}(2,k)\right) \geq 4, & \text{otherwise.} \end{cases}$$

Note that all the possible k modulo 7^2 have been checked, and the results are grouped in the formula above. It follows from Proposition 4.25 that

$$v_7\left(C_{[2]}(2,k)\right) > 2 \cdot v_7\left(C_{[2]}(1,k)\right)$$

for every k, provided that $v_7(k-10) < 2$.

The case $m = 3$

Using Koike's formula we can compute $C_{[4]}(3,k)$:

$$C_{[4]}(3,k) \equiv 7^8 k(2 + 4k + 3k^2 + 3k^4 + k^5 + 6k^6 + 2k^7) \mod 7^9.$$

By inspection we see that

$$\begin{cases} v_7\left(C_{[4]}(3,k)\right) \geq 9, & \text{if } k \equiv 3 \mod 7 \\ v_7\left(C_{[4]}(3,k)\right) \geq 8, & \text{otherwise.} \end{cases}$$

It follows from Proposition 4.25 that

$$v_7\left(C_{[4]}(3,k)\right) > 3 \cdot v_7\left(C_{[4]}(1,k)\right)$$

for every k, provided that $v_7(k-10) < 2$.

The cases $m = 4, 5, , \ldots, 15$

Using Koike's formula as above, we get that for each k such that $v_7(k) \geq 0$:

$$\begin{aligned} C_{[4]}(4, k) &\equiv 0 \mod 7^{12}, \\ C_{[4]}(5, k) &\equiv 0 \mod 7^{15}, \\ C_{[4]}(6, k) &\equiv 0 \mod 7^{18}, \\ C_{[4]}(7, k) &\equiv 0 \mod 7^{21}, \\ C_{[4]}(8, k) &\equiv 0 \mod 7^{24}, \\ C_{[4]}(9, k) &\equiv 0 \mod 7^{27}, \\ C_{[4]}(10, k) &\equiv 0 \mod 7^{30}, \\ C_{[4]}(11, k) &\equiv 0 \mod 7^{33}, \\ C_{[4]}(12, k) &\equiv 0 \mod 7^{36}, \\ C_{[4]}(13, k) &\equiv 0 \mod 7^{39}, \\ C_{[4]}(14, k) &\equiv 0 \mod 7^{42}, \\ C_{[4]}(15, k) &\equiv 0 \mod 7^{45}. \end{aligned}$$

This implies that, if $m \in \{4, 5, \ldots, 15\}$, then

$$v_7\left(C_{[4]}(m, k)\right) \geq 3m > m \cdot v_7\left(C_{[4]}(1, k)\right)$$

and the inequality is verified for all m.
We summarize this discussion with the following statement.

Proposition 4.26. *Assume Conjecture 4.18. Let $k \in \mathbb{C}_7$, $v_7(k) \geq 0$ and assume that $k \in \mathcal{W}_{[4]}$. Furthermore assume that $v_7(k - 10) < 2$. Then the first segment of the Newton polygon of the characteristic power series of the U_7^* restricted to cusp forms operator has length one. Furthermore, the slope of this segment is*

$$\alpha_{[4]}^{(7)}(k) = 1 + v_7(k - 10).$$

Remark. The result is optimal, in the sense that the computations above imply that if $v_7(k - 10) \geq 2$ then the minimal slope is at least 3 and the dimension of the cuspidal space is strictly bigger then one.

4.3.4 p-adic analytic families and congruences when $[a] \neq [0]$

In Sections 4.3.1, 4.3.2 and 4.3.3 we proved under what conditions on the weight k, for the primes $p = 5, 7$, the first segment of the Newton polygon of the characteristic power series $P_{[a]}(k, T)$ has length one. Using Propositions 4.3 and 4.1, we can rephrase the results with the following statements.

Proposition 4.27. *Assume Conjecture 4.18. Let $p = 5$. Let $k \in \mathbb{C}_5$, $v_5(k) \geq 0$ and assume that k lies in the the connected component $\mathcal{W}_{[2]}$ of the weight space associated to the class $[2]$ modulo 4. Furthermore assume that $v_5((k - 10)(k - 14)) < 2$. Then, there is a unique root λ^{-1} of $P_{[2]}(k, T)$ and such that*

$$v_5(\lambda) = \alpha_{[2]}^{(5)}(k) = 2 + v_5((k - 10)(k - 14)).$$

Furthermore λ is an eigenvalue of the operator U_5^* and the dimension of the eigenspace of cusp forms corresponding to λ is 1.

Proposition 4.28. *Assume Conjecture 4.18. Let $p = 7$. Let $k \in \mathbb{C}_7$, $v_7(k) \geq 0$ and assume that k lies in the the connected component $\mathcal{W}_{[2]}$ of the weight space associated to the class* [2] *modulo* 6*. Furthermore assume that $v_7(k-8) < 1$ and that $v_7(k-14) < 2$. Then, there is a unique root λ^{-1} of $P_{[2]}(k,T)$ and such that*

$$v_7(\lambda) = \alpha_{[2]}^{(7)}(k) = 2 + v_7((k-8)(k-14)).$$

Furthermore λ is an eigenvalue of the operator U_7^* and the dimension of the eigenspace of cusp forms corresponding to λ is 1.

Proposition 4.29. *Assume Conjecture 4.18. Let $p = 7$. Let $k \in \mathbb{C}_7$, $v_7(k) \geq 0$ and assume that k lies in the the connected component $\mathcal{W}_{[4]}$ of the weight space associated to the class* [4] *modulo* 6*. Furthermore assume that $v_7(k-10) < 2$. Then, there is a unique root λ^{-1} of $P_{[4]}(k,T)$ and such that*

$$v_7(\lambda) = \alpha_{[4]}^{(7)}(k) = 1 + v_7(k-10).$$

Furthermore λ is an eigenvalue of the operator U_7^* and the dimension of the eigenspace of cusp forms corresponding to λ is 1.

We now show results analogous to those of Section 4.2.4 for the other classes $[a]$ modulo $p-1$.

