PhD thesis

University of Copenhagen – Department of Mathematical Sciences

Automorphic Forms - Multiplier Systems and Taylor Coefficients

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Abstract

The Taylor coefficients of weight k Eisenstein series wrt. $SL_2(\mathbb{Z})$ are related to values of L-functions for Hecke characters in the point k. We show some congruences for Taylor coefficients of Eisenstein series of weight 4 and 6 and use them to establish congruences for values of L-functions for Hecke characters in the points 4 and 6.

It is well known, that all zeros of the Eisenstein series E_k wrt. $SL_2(\mathbb{Z})$ in the standard fundamental domain has modulus 1. We show that this is also true for $\vartheta^n E_k$, where ϑ is a certain differential operator.

We then proceed to study logarithms of multiplier systems. For automorphic forms wrt. Hecke triangle groups and Fuchsian groups with no elliptic elements and genus 0, we show that some logarithms of multiplier systems can be interpreted as a linking number.

Finally we show a "twisted" version of the prime geodesics theorem, and use this to show some results about the distribution of prime geodesics wrt. logarithms of multiplier systems.

Resumé

Taylor-koefficienter for vægt k Eisenstein-rækker mht. $SL_2(\mathbb{Z})$ er relateret til værdien af L-funktioner for Hecke-karakterer i k. Vi viser nogle kongruenser for Taylor koefficienter for Eisenstein-rækkerne af vægt 4 og 6, og får dermed også kongruenser for L-funktioner for Hecke-karakterer i punkterne 4 og 6.

Det er velkendt, at alle nulpunkter for Eisenstein rækker E_k mht. $SL_2(\mathbb{Z})$ har modulus 1. Vi viser, at dette også gælder for $\vartheta^n E_k$, hvor ϑ er en given differentialoperator.

Derefter studerer we logaritmer af multiplikator systemer. For automorfe former mht. Hecke trekantsgrupper og genus 0 Fuchs-grupper uden elliptiske elementer viser vi, at sådanne logaritmer af multiplikator systemer kan fortolkes som et linkingtal.

Endelig beviser vi en "twisted" version primgeodætsætningen og bruger denne til at vise nogle resultater om fordelingen af primgeodæter mht. logartimer af multiplikator systemer.

Contents

Chapter 1

Introduction

1.1 Taylor Coefficients for Eisenstein Series

For any z_0 in the upper half-plane, we can define

$$\sigma_{z_0} := \begin{pmatrix} -\overline{z}_0 & z_0 \\ -1 & 1 \end{pmatrix}.$$

σ_{z_0} acts as a Möbius transformation on $\mathbb{H}$, it maps the unit disc $\mathbb{D}$ bijectively to the upper half-plane $\mathbb{H}$, and $\sigma_{z_0}0 = z_0$. For any function $f : \mathbb{H} \to \mathbb{C}$, and even integer k, we can therefore define a function $f|_k\sigma_z : \mathbb{D} \to \mathbb{C}$ given by

$$(f|_k\sigma_{z_0})(z) = \frac{(\det \sigma_{z_0})^{k/2} f(\sigma_{z_0} z)}{(j_{\sigma_{z_0}}(z))^k},$$

where $j_\gamma(z) = cz + d$, when $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

The function $f|_k\sigma_{z_0}$ is holomorphic, if f is holomorphic, and so it has a Taylor expansion in 0. This Taylor series is convergent in all of $\mathbb{D}$ (opposed to Taylor expansions of f, which will only converge in all of $\mathbb{H}$, if f can be continued analytically to all of $\mathbb{C}$). If f is a weight k modular form wrt. $SL_2(\mathbb{Z})$, and we set $z_0 = i$, then for the Taylor expansion of $f|_k\sigma_i$

$$(f|_k\sigma_i)(z) = \sum_{n=0}^{\infty} c(n, f) z^n,$$

we have

$$4 \nmid (2n + k) \Rightarrow c(n, f) = 0.$$

For even $k \geq 4$ let $E_k(z)$ be the holomorphic weight k Eisenstein series wrt. $SL_2(\mathbb{Z})$, that is

$$E_k(z) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)=1}} \frac{1}{(mz + n)^k},$$

and let

$$(E_k|_k\sigma_i)(z) = \sum_{n=0}^{\infty} c(n, E_k) z^n.$$

For $4|(k+2n)$, we then have

$$\begin{aligned} c(n, E_k) &= \frac{(k+n-1)! i^{2-k/2}}{n! B_k(\sqrt{2}\pi)^k} \sum_{\lambda\in\mathbb{Z}[i]\setminus\{0\}} \frac{(\overline{\lambda}/\lambda)^{k/2+n}}{|\lambda|^k} \\ &= \frac{4(k+n-1)! i^{2-k/2}}{n! B_k(\sqrt{2}\pi)^k} L(k, \psi_{k/2+n}), \end{aligned}$$

where B_k is the k'th Bernoulli number, and $L(\cdot, \psi_{k/2+n})$ is the L-function for the Hecke character $\psi_{k/2+n}$.

Now we define

$$\mathcal{A}_n := \frac{-(2\pi)^{2n+6} 12^{n/2-1} n!}{(\Gamma(1/4))^{4n+8}} c(n, E_4)$$

and

$$\mathcal{B}_n := \frac{i(2\pi)^{2n+9} 12^{(n-3)/2} n!}{(\Gamma(1/4))^{4n+12}} c(n, E_6).$$

We can prove the following congruence for $\mathcal{A}_n$ and $\mathcal{B}_n$ (which gives us congruences for $c(n, E_4)$ and $c(n, E_6)$ and for $L(4, \psi_{n+2})$ and $L(6, \psi_{n+3})$).

Theorem A. *The numbers $\mathcal{A}_n$ and $\mathcal{B}_n$ are integers, and we have*

$$\mathcal{A}_{2n+1} = \mathcal{B}_{2n} = 0.$$

For $n \geq 1$, we have

$$\mathcal{A}_n \equiv \mathcal{A}_{n+72} \pmod{13},$$

and if n is even, then $13 \nmid \mathcal{A}_n$. For $m \geq 0$, we have

$$\begin{aligned} \mathcal{B}_{4m+1} &\equiv 1 \pmod{5} \\ \mathcal{B}_{4m+3} &\equiv 3 \pmod{5}. \end{aligned}$$

To prove Theorem A we use the differential operator

$$\vartheta_k = \frac{1}{2\pi i}\frac{d}{dz} - \frac{kE_2}{12},$$

where E_2 is the weight 2 holomorphic Eisenstein series. This operator sends weight k holomorphic modular forms to weight $k+2$ holomorphic modular forms, and hence the operator

$$\vartheta_k^n := \vartheta_{k+2n-2} \circ \vartheta_{k+2n-4} \circ \cdots \circ \vartheta_k$$

sends weight k holomorphic modular forms to weight $k+2m$ holomorphic modular forms.

In 1970 F. K. C. Rankin and H. P. F. Swinnerton-Dyer ([13]) proved that for $k \geq 4$ all zeros of E_k in the standard fundamental domain

$$\mathcal{F} = \{z \in \mathbb{H} \mid \Re z \leq 1/2,\ |z| \geq 1\}$$

are located on the arc

$$A = \{\exp(it) \mid t \in [\pi/3, 2\pi/3]\}.$$

Using this we prove the following theorem.

Theorem B. *For $k \geq 4$ and $n \geq 1$ all the zeros of the function $\vartheta^n E_k$ in $\mathcal{F}$ are located on A.*

1.2 Interpretations of Logarithms of Multiplier Systems

Let Γ be a Fuchsian group. If a holomorphic function $f : \mathbb{H} \to \mathbb{C}$, transforms like

$$f(\gamma z) = \nu(\gamma)(j_\gamma(z))^k f(z),$$

for a all $\gamma \in \Gamma$ and $z \in \mathbb{H}$, and f is also holomorphic in the cusp, we say that f is a weight k automorphic form with multipler system ν wrt. Γ. If $\nu \equiv 1$, we say that f is a weight k modular form wrt. Γ.

Let f be a weight k holomorphic automorphic form with multiplier system ν wrt. Γ. If f has no zeros in $\mathbb{H}$, then it has a holomorphic logarithm, and by taking logarithms in the transformation equation we get

$$(\log f)\left(\frac{az+b}{cz+d}\right) = (\log f)(z) + k\log(cz+d) + 2\pi ik\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1.1}$$

where $\log f$ is a holomorphic logarithm, $\log(cz+d)$ is the main logarithm (i.e. the logarithm that has imaginary part in $(-\pi,\pi]$) of $cz+d$, and $\exp(2\pi ik\Phi) = \nu$. Since f is zero free, the power $f^t = \exp(t(\log f))$ is well defined and holomorphic for any $t \in \mathbb{R}$. Furthermore if $t \geq 0$ f^t is a weight tk automorphic form with multiplier system $\nu^t = \exp(2\pi itk\Phi)$ wrt. Γ (if $t < 0$ f^t will have a pole in a cusp).

An example of such an zero free automorphic form is the Dedekind eta function η, which is a weight $1/2$ automorphic form with multiplier system ν_η wrt. $\Gamma = SL_2(\mathbb{Z})$. In [2] É. Ghys gives an interpretation, for any hyperbolic $\gamma \in SL_2(\mathbb{Z})$ with positive trace, of the logarithm $\pi i\Phi_\eta(\gamma)$ of $\nu_\eta(\gamma)$ as a linking number of a trefoil knot and a closed curve (i.e. the number of times the curve winds around the trefoil knot).

Inspired by this we consider Fuchsian groups Γ for which $\Gamma\backslash\mathbb{H}$ has finite volume, genus zero and one or more cusps. For such a group there exists modular forms, which are zero free in $\mathbb{H}$ (they do however have zeros in one or more cusps). If $n \geq 3$ is an integer, and we define the Hecke triangle group H_n, to be the group generated by the 2 matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2\cos(\pi/n) \\ 0 & 1 \end{pmatrix},$$

then $H_n\backslash\mathbb{H}$ has volume $\pi(n-2)/n$, genus 0 and one cusp, so H_n is a group of the desired type.

The point $\rho_n = \exp(\pi i/n)$ is elliptic of order n wrt. to H_n and i are elliptic of order 2, and there exists automorphic forms g and h wrt. H_n, such that g has a simple zero in ρ_n and h has a simple zero in i, and the only other zeros are H_n-translates of these. We define a function $\widetilde{\Lambda} : SL_2(\mathbb{R}) \to S^3$ (where S^3 is the unit sphere in $\mathbb{C}^2$) by

$$\widetilde{\Lambda}(\sigma) = \left(\frac{g(\sigma i)}{(tj_\sigma(i))^{4/(n-2)}}, \frac{h(\sigma i)}{(tj_\sigma(i))^{2n/(n-2)}} \right),$$

where t depends on σ and is given such that $\widetilde{\Lambda}(\sigma) \in S^3$. Furthermore we define an equivalence relation $\sim$ on S^3, given by $(z_1, z_2) \sim (\zeta z_1, \zeta z_2)$ for all $n-2$'nd roots of unity ζ, and we let κ be the set $\{(z_1, z_2) \in S^3 \mid z_1^n = z_2^2\}$. We then show that the function $\widetilde{\Lambda}_0 : H_n\backslash SL_2(\mathbb{R}) \to (S^3\backslash\kappa)/\sim$, given by

$$\widetilde{\Lambda}_0(H_n\sigma) = \{x \in S^3 \mid x \sim \widetilde{\Lambda}(\sigma)\},$$

is well defined and a homeomorphism.

Any hyperbolic element of H_n corresponds to a geodesic $f_\gamma : \mathbb{R} \to SL_2(\mathbb{R})$ ($PSL_2(\mathbb{R})$ is a realization of the unit tangent bundle on $\mathbb{H}$, so it also corresponds to a geodesic on this unit tangent bundle), with $f_\gamma(t) = \gamma f_\gamma(t+l)$ for some $l > 0$, and $\mathcal{B}_\gamma : [0, l(n-2)] \to S^3\backslash\kappa$ given by

$$\mathcal{B}_\gamma(t) = \widetilde{\Lambda}(f_{\gamma^{n-2}}(t)),$$

is a closed curve. If n is odd, then the set κ is (the image of) one knot, and hence it makes sense to talk about the linking number $\mathrm{link}(\kappa, \mathcal{B}_\gamma)$ of κ and $\mathcal{B}_\gamma$. When n is even, then κ is two knots κ_1, κ_2, and we define

$$\mathrm{link}(\kappa, \mathcal{B}_\gamma) = \mathrm{link}(\kappa_1, \mathcal{B}_\gamma) + \mathrm{link}(\kappa_2, \mathcal{B}_\gamma).$$

Now let D be a weight k modular form wrt. H_n, that only has zeros in the cusp, and define $\Phi : H_n \to \mathbb{Q}$ as in (1.1). Because H_n only has one cusp any weight k' modular forms, that only has zeros in the cusp, are on the form $\alpha D^{k'/k}$

for some $\alpha \in \mathbb{C}$, so Φ is independent on which modular form we choose (as long as it has no zeros in $\mathbb{H}$).

We prove the following theorem, which displays a connection between multiplier systems values in hyperbolic elements and the closed geodesics corresponding to the hyperbolic elements.

Theorem C. *Let $\gamma \in H_n$ be hyperbolic and have positive trace, then*

$$\mathit{link}(\kappa, \mathcal{B}_\gamma) = 4n\Phi(\gamma).$$

For $n = 3$ $H_n = SL_2(\mathbb{Z})$ and $\Phi = \Phi_\eta$. This case of Theorem C, is proved by Ghys in [2].

If Γ is a Fuchsian group with no elliptic elements and $\Gamma\backslash\mathbb{H}$ has finite area and genus zero, then we can make an interpretation much like the one in Theorem C of logarithms of multiplier systems.

To do this we show that $\Gamma\backslash\mathbb{H}$ has at least 3 cusps. We denote the cusps $a_1, \ldots, a_h$ and define weight 2 modular forms $F_1, \ldots, F_h$, such that F_1 has all its zeros in a_1, while for $j \neq 1$ F_j has a simple zero in a_j and the rest of its zeros in a_1. We define Φ_j to be such that (1.1) holds for F_j and Φ_j. We then use F_1 and F_2 to define a homeomorphism, which we also call $\widetilde{\Lambda}$, between $\Gamma\backslash SL_2(\mathbb{R})$ and

$$S^3 \backslash \bigcup_{j=1}^{h} \kappa_j,$$

where κ_j is a knot, that corresponds to the cusp a_j.

If we let $\mathcal{A}_\gamma$ be the closed curve, that $\widetilde{\Lambda}$ maps the closed geodesic associated with γ to, then we can prove the following theorem.

Theorem D. *Let $\gamma \in \Gamma$ be hyperbolic and have positive trace, then*

$$\mathit{link}(\kappa_j, \mathcal{A}_\gamma) = 2\Phi_j(\gamma). \tag{1.2}$$

Since F_j has all but one zero in a_1, it matters which cusp we choose to label a_1. So $\{F_j \mid 1 \leq j \leq h\}$ depends on, which cusp we label a_1, and $\{\Phi_j \mid 1 \leq j \leq h\}$ depends on $\{F_j \mid 1 \leq j \leq h\}$. Hence the value of the right hand side of (1.2) changes, if we change, which cusp is a_1 (this also changes $\widetilde{\Lambda}$, and hence it changes $\mathcal{A}_\gamma$ and the left hand side of (1.2))

1.3 Distributions wrt. Logarithms of Multiplier Systems

For any Fuchsian group Γ and hyperbolic element $\gamma \in \Gamma$, there is an associated closed geodesic in $\Gamma\backslash\mathbb{H}$ with length

$$l(\gamma) = 2\log\left(\frac{|\operatorname{Tr}\gamma| + \sqrt{(\operatorname{Tr}\gamma)^2 - 4}}{2}\right).$$

Two hyperbolic elements $\gamma, \tau \in \Gamma$ have the same associated geodesic if and only if $\pm\tau$ is in the conjugacy class $[\gamma]$, that is

$$\pm\tau \in \{\sigma\gamma\sigma^{-1} \mid \sigma \in \Gamma\}.$$

If there is no $\tau \in \Gamma$ and $n \geq 2$, such that $\gamma = \tau^n$, then we say that γ is primitive, and we say, that the associated geodesic is a prime geodesic. The prime geodesics are the closed geodesics, that are not periodic (we can think of this as they "go once around"), while the geodesic associated with γ^n, will "go n times around" the geodesic associated with γ.

We will denote the set of conjugacy classes of primitive hyperbolic matrices with positive trace Γ'. So there is a one-to-one correspondence between the elements of Γ' and the prime geodesics. When $\Gamma\backslash\mathbb{H}$ has finite area, then the prime geodesics theorem gives the following estimate on the number of prime geodesics of bounded length on $\Gamma\backslash\mathbb{H}$

$$\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq T}} l(\gamma) \sim e^T.$$

In [18] and [10] Sarnak and Mozzochi gives an estimate of

$$\sum_{\substack{[\gamma]\in SL_2(\mathbb{Z})' \\ l(\gamma)\leq T}} \nu_\eta^k l(\gamma),$$

which depends on the power k. This can be seen as a "twisted" version of the prime geodesic theorem, and Sarnak and Mozzochi use this to prove a distribution result for the prime geodesics on $SL_2(\mathbb{Z})\backslash\mathbb{H}$.

Now let Γ be a cofinite Fuchsian group, i.e. let $\Gamma\backslash\mathbb{H}$ have finite area, and let f be a zero free modular form wrt. Γ. Then we can define Φ as in (1.1), and let $\nu^t := \exp(2\pi i t\Phi)$. We will assume that f^r is not modular (i.e. does not have trivial multiplier system) for $0 < r < 1$ and we define N to be the weight of f. Inspired by the work of Mozzochi and Sarnak, we use Selberg's trace formula for a suitable pair of test functions to prove following estimate (which can be viewed as a "twisted" version of the prime geodesics theorem).

Theorem E. *There exists a $\delta > 0$, such that for $k \in (-1, N-1]$*

$$\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq T}} \nu^k(\gamma) l(\gamma) = \begin{cases} \frac{e^{T(1-|k|/2)}}{1-|k|/2} + O(e^{T(1-\delta)}L(\nu^k)) & \text{if } |k| \leq 1/2 \\ O(e^{T(1-\delta)}L(\nu^k)) & \text{otherwise} \end{cases}$$

The function $L(\nu^k)$ in the theorem grows like $-\log|k|$, when k approaches 0 (but is 0 in $k = 0$), and might grow in a similar fashion, when k approaches some (finitely many) other points in $(-1, N-1]$, but it is otherwise bounded. The constant δ depends (only) on Γ, and so does the implied constant in the error term.

By integrating the expression from Theorem E wrt. $\exp(-2\pi ikn/N)dk$, we get a version of the prime geodesic theorem, where we only sum over prime geodesics, with a specific Φ-value.

Theorem F. *There exists a $\delta > 0$, such that for $n \in \mathbb{Z}$*

$$\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq\log x \\ N\Phi(\gamma)=n}} l(\gamma) = \frac{4}{N}\int_2^x \frac{\log y}{(4\pi n/N)^2 + (\log y)^2}\, dy + O(x^{1-\delta}).$$

Again δ depends (only) on Γ, and so does the implied constant in the error term.

As a consequence of this theorem we get an asymptotic relation between the number of prime geodesics of bounded length and the number of prime geodesics of bounded length with a given Φ-value.

Theorem G. *For $n \in \mathbb{Z}$*

$$\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq x \\ N\Phi(\gamma)=n}} 1 \sim \frac{4}{Nx}\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq x}} 1.$$

We can use these results to prove that the prime geodesics are asymtotically Cauchy distributed wrt. Φ/l, that is, we can show the following.

Theorem H. *For $x \in \mathbb{R}$ we have*

$$\lim_{t\to\infty}\frac{1}{\pi(t)}\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq t \\ \Phi(\gamma)\leq xl(\gamma)}} 1 = \frac{\arctan(4\pi x)}{\pi} + \frac{1}{2}. \tag{1.3}$$

Note that, when Φ can be interpreted as a linking number, Φ/l is the number of times the geodesic winds around the knot divided by the length of the geodesic.

The $\Gamma = SL_2(\mathbb{Z})$ case of Theorem E-H was already proved by Sarnak and Mozzochi in [10] and [18].

If the limit

$$d(A) = \lim_{M\to\infty}\frac{\#\{n \in A \mid |n| \leq M\}}{2M+1}$$

exists for a subset $A \subset \mathbb{Z}$, then we say that A has natural density $d(A)$, and we conclude the thesis by proving the following theorem about such A.

Theorem I. *If $A \subset \mathbb{Z}$ has natural density $d(A)$, then*

$$\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le T \\ N\Phi(\gamma)\in A}} 1 \sim d(A) \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le T}} 1,$$

when $T \to \infty$.

Chapter 2

Prerequisites

We start out by recalling some properties of Fuchsian groups and automorphic forms. Some general references for this section is [5], [7], [9], [15] and [16].

2.1 Hyperbolic Geometry and Fuchsian Groups

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$ be the upper half plane, and equip it with the Poincaré metric ds given by

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

(where $z = x + iy$). The geodesics on $\mathbb{H}$ is then the vertical half lines and semicircles with center on the real axis.

The isometries on $\mathbb{H}$ is the functions on the form

$$z \mapsto \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{R},\ ad - bc = 1$$

or

$$z \mapsto \frac{a\overline{z}+b}{c\overline{z}+d}, \qquad a, b, c, d \in \mathbb{R},\ ad - bc = -1.$$

The first type of isometries are called Möbius transformations and are of special interest to us. We note that Möbius transformations correspond to matrices in $SL_2(\mathbb{R})$, so for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}),$$

we define

$$\gamma z = \frac{az+b}{cz+d}.$$

We see that γ_1 and γ_2 gives us the same Möbius transformation, if and only if $\gamma_1 = \pm\gamma_2$, so we can identify the set of Möbius transformations with

$PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$, where I is the identity matrix. Furthermore a simple calculation shows that for $\gamma_1, \gamma_2 \in SL_2(\mathbb{R})$ and $z \in \mathbb{H}$, we have

$$(\gamma_1\gamma_2)z = \gamma_1(\gamma_2 z).$$

So this identification of matrices with Möbius transformations is a homomorphism with kernel $\{\pm I\}$.

In the following we will often not distinguish between the matrix $\gamma \in SL_2(\mathbb{R})$, the equivalence class $\{\gamma, -\gamma\} \in PSL_2(\mathbb{R})$ and the corresponding transformation. Sometimes it is however important to make a distinction between γ and $-\gamma$ (for instance when we discuss multiplier systems).

The Möbius transformation for $\gamma(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in SL_2(\mathbb{R})$ is naturally extended to a bijection on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, by

$$\gamma z = \begin{cases} a/c & \text{if } z = \infty \\ \infty & \text{if } z = -d/c \\ \frac{az+b}{cz+d} & \text{otherwise} \end{cases}.$$

One can easily show that

i) The Möbius transformation γ has two fix points if $|Tr\gamma| < 2$. One in the upper half plane and its conjugate.

ii) The Möbius transformation γ has one fix points if $|Tr\gamma| = 2$ and $\gamma \neq \pm I$. This fix point is on $\mathbb{R} \cup \{\infty\}$.

iii) The Möbius transformation γ has two fix points if $|Tr\gamma| > 2$. Both of these are located on $\mathbb{R} \cup \{\infty\}$.

We call a matrix/transformation elliptic, if it is of type i), parabolic if it is of type ii), and hyperbolic if it of type iii). We call a point fixed by an elliptic matrix E for an elliptic point, and we say that it has order m, if m is the smallest positive integer such that $E^m = \pm I$.

The hyperbolic measure μ on $\mathbb{H}$ is given by

$$d\mu(z) = \frac{dxdy}{y^2}.$$

This measure is invariant under Möbius transformations.

A Fuchsian group Γ is a discrete subgroup of $SL_2(\mathbb{R})$ (to define discreteness we can identify $SL_2(\mathbb{R})$ with a subspace of $\mathbb{R}^4$). We say that a measurable subset $\mathcal{F}$ of $\mathbb{H}$ is a fundamental domain for Γ if $\Gamma\mathcal{F} = \{\gamma z \mid \gamma \in \Gamma, z \in \mathcal{F}\} = \mathbb{H}$, and

$$\gamma z_1 = z_2 \Rightarrow z_1, z_2 \in \partial\mathcal{F}$$

for $\gamma \in \Gamma\backslash\{\pm 1\}$ and $z_1, z_2 \in \mathcal{F}$. Since μ is invariant under Möbius transformations, we get that $\mu(\mathcal{F}) = \mu(\mathcal{G})$ if both $\mathcal{F}$ and $\mathcal{G}$ are fundamental domains for Γ, and we define $\mu(\Gamma\backslash\mathbb{H}) := \mu(\mathcal{F})$.

We say that Γ is cocompact, if it has a compact fundamental domain, and that Γ is cofinite if $\mu(\Gamma\backslash\mathbb{H}) < \infty$. If Γ is cofinite, there exists a hyperbolic polygon, which is a fundamental domain for Γ, this implies that Γ is finitely generated. If Γ is cocompact, all of the vertices of this polygon is in $\mathbb{H}$, but if Γ is not cocompact, one or more of these vertices will be in $\mathbb{R}\cup\{\infty\}$. Such a vertex a is called a cusp, and its stabilizer $\Gamma_a = \{\gamma \in \Gamma \mid \gamma a = a\}$ is generated by a parabolic matrix. We will consider two cusps a, b to be equivalent, if they are Γ equivalent, i.e. if there exists a $\gamma \in \Gamma$ such that $\gamma a = b$. The cusps are exactly the (Γ-equivalence classes of) points in $\mathbb{R}\cup\{\infty\}$, that are fixed by some element of Γ.

If Γ is a Fuchsian group, then

$$\pm\Gamma = \{\gamma \mid \gamma \in \Gamma \ \vee\ -\gamma \in \Gamma\}$$

is a Fuchsian group, and it generates the same set of Möbius transformations. Because of this the difference between Γ and $\pm\Gamma$ is relatively small, but some results are easier to state if we assume, that $-I \in \Gamma$. Hence we will in the rest of the thesis only study Fuchsian groups, that contains $-I$, and when we write Fuchsian group, it will be implicit, that this means Fuchsian group containing $-I$.

As mentioned earlier cofinite Fuchsian groups are finitely generated, and the following theorem by Fricke and Klein (see [8] p. 42) gives us some information about a set of generators.

Theorem 2.1.1. *Let Γ be a cofinite Fuchsian group. Let g denote the genus of the surface $\Gamma\backslash\mathbb{H}$, h the number of its cusps, and let r be the number of conjugacy classes of elliptic matrices in Γ. Then $g, h, r < \infty$, and Γ is generated by $-I$, $2g$ hyperbolic matrices $A_1, \dots, A_g, B_1, \dots, B_g$, r elliptic matrices (one from each conjugacy class) $E_1, \dots, E_r$ and h parabolic matrices (one from each conjugacy class) $P_1, \dots, P_h$. These matrices satisfy the identity*

$$[A_1, B_1]\cdots[A_g, B_g]E_1\cdots E_r P_1\cdots P_h = I,$$

where $[A_j, B_j]$ denotes the commutator $A_jB_jA_j^{-1}B_j^{-1}$.

There is an important formula about the area of fundamental domains, called the Gauss-Bonnet formula (see [8] p. 43), which states that

$$\frac{\mu(\Gamma\backslash\mathbb{H})}{2\pi} = 2g - 2 + h + \sum_{j=1}^{r}(1 - e_r^{-1}). \tag{2.1}$$

Here g, h, r are as in Theorem 2.1.1, and $e_1, \dots, e_r$ are the smallest positive integers such that $E_j^{e_j} = \pm I$.

The Gauss defect formula is another important formula for calculating hyperbolic areas. This formula states that a hyperbolic triangle T (i.e. the area between three hyperbolic geodesics, that pairwise intersects each other in $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$), is given by

$$\mu(T) = \pi - \alpha - \beta - \gamma, \tag{2.2}$$

where α, β, γ is the angles of T.

2.2 Automorphic Forms and Functions

Let Γ be a Fuchsian group. If Γ has a cusp in $a \in \mathbb{R} \cup \{\infty\}$, with stabilizer Γ_a, then Γ_a is generated by $-I$ and some matrix γ_a, where $\operatorname{Tr}\gamma_a = 2$. We then have a matrix $\sigma_a \in SL_2(\mathbb{R})$ such that $\sigma_a(a) = \infty$ and $\sigma_a \gamma_a \sigma_a^{-1} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. If we for $f : \mathbb{H} \to \mathbb{C}$, have $f(\gamma_a z) = f(z)$, then we see that $f \circ \sigma_a^{-1}$ is 1-periodic. So $f \circ \sigma_a^{-1}$ has a Fourier expansion, if f is sufficiently nice. We write this expansion as

$$f(\sigma_a^{-1} z) = \sum_{n \in \mathbb{Z}} b_n \exp(2\pi i n z). \tag{2.3}$$

If there is $m \in \mathbb{Z}$, such that $b_n = 0$ for $n < m$, we say that f is meromorphic at a, and if $b_n = 0$ for all negative n we say that f is holomorphic at a.

If $f : \mathbb{H} \to \mathbb{C}$ is meromorphic, and

$$f(\gamma z) = f(z), \qquad \text{for } z \in \mathbb{H} \text{ and } \gamma \in \Gamma,$$

then we call f an automorphic function (with respect to Γ), if f is also meromorphic in the cusps of Γ. If f has a zero (resp. a pole) at z_0 of order m, then for any $\gamma \in \Gamma$, f has a zero (resp. a pole) of order m at γz. Hence, if $f \not\equiv 0$, we can define a function μ_f on $\Gamma \backslash \mathbb{H}$, given by

$$\mu_f(\Gamma z) = \begin{cases} m & \text{if } f \text{ has a zero of order } m \text{ at } z \\ -m & \text{if } f \text{ has a pole of order } m \text{ at } z \\ 0 & \text{otherwise} \end{cases}.$$

Let $f \not\equiv 0$ have the Fourier expansion (2.3) in a, and let $m \in \mathbb{Z}$ be such that $b_m \neq 0$ and $b_n = 0$ for $n < m$. We see that for $\gamma \in \Gamma$, we have $\sigma_{\gamma a} = \sigma_a \gamma^{-1}$, and since f is Γ-invariant, we get the same Fourier expansion in γa. So we can define $\mu_f(\Gamma a)$ to be m, and if $m > 0$ (resp. $m < 0$) we say that f has a zero (resp. pole) of order m (resp $-m$) in a.

Let $z \in \mathbb{H}$ be a elliptic point i.e. a fix point for some elliptic matrix $\gamma_z \in \Gamma$. We define $ord(z)$ to be the order of γ_z (that is, $ord(z)$ is the smallest $n \in \mathbb{N}$ such that $\gamma_z^n = \pm I$). For all non-elliptic points $z \in \mathbb{H}$ we define $ord(z) = 1$. For any $z \in \mathbb{H}$ and $\gamma \in \Gamma$ we have $ord(\gamma z) = ord(z)$, so we can define $ord(\Gamma z) = ord(z)$. Hence the following is well defined, for $f \not\equiv 0$,

$$Deg(f) = \sum_{a \in \Gamma \backslash \mathbb{R} \cup \{\infty\} \text{ cusp}} \mu_f(a) + \sum_{z \in \Gamma \backslash \mathbb{H}} \frac{\mu_f(z)}{ord(z)},$$

and it can be shown that $Deg(f) = 0$ for all non-zero automorphic functions. Especially any non-zero automorphic function that is holomorphic (in $\mathbb{H}$ and in the cusps), will be zero free.

We now change the condition $f(\gamma z) = f(z)$ slightly. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we define $j_\gamma : \mathbb{C} \to \mathbb{C}$ by $j_\gamma(z) = cz + d$, and note that

$$j_{\gamma_1\gamma_2}(z) = j_{\gamma_1}(\gamma_2 z) j_{\gamma_2}(z). \tag{2.4}$$

Let $f : \mathbb{H} \to \mathbb{C}$ be holomorphic, and given such that

$$f(\gamma z) = \nu(\gamma)(j_\gamma(z))^k f(z) \tag{2.5}$$

for all $z \in \mathbb{H}$ and $\gamma \in \Gamma$, where $k \in \mathbb{R}$ (we define $(j_\gamma(z))^k = \exp(k \log(j_\gamma(z))$, where log is the main logarithm), and ν is a function on Γ taking values in $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. If $f \not\equiv 0$ and $\gamma_1, \gamma_2 \in \Gamma$, we see that

$$\nu(\gamma_1\gamma_2) = \frac{\nu(\gamma_1)(j_{\gamma_1}(\gamma_2 z))^k f(\gamma_2 z)}{(j_{\gamma_1\gamma_2}(z))^k f(z)} = \frac{\nu(\gamma_1)\nu(\gamma_2)(j_{\gamma_1}(\gamma_2 z))^k (j_{\gamma_2}(z))^k}{(j_{\gamma_1\gamma_2}(z))^k},$$

and

$$f(z) = f((-I)z) = \nu(-I)\exp(k\pi i) f(z).$$

We note that

$$\frac{(j_{\gamma_1}(\gamma_2 z))^k (j_{\gamma_2}(z))^k}{(j_{\gamma_1\gamma_2}(z))^k} = \exp(ki(\arg j_{\gamma_1}(\gamma_2 z) + \arg j_{\gamma_2}(z) - \arg j_{\gamma_1\gamma_2}(z)))$$

and that by (2.4)

$$\omega(\gamma_1, \gamma_2) := \frac{\arg j_{\gamma_1}(\gamma_2 z) + \arg j_{\gamma_2}(z) - \arg j_{\gamma_1\gamma_2}(z)}{2\pi}$$

is an integer between -1 and 1, and since it is continuous in z, it is independent of $z \in \mathbb{H}$. So if we define

$$w_k(\gamma_1, \gamma_2) = \exp(2\pi i k \omega(\gamma_1, \gamma_2))$$

the following holds for ν

$$\nu(-I) = \exp(-k\pi i), \tag{2.6}$$
$$\nu(\gamma_1\gamma_2) = w_k(\gamma_1, \gamma_2)\nu(\gamma_1)\nu(\gamma_2) \tag{2.7}$$

A function such as ν is called a multiplier system.

Definition 2.2.1. *Let $k \in \mathbb{R}$, let Γ be a Fuchsian group, and let $\nu : \Gamma \to S^1$. If(2.6) and (2.7) holds for ν and k, when $\gamma_1, \gamma_2 \in \Gamma$, we say that ν is a weight k multiplier system on Γ.*

Note that if ν is a weight k multiplier system, then it is also a weight $k+2n$ multiplier system for all $n \in \mathbb{Z}$.

When calculating values of multiplier systems it is useful, with some formulas for ω. In § 2.6 in [7] we find the following formulas

$$\begin{aligned}
\omega(AB, C) + \omega(A, B) &= \omega(A, BC) + \omega(B, C) && (2.8)\\
\omega(DA, B) &= \omega(A, BD) = \omega(A, B) && (2.9)\\
\omega(ADA^{-1}, A) &= \omega(A, A^{-1}DA) = 0 && (2.10)
\end{aligned}$$

when $A, B, C, D \in SL_2(\mathbb{R})$, and D is on the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$.

Let a be a cusp of Γ, and let $f : \mathbb{H} \to \mathbb{C}$ be a function, for which (2.5) holds. We define $f|_\gamma(z) = (j_\gamma(z))^{-k} f(\gamma z)$, and we let $\kappa_a \in [0, 2\pi)$ be such that $\exp(\kappa_a i) = \nu(\gamma_a)$. The function $z \mapsto \exp(-\kappa_a i z) f|_{\sigma_a^{-1}}(z)$ is then 1-periodic, and it has a Fourier expansion

$$\exp(-\kappa_a i z) f|_{\sigma_a^{-1}}(z) = \sum_{n \in \mathbb{Z}} b_n \exp(2\pi i n z).. \tag{2.11}$$

If $b_n = 0$ in (2.11) for $n < 0$, we say that f is holomorphic in a. We can show that, if f is holomorphic in a, then f is holomorphic in γa for all $\gamma \in \Gamma$.

We can now state the definition of holomorphic automorphic forms.

Definition 2.2.2. *We say that $f : \mathbb{H} \to \mathbb{C}$ is a weight k (classical) holomorphic automorphic form with multiplier system ν wrt. Γ, if the following holds:*

i) f is holomorphic in $\mathbb{H}$,

ii) f is holomorphic in the cusps,

iii) equation (2.5) holds for all $z \in \mathbb{H}$ and $\gamma \in \Gamma$.

If f is a weight $k \in \mathbb{Z}$ modular form with trivial multiplier system (i.e. $\nu(\gamma) = 1$ for all $\gamma \in \Gamma$), we call f a modular form. If ν is trivial, (2.6) implies that k is even.

Let f be an automorphic form and a a cusp, and write $f|_{\sigma_a}$ on the form (2.11). We say that f has a zero of order m in a if $b_m \neq 0$ and $b_n = 0$ for $n < m$, and it can easily be shown, that f has a zero of order m in a, if and only if f has a zero of order m in γa for any $\gamma \in \Gamma$. Furthermore if f has a zero of order m in $z \in \mathbb{H}$, then it follows from (2.5), that f has a zero of order m in γz for all $\gamma \in \Gamma$. So we can define μ_f and $Deg(f)$ just as we did for automorphic functions (but μ_f will not assume negative values for automorphic forms).

For cofinite Fuchsian groups we have the following two theorems about the number of zeros of the modular forms and the number of modular forms (see [20] Proposition 2.16. p. 39 and Theorem 2.23. p. 46).

Theorem 2.2.3. *Let Γ be a cofinite Fuchsian group. Let g denote the genus of the surface $\Gamma\backslash\mathbb{H}$, h the number of its cusps, and let r be the number of conjugacy classes of elliptic matrices in Γ. If $f \neq 0$ is a weight k modular form, then*

$$Deg(f) = \frac{k}{2}\left(2g - 2 + h + \sum_{j=1}^{r}(1 - e_r^{-1})\right)$$

Theorem 2.2.4. *Let Γ be a cofinite Fuchsian group. Let g denote the genus of the surface $\Gamma\backslash\mathbb{H}$, h the number of its cusps, and let r be the number of conjugacy classes of elliptic matrices in Γ. If G_k denotes the space of weight k modular forms, then for even k*

$$Dim(G_k) = \begin{cases} (k-1)(g-1) + \frac{kh}{2} + \sum_{j=1}^{r} \left\lfloor \frac{k(1-e_r^{-1})}{2} \right\rfloor & \text{if } k > 2 \\ g + h - 1 & \text{if } k = 2 \text{ and } h > 0 \\ g & \text{if } k = 2 \text{ and } h = 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k < 0 \end{cases}.$$

We note that if we combine Theorem 2.2.3 with formula (2.1), then we get

$$Deg(f) = k \cdot \frac{\mu(\Gamma\backslash\mathbb{H})}{4\pi}.$$

Especially the degree is positive, if the weight is positive.

G_0 is the space of holomorphic automorphic functions. It contains the constant functions and according to Theorem 2.2.4, these are the only holomorphic automorphic functions.

2.3 Spectral Theory of Automorphic Forms

This section is concerned with functions that transforms almost like holomorphic automorphic forms. That is, functions $f : \mathbb{H} \to \mathbb{C}$, for which we have a multiplier system ν on a cofinite Fuchsian group Γ, and a weight $k \in \mathbb{R}$, such that for $z \in \mathbb{H}$ and $\gamma \in \Gamma$

$$f(\gamma z) = \nu(\gamma)\left(\frac{j_\gamma(z)}{|j_\gamma(z)|}\right)^k f(z) \tag{2.12}$$

(as in the previous sections x^k means $\exp(k \log x)$, where log is the main logarithm).

If (2.12) holds for $f : \mathbb{H} \to \mathbb{C}$ and $\mathcal{F}_1$ and $\mathcal{F}_2$ are two fundamental domains wrt. Γ, then

$$\int_{\mathcal{F}_1} |f|^2 \, d\mu = \int_{\mathcal{F}_2} |f|^2 \, d\mu,$$

and hence it makes sense to define

$$||f|| = \left(\int_{\mathcal{F}} |f|^2 \, d\mu \right)^{1/2},$$

for any fundamental domain $\mathcal{F}$ wrt. Γ. We define $\mathcal{H}(\Gamma, \nu, k)$ to be the set of functions f for which (2.12) holds, and $||f|| < \infty$. We can define an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}(\Gamma, \nu, k)$ by

$$\langle f, g \rangle = \int_{\mathcal{F}} f \bar{g} \, d\mu.$$

The following 3 differential operators are of interest

$$\begin{aligned} K_k &= iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{k}{2} = (z - \bar{z})\frac{\partial}{\partial z} + \frac{k}{2} \\ \Lambda_k &= iy\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \frac{k}{2} = (z - \bar{z})\frac{\partial}{\partial \bar{z}} + \frac{k}{2} \\ \Delta_k &= y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - iky\frac{\partial}{\partial x}. \end{aligned}$$

We call K_k the (weight k) Maass raising operator, Λ_k the (weight k) Maass lowering operator and Δ_k the (weight k) Laplacian. If f is C^1, then

$$K_k \left(\left(\frac{j_\gamma(z)}{|j_\gamma(z)|} \right)^{-k} f(\gamma z) \right) = \left(\frac{j_\gamma(z)}{|j_\gamma(z)|} \right)^{-k-2} (K_k f)(\gamma z) \tag{2.13}$$

$$\Lambda_k \left(\left(\frac{j_\gamma(z)}{|j_\gamma(z)|} \right)^{-k} f(\gamma z) \right) = \left(\frac{j_\gamma(z)}{|j_\gamma(z)|} \right)^{-k+2} (\Lambda_k f)(\gamma z). \tag{2.14}$$

Hence if (2.12) holds for f, then (2.12) with k replaced by $k+2$ (resp. $k-2$) holds for $K_k f$ (resp. $\Lambda_k f$). So the raising operator increases the weight by 2 and the lowering operator decreases the weight by 2.

If $f \in \mathcal{H}(\Gamma, \nu, k)$, $g \in \mathcal{H}(\Gamma, \nu, k+2)$ and f, g are C^1 (in x, y), then we have the following identity

$$\langle K_k f, g \rangle = \langle f, \Lambda_{k+2} g \rangle. \tag{2.15}$$

Since

$$-\Delta_k = \Lambda_{k+2} K_k - \frac{k}{2}\left(1 + \frac{k}{2}\right) = K_{k-2}\Lambda_k + \frac{k}{2}\left(1 - \frac{k}{2}\right), \tag{2.16}$$

(see [15] formula (3.4) p. 305) this means that if $f, g \in \mathcal{H}(\Gamma, \nu, k)$ are C^2, then

$$\langle \Delta_k f, g \rangle = \langle f, \Delta_k g \rangle.$$

So if we define the subset $\mathcal{D}(\Gamma,\nu,k) \subset \mathcal{H}(\Gamma,\nu,k)$ by

$$\mathcal{D}(\Gamma,\nu,k) = \{f \in \mathcal{H}(\Gamma,\nu,k) \mid f \text{ is } C^2, \Delta_k f \in \mathcal{H}(\Gamma,\nu,k)\}$$

then Δ_k is symmetric on $\mathcal{D}(\Gamma,\nu,k)$.

The subset $\mathcal{D}(\Gamma,\nu,k)$ is dense in $\mathcal{H}(\Gamma,\nu,k)$, and when we consider $\mathcal{D}(\Gamma,\nu,k)$ to be the domain of Δ_k, then this operator is essentially self-adjoint (see [15] Satz 3.2 p. 310). So $\Delta_k^* = \Delta_k^{**} = \overline{\Delta_k}$, and we can extend Δ_k to be an operator on $\mathcal{H}(\Gamma,\nu,k)$, by defining $\Delta_k := \Delta_k^*$.

When $f,g \in \mathcal{D}(\Gamma,\nu,k)$, then by formula (2.15) and (2.16)

$$\begin{aligned} \langle f, -\Delta_k g\rangle &= \langle K_k f, K_k g\rangle - \frac{k}{2}\left(1+\frac{k}{2}\right)\langle f,g\rangle, && (2.17)\\ \langle f, -\Delta_k g\rangle &= \langle \Lambda_k f, \Lambda_k g\rangle + \frac{k}{2}\left(1-\frac{k}{2}\right)\langle f,g\rangle. && (2.18)\end{aligned}$$

So

$$\langle f, -\Delta_k f\rangle \geq \frac{|k|}{2}\left(1-\frac{|k|}{2}\right)||f||^2$$

for $f \in \mathcal{D}(\Gamma,\nu,k)$, and when we take the closure, we get that the same inequality holds for $f \in \mathcal{H}(\Gamma,\nu,k)$. Hence the spectrum of $-\Delta_k$ is contained in $[|k|/2(1-|k|/2),\infty)$.

All the eigenfunctions of $-\Delta_k$ are in $\mathcal{D}(\Gamma,\nu,k)$ (see [15] Satz 5.7a p. 325). By (2.17) and (2.18) the smallest possible eigenvalue is $|k|/2(1-|k|/2)$, and we can only obtain this eigenvalue, if we for the corresponding eigenfunction f have $\Lambda_k f \equiv 0$ or $K_k f \equiv 0$. On the other hand it follows from formula (2.16), that if $\Lambda_k f \equiv 0$ or $K_k f \equiv 0$, then f is an eigenfunction with this eigenvalue.

If $\Lambda_k f \equiv 0$, then

$$\begin{aligned} 0 &= iy\frac{\partial f}{\partial x}(z) - y\frac{\partial f}{\partial y}(z) + \frac{k}{2}f(z)\\ &= iy^{1+k/2}\frac{\partial}{\partial x}(y^{-k/2}f(z)) - y^{1+k/2}\frac{\partial}{\partial y}(y^{-k/2}f(z))\\ &= y^{1+k/2}\frac{\partial}{\partial \bar{z}}(y^{-k/2}f(z)),\end{aligned}$$

so $y^{-k/2}f(z)$ is holomorphic. If (2.12) holds for f, then (2.5) holds for $y^{-k/2}f(z)$, since $\Im(\gamma z) = y|j_\gamma(z)|^{-2}$ for $\gamma \in SL_2(\mathbb{R})$. If furthermore $||f|| < \infty$, then $y^{-k/2}f(z)$ will be holomorphic in the cusps, and hence it will be a holomorphic automorphic form. On the other hand if $y^{-k/2}f(z)$ is weight k holomorphic automorphic form wrt. Γ and with multiplier system ν, then $f \in \mathcal{D}(\Gamma,\nu,k)$, and $\Lambda_k f \equiv 0$ (by reversing the arguments).

Similar arguments show that $K_k f \equiv 0$ and $f \in \mathcal{D}(\Gamma, \nu, k)$, if and only if $y^{k/2}\overline{f(z)}$ is a weight $-k$ holomorphic automorphic form wrt. Γ and with multiplier system $\bar{\nu}$.

We let $\lambda_0 \leq \lambda_1 \leq \ldots$ be the eigenvalues of $-\Delta_k$ (on $\mathcal{H}(\Gamma, \nu, k)$), where we have $\lambda = \lambda_n$ for m different n's if λ is an eigenvalue of multiplicity m. We define $r_n := (\lambda_n - 1/4)^{1/2}$, so that

$$\left(\frac{1}{2} + ir_n\right)\left(1 - \left(\frac{1}{2} + ir_n\right)\right) = \left(\frac{1}{2} + ir_n\right)\left(\frac{1}{2} - ir_n\right) = \lambda_n.$$

Let Γ_a, γ_a and σ_a be as in the start of section 2.2, and let $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. If ν is a weight $k \in \mathbb{R}$ multiplier system on Γ, $z \in \mathbb{H}$, $s \in \mathbb{C}$ and $\Re s > 1$, then we define

$$E_a(z, s, \nu, k) = \sum_{\gamma \in \Gamma_a \backslash \Gamma} \overline{w_k(\sigma_a, \gamma)\nu(\gamma)\left(\frac{j_{\sigma_a \gamma}(z)}{|j_{\sigma_a \gamma}(z)|}\right)^k} (\Im(\sigma_a \gamma z))^s, \tag{2.19}$$

if a is singular wrt. ν, i.e. if $\nu(\gamma_a) = 1$ (we call a non-singular cusp regular). Here the sum makes sense since

$$\begin{aligned}
\Im(\sigma_a \gamma_a \gamma z) &= \Im(T\sigma_a \gamma z) = \Im(1 + \sigma_a \gamma z) = \Im(\sigma_a \gamma z), \\
j_{\sigma_a \gamma_a \gamma}(z) &= j_{T\sigma_a \gamma}(z) = j_{\sigma_a \gamma}(z)
\end{aligned}$$

and by (2.8), (2.9) and (2.10)

$$\begin{aligned}
\omega(\sigma_a, \gamma_a \gamma) + \omega(\gamma_a, \gamma) &= \omega(\sigma_a \gamma_a, \gamma) + \omega(\sigma_a, \gamma_a) \\
&= \omega(T\sigma_a, \gamma) + \omega(\sigma_a, \sigma_a^{-1} T \sigma_a) \\
&= \omega(\sigma_a, \gamma),
\end{aligned}$$

so the terms in (2.19) do not change if we replace γ by $\gamma_a^n \gamma$ (i.e. takes another representative in $\Gamma_a \gamma$).

It can easily be shown that formula (2.12) holds for $E_a(\cdot, s, \nu, k)$.

If a and b are (not necessarily distinct) singular cusps, and we define a multiplier system ν_{ab} on $\sigma_a \Gamma \sigma_b^{-1}$ by

$$\nu_{ab}(\gamma) = \nu(\sigma_a^{-1} \gamma \sigma_b) w_k(\sigma_a, \sigma_a^{-1} \gamma \sigma_b) w_k(\gamma \sigma_b, \sigma_b^{-1})$$

then we can rewrite (2.19) in the following way

$$\begin{aligned}
\left(\frac{j_{\sigma_b^{-1}}(z)}{|j_{\sigma_b^{-1}}(z)|}\right)^{-k} E_a(\sigma_b^{-1} z, s, \nu, k) &= \sum_{\gamma \in \Gamma_\infty \backslash \sigma_a \Gamma \sigma_b^{-1}} \overline{\nu_{ab}(\gamma)\left(\frac{j_\gamma(z)}{|j_\gamma(z)|}\right)^k} (\Im(\gamma z))^s \\
&= E_\infty(z, s, \nu_{ab}, k).
\end{aligned}$$

We have $K_k y^s = (k/2+s)y^s$ and $\Lambda_k y^s = (k/2-s)y^s$, so by (2.13) and (2.14)

$$\begin{aligned} K_k E_a(\sigma_b^{-1}z,s,\nu,k) &= \left(\frac{k}{2}+s\right) E_a(\sigma_b^{-1}z,s,\nu,k+2) \\ \Lambda_k E_a(\sigma_b^{-1}z,s,\nu,k) &= \left(\frac{k}{2}-s\right) E_a(\sigma_b^{-1}z,s,\nu,k-2), \end{aligned}$$

and hence by (2.16)

$$-\Delta_k E_a(\sigma_b^{-1}z,s,\nu,k) = s(1-s)E_a(\sigma_b^{-1}z,s,\nu,k).$$

For $\Re s > 1$, the function

$$\left(\frac{j_{\sigma_b^{-1}}(z)}{|j_{\sigma_b^{-1}}(z)|}\right)^{-k} E_a(\sigma_b^{-1}z,s,\nu,k) = E_\infty(z,s,\nu_{ab},k)$$

is 1-periodic in z, and it has a Fourier expansion. This Fourier expansion is given by

$$E_\infty(z,s,\nu_{ab},k) = \delta_{ab}y^s + \varphi_{ab}(s,\nu)y^{1-s} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \varphi_{ab,n}(y,s,\nu)e^{2\pi i n x},$$

where

$$\varphi_{ab}(s,\nu) = \frac{\pi 4^{1-s}\Gamma(2s-1)}{\Gamma(s+k/2)\Gamma(s-k/2)} e^{-ik\pi/2} \sum_{\substack{\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\sigma_a\Gamma\sigma_b^{-1}/\Gamma_\infty \\ c>0}} \frac{\overline{\nu_{ab}\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)}}{c^{2s}} \tag{2.20}$$

(see [5] formula (5.20) and (5.22) p. 368), and $\varphi_{ab,n}(y,s,\nu)$ is holomorphic in s (see [5] formula (5.23) p. 369 for specific expressions). By formula (5.21) on p.368 in [5], we have for $\Re s > 1$,

$$E_\infty(z,s,\nu_{aa},k) = y^s + O(y^{1-s}),$$

and hence

$$||E_a(\cdot,s,\nu,k)|| = ||E_\infty(\cdot,s,\nu_{ab},k)|| = \infty.$$

So (2.12) holds for $E_a(\cdot,s,\nu,k)$, and it is an eigenfunction of $-\Delta_k$, but it is not in $\mathcal{H}(\Gamma,\nu,k)$.

The functions $\varphi_{ab}(s,\nu)$ and $E_a(z,s,\nu,k)$ are holomorphic for $\Re s > 1$, and they can be meromorphically extended to all $s\in\mathbb{C}$ (see [16] p. 293). Let $a_1,\ldots,a_{K_0}$ be all the singular cusps (wrt. ν) and define $\Phi(s,\nu)$ to be the matrix given by

$$\Phi(s,\nu) = (\varphi_{a_i a_j}(s,\nu))_{1\le i,j\le K_0}.$$

When we let $\mathcal{E}(z,s,\nu,k)$ be the vector

$$\begin{pmatrix} E_{a_1}(z,s,\nu,k) \\ \vdots \\ E_{a_{K_0}}(z,s,\nu,k) \end{pmatrix},$$

we get

$$\mathcal{E}(z,s,\nu,k) = \Phi(s,\nu)\mathcal{E}(z,1-s,\nu,k) \tag{2.21}$$

(see [16] formula (10.19) p. 296).

The matrix $\Phi(s,\nu)$ is called the scattering matrix, and we define

$$\varphi(s,\nu) = \det \Phi(s,\nu),$$

and call $\varphi(s,\nu)$ the scattering determinant. Formula (2.21) gives us

$$\Phi(s,\nu)\Phi(1-s,\nu) = I, \tag{2.22}$$

where I as usual denotes the identity matrix (but in this case the $K_0 \times K_0$-identity matrix), and we have (see [5] formula (5.24) p. 369)

$$\varphi_{ab}(s,\nu) = \overline{\varphi_{ba}(\bar{s},\nu)}. \tag{2.23}$$

Hence for $s = 1/2 + it$ we get

$$\varphi\left(\frac{1}{2}+it,\nu\right) = \left(\varphi\left(\frac{1}{2}-it,\nu\right)\right)^{-1} = \left(\overline{\varphi\left(\frac{1}{2}+it,\nu\right)}\right)^{-1},$$

so

$$\left|\varphi\left(\frac{1}{2}+it,\nu\right)\right| = 1. \tag{2.24}$$

Furthermore by (2.22) and (2.23)

$$\Phi\left(\frac{1}{2}+it,\nu\right)\overline{\Phi\left(\frac{1}{2}+it,\nu\right)^t} = \Phi\left(\frac{1}{2}+it,\nu\right)\Phi\left(\frac{1}{2}-it,\nu\right) = I,$$

where Φ^t is the transposed of Φ. So by considering the entries in the diagonal, we get

$$\sum_b \left|\varphi_{ab}\left(\frac{1}{2}+it,\nu\right)\right|^2 = 1. \tag{2.25}$$

If $E_a(z_0,s,\nu,k)$ has a pole in $s=s_0$, then $E_a(z,s,\nu,k)$ has a pole in $s=s_0$ for all $z \in \mathbb{H}$, and $\Re s_0 \neq 1/2$, furthermore if $\Re s_0 > 1/2$ (because of (2.21) it is enough to study the zeros and the poles s with $\Re s \geq 1/2$), then the pole is simple (for all $z \in \mathbb{H}$), $s_0 \in (1/2, 1]$ and $\varphi_{aa}(s)$ also has a simple pole in $s = s_0$ (see [16] Satz 10.3 p. 297 and Satz 10.4 p. 299). Conversely if $\varphi_{aa}(s)$ has a pole in $s = s_0$, with $\Re s_0 > 1/2$, then it follows from the Maass-Selberg relations (see [16] Lemma 11.2 p. 300-301), that $E_a(z, s, \nu, k)$ has a pole in $s = s_0$.

The residue $h_{a,s_0}(z)$ of $E_a(z, s, \nu, k)$ in $s = s_0$ is in $\mathcal{D}(\Gamma, \nu, k)$, and h_{a,s_0} is an eigenfunction of $-\Delta_k$ with eigenvalue $s_0(1 - s_0)$ (see [16] Satz 11.2 a) p. 302). If $h_{a,s_0}(z)$ are the residue of $E_a(z, s, \nu, k)$ in $s = s_0$, and $h_{b,s_0}(z)$ is the residue of $E_b(z, s, \nu, k)$ in $s = s_0$, then

$$\langle h_{a,s_0}, h_{b,s_0} \rangle = \mathrm{Res}\left(\varphi_{ab}(s), s = s_0\right) \tag{2.26}$$

(see [16] Satz 11.2 b) p. 302).

If $\kappa : \mathbb{H}^2 \to \mathbb{C}$ is a suitable nice function, then

$$\int_{\mathcal{F}} \sum_{\gamma \in \Gamma} \nu(\gamma) \left(\frac{j_\gamma(z)}{|j_\gamma(z)|} \right)^k \kappa(z, \gamma z)\, d\mu(z), \tag{2.27}$$

is well defined (as usual $\mathcal{F}$ is a fundamental domain for Γ). If we divide Γ into conjugacy classes, we can rewrite (2.27) to the following

$$\sum_{C} \int_{\mathcal{F}} \sum_{\gamma \in C} \nu(\gamma) \left(\frac{j_\gamma(z)}{|j_\gamma(z)|} \right)^k \kappa(z, \gamma z)\, d\mu(z),$$

where the first sum is over the conjugacy classe $C \subset \Gamma$. Dividing this sum into the four cases of the identity, hyperbolic, parabolic and elliptic matrices, lets us express (2.27) in terms that relates to the geometry of $\Gamma \backslash \mathbb{H}$.

The function

$$K(z) = \sum_{\gamma \in \Gamma} \nu(\gamma) \left(\frac{j_\gamma(z)}{|j_\gamma(z)|} \right)^k \kappa(z, \gamma z)$$

is in $\mathcal{H}(\Gamma, \nu, k)$ and can be expressed as a linear combination of eigenfunctions of $-\Delta_k$ plus some integrals over Eisenstein series. We can use this to express (2.27) in terms related to the spectrum of $-\Delta_k$. Combining these expressions we get

Selberg's trace formula (see [5] Theorem 6.3 p. 412-413)

$$\begin{aligned}
\sum_{n=0}^{\infty} h(r_n) &= \frac{\mu(\mathcal{F})}{4\pi}\int_{\mathbb{R}} rh(r)\frac{\sinh(2\pi r)dr}{\cosh(2\pi r)+\cos(\pi k)} && (2.28)\\
&+\frac{\mu(\mathcal{F})}{4\pi}\sum_{\substack{l \text{ odd}\\ 1\le l\le |k|}} (|k|-l)h\left(\frac{i(|k|-l)}{2}\right) && (2.29)\\
&+\sum_{\substack{[\gamma]\\ \operatorname{Tr}\gamma>2}} \frac{\nu(\gamma)l(\gamma_0)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}}g(l(\gamma)) && (2.30)\\
&+\sum_{\substack{\{R\}\\ \operatorname{Tr}R<2\\ 0<\theta(R)<\pi}} \frac{\nu(R)ie^{i(k-1)\theta}}{4M_R\sin\theta}\int_{\mathbb{R}} g(u)e^{(k-1)u/2}\frac{(e^u-e^{2i\theta})du}{\cosh u-\cos(2\theta)} && (2.31)\\
&-g(0)\sum_{\alpha_j\neq 0}\log|1-e^{2\pi i\alpha_j(\nu)}| && (2.32)\\
&+\frac{1}{2}\sum_{\alpha_j(\nu)\neq 0}\left(\tfrac{1}{2}-\alpha_j(\nu)\right)\mathrm{PV}\int_{-\infty}^{\infty} g(u)e^{(k-1)u/2}\frac{(e^u-1)du}{\cosh u-1} && (2.33)\\
&+K_0\int_0^{\infty}\frac{g(u)(1-\cosh(\frac{k}{2}u))}{e^{u/2}-e^{-u/2}}du && (2.34)\\
&-K_0\left(g(0)\log 2+\frac{1}{2\pi}\int_{\mathbb{R}} h(r)\frac{\Gamma'(1+ir)}{\Gamma(1+ir)}dr\right) && (2.35)\\
&+\frac{1}{4}h(0)\operatorname{Tr}(I-\Phi(\tfrac{1}{2},\nu)) && (2.36)\\
&+\frac{1}{4\pi}\int_{\mathbb{R}} h(t)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\nu\right)dt. && (2.37)
\end{aligned}$$

Here h is any holomorphic even function defined on

$$\left\{z\in\mathbb{C}\;\middle|\;|y|<\max\left\{\frac{|k|-1}{2},\frac{1}{2}\right\}+\delta\right\},$$

for some $\delta>0$, such that $h(r)=O((1+r)^{-2-\delta})$, and

$$g(u)=\frac{1}{2\pi}\int_{\mathbb{R}} h(r)e^{-iru}\,dr$$

(h and g comes from the Selberg/Harish-Chandra transform of κ). If there are no singular cusps, then the scattering matrix is not defined, and we define the terms (2.36) and (2.37) to be 0 in this case.

The term (2.37) and the left hand side of (2.28), comes from (2.27) expressed in terms of the spectrum of $-\Delta_k$, and hence we call these terms the spectral terms. The other terms originates from (2.27) expressed in terms related to the geometry of $\Gamma\backslash\mathbb{H}$, and hence we call them geometric terms.

The term (2.30) is related to the hyperbolic conjugacy classes, and hence we refer to it as the hyperbolic term. The γ_0, that occurs in this term, is defined to be the primitive hyperbolic matrix, which γ is a positive power of. The functions l and N are the length and norm, which we will give a definition of in the next section.

The term (2.31) is related to elliptic conjugacy classes, and we call it the elliptic term. Here $\theta = \theta(R)$ is defined to be in $(0, 2\pi)$, such that R is a $SL_2(\mathbb{R})$-conjugate of

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

For a cusp a we define $\alpha_a(\nu) \in [0, 1)$ to be given by $\nu(\gamma_a) = \exp(2\pi i \alpha_a(\nu))$, if b is a cusp equivalent to a, then $\alpha_a(\nu) = \alpha_b(\nu)$ so if we call the equivalence classes of cusps $c_1, \ldots, c_h$, we can define $\alpha_j(\nu) := \alpha_a(\nu)$, where a is some cusp in c_j. So in (2.32) and (2.33) we sum over the regular cusps.

2.4 Closed Geodesics on $\Gamma\backslash\mathbb{H}$

We can identify the group $PSL_2(\mathbb{R})$ with $S\mathbb{H}$ (the unit tangent bundle on $\mathbb{H}$). The standard way to do this is to use the homeomorphism

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\frac{ai+b}{ci+d}, (ci+d)^{-2}\zeta\right),$$

where ζ is the unit vector at i up along the imaginary axis. For $t \in \mathbb{R}$ the matrix

$$\varphi_t = \begin{pmatrix} \exp(t/2) & 0 \\ 0 & \exp(-t/2) \end{pmatrix},$$

sends i to $\exp(t)i$. So the family $\{\varphi_t\}_{t\in\mathbb{R}}$ moves i along the geodesic through i in direction ζ, and $\mathrm{dist}(i, \varphi_t i) = |t|$.

Since Möbius transformations sends geodesics to geodesics and preserves distances, we see that $f_\gamma : \mathbb{R} \to \mathbb{H}$, given by

$$f_\gamma(t) = \gamma\phi_t i, \tag{2.38}$$

is a geodesic for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$, and $\mathrm{dist}(\gamma i, \gamma\varphi_t i) = |t|$. Furthermore the geodesic (2.38) goes through γi in direction $(ci+d)^{-2}\zeta$. For any given point in $\mathbb{H}$ and any direction, there is a unique geodesic going through this point in this direction, so we can write any geodesic in the form (2.38).

Now let Γ be a Fuchsian group, and let $\gamma \in \Gamma$ be hyperbolic. Then γ can be diagonalized, i.e.

$$A^{-1}\gamma A = \pm\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \tag{2.39}$$

for $\lambda = \frac{1}{2}(|Tr\gamma| + \sqrt{(Tr\gamma)^2 - 4}) > 1$ and some $A \in SL_2(\mathbb{R})$. The A in (2.39) is not unique, but if A_1 and A_2 are two such A's, then $A_1 = A_2\phi_t$ for some $t \in \mathbb{R}$.

Since $\phi_{t+s} = \phi_t\phi_s$, we have

$$\gamma^n A\phi_t = (A\phi_{2\log\lambda}A^{-1})^n A\phi_t = A\phi_{2n\log\lambda+t},$$

and hence the points $A\phi_t i$ and $A\phi_{2n\log\lambda+t} i$ on the geodesic $\{A\phi_s i \mid s \in \mathbb{R}\}$ are Γ-equivalent.

In the following we will save some notation and assume, that the sign in (2.39) is positive.

The hyperbolic metric on $\mathbb{H}$ induces a metric on $\Gamma\backslash\mathbb{H}$, hence $\mathcal{G}_\sigma : \mathbb{R} \to \Gamma\backslash\mathbb{H}$, given by

$$\mathcal{G}_\sigma(t) = \{\tau\sigma\phi_t i \mid \tau \in \Gamma\}, \tag{2.40}$$

is a geodesic, for any $\sigma \in SL_2(\mathbb{R})$. When σ is the A in (2.39), we get

$$\{\tau A\phi_t i \mid \tau \in \Gamma\} = \{\tau\gamma A\phi_t i \mid \tau \in \Gamma\} = \{\tau A\phi_{t+2\log\lambda} i \mid \tau \in \Gamma\},$$

so the geodesic $\mathcal{C}_\gamma : [0, 2\log\lambda] \to \Gamma\backslash\mathbb{H}$, given by

$$\mathcal{C}_\gamma(t) = \{\tau A\phi_t i \mid \tau \in \Gamma\} \tag{2.41}$$

is a closed curve.