Theorem 4.30. *Assume Conjecture 4.18. Let $p = 5$ and let $X_{[2]}^5$ be a subset of the connected component $\mathcal{W}_{[2]}$ of the weight space associated to the class* [2] *modulo* 4*; furthermore assume that $X_{[2]}^5$ is a strict neighborhood of*

$$\bar{X}_{[2]}^5 = B[0,1] \setminus \left(B\left(10, 5^{-2+\varepsilon_1}\right) \cup B\left(14, 5^{-2+\varepsilon_2}\right)\right) \subset \mathcal{W}_{[2]},$$

for some $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_{>0}$.
Then, for every $n \geq 2$, there is an analytic function $a_n(k)$ on $X_{[2]}^5$ such that the formal power series

$$F_{[2]}^{(k)}(q) = q + \sum_{n=2}^{\infty} a_n(k) q^n$$

specializes, for $k_0 \in \mathbb{Z}$, to the q-expansion of an overconvergent normalized 5*-adic cuspidal eigenform form of tame level* 1*, generalized weight $(k_0, [k_0])$ and slope*

$$\alpha_{[2]}^{(5)}(k_0) = 2 + v_5((k_0-10)(k_0-14)).$$

Furthermore $\alpha_{[2]}^{(5)}(k_0)$ is the minimal possible slope among overconvergent 5*-adic cuspidal eigenforms of generalized weight $(k_0, [k_0])$ and $F_{[2]}^{(k_0)}(q)$ is the unique normalized overconvergent form with this slope.*
Moreover, if k_0 and k_1 are in $\mathbb{Z}$, $k_0 \equiv 2 \mod 4$, $k_1 \equiv 2 \mod 4$, $k_0 \geq 4$ and $k_1 \geq 4$, the forms

$$F_{[2]}^{(k_0)}(q) \quad \text{and} \quad F_{[2]}^{(k_1)}(q)$$

are two classical cuspidal eigenforms of weight k_0 and k_1 over $\Gamma_0(5)$ and we have that uniformly for $n \geq 2$

$$\begin{aligned} v_5\left(a_n(k_0) - a_n(k_1)\right) &\geq v_5(k_0 - k_1) + 1 \\ &+ \min\{0, 1 - v_5\left((k_0 - 10)(k_1 - 10)\right), 1 - v_5\left((k_0 - 14)(k_1 - 14)\right)\}. \end{aligned}$$

Proof. The proof is essentially the same as for Theorem 4.16, but using the results of Proposition 4.27. In order to apply Lemma 4.15, we take $H < 0$, $S = \{10, 14\}$ and $h(10) = h(14) = 2$. □

Remark. Following the notation of [Col97, B3.5], we hence proved that $M \leq 0$ when the 5-adic slope is 2, i.e when $v_5((k_0 - 10)(k_0 - 14)) = 0$. The result is optimal, in the sense that the we are able to find integers k_0 and k_1 in $2 + 4\mathbb{Z}$ such that the inequality of Theorem 4.30 is actually an equality, hence $M = 0$. The theorem also implies that $M \leq 1$ when the 5-adic slope is 3, i.e. when $v_5((k_0 - 10)(k_0 - 14)) = 1$. On the other hand, numerical experiments suggest that we could also have $M = 0$ when the slope is 3. See Appendix C for the details.

Theorem 4.31. *Assume Conjecture 4.18. Let $p = 7$ and let $X^7_{[2]}$ be a subset of the connected component $\mathcal{W}_{[2]}$ of the weight space associated to the class* [2] *modulo* 6*; furthermore assume that $X^7_{[2]}$ is a strict neighborhood of*

$$\bar{X}^7_{[2]} = B[0, 1] \setminus \left(B\left(8, 7^{-1+\varepsilon_1}\right) \cup B\left(14, 7^{-2+\varepsilon_2}\right)\right) \subset \mathcal{W}_{[2]},$$

for some $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_{>0}$.
Then, for every $n \geq 2$, there is an analytic function $a_n(k)$ on $X^7_{[2]}$ such that the formal power series

$$F^{(k)}_{[2]}(q) = q + \sum_{n=2}^{\infty} a_n(k) q^n$$

specializes, for $k_0 \in \mathbb{Z}$, to the q-expansion of an overconvergent normalized 7*-adic cuspidal eigenform form of tame level* 1*, generalized weight $(k_0, [k_0])$ and slope*

$$\alpha^{(7)}_{[2]}(k_0) = 2 + v_7((k_0 - 8)(k_0 - 14)).$$

Furthermore $\alpha^{(7)}_{[2]}(k_0)$ is the minimal possible slope among overconvergent 7*-adic cuspidal eigenforms of generalized weight $(k_0, [k_0])$ and $F^{(k_0)}_{[2]}(q)$ is the unique normalized overconvergent form with this slope.*
Moreover, if k_0 and k_1 are in $\mathbb{Z}$, $k_0 \equiv 2 \mod 6$, $k_1 \equiv 2 \mod 6$, $k_0 \geq 5$ and $k_1 \geq 5$, the forms

$$F^{(k_0)}_{[2]}(q) \quad \textit{and} \quad F^{(k_1)}_{[2]}(q)$$

are two classical cuspidal eigenforms of weight k_0 and k_1 over $\Gamma_0(7)$ and we have that uniformly for $n \geq 2$

$$v_7\left(a_n(k_0) - a_n(k_1)\right) \geq v_7(k_0 - k_1) + 1 + \min\{0, 1 - v_7\left((k_0 - 14)(k_1 - 14)\right)\}.$$

Proof. The proof is essentially the same as for Theorem 4.16, but using the results of Proposition 4.28. In order to apply Lemma 4.15, we take $H < 0$, $S = \{14\}$ and $h(14) = 2$, as the condition $k_0 \in X^7_{[2]}$ implies $v_7(k_0 - 8) = 0$ when we specialize to $k_0 \in \mathbb{Z}$. □

Remark. Following the notation of [Col97, B3.5], we hence proved that $M \leq 0$ when the 7-adic slope is 2, i.e when $v_7((k_0-8)(k_0-14)) = 0$. The result is optimal, in the sense that the we are able to find integers k_0 and k_1 in $2+6\mathbb{Z}$ such that the inequality of Theorem 4.31 is actually an equality, hence $M = 0$. The theorem also implies that $M \leq 1$ when the 7-adic slope is 3, i.e. when $v_7(k_0-14) = 1$. On the other hand, numerical experiments suggest that we could also have $M = 0$ when the slope is 3. See appendix C for the details.

Theorem 4.32. *Assume Conjecture 4.18. Let $p = 7$ and let $X^7_{[4]}$ be a subset of the connected component $\mathcal{W}_{[4]}$ of the weight space associated to the class $[4]$ modulo 6; furthermore assume that $X^7_{[4]}$ is a strict neighborhood of*

$$\bar{X}^7_{[4]} = B[0,1] \setminus B\left(10, 7^{-2+\varepsilon}\right) \subset \mathcal{W}_{[4]},$$

for some $\varepsilon \in \mathbb{R}_{>0}$.
Then, for every $n \geq 2$, there is an analytic function $a_n(k)$ on $X^7_{[4]}$ such that the formal power series

$$F^{(k)}_{[4]}(q) = q + \sum_{n=2}^{\infty} a_n(k) q^n$$

specializes, for $k_0 \in \mathbb{Z}$, to the q-expansion of an overconvergent normalized 7-adic cuspidal eigenform form of tame level 1, generalized weight $(k_0, [k_0])$ and slope

$$\alpha^{(7)}_{[4]}(k_0) = 1 + v_7(k_0 - 10).$$

Furthermore $\alpha^{(7)}_{[4]}(k_0)$ is the minimal possible slope among overconvergent 7-adic cuspidal eigenforms of generalized weight $(k_0, [k_0])$ and $F^{(k_0)}_{[4]}(q)$ is the unique normalized overconvergent form with this slope.
Moreover, if k_0 and k_1 are in $\mathbb{Z}$, $k_0 \equiv 4 \mod 6$, $k_1 \equiv 4 \mod 6$, $k_0 \geq 4$ and $k_1 \geq 4$, the forms

$$F^{(k_0)}_{[4]}(q) \quad \textit{and} \quad F^{(k_1)}_{[4]}(q)$$

are two classical cuspidal eigenforms of weight k_0 and k_1 over $\Gamma_0(7)$ and we have that uniformly for $n \geq 2$

$$v_7\left(a_n(k_0) - a_n(k_1)\right) \geq v_7(k_0 - k_1) + 1 + \min\{0, 1 - v_7\left((k_0 - 10)(k_1 - 10)\right)\}.$$

Proof. The proof is essentially the same as for Theorem 4.16, but using the results of Proposition 4.29. In order to apply Lemma 4.15, we take $H < 0$, $S = \{10\}$ and $h(10) = 2$. □

Remark. Following the notation of [Col97, B3.5], we hence proved that $M \leq 0$ when the 7-adic slope is 1, i.e when $v_7(k_0 - 10) = 0$. The result is optimal, in the sense that the we are able to find integers k_0 and k_1 in $4+6\mathbb{Z}$ such that the inequality of Theorem 4.32 is actually an equality, hence $M = 0$. The theorem also implies that $M \leq 1$ when the 7-adic slope is 2, i.e. when $v_7(k_0 - 10) = 1$. On the other hand, numerical experiments suggest that we could also have $M = 0$ when the slope is 2. See Appendix C for the details.

Appendix A

Coefficients of the modular equations

Here we display for the primes $p = 2, 3, 5, 7, 13$ the polynomials $a_i^{(p)}(y)$, computed as described in subsection 2.1.

$$a_1^{(2)}(y) = -2^{12}y^2 - 2^4 \cdot 3y.$$

$$\begin{aligned}
a_1^{(3)}(y) &= -3^{12}y^3 - 4 \cdot 3^8 y^2 - 10 \cdot 3^3 y, \\
a_2^{(3)}(y) &= -3^6 y^2 - 4 \cdot 3^2 y.
\end{aligned}$$

$$\begin{aligned}
a_1^{(5)}(y) &= -5^{12}y^5 - 6 \cdot 5^{10}y^4 - 63 \cdot 5^7 y^3 - 52 \cdot 5^5 y^2 - 63 \cdot 5^2 y, \\
a_2^{(5)}(y) &= -5^9 y^4 - 6 \cdot 5^7 y^3 - 63 \cdot 5^4 y^2 - 52 \cdot 5^2 y, \\
a_3^{(5)}(y) &= -5^6 y^3 - 6 \cdot 5^4 y^2 - 63 \cdot 5y, \\
a_4^{(5)}(y) &= -5^3 y^2 - 6 \cdot 5y.
\end{aligned}$$

$$\begin{aligned}
a_1^{(7)}(y) = &- 7^{12}y^7 - 4 \cdot 7^{11}y^6 - 46 \cdot 7^9 y^5 - 272 \cdot 7^7 y^4 - 845 \cdot 7^5 y^3 \\
&- 176 \cdot 7^4 y^2 - 82 \cdot 7^2 y, \\
a_2^{(7)}(y) = &- 7^{10}y^6 - 4 \cdot 7^9 y^5 - 46 \cdot 7^7 y^4 - 272 \cdot 7^5 y^3 - 845 \cdot 7^3 y^2 - 176 \cdot 7^2 y, \\
a_3^{(7)}(y) = &- 7^8 y^5 - 4 \cdot 7^7 y^4 - 46 \cdot 7^5 y^3 - 272 \cdot 7^3 y^2 - 845 \cdot 7y, \\
a_4^{(7)}(y) = &- 7^6 y^4 - 4 \cdot 7^5 y^3 - 46 \cdot 7^3 y^2 - 272 \cdot 7y, \\
a_5^{(7)}(y) = &- 7^4 y^3 - 4 \cdot 7^3 y^2 - 46 \cdot 7y, \\
a_6^{(7)}(y) = &- 7^2 y^2 - 4 \cdot 7y.
\end{aligned}$$