When $\mathrm{Tr}\,\gamma = \lambda+\lambda^{-1}$, then the geodesic $\mathcal{C}_\gamma$ has length $2\log\lambda$. So for hyperbolic γ with trace $\lambda + \lambda^{-1}$, we define the length

$$l(\gamma) := 2\log\lambda,$$

and norm

$$N(\gamma) := \exp(l(\gamma)) = \lambda^2.$$

If $f, g : [0, a] \to \Gamma\backslash\mathbb{H}$ are closed geodesics and

$$f(t) = \begin{cases} g(t+b) & \text{if } t+b \le a \\ g(t+b-a) & \text{if } t+b > a \end{cases}$$

for some $b \in [0, a]$ (i.e. f is g except that it starts and ends in another point), then we will consider f and g to be the same closed geodesic. So to each hyperbolic element $\gamma = A\phi_l A^{-1}$ in Γ, we can associate a unique closed geodesic $\mathcal{C}_\gamma$ of length l.

Conversely if we have a curve $\mathcal{C}$ on $\Gamma\backslash\mathbb{H}$, then we can lift $\mathcal{C}$ to $\mathbb{H}$, so we get the set

$$\mathcal{C}_{\mathbb{H}} = \{z \in \mathbb{H} \mid \Gamma z \in \mathcal{C}\}.$$

We can choose a $z_0 \in \mathcal{C}_{\mathbb{H}}$, and a neighborhood U of z_0. Then ΓU is a neighborhood of Γz_0 in $\Gamma\backslash\mathbb{H}$, and Γz is on a segment of $\mathcal{C}_{\mathbb{H}}$, which is contained in ΓU. By lifting this segment to $\mathbb{H}$ we get infinitely many curves in $\mathbb{H}$ (one for each element in Γ). If we assume, that $\mathcal{C}$ is a geodesic on $\Gamma\backslash\mathbb{H}$, then these curves becomes segments of geodesics on $\mathbb{H}$. If we name the geodesic through z_0 $\mathcal{C}_0$, then the other geodesics are on the form $\tau\mathcal{C}_0$ for $\tau \in \Gamma$, and $\Gamma\mathcal{C}_0 = \mathcal{C}_{\mathbb{H}}$. If we now assume $\mathcal{C}$ to be closed and of length l, we can choose a point z_1 on $\mathcal{C}_0$, and we denote by z_2 the point that is at distance l from z_1 along $\mathcal{C}_0$ (in direction of the orientation). Moving from z_1 to z_2 then corresponds to move l along $\mathcal{C}$, i.e. once around $\mathcal{C}$, from Γz_1. But then $\Gamma z_1 = \Gamma z_2$, so two points on $\mathcal{C}_0$, that are the l apart, are Γ equivalent. Since this is true for any two points at distance l on $\mathcal{C}_0$, it follows by continuity of the geodesic and discreteness of Γ, that there is a $\gamma \in \Gamma$ that moves any point on $\mathcal{C}_0$ l along $\mathcal{C}_0$. Furthermore $\mathcal{C}_0$ is on the form (2.38), so $\mathcal{C}$ is on the form (2.41) for some $A \in SL_2(\mathbb{R})$, and $\lambda = \exp(l/2)$.

So to any hyperbolic matrix $\gamma \in \Gamma$ we can associate a closed geodesic $\mathcal{C}_\gamma$ on $\Gamma\backslash\mathbb{H}$, and for any closed geodesic $\mathcal{C}$ there is at least one hyperbolic matrix γ, such that $\mathcal{C} = \mathcal{C}_\gamma$.

If $\gamma = A\phi_l A^{-1}$, then for $n \in \mathbb{N}$, we have $\gamma^n = A\phi_{ln}A^{-1}$, and hence

$$\mathcal{C}_{\gamma^n}(t + jl) \;=\; \mathcal{C}_{\gamma^n}(t) \;=\; \mathcal{C}_\gamma(t),$$

for any $j \in \{0, \ldots, n-1\}$. So $\mathcal{C}_{\gamma^n}$ ”runs around” $\mathcal{C}_\gamma$ n times. If γ is a hyperbolic matrix, and γ is not a positive power of another matrix, then we say that γ is primitive, and we say that $\mathcal{C}_\gamma$ is a prime geodesic.

If $\gamma = A\phi_l A^{-1}$ and $\mathcal{C}_\gamma$ runs around the same geodesic $\mathcal{C}_0$ $n > 1$ times, then we must have $0 < t_1 < t_2 < l$, such that

$$\mathcal{C}_\gamma(t_1) \;=\; \mathcal{C}_\gamma(t_2),$$

and small neighborhoods $U_1 \ni t_1$ and $U_2 \ni t_2$, such that

$$\mathcal{C}_\gamma(U_1) = \mathcal{C}_\gamma(U_2).$$

So

$$A\phi_{t_1} i = \gamma_0 A\phi_{t_2} i$$

for some $\gamma_0 \in \Gamma$, and for $s \in U_1$ there is $r \in U_2$ such that

$$A\phi_s i = \sigma A\phi_r i$$

for some $\sigma \in \Gamma$.

Since we can choose U_1 and U_2 arbitrarily small, and since Γ is discrete and Möbius transformations are continuous, we see that $\sigma = \gamma_0$ for all $s \in U_1$. So γ_0 moves all points $A\phi_s i$, with $s \in U_1$ a fixed distance $l_0 < l$ (since it is an isometry) along the geodesic $t \mapsto A\phi_t i$, and hence it does so for all $s \in \mathbb{R}$, so it is hyperbolic (it fixes both endpoints of the geodesic), and on the form $A\phi_{l_0}A^{-1}$. Since $\mathcal{C}_\gamma$ runs around the same geodesic $\mathcal{C}_0$ n times, we have $l = l_0 n$. So $\gamma = \gamma_0^n$, which means γ

is not primitive. Hence the prime geodesics are the closed geodesics, that "goes once around".

If $\gamma_1, \gamma_2 \in \Gamma$ are hyperbolic matrices with $\mathcal{C}_{\gamma_1} = \mathcal{C}_{\gamma_2}$, then we have $\gamma_1 = A_1 \phi_l A_1^{-1}$ and $\gamma_2 = A_2 \phi_l A_2^{-1}$, for some $l > 0$ and $A_1, A_2 \in SL_2(\mathbb{R})$. We note that $A_1 i = \sigma A_2 \phi_t i$ for some $\sigma \in \Gamma$ and $t \in [0, l)$, so if we move along the geodesic, we get $A_1 \phi_s i = \sigma A_2 \phi_{t+s} i$, for $s \in [0, l-t]$. Hence $A_1 = \sigma A_2 \phi_t$ and since $\gamma_2 = (A_2 \phi_t) \phi_l (A_2 \phi_t)^{-1}$, we see that

$$\gamma_1 = A_1 \phi_l A_1^{-1} = (\sigma A_2 \phi_t) \phi_l (\sigma A_2 \phi_t)^{-1} = \sigma \gamma_2 \sigma^{-1}.$$

In other words γ_1 is a Γ-conjugate of γ_2.

If $\gamma, \sigma \in \Gamma$ and γ is hyperbolic, then we can write $\gamma = A \phi_l A^{-1}$. We then have $\sigma \gamma \sigma^{-1} = \sigma A \phi_l (\sigma A)^{-1}$, and we see that for $t \in [0, l]$

$$\mathcal{C}_{\sigma\gamma\sigma^{-1}}(t) = \{\tau \sigma A \phi_t i \mid \tau \in \Gamma\} = \{\tau A \phi_t i \mid \tau \in \Gamma\} = \mathcal{C}_\gamma(t).$$

We have shown, that there is a one-to-one correspondence between closed geodesics on $\Gamma \backslash \mathbb{H}$ and conjugacy classes $[\gamma] = \{\sigma\gamma\sigma^{-1} \mid \sigma \in \Gamma\}$ of hyperbolic matrices $\gamma \in \Gamma$ with positive trace.

Selberg's trace formula gives us a correspondence between conjugacy classes of elements in Γ and the spectrum of the operator Δ_k. The elements in Γ (and conjugacy classes of the elements) can be divided into the identity, elliptic elements, parabolic elements, and hyperbolic elements. To each elliptic elements corresponds an elliptic point, to each conjugacy class of parabolic elements corresponds a cusp, and to conjugacy classes of the hyperbolic elements corresponds closed geodesics. So we can also view the trace formula as a correspondence between the spectrum of Δ_k and the geometry of $\Gamma \backslash \mathbb{H}$. Especially will (2.30), the term we get from the hyperbolic matrices, be a sum over the closed geodesics on $\Gamma \backslash \mathbb{H}$, where we sum expressions given by the length of the geodesics and the multiplier system. The length of the geodesic is obviously closely related to the geodesic itself, but it is not clear, how to interpret the multiplier system as something concerning the geodesic. In section 4 we will suggest how the multiplier systems value for a hyperbolic matrix, can be interpreted as a number (rather) closely related to the geodesic.

For now we will however ignore the multiplier system, by simply letting it be 1 for every element in Γ. For this to make sense, the weight k must be even, so we will let $k = 0$. We can choose the g in the trace formula to depend on $s \in \{z \in \mathbb{C} \mid \Re z > 1\}$ in such a way, that (2.30) becomes a function in s, that is much like the logarithm of Riemann's zeta function (and is known as Selberg's zeta function), but instead of being a sum over prime numbers, it is a sum over prime geodesics. We can then proceed as in the proof of the prime number theorem (for a different approach see chapter 10.8-9 p. 152-156 in [8]), to get an estimate on the number of closed geodesics of a certain length. This estimate is known as the prime geodesic theorem (see p. 155 of [8]).

Theorem 2.4.1 (Prime geodesic theorem). *Let $1 = s_1 > s_2 \geq \cdots \geq s_n > 1/2$ be given, such that the discrete spectrum of Δ_0 intersected with $[0, 1/4)$ is given by $\{s_1(1-s_1), \ldots, s_n(1-s_n)\}$, then*

$$\sum_{\substack{[\gamma]\in\Gamma' \\ N(\gamma)\leq X}} l(\gamma) = \sum_j s_j^{-1} X^{s_j} + O_\Gamma(X^{3/4}). \tag{2.42}$$

We define

$$\pi(t) := \#\{[\gamma] \in \Gamma' \mid l(\gamma) \leq t\}.$$

By using partial summation on Theorem 2.4.1 we get the following corollary.

Corollary 2.4.2. *We have*

$$\pi(\log X) = li(X) + O_\Gamma(X^{s_2} + X^{3/4}), \tag{2.43}$$

where s_2 is as in Theorem 2.4.1.

Chapter 3

Taylor Coefficients for Eisenstein Series

3.1 Modular Forms wrt. $SL_2(\mathbb{Z})$

When no other reference is given, the results in this section can be found in Chapter 2 and 5 in Don Zagier's "Elliptic Modular Forms and Their Applications", which is the first part of [1].

For even $k \geq 4$ we define the Eisenstein Series of weight k wrt. $SL_2(\mathbb{Z})$ $E_k : \mathbb{H} \to \mathbb{C}$ by

$$E_k(z) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)=1}} (mz+n)^{-k}, \tag{3.1}$$

where (m,n) denotes the greatest common divisor of m and n. E_k is a weight k holomorphic modular form wrt. $SL_2(\mathbb{Z})$, and any holomorphic modular form wrt. $SL_2(\mathbb{Z})$ can be written as a polynomial in E_4 and E_6.

In the rest of this chapter we will omit "wrt. $SL_2(\mathbb{Z})$" and just write "modular form", when we consider modular forms wrt. $SL_2(\mathbb{Z})$.

We would like to define a weight 2 Eisenstein Series, but (3.1) does not make sense for $k = 2$, since the sum is not absolutely convergent in this case. We can however define $E_2 : \mathbb{H} \to \mathbb{C}$ by

$$E_2(z) = \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_{\substack{n \in \mathbb{Z} \\ (m,n)=1}} (mz+n)^{-2}.$$

The function E_2 is holomorphic and has a transformation equation much like the one for modular forms. That is

$$E_2(\gamma z) = (cz+d)^2 E_2(z) + \frac{6}{\pi i} c(cz+d) \qquad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Since

$$\left(\Im\left(\frac{az+b}{cz+d}\right)\right)^{-1} = \frac{|cz+d|^2}{(ad-bc)\Im z} = \frac{(cz+d)^2}{(ad-bc)\Im z} - \frac{2ic(cz+d)^2}{(ad-bc)},$$

we see that $E_2^*: \mathbb{H} \to \mathbb{C}$ defined by

$$E_2^*(x+iy) = E_2(x+iy) - \frac{3}{\pi y},$$

transforms like a weight 2 modular form. E_2^* is however not holomorphic.

We define a differential operator D by $D := \frac{1}{2\pi i}\frac{d}{dz}$. If f is a weight k modular form, we have

$$(j_\gamma(z))^{-2} Df(\gamma z) = D(f \circ \gamma)(z) = D(f \cdot j_\gamma^k)(z) = Df(z) j_\gamma^k(z) + \frac{ck}{2\pi i} f(z) j_\gamma^{k-1},$$

when $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. So, much like E_2, Df transforms in the following way

$$Df(\gamma z) = Df(z) j_\gamma^{k+2}(z) + \frac{ck}{2\pi i} f(z) j_\gamma^{k+1}.$$

Because of this it makes sense to define two other (families of) differential operators. For $k \in \mathbb{N}$ define ∂_k and ϑ_k by

$$\partial_k f(z) = Df(z) - \frac{k}{4\pi \Im z} f(z), \qquad \vartheta_k f = Df - \frac{k}{12} E_2 f.$$

We then see that ∂_k takes almost holomorphic (understood as polynomials in $\Im z$ with holomorphic functions as coefficients) modular forms of weight k to almost holomorphic modular forms of weight $k+2$, and ϑ_k takes holomorphic modular forms of weight k to holomorphic modular forms of weight $k+2$. If f is a weight k modular form, we will save notation by writing ∂f and ϑf instead of $\partial_k f$ and $\vartheta_k f$.

Since ϑ takes holomorphic modular forms of weight k to holomorphic modular forms of $k+2$, we have $\vartheta E_4 = cE_6$ for some $c \in \mathbb{C}$. By comparing the constant terms in the Fourier expansions of ϑE_4 and E_6, we see that $c = -1/3$. In the same way we see that $\vartheta E_6 = -E_4^2/2$. Since $\vartheta(fg) = f\vartheta g + g\vartheta f$, and since holomorphic modular forms are polynomials in E_4 and E_6, we can write ϑ in the following way

$$\vartheta = \frac{-E_6}{3}\frac{\partial}{\partial E_4} - \frac{E_4^2}{2}\frac{\partial}{\partial E_6}. \tag{3.2}$$

If f is a weight k modular form, then we define

$$\begin{aligned} \partial^n f &:= \partial_{k+2n-2} \cdots \partial_{k+2} \partial_k f, \\ \vartheta^n f &:= \vartheta_{k+2n-2} \cdots \vartheta_{k+2} \vartheta_k f. \end{aligned}$$

So ∂^n takes almost holomorphic modular forms of weight k to almost holomorphic modular forms of weight $k+2n$, and ϑ^n takes holomorphic modular forms of weight k to holomorphic modular forms of weight $k+2n$.

It turns out, that there is a (in some sense) better way to define powers of ϑ than ϑ^n. For a modular form f of weight k, we define $\vartheta^{[n]}f$ by $\vartheta^{[0]}f = f$, $\vartheta[1]f = \vartheta_k f$ and for $n \geq 1$

$$\vartheta^{[n+1]}f = \vartheta(\vartheta^{[n]}f) - n(k+n-1)\frac{E_4}{144}\vartheta^{[n-1]}f.$$

Formula (4.3) in [11] gives us the following relation between the derivatives ∂^m and $\vartheta^{[m]}$

$$\partial^n f = \sum_{m=0}^{n} \frac{n!}{m!}\binom{n+k-1}{m+k-1}\left(\frac{E_2^*}{12}\right)^{n-m}\vartheta^{[m]}f.$$

For $f : \mathbb{H} \to \mathbb{C}$, $\sigma \in GL_2(\mathbb{C})$ and $k \in \mathbb{Z}$ even, we define $f|_k\sigma$ to be

$$(f|_k\sigma)(z) = \frac{\det(\sigma)^{k/2} f(\sigma z)}{j_\sigma^k(z)}$$

(for $\sigma \in SL_2(\mathbb{R})$ this is consistent with the way we defined $f|_k\sigma$ in section 2.2). For $z_0 = x_0 + iy_0 \in \mathbb{H}$ define

$$\sigma_{z_0} = \begin{pmatrix} -\overline{z_0} & z_0 \\ -1 & 1 \end{pmatrix}.$$

We then have $\sigma_{z_0}0 = z_0$, and $z \mapsto \sigma_{z_0}z$ is a holomorphic bijection from the unit disc $\mathbb{D}$ to $\mathbb{H}$.

Now let f be a weight k holomorphic modular form. Then $f|_k\sigma_{z_0}$ is holomorphic in $\mathbb{D}$, and so it has a Taylor expansion around 0

$$(f|_k\sigma_{z_0})(z) = \sum_{n=0}^{\infty} c_{z_0}(n,f)z^n. \tag{3.3}$$

We then have

$$f(z) = \frac{(z_0-\overline{z_0})^{k/2}}{(z-\overline{z_0})^k}\sum_{n=0}^{\infty} c_{z_0}(n,f)(\sigma_{z_0}^{-1}z)^n.$$

If z_0 is an elliptic point (i.e. z_0 is a $SL_2(\mathbb{Z})$-translate of i or $\rho\exp(\pi i/3)$), and

$$\Gamma_{z_0} := \{\gamma \in SL_2(\mathbb{Z}) \mid \gamma z_0 = z_0\} = \langle\gamma_0\rangle,$$

then

$$\sigma_{z_0}^{-1}\gamma_0\sigma_{z_0} = \begin{pmatrix} j_{\gamma_0}^{-1}(z_0) & 0 \\ 0 & j_{\gamma_0}(z_0) \end{pmatrix},$$

furthermore $j_{\gamma_0}(z_0) = \pm 1$ if $z_0 \in SL_2(\mathbb{Z})i$, and $j_{\gamma_0}(z_0) = \rho^{\pm 1}$ if $z_0 \in SL_2(\mathbb{Z})\rho$ (see [6] section 4.1).

Since

$$(f|_k\sigma_{z_0})(\sigma_{z_0}^{-1}\gamma\sigma_{z_0}z) = \frac{\det(\sigma_{z_0})^{k/2}f(\gamma\sigma_{z_0}z)}{j^k_{\sigma_{z_0}}(\sigma_{z_0}^{-1}\gamma\sigma_{z_0}z)} = j^k_{\sigma_{z_0}^{-1}\gamma\sigma_{z_0}}(z)(f|_k\sigma_{z_0})(z),$$

we see that

$$\begin{aligned} j_{\gamma_0}^{-k}(z_0)(f|_k\sigma_{z_0})\left(\frac{z}{j_{\gamma_0}^2(z_0)}\right) &= j_{\gamma_0}^{-k}(z_0)(f|_k\sigma_{z_0})(\sigma_{z_0}^{-1}\gamma_0\sigma_{z_0}z) \\ &= j_{\gamma_0}^{-k}(z_0)j^k_{\sigma_{z_0}^{-1}\gamma_0\sigma_{z_0}}(z)(f|_k\sigma_{z_0})(z) \\ &= (f|_k\sigma_{z_0})(z). \end{aligned}$$

By Cauchy's integral formula we have

$$c_{z_0}(n,f) = \frac{1}{2\pi i}\int_{S_r}\frac{f|_k\sigma_{z_0}(z)}{z^{n+1}}dz$$

where $r>0$ is small, and S_r is the cirkel with radius r and center 0. Hence

$$\begin{aligned} c_{z_0}(n,f) &= \frac{1}{2\pi i}\int_{S_r}\frac{f|_k\sigma_{z_0}(z)}{z^{n+1}}dz = \frac{1}{2\pi i}\int_{S_r}\frac{f|_k\sigma_{z_0}(j_{\gamma_0}^{-2}(z_0)z)}{j^k_{\gamma_0}(z_0)z^{n+1}}dz \\ &= \frac{1}{2\pi i j_{\gamma_0}^{k+2n}(z_0)}\int_{S_r}\frac{f|_k\sigma_{z_0}(z)}{z^{n+1}}dz = j_{\gamma_0}^{-k-2n}(z_0)c_{z_0}(n,f) \end{aligned}$$

So if $z_0 \in SL_2(\mathbb{Z})i$ and 4 does not divide $k+2n$, then $c_{z_0}(n,f)=0$, and likewise if $z_0 \in SL_2(\mathbb{Z})\rho$ and 6 does not divide $k+2n$, then $c_{z_0}(n,f)=0$.

Let $z_0 \in \mathbb{H}$ and f be a holomorphic modular form. According to [11] formula (3.7) we have

$$c_{z_0}(n,f) = \sum_{m=0}^{n}\binom{n+k-1}{m+k-1}\frac{(z_0-\overline{z_0})^{m+k/2}}{m!}f^{(m)}(z_0),$$

and by formula (3.9) in [11]

$$\partial^n f(z) = \frac{n!}{(-4\pi\Im z)^n}\sum_{m=0}^{n}\binom{n+k-1}{m+k-1}\frac{(2i\Im z)^m}{m!}f^{(m)}(z_0),$$

so

$$c_{z_0}(n,f) = \frac{(2\pi i)^n(2i\Im z_0)^{n+k/2}}{n!}\partial^n f(z_0). \tag{3.4}$$

3.2 Values of L-Functions

In [1] (p. 89-90) Zagier shows a connection between $\partial^n E_k(i)$, with ∂^m and E_k as in section 3.1, and L-series for Hecke characters. More precisely he shows, that for $4|k+2n$

$$\begin{aligned}\partial^n E_k(i) &= \frac{(k+n-1)!}{2\zeta(k)(-4\pi)^n(k-1)!}\sum_{\lambda\in\mathbb{Z}[i]\setminus\{0\}}\frac{\overline{\lambda}^n}{\lambda^{k+n}}\\ &= \frac{(k+n-1)!}{2\zeta(k)(-4\pi)^n(k-1)!}\sum_{\lambda\in\mathbb{Z}[i]\setminus\{0\}}\frac{(\overline{\lambda}/\lambda)^{k/2+n}}{|\lambda|^k}\\ &= \frac{2(k+n-1)!}{\zeta(k)(-4\pi)^n(k-1)!}\sum_{\mathfrak{a}}\frac{\psi_{k/2+n}(\mathfrak{a})}{N(\mathfrak{a})^k},\end{aligned}$$

where the last sum runs over the ideals $\mathfrak{a}$ of $\mathbb{Z}[i]$, N is the norm, and $\psi_{k/2+n}$ is the Hecke character given by $\psi_{k/2+n}(\mathfrak{a}) = (\overline{\lambda}/\lambda)^{k/2+n}$, where λ is a generator of $\mathfrak{a}$ (this is independent of the choice of λ, since $4|k+2n$).

We know that if 4 does not divide $k+2n$, then $c_i(n, E_k) = 0$, and hence by (3.4) $\partial^n E_k(i) = 0$. Another way to see this is by noting

$$\frac{(a-bi)^n}{(a+bi)^{k+n}} = \frac{(a-bi)^{k+2n}}{(a^2+b^2)^{k+n}} = -\frac{(ai+b)^{k+2n}}{(a^2+b^2)^{k+n}} = -\frac{(b+ai)^n}{(b-ai)^{k+n}},$$

for $k+2n \equiv 2 \pmod 4$. So the terms cancel out in the sum over $\mathbb{Z}[i]\setminus\{0\}$.

So when $k+2n \equiv 2 \pmod 4$, $\partial^n E_k(i) = c_i(n, E_k) = 0$. But what happens, when $4|k+2n$?

For

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$Si = i$, and hence

$$E_2^*(i) = E_2^*(Si) = i^2 E_2^*(i).$$

So $E_2^*(i) = 0$. Hence

$$\partial^n E_k(i) = \sum_{m=0}^{n}\frac{n!}{m!}\binom{n+k-1}{m+k-1}\left(\frac{E_2^*(i)}{12}\right)^{n-m}\vartheta^{[m]}E_k(i) = \vartheta^{[n]}E_k(i).$$

So we can study the holomorphic modular form $\vartheta^{[n]}E_k$ instead of the almost holomorphic modular form $\partial^n E_k$.

If f is a weight k holomorphic modular form, then f is a polynomial in E_4 and E_6. So we can write

$$f = \sum_{\substack{a,b\geq 0\\ 4a+6b=k}} c(a,b)E_4^a E_6^b.$$

We can then use (3.2) and get

$$\begin{aligned} -12\vartheta f &= \sum_{4a+6b=k} c(a,b)(6bE_6^{b-1}E_4^{a+2} + 4aE_6^{b+1}E_4^{a-1}) \\ &= kE_6E_4^{-1}(-12)^{-n}f - 6(E_6^2E_4^{-3}-1)\sum_{4a+6b=k+2n} c(a,b)bE_6^{b-1}E_4^{a+2}. \end{aligned}$$

For $n \geq 0$ we write

$$(-12)^n\vartheta^{[n]}f = \sum_{\substack{a,b\geq 0\\ 4a+6b=k+2n}} c(a,b)E_4^aE_6^b, \tag{3.5}$$

and define a family of complex polynomials $\{p_n\}_{n=0}^\infty$, by

$$p_n(t) = \sum_{\substack{a,b\geq 0\\ 4a+6b=k+2n}} c(a,b)t^b. \tag{3.6}$$

So

$$(-12)^n\vartheta^{[n]}f = E_4^{(k+2n)/4}p_n(E_6E_4^{-3/2}).$$

Since $E_6(i) = 0$ and $E_4(i) = 12\Omega_{-4}^4$, with $\Omega_{-4} = \Gamma(1/4)^2/(4\pi^{3/2})$ (see [11] section 5.1) we especially have

$$\vartheta^{[n]}f(i) = (-1)^n 12^{k/4-n/2}\Omega_{-4}^{k+2n}p_n(0). \tag{3.7}$$

We can use the definition of $\vartheta^{[n]}$ and our calculation of $-12\vartheta f$ to get a recurrence relation on the p_n's

$$\begin{aligned} p_1(t) &= ktp_0(t) - 6(t^2-1)p_0'(t), \\ p_{n+1}(t) &= (2n+k)tp_n(t) - 6(t^2-1)p_n'(t) - n(n+k-1)p_{n-1}(t) \quad \text{for } n \geq 1. \end{aligned}$$

Note that this implies, that $p_n(t) \in \mathbb{Z}[t]$ for all $n \in \mathbb{N}$, if $p_0 \in \mathbb{Z}[t]$.

For $m \in \mathbb{N}$ we define an equivalence relation $\equiv_m$ on $\mathbb{Z}[t]$, by

$$\sum_n a_nt^n \equiv_m \sum_n b_nt^n,$$

if $a_n \equiv b_n \pmod m$ for $0 \leq n \leq m-1$. So there are m^m equivalence classes wrt. $\equiv_m$, and we see from the recurrence relation on $\{p_n\}_n$, that if we know, which equivalence classes $p_n(t)$ and $p_{n-1}(t)$ are in, we can calculate which class $p_{n+1}(t)$ is in. Furthermore if $p_{n_0}(t) \equiv_m p_{n_0+mn_1}(t)$ and $p_{n_0+1}(t) \equiv_m p_{n_0+mn_1+1}(t)$, then $p_n(t) \equiv_m p_{n+n_1m}(t)$ for $n \geq n_0$. Especially $p_n(0) \equiv p_{n+n_1m}(0) \pmod m$ for $n \geq n_0$.

If $f = E_4$, then $p_0(t) = 1$ and hence $p_1(t) = 4t$. Running the following commands in Maple calculates polynomials that are $\equiv_{13}$-equivalent to the first 1000 p_n's

```
p[0] := 1; p[1] := 4*t; for i from 2 to 1000 do
p[i] := `mod`(simplify((2*i+2)*t*p[i-1]
-6*(t^2-1)*(diff(p[i-1], t))-(i-1)*(i+2)*p[i-2]), 13) end do:
for i from 0 to 1000 do q[i] := taylor(p[i], t = 0, 13) end do;
```

We see that

$$\begin{aligned} p_{10}(t) &\equiv_{13} p_{946}(t) \equiv_{13} 4t^4 + 11t^2 + 11 \\ p_{11}(t) &\equiv_{13} p_{947}(t) \equiv_{13} 7t^3 + 6t, \end{aligned}$$

and since $936 = 72 \cdot 13$, this means that $p_n(t) \equiv_{13} p_{n+936}(t)$ for $n \geq 10$. Especially

$$p_n(0) \equiv p_{n+936}(0) \pmod{13} \tag{3.8}$$

for $n \geq 10$.

After having run the previous commands in Maple, we can run these commands

```
for i from 1 to 1000 do n[i]:=coeff(p[i],t,0) end do;
sum('if'(n[j+72]-n[j]=0, 0, 1), j= 1 .. 928);
```

The first command finds $p_n(0) \pmod{13}$ for $1 \leq n \leq 1000$, and the second calculates that

$$\sum_{n=1}^{928} (1 - \delta_{[p_{n+72}(0)]_{13}[p_n(0)]_{13}}) = 0,$$

where $\delta_{[p_{n+72}(0)]_{13}[p_n(0)]_{13}} = 1$ if $p_{n+72}(0) \equiv p_n(0) \pmod{13}$ and 0 otherwise. Hence

$$p_n(0) \equiv p_{n+72}(0) \pmod{13},$$

for $1 \leq n \leq 928$ (we do however have $p_0(0) = 1$ and $p_{72}(0) \equiv 11 \pmod{13}$). When we combine this with (3.8), we see, that for any $n, m \in \mathbb{N}$ we have $p_n(0) \equiv p_m(0) \pmod{13}$, when $n \equiv m \pmod{72}$.

If we run the following command in Maple

```
`mod`(coeff(product(p[2*n], n = 1 .. 36), t, 0),13);
```

we see, that

$$\prod_{n=1}^{36} p_{2n}(0) \equiv 5 \pmod{13}.$$

Especially 13 does not divide $p_n(0)$, when $n \leq 72$ is even (when n is odd $p_n(0) = 0$, since $\vartheta^{[n]} E_4(i) = 0$).

So we have proven, that for all $n \geq 0$,

$$p_n(0) = (-1)^n 12^{n/2-1} \Omega_{-4}^{-4-2n} \vartheta^{[n]} E_4(i) = (-1)^n 12^{n/2-1} \Omega_{-4}^{-4-2n} \partial^n E_4(i)$$

is an integer. If n is odd, we trivially have $p_n(0) = 0$, but when n is even, 13 does not divide $p_n(0)$, and $p_n(0) \equiv p_m(0) \pmod{13}$, when $n \equiv m \pmod{72}$.

By combining this with the relation between L-series and $\partial^n E_k(i)$, and by using that $\zeta(4) = \pi^4/90$, we see that

$$\begin{aligned} p_n(0) = (-1)^n 12^{n/2-1}\Omega_{-4}^{-4-2n}\partial^n E_4(i) &= \frac{40\cdot\Omega_{-4}^{-4-2n}\cdot 3^{n/2}(3+n)!}{(2\pi)^{n+4}}\sum_{\mathfrak{a}}\frac{\psi_{2+n}(\mathfrak{a})}{N(\mathfrak{a})^4} \\ &= \frac{40\cdot\Omega_{-4}^{-4-2n}\cdot 3^{n/2}(3+n)!}{(2\pi)^{n+4}}L(\psi_{2+n},4). \end{aligned}$$

So we have proven the following.

Theorem 3.2.1. *For even $n \geq 0$, define*

$$\mathcal{A}_n := \frac{40\cdot\Omega_{-4}^{-4-2n}\cdot 3^{n/2}(3+n)!}{(2\pi)^{n+4}}L(\psi_{2+n},4).$$

Then $\mathcal{A}_n \in \mathbb{Z}$, and $\mathcal{A}_n$ is not divisible by 13. Furthermore for $n \geq 2$ we have

$$\mathcal{A}_n \equiv \mathcal{A}_{n+72} \pmod{13}.$$

If we look at E_6 instead of E_4, we can in a similar way prove the following.

Theorem 3.2.2. *For odd $n \in \mathbb{N}$, define*

$$\mathcal{B}_n := \frac{14\cdot\Omega_{-4}^{-6-2n}3^{(n+1)/2}(n+5)!}{(2\pi)^{n+6}}L(\psi_{3+n},6).$$

Then $\mathcal{B}_n \in \mathbb{Z}$, and for $m \geq 0$ we have

$$\begin{aligned} \mathcal{B}_{4m+1} &\equiv 1 \pmod{5}, \\ \mathcal{B}_{4m+3} &\equiv 3 \pmod{5}. \end{aligned}$$

Proof. If we let $f = E_6$, and we define p_n by (3.5) and(3.6), we get $p_0(t) = t$ and $p_1(t) = 6$.

We can then proceed, almost as we did for E_4. Running the following commands in Maple

```
p[0] := t; p[1] := 6; for i from 2 to 27 do
p[i] := `mod`(simplify((2*i+4)*t*p[i-1]
-6*(t^2-1)*(diff(p[i-1], t))-(i-1)*(i+4)*p[i-2]), 5) end do:
```

gives us $p_5(t) \equiv_5 p_{25}(t)$, and $p_6 \equiv_5 p_{26}(t)$, $p_{4n+1}(0) \equiv 1 \pmod{5}$ and $p_{4n+3}(0) \equiv 3 \pmod{5}$ for $0 \leq n \leq 6$.