$$
\begin{aligned}
a_1^{(13)} =& -13^{12}y^{13}-2\cdot 13^{12}y^{12}-25\cdot 13^{11}y^{11}-196\cdot 13^{10}y^{10}-1064\cdot 13^{9}y^{9}\\
&-4180\cdot 13^{8}y^{8}-12086\cdot 13^{7}y^{7}-25660\cdot 13^{6}y^{6}-3014\cdot 13^{6}y^{5}\\
&-41140\cdot 13^{4}y^{4}-27272\cdot 13^{3}y^{3}-9604\cdot 13^{2}y^{2}-1165\cdot 13y,\\
a_2^{(13)} =& -13^{11}y^{12}-2\cdot 13^{11}y^{11}-25\cdot 13^{10}y^{10}-196\cdot 13^{9}y^{9}-1064\cdot 13^{8}y^{8}\\
&-4180\cdot 13^{7}y^{7}-12086\cdot 13^{6}y^{6}-25660\cdot 13^{5}y^{5}-3014\cdot 13^{5}y^{4}\\
&-41140\cdot 13^{3}y^{3}-27272\cdot 13^{2}y^{2}-9604\cdot 13y,\\
a_3^{(13)} =& -13^{10}y^{11}-2\cdot 13^{10}y^{10}-5^{2}13^{9}y^{9}-2^{2}7^{2}13^{8}y^{8}-2^{3}7\cdot 13^{7}19y^{7}\\
&-2^{2}5\cdot 11\cdot 13^{6}19y^{6}-2\cdot 13^{5}6043y^{5}-2^{2}5\cdot 13^{4}1283y^{4}-2\cdot 11\cdot 13^{4}137y^{3}\\
&-2^{2}5\cdot 11^{2}13^{2}17y^{2}-2^{3}7\cdot 13\cdot 487y\\
a_4^{(13)} =& -13^{9}y^{10}-2\cdot 13^{9}y^{9}-5^{2}13^{8}y^{8}-2^{2}7^{2}13^{7}y^{7}-2^{3}7\cdot 13^{6}19y^{6}\\
&-2^{2}5\cdot 11\cdot 13^{5}19y^{5}-2\cdot 13^{4}6043y^{4}-2^{2}5\cdot 13^{3}1283y^{3}-2\cdot 11\cdot 13^{3}137y^{2}\\
&-2^{2}5\cdot 11^{2}13\cdot 17y,\\
a_5^{(13)} =& -13^{8}y^{9}-2\cdot 13^{8}y^{8}-5^{2}13^{7}y^{7}-2^{2}7^{2}13^{6}y^{6}-2^{3}7\cdot 13^{5}19y^{5}\\
&-2^{2}5\cdot 11\cdot 13^{4}19y^{4}-2\cdot 13^{3}6043y^{3}-2^{2}5\cdot 13^{2}1283y^{2}-2\cdot 11\cdot 13^{2}137y,\\
a_6^{(13)} =& -13^{7}y^{8}-2\cdot 13^{7}y^{7}-5^{2}13^{6}y^{6}-2^{2}7^{2}13^{5}y^{5}-2^{3}7\cdot 13^{4}19y^{4}\\
&-2^{2}5\cdot 11\cdot 13^{3}19y^{3}-2\cdot 13^{2}6043y^{2}-2^{2}5\cdot 13\cdot 1283y,\\
a_7^{(13)} =& -13^{6}y^{7}-2\cdot 13^{6}y^{6}-5^{2}13^{5}y^{5}-2^{2}7^{2}13^{4}y^{4}-2^{3}7\cdot 13^{3}19y^{3}\\
&-2^{2}5\cdot 11\cdot 13^{2}19y^{2}-2\cdot 13\cdot 6043y,\\
a_8^{(13)} =& -13^{5}y^{6}-2\cdot 13^{5}y^{5}-5^{2}13^{4}y^{4}-2^{2}7^{2}13^{3}y^{3}-2^{3}7\cdot 13^{2}19y^{2}\\
&-2^{2}5\cdot 11\cdot 13\cdot 19y,\\
a_9^{(13)} =& -13^{4}y^{5}-2\cdot 13^{4}y^{4}-5^{2}13^{3}y^{3}-2^{2}7^{2}13^{2}y^{2}-2^{3}7\cdot 13\cdot 19y,\\
a_{10}^{(13)} =& -13^{3}y^{4}-2\cdot 13^{3}y^{3}-25\cdot 13^{2}y^{2}-196\cdot 13y,\\
a_{11}^{(13)} =& -13^{2}y^{3}-2\cdot 13^{2}y^{2}-25\cdot 13y,\\
a_{12}^{(13)} =& -13y^{2}-2\cdot 13y.
\end{aligned}
$$

Appendix B

Koike's formula

In this appendix, let p be an odd prime. Similar constructions can be done also for $p = 2$, see [Koi75, Section 3].
The Eichler-Selberg trace formula on $\mathrm{SL}_2(\mathbb{Z})$ on the classical space of cusp forms S_k for the Hecke operator T_m is given by Zagier in [Lan95, Appendix to chapter III]:

$$\operatorname{Tr}\left(T_m^{(k)}\right) = -\frac{1}{2}\sum_{u\in\mathbb{Z}} P_k(u,m)H(u^2-4m) - \frac{1}{2}\sum_{dd'=m} \min\{d,d'\}^{k-1}.$$

With the following notations:
if $D < 0$, $H(D)$ is the Hurwitz class number, that is, the number of equivalence classes of positive definite binary quadratic forms with discriminant D with certain weights as described in [Lan95, p.47] and normalized such that

$$H(0) = -\frac{1}{12};$$

the expression $P_k(u,m)$ is given by

$$P_k(u,m) = \frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}}$$

where ρ and $\bar{\rho}$ are the roots of the polynomial

$$x^2 - ux + m.$$

Let $k \in 2\mathbb{Z}$. Koike proved in [Koi75] that the trace of the p-adic $U_p^{(k)}$ operator and its powers acting on the p-adic Banach space subspace of cuspidal form of level 1 can be obtained by taking the p-adic limit of the trace of the classical operators.
In the formula above, set $m = p^n$ and let $[a]$ be the congruence class of k modulo $p-1$. Consider now a sequence of weights $\{k_i\}_{i\in\mathbb{N}}$ such that

$$k_i \equiv k \mod p^{n_i}(p-1)$$

with $n_i \to \infty$ as i tend to infinity. Then the sequence $\{k_i\}$ converges p-adically to $(k,[a])$ in the connected component $W_{[a]}$ of the weight space. It is clear that, in $\mathbb{Z}$, the integers k_i tend to ∞.

We can now consider the limit

$$\lim_{i\to\infty}\left(T_{p^n}^{(k_i)}\right).$$

When we take the p-adic limit, we have that

$$\lim_{i\to\infty} -\frac{1}{2}\sum_{dd'=m}\min\{d,d'\}^{k_i-1} = \lim_{i\to\infty} -\frac{1}{2}\left(1^{k_i-1}+p^{k_i-1}+\cdots+1^{k_i-1}\right) = -1,$$

because all the terms different from 1 have strictly positive valuation and hence their powers tend to 0 p-adically. Similarly, if $p|u$, as ρ and $\bar{\rho}$ are roots of the equation $x^2-ux+p^n=0$, it means that both ρ and $\bar{\rho}$ must have positive valuation and their powers tend to 0 in the limit. If $p\nmid u$, let $\gamma(u)$ be the only root of $x^2-ux+p^n=0$ that is a p-adic unit; then $\overline{\gamma(u)}$ then must have positive valuation. Moreover we have that

$$\lim_{i\to\infty}\gamma(u)^{k_i-1}=\gamma(u)^k$$

exists because we can write

$$\gamma(u)^{k_i-1}=\tau(\gamma(u))^a\left\langle\gamma(u)\right\rangle^{k_i-1}$$

and we have that the quotient

$$\lim_{i\to\infty}\frac{\left\langle\gamma(u)\right\rangle^{k-1}}{\left\langle\gamma(u)\right\rangle^{k_i-1}} = \lim_{i\to\infty}\left\langle\gamma(y)\right\rangle^{k-k_i}$$

converges to 1 as it is raised to powers divisible by higher and higher powers of p. Next note that, as we are dealing with forms in level 1, the exponents k_i-1 must be odd, and we have that

$$\frac{\gamma(-u)^{k_i-1}-\overline{\gamma(-u)}^{k_i-1}}{\gamma(-u)-\overline{\gamma(-u)}} = \frac{\gamma(u)^{k_i-1}-\overline{\gamma(u)}^{k_i-1}}{\gamma(u)-\overline{\gamma(u)}}.$$