Hence by (3.7), we have

$$p_n(0) = (-1)^n 12^{(n-3)/2}\Omega_{-4}^{-6-2n}\vartheta^{[n]}E_6(i) = (-1)^n 12^{(n-3)/2}\Omega_{-4}^{-6-2n}\partial^n E_6(i)$$

is 0 if n is odd, congruent to 1 (mod 5) if $n \equiv 1 \pmod 4$, and congruent to 3 (mod 5) if $n \equiv 3 \pmod 4$. Since $\zeta(6) = \pi^6/945$, we have

$$\begin{aligned}(-1)^n 12^{(n-3)/2}\Omega_{-4}^{-6-2n}\partial^n E_6(i) &= \frac{14 \cdot \Omega_{-4}^{-6-2n} 3^{(n+1)/2}(n+5)!}{(2\pi)^{n+6}} \sum_{\mathfrak{a}} \frac{\psi_{3+n}(\mathfrak{a})}{N(\mathfrak{a})^6} \\ &= \frac{14 \cdot \Omega_{-4}^{-6-2n} 3^{(n+1)/2}(n+5)!}{(2\pi)^{n+6}} L(\psi_{3+n}, 6).\end{aligned}$$

So $\mathcal{B}_n = p_n(0)$, which proves the theorem. □

A similar approach works for Eisenstein series of higher weight. From $k = 12$ the Eisenstein series E_k will not in general (if ever) have integer coefficients, when written as a polynomial in E_4 and E_6. It will however have rational coefficients, so we can multiply with a suitable constant to get something with integer coefficients. By proceeding as in this section, we can then construct congruences modulo p (for some prime p) for polynomials related to $\vartheta^{[n]}E_k(i)$ and $L(\psi_{k/2+n}, k)$. However, we might not be able to find a p, such that these values are non-zero modulo p.

We could also consider Taylor coefficients for E_k in ρ instead of i, which in the same way, could give us some congruences for some other L-functions.

3.3 Zeros of $\vartheta^n E_k$

Instead of studying of $\vartheta^{[n]}E_k$ we can study zeros of $\vartheta^n E_k$.

In [13] F. K. C. Rankin and H. P. F. Swinnerton-Dyer proved that for any even $k \geq 4$ all zeros of E_k are in the set

$$\left\{ \gamma e^{ti} \,\middle|\, \gamma \in SL_2(\mathbb{Z}),\, t \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right] \right\}.$$

They did this by showing, that $e^{itk/2}E_k(e^{it}) \in \mathbb{R}$ for $t \in (0, \pi)$, and $e^{itk/2}E_k(e^{it}) = 2\cos(tk/2) + R_1(t)$, with $R_1(t) < 2$ for $t \in [\pi/2, 2\pi/3]$. So if $t \in [\pi/2, 2\pi/3]$, then $e^{itk/2}E_k(e^{it})$ is positive when $tk/(2\pi)$ is an even integer, and negative when $tk/(2\pi)$ is odd. If we define $\theta = tk/(2\pi)$, we see that $t \in [\pi/2, 2\pi/3]$ is equivalent to $\theta \in [k/4, k/3]$, and hence $E_k(e^{it})$ has at least $\#([k/4, k/3] \cap \mathbb{N}) - 1$ different zeros $t \in (\pi/2, 2\pi/3)$.

If we define $s \in \{0, 4, 6, 8, 10, 14\}$ by $s \equiv k \pmod{12}$, we see (by considering each possible value of s separately), that

$$\#([k/4, k/3] \cap \mathbb{N}) - 1 = \frac{k-s}{12}.$$

By Theorem 2.2.3 we have $Deg(E_k) = k/12$, and since all points in $\{\exp(it) \mid t \in [\pi/2, 2\pi/3]\}$ are $SL_2(\mathbb{Z})$ inequivalent, we see, that

$$Deg(E_k) \geq \sum_{x \in (\pi/2, 2\pi/3)} \mu_{E_k}(e^{ix}) + \frac{\mu_{E_k}(i)}{2} + \frac{\mu_{E_k}(\rho)}{3} \geq \frac{k-s}{12} + \frac{\mu_{E_k}(i)}{2} + \frac{\mu_{E_k}(\rho)}{3}.$$

If a, b, c are non-negative integers and

$$a + \frac{b}{2} + \frac{c}{3} = \frac{s}{12},$$

we see, that

$$(a,b,c) = \begin{cases} (0,0,0) & \text{if } s = 0 \\ (0,0,1) & \text{if } s = 4 \\ (0,1,0) & \text{if } s = 6 \\ (0,0,2) & \text{if } s = 8 \\ (0,1,1) & \text{if } s = 10 \\ (0,1,2) & \text{if } s = 14 \end{cases}.$$

Hence we must have

$$\frac{\mu_{E_k}(i)}{2} + \frac{\mu_{E_k}(\rho)}{3} = \frac{s}{12}.$$

So all E_k's zeros are in $\{\gamma(e^{it}) \mid \gamma \in SL_2(\mathbb{Z}),\ t \in [\pi/2, 2\pi/3]\}$, and they are all simple, except if $k \equiv 2 \pmod 6$, then there is a double zero in ρ (and $SL_2(\mathbb{Z})$-translates of ρ).

It turns out, that this can be generalized to $\vartheta^n E_k$, and that the zeros of $\vartheta^n E_k$ and $\vartheta^{n+1} E_k$ interlaces.

Theorem 3.3.1. *For $k \geq 4$ even and $n \geq 0$ the modular form $\vartheta^n E_k$ has only zeros in $\{\gamma(e^{it}) \mid \gamma \in SL_2(\mathbb{Z}),\ t \in [\pi/2, 2\pi/3]\}$. Except for a possible double zero in $SL_2(\mathbb{Z})\rho$, all these zeros are simple, and if $\pi/2 \leq t_1 < t_2 \leq 2\pi/3$ and $\vartheta^n E_k(e^{it_1}) = \vartheta^n E_k(e^{it_2}) = 0$, then $\vartheta^{n+1} E_k(e^{it}) = 0$ for some $t \in (t_1, t_2)$.*

Proof. Since E_k is a modular form wrt. $SL_2(\mathbb{Z})$, it can be expressed as a polynomial in E_4 and E_6, so we have

$$E_k = \sum_{4a+6b=k} c(a,b) E_4^a E_6^b.$$

Hence

$$e^{itk/2} E_k(e^{it}) = \sum_{4a+6b=k} c(a,b) (e^{2it} E_4)^a (e^{3it} E_6)^b,$$

and since $e^{itk/2} E_k(e^{it}), e^{2it} E_4(e^{it}), e^{3it} E_6(e^{it}) \in \mathbb{R}$ for $t \in [\pi/2, 2\pi/3]$, we see that the $c(a,b)$'s are real. By (3.2) $\vartheta^n E_k$ have real coefficient, when expressed as a polynomial in E_4 and E_6. So $e^{it(k/2+n)} \vartheta^n E_k(e^{it}) \in \mathbb{R}$ for $t \in [\pi/2, 2\pi/3]$.

We define $F_{k,n} : [\pi/2, 2\pi/3] \to \mathbb{R}$ by

$$F_{k,n}(t) = e^{it(k/2+n)} \vartheta^n E_k(e^{it}).$$

We then have

$$\begin{aligned} F_{k,n+1}(t) &= e^{it(k/2+n+1)}\vartheta\vartheta^n E_k(e^{it}) \\ &= \frac{-e^{it(k/2+n)}}{2\pi}\frac{d}{dt}(\vartheta^n E_k(e^{it})) - \frac{(k+2n)E_2(e^{it})e^{it(k/2+n+1)}}{12}\vartheta^n E_k(e^{it}) \\ &= \frac{-1}{2\pi}F'_{k,n}(t) + \left(\frac{i}{4\pi} - \frac{E_2(e^{it})e^{it}}{12}\right)(k+2n)F_{k,n}(t). \end{aligned}$$

If $t_1 < \cdots < t_\nu$ are the different zeros of $F_{k,n}$, then we see that $2\pi F_{k,n+1}(t_j) = -F'_{k,n}(t_j)$, for $j = 1, \ldots, \nu$. Especially $F_{k,n+1}(t_j) \neq 0$ unless $j = \nu$ and $k+2n \equiv 2 \pmod 6$.

If $1 \leq j < \nu$, then t_j is a simple zero of $F_{k,n}$, so $F'_{k,n}(t_j) < 0$ if $F_{k,n}(t_j + \epsilon) < 0$ for small ϵ and $F'_{k,n}(t_j) > 0$ if $F_{k,n}(t_j + \epsilon) > 0$. If $t_{j+1} \neq 2\pi/3$ or $k + 2n \not\equiv 2 \pmod 6$, then t_{j+1} is a simple zero of $F_{k,n}$ and hence $F'_{k,n}(t_{j+1}) < 0$ if $F_{k,n}(t_{j+1} - \epsilon) > 0$ and $F'_{k,n}(t_{j+1}) > 0$ if $F_{k,n}(t_{j+1} - \epsilon) < 0$.

Since $F_{k,n}$ does not have any zeros in (t_j, t_{j+1}), $F_{k,n}(t_j + \epsilon)$ and $F_{k,n}(t_{j+1} - \epsilon)$ have the same sign, and hence $F'_{k,n}(t_j)$ and $F'_{k,n}(t_{j+1})$ have opposite signs, unless $k + 2n \equiv 2 \pmod 6$ and $j + 1 = \nu$. Hence $F_{k,n+1}$ has a zero in $(t_j, t_j + 1)$ unless $k + 2n \equiv 2 \pmod 6$ and $j + 1 = \nu$.

If $k + 2n \equiv 2 \pmod 6$, then $F_{k,n}$ has a double zero in $t_\nu = 2\pi/3$, and so $F'_{k,n}$ has a simple zero in $2\pi/3$. So for t close to $2\pi/3$ $F_{k,n+1}(t)$ is approximately $-(2\pi)^{-1}F'_{k,n}(t)$, especially $F_{k,n+1}(t)$ and $F'_{k,n}(t)$ has opposite signs.

If $F'_{k,n}(t_{\nu-1}) > 0$, then $F_{k,n}(t) > 0$ for $t \in (t_{\nu-1}, 2\pi/3)$, and since $F_{k,n}(2\pi/3) = 0$, $F_{k,n}(t)$ is descending for t close to $2\pi/3$, so $F'_{k,n}(t) < 0$. So

$$F_{k,n+1}(t_{\nu-1}) = \frac{-1}{2\pi}F'_{k,n}(t_{\nu-1}) < 0,$$

and $F_{k,n+1}(t) > 0$ for t close to $2\pi/3$ since $-2\pi F_{k,n+1}(t) \approx F_{k,n}(t) < 0$, and hence $F_{k,n+1}$ has a zero in $(t_{\nu-1}, t_\nu)$.

Likewise if $F'_{k,n}(t_{\nu-1}) < 0$, then $F_{k,n+1}(t_{\nu-1}) > 0$ and $-2\pi F_{k,n+1}(t) \approx F_{k,n}(t) > 0$ for t close to $2\pi/3$. So $F_{k,n+1}$ has a zero in $(t_{\nu-1}, 2\pi/3)$.

So we have proven the interlacing property stated in the theorem. To prove the rest of the theorem we proceed by induction. Rankin and Swinnerton-Dyer's result tells us, that the theorem is true for $n = 0$. Now assume that it is true for some fixed n. If $k + 2n \equiv 2 \pmod{12}$, then $\vartheta^n E_k$ has a (simple) zero in i and a (double) zero in ρ, and $(2n + k - 14)/12$ other zeros on the arc between these two points. Hence by the interlacing property $\vartheta^{n+1}E_k$ has $(2n + k - 2)/12$ zeros in $\{\exp(it) \mid t \in (\pi/2, 2\pi/3)\}$, and since it is a modular form of weight $k + 2n + 2 \equiv 4 \pmod{12}$, it also has a simple zero in ρ. Since

$$Deg(\vartheta^{n+1}E_k) = \frac{k+2n+2}{12} = \frac{k+2n-2}{12} + \frac{1}{3},$$

these are all the zeros of $\vartheta^{n+1}E_k$, and they are all simple, and so the theorem holds for $\vartheta^{n+1}E_k$ (if $k + 2n \equiv 2 \pmod{12}$).

Similar considerations for $k + 2n \equiv 0, 4, 6, 8, 10 \pmod{12}$ shows that the theorem is true for $\vartheta^{n+1}E_k$, regardless of which congruence class $k + 2n$ is in. This completes the induction and the proof. $\square$

Chapter 4

An Interpretation of some Multiplier Systems

4.1 Zero Free Automorphic Forms

Let $f : \mathbb{H} \to \mathbb{C}\backslash\{0\}$ be a weight $k_0 > 0$ holomorphic automorphic form wrt. a cofinite Fuchsian group Γ, and multiplier system ν, and assume that f has no zeros in $\mathbb{H}$. Since f is zero free, there is a holomorphic logarithm F of f, and hence we can define a function $\Phi : \Gamma \to \mathbb{R}$ by

$$F(\gamma z) = F(z) + k_0 \log(j_\gamma(z)) + 2\pi i k_0 \Phi(\gamma),$$

for some $z \in \mathbb{H}$ (the definition is independent of which $z \in \mathbb{H}$ we choose, and which logarithm F we choose). If we assume that Φ only takes rational values, it follows from (2.7), that since Γ is cofinite and hence finitely generated, there is an $m \in \mathbb{N}$ such that $m\Phi(\gamma) \in \mathbb{Z}$ for all $\gamma \in \Gamma$. Let N be the smallest such m. Since $\Phi(-I) = 1/2$ modulo 1, we know that N is even.

We can define powers of ν by $\nu^t = \exp(2\pi i t k_0 \Phi)$. Then $\exp(kF/k_0)$ is a weight k automorphic form with multiplier system ν^{k/k_0}.

We see, that $f_N := f^{N/k_0} = \exp(NF/k_0)$ is a modular form of weight N, and hence it has positive degree. So f_N is zero free (in $\mathbb{H}$) but has positive degree, so it must have a zero in a cusp, and the sum of the multiplicities of the zeros in the cusps must be the degree. This implies, that if Γ is cocompact, we do not have this type of automorphic forms, since there are no cusps, and hence no zero free modular forms.

If $\Gamma\backslash\mathbb{H}$ has genus 0, and $e_1, \ldots, e_r$ is the orders of the elliptic matrices, we can choose $k_0/2$ to be the lowest common multiple of $e_1, \ldots, e_r$. Then by Theorem 2.2.4 we have

$$Dim(G_{k_0}) = 1 - k_0 + \frac{k_0 h}{2} + \frac{k_0}{2}\sum_{j=1}^{r}(1 - e_r^{-1}).$$

We can choose a cusp a and a basis $f_1, \ldots, f_{Dim(G_{k_0})}$ for G_{k_0}. By writing the f_j's Fourier expansions in a, and solving a system of $Dim(G_{k_0}) - 1$ linear equations with $Dim(G_{k_0})$ variables, we get a linear combination $f \not\equiv 0$ of the f_j's, with the first $Dim(G_{k_0}) - 1$ Fourier coefficients in a equal to 0. So f has a zero in a of order at least $Dim(G_{k_0}) - 1$. Since $f \in G_{k_0}$ and $g = 0$ we have by Theorem 2.2.3

$$Deg(f) = \frac{k_0}{2}\left(-2 + h + \sum_{j=1}^{r}(1 - e_r^{-1})\right) = Dim(G_{k_0}) - 1.$$

Hence all f's zeros are in a.

So f is zero free in $\mathbb{H}$, and so we can take powers of f (note that even though the multiplier system for f is trivial, this will not in general be the case for the powers of f).

An explicit construction of a holomorphic logarithm of f can be found in [3], this construction also works for $g \neq 0$, but it is not clear whether it produces something, where Φ takes rational values on hyperbolic elements.

4.2 Hecke Triangle Groups and Knots

In [2] É. Ghys makes a connection between the logarithm of the multiplier system for Dedekinds eta function and the linking number of prime geodesics with a certain knot. In this section we will make a generalization of this, to multiplier systems on Hecke Triangle groups.

For integer $n \geq 3$ we define $\lambda_n = 2\cos(\pi/n)$ and

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T_n := \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix}.$$

The Hecke triangle group H_n is the group generated by S and T_n (note that $H_3 = SL_2(\mathbb{Z})$). It can be shown that,

$$\mathcal{F}_n := \left\{ z \in \mathbb{H} \mid |z| \geq 1, \, |\Re z| \leq \frac{\lambda_n}{2} \right\}$$

is a fundamental domain for H_n.

We have $S^2 = -I$, and

$$T_n S = \begin{pmatrix} \lambda_n & -1 \\ 1 & 0 \end{pmatrix}.$$

The Chebychev polynomials of the second kind are given by

$$U_{-1} = 0, \, U_0(x) = 1, \, U_m(x) = 2xU_{m-1}(x) - U_{m-2}(x) \text{ for } m \in \mathbb{N}.$$

An equivalent way to define U_m is, by

$$U_m(\cos(t)) = \frac{\sin((m+1)t)}{\sin t}.$$

We can show by induction, that for $m \in \mathbb{N}$

$$\begin{pmatrix} 2x & -1 \\ 1 & 0 \end{pmatrix}^m = \begin{pmatrix} U_m(x) & -U_{m-1}(x) \\ U_{m-1}(x) & -U_{m-2}(x) \end{pmatrix}. \tag{4.1}$$

If we replace x by $\cos(\pi/n)$ in (4.1), we see that $(T_nS)^n = -I$, and that for $0 < m < n$ $(T_nS)^m \neq \pm I$.

So T_nS is an elliptic matrix of order n, and we note that T_nS fixes $\rho_n := \exp(i\pi/n)$.

The set $\mathcal{F}_n$ is a hyperbolic triangle, with vertices in ∞, ρ_n and $-\overline{\rho}_n$. The angle in ∞ is 0, and the angles in ρ_n and $-\overline{\rho}_n$ are π/n, so by formula (2.2)

$$\mu(\mathcal{F}_n) = \frac{\pi(n-2)}{n}.$$

Since $\mathcal{F}_n$ has one cusp (in ∞), and we have two elliptic matrices in H_n of order 2 and n, formula (2.1) tells us that

$$\mu(H_n \backslash \mathbb{H}) \geq 2\pi\left(-2 + 1 + \frac{1}{2} + \frac{n-1}{n}\right) = \frac{\pi(n-2)}{n},$$

with equality if and only if $g = 0$ and we only have these two conjugacy classes of elliptic matrices. Since we have $\mu(\mathcal{F}_n) = \mu(H_n \backslash \mathbb{H})$, this must be the case.

We can now use Theorem 2.2.4 to see that there exists modular forms wrt. H_n of weight 4 and 6 (one of each). These forms are unique up to multiplication by a constant, and in Lemma 4.2.1 we show, that they do not have zeros in the cusp. Hence there is a unique weight 4 modular form (wrt. H_n) E_4, for which the constant coefficient in the Fourier expansion is 1. Likewise there is a unique weight 6 modular form E_6, for which the constant coefficient in the Fourier expansion is 1.

Lemma 4.2.1. *The modular form E_4 has a zero of multiplicity $n-2$ in ρ_n, E_6 has a zero of multiplicity 1 in i and a zero of multiplicity $n-3$ in ρ_n. If we define*

$$\begin{aligned} G &:= E_6^2 - E_4^3, \\ D &:= G^{n-2} E_4^{-2n+6}, \\ H &:= G^{n-3} E_4^{-2n+9}, \end{aligned}$$

then D has a zero of multiplicity $n-2$ in the cusp, while H has a zero of multiplicity $n-3$ in the cusp and a zero of multiplicity n in ρ_n. All other zeros of E_4, E_6, D, H in $\mathbb{H} \cup \{\infty\}$ are H_n-translates of these zeros.

Proof. By Theorem 2.2.3 E_4 and E_6 has degrees

$$\begin{aligned} Deg(E_4) &= \frac{4\mu(\mathcal{F}_n)}{4\pi} = \frac{n-2}{n}, \\ Deg(E_6) &= \frac{6\mu(\mathcal{F}_n)}{4\pi} = \frac{3(n-2)}{2n}. \end{aligned}$$

If we differentiate the transformation formula for modular forms

$$f(\gamma z) = (j_\gamma(z))^k f(z)$$

μ times, we get that

$$f^{(\mu)}(\gamma z) = (j_\gamma(z))^{k+2\mu} f^{(\mu)}(z) + \sum_{m=0}^{\mu-1} p_m(z) f^{(m)}(z), \tag{4.2}$$

for some polynomials $p_0, \ldots, p_{n-1}$ (depending on the choice of γ). If $\gamma = S$, $z = i$ and $\mu = \mu_f(i)$ is the multiplicity of f's zero in i ($\mu = 0$ if i is not a zero of f), formula (4.2) becomes

$$f^{(\mu)}(i) = i^{k+2\mu} f^{(\mu)}(i),$$

so $4|(k + 2\mu_f(i))$.

Likewise if $\gamma = T_n S$, $z = \rho_n$ and $\mu = \mu_f(\rho_n)$ is the multiplicity of f's zero in ρ_n, formula (4.2) becomes

$$f^{(\mu)}(\rho_n) = \exp(i\pi(k + 2\mu)/n) f^{(\mu)}(\rho_n),$$

so $2n|(k + 2\mu_f(\rho_n))$.

Combining this with the degrees of E_4 and E_6 we see, that E_4 has a zero of multiplicity $n-2$ in ρ_n, and all other zeros are H_n-translates of ρ_n, and E_6 has a zero of multiplicity 1 in i, a zero of multiplicity $n-3$ in ρ_n and no other zeros (except for H_n-translates).

It follows from the definition of G, E_4, E_6, that G has a zero in ∞. Since E_4^3 has a zero of order $3n-6$ in ρ_n, andE_6^2 has a zero of order $2n-6$ in ρ_n, G has a zero of order $2n-6$ in ρ_n. By Theorem 2.2.3 and 2.1 we have

$$Deg(G) = \frac{12\mu(\mathcal{F}_n)}{4\pi} = \frac{3(n-2)}{n} = 1 + \frac{2n-6}{n}.$$

So G's only zeros are in the cusp and in the H_n translates of ρ_n.

From the definition of D and H, it now follows, that the zeros of D and H are as stated in the lemma. □

We can choose a holomorphic $n-2$'nd root g of E_4 in the following way. Choose ξ to be an $n-2$'nd root of $E_4(i)$. For $N \in \mathbb{N}$ define B_N to be the ball consisting of all points in $\mathbb{H}$, with hyperbolic distance to i less than N.

There are a finite number of H_n-translates of ρ_n in B_N. If we denote these translates $\gamma_1\rho_n, \ldots, \gamma_{M_N}\rho_n$, we can for $z \in B_N$ write $E_4(z)$ in the following way

$$E_4(z) = \exp(\psi_N(z)) \prod_{j=1}^{M_N} (z - \gamma_j \rho_n)^{n-2},$$

where ψ_N is some holomorphic function on B_N. Hence we can choose $m \in \mathbb{N}$ (depending on N) such that

$$\xi = \exp\left(\frac{\psi_N(i) + 2\pi i m}{n-2}\right) \prod_{j=1}^{M_N} (i - \gamma_j \rho_n),$$

and define a holomorphic function $g_N : B_N \to \mathbb{C}$ by

$$g_N(z) = \exp\left(\frac{\psi_N(z) + 2\pi i m}{n-2}\right) \prod_{j=1}^{M_N} (z - \gamma_j \rho_n).$$

So $g_N^{n-2}(z) = E_4(z)$ for $z \in B_N$ and $g_N(i) = \xi$. Hence for $N_1 < N_2$ and $z \in B_{N_1}$ we have $g_{N_1}(z) = g_{N_2}(z)$, and it makes sense to define $g(z)$ to be $g_N(z)$, for any $N \in \mathbb{N}$, that is greater than the (hyperbolic) distance from i to z.

The function g has a simple zero in $\gamma\rho_n$ for any $\gamma \in H_n$, and since we have

$$g^{n-2}(\gamma z) = E_4(\gamma z) = j_\gamma^4(z) E_4(z) = ((j_\gamma(z))^{4/(n-2)} g(z))^{n-2},$$

we see that

$$g(\gamma z) = \nu(\gamma)(j_\gamma(z))^{4/(n-2)} g(z),$$

for some $n-2$'nd root of unity $\nu(\gamma)$ ($\nu(\gamma)$ is continuous as a function of z, so it is independent on z). So g is an automorphic form of weight $4/(n-2)$ with multiplier system ν. If we define $h := E_6 g^{-n+3}$, we see that h is a weight $2n/(n-2)$ automorphic form with multiplier system ν, and that h has simple zeros in γi for $\gamma \in \Gamma$, and no other zeros.

We let $GL_2^+(\mathbb{R})$ be the real 2×2-matrices with positive determinant and define $\Lambda : GL_2^+(\mathbb{R}) \to \mathbb{C}^2$ by

$$\Lambda(\sigma) = \left(\frac{g(\sigma i)}{(j_\sigma(i))^{4/(n-2)}}, \frac{h(\sigma i)}{(j_\sigma(i))^{2n/(n-2)}} \right).$$

If we define $\sim$ to be the equivalence relation on $\mathbb{C}^2$ given by $(z_1, z_2) \sim (z_3, z_4)$ if and only if, there is a $n-2$'nd root of unity ζ, such that $z_1 = \zeta z_3$ and $z_2 = \zeta z_4$, then we get the following lemma.

Lemma 4.2.2. *For $\sigma_1, \sigma_2 \in GL_2^+(\mathbb{R})$, $\Lambda(\sigma_1) \sim \Lambda(\sigma_2)$ if and only if $\sigma_1\sigma_2^{-1} \in H_n$.*

Proof. If $\sigma \in GL_2^+(\mathbb{R})$ and $\gamma \in H_n$, then we see that

$$\begin{aligned}
\frac{g(\gamma\sigma i)}{(j_{\gamma\sigma}(i))^{4/(n-2)}} &= \nu(\gamma)\left(\frac{j_\gamma(\sigma i)}{j_{\gamma\sigma}(i)}\right)^{4/(n-2)} g(\sigma i) \\
&= \nu(\gamma)\exp\left(\frac{8\pi i}{n-2}\omega(\gamma,\sigma)\right)\frac{g(\sigma i)}{(j_\sigma(i))^{4/(n-2)}}, \\
\frac{h(\gamma\sigma i)}{(j_{\gamma\sigma}(i))^{2n/(n-2)}} &= \nu(\gamma)\left(\frac{j_\gamma(\sigma i)}{j_{\gamma\sigma}(i)}\right)^{2n/(n-2)} h(\sigma i) \\
&= \nu(\gamma)\exp\left(\frac{4n\pi i}{n-2}\omega(\gamma,\sigma)\right)\frac{h(\sigma i)}{(j_\sigma(i))^{2n/(n-2)}} \\
&= \nu(\gamma)\exp\left(\frac{8\pi i}{n-2}\omega(\gamma,\sigma)\right)\frac{h(\sigma i)}{(j_\sigma(i))^{2n/(n-2)}}.
\end{aligned}$$

So

$$\Lambda(\gamma\sigma) \sim \Lambda(\sigma),$$

which proves the "if" part.

Now define $J = H/D$. Then by Lemma 4.2.1 J is an automorphic function with a simple pole in the cusp, and a zero of multiplicity n in $\gamma\rho_n$, for $\gamma \in H_n$, and these are all the poles and zeros. For any $z_0 \in \mathbb{C}$, $J - z_0$ is an automorphic function, which has a single simple pole in the cusp, and hence

$$J^{-1}(z_0) = (J - z_0)^{-1}(0) = H_n z,$$

for some $z \in \mathbb{H}$. So $H_n z \mapsto J(z)$ is a bijection between $H_n\backslash\mathbb{H}$ and $\mathbb{C}$.

We have

$$J = \frac{E_4^3}{G} = \frac{E_4^3}{E_6^2 - E_4^3} = \frac{g^{3n-6}}{(hg^{n-3})^2 - g^{3n-6}} = \frac{g^n}{h^2 - g^n}.$$

If we choose $\sigma_1, \sigma_2 \in GL_2^+(\mathbb{R})$ such that $\Lambda(\sigma_1) = (z_1, z_2) = \zeta\Lambda(\sigma_2)$, with ζ a $n-2$'nd root of unity, then we see that

$$J(\sigma_1 i) = \frac{g^n(\sigma_1 i)\zeta^{-n}(j_{\sigma_1}(i))^{-4n/(n-2)}}{(h^2(\sigma_1 i) - g^n(\sigma_1 i))\zeta^{-n}(j_{\sigma_1}(i))^{-4n/(n-2)}} = J(\sigma_2 i)$$

so $\sigma_1 i = \gamma\sigma_2 i$ for some $\gamma \in H_n$. So if $\Lambda(\sigma_1) \sim \Lambda(\sigma_2)$, we see that $\sigma_1 i \in H_n\sigma_2 i$.

If $\sigma_1 i = \gamma\sigma_2 i$ then $\sigma_2^{-1}\gamma^{-1}\sigma_1$ fixes i and is hence on the form

$$\begin{pmatrix} \lambda\cos\theta & -\lambda\sin\theta \\ \lambda\sin\theta & \lambda\cos\theta \end{pmatrix}, \tag{4.3}$$

for some $\lambda > 0$ and $\theta \in \mathbb{R}$. We see, that

$$\begin{aligned}
\frac{g(\sigma_1 i)}{(j_{\sigma_1}(i))^{4/(n-2)}} &= \frac{g(\gamma\sigma_2 i)}{(j_{\gamma\sigma_2\sigma_2^{-1}\gamma^{-1}\sigma_1}(i))^{4/(n-2)}} \\
&= \frac{\nu(\gamma)(j_\gamma(\sigma_2 i))^{4/(n-2)} g(\sigma_2 i)}{(j_{\gamma\sigma_2}(\sigma_2^{-1}\gamma^{-1}\sigma_1 i) j_{\sigma_2^{-1}\gamma^{-1}\sigma_1}(i))^{4/(n-2)}} \\
&= \frac{\nu(\gamma)(j_\gamma(\sigma_2 i))^{4/(n-2)} g(\sigma_2 i)}{(j_{\gamma\sigma_2}(i)(i\lambda\sin\theta + \lambda\cos\theta))^{4/(n-2)}} \\
&= \frac{\nu(\gamma)\exp(8\pi i\omega(\gamma,\sigma_2)/(n-2))}{(i\lambda\sin\theta + \lambda\cos\theta)^{4/(n-2)}} \cdot \frac{g(\sigma_2 i)}{(j_{\sigma_2}(i))^{4/(n-2)}},
\end{aligned}$$

and likewise

$$\frac{h(\sigma_1 i)}{(j_{\sigma_1}(i))^{2n/(n-2)}} = \frac{\nu(\gamma)\exp(8\pi i\omega(\gamma,\sigma_2)/(n-2))}{(i\lambda\sin\theta + \lambda\cos\theta)^{2n/(n-2)}} \cdot \frac{h(\sigma_2 i)}{(j_{\sigma_2}(i))^{2n/(n-2)}}.$$

Since $\Lambda(\sigma_1) = \zeta\Lambda(\sigma_2)$, with $|\zeta| = 1$, we see that $\lambda = 1$, and

$$\begin{aligned}
\frac{g(\sigma_1 i)}{(j_{\sigma_1}(i))^{4/(n-2)}} &= \frac{\zeta\nu(\gamma)\exp(8\pi i\omega(\gamma,\sigma_2)/(n-2))}{(i\lambda\sin\theta + \lambda\cos\theta)^{4/(n-2)}} \cdot \frac{g(\sigma_1 i)}{(j_{\sigma_1}(i))^{4/(n-2)}}, \\
\frac{h(\sigma_1 i)}{(j_{\sigma_1}(i))^{4/(n-2)}} &= \frac{\zeta\nu(\gamma)\exp(8\pi i\omega(\gamma,\sigma_2)/(n-2))}{(i\lambda\sin\theta + \lambda\cos\theta)^{2n/(n-2)}} \cdot \frac{h(\sigma_1 i)}{(j_{\sigma_1}(i))^{4/(n-2)}}.
\end{aligned}$$

So if $\sigma_1 i \notin H_n(\{i, \rho_n\})$, then $g(\sigma_1 i), h(\sigma_1 i) \neq 0$, and hence

$$1 = \frac{\zeta\nu(\gamma)\exp(8\pi i\omega(\gamma,\sigma_2)/(n-2))}{\zeta\nu(\gamma)\exp(8\pi i\omega(\gamma,\sigma_2)/(n-2))} = \frac{(i\sin\theta + \cos\theta)^{2n/(n-2)}}{(i\sin\theta + \cos\theta)^{4/(n-2)}} = (i\sin\theta + \cos\theta)^2.$$

Hence $\sigma_2^{-1}\gamma^{-1}\sigma_1 = \pm I$, and so $\sigma_1\sigma_2^{-1} = \pm\gamma \in H_n$.

If $\sigma_1 i \in H_n i$, we see that $g(\sigma_1 i) \neq 0$. Then we have

$$1 = \zeta^{n-2} = \left(\frac{\nu(\gamma)\exp(8\pi i\omega(\gamma,\sigma_2)/(n-2))}{(i\sin\theta + \cos\theta)^{4/(n-2)}}\right)^{n-2} = e^{4i\theta}.$$

So $2\theta/\pi \in \mathbb{Z}$ and hence $\sigma_2^{-1}\gamma^{-1}\sigma_1 = S^m$ for some $m \in \mathbb{Z}$. We know that $\sigma_1 i = \gamma_1 i$ for some $\gamma_1 \in H_n$, and hence $\gamma_1^{-1}\sigma_1$ is on the form (4.3). Since matrices on the form (4.3) commutes, we see

$$\gamma_1^{-1}\gamma\sigma_2 = \gamma_1^{-1}\sigma_1 S^{-m} = S^{-m}\gamma_1^{-1}\sigma_1,$$

so $\sigma_1\sigma_2^{-1} = \gamma_1 S^m \gamma_1^{-1}\gamma \in H_n$.

Likewise if $\sigma_1 i = \gamma_1\rho_n$ for some $\gamma_1 \in H_n$, then $h(\sigma_1 i) \neq 0$, and $\sigma_2^{-1}\gamma^{-1}\sigma_1$ is on the form (4.3), with $\lambda = 1$ and $n\theta/\pi \in \mathbb{Z}$. Now let

$$\tau = \begin{pmatrix} \sin(\pi/n) & \cos(\pi/n) \\ 0 & 1 \end{pmatrix},$$

then $\tau^{-1}\gamma_1^{-1}\sigma_1$ is on the form (4.3), so

$$\tau^{-1}\gamma_1^{-1}\gamma\sigma_2 = (\tau^{-1}\gamma_1^{-1}\sigma_1)(\sigma_2^{-1}\gamma^{-1}\sigma_1)^{-1} = (\sigma_2^{-1}\gamma^{-1}\sigma_1)^{-1}\tau^{-1}\gamma_1^{-1}\sigma_1.$$

Hence $\sigma_1 = \gamma_1\tau(\sigma_2^{-1}\gamma^{-1}\sigma_1)\tau^{-1}\gamma_1^{-1}\gamma\sigma_2$, and since

$$\begin{aligned}\tau\begin{pmatrix}\cos(m\pi/n) & -\sin(m\pi/n)\\ \sin(m\pi/n) & \cos(m\pi/n)\end{pmatrix}\tau^{-1} &= \left(\tau\begin{pmatrix}\cos(\pi/n) & -\sin(\pi/n)\\ \sin(\pi/n) & \cos(\pi/n)\end{pmatrix}\tau^{-1}\right)^m\\ &= (T_nS)^m,\end{aligned}$$

we have $\sigma_1\sigma_2^{-1} = \gamma_1(T_nS)^m\gamma_1^{-1}\gamma \in H_n$.

So we have proved for all $\sigma_1, \sigma_2 \in GL_2^+(\mathbb{R})$, that $\sigma_1\sigma_2^{-1} \in H_n$ if $\Lambda(\sigma_1) \sim \Lambda(\sigma_2)$. □

Due to Lemma 4.2.2, we can define a function $\Lambda_0 : H_n\backslash GL_2^+(\mathbb{R}) \to \mathbb{C}^2/\sim$ by

$$\Lambda_0(H_n\sigma) = \{e^{2\pi im/(n-2)}\Lambda(\sigma) \mid m \in \mathbb{Z}\}.$$

We have the following lemma about Λ_0.

Lemma 4.2.3. *The function Λ_0 maps $H_n\backslash GL_2^+(\mathbb{R})$ homeomorphically to $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^n \neq z_2^2\}/\sim$.*

Proof. It follows from Lemma 4.2.2, that Λ_0 is injective.

If $\Lambda(\sigma) = (z_1, z_2)$, then we have

$$J(\sigma i) = \frac{g^n}{h^2 - g^n}(\sigma i) = \frac{g^n(\sigma i)(j_\sigma(i))^{4n/(n-2)}}{(h^2(\sigma i) - g^n(\sigma i))(j_\sigma(i))^{4n/(n-2)}} = \frac{z_1^n}{z_2^2 - z_1^n},$$

and since J's only pole is in the cusp, this shows that $z_1^n \neq z_2^2$.

On the other hand, if $z_1, z_2 \in \mathbb{C}\backslash\{0\}$ and $z_1^n \neq z_2^2$, then there is a $z \in \mathbb{H}$, such that

$$J(z) = \frac{z_1^n}{z_2^2 - z_1^n}.$$

Since $z_1, z_2 \neq 0$, $J(z) \neq 0, -1$, and hence $g(z), h(z) \neq 0$. So we can define $z_4 := h(z)z_1/(g(z)z_2)$, and let $a, b, c, d \in \mathbb{R}$ be such that, $ci + d$ is a square root of z_4, and $ai + b = z(ci + d)$. Then $\sigma := \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ is a 2×2-matrix that sends i to $z \in \mathbb{H}$, and hence $\sigma \in GL_2^+$.

We then see that

$$\frac{g^n}{h^2 - g^n}(z) = J(z) = \frac{z_1^n}{z_2^2 - z_1^n},$$

and hence $z_2^2 g^n(z) = h^2(z)z_1^n$. This gives us

$$\frac{g(z)^{n-2}}{(ci + d)^4} = \frac{g^n(z)z_2^2}{h^2(z)z_1^2} = z_1^{n-2},$$

so $g(z)/(ci+d)^{4/(n-2)} = \zeta z_1$ for some $n-2$'nd root of unity ζ. We can then conclude, that

$$\frac{h(z)}{(ci+d)^{2n/(n-2)}} = \frac{h(z)\zeta z_1}{(ci+d)^2 g(z)} = \zeta z_2,$$

and hence

$$\Lambda(\sigma) \;=\; \left(\frac{g(z)}{(ci+d)^{4/(n-2)}}, \frac{h(z)}{(ci+d)^{2n/(n-2)}}\right) \sim (z_1, z_2).$$

If $z_2 \neq z_1 = 0$, then we can choose $a, b, c, d \in \mathbb{R}$, such that $ci+d$ is a $2n$'th root of $z_2^{2-n} h^{n-2}(\rho_n)$ and $ai+b = \rho_n(ci+d)$. Then $\sigma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$, and

$$\frac{h^{n-2}(\sigma i)}{(ci+d)^{2n}} = z_2^{n-2}.$$

So $\Lambda(\sigma) \sim (z_1, z_2)$.

Likewise if $z_1 \neq z_2 = 0$, then we can choose $a, b, c, d \in \mathbb{R}$, such that $ci+d$ is a 4'th root of $z_1^{2-n} g^{n-2}(i)$ and $ai+b = i(ci+d)$. Then $\sigma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$, and

$$\frac{g^{n-2}(\sigma i)}{(ci+d)^4} = z_1^{n-2}.$$

So $\Lambda(\sigma) = (z_1, z_2)$.

This shows that Λ_0 maps $H_n \backslash GL_2^+(\mathbb{R})$ surjectively to $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^n \neq z_2^2\}/\sim$.

Λ_0 is continuous because Λ is continuous. To see that Λ_0^{-1} is continuous choose $(x, y), (s, t) \in \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^n \neq z_2^2\}$, such that (x, y) is "close to" $(\zeta s, \zeta t)$ for some $n-2$'nd root of unity ζ, and let σ_1 and σ_2 be such that

$$\Lambda(\sigma_1) = (x, y), \qquad \Lambda(\sigma_2) = (s, t).$$

Then

$$J(\sigma_1) = \frac{x^n}{y^2 - x^n},$$

is close to

$$J(\sigma_2) = \frac{s^n}{t^2 - s^n} = \frac{(\zeta s)^n}{(\zeta t)^2 - (\zeta s)^n},$$

and hence σ_1 is close to $\gamma\sigma_2$ for some $\gamma \in H_n$. □

Due to the identification of $PSL_2(\mathbb{R})$ with the unit tangent bundle on the hyperbolic plane, $H_n \backslash SL_2(\mathbb{R})$ can be identified with the unit tangent bundle on $H_n \backslash \mathbb{H}$. Hence if we restrict Λ_0 to $H_n \backslash SL_2(\mathbb{R})$, Lemma 4.2.3 gives an identification of the unit tangent bundle on $H_n \backslash \mathbb{H}$, with some subset of $\mathbb{C}^2/\sim$. This subset is however not particularly nice.

We note that if $\Lambda(\sigma) = (z_1, z_2)$, and $t > 0$, then

$$\Lambda\left(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\sigma\right) = \left(\frac{z_1}{t^{4/(n-2)}}, \frac{z_2}{t^{2n/(n-2)}}\right). \tag{4.4}$$

Hence it is natural to define a function $\widetilde{\Lambda} : SL_2(\mathbb{R}) \to \mathbb{S}^3$ by

$$\widetilde{\Lambda}(\sigma) = \Lambda\left(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\sigma\right),$$

for $t = t(\sigma) > 0$ such that $\Lambda((\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix})\sigma) \in \mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$.

We define

$$\kappa := \{(z_1, z_2) \in S^3 \mid z_1^n = z_2^2\},$$

so the image of $\widetilde{\Lambda}$ is contained in $S^3 \backslash \kappa$.

Just like we defined Λ_0, we can define $\widetilde{\Lambda}_0 : H_n \backslash SL_2(\mathbb{R}) \to (\mathbb{S}^3 \backslash \kappa)/\sim$, by

$$\widetilde{\Lambda}_0(H_n\sigma) = \{e^{2\pi i m/(n-2)}\widetilde{\Lambda}(\sigma) \mid m \in \mathbb{Z}\}.$$

We then get

Theorem 4.2.4. *The function $\widetilde{\Lambda}_0$ is a homeomorphism.*

Proof. We note that for $\sigma \in SL_2(\mathbb{R})$

$$\widetilde{\Lambda}_0(H_n\sigma) \;=\; \{e^{2\pi i m/(n-2)}\widetilde{\Lambda}(\sigma) \mid m \in \mathbb{Z}\} \;=\; \Lambda_0\left(H_n \begin{pmatrix} t(\sigma) & 0 \\ 0 & t(\sigma) \end{pmatrix}\sigma\right).$$

The function $\Psi : H_n \backslash SL_2(\mathbb{R}) \to \Lambda_0^{-1}((S^3 \backslash \kappa)/\sim)$, given by

$$\Psi(H_n\sigma) = H_n \begin{pmatrix} t(\sigma) & 0 \\ 0 & t(\sigma) \end{pmatrix}\sigma$$

is continuous, since $t(\sigma)$ is continuous as a function of σ. If $\sigma_1, \sigma_2 \in SL_2(\mathbb{R})$, then

$$\begin{aligned} \Psi(H_n\sigma_1) \;=\; \Psi(H_n\sigma_2) \;&\Rightarrow\; H_n\sigma_1 = \begin{pmatrix} t(\sigma_2)/t(\sigma_1) & 0 \\ 0 & t(\sigma_2)/t(\sigma_1) \end{pmatrix} H_n\sigma_2 \\ &\Rightarrow\; H_n\sigma_1 \;=\; H_n\sigma_2, \end{aligned}$$

so Ψ is injective.

We see that Ψ^{-1}, is given by

$$\Psi^{-1}(H_n\sigma) \;=\; H_n \begin{pmatrix} (\det\sigma)^{-1/2} & 0 \\ 0 & (\det\sigma)^{-1/2} \end{pmatrix}\sigma,$$

and hence that it is continuous. Since

$$H_n \begin{pmatrix} (\det\sigma)^{-1/2} & 0 \\ 0 & (\det\sigma)^{-1/2} \end{pmatrix}\sigma,$$

is well defined for any $\sigma \in GL_2^+(\mathbb{R})$, and

$$\Psi\left(H_n\begin{pmatrix}(\det\sigma)^{-1/2} & 0\\ 0 & (\det\sigma)^{-1/2}\end{pmatrix}\sigma\right) = H_n\sigma,$$

when $H_n\sigma \in \Lambda_0^{-1}((S^3\backslash\kappa)/\sim)$, by definition of Ψ, Ψ is surjective.

So Ψ is a homeomorphism and so is Λ_0 by Lemma 4.2.3, so $\widetilde{\Lambda}_0 = \Lambda_0 \circ \Psi$ is also a homeomorphism. $\square$

The set $\kappa = \{(z_1, z_2) \in \mathbb{S}^3 \mid z_1^n = z_2^2\}$, can be rewritten to

$$\{(r^2\exp(2\pi i x), r^n\exp(2\pi i y)) \mid nx \equiv 2y \pmod 1\},$$

where $r > 0$ is given by $r^4 + r^{2n} = 1$. Hence for n odd κ is the knot

$$\{(r^2\exp(4\pi i x), r^n\exp(2n\pi i x)) \mid x \in [0,1]\},$$

so κ is a knot, that goes twice around a torus in one direction and n times around in the other (this is sometimes called a $(2,n)$-torus knot).

For n even we get

$$\kappa = \{(r^2e^{2\pi i x}, r^ne^{n\pi i x}) \mid x \in [0,1]\} \cup \{(r^2e^{2\pi i x}, -r^ne^{n\pi i x}) \mid x \in [0,1]\} = \kappa_1 \cup \kappa_2.$$

So κ is a link of two trivial knots. If we define $f : \mathbb{S}^3 \to \mathbb{C}$ by $f(z_1, z_2) = z_1^{n/2} + z_2$, then κ_2 is the preimage $f^{-1}(0)$. Hence the linking number of these two knots is the winding number (around 0) of f taken on κ_1 (or minus the winding number depending on, which orientations we choose for the knots). This winding number is

$$\frac{1}{2\pi i}\int_0^1 \frac{2r^n(2\exp(n\pi i x))n\pi i}{2r^n(2\exp(n\pi i x))}\,dx = \frac{n}{2}.$$

If $\gamma \in H_n$ is hyperbolic with positive trace, then we can write γ in the following way

$$\gamma = A\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}A^{-1} = A\phi_{2\log\lambda}A^{-1}, \qquad A = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{R}),$$

with $\lambda > 1$. We then have a closed geodesic $\mathcal{C}_\gamma : [0, 2\log\lambda] \to H_n\backslash SL_2(\mathbb{R})$ given by

$$\mathcal{C}_\gamma(t) = H_nA\phi_t.$$

So $\mathcal{A}_\gamma : [0, 2\log\lambda] \to (\mathbb{S}^3\backslash\kappa)/\sim$ given by

$$\mathcal{A}_\gamma(t) = \widetilde{\Lambda}_0(H_nA\phi_t)$$

is a closed curve.

Furthermore for $t \in \mathbb{R}$, there is some $n-2$'nd root of unity ζ, such that

$$\widetilde{\Lambda}(A\phi_t) = \zeta\widetilde{\Lambda}(A\phi_{t+2\log\lambda}).$$

Since $t \mapsto \widetilde{\Lambda}_0(A\phi_t)$ is continuous, ζ is continuous in and hence independent of t. So

$$\widetilde{\Lambda}(A\phi_t) = \zeta^{n-2}\widetilde{\Lambda}(A\phi_{t+2(n-2)\log\lambda}) = \widetilde{\Lambda}(A\phi_{t+2(n-2)\log\lambda}),$$

and the curve $\mathcal{B}_\gamma : [0, 2(n-2)\log\lambda] \to \mathbb{S}^3\backslash\kappa$

$$\mathcal{B}_\gamma(t) = \widetilde{\Lambda}(A\phi_t)$$

is closed.

The modular form D from Lemma 4.2.1 has weight $4n$, and all its zeros are in the cusp, hence D has a holomorphic logarithm d, and we can define a function $\Phi : H_n \to \mathbb{Q}$ by

$$d(\gamma z) = d(z) + 4n\log(j_\gamma(z)) + 8n\pi i\Phi(\gamma).$$

Then

$$z \mapsto \exp\left(\frac{k}{4n}d(z)\right),$$

is a weight k automorphic form wrt. H_n, with multiplier system $\exp(2\pi ik\Phi)$.

We have the following theorem about $\mathcal{B}_\gamma$ and Φ.

Theorem 4.2.5. *Let $\gamma \in H_n$ be hyperbolic and have positive trace, then the linking number of κ and $\mathcal{B}_\gamma$ is $4n\Phi(\gamma)$.*

If n is even, then we mean the linking number of κ_1 and $\mathcal{B}_\gamma$ plus the linking number of κ_2 and $\mathcal{B}_\gamma$, when we write the linking number of κ and $\mathcal{B}_\gamma$.

Proof. If we define $f : \mathbb{S}^3 \to \mathbb{C}$ by

$$f(z_1, z_2) = z_1^n - z_2^2,$$

then $f^{-1}(0) = \kappa$. Hence the linking number of $\mathcal{B}_\gamma$ and κ is the winding number around 0 of $f \circ \mathcal{B}_\gamma$ (this defines an orientation on κ). This winding number is

$$\frac{1}{2\pi}\left(\Im\log(f\circ\mathcal{B}_\gamma)(2(n-2)\log\lambda) - \Im\log(f\circ\mathcal{B}_\gamma)(0)\right),$$

when $\log(f \circ \mathcal{B}_\gamma)$ is a continuous logarithm.

We have

$$\Im\log(f\circ\mathcal{B}_\gamma)(t) = \arg f(\widetilde{\Lambda}(A\phi_t)) = \arg f(\Lambda(A\phi_t)),$$

and

$$f(\Lambda(A\phi_t)) = (g^n(A\phi_t i) - h^2(A\phi_t i))(j_{A\phi_t}(i))^{-4n/(n-2)}.$$

Since $(g^n - h^2)^{n-2}$ is a modular form of weight $4n$, with a zero of order $n-2$ in the cusp, we have $(g_n - h^2)^{n-2} = \alpha D$, for some $\alpha \in \mathbb{C}\backslash\{0\}$. Hence we can take holomorphic logarithms

$$\begin{aligned}\log(f(\Lambda(A\phi_t))) &= \log(g^n - h^2)(A\phi_t i) - \frac{4n}{n-2}\log(j_{A\phi_t}(i)) \\ &= \frac{1}{n-2}\left(d(A\phi_t i) + \log\alpha\right) - \frac{4n}{n-2}\log(j_{A\phi_t}(i)).\end{aligned}$$

We then get

$$\begin{aligned}(n-2)\log(f(\Lambda(A\phi_{t+2\log\lambda}))) &= d(A\phi_{t+2\log\lambda} i) - 4n\log(j_{A\phi_{t+2\log\lambda}}(i)) + \log\alpha \\ &= d(\gamma A\phi_t i) - 4n\log(j_{\gamma A\phi_t}(i)) + \log\alpha \\ &= (n-2)\log(f(\Lambda(A\phi_t))) + 8n\pi i(\Phi(\gamma) + \omega(\gamma, A\phi_t)),\end{aligned}$$

and

$$\omega(\gamma, A\phi_t) = \omega(A\phi_t\phi_{2\log\lambda}(A\phi_t)^{-1}, A\phi_t) = 0$$

by formula (2.10). Hence

$$\log(f(\Lambda(A\phi_t))) = \log(f(\Lambda(A\phi_{t-2\log\lambda}))) + \frac{8n\pi i\Phi(\gamma)}{n-2},$$

and we can calculate the linking number of κ and $\mathcal{B}_\gamma$

$$\operatorname{link}(\kappa, \mathcal{B}_\gamma) = \frac{1}{2\pi i}(\log(f(\Lambda(A\phi_{2(n-2)\log\lambda}))) - \log(f(\Lambda(A\phi_0)))) = 4n\Phi(\gamma).$$

□

The function Φ is a logarithm of the multiplier system for the zero free automorphic form $h^2 - g^n$ divided by $2\pi i k$, where $k = 4n/(n-2)$ is the weight of $h^2 - g^n$. Any zero free automorphic form wrt. H_n that is a power of some modular form wrt. H_n, will be a power of D and hence of $h^2 - g^n$, so if we in the same way take a normalized logarithm of its multiplier system, we will again get Φ. So Φ is the normalized logarithm of all "suitably nice" zero free automorphic forms wrt. H_n.

One way to calculate linking numbers between two knots is to look at a surface which have boundary given by the first knot, such a surface is called a Seifert surface, and take the number of times the other knot passes through this surface in one direction, and subtract the number of times it passes through in the other direction. So if we do this for κ and $\mathcal{B}_\gamma$, we get $4n\Phi(\gamma)$. If we move this from S^3

to $S^3/\sim$, κ becomes $\kappa/\sim$, the Seifert surface becomes a surface with boundary $\kappa/\sim$, and $\mathcal{B}_\gamma$ becomes the closed curve that goes $n-2$ times around $\mathcal{A}_\gamma$. If we for two closed curves in $S^3/\sim$ defines their linking number in the same way as for S^3, i.e. we see how many times the one curves goes trough a surface with boundary given by the other curve in each direction and subtract these numbers, then we can calculate a linking number for $\mathcal{A}_\gamma$ and $\kappa/\sim$ in the following way.

Let $\mathcal{S}$ be a Seifert surface in S^3 with boundary κ. Then

$$\mathcal{S}_0 := \{\{(e^{2\pi ij/(n-2)}z_1, e^{2\pi ij/(n-2)}z_2) \mid j \in \mathbb{N}\} \mid (z_1, z_2) \in S^3\},$$

is a surface in $S^3/\sim$, with boundary $\kappa/\sim$, and hence the linking number between $\kappa/\sim$ and $\mathcal{A}_\gamma$ is determined by how $\mathcal{A}_\gamma$ passes through $\mathcal{S}_0$. To calculate this, we lift $\mathcal{A}_\gamma$ back to S^3, and see how many times it intersects any of the surfaces

$$\mathcal{S}_j := \{(e^{2\pi ij/(n-2)}z_1, e^{2\pi ij/(n-2)}z_2) \mid (z_1, z_2) \in S^3\},$$

for $j \in \{1, 2, \ldots, n-2\}$ in each direction.

Any of the functions

$$\mathcal{B}_\gamma|_{[2(j-1)\log\lambda, 2j\log\lambda]},$$

for $j \in \{1, 2, \ldots, n-2\}$ will be a lift of $\mathcal{A}_\gamma$, and hence $(n-2)\text{link}(\kappa/\sim, \mathcal{A}_\gamma)$ is the number of times one of these curves passes through one of the $\mathcal{S}_j$ surfaces in the positive direction minus the number of times it passes through in the negative direction. In other words it is the number of times $\mathcal{B}_\gamma$ passes through one of the $n-2$ Seifert surfaces in the positive direction minus the number of times it passes through in the negative direction, so it is $(n-2)\text{link}(\kappa, \mathcal{B}_\gamma)$. Hence

$$\text{link}\,(\kappa/\sim, \mathcal{A}_\gamma) = \text{link}\,(\kappa, \mathcal{B}_\gamma) = 4n\Phi(\gamma).$$

So by Theorem 4.2.4 the unit tangent bundle on $H_n\backslash\mathbb{H}$ is homeomorphic to $S^3/\sim$, with a knot removed. This homeomorphism sends the geodesic associated with γ to $\mathcal{A}_\gamma$, and because of Theorem 4.2.5 we can see that the $4n$ times the normalized logarithm of multiplier systems $\Phi(\gamma)$ for "nice" zero free automorphic forms wrt. H_n, tells us the number of times $\mathcal{A}_\gamma$ "winds around" the removed knot.

4.3 Groups with no Elliptic Elements

As in the previous sections we let g be the genus of $\Gamma\backslash\mathbb{H}$, h be the number of cusps, and r be the number of conjugacy classes of elliptic elements in Γ. We will assume that $g = r = 0$, and we will denote the cusps $a_1, \ldots, a_h$. For such groups there are an interpretation of logarithms of some multiplier systems for such Γ's, which is very similar to the one for Hecke triangle groups.

We have by (2.1)

$$0 < \frac{\mu(\Gamma\backslash\mathbb{H})}{2\pi} = -2 + h,$$

and hence $h \geq 3$. We can use Theorem 2.2.4, to show that for G_2, i.e. the space of modular forms of weight 2 wrt. Γ, we have

$$Dim(G_2) \;=\; h-1 \;\geq\; 2.$$

Furthermore if F is a weight 2 modular form wrt. Γ, then Theorem 2.2.3, tells us that $Deg(F) = h-2$. Since there are $h-1$ linearly independent modular forms of weight 2, we can use basic linear algebra to create weight 2 modular forms F_1, F_2 wrt. Γ, such that F_1 has all its $h-2$ zeros in the cusp a_1, and F_2 has $h-3$ zeros in a_1 and 1 zero in a_2.

Now define $\Lambda : GL_2^+(\mathbb{R}) \to \mathbb{C}^2$ by

$$\Lambda(\sigma) = \left(\frac{F_1(\sigma i)}{(j_\sigma(i))^2}, \frac{F_2(\sigma i)}{(j_\sigma(i))^2}\right).$$

We then get the following lemma.

Lemma 4.3.1. *For $\sigma_1, \sigma_2 \in GL_2^+(\mathbb{R})$, $\Lambda(\sigma_1) = \Lambda(\sigma_2)$ if and only if $\sigma_1\sigma_2^{-1} \in \Gamma$.*

Proof. If $\sigma \in GL_2^+(\mathbb{R})$ and $\gamma \in \Gamma$, then we see that for $s = 1, 2$

$$\frac{F_s(\gamma\sigma i)}{(j_{\gamma\sigma}(i))^2} \;=\; \frac{(j_\gamma(\sigma i))^2 F_s(\gamma\sigma i)}{(j_\gamma(\sigma i) j_\sigma(i))^2} \;=\; \frac{F_s(\sigma i)}{(j_\sigma(i))^2}.$$

So

$$\Lambda(\gamma\sigma) = \Lambda(\sigma),$$

which proves the "if" part.

Now define $J = F_2/F_1$. Then J is an automorphic function with a simple pole in a_1, and a simple zero in a_2, and these are all the poles and zeros. For any $z_0 \in \mathbb{C}$, $J - z_0$ is an automorphic function, which has a single simple pole in a_1, and hence

$$J^{-1}(z_0) = (J - z_0)^{-1}(0) = X,$$

for some $X \in \Gamma\backslash\mathbb{H} \cup \{a_2, \ldots, a_h\}$ (when we define $J(a_i)$ to be the constant term in the Fourier expansion in a_i). So $\Gamma z \mapsto J(z)$ is a bijection between $\Gamma\backslash\mathbb{H}$ and $\mathbb{C}\backslash\{J(a_2), \ldots, J(a_h)\}$. Hence if $J(\sigma_1 i) = J(\sigma_2 i)$, then there exists $\gamma \in \Gamma$ such that $\sigma_1 i = \gamma\sigma_2 i$. Since

$$J(\sigma i) = \frac{F_2(\sigma i)}{(j_\sigma(i))^2}\left(\frac{F_1(\sigma i)}{(j_\sigma(i))^2}\right)^{-1}.$$

we see that if $\Lambda(\sigma_1) = \Lambda(\sigma_2)$, then $J(\sigma_1 i) = J(\sigma_2 i)$.

So if we assume that $\Lambda(\sigma_1) = \Lambda(\sigma_2)$, we have $\sigma_1 i = \gamma\sigma_2 i$, for some $\gamma \in \Gamma$. So $\sigma_2^{-1}\gamma^{-1}\sigma_1$ fixes i and is hence on the form (4.3). We have

$$\begin{aligned}
\frac{F_1(\sigma_1 i)}{(j_{\sigma_1}(i))^2} &= \frac{F_1(\gamma\sigma_2 i)}{(j_{\gamma\sigma_2\sigma_2^{-1}\gamma^{-1}\sigma_1}(i))^2} \\
&= \frac{(j_\gamma(\sigma_2 i))^2 F_1(\sigma_2 i)}{(j_{\gamma\sigma_2}(\sigma_2^{-1}\gamma^{-1}\sigma_1 i) j_{\sigma_2^{-1}\gamma^{-1}\sigma_1}(i))^2} \\
&= \frac{(j_\gamma(\sigma_2 i))^2 F_1(\sigma_2 i)}{(j_{\gamma\sigma_2}(i)(i\lambda\sin\theta + \lambda\cos\theta))^2} \\
&= \frac{F_1(\sigma_2 i)}{(j_{\sigma_2}(i)(i\lambda\sin\theta + \lambda\cos\theta))^2}.
\end{aligned}$$

Since $\Lambda(\sigma_1) = \Lambda(\sigma_2)$, we see that

$$(i\lambda\sin\theta + \lambda\cos\theta)^2 = 1,$$

and hence $\sigma_2^{-1}\gamma^{-1}\sigma_1 = \pm I$, so $\sigma_1\sigma_2^{-1} = \pm\gamma \in \Gamma$. $\square$

Due to Lemma 4.3.1, we can define a function $\Lambda_0 : \Gamma\backslash GL_2^+(\mathbb{R}) \to \mathbb{C}^2$, given by

$$\Lambda_0(\Gamma\sigma) = \Lambda(\sigma).$$

We have the following lemma about Λ_0.

Lemma 4.3.2. *The function Λ_0 maps $\Gamma\backslash GL_2^+(\mathbb{R})$ homeomorphically to*

$$\Omega = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; z_1 \neq 0, \frac{z_2}{z_1} \notin \{J(a_2), \ldots, J(a_h)\} \right\}.$$

Proof. It follows from Lemma 4.3.1, that Λ_0 is injective.

If $\Lambda(\sigma) = (z_1, z_2)$, then we have

$$J(\sigma i) = \frac{F_2(\sigma i)(j_\sigma(i))^2}{F_1(\sigma i)(j_\sigma(i))^2} = \frac{z_2}{z_1},$$

and since $\Gamma z \mapsto J(z)$ is a bijection between $\Gamma\mathbb{H}$ and $\mathbb{C}\backslash\{J(a_2), \ldots, J(a_h)\}$, we see that $z_1 \neq 0$ and $z_2/z_1 \notin \{J(a_2), \ldots, J(a_h)\}$. Since $J(a_2) = 0$ this is equivalent to $z_1, z_2 \neq 0$ and $z_2/z_1 \notin \{J(a_3), \ldots, J(a_h)\}$.

On the other hand, if $z_1, z_2 \in \mathbb{C}\backslash\{0\}$ and $z_2/z_1 \notin \{J(a_3), \ldots, J(a_h)\}$, then there is a $z \in \mathbb{H}$, such that $J(z) = z_2/z_1$. Since $z_1, z_2 \neq 0$, $F_1(z), F_2(z) \neq 0$ and there is $z_0 \in \mathbb{C}\backslash\{0\}$, such that

$$z_0 = \frac{F_1(z)}{z_1} = \frac{F_2(z)}{z_2}.$$

So we can let $a, b, c, d \in \mathbb{R}$ be such that, $ci + d$ is a square root of z_0, and $ai + b = z(ci + d)$. Then $\sigma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a 2×2-matrix that sends i to $z \in \mathbb{H}$, and hence $\sigma \in GL_2^+$. We then see that

$$\Lambda(\sigma) \;=\; \left(\frac{F_1(z)}{z_0}, \frac{F_2(z)}{z_0}\right) \;=\; (z_1, z_2).$$

This shows that Λ_0 maps $\Gamma\backslash GL_2^+(\mathbb{R})$ surjectively to Ω.

Λ_0 is continuous because Λ is continuous. To see that Λ_0^{-1} is continuous choose $(x, y), (s, t) \in \Omega$, such that (x, y) is "close to" (s, t), and let σ_1 and σ_2 be such that

$$\Lambda(\sigma_1) = (x, y), \qquad \Lambda(\sigma_2) = (s, t).$$

Then $J(\sigma_1) = y/x$ is close to $J(\sigma_2) = t/s$, and hence σ_1 is close to $\gamma\sigma_2$ for some $\gamma \in \Gamma$. □

Due to the identification of $PSL_2(\mathbb{R})$ with the unit tangent bundle on the hyperbolic plane, $\Gamma\backslash SL_2(\mathbb{R})$ can be identified with the unit tangent bundle on $\Gamma\backslash\mathbb{H}$. Hence if we restrict Λ_0 to $\Gamma\backslash SL_2(\mathbb{R})$, Lemma 4.3.2 gives an identification of the unit tangent bundle on $\Gamma\backslash\mathbb{H}$, with some subset $\Lambda_0(\Gamma\backslash SL_2(\mathbb{R})) \subset \mathbb{C}^2$.

We define $||\cdot||$ to be the norm on $\mathbb{C}^2$ given by

$$||(z_1, z_2)|| = \sqrt{|z_1|^2 + |z_2|^2},$$

so $S^3 = \{x \in \mathbb{C}^2 \mid ||x|| = 1\}$.

If $\sigma \in GL_2^+(\mathbb{R})$ and $\Lambda(\sigma) = (z_1, z_2)$, then $z_1 \neq 0$, and hence $||\Lambda(\sigma)|| \neq 0$. So it makes sense to define functions $\widetilde{\Lambda} : GL_2^+(\mathbb{R}) \to S^3 \cap \Omega$ and $\widetilde{\Lambda}_0 : \Gamma\backslash GL_2^+(\mathbb{R}) \to S^3 \cap \Omega$, by

$$\begin{aligned} \widetilde{\Lambda}(\sigma) &= \left(\frac{F_1(\sigma(i))}{(j_\sigma(i))^2||\Lambda(\sigma)||}, \frac{F_2(\sigma(i))}{(j_\sigma(i))^2||\Lambda(\sigma)||}\right) \\ \widetilde{\Lambda}_0(\Gamma\sigma) &= \widetilde{\Lambda}(\sigma). \end{aligned}$$

We will now prove that the function $\widetilde{\Lambda}_0$ is a homeomorphism. Hence this shows, that the unit tangent bundle on $\Gamma\backslash\mathbb{H}$ is homeomorphic to $S^3 \cap \Omega$ (which seems like a nicer set than $\Lambda_0(\Gamma\backslash\mathbb{H})$).