Finally note that

$$\frac{\gamma(u)^{k-1}}{\gamma(u)-\overline{\gamma(u)}} = \frac{\gamma(u)^{k-1}}{\gamma(u)-\frac{p^n}{\gamma(u)}} = \frac{\gamma(u)^k}{\gamma(u)^2-p^n}.$$

Hence we just showed that

$$\operatorname{Tr}\left(U_p^{(k)}\right)^n = \lim_{i\to\infty}\left(T_{p^n}^{(k_i)}\right) = -\sum_{\substack{0\le u<\sqrt{p^n}\\(u,p)=1}} \mathrm{H}(u^2-4p^n)\cdot\frac{(\gamma(u))^k}{\gamma(u)^2-p^n}-1,$$

because $H(D)=0$ if $D>0$.

A PARI/GP [PG17b] script allow to easily compute the trace of $(U_p)^n$ and hence the characteristic power series $P_{[a]}(k,T)$ of the $U_p^{(k)}$ operator acting on the subspace of overconvergent p-adic cusp forms of level 1:

$$P_{[a]}(k,T)=\exp\left(-\sum_{n=1}^{\infty}\left(\frac{\operatorname{Tr}\left(U_p^{(k)}\right)^n T^n}{n}\right)\right). \tag{B.1}$$

A slightly slower implementation of the formula is also present in [Smi00], where it was mostly taken from [CST98].

Remark. We can compute the coefficients of $P_{[a]}(k,T)$ up to an arbitrarily large p-adic precision, hence after the computations we must work with them modulo specific powers of p.

Remark. If k is left as a variable, the script computes the generic coefficients of the characteristic power series, where $\gamma(u)^k$ is computed as a formal power series in k, up to an arbitrarily large precision. In this case, it is of crucial importance to specify beforehand what is the class $[a]$ associated to the desired connected component $W_{[a]}$ of the weight space, as from a practical point of view we have

$$\gamma(u)^k = \tau(\gamma(u))^a \left(\frac{\gamma(u)}{\tau(\gamma(u))}\right)^k = \tau(\gamma(u))^a \left\langle\gamma(u)\right\rangle^k ,$$

where $\tau(\gamma) \in \boldsymbol{\mu}_{p-1}$ is the Teichmüller of γ and $\langle\gamma\rangle$ its one-unit part.

It's clear that the computation of the trace of $(U_p)^n$ is exponential in n, hence we are only able to compute the coefficients $C_{[a]}(m,k)$ of the characteristic power series for small m and primes p that are not too large.

Appendix C

Numerical experiments

The verification of this appendix have been performed using *The L-functions and Modular Forms Database* [LMF17] and the new modular form package for PARI/GP present in [PG17a].

C.1 Results verification

Verification when $p = 5$ and $[a] = [0]$

In this section, let k_0 and k_1 be in $4\mathbb{Z}$ with the additional condition that $v_5(k_0 - 8) \leq 1$ and $v_5(k_1 - 8) \leq 1$. As it only makes sense to investigate the congruences between forms in the same family, we will assume that

$$v_5(k_0 - 8) = v_5(k_1 - 8).$$

In this case the minimal slope predicted by Theorem 4.16 is

$$\alpha^{(5)}_{[0]}(k_i) = 1 + v_5(8 - k_i) \leq 2.$$

Remark. By Theorem 4.16, if $k_i \geq 4$, the $5-$adic family should specialize to a classical modular form over $\Gamma_0(5)$. We perform such a search for oldforms (i.e. in level 1) and newforms (i.e. in level 5).

For $k_i = 4, 12, 16, 20, 24, 32, 36, 40, 64$ we find a unique eigenform of the predicted slope $\alpha(k_i)^{(5)}_{[0]} = 1$, either in level 1 or in level 5.

Remark. In particular, we find that for $k_i = 4$ the eigenform of minimal slope is a newform. For $k_i = 12, 16, 20, 24, 32, 36, 40, 64$ the unique form of minimal slope is an oldform.
For $k_i = 28, 48$ we find a unique cusp form of the predicted slope $\alpha(k_i)^{(5)}_{[0]} = 2$, in both instances the eigenform is in level 1.

For $64 \leq k_i \leq 240$ and for $k_i = 512$ we verify that there is a unique oldform of minimal slope $\alpha^{(5)}_{[0]}(k_i)$, but the size of the weights do not allow to check easily among newforms in this cases.

Remark. In the following, we use the oldforms of weights $64 \leq k_i \leq 240$ and for $k_i = 512$ of minimal slope despite this lack of verification among newforms.

Then we investigate the two inequalities for $i = 0, 1$ given by Theorem 4.16:

$$\begin{cases} v_5\left(a_n(k_0) - a_n(k_1)\right) \geq v_5(k_0 - k_1) + 1 & \text{if } v_5(k_i - 8) = 0, \\ v_5\left(a_n(k_0) - a_n(k_1)\right) \geq v_5(k_0 - k_1) & \text{if } v_5(k_i - 8) = 1. \end{cases}$$

The first inequality holds true for the first 200 coefficients of the q-expansions obtained with every possible choice of k_0, k_1 such that
$v_5(k_0 - 8) = v_5(k_1 - 8) = 0$ and such that

$$4 \leq k_i \leq 240.$$

Moreover, the inequality holds true for the first 50 coefficients of the q-expansion of the forms obtained by picking $k_0 = 512$ and k_1 in the same range as above.
Finally, the second inequality holds true for the first 200 coefficients of the q-expansions obtained with every possible choice of k_0, k_1 such that
$v_5(k_0 - 8) = v_5(k_1 - 8) = 1$ and such that

$$28 \leq k_i \leq 228.$$

Remark. The above inequality is optimal whenever $\alpha^{(5)}_{[0]}(k_i) = 1$. This has been checked for all pairs (k_0, k_1).

Remark. Whenever $\alpha^{(5)}_{[0]}(k_i) = 2$ (or, equivalently, if $v_5(k_i - 8) = 1$), the forms of weight k_0 and k_1 are observed to be congruent modulo $5^{v_5(k_0 - k_1)+1}$ instead of $5^{v_5(k_0 - k_1)}$. This have been checked for all such pairs in that range.