Theorem 4.3.3. *The function $\widetilde{\Lambda}_0$ is a homeomorphism.*

Proof. We note that for $\sigma \in SL_2(\mathbb{R})$

$$\begin{aligned} \widetilde{\Lambda}_0(\Gamma\sigma) = \widetilde{\Lambda}(\sigma) &= \Lambda\left(\begin{pmatrix} \sqrt{||\Lambda(\sigma)||} & 0 \\ 0 & \sqrt{||\Lambda(\sigma)||} \end{pmatrix}^{-1} \sigma\right) \\ &= \Lambda_0\left(\Gamma \begin{pmatrix} \sqrt{||\Lambda(\sigma)||} & 0 \\ 0 & \sqrt{||\Lambda(\sigma)||} \end{pmatrix}^{-1} \sigma\right). \end{aligned}$$

The function $\Psi : \Gamma\backslash SL_2(\mathbb{R}) \to \Lambda_0^{-1}(S^3 \cap \Omega)$, given by

$$\Psi(\Gamma\sigma) = \Gamma \begin{pmatrix} \sqrt{||\Lambda(\sigma)||} & 0 \\ 0 & \sqrt{||\Lambda(\sigma)||} \end{pmatrix}^{-1} \sigma$$

is continuous, since $||\Lambda(\sigma)||$ is continuous as a function of σ. If $\sigma_1, \sigma_2 \in SL_2(\mathbb{R})$, then

$$\begin{aligned} \Psi(\Gamma\sigma_1) \;=\; \Psi(\Gamma\sigma_2) \;&\Rightarrow\; \Gamma\sigma_1 = \begin{pmatrix} \left(\frac{||\Lambda(\sigma_1)||}{||\Lambda(\sigma_2)||}\right)^{1/2} & 0 \\ 0 & \left(\frac{||\Lambda(\sigma_1)||}{||\Lambda(\sigma_2)||}\right)^{1/2} \end{pmatrix} \Gamma\sigma_2 \\ &\Rightarrow\; \Gamma\sigma_1 \;=\; \Gamma\sigma_2, \end{aligned}$$

so Ψ is injective.

We see that Ψ^{-1}, is given by

$$\Psi^{-1}(\Gamma\sigma) \;=\; \Gamma \begin{pmatrix} \sqrt{\det\sigma} & 0 \\ 0 & \sqrt{\det\sigma} \end{pmatrix}^{-1} \sigma,$$

so Ψ^{-1} is continuous.

Since

$$\left|\left|\Lambda\left(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}^{-1} \sigma\right)\right|\right| = t^{-2}||\Lambda(\sigma)||,$$

we have

$$\Psi\left(\Gamma \begin{pmatrix} \sqrt{\det\sigma} & 0 \\ 0 & \sqrt{\det\sigma} \end{pmatrix}^{-1} \sigma\right) \;=\; \Gamma \begin{pmatrix} \sqrt{||\Lambda(\sigma)||} & 0 \\ 0 & \sqrt{||\Lambda(\sigma)||} \end{pmatrix}^{-1} \sigma.$$

So when $\Gamma\sigma \in \Lambda_0^{-1}(S^3 \cap \Omega)$, we have

$$\Psi\left(\Gamma \begin{pmatrix} \sqrt{\det\sigma} & 0 \\ 0 & \sqrt{\det\sigma} \end{pmatrix}^{-1} \sigma\right) \;=\; \Gamma\sigma,$$

so Ψ is surjective.

So Ψ is a homeomorphism and so is Λ_0 by Lemma 4.3.2, so $\widetilde{\Lambda}_0 = \Lambda_0 \circ \Psi$ is also a homeomorphism. $\square$

We define $\kappa_1 : [0, 2\pi] \to S^3$ by

$$\kappa_1(t) = (0, \exp(it)),$$

and for $j = 2, \ldots, h$ we define $\kappa_j : [0, 2\pi] \to S^3$ by

$$\kappa_j(t) = (r_j \exp(it), J(a_j) r_j \exp(it)),$$

where

$$r_j := (1 + |J(a_j)|^2)^{-1/2}.$$

So for all $j \in \{1, \dots, h\}$ κ_j is homotopic to a circle, and

$$S^3 \backslash \Omega = \bigcup_{j=1}^{h} \kappa_j([0, 2\pi]).$$

If $\gamma \in \Gamma$ is hyperbolic with positive trace, then we can write γ in the following way

$$\gamma = A \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} A^{-1} = A \phi_{2 \log \lambda} A^{-1}, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}),$$

with $\lambda > 1$. We then have a closed geodesic $\mathcal{C}_\gamma : [0, 2 \log \lambda] \to \Gamma \backslash SL_2(\mathbb{R})$ given by

$$\mathcal{C}_\gamma(t) = \Gamma A \phi_t.$$

So $\mathcal{A}_\gamma : [0, 2 \log \lambda] \to \mathbb{S}^3 \backslash \Omega$ given by

$$\mathcal{A}_\gamma(t) = \widetilde{\Lambda}_0(\Gamma A \phi_t)$$

is a closed curve.

For each cusp a_j $j \neq 1$, there exists a weight 2 modular form F_j, which have a simple zero in a_j, and the rest in a_1 (by the same argument that showed the existence of F_2). For $j = 1, \dots, h$ F_j has a holomorphic logarithm d_j, and we can define a function $\Phi_j : \Gamma \to \mathbb{Q}$ by

$$d_j(\gamma z) = d_j(z) + 2 \log(j_\gamma(z)) + 4\pi i \Phi_j(\gamma).$$

So $\exp(2\pi i k \Phi_j)$ is the multiplier system for $D_j^{k/2} = \exp(k d_j / 2)$

We have the following theorem about $\mathcal{A}_\gamma$ and Φ_j.

Theorem 4.3.4. *Let $\gamma \in \Gamma$ be hyperbolic and have positive trace, and let $1 \leq j \leq h$, then the linking number of κ_j and $\mathcal{A}_\gamma$ is $2\Phi_j(\gamma)$.*

Proof. If we define $f_1 : S^3 \to \mathbb{C}$ by

$$f_1(z_1, z_2) = z_1,$$

and for $j = 2, \dots, h$ define $f_j : S^3 \to \mathbb{C}$ by

$$f_j(z_1, z_2) = z_1 J(a_j) - z_2,$$

then $f_j^{-1}(0) = \kappa_j$. Hence the linking number of $\mathcal{A}_\gamma$ and κ_j is the winding number around 0 of $f_j \circ \mathcal{A}_\gamma$. This winding number is

$$\frac{1}{2\pi} \left(\Im \log(f_j \circ \mathcal{A}_\gamma)(2 \log \lambda) - \Im \log(f \circ \mathcal{A}_\gamma)(0) \right)$$

when $\log(f_j \circ \mathcal{A}_\gamma)$ is a continuous logarithm.

For $j = 1, \dots, h$ we have.

$$\Im \log(f_j \circ \mathcal{A}_\gamma)(t) = \arg f_j(\widetilde{\Lambda}(A\phi_t)) = \arg f_j(\Lambda(A\phi_t)) = \Im \log(f_j(\Lambda(A\phi_t))).$$

Furthermore

$$\begin{aligned} f_1(\Lambda(A\phi_t)) &= \frac{F_1(A\phi_t i)}{(j_{A\phi_t}(i))^2}, \\ f_j(\Lambda(A\phi_t)) &= \frac{J(a_j)F_1(A\phi_t i) - F_2(A\phi_t i)}{(j_{A\phi_t}(i))^2} \end{aligned}$$

for $j \neq 1$.

We have

$$\begin{aligned} \log(f_1(\Lambda(A\phi_t))) &= \log F_1(A\phi_t i) - 2\log(j_{A\phi_t}(i)) \\ &= d_1(A\phi_t i) - 2\log(j_{A\phi_t}(i)). \end{aligned}$$

So we get

$$\begin{aligned} \log(f_1(\Lambda(A\phi_{t+2\log\lambda}))) &= d_1(A\phi_{t+2\log\lambda} i) - 2\log(j_{A\phi_{t+2\log\lambda}}(i)) \\ &= d_1(\gamma A\phi_t i) - 2\log(j_{\gamma A\phi_t}(i)) \\ &= \log(f_1(\Lambda(A\phi_t))) + 4\pi i(\Phi_1(\gamma) + \omega(\gamma, A\phi_t)), \end{aligned}$$

and

$$\omega(\gamma, A\phi_t) = \omega(A\phi_t \phi_{2\log\lambda}(A\phi_t)^{-1}, A\phi_t) = 0$$

by formula (2.10). Hence

$$\operatorname{link}(\kappa_1, \mathcal{A}_\gamma) = \frac{1}{2\pi i}(\log(f_1(\Lambda(A\phi_{2\log\lambda}))) - \log(f_1(\Lambda(A\phi_0)))) = 2\Phi_1(\gamma).$$

For $j \neq 1$ $J(a_j)F_1 - F_2$ is a modular form of weight 2, with a zero of order 1 in a_j and a zero of order $h-3$ in the a_j, so

$$J(a_j)F_1 - F_2 = \alpha D_j$$

for some $\alpha \in \mathbb{C}\backslash\{0\}$, and we can assume without loss of generality that $\alpha = 1$. Hence we can take holomorphic logarithms

$$\begin{aligned} \log(f_j(\Lambda(A\phi_t))) &= \log(J(a_j)F_1 - F_2)(A\phi_t i) - 2\log(j_{A\phi_t}(i)) \\ &= d_j(A\phi_t i) - 2\log(j_{A\phi_t}(i)). \end{aligned}$$

We then get

$$\begin{aligned} \log(f_j(\Lambda(A\phi_{t+2\log\lambda}))) &= d_j(\gamma A\phi_t i) - 2\log(j_{\gamma A\phi_t}(i)) \\ &= \log(f_j(\Lambda(A\phi_t))) + 4\pi i \Phi_j(\gamma). \end{aligned}$$

Hence

$$\mathrm{link}(\kappa_j, \mathcal{A}_\gamma) = \frac{1}{2\pi i}(\log(f_j(\Lambda(A\phi_{2\log\lambda}))) - \log(f_j(\Lambda(A\phi_0)))) = 2\Phi_j(\gamma).$$

□

So just as for Hecke triangle groups we see, that we have a homeomorphism between $\Gamma\backslash SL_2(\mathbb{R})$ and the sphere with some knots removed, and that the linking number of one of these knots and the image of a closed geodesic, is given by a logarithm of a multiplier system. We can however choose this homeomorphism in different ways by changing the numbering of the cusps, and we note, that if $h \geq 4$, then it is important which cusp we label a_1, since the corresponding function F_1 has all its zeros in a_1, while F_j only has one zero in cusp a_j for $j \neq 1$. The Φ_j's depends on the F_j's, and so the linking number of $\mathcal{A}_\gamma$ and the knot κ_j corresponding to a certain cusp depends on which cusp we have labeled a_1.

So we can give a geometric interpretation of some logarithms of multiplier systems as linking numbers, but for $h > 3$ there are h different such interpretations, that are equally valid. It seems we could avoid this problem by taking a $h-2$'nd root g of F_1, and choose a homeomorphism that used g and $F_2 g^{-h+3}$ instead of F_1 and F_2. This homeomorphism and the corresponding results would be (even more) similar, to the homeomorphism we used, and the results we got for Hecke triangle groups.

It is however not clear, that such a homeomorphism gives a better interpretation, but maybe it is more general. While the construction of the homeomorphism for Hecke triangle groups uses some properties of these groups, it seems, that a similar construction should be possible for many other cofinite groups.

Chapter 5

Distribution of Prime Geodesics

In this chapter we use Selberg's trace formula to prove a twisted version of the prime geodesic theorem, and then use this theorem to a show distribution result. Before we can use the trace formula, we do however need some results related to the spectral terms.

5.1 A Weyl Law

It is very well known, that for general groups and multiplier systems we can estimate

$$\sum_{\lambda_n(\nu)\leq U^2+1/4} 1 - \frac{1}{4\pi}\int_{-U}^{U} \frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\nu\right) dt \quad \sim \quad \frac{\mu(\mathcal{F})}{4\pi}U^2 \tag{5.1}$$

(see [5] (ii) p. 414), but we do not know how the error term depends on the weight and the multiplier system.

We are going to investigate this by making a similar estimate on

$$\sum_{\lambda_n\leq U^2+1/4} 1 + \int_{-U}^{U} \left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\nu\right)\right| dt,$$

but one that not only depends on U, but also on the multiplier system. Since ν is a multiplier system of weight k, if and only if it is a multiplier system of weight $k+2$, we will only consider $|k|\leq 1$ (it is however quite easy to extend the results to all $k\in\mathbb{R}$).

Theorem 5.1.1. *For $U>0$ and ν a multiplier system of weight $k\in[-1,1]$, we have the following estimate*

$$\sum_{\lambda_n(\nu)\leq U^2+1/4} 1 \quad \ll \quad U^2+(U+1)L(\nu), \tag{5.2}$$

$$\int_{-U}^{U} \left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\nu\right)\right| dt \quad \ll \quad U^2+(U+1)L(\nu), \tag{5.3}$$

where

$$L(\nu) = 1 + \sum_{\alpha_j \neq 0} \log(\alpha_j(\nu)^{-1}), \tag{5.4}$$

and the implied constants are independent of ν.

Before we prove this theorem we need some lemmas.

Lemma 5.1.2. *If* $\varphi(s,\nu)$ *has a pole of order* n *in* $s = s_0$*, with* $\Re s_0 > 1/2$*, then* $-\Delta_k$ *has* n *linearly independent eigenfunctions in* $\mathcal{D}(\Gamma,\nu,k)$ *with eigenvalue* $s_0(1-s_0)$.

Proof. Let $h_{a,s_0}(z)$ be the residue of $E_a(z,s,\nu,k)$ in $s = s_0$. If all these residues are identically 0 then none of the Eisenstein series have a pole in s_0, and hence none of the functions φ_{ab} has a residue in s_0, so s_0 is not a pole of φ.

If one or more of the h_{a,s_0}'s are not identically zero, then we can choose $a_1, \ldots, a_n$, such that $h_{a_1,s_0}, \ldots, h_{a_n,s_0}$ is a basis for the space spanned by all the h_{a,s_0}'s. By (2.26) we can make row operations on $\Phi(s,\nu)$ and get a matrix, that only has poles in $s = s_0$ in the entries in line $a_1, \ldots, a_n$. Since $\varphi(s,\nu)$ is the determinant of this matrix, $\varphi(s,\nu)$ has a pole of order at most n or no pole at all in $s = s_0$. Since $h_{a_1,s_0}, \ldots, h_{a_n,s_0}$ are n linearly independent eigenfunctions in $\mathcal{D}(\Gamma,\nu,k)$ with eigenvalue $s_0(1-s_0)$, this proves the lemma. □

By (2.20) the entries in the scattering matrix is some Γ-factors times a Dirichlet series with coefficients depending (only) on ν. Hence the same is true for the scattering determinant, and we can write

$$\varphi(s,\nu) = \left(\frac{\sqrt{\pi}4^{1-s}\Gamma(2s-1)}{\Gamma(s+k/2)\Gamma(s-k/2)}\right)^{K_0(\nu)} \sum_{n=1}^{\infty} \frac{a_n(\nu)}{b_n(\nu)^s},$$

where $0 < b_1(\nu) < b_2(\nu) < \ldots$ and $\{a_n(\nu)\}_{n\in\mathbb{N}} \subset \mathbb{C}$. Note that the $b_n(\nu)$'s only depends on, which cusps are singular wrt. ν. If n_0 is the smallest n such that $a_n(\nu) \neq 0$, then we define $b(\nu) := b_{n_0}(\nu)$. There are only finitely many possibilities for which cusps are singular, and hence

$$0 < \min_{\nu} b_1(\nu) \leq \inf_{\nu} b(\nu).$$

In other words $b(\nu)$ is bounded from below.

Inspired by [19] (p. 655-656) we show the following.

Lemma 5.1.3. *Let* $\sigma_1(\nu), \ldots, \sigma_{N(\nu)}(\nu)$ *be the poles of* $\varphi(\cdot,\nu)$ *in the right half-plane*

$$H_{1/2} := \{s \in \mathbb{C} \mid \Re s \geq 1/2\}$$

(such that if s_0 is a pole of order n, then $s_0 = \sigma_j(\nu)$ for n j's), and for $s \in \mathbb{C}$ define

$$\varphi^*(s,\nu) := b(\nu)^{s-1/2} \prod_{j=1}^{N(\nu)} \frac{\sigma_j(\nu) - s}{\sigma_j(\nu) + s - 1} \varphi(s,\nu). \tag{5.5}$$

Then we have

$$\frac{\varphi^{*\prime}}{\varphi^*}\left(\frac{1}{2} + it, \nu\right) < 0,$$

for all $t \in \mathbb{R}$.

Proof. Fix ν, and define $\varphi(s) := \varphi(s,\nu)$, $b := b(\nu)$ etc..

We note that, by (5.5) $\varphi^*(s)$ is holomorphic for $s \in H_{1/2}$, and by (2.22) and (2.24) we have

$$\begin{aligned} \varphi^*(s)\varphi^*(1-s) &= 1, & (5.6)\\ |\varphi^*(1/2 + it)| &= 1. & (5.7) \end{aligned}$$

Let $\delta > 0$. For $1/2 \le \Re s \le 3/2$, and $|\Im s| \ge \delta$, $\varphi(s)$ is bounded (see [5] equation (5.46) p. 381), and hence $\varphi^*(s)$ is bounded for $1/2 \le \Re s \le 3/2$. For $\Re s \ge 3/2$ we have by Stirling's formula ([8] formula (B.7) p. 198)

$$\frac{\Gamma(s-1/2)\Gamma(s)}{\Gamma(s+k/2)\Gamma(s-k/2)} \sim \left(\frac{e}{s}\right)^{1/2} \left(\frac{s(s-1/2)}{(s+k/2)(s-k/2)}\right)^s,$$

for $\Re s \to \infty$, and

$$\frac{\Gamma(s-1/2)\Gamma(s)}{\Gamma(s+k/2)\Gamma(s-k/2)} \ll s^{-1/2} \left(\frac{s(s-1/2)}{(s+k/2)(s-k/2)}\right)^s,$$

for $\Re s \ge 3/2$.

We see that

$$\begin{aligned} \left(\frac{s(s-1/2)}{(s+k/2)(s-k/2)}\right)^s &= \left(\frac{s}{s+k/2}\right)^s \left(\frac{s-1/2}{s-k/2}\right)^s \\ &= \left(1 - \frac{k/2}{s+k/2}\right)^s \left(1 + \frac{k/2-1/2}{s-k/2}\right)^s, \end{aligned}$$

which tends to $e^{-k/2} \cdot e^{k/2-1/2} = e^{-1/2}$, when $s \to \infty$, and is bounded for $\Re s \ge 3/2$.

Since

$$b^{s-1/2} \sum_{n=1}^{\infty} \frac{a_n}{b_n^s} = \frac{a_{n_0}}{\sqrt{b}} + \frac{1}{\sqrt{b}} \sum_{n=n_0+1}^{\infty} \frac{a_n}{(b_n/b)^s} = \frac{a_{n_0}}{\sqrt{b}} + O\left(\sum_{n=n_0+1}^{\infty} \frac{|a_n|}{(b_n/b)^\sigma}\right),$$

for $\Re s = \sigma \geq 3/2$, we have $\varphi^*(s)$ is bounded for $\Re s \geq 1/2$, and

$$\varphi^*(s) \sim \frac{a_{n_0}}{\sqrt{b}} s^{-K_0/2} \qquad \text{for } \Re s \to \infty.$$

The Möbius transformation

$$s \mapsto \frac{s-1}{s} = 1 - \frac{1}{s},$$

sends $H_{1/2}$ bijectively to the unit disc $\overline{\mathbb{D}}$, and its inverse is

$$z \mapsto \frac{1}{1-z}.$$

Since φ^* is bounded in the right half plane, the function $\widetilde{\varphi} : \mathbb{D} \to \mathbb{C}$ given by

$$\widetilde{\varphi}(z) = \varphi^* \left(\frac{1}{1-z} \right),$$

is bounded on the unit disc, and since $|\varphi^*(s)| = 1$ for $\Re s = 1/2$,

$$\lim_{r \uparrow 1} |\widetilde{\varphi}(re^{it})| = 1 \tag{5.8}$$

for $t \in (0, 2\pi)$. Hence $\widetilde{\varphi}$ is an inner function, and it can be written in the form

$$\widetilde{\varphi}(z) = cB(z) \exp\left(- \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \, d\mu(t) \right),$$

where $|c| = 1$, B is the Blaschke product with the same zeros as $\widetilde{\varphi}$, and μ is a positive Borel measure on $(-\pi, \pi]$, which is singular with respect to Lebesgue measure (see [17] 17.15 p. 342).

The Blaschke product $B : \mathbb{D} \to \mathbb{C}$ is given by

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \cdot \frac{|\alpha_n|}{\alpha_n},$$

where $\alpha_1, \alpha_2, \ldots$ are the zeros (counted with multiplicity) of $\widetilde{\varphi}$ in $\mathbb{D} \backslash \{0\}$, and $k = 0$ if $\widetilde{\varphi}(0) \neq 0$, and k is the order of the zero in $z = 0$ otherwise (see [17] 15.21 Theorem p. 310). Note that

$$\frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \cdot \frac{|\alpha_n|}{\alpha_n}$$

is holomorphic in the open disc with center in 0 and radius $|\alpha_n|^{-1}$, and that it has modulus 1 for $|z| = 1$. So for any $z \in \mathbb{D}$, all the factors of $B(z)$ has modulus (strictly) smaller than 1, and hence $|B(z)| < 1$ (unless $\widetilde{\varphi}$ is zero free, in which case there are no factors and $B \equiv 1$). So

$$|\widetilde{\varphi}(z)| \leq \exp\left(- \int_{-\pi}^{\pi} \Re \frac{e^{it} + z}{e^{it} - z} \, d\mu(t) \right).$$

For $z = re^{i\theta} \in \mathbb{D}$ we have

$$\int_{-\pi}^{\pi} \Re \frac{e^{it}+z}{e^{it}-z}\,dt = \int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(t-\theta)}\,dt = 2\pi.$$

Since μ is singular with respect to Lebesgue measure we have

$$\lim_{\epsilon\downarrow 0} \frac{\mu(\theta-\epsilon,\theta+\epsilon)}{2\epsilon} = \infty \tag{5.9}$$

for almost all $\theta \in (-\pi,\pi)$ wrt. μ (see [17] 7.15 Theorem p. 143). If (5.9) holds for $\theta \in (-\pi,\pi)$ and $M \in \mathbb{N}$, we can choose $\delta > 0$, such that for $\epsilon \leq \delta$, we have $\mu(\theta-\epsilon,\theta+\epsilon) > 2\epsilon M$. Hence

$$\begin{aligned}
\lim_{r\uparrow 1} \int_{-\pi}^{\pi} \Re \frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\,d\mu(t) &= \lim_{r\uparrow 1} \int_{\theta-\delta}^{\theta+\delta} \frac{1-r^2}{1+r^2-2r\cos(t-\theta)}\,d\mu(t) \\
&\geq M \lim_{r\uparrow 1} \int_{\theta-\delta}^{\theta+\delta} \frac{1-r^2}{1+r^2-2r\cos(t-\theta)}\,dt \\
&= M \lim_{r\uparrow 1} \int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(t-\theta)}\,dt = 2\pi M.
\end{aligned}$$

Since this holds for arbitrary M, we have

$$\lim_{r\uparrow 1} \int_{-\pi}^{\pi} \Re \frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\,d\mu(t) = \infty,$$

and hence

$$\lim_{r\uparrow 1} |\widetilde{\varphi}(re^{i\theta})| \leq \lim_{r\uparrow 1} \exp\left(-\int_{-\pi}^{\pi} \Re \frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\,d\mu(t)\right) = 0.$$

By (5.8), this means that $\theta = 0$.

So μ can be written on the form

$$\mu = m_\pi \delta_\pi + m_0 \delta_0,$$

where δ_x is the Dirac measure with mass in x, and $m_\pi, m_0 \geq 0$. But

$$\begin{aligned}
1 = \lim_{r\uparrow 1} |\widetilde{\varphi}(-r)| &\leq \lim_{r\uparrow 1} \exp\left(-\int_{-\pi}^{\pi} \Re \frac{e^{it}-r}{e^{it}+r}\,d\mu(t)\right) \\
&\leq \lim_{r\uparrow 1} \exp\left(-m_\pi \frac{1+r}{1-r}\right),
\end{aligned}$$

so $m_\pi = 0$, and $\mu = m_0 \delta_0$.

Hence

$$|\widetilde{\varphi}(r)| \leq \exp\left(-m_0 \frac{1+r}{1-r}\right) = \exp\left(m_0 - 2m_0 \frac{1}{1-r}\right),$$

but

$$|\widetilde{\varphi}(r)| = \left|\varphi^*\left(\frac{1}{1-r}\right)\right| \sim \frac{|a_{n_0}|}{\sqrt{b}}\left(\frac{1}{1-r}\right)^{-K_0/2} \quad \text{for } r \uparrow 1,$$

so $m_0 = 0$. Hence μ is the zero measure and $\widetilde{\varphi} = cB$.

A point $\alpha \in \mathbb{D}$ is a zero of B if and only if it is on the form

$$\alpha = \frac{\rho - 1}{\rho},$$

for some zero $\rho \in H_{1/2}$ of φ. Hence we have

$$\begin{aligned}
\varphi^*(s) = cB\left(\frac{s-1}{s}\right) &= c\left(\frac{s-1}{s}\right)^k \prod_{\rho\neq 1} \frac{(\rho-1)/\rho - (s-1)/s}{1-(\bar{\rho}-1)(s-1)/(\bar{\rho}s)} \cdot \frac{|(\rho-1)/\rho|}{(\rho-1)/\rho} \\
&= c\left(\frac{s-1}{s}\right)^k \prod_{\rho\neq 1} \frac{1/s - 1/\rho}{1/s + 1/\bar{\rho} - 1/(\bar{\rho}s)} \cdot \frac{|(\rho-1)/\rho|}{(\rho-1)/\rho} \\
&= c\left(\frac{s-1}{s}\right)^k \prod_{\rho\neq 1} \frac{\overline{\rho(\rho-1)}}{|\rho(\rho-1)|} \cdot \frac{\rho - s}{\bar{\rho}+s-1},
\end{aligned}$$

where the product is over the zeros of φ in $H_{1/2}$. If ρ is a zero of φ, then so is $\bar{\rho}$ by (2.23), and we have

$$\left(\frac{\overline{\rho(\rho-1)}}{|\rho(\rho-1)|} \cdot \frac{\rho - s}{\bar{\rho}+s-1}\right)\left(\frac{\rho(\rho-1)}{|\rho(\rho-1)|} \cdot \frac{\bar{\rho} - s}{\rho+s-1}\right) = \frac{\rho - s}{\bar{\rho}+s-1} \cdot \frac{\bar{\rho} - s}{\rho+s-1}.$$

So we have

$$\varphi^*(s) = c(-1)^k \prod_n \frac{\rho_n - s}{\bar{\rho}_n + s - 1} = c(-1)^k \prod_n \frac{\rho_n - s}{\rho_n + s - 1}, \tag{5.10}$$

where the ρ_n's are the zeros of φ in $H_{1/2}$, and they are ordered such that if $\Im\rho_n > 0$, then $\rho_n = \bar{\rho}_{n+1}$.

Taking logarithmic derivatives in (5.10) we see, that

$$\begin{aligned}
\frac{\varphi^{*\prime}}{\varphi^*}(s) &= \sum_n \frac{d}{ds}(\log(\rho_n - s) - \log(\bar{\rho}_n + s - 1)) \\
&= \sum_n \left(\frac{-1}{\rho_n - s} - \frac{1}{\bar{\rho}_n + s - 1}\right) \\
&= \sum_n \frac{1 - 2\Re\rho}{(\rho_n - s)(\bar{\rho}_n + s - 1)}.
\end{aligned}$$

Setting $s = 1/2 + it$, we get

$$\begin{aligned}\frac{\varphi^{*\prime}}{\varphi^*}\left(\frac{1}{2}+it\right) &= \sum_n \frac{1-2\Re\rho_n}{(\rho_n - 1/2 - it)(\bar{\rho}_n - 1/2 + it)} \\ &= \sum_n \frac{1-2\Re\rho_n}{|1/2+it-\rho_n|^2} < 0.\end{aligned}$$

□

We note that by (2.22), we have $\varphi(1/2) = \pm 1$, and that by (5.10)

$$c(-1)^k = \varphi^*\left(\frac{1}{2}\right) = \varphi\left(\frac{1}{2}\right).$$

So

$$\varphi(s) = \varphi\left(\frac{1}{2}\right) b^{1/2-s} \prod_{j=1}^{N(\nu)} \frac{\sigma_j + s - 1}{\sigma_j - s} \prod_n \frac{\rho_n - s}{\bar{\rho}_n + s - 1}. \tag{5.11}$$

We can use the trace formula to get a "smooth version" of Theorem 5.1.1, which we will use to prove the theorem.

Lemma 5.1.4. *If we for $U > 0$ define $H_U : \mathbb{C} \to \mathbb{C}$ by $H_U(x) = e^{-(x/U)^2}$, then for $U \geq 2$ and $|k| \leq 1$, we have*

$$\sum_{n=0}^{\infty} H_U(r_n(\nu)) - \frac{1}{4\pi}\int_{\mathbb{R}} H_U(t)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\nu\right) dt \ll U^2 + UL(\nu), \tag{5.12}$$

where the implied constant is independent of ν.

Proof. We are going to use the trace formula with $h = H_U$, and estimate the geometric terms. The corresponding g will be $G_U(x) = \frac{U}{\sqrt{4\pi}} e^{-(xU/2)^2}$.

We first estimate (2.28). To do so we note that $|\sinh(x)/\cosh(x)| \to 1$ when $x \to \pm\infty$, and that $\cosh(x) - 1$ is positive on $\mathbb{R}$ except in $x = 0$, where it has a double zero, so since $\sinh(x)$ has a zero at $x = 0$ can

$$\frac{x\sinh(2\pi x)}{\cosh(2\pi x) + \cos(\pi k)},$$

be continuously extended to $(x, k) = (0, \pm 1)$, and it is thus uniformly bounded for $|x| \leq 1$. Hence

$$\begin{aligned}\frac{\mu(\mathcal{F})}{4\pi}\int_{\mathbb{R}} rH_U(r)\frac{\sinh(2\pi r)dr}{\cosh(2\pi r)+\cos(\pi k)} &\ll \int_{\mathbb{R}} (|r|+1)e^{-(r/U)^2}\,dr \\ &= 2\int_0^{\infty} (r+1)e^{-(r/U)^2}\,dr \\ &= 2U\int_0^{\infty}(Ux+1)e^{-x^2}\,dx \ll U^2.\end{aligned}$$

Since we only consider $|k| \leq 1$, the second term is 0.

When estimating (2.30) we get

$$\sum_{\substack{[\gamma] \\ \operatorname{Tr}\gamma>2}} \frac{\nu(\gamma)l(\gamma_0)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}} G_U(l(\gamma)) \ll \sum_{j=1}^{\infty} \sum_{[\gamma]\in\Gamma'} \frac{l(\gamma)G_U(jl(\gamma))}{N(\gamma)^{j/2}-N(\gamma)^{-j/2}},$$

where Γ' denotes the set of conjugacy classes of primitive hyperbolic matrices with positive trace in Γ. We see that

$$G_U(x) \leq Ue^{-x^2} \ll \frac{U}{\exp(x/2)x^2},$$

so when we apply partial summation and the prime geodesic theorem (Corollary 2.4.2) to the $j = 1$ term, we get

$$\begin{aligned}
\sum_{[\gamma]\in\Gamma'} \frac{l(\gamma)G_U(l(\gamma))}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}} &\ll \sum_{[\gamma]\in\Gamma'} \frac{U}{(N(\gamma)-1)l(\gamma)} \\
&= \lim_{A\to\infty} \frac{U}{(A-1)\log A}\pi(\log A) + U\int_{N_\Gamma}^{\infty} \frac{(\log t + 1 - t^{-1})\pi(\log t)}{((t-1)\log t)^2}\, dt \\
&\ll \lim_{A\to\infty} \frac{U}{(\log A)^2} + U\int_{N_\Gamma}^{\infty} \frac{1}{t(\log t)^2}\, dt \\
&= U\int_{\log N_\Gamma}^{\infty} \frac{1}{x^2}\, dx \ll U,
\end{aligned}$$

where $N_\Gamma > 1$ is the minimal norm of any hyperbolic element of Γ. To estimate the $j \geq 2$ part, we note that

$$\begin{aligned}
\sum_{j=2}^{\infty} G_U(jl(\gamma)) &\ll U\sum_{j=2}^{\infty} e^{-(jl(\gamma))^2} \leq U\sum_{j=4}^{\infty} (e^{-(l(\gamma))^2})^j \\
&= \frac{Ue^{-4(l(\gamma))^2}}{1-e^{-(l(\gamma))^2}} \ll U(l(\gamma))^{-2}.
\end{aligned}$$

So

$$\begin{aligned}
\sum_{j=2}^{\infty} \sum_{[\gamma]\in\Gamma'} \frac{l(\gamma)G_U(jl(\gamma))}{N(\gamma)^{j/2}-N(\gamma)^{-j/2}} &\ll \sum_{[\gamma]\in\Gamma'} \frac{l(\gamma)}{N(\gamma)} \sum_{j=2}^{\infty} G_U(jl(\gamma)) \\
&\ll \sum_{[\gamma]\in\Gamma'} \frac{U}{N(\gamma)l(\gamma)},
\end{aligned}$$

and by partial summation

$$\begin{aligned}
\sum_{[\gamma]\in\Gamma'} \frac{U}{N(\gamma)l(\gamma)} &= \lim_{A\to\infty} \frac{U\pi(\log A)}{A\log A} + U\int_{N_\Gamma}^{\infty} \frac{(\log t+1)\pi(t)}{(t\log t)^2} \\
&\ll \lim_{A\to\infty} \frac{U}{(\log A)^2} + U\int_{N_\Gamma}^{\infty} \frac{1}{t(\log t)^2} \ll U.
\end{aligned}$$

So (2.30) is $O(U)$.