Let f_{k_0} be the cuspidal eigenform of minimal slope in weight k_0, where k_0 is taken as above. In particular the following congruences have been observed for the first 200 coefficients (resp. for the first 50 coefficients of the congruence involving f_{512}):

- $f_{12} \equiv f_4 \mod 5$ as expected, since $v_5(12 - 4) = 0$;
- $f_{12} \equiv f_{32} \mod 5^2$ as expected, since $v_5(12 - 32) = 1$;
- $f_{12} \equiv f_{112} \mod 5^3$ as expected, since $v_5(12 - 112) = 2$;
- $f_{12} \equiv f_{512} \mod 5^4$ as expected, since $v_5(12 - 512) = 3$;
- $f_{28} \equiv f_{48} \mod 5^2$, but the inequality only predicts congruence $\mod 5$;
- $f_{28} \equiv f_{128} \mod 5^3$, but the inequality only predicts congruence $\mod 5^2$.

Note moreover that f_{12} is the usual Δ.

Verification when $p = 7$ and $[a] = [0]$

In this section, let k_0 and k_1 be in $6\mathbb{Z}$ with the additional condition that $v_7(k_0 - 6) = 0$ and $v_7(k_1 - 6) = 0$.
In this case the minimal slope predicted by Theorem 4.17 is

$$\alpha^{(7)}_{[0]}(k_i) = 1.$$

Remark. By Theorem 4.17, if $k_i \geq 3$, the $7-$adic family should specialize to a classical modular form over $\Gamma_0(7)$. We perform such a search for oldforms (i.e. in level 1) and newforms (i.e. in level 7).

For $k_i = 12, 18, 24, 30, 36, 42, 54, 60$ we find a unique eigenform of the predicted slope 1, always in level 1.
For $66 \leq k_i \leq 240$ and for $k_i = 306$ we verify that there is a unique oldform of minimal slope 1, but the size of the weights do not allow to check easily among newforms in this cases.

Remark. In the following, we use the oldforms of weights $66 \leq k_i \leq 240$ and for $k_i = 306$ of minimal slope despite this lack of verification among newforms.

Then we investigate the inequality for $i = 0, 1$ given by Theorem 4.17:

$$v_7\left(a_n(k_0) - a_n(k_1)\right) \geq v_7(k_0 - k_1) + 1 \text{ if } v_7(k_i - 6) = 0.$$

The inequality holds true for the first 200 coefficients of the q-expansions obtained with every possible choice of k_0, k_1 such that
$v_7(k_0 - 6) = v_5(k_1 - 6) = 0$ and such that

$$12 \leq k_i \leq 240.$$

Moreover, the inequality holds true for the first 80 coefficients of the q-expansion of the form obtained by picking $k_0 = 306$ and k_1 as above.

Remark. The above inequality is optimal. This has been checked for all pairs (k_0, k_1) in that range.

Let f_{k_0} be the cuspidal eigenform of minimal slope 1 in weight k_0, where k_0 is taken as above. In particular the following congruences have been observed for the first 200 coefficients:

- $f_{12} \equiv f_{18} \mod 7$ as expected, since $v_7(12 - 18) = 0$;
- $f_{12} \equiv f_{54} \mod 7^2$ as expected, since $v_7(12 - 54) = 1$;
- $f_{12} \equiv f_{306} \mod 5^3$ as expected, since $v_5(12 - 306) = 2$.

Note moreover that f_{12} is the usual Δ.

Remark. To compute rapidly the slope of modular forms defined over finite extension K of $\mathbb{Q}$ and their q-expansion, especially for very big weights, we use the following method.

- We obtain the monic polynomial $G(x)$ with coefficients in $\mathbb{Z}$ such that K is the splitting field of G;
- We compute the eigenvalue $a_p(x)$ of the q-expansion of the forms, it is expressed as a polynomial in the roots of G;
- Using p-adic approximation we compute up to a high p-adic precision the p-adic roots of $G(x)$;
- We evaluate $a_p(x)$ to each of the p-adic roots and compute in every case the p-adic valuation.

- Once we have verified that there is only one instance with the minimal predicted valuation, we compute the rest of the q-expansion of the form by evaluating all the coefficients to this specific approximated p-adic root of $G(x)$.

The use of p-adic approximation is much faster than the computations of the ideals $\mathfrak{p}$ lying over p in K and the evaluation of $a_p(x)$ at these prime ideals: our technique essentially amounts to making a suitable explicit choice of embedding $K \hookrightarrow \mathbb{Q}_p$. Another advantage is the fact that confronting q-expansions of forms defined over different number fields is now immediate as we see them all directly with p-adic coefficients. We were not able to find forms of minimal slope whose p-adic field of definition is bigger then $\mathbb{Q}_p$.

C.2 Conjecture verification

The Theorems 4.30, 4.31 and 4.32 are a consequence of Conjecture 4.18 and they can be easily verified for many weights, as shown in this section.

Verification when $p = 5$ and $[a] = [2]$

In this section, let k_0 and k_1 be in $2 + 4\mathbb{Z}$ with the additional condition that $v_5((k_0 - 10)(k_0 - 14)) \leq 1$ and $v_5((k_1 - 10)(k_1 - 14)) \leq 1$. As it only makes sense to investigate the congruences between forms in the same family, we will assume that

$$v_5((k_0 - 10)(k_0 - 14)) = v_5((k_1 - 10)(k_1 - 14)).$$

In this case the minimal slope predicted by Theorem 4.30 is

$$\alpha^{(5)}_{[2]}(k) = 1 + v_5((k_i - 10)(k_i - 14)) \leq 3.$$

Remark. By Theorem 4.30, if $k_i \geq 5$, the $5-$adic family should specialize to a classical modular form over $\Gamma_0(5)$. We perform such a search for oldforms (i.e. in level 1) and newforms (i.e. in level 5).

For $k_i = 6, 18, 22, 26, 38, 42, 46, 58, 62$ we find a unique eigenform of the predicted slope $\alpha(k_i)^{(5)}_{[2]} = 2$, either in level 1 or in level 5.

Remark. In particular, we find that for $k_i = 6$ the eigenform of minimal slope is a newform. For $k_i = 18, 22, 26, 38, 42, 46, 58, 62$ the unique form of minimal slope is an oldform.

For $k_i = 30, 34, 50, 54$ we find a unique cusp form of the predicted slope $\alpha(k_i)^{(5)}_{[2]} = 3$; in all four instances the eigenform is in level 1.
For $66 \leq k_i \leq 238$ we verify that there is a unique oldform of minimal slope $\alpha^{(5)}_{[2]}(k_i)$, but the size of the weights do not allow to check easily among newforms in this cases.

Remark. In the following, we use the oldforms of weights $66 \leq k_i \leq 238$ of minimal slope despite this lack of verification among newforms.

Then we investigate the two inequalities for $i = 0, 1$ given by Theorem 4.30:

$$v_5\left(a_n(k_0) - a_n(k_1)\right) \geq v_5(k_0 - k_1) + 1 \text{ if } v_5((k_i - 10)(k_i - 14)) = 0,$$

and

$$\begin{aligned} v_5\left(a_n(k_0) - a_n(k_1)\right) &\geq v_5(k_0 - k_1) + 1 \\ &\quad + \min\{0, 1 - v_5((k_0 - 10)(k_1 - 10)), 1 - v_5((k_0 - 14)(k_1 - 14))\} \end{aligned}$$

if $v_5((k_i - 10)(k_i - 14)) = 1$.
The first inequality holds true for the first 200 coefficients of the q-expansions obtained with every possible choice of k_0, k_1 such that
$v_5((k_0 - 10)(k_0 - 14)) = v_5((k_1 - 10)(k_1 - 14)) = 0$ and such that

$$6 \leq k_i \leq 238.$$

Finally, the second inequality holds true for the first 200 coefficients of the q-expansions obtained with every possible choice of k_0, k_1 such that
$v_5((k_0 - 10)(k_0 - 14)) = v_5((k_1 - 10)(k_1 - 14)) = 1$ and such that

$$30 \leq k_i \leq 234.$$

Remark. The above inequality is optimal whenever $\alpha^{(5)}_{[2]}(k_i) = 2$. This has been checked for all pairs (k_0, k_1) in that range.

Remark. Whenever $\alpha^{(5)}_{[2]}(k_i) = 3$ (or, equivalently, if $v_5((k_i - 10)(k_i - 14)) = 1$), the forms of weight k_0 and k_1 are observed to be congruent modulo $5^{v_5(k_0 - k_1)+1}$. On the other hand, when

$$v_5((k_0 - 10)(k_1 - 10)) = 2$$

or when

$$v_5((k_0 - 14)(k_1 - 14)) = 2,$$

Theorem 4.30 only predicts congruence modulo $5^{v_5(k_0 - k_1)}$. This have been checked for all such pairs.

Let f_{k_0} be the cuspidal eigenform of minimal slope in weight k_0, where k_0 is taken as above. In particular the following congruences have been observed for the first 200 coefficients:

- $f_{18} \equiv f_6 \mod 5$ as expected, since $v_5(18 - 6) = 0$;
- $f_{18} \equiv f_{38} \mod 5^2$ as expected, since $v_5(18 - 38) = 1$;
- $f_{18} \equiv f_{118} \mod 5^3$ as expected, since $v_5(18 - 218) = 2$;
- $f_{30} \equiv f_{34} \mod 5$, as expected,, since $v_5(30 - 34) = 0$;
- $f_{30} \equiv f_{50} \mod 5^2$, but the inequality only predicts congruence $\mod 5$;
- $f_{34} \equiv f_{54} \mod 5^2$, but the inequality only predicts congruence $\mod 5$;
- $f_{30} \equiv f_{230} \mod 5^3$, but the inequality only predicts congruence $\mod 5^2$.

In other words, the relevance of the term

$$\min\{0, 1 - v_5((k_0 - 10)(k_1 - 10)), 1 - v_5((k_0 - 14)(k_1 - 14))\}$$

in the formula above is never observed.

Verification when $p = 7$ and $[a] = [2]$

In this section, let k_0 and k_1 be in $2 + 6\mathbb{Z}$ with the additional conditions that $v_7(k_0 - 8) = v_7(k_1 - 8) = 0$, $v_7(k_0 - 14) \le 1$ and $v_7(k_1 - 14) \le 1$. As it only makes sense to investigate the congruences between forms in the same family, we will assume that

$$v_7(k_0 - 14) = v_7(k_1 - 14).$$

In this case the minimal slope predicted by Theorem 4.31 is

$$\alpha_{[2]}^{(7)}(k_i) = 2 + v_7(k_i - 14) \le 3.$$

Remark. By Theorem 4.31, if $k_i \ge 5$, the 7−adic family should specialize to a classical modular form over $\Gamma_0(7)$. We perform such a search for oldforms (i.e. in level 1) and newforms (i.e. in level 7).

For $k_i = 20, 26, 32, 38, 44, 62$ we find a unique eigenform of the predicted slope $\alpha(k_i)_{[2]}^{(7)} = 2$, always in level 1. For $k_i = 56$ we find a unique oldform of the predicted slope $\alpha(k_i)_{[2]}^{(7)} = 3$.
For $68 \le k_i \le 236$ and for $k_i = 350$ we verify that there is a unique oldform of minimal slope $\alpha_{[2]}^{(7)}(k_i)$, but the size of the weights do not allow to check easily among newforms in this cases.

Remark. In the following, we use the oldforms of weights $68 \le k_i \le 236$ and for $k_i = 350$ of minimal slope despite this lack of verification among newforms.

Then we investigate the two inequalities for $i = 0, 1$ given by Theorem 4.31:

$$\begin{cases} v_7\left(a_n(k_0) - a_n(k_1)\right) \ge v_7(k_0 - k_1) + 1 & \text{if } v_7(k_i - 14) = 0, \\ v_7\left(a_n(k_0) - a_n(k_1)\right) \ge v_7(k_0 - k_1) & \text{if } v_7(k_i - 14) = 1. \end{cases}$$

The first inequality holds true for the first 200 coefficients of the q-expansions obtained with every possible choice of k_0, k_1 such that
$v_7(k_0 - 14) = v_7(k_1 - 14) = 0$ and such that

$$20 \le k_i \le 236.$$

Finally, the second inequality holds true for the first 200 coefficients of the q-expansions obtained with every possible choice of k_0, k_1 such that
$v_7(k_0 - 14) = v_7(k_1 - 14) = 1$ and such that

$$56 \le k_i \le 224 \text{ and } k_i = 350.$$

Remark. The above inequality is optimal whenever $\alpha_{[2]}^{(7)}(k_i) = 2$. This has been checked for all pairs (k_0, k_1) in that range.