We estimate (2.31) in the following way

$$\sum_{\substack{\{R\} \\ \operatorname{Tr} R<2 \\ 0<\theta(R)<\pi}} \frac{\nu(R) i e^{i(k-1)\theta}}{4M_R \sin\theta} \int_{\mathbb{R}} G_U(u) e^{(k-1)u/2} \frac{e^u - e^{2i\theta}}{\cosh u - \cos(2\theta)} du$$

$$\ll U \int_{\mathbb{R}} e^{-u^2+(k-1)u/2}(e^u+1)du \ll U.$$

In the same way we see that (2.33) and (2.34) is $O(U)$.

We estimate (2.32) by

$$\begin{aligned} G_U(0) \sum_{\alpha_j \neq 0} \log|1-e^{2\pi i \alpha_j(\nu^k)}| &\ll U \sum_{\alpha_j\neq 0} |\log(|2\pi i \alpha_j(\nu^k)|)| \\ &\ll U\left(1+\sum_{\alpha_j\neq 0}\log(\alpha_j(\nu^k)^{-1})\right). \end{aligned}$$

With the help of formula (B.11) on p. 199 in [8], we can estimate (2.35)

$$\begin{aligned} &K_0\left(G_U(0)\log 2 + \frac{1}{2\pi}\int_{\mathbb{R}} H_U(r)\frac{\Gamma'(1+ir)}{\Gamma(1+ir)}dr\right) \\ &\qquad \ll U + \int_{\mathbb{R}} e^{-(r/U)^2}\left|\frac{\Gamma'(1+ir)}{\Gamma(1+ir)}\right| dr \\ &\qquad = U + U\int_{\mathbb{R}} e^{-r^2}\left|\frac{\Gamma'(1+iUr)}{\Gamma(1+iUr)}\right| dr \\ &\qquad \ll U + U\int_{\mathbb{R}} e^{-r^2}(|\log(1+iUr)| + |1+iUr|^{-1})dr \\ &\qquad \ll U + U\int_{\mathbb{R}} e^{-r^2}(1+|Ur|)dr \ll U^2. \end{aligned}$$

The term (2.36) is bounded since the entries of $\Phi(\frac{1}{2})$ are bounded. By collecting all these terms we get the desired estimate. □

We are now going to use these two lemmas to prove Theorem 5.1.1.

Proof of Theorem 5.1.1. We look at the relation between the logarithmic derivative of φ^* and φ,

$$
\begin{aligned}
-\frac{\varphi^{*\prime}}{\varphi^*}(s,\nu) &= -\frac{\varphi'}{\varphi}(s,\nu) - \frac{d}{ds}\left(\log\left(b(\nu)^{s-1/2}\prod_{j=1}^{N(\nu)}\frac{\sigma_j(\nu)-s}{\sigma_j(\nu)+s-1}\right)\right) \\
&= -\frac{\varphi'}{\varphi}(s,\nu) - \log b(\nu) + \sum_{j=1}^{N(\nu)}\left(\frac{1}{\sigma_j(\nu)-s} + \frac{1}{\sigma_j(\nu)+s-1}\right) \\
&= -\frac{\varphi'}{\varphi}(s,\nu) - \log b(\nu) + \sum_{j=1}^{N(\nu)}\frac{2\sigma_j(\nu)-1}{(\sigma_j(\nu)-s)(\sigma_j(\nu)+s-1)}.
\end{aligned}
$$

So for $s = 1/2 + it$, we have by Lemma 5.1.3

$$
\begin{aligned}
0 &< -\frac{\varphi^{*\prime}}{\varphi^*}\left(\frac{1}{2}+it,\nu\right) \\
&= -\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\nu\right) - \log b(\nu) + 2\sum_{j=1}^{N(\nu)}\frac{\sigma_j(\nu)-1/2}{(\sigma_j(\nu)-1/2)^2+t^2}.
\end{aligned}
$$

For $a > 0$ we have

$$
\int_{\mathbb{R}}\frac{a}{a^2+t^2}\,dt = \int_{\mathbb{R}}\frac{a^2}{a^2+(at)^2}\,dt = \int_{\mathbb{R}}\frac{1}{1+t^2}\,dt = \pi,
$$

and hence

$$
\begin{aligned}
\sum_{j=1}^{N(\nu)}\int_{\mathbb{R}}H_U(t)\frac{\sigma_j(\nu)-1/2}{(\sigma_j(\nu)-1/2)^2+t^2}dt &\le \sum_{j=1}^{N(\nu)}\int_{\mathbb{R}}\frac{\sigma_j(\nu)-1/2}{(\sigma_j(\nu)-1/2)^2+t^2}dt \;=\; \pi N(\nu), \\
\int_{\mathbb{R}}H_U(t)\log b(\nu)\,dt &= \log b(\nu)\int_{\mathbb{R}}UH_1(t)\,dt \;=\; U\sqrt{\pi}\log b(\nu).
\end{aligned}
$$

Since the left hand sides of (5.2) and (5.3) are increasing, it is enough to show the theorem for $U \ge 2$. So assume that $U \ge 2$, and define $H_U^* : \mathbb{C} \to \mathbb{C}$ by

$$
H_U^*(x) = \begin{cases} H_U(x) & \text{for } x \in \mathbb{R} \\ H_U(x) - \frac{1}{2} & \text{for } x \notin \mathbb{R} \end{cases}.
$$

If we assume that $b(\nu) \le 1$ (and remember that $b(\nu)$ is bounded from below), then Lemma 5.1.2 and 5.1.4 gives us

$$
\begin{aligned}
0 &< \sum_{n=0}^{\infty}H_U^*(r_n(\nu)) - \frac{1}{4\pi}\int_{\mathbb{R}}H_U(t)\frac{\varphi^{*\prime}}{\varphi^*}\left(\frac{1}{2}+it,\nu\right)dt \\
&\le \sum_{n=0}^{\infty}H_U^*(r_n(\nu)) + \frac{N(\nu)}{2} - \frac{1}{4\pi}\int_{\mathbb{R}}H_U(t)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\nu\right)dt - \frac{U\log b(\nu)}{2\sqrt{\pi}} \\
&\le \sum_{n=0}^{\infty}H_U(r_n(\nu)) - \frac{1}{4\pi}\int_{\mathbb{R}}H_U(t)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\nu\right)dt - \frac{U\log b(\nu)}{2\sqrt{\pi}} \\
&\ll U^2 + UL(\nu).
\end{aligned}
$$

Hence

$$\begin{aligned}
0 &< \frac{1}{e}\left(\sum_{\lambda_n(\nu)\le U^2+1/4} 1 - \frac{1}{4\pi}\int_{-U}^{U}\frac{\varphi^{*\prime}}{\varphi^*}\left(\frac{1}{2}+it,\nu\right)dt\right)\\
&\le \sum_{n=0}^{\infty} H_U^*(r_n(\nu)) - \frac{1}{4\pi}\int_{\mathbb{R}} H_U(t)\frac{\varphi^{*\prime}}{\varphi^*}\left(\frac{1}{2}+it,\nu\right)dt\\
&\ll U^2 + UL(\nu).
\end{aligned}$$

Which proves that (5.2) holds for $b(\nu) \le 1$.

We can now make the following estimate

$$\begin{aligned}
0 &< \int_{-U}^{U}\left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\nu\right)\right|dt\\
&\le \int_{\mathbb{R}} 2\sum_{j=1}^{N(\nu)}\frac{\sigma_j(\nu)-1/2}{(\sigma_j(\nu)-1/2)^2+t^2}dt - \int_{-U}^{U}\frac{\varphi^{*\prime}}{\varphi^*}\left(\frac{1}{2}+it,\nu\right)+\log b(\nu)\,dt\\
&\le 2\pi\sum_{\lambda_n(\nu)\le 1/4} 1 - \int_{-U}^{U}\frac{\varphi^{*\prime}}{\varphi^*}\left(\frac{1}{2}+it,\nu\right)dt + O(U)\\
&= O(U^2+UL(\nu)).
\end{aligned}$$

Hence we have proved Theorem 5.1.1 for $b(\nu) \le 1$.

For $b(\nu) > 1$, we can make almost the same argument if we replace $-\varphi^{*\prime}/\varphi^*$ by $-\varphi^{*\prime}/\varphi^* + \log b(\nu)$. By Lemma 5.1.2 and 5.1.4

$$\begin{aligned}
0 &< \sum_{n=0}^{\infty} H_U^*(r_n(\nu)) + \frac{1}{4\pi}\int_{\mathbb{R}} H_U(t)\left(-\frac{\varphi^{*\prime}}{\varphi^*}\left(\frac{1}{2}+it,\nu\right)+\log b(\nu)\right)dt\\
&\le \sum_{n=0}^{\infty} H_U(r_n(\nu)) - \frac{1}{4\pi}\int_{\mathbb{R}} H_U(t)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\nu\right)dt\\
&\ll U^2 + UL(\nu).
\end{aligned}$$

Hence

$$\begin{aligned}
0 &< \frac{1}{e}\left(\sum_{\lambda_n(\nu)\le U^2+1/4} 1 + \frac{1}{4\pi}\int_{-U}^{U} -\frac{\varphi^{*\prime}}{\varphi^*}\left(\frac{1}{2}+it,\nu\right)+\log b(\nu)dt\right)\\
&\le \sum_{n=0}^{\infty} H_U^*(r_n(\nu)) + \frac{1}{4\pi}\int_{\mathbb{R}} H_U(t)\left(-\frac{\varphi^{*\prime}}{\varphi^*}\left(\frac{1}{2}+it,\nu\right)+\log b(\nu)\right)dt\\
&\ll U^2 + UL(\nu).
\end{aligned}$$

Which proves that (5.2) holds for $b(\nu) \ge 1$.

Finally

$$\begin{aligned}
0 &< \int_{-U}^{U} \left| \frac{\varphi'}{\varphi}\left(\frac{1}{2}+it,\nu\right)\right| dt \\
&\leq \int_{\mathbb{R}} 2\sum_{j=1}^{N(\nu)} \frac{\sigma_j(\nu)-1/2}{(\sigma_j(\nu)-1/2)^2+t^2} dt + \int_{-U}^{U} -\frac{\varphi^{*\prime}}{\varphi^*}\left(\frac{1}{2}+it,\nu\right) + \log b(\nu)\, dt \\
&\leq 2\pi \sum_{\lambda_n(\nu)\leq 1/4} 1 - \int_{-U}^{U} \frac{\varphi^{*\prime}}{\varphi^*}\left(\frac{1}{2}+it,\nu\right) + \log b(\nu)\, dt \\
&= O(U^2+UL(\nu)).
\end{aligned}$$

Which proves that (5.3) holds for $b(\nu) \geq 1$. □

5.2 Continuity of Small Eigenvalues

We recall some properties of the Laplace transform $\mathscr{L}(f)$ of a continuous function $f : \mathbb{R}_+ \to \mathbb{C}$ defined by

$$\mathscr{L}(f)(z) = \int_0^\infty e^{-zt} f(t)\, dt, \tag{5.13}$$

if f is sufficiently nice, so (5.13) converges absolutely in a half plane $\Re z > a_0$ (for details on the Laplace transform see [21]). For $a > a_0$ we have

$$f(u) = \frac{1}{2\pi}\int_{a-i\infty}^{a+i\infty} e^{zu}\mathscr{L}(f)(z)\, dz$$

(see [21] Theorem 7.3 p. 66), and for $\rho > 0$

$$\frac{1}{2\pi}\int_{a-i\infty}^{a+i\infty} e^{zu}\frac{\mathscr{L}(f)(z)}{z^\rho}\, dz = \begin{cases} f_\rho(u) & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases}$$

(see [21] Theorem 8.1 p. 73 and Theorem 8.2 p. 74), where

$$f_\rho(u) = \int_0^u \frac{(u-t)^{\rho-1}}{\Gamma(\rho)} f(t)\, dt.$$

We use the idea from [14] section 3.3 to show a similar result about continuity of small eigenvalues as functions of the weight.

Theorem 5.2.1. *Let I be an open interval containing 0, and for $k \in I$ let $\nu_k : \Gamma \to \mathbb{S}^1$ be a multiplier system of weight k, such that $\nu_k(\gamma)$ is continuous as a function of the weight k, for any fixed $\gamma \in \Gamma$. Denote the eigenvalues corresponding to ν_k by $\lambda_0(k), \lambda_1(k), \dots,$ and let $T < 1/4$ be such that $T \neq \lambda_n(0)$ for all n. Then there exists $\epsilon > 0$ such that $|\{\lambda_n(k) < T\}|$ is constant for $|k| \leq \epsilon$.*

Proof. We use the trace formula for $h_z(r) = e^{-zr^2}$, where $z \in \mathbb{C}$ has positive real part (this gives us $g_z(x) = (4\pi z)^{-1/2}\exp(-x^2/(4z))$). We let $f(u) = 1$ and $T < 1/4$, and multiply in the trace formula with $\mathscr{L}(f)(z)e^{z(T-1/4)}/z$, so that the left hand side becomes

$$\sum_n \frac{e^{z(T-\lambda_n(k))}\mathscr{L}(f)(z)}{z}.$$

We then integrate with respect to $(2\pi i)^{-1}dz$ from $z = a - i\infty$ to $z = a + i\infty$ (for some positive a), so the left hand side becomes

$$\sum_{\lambda_n(k)\leq T} f_1(T-\lambda_n(k)) = \sum_{\lambda_n(k)\leq T}\int_0^{T-\lambda_n(k)} 1\,du = \sum_{\lambda_n(k)\leq T}(T-\lambda_n(k)).$$

When we make the multiplication and integration in (2.28), we get

$$\begin{aligned}
\frac{1}{2\pi i}\cdot\frac{\mu(\mathcal{F})}{4\pi}\int_{a-i\infty}^{a+i\infty}\frac{\mathscr{L}(f)(z)e^{z(T-1/4)}}{z}\int_{\mathbb{R}} rh_z(r)\frac{\sinh(2\pi r)dr}{\cosh(2\pi r)+\cos(\pi k)}\,dz &= \\
\frac{1}{2\pi i}\cdot\frac{\mu(\mathcal{F})}{4\pi}\int_{\mathbb{R}} r\frac{\sinh(2\pi r)}{\cosh(2\pi r)+\cos(\pi k)}\int_{a-i\infty}^{a+i\infty}\frac{\mathscr{L}(f)(z)e^{z(T-1/4)}}{z}h_z(r)\,dz\,dr &= 0,
\end{aligned}$$

since

$$\int_{a-i\infty}^{a+i\infty}\frac{\mathscr{L}(f)(z)e^{z(T-1/4)}}{z}h_z(r)\,dz = \int_{a-i\infty}^{a+i\infty}\frac{\mathscr{L}(f)(z)e^{z(T-1/4-r^2)}}{z}\,dz = 0.$$

Likewise we get 0 from (2.32), (2.35), (2.36) and (2.37) (where we use $g_z(0) = (4\pi z)^{-1/2}$). If we look at weight $k \in [-1,1]$ (2.29) is zero.

By [5] p. 401-402 we can rewrite (2.31) to

$$\begin{aligned}
&\sum_{\substack{\{R\}\\ \operatorname{Tr} R<2\\ 0<\theta(R)<\pi}}\frac{\nu(R)}{4M_R\sin\theta}\int_{\mathbb{R}} h_z(r)\frac{\cosh((2r(\pi-\theta))+e^{ik\pi}\cosh(2r\theta)}{\cosh(2\pi r)-\cos(\pi k)}dr \\
&+\sum_{\substack{\{R\}\\ \operatorname{Tr} R<2\\ 0<\theta(R)<\pi}}\frac{\nu(R)}{2M_R\sin\theta}\operatorname{sign}(k)\sum_{\substack{l\text{ odd}\\ 1\leq l\leq|k|}} i\exp\left(i(k-l\operatorname{sign}(k))\theta\right)h\left(\frac{i(|k|-l)}{2}\right)
\end{aligned}$$

and (2.33) to

$$\begin{aligned}
&\sum_{\alpha_j\neq 0}\left(\frac{1}{2}-\alpha_j\right)\frac{1}{2}\int_{-\infty}^{\infty} h_z(r)\frac{\sin(k\pi)}{\cosh(2\pi r)+\cos(\pi k)}dr \\
&+\sum_{\alpha_j\neq 0}\left(\frac{1}{2}-\alpha_j\right)\operatorname{sign}(k)\sum_{\substack{l\text{ odd}\\ 1\leq l\leq|k|}} h\left(\frac{i(|k|-l)}{2}\right)
\end{aligned}$$

so these terms also becomes zero after multiplication and integration, when $|k| \leq 1$.

Hence for small weight

$$\sum_{\lambda_n \leq T} (T - \lambda_n) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\mathscr{L}(f)(z) e^{z(T-1/4)}}{z} F_k(z)\, dz$$

where

$$F_k(z) = \sum_{\substack{[\gamma] \\ \mathrm{Tr}\,\gamma > 2}} \frac{\nu_k(\gamma) l(\gamma_0) g_z(l(\gamma))}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} + K_0(k) \int_0^\infty \frac{g_z(u)(1 - \cosh(\frac{k}{2}u))}{e^{u/2} - e^{-u/2}} du.$$

We see that

$$\frac{\mathscr{L}(f)(z) e^{z(T-1/4)}}{z} \cdot \frac{g_z(u)(1 - \cosh(\frac{k}{2}u))}{e^{u/2} - e^{-u/2}} \to 0$$

uniformly for $z \in]a - i\infty, a + i\infty[$ and $u \in \mathbb{R}_+$, when $k \to 0$. Hence by dominated convergence

$$\int_{a-i\infty}^{a+i\infty} \int_0^\infty \frac{\mathscr{L}(f)(z) e^{z(T-1/4)}}{z} \cdot \frac{g_z(u)(1 - \cosh(\frac{k}{2}u))}{e^{u/2} - e^{-u/2}}\, du\, dz$$

is continuous as a function of k in $k = 0$, and it is zero at $k = 0$. Since $K_0(k)$ is constant for $k \in J \backslash \{0\}$, if J is a small interval around 0, we see that

$$\int_{a-i\infty}^{a+i\infty} K_0 \int_0^\infty \frac{\mathscr{L}(f)(z) e^{z(T-1/4)}}{z} \cdot \frac{g_z(u)(1 - \cosh(\frac{k}{2}u))}{e^{u/2} - e^{-u/2}}\, du\, dz,$$

is continuous in $k = 0$.

Likewise dominated convergence implies that

$$\sum_{\substack{[\gamma] \\ \mathrm{Tr}\,\gamma > 2}} \frac{\nu_k(\gamma) l(\gamma_0)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\mathscr{L}(f)(z) e^{z(T-1/4)}}{z} g_z(l(\gamma))\, dz,$$

is continuous at $k = 0$. So $\sum_{\lambda_n(k) \leq T} (T - \lambda_n(k))$ is continuous at $k = 0$.

Defining $N(T, k) = \sum_{\lambda_n(k) \leq T} (T - \lambda_n(k))$, we see that for $0 < \epsilon < 1/4 - T$, we have

$$\frac{N(T, k) - N(T - \epsilon, k)}{\epsilon} \leq |\{\lambda_n(k) \leq T\}| \leq \frac{N(T + \epsilon, k) - N(T, k)}{\epsilon}.$$

Since $N(T, k)$ is continuous in $k = 0$, this gives us

$$\begin{aligned}
\liminf_{k \to 0} |\{\lambda_n(k) \leq T\}| &\geq \frac{N(T, 0) - N(T - \epsilon, 0)}{\epsilon} \\
\limsup_{k \to 0} |\{\lambda_n(k) \leq T\}| &\leq \frac{N(T + \epsilon, 0) - N(T, 0)}{\epsilon}.
\end{aligned}$$

For ϵ close to zero, the right hand sides are $|\{\lambda_n(0) \leq T\}|$, if $T \neq \lambda_n(0)$ for all n, so

$$\limsup_{k\to 0} |\{\lambda_n(k) \leq T\}| \leq \{\lambda_n(0) \leq T\}| \leq \liminf_{k\to 0} |\{\lambda_n(k) \leq T\}|.$$

But this means, that $\{\lambda_n(k) \leq T\}|$ is continuous in $k = 0$ (if $T \neq \lambda_n(0)$), so this proves the theorem. □

Another way to state the theorem would be, that if ν_k is as stated in the theorem, and $\lambda_0(k) \leq \lambda_1(k) \leq \dots$ are the corresponding eigenvalues, then $\lambda_n(k)$ is continuous in $k = 0$, if $\lambda_n(0) < 1/4$. We get the following corollary to Theorem 5.2.1

Corollary 5.2.2. *Let $I \subset [-1, 1]$ be closed, and for $k \in I$ let $\nu_k : \Gamma \to \mathbb{S}^1$ be a multiplier system of weight k, such that $\nu_k(\gamma)$ is continuous as a function of the weight k, for any fixed $\gamma \in \Gamma$. Denote the eigenvalues corresponding to ν_k by $\lambda_0(k) \leq \lambda_1(k) \leq \dots$. If $0 \in I$ and $\nu_0 \not\equiv 1$ or if $0 \notin I$, then there exists a $\varepsilon > 0$ so $\lambda_0(k) \geq \varepsilon$ for all $k \in I$. If $0 \in I$ and $\nu_0 \equiv 1$, then $\lambda_0(0) = 0$ and there exists $\varepsilon > 0$ so $\lambda_1(k) \geq \varepsilon$ for all $k \in I$.*

Proof. It follows from (2.17) and (2.18), that

$$\lambda_0(k) \geq \frac{|k|}{2}\left(1 - \frac{|k|}{2}\right),$$

and that if f is the eigenfunction corresponding to $\lambda_0(k)$, then equality holds exactly when $y^{k/2}\overline{f(z)}$ is holomorphic and $k < 0$, $y^{-k/2}f(z)$ is holomorphic and $k > 0$, or $k = 0$ and $f, \overline{f}$ are holomorphic. So if $0 \in I$, then $\lambda_0(0) = 0$ if there exists an eigenfunction f of Δ_0 such that f and $\overline{f}$ is holomorphic, and otherwise $\lambda_0(0)$ is positive. But the only way, that both f and $\overline{f}$ can be holomorphic, is, if f is constant, and the constant functions are automorphic forms if and only if, the multiplier system is constant (and the weight is 0). So if $\nu_0 \equiv 1$, then $0 = \lambda_0(0) < \lambda_0(1)$, and if $\nu_0 \not\equiv 1$ is $0 < \lambda_0(0)$.

If $0 \in I$ and $\nu_0 \equiv 1$, let $\lambda_1(0) = 2\varepsilon'$. By Theorem 5.2.1 there exists $\delta \in (0, 1)$ such that $\lambda_1(k) \geq \varepsilon'$ for $|k| \leq \delta$. For $\delta \leq |k| \leq 1$ we have

$$\lambda_1(k) \geq \lambda_0(k) \geq \frac{|k|}{2}\left(1 - \frac{|k|}{2}\right) \geq \frac{\delta}{2}\left(1 - \frac{\delta}{2}\right) = \varepsilon'',$$

so setting $\varepsilon = \min\{\varepsilon', \varepsilon''\}$ gives the desired result for $\nu_0 \equiv 1$. If we let $\nu_0 \not\equiv 1$ and $\lambda_0(0) = 2\varepsilon'$ the same argument yields the other part of the proof, for $0 \in I$. If $0 \notin I$, define $\delta = \min_{x\in I} |x|$, and $\varepsilon = \varepsilon''$. □

5.3 A "Twisted" Prime Geodesic Theorem

To each hyperbolic element $\gamma \in \Gamma$ of norm $N(\gamma)$ corresponds a geodesic on $\Gamma\backslash\mathbb{H}$ of length $l(\gamma) = \log(N(\gamma))$. We want to investigate sums over prime geodesics of length at most T for some $T > 0$. To do so, we would like to use the trace formula with $g = \mathbf{1}_{[-T,T]}$, since we then only would sum over hyperbolic geodesics of length at most T. We can however not choose $g = \mathbf{1}_{[-T,T]}$, since this is not smooth, so we have to smooth it out. To do so we convolute with a cut-off function K_ϵ supported on $[-\epsilon,\epsilon]$, where ϵ is some positive constant.

Following Sarnak (see [18]), we let $K : \mathbb{R} \to [0,\infty[$ be a smooth even function, such that $K(t) = 0$, for $|t| \geq 1$, and

$$\int_{\mathbb{R}} K(t)dt = 1.$$

For $T > 1$ and $0 < \epsilon < 1$ define

$$\begin{aligned} K_\epsilon(t) &= \frac{1}{\epsilon} K\left(\frac{t}{\epsilon}\right) \\ g_{T,\epsilon}(t) = g_T(t) = g(t) &= \frac{1}{2\pi}(\mathbf{1}_{[-T,T]} * K_\epsilon)(t) \\ h_{T,\epsilon}(t) = h_T(t) = h(t) &= 2\pi\hat{g}(t) = \frac{2\sin(Tt)}{t}\hat{K}(\epsilon t), \end{aligned}$$

so that h and g can be used in the trace formula. Then K_ϵ is supported on $[-\epsilon,\epsilon]$ and $\int K_\epsilon = 1$. So $g_{T,\epsilon}(t)$ is supported on $[-T-\epsilon, T+\epsilon]$, it is $(2\pi)^{-1}$ for $|t| \leq T-\epsilon$ and is between 0 and $(2\pi)^{-1}$ for all t.

Estimations on the terms in the trace formula with $h_{T,\epsilon}$ and $g_{T,\epsilon}$ gives us the following lemma.

Lemma 5.3.1. *For $h_{T,\epsilon}$ and $g_{T,\epsilon}$ defined as above we have*

$$\sum_{n=0}^{\infty} h_{T,\epsilon}(r_n) = \sum_{\substack{[\gamma] \\ Tr\gamma>2}} \frac{\nu(\gamma)l(\gamma_0)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} g_{T,\epsilon}(l(\gamma)) + O(L(\nu)(T+\epsilon^{-1})),$$

where the implied constant is independent on the multiplier system ν, and L is defined by (5.4).

Proof. Since a multiplier system of weight k is also a multiplier system of weight $k \pm 2$, we can assume that $|k| \leq 1$. We are going to use the trace formula, so we have to estimate the terms (2.28)-(2.29) and (2.31)-(2.37).

Since $|k| \leq 1$ (2.29) is zero.

We now estimate (2.28),

$$\begin{aligned} \frac{\mu(\mathcal{F})}{4\pi}\int_{\mathbb{R}} rh(r)\frac{\sinh(2\pi r)dr}{\cosh(2\pi r)+\cos(\pi m)} &\ll \int_{\mathbb{R}} \left| h(r)\frac{r\sinh(2\pi r)}{\cosh(2\pi r)-1}\right| dr \\ &\ll \int_{\mathbb{R}} |h(r)|(|r|+1)\,dr. \end{aligned}$$

By definition of h we have

$$\begin{aligned}\int_{\mathbb{R}} |h(r)|(|r|+1)\,dr &\ll T\int_{-1}^{1}\left|\frac{\sin(Tr)}{Tr}\hat{K}(\epsilon r)\right|\,dr + 2\int_{1}^{\infty}\left|\sin(Tr)\hat{K}(\epsilon r)\right|\,dr \\ &\ll T+\epsilon^{-1}\int_{\epsilon}^{\infty}|\hat{K}(r)|\,dr \ll T+\epsilon^{-1}.\end{aligned}$$

For fixed θ is

$$e^{(k-1)u/2}\frac{e^u-e^{2i\theta}}{\cosh u-\cos(2\theta)}$$

uniformly bounded when $|k|\le 1$ and $u\in\mathbb{R}$, so (2.31) can be estimated by

$$\begin{aligned}&\sum_{\substack{\{R\}\\ \operatorname{Tr} R<2\\ 0<\theta(R)<\pi}} \frac{\nu(R)ie^{i(k-1)\theta}}{4M_R\sin\theta}\int_{\mathbb{R}} g(u)e^{(k-1)u/2}\frac{e^u-e^{2i\theta}}{\cosh u-\cos(2\theta)}\,du \\ &\qquad\ll \sum_{\substack{\{R\}\\ \operatorname{Tr} R<2\\ 0<\theta(R)<\pi}} (4M_R\sin\theta)^{-1}\int_{\mathbb{R}} g(u)\,du \ll T.\end{aligned}$$

Likewise (2.33) and (2.34) is $O(T)$.

We note that $g(0)=1$, and estimate (2.32) by

$$\left|g(0)\sum_{\alpha_j(\nu)\neq 0}\log|1-e^{2\pi i\alpha_j(\nu)}|\right| \le \sum_{\alpha_j(\nu)\neq 0}\left|\log|1-e^{2\pi i\alpha_j(\nu)}|\right| \ll L(\nu).$$

Since $\Gamma'(s)/\Gamma(s) = \log s-(2s)^{-1}+O(|s|^{-2})$ uniformly on vertical the line $\Re s=1$ (see [8] (B.11) p. 199), is

$$\begin{aligned}&K_0\left(g(0)\log 2+\frac{1}{2\pi}\int_{\mathbb{R}} h(r)\frac{\Gamma'(1+ir)}{\Gamma(1+ir)}dr\right) \\ &\qquad\ll 1+\int_{\mathbb{R}}|\log(1+ir)h(r)|\,dr+\int_{-1}^{1}|h(r)|\,dr \\ &\qquad\ll T+\int_{\mathbb{R}}\hat{K}(\epsilon r)\,dr \\ &\qquad\ll T+\int_{\mathbb{R}}(1+(\epsilon r)^2)^{-1}\,dr \\ &\qquad\ll T+\epsilon^{-1}\int_{\mathbb{R}}(1+t^2)^{-1}\,dt \ll T+\epsilon^{-1},\end{aligned}$$

since $\hat{K}$ is a Schwartz-function.

(2.36) is trivially estimated by

$$\left|\frac{1}{4}h(0)\mathrm{Tr}\,(I-\Phi(\tfrac{1}{2}))\right| \le K_0 h(0)/2 = K_0 T.$$

So the only thing left to do, is estimating (2.37). To do this we use Theorem 5.1.1. We see that

$$\int_{-1}^{1} h(r)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\,dr \le 2T\int_{-1}^{1}\left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\right|\,dr \ll TL(\nu).$$

Furthermore

$$\begin{aligned}\int_{1}^{\epsilon^{-1}} h(r)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\,dr &\ll \int_{1}^{\epsilon^{-1}}\frac{1}{r}\left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\right|\,dr\\ &= \epsilon\int_{1}^{\epsilon^{-1}}\left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\right|\,dr + \int_{1}^{\epsilon^{-1}} r^{-2}\int_{1}^{r}\left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ix\right)\right|\,dx\,dr\\ &\ll \epsilon^{-1}L(\nu),\end{aligned}$$

and likewise for the integral over $[-\epsilon^{-1},-1]$. Finally

$$\begin{aligned}&\int_{\epsilon^{-1}}^{\infty} h(r)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\,dr\\ &= \sup_{x>0}(\hat{K}(x)x^2)\int_{\epsilon^{-1}}^{\infty}\epsilon^{-2}r^{-3}\left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\right|\,dr\\ &\ll \lim_{A\to\infty} A^{-3}\int_{\epsilon^{-1}}^{A}\epsilon^{-2}\left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\right|\,dr + \int_{\epsilon^{-1}}^{\infty} r^{-4}\epsilon^{-2}\int_{\epsilon^{-1}}^{r}\left|\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ix\right)\right|\,dx\,dr\\ &\ll \epsilon^{-1}L(\nu),\end{aligned}$$

here we have again used that $\hat{K}$ is a Schwartz function. Since we can make the same estimate for $]-\infty,-\epsilon^{-1}]$, is

$$\int_{\mathbb{R}} h(r)\frac{\varphi'}{\varphi}\left(\frac{1}{2}+ir\right)\,dr = O(\epsilon^{-1}L(\nu)).$$

This proves the lemma. □

We chose g so that it was almost an indicator function, so that the hyperbolic term in the trace formula (2.30) would approximately be a sum over the elements of Γ' (the set of conjugacy classes of primitive hyperbolic matrices with positive trace in Γ) of length at most T. The following lemma formalizes this.