Remark. Whenever $\alpha_{[2]}^{(7)}(k_i) = 3$ (or, equivalently, if $v_7(k_i - 14) = 1$), the forms of weight k_0 and k_1 are observed to be congruent modulo $7^{v_7(k_0 - k_1)+1}$. On the other hand, when

$$v_7((k_0 - 14)(k_1 - 14)) = 2,$$

Theorem 4.31 only predicts congruence modulo $7^{v_7(k_0 - k_1)}$. This have been checked for all such pairs.

Let f_{k_0} be the cuspidal eigenform of minimal slope in weight k_0, where k_0 is taken as above. In particular the following congruences have been observed for the first 200 coefficients:

- $f_{20} \equiv f_{26} \mod 7$ as expected, since $v_7(20-26) = 0$;
- $f_{20} \equiv f_{62} \mod 7^2$ as expected, since $v_7(20-62) = 1$;
- $f_{56} \equiv f_{98} \mod 7^2$, but the inequality only predicts congruence $\mod 7$;
- $f_{56} \equiv f_{350} \mod 7^3$, but the inequality only predicts congruence $\mod 7^2$.

Verification when $p = 7$ and $[a] = [4]$

In this section, let k_0 and k_1 be in $4 + 6\mathbb{Z}$ with the additional conditions that $v_7(k_0 - 10) \leq 1$ and $v_7(k_1 - 10) \leq 1$. As it only makes sense to investigate the congruences between forms in the same family, we will assume that

$$v_7(k_0 - 10) = v_7(k_1 - 10).$$

In this case the minimal slope predicted by Theorem 4.32 is

$$\alpha^{(7)}_{[4]}(k_i) = 1 + v_7(k_i - 10) \leq 2.$$

Remark. By Theorem 4.32, if $k_i \geq 4$, the 7−adic family should specialize to a classical modular form over $\Gamma_0(7)$. We perform such a search for oldforms (i.e. in level 1) and newforms (i.e. in level 7).

For $k_i = 4, 16, 22, 28, 34, 40, 46, 58, 64$ we find a unique eigenform of the predicted slope $\alpha(k_i)^{(7)}_{[4]} = 1$, either in level 1 or in level 7.

Remark. In particular, we find that for $k_i = 4$ the eigenform of minimal slope is a newform. For $k_i = 16, 22, 28, 34, 40, 46, 58, 64$ the unique form of minimal slope is an oldform.

For $k_i = 52$ we find a unique oldform of the predicted slope $\alpha(k_0)^{(7)}_{[4]} = 2$.
For $70 \leq k_i \leq 238$ and for $k_i = 310$ we verify that there is a unique oldform of minimal slope $\alpha^{(7)}_{[4]}(k)$, but the size of the weights do not allow to check easily among newforms in this cases.

Remark. In the following, we use the oldforms of weights $70 \leq k_0 \leq 238$ and for $k_0 = 310$ of minimal slope despite this lack of verification among newforms.

Then we investigate the two inequalities for $i = 0, 1$ given by Theorem 4.32:

$$\begin{cases} v_7\left(a_n(k_0) - a_n(k_1)\right) \geq v_7(k_0 - k_1) + 1 & \text{if } v_7(k_i - 10) = 0, \\ v_7\left(a_n(k_0) - a_n(k_1)\right) \geq v_7(k_0 - k_1) & \text{if } v_7(k_i - 10) = 1. \end{cases}$$

The first inequality holds true for the first 200 coefficients of the q-expansions obtained with every possible choice of k_0, k_1 such that
$v_7(k_0 - 10) = v_7(k_1 - 10) = 0$ and such that

$$4 \leq k_i \leq 238 \text{ and } k_i = 310.$$

Finally, the second inequality holds true for the first 200 coefficients of the q-expansions obtained with every possible choice of k_0, k_1 such that $v_7(k_0 - 10) = v_7(k_1 - 10) = 1$ and such that

$$52 \leq k_i \leq 220$$

Remark. The above inequality is optimal whenever $\alpha^{(7)}_{[4]}(k_i) = 1$. This has been checked for all pairs (k_0, k_1).

Remark. Whenever $\alpha^{(7)}_{[4]}(k_i) = 2$ (or, equivalently, if $v_7(k_i - 10) = 1$), the forms of weight k_0 and k_1 are observed to be congruent modulo $7^{v_7(k_0 - k_1)+1}$. On the other hand, when

$$v_7((k_0 - 10)(k_1 - 10)) = 2,$$

Theorem 4.32 only predicts congruence modulo $7^{v_7(k_0 - k_1)}$. This have been checked for all such pairs.

Let f_{k_0} be the cuspidal eigenform of minimal slope in weight k_0, where k_0 is taken as above. In particular the following congruences have been observed for the first 200 coefficients:

- $f_{16} \equiv f_4 \mod 7$ as expected, since $v_7(16 - 4) = 0$;
- $f_{16} \equiv f_{58} \mod 7^2$ as expected, since $v_7(16 - 58) = 1$;
- $f_{16} \equiv f_{310} \mod 7^3$, as expected, since $v_7(16 - 310) = 2$;
- $f_{52} \equiv f_{94} \mod 7^2$, but the inequality only predicts congruence $\mod 7$.

C.3 Direct verification of the conjecture

We can also show more direct numerical evidence for Conjecture 4.18. Let $p \in \{5, 7, 13\}$ and let $t_p(z)$ be the generator of the function field of $X_0(p)$, as introduced in Chapter 2. Note that for every such p the q-expansion of t_p has the form

$$t_p = q + \sum_{i=2}^{\infty} a_i q^i$$

for some $a_i \in \mathbb{Q}$. Let e_1 be the p-adic modular function introduced in Section 4.3 defined by the quotient

$$e_1 = \frac{E_1}{V(E_1)}.$$

As e_1 is a p-adic modular function, it must have a t_p-expansion of the form

$$e_1 = 1 + t_p + \sum_{i=2}^{\infty} b_i (t_p)^i.$$

With PARI/GP [PG17b] we can recursively compute the coefficients b_i for arbitrarily large i from the q-expansions of t_p and e_1.

By Proposition 2.6 and the following Remark, we can check the overconvergence rate of e_1 by looking at the p-adic valuation of the coefficients b_i.
In particular, if e_1 is r-overconvergent for every r such that

$$0 \leq v_p(r) < \frac{1}{p+1},$$

then the following inequality must hold for every $i \in \mathbb{N}$:

$$v_p(b_i) \geq \frac{12}{p^2-1} i = \gamma_p i. \tag{C.1}$$

It is then clear that we can produce direct evidence for Conjecture 4.18 by checking the inequality (C.1).
Let $p \in \{5, 7, 13\}$; for each one of these primes we verified the inequality (C.1) for $i \leq 10^3$.

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