Lemma 5.3.2. *For $h_{T,\epsilon}$ and $g_{T,\epsilon}$ defined as above we have, that the hyperbolic term in the trace formula* (2.30) *is*

$$\frac{1}{2\pi}\sum_{\substack{[\gamma]\in\Gamma'\\ l(\gamma)\le T}}\frac{\nu(\gamma)l(\gamma)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}} + O(e^{T/2}\epsilon + e^{T/4}), \tag{5.14}$$

where the implied constant is independent on the multiplier system ν.

Proof. We rewrite (2.30)

$$\sum_{\substack{[\gamma]\\ \operatorname{Tr}\gamma>2}} \frac{\nu(\gamma)g(l(\gamma))l(\gamma_0)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}} = \sum_{j=1}^{\infty}\sum_{[\gamma]\in\Gamma'} \frac{\nu(\gamma^j)g(jl(\gamma))l(\gamma)}{N(\gamma)^{j/2}-N(\gamma)^{-j/2}}.$$

We first want to estimate the $j\geq 2$ part, so that we are left with a sum over the primitive geodesics.

Let N_t be the norm of matrices with trace t and let $m(t)$ be the number of primitive conjugation classes with trace t, then

$$\begin{aligned}\sum_{j=2}^{\infty}\sum_{[\gamma]\in\Gamma'} \frac{\nu(\gamma^j)g(jl(\gamma))l(\gamma)}{N(\gamma)^{j/2}-N(\gamma)^{-j/2}} &\ll \sum_t \frac{m(t)\log N_t}{N_t}\sum_{j=2}^{\infty} g(j\log N_t)\\ &\ll \sum_{2\log N_t\leq T+\epsilon} \frac{m(t)\log N_t}{N_t}\cdot\frac{T}{\log N_t},\end{aligned}$$

here the implied constant only depends on Γ. By partial summation and the prime geodesics theorem we get

$$T\sum_{2\log N_t\leq T+\epsilon}\frac{m(t)}{N_t} = T\pi\left(\frac{T+\epsilon}{2}\right)e^{-(T+\epsilon)/2} + T\int_1^{(T+\epsilon)/2}\pi(t)e^{-t}\,dt = O(T\log T).$$

This takes care of the non-primitive part of (2.30). To estimate the primitive part, we use that g is defined so it is almost an indicator function, so

$$\begin{aligned}&\sum_{[\gamma]\in\Gamma'}\frac{\nu(\gamma)l(\gamma)g(\log N(\gamma))}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}} - \frac{1}{2\pi}\sum_{\substack{[\gamma]\in\Gamma'\\ l(\gamma)\leq T}}\frac{\nu(\gamma)l(\gamma)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}}\\ &\qquad\ll \sum_{\substack{[\gamma]\in\Gamma'\\ T-\epsilon<l(\gamma)\leq T+\epsilon}}\frac{l(\gamma)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}}.\end{aligned}$$

We then use (2.42) on this estimate

$$\begin{aligned}\sum_{\substack{[\gamma]\in\Gamma'\\ T-\epsilon<l(\gamma)\leq T+\epsilon}}\frac{l(\gamma)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}} &\ll e^{-(T-\epsilon)/2}\sum_{\substack{[\gamma]\in\Gamma'\\ T-\epsilon<l(\gamma)\leq T+\epsilon}} l(\gamma)\\ &\ll e^{-(T-\epsilon)/2}\sum_{1/2<s_j\leq 1} s_j^{-1}\left(e^{s_j(T+\epsilon)}-e^{s_j(T-\epsilon)}\right)+e^{T/4+5\epsilon/4}\\ &\ll \sum_{1/2<s_j\leq 1} s_j^{-1}e^{(T+\epsilon)/2}\left(e^{s_j\epsilon}-e^{-s_j\epsilon}\right)+e^{T/4}\\ &\ll e^{T/2}\epsilon+e^{T/4},\end{aligned}$$

which proves the lemma. $\square$

If we have a zero free automorphic form f of weight $k_0 > 0$ wrt. Γ with multiplier system ν, we can take logarithms F and Φ like we did in section 4.1. In this way we can get powers of f and ν, and since f^{1/k_0} has weight 1, we can without loss of generality assume that f has weight 1. So we get, that $\nu^k = \exp(2\pi i k\Phi)$ is a weight k multiplier system, and hence also a weight $k+2n$ multiplier system for any $n \in \mathbb{Z}$. Especially if we for $k \in \mathbb{R}$ define k' to be the number in $(-1,1]$, such that $k' \equiv k$ modulo 2, then ν^k is a weight k' multiplier system.

We will assume that Φ only takes rational values, so there is an even $m \in \mathbb{N}$, such that $m\Phi$ only takes integer values. We let N be the smallest such N.

Lemma 5.3.2 gives us an estimate of the right hand side of the trace formula for the test functions $g_{T,\epsilon}$ and $h_{T,\epsilon}$, and we want to combine this estimate with an estimate on the left hand side for the multiplier systems ν^k.

Before we do so, we define $I := (-1, N-1]$ and $P : (1,\infty) \times I \to \mathbb{C}$ given by

$$P(T,k) := \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le T}} \frac{\nu^k(\gamma) l(\gamma)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}}.$$

Lemma 5.3.3. *There exists $c \in [1/4, 1/2)$, such that*

$$P(T,k) = \begin{cases} \frac{2e^{T(1-|k|)/2}}{1-|k|} + O(L(\nu^k)e^{Tc}) & \text{if } |k| \le 1/2 \\ O(L(\nu^k)e^{Tc}) & \text{otherwise} \end{cases}, \tag{5.15}$$

with the implied constant only depending on the Fuchsian group Γ.

Proof. We split I into the intervals $I_n = (2n-1, 2n+1]$, for $0 \le n \le N/2$. We fix an n, and for $k \in I_n$ consider the multiplier system ν^k, to be of weight k'. By Corollary 5.2.2 and the definition of N there exists $\varepsilon_n \in (0, 3/16]$, such that $\lambda_1(k) \ge \varepsilon_n$ for $k \in I_n$ if $n = 0$ and $\lambda_0(k) \ge \varepsilon_n$ otherwise (where $\lambda_0(k) \le \lambda_1(k)$ are the two smallest eigenvalues for ν^k). Since $\varepsilon_n \in (0, 3/16]$, we can define $R_n \in (0, 1/2]$, by

$$\varepsilon_n = \frac{R_n}{2}\left(1 - \frac{R_n}{2}\right).$$

This means that

$$\frac{1}{4} + \left(\frac{i}{2} - \frac{iR_n}{2}\right)^2 = -\left(\frac{R_n}{2}\right)^2 + \frac{R_n}{2} = \varepsilon_n,$$

and we see that $1 - R_n$ is decreasing as a function of ε_n.

Since

$$\begin{aligned} \hat{K}(\epsilon i r) &= \frac{1}{2\pi}\int_{-1}^{1} \exp(-\epsilon r t) K(t)\, dt \le \frac{1}{2\pi}\exp(\epsilon r) = \frac{1}{2\pi} + O(\epsilon r) \\ \hat{K}(\epsilon i r) &= \frac{1}{2\pi}\int_{-1}^{1} \exp(-\epsilon r t) K(t)\, dt \ge \frac{1}{2\pi}\exp(-\epsilon r) = \frac{1}{2\pi} + O(\epsilon r) \end{aligned}$$

for $0 \le r \le 1/2$, so for $0 \le r \le 1/2$ we have

$$\begin{aligned} h(ir) &= \frac{e^{Tr} - e^{-Tr}}{r}\hat{K}(\epsilon ir) \\ &= \begin{cases} e^{Tr}/(2\pi r) + O(e^{Tr}\epsilon + e^{-Tr}/r) & \text{for } 1/T \le r \le 1/2 \\ O(T) & \text{for } 0 \le r < 1/T \end{cases} \end{aligned}$$

So according to (5.2)

$$\begin{aligned} \sum_{\varepsilon_n \le \lambda_m(k) \le 1/4} h(r_m) &\ll \left(T + \frac{2e^{T(1/2 - R_n/2)}}{1 - R_n}\right) \sum_{\varepsilon_n \le \lambda_m(k) \le 1/4} 1 \\ &\ll (T + e^{T(1-R_n)/2}) L(\nu^k). \end{aligned}$$

We now use (5.2) to estimate the contribution from the $\lambda_m(k) > 1/4$. We start with the $r_m(k) \le 1$, which is easily estimated by

$$\begin{aligned} \sum_{1/4 \le \lambda_m(k) \le 5/4} h(r_m(k)) &= \sum_{1/4 \le \lambda_m(k) \le 5/4} \frac{2\sin(Tr_m(k)}{r_m}\hat{K}(\epsilon r_m(k)) \\ &\le 2T \sum_{1/4 \le \lambda_m(k) \le 5/4} \frac{\sin(Tr_m(k))}{Tr_m(k)} \ll TL(\nu^k). \end{aligned}$$

The next part we estimate by

$$\begin{aligned} \sum_{1 \le r_m(k) \le \epsilon^{-1}} h(r_m(k)) &\le \sum_{1 \le r_m(k) \le \epsilon^{-1}} \frac{2}{r_m(k)} \\ &= \epsilon \sum_{1 \le r_m(k) \le \epsilon^{-1}} 1 + \int_1^{1/\epsilon} t^{-2} \sum_{1 \le r_m(k) \le t} 1\, dt \\ &\ll \epsilon^{-1} + L(\nu^k) + \int_1^{1/\epsilon} L(\nu^k)\, dt \ll \epsilon^{-1} L(\nu^k). \end{aligned}$$

To estimate the contribution from the large eigenvalues we use that $\hat{K}$ is a Schwartz function, so

$$\begin{aligned} \sum_{\epsilon^{-1} < r_m(k)} h(r_m(k)) &= 2\epsilon \sum_{1 < \epsilon r_m(k)} \frac{\sin(Tr_m(k))}{\epsilon r_m(k)}\hat{K}(\epsilon r_m(k)) \\ &\ll \epsilon \sum_{1 < \epsilon r_m(k)} (\epsilon r_m(k))^{-3} \\ &= \epsilon \lim_{A \to \infty} \left(A^{-3} \sum_{1 < \epsilon r_m(k)} 1 \right) + \epsilon \int_1^\infty t^{-4} \sum_{1 \le \epsilon r_m(k) \le t} 1\, dt \\ &\ll \epsilon \int_1^\infty t^{-2}\epsilon^{-2} + t^{-3}\epsilon^{-1} L(\nu^k)\, dt \ll \epsilon^{-1} L(\nu^k). \end{aligned}$$

So if we define

$$c := \frac{\max_n(1 - R_n)}{2},$$

then

$$\sum_{\varepsilon_n \leq \lambda_m(k)} h(r_m(k)) \ll (e^{Tc} + \epsilon^{-1}) L(\nu^k).$$

for all n. This result combined with Lemma 5.3.1 and 5.3.2 proves that

$$P(T,k) = \begin{cases} 2\pi h((\lambda_0(k) - 1/4)^{1/2}) + R(T,\epsilon,k) & \text{if } |k| \leq 1 \text{ and } \lambda_0(k) \leq \varepsilon_0 \\ R(T,\epsilon,k) & \text{otherwise} \end{cases},$$

where

$$R(T,\epsilon,k) = O(e^{T/2}\epsilon + L(\nu^k)(e^{Tc} + \epsilon^{-1})).$$

If $0 \leq k \leq 1$ then $k' = k$ and we have

$$-\Delta_{k'}(f(z)y^{k/2}) = \frac{k}{2}\left(1 - \frac{k}{2}\right) f(z)y^{k/2}.$$

If $-1 < k \leq 0$ then $k' = k$ and we have

$$-\Delta_{k'}(\overline{f(z)}y^{-k/2}) = -\frac{k}{2}\left(1 + \frac{k}{2}\right) f(z)y^{k/2}.$$

So for $k \in I_0 = (-1, 1]$, we have

$$\lambda_0(k) = \frac{|k|}{2}\left(1 - \frac{|k|}{2}\right).$$

We see that

$$\left(\frac{1 - |k|}{2}\right)^2 = 1/4 - \frac{|k|}{2}\left(1 - \frac{|k|}{2}\right)$$

so for $|k| \leq 1/2$

$$\begin{aligned} 2\pi h((\lambda_0(k) - 1/4)^{1/2}) &= 2\pi h\left(\frac{i - i|k|}{2}\right) \\ &= \frac{2e^{T(1-|k|)/2}}{1 - |k|} + O(e^{T(1-|k|)/2}\epsilon + T). \end{aligned}$$

For $1/2 < |k| \leq 1$ we have

$$\lambda_0(k) = \frac{1}{4} - \left(\frac{1 - |k|}{2}\right)^2 > \frac{1}{4} - \left(\frac{1}{4}\right)^2 = \frac{3}{16} \geq \varepsilon_0.$$

By setting $\epsilon = e^{-T/4}$ we get

$$P(T,k) = \begin{cases} \frac{2e^{T(1-|k|)/2}}{1-|k|} + O(L(\nu^k)e^{Tc}) & \text{if } |k| \le 1/2 \text{ and } \lambda_0(k) \le \varepsilon_0 \\ O(L(\nu^k)e^{Tc}) & \text{otherwise} \end{cases}.$$

Since

$$\frac{2e^{T(1-|k|)/2}}{1-|k|} = O(e^{Tc}),$$

when $|k| \le 1/2$ and $\lambda_0(k) \ge \varepsilon_0$, this proves the lemma. $\square$

Note that we in the proof of the lemma, showed that if $R \le 1/4$ and $R(1-R) \le \lambda_1(k)$, for all $k \in I$, and $R(1-R) \le \lambda_0(k)$, when $|k| \ge 1$, then $c = 1/2 - R$ can be used in (5.15).

Lemma 5.3.3 leads us to the following "twisted" version of the prime geodesic theorem.

Theorem 5.3.4. *There exists $\delta \in (0, 1/4]$, such that for $k \in (-1, N-1]$ we have*

$$\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le T}} \nu^k(\gamma) l(\gamma) = \begin{cases} \frac{e^{T(1-|k|/2)}}{1-|k|/2} + O(e^{T(1-\delta)}L(\nu^k)) & \text{if } |k| \le 1/2 \\ O(e^{T(1-\delta)}L(\nu^k)) & \text{otherwise} \end{cases}. \tag{5.16}$$

Proof. By partial summation we get

$$\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le T}} \nu^k(\gamma) l(\gamma) = (e^{T/2} - e^{-T/2}) P(T,k) - \int_{N_\Gamma}^{e^T} \frac{1+x^{-1}}{2\sqrt{x}} P(\log x, k)\, dx.$$

Since

$$\begin{aligned} (e^{T/2} - e^{-T/2}) \frac{2e^{T(1-|k|)/2}}{1-|k|} &= \frac{2e^{T(1-|k|/2)}}{1-|k|} + O(1), \\ (e^{T/2} - e^{-T/2}) L(\nu^k) e^{Tc} &= O(L(\nu^k)e^{T(1/2+c)}), \\ \int_{N_\Gamma}^{e^T} \frac{1+x^{-1}}{2\sqrt{x}} L(\nu^k) x^c dx &= O(L(\nu^k)e^{T(1/2+c)}), \\ \int_{N_\Gamma}^{e^T} \frac{1+x^{-1}}{2\sqrt{x}} \frac{2x^{(1-|k|)/2}}{1-|k|} dx &= \int_{N_\Gamma}^{e^T} \frac{x^{-|k|/2}}{1-|k|} dx + O(1) \\ &= \frac{e^{T(1-|k|/2)}}{(1-|k|)(1-|k|/2)} + O(1), \end{aligned}$$

we get for $|k| > 1/2$

$$\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le T}} \nu^k(\gamma) l(\gamma) = O(L(\nu^k)e^{T(1/2+c)}),$$

and for $|k| \leq 1/2$

$$\begin{aligned}\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq T}} \nu^k(\gamma)l(\gamma) &= \frac{2e^{T(1-|k|/2)}}{1-|k|} - \frac{e^{T(1-|k|/2)}}{(1-|k|)(1-|k|/2)} + O(L(\nu^k)e^{T(1/2+c)}) \\ &= \frac{e^{T(1-|k|/2)}}{1-|k|/2} + O(L(\nu^k)e^{T(1/2+c)}).\end{aligned}$$

If we define $\delta = 1/2 - c$, then we get that $\delta \in (0, 1/4]$ and $1/2 + c = 1 - \delta$, so

$$\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq T}} \nu^k(\gamma)l(\gamma) = \begin{cases} \frac{e^{T(1-|k|/2)}}{1-|k|/2} + O(e^{T(1-\delta)}L(\nu^k)) & \text{if } |k| \leq 1/2 \\ O(e^{T(1-\delta)}L(\nu^k)) & \text{otherwise} \end{cases}. \tag{5.17}$$

□

Since we can use $c = 1/2 - \delta$ in Lemma 5.3.3 if $\delta \leq 1/4$ and $\delta(1-\delta) \leq \lambda_1(k)$, for all $k \in I$, and $\delta(1-\delta) \leq \lambda_0(k)$, when $|k| \geq 1$, we can use

$$1/2 - (1/2 - \delta) = \delta$$

in Theorem 5.3.4.

5.4 Prime Geodesics Distributed wrt. a Multiplier System

A simple integration of (5.16) gives us the following theorem.

Theorem 5.4.1. *For $n \in \mathbb{Z}$ we have*

$$\sum_{\substack{[\gamma]\in\Gamma' \\ N(\gamma)\leq x \\ N\Phi(\gamma)=n}} l(\gamma) = \frac{4}{N}\int_2^x \frac{\log y}{(4\pi n/N)^2 + (\log y)^2}\,dy + O(x^{1-\delta}), \tag{5.18}$$

for $\delta \in (0, 1/4]$ such that $\delta(1-\delta) \leq \lambda_0(k)$ for all $k \in (1, N-1]$, and $\delta(1-\delta) \leq \lambda_1(k)$ for all $k \in (-1, 1]$.

Proof. For a fixed $n \in \mathbb{Z}$, we integrate (5.16) wrt. $e(-kn/N)dk$ (where $e(x) =$

$\exp(2\pi i x))$, over $(-1, N-1]$. The left hand side becomes

$$\begin{aligned}
&\int_{-1}^{N-1} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq T}} l(\gamma) e(k(\Phi(\gamma)-n/N))dk \\
= &\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq T}} l(\gamma) \int_{-1}^{N-1} e(k(\Phi(\gamma)-n/N))dk \\
= &\; N \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq T \\ N\Phi(\gamma)=n}} l(\gamma).
\end{aligned}$$

From the error term we get

$$\int_{-1}^{N-1} e^{T(1-\delta)} L(\nu^k) e(-kn/N)dk \ll e^{T(1-\delta)}.$$

To integrate the main term on the right hand side we use that $\lambda_0(k) = 0$ if and only if $\nu^k \equiv 1$. Hence

$$\lambda_0(k) = 0 \Leftrightarrow \nu^k \equiv 1 \Leftrightarrow N|k.$$

Especially for $k \in (-1, N-1]$, $\lambda_0(k) = 0$ if and only if $k = 0$.

Now we can integrate the main term on the right hand side, and finish the proof.

$$\begin{aligned}
\int_{-1/2}^{1/2} \frac{e^{T(1-|k|/2)}}{1-|k|/2} e\left(\frac{-kn}{N}\right) dk &= 2\Re \int_0^{1/2} \frac{e^{T(1-k/2)}}{1-k/2} e\left(\frac{kn}{N}\right) dk \\
&= 2\Re \int_0^{1/2} \int_2^{e^T} y^{-k/2} dy\, e\left(\frac{kn}{N}\right) dk + O(1) \\
&= 2\Re \int_2^{e^T} \int_0^{1/2} e^{k(2\pi i n/N - \log y/2)} dk\, dy + O(1) \\
&= 2\Re \int_2^{e^T} \frac{e^{\pi i n/N - \log y/4} - 1}{2\pi i n/N - \log y/2} dy + O(1) \\
&= 4\Re \int_2^{e^T} \frac{4\pi i n/N + \log y}{(4\pi n/N)^2 + (\log y)^2} dy + O\left(\int_2^{e^T} \frac{dy}{y^{1/4}}\right) \\
&= 4 \int_2^{e^T} \frac{\log y}{(4\pi n/N)^2 + (\log y)^2} dy + O(e^{3T/4}).
\end{aligned}$$

By combining these estimates, and substituting e^T by x, we get (5.18). $\square$

If $n = 0$, we have

$$\int_2^x \frac{\log y}{(4\pi n/N)^2 + (\log y)^2}\, dy = \int_2^x \frac{1}{\log y}\, dy = li(x),$$

and for fixed n, we have

$$\int_2^x \frac{\log y}{(4\pi n/N)^2 + (\log y)^2}\, dy \sim li(x).$$

Because of this likeness, we define

$$li(p, x) = \int_2^x \frac{\log y}{(4\pi p)^2 + (\log y)^2}\, dy.$$

So the main term in (5.18) becomes $4li(n/N, x)/N$.

We can use (5.18) to give some estimates on the number of prime geodesics of a certain Φ-value and bounded length.

Corollary 5.4.2. *For $n \in \mathbb{Z}$ we have*

$$\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ N\Phi(\gamma)=n}} 1 = \frac{4}{Nt} li\left(\frac{n}{N}, e^t\right) + O\left(\frac{e^t}{t^3}\right), \tag{5.19}$$

$$\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ N\Phi(\gamma)=n}} 1 = \frac{4\pi(t)}{Nt}\left(1 - \left(\frac{4\pi n}{Nt}\right)^2 + O\left(\frac{n^4}{t^4} + \frac{n^2}{t^3} + \frac{1}{t}\right)\right), \tag{5.20}$$

where $\pi(t)$ in (5.20) *is the number of prime geodesics, which has length at most t.*

Note that we have the constant π as well as the function that count the number of prime geodesics of length at most t $\pi(t)$ in these formulas.

Proof. By partial integration and equation (5.18) we get

$$\begin{aligned}
\sum_{\substack{[\gamma]\in\Gamma' \\ N(\gamma)\le x \\ N\Phi(\gamma)=n}} 1 &= \frac{1}{\log x} \sum_{\substack{[\gamma]\in\Gamma' \\ N(\gamma)\le x \\ N\Phi(\gamma)=n}} l(\gamma) + \int_{N_\Gamma}^x \frac{1}{t(\log t)^2} \sum_{\substack{[\gamma]\in\Gamma' \\ N(\gamma)\le t \\ N\Phi(\gamma)=n}} l(\gamma) dt \\
&= \frac{4li(n/N, x)}{N \log x} + \int_{N_\Gamma}^x \frac{4}{Nt(\log t)^2} li\left(\frac{n}{N}, t\right) dt + O(x^{1-\delta}).
\end{aligned}$$

Estimating the second term we get

$$\int_{N_\Gamma}^x \frac{4}{Nt(\log t)^2} li\left(\frac{n}{N}, t\right) dt \ll \int_{N_\Gamma}^x \frac{1}{t(\log t)^2} li(t) dt \ll \int_{N_\Gamma}^x \frac{1}{(\log t)^3} dt.$$

By partial integration we see that for $m > 0$, we have

$$\begin{aligned}\int_2^x (\log t)^{-m} dt &= \frac{x}{(\log x)^m} - \frac{2}{2^m} + \int_2^x t \cdot \frac{m}{t(\log t)^{m+1}} dt \\ &= \frac{x}{(\log x)^m} - \frac{2}{2^m} + m \int_{\log N_\Gamma}^{\log x} \frac{e^u}{u^4} du \ll_m \frac{x}{(\log x)^3}.\end{aligned}$$

When we combine these estimates, and substitutes x by e^t, we get (5.19).

We can (again) use intagration by parts to estimate $li(p, x)$.

$$\begin{aligned}li\left(\frac{p}{4\pi}, x\right) &= \frac{x \log x}{p^2 + (\log x)^2} - \int_2^x y \cdot \frac{(p^2 - (\log y)^2)/y}{(p^2 + (\log y)^2)^2} dy + O(1) \\ &= \frac{x \log x}{p^2 + (\log x)^2} + O\left(\int_2^x \frac{1}{(\log y)^2} dy\right) \\ &= \frac{x \log x}{p^2 + (\log x)^2} + O\left(\frac{x}{(\log x)^2}\right).\end{aligned}$$

This estimate and (5.19) gives us

$$\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq t \\ N\Phi(\gamma)=n}} 1 = \frac{4e^t}{N((4\pi n/N)^2 + t^2)} + O\left(\frac{e^t}{t^3}\right). \tag{5.21}$$

For $0 < r < 1$ we have, that for all $|t| \leq r$ is $(1+t)^{-1} = 1 - t + O(t^2)$. Hence (5.21) gives us

$$\begin{aligned}\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq t \\ N\Phi(\gamma)=n}} 1 &= \frac{4e^t}{Nt^2}\left(\frac{1}{1 + (\frac{4\pi n}{Nt})^2} + O\left(\frac{1}{t}\right)\right) \\ &= \frac{4e^t}{Nt^2}\left(1 - \left(\frac{4\pi n}{Nt}\right)^2 + O\left(\left(\frac{n}{t}\right)^4 + \frac{1}{t}\right)\right).\end{aligned}$$

By (2.43) we get

$$\frac{e^t}{t} = \pi(t) + O\left(\frac{e^t}{t^2}\right) = \pi(t)\left(1 + O(t^{-1})\right).$$

So

$$\begin{aligned}\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq t \\ N\Phi(\gamma)=n}} 1 &= \frac{4\pi(t)}{Nt}\left(1 + O\left(\frac{1}{t}\right)\right)\left(1 - \left(\frac{4\pi n}{Nt}\right)^2 + O\left(\left(\frac{n}{t}\right)^4 + \frac{1}{t}\right)\right) \\ &= \frac{4\pi(t)}{Nt}\left(1 - \left(\frac{4\pi n}{Nt}\right)^2 + O\left(\frac{n^4}{t^4} + \frac{n^2}{t^3} + \frac{1}{t}\right)\right),\end{aligned}$$

which is (5.20). $\square$

The formula (5.20) tells us that the number of prime geodesics of length at most t and with a given Φ-value n/N is asymptotic equivalent to $4\pi(t)/(Nt)$, for all $n \in \mathbb{Z}$. However (5.20) also tells us, that if $n_1, n_2 \in \mathbb{Z}$ and $|n_1| < |n_2|$, then for large t

$$\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ N\Phi(\gamma)=n_1}} 1 > \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ N\Phi(\gamma)=n_2}} 1.$$

We can use formula (5.20) to prove that the prime geodesics are asymptotically Cauchy distributed wrt. the value of Φ/l. More precisely we have the following theorem.

Theorem 5.4.3. *For $x \in \mathbb{R}$ we have*

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ \Phi(\gamma)\le xl(\gamma)}} 1 = \frac{\arctan(4\pi x)}{\pi} + \frac{1}{2}. \tag{5.22}$$

Proof. By (5.21) we have

$$\begin{aligned} t^2 e^{-t} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ \Phi(\gamma)\le xt}} 1 &= t^2 e^{-t} \sum_{n\le xtN} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ N\Phi(\gamma)=n}} 1 \\ &= \frac{4}{N} \sum_{n\le xtN} \left(1 + \left(\frac{4\pi n}{Nt}\right)^2\right)^{-1} + O(t^{-1}) \\ &= \frac{4}{N} \int_{-\infty}^{xtN} \left(1 + \left(\frac{4\pi u}{Nt}\right)^2\right)^{-1} du + O(1) \\ &= \frac{t}{\pi} \int_{-\infty}^{4\pi x} (1+s^2)^{-1} ds + O(1) \\ &= \frac{t\arctan(4\pi x)}{\pi} + \frac{t}{2} + O(1). \end{aligned}$$

So

$$(\pi(t))^{-1} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ \Phi(\gamma)\le xt}} 1 \sim te^{-1} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ \Phi(\gamma)\le xt}} 1 \sim \frac{\arctan(4\pi x)}{\pi} + \frac{1}{2}.$$

So to prove the theorem, we need to prove that

$$\lim_{t\to\infty} (\pi(t))^{-1} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ xl(\gamma)<\Phi(\gamma)\le xt}} 1 = 0. \tag{5.23}$$

Let $\epsilon > 0$, then

$$\begin{aligned}
\limsup_{t\to\infty} \frac{\sum_{xl(\gamma)<\Phi(\gamma)\le xt} 1}{\pi(t)} &\le \limsup_{t\to\infty} (\pi(t))^{-1} \left(\pi(t(1-\epsilon)) + \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ xt(1-\epsilon)<\Phi(\gamma)\le xt}} 1 \right) \\
&= \limsup_{t\to\infty} \frac{e^{t(1-\epsilon)}}{e^t(1-\epsilon)} + \frac{\arctan 4\pi x - \arctan 4\pi x(1-\epsilon)}{\pi} \\
&= \frac{\arctan 4\pi x - \arctan 4\pi x(1-\epsilon)}{\pi}.
\end{aligned}$$

Since this is true for all $\epsilon > 0$, equation (5.23) is true. □

For $A \subset \mathbb{Z}$ define $\pi_A : \mathbb{R}_+ \to \mathbb{N} \cup \{0\}$ by

$$\pi_A(t) = \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ N\Phi(\gamma)\in A}} 1.$$

If the quantity

$$d(A) := \lim_{M\to\infty} \frac{\#\{n \in A \mid |n| \le M\}}{2\lfloor M \rfloor + 1}$$

is well defined, we say that A has natural density $d(A)$.

Theorem 5.4.4. *Let $A \subset \mathbb{Z}$ have natural density $d(A)$, then*

$$\lim_{t\to\infty} \frac{\pi_A(t)}{\pi(t)} = d(A).$$

Proof. For $K > 0$, we have

$$\frac{1}{\pi(t)} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ N\Phi(\gamma)\in A \\ |N\Phi(\gamma)|>Kt}} 1 \le \frac{1}{\pi(t)} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ |N\Phi(\gamma)|>Kt}} 1 \to \frac{1}{2} - \frac{2\arctan(4\pi K/N)}{\pi}.$$

For $\epsilon > 0$, we can choose K so large, that $\frac{1}{2} - 2\arctan(4\pi K/N)/\pi < \epsilon$, and hence

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ N\Phi(\gamma)\in A \\ |N\Phi(\gamma)|>Kt}} 1 \le \lim_{t\to\infty} \frac{1}{\pi(t)} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\le t \\ |N\Phi(\gamma)|>Kt}} 1 < \epsilon. \tag{5.24}$$

By definition of $d(A)$, there exists $M_1 \in \mathbb{N}$ so that

$$\left| \frac{\#\{n \in A \mid |n| \leq M\}}{\#\{n \in \mathbb{Z} \mid |n| \leq M\}} - d(A) \right| < \epsilon, \tag{5.25}$$

for $M \geq M_1$.

The equations (5.20) gives us

$$\frac{1}{\pi(t)} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq t \\ N\Phi(\gamma)\in A \\ |N\Phi(\gamma)|\leq M_1}} 1 \leq \frac{1}{\pi(t)} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq t \\ |N\Phi(\gamma)|\leq M_1}} 1 = O\left(\frac{M_1}{t}\right),$$

so

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq t \\ N\Phi(\gamma)\in A \\ |N\Phi(\gamma)|\leq M_1}} 1 = \lim_{t\to\infty} \frac{1}{\pi(t)} \sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq t \\ |N\Phi(\gamma)|\leq M_1}} 1 = 0. \tag{5.26}$$

So all there is left, is to estimate the contribution, from the geodesics with Φ-value between M_1/N and Kt/N. Equation (5.19) gives us

$$\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq t \\ N\Phi(\gamma)\in A \\ M_1<|N\Phi(\gamma)|\leq Kt}} 1 = \frac{4}{Nt} \sum_{\substack{n\in A \\ M_1<|n|\leq Kt}} li\left(\frac{n}{N}, e^t\right) + O\left(\frac{e^t}{t^2}\right).$$

If we let $li'(x,t)$ denote the derivative of li wrt. x, and define

$$A_t := \{n \in A \mid |n| \leq t\},$$

then partial summation gives us

$$\begin{aligned}
\sum_{\substack{n\in A_{Kt}\\ M_1<|n|}} li\left(\frac{n}{N},e^t\right) &= li\left(\frac{Kt}{N},e^t\right)\sum_{\substack{n\in A_{Kt}\\ M_1<|n|}} 1-\frac{1}{N}\int_{M_1}^{Kt} li'\left(\frac{x}{N},e^t\right)\sum_{\substack{n\in A_x\\ M_1<|n|}} 1\,dx \\
&= li\left(\frac{Kt}{N},e^t\right)\left(\sum_{n\in A_{Kt}} 1-\sum_{n\in A_{M_1}} 1\right) \\
&\quad -\frac{1}{N}\int_{M_1}^{Kt} li'\left(\frac{x}{N},e^t\right)\left(\sum_{n\in A_x} 1-\sum_{n\in A_{M_1}} 1\right)\,dx \\
&= d(A)\left(li\left(\frac{Kt}{N},e^t\right)\left(\sum_{M_1<|n|\le Kt} 1+O(\epsilon)\right)\right. \\
&\quad \left.-\frac{1}{N}\int_{M_1}^{Kt} li'\left(\frac{x}{N},e^t\right)\left(\sum_{M_1<|n|\le Kt} 1+O(\epsilon)\right)\,dx\right) \\
&= d(A)\sum_{M_1<|n|\le Kt} li\left(\frac{n}{N},e^t\right)+O\left(li\left(\frac{M_1}{N},e^t\right)\epsilon\right) \\
&= d(A)\sum_{M_1<|n|\le Kt} li\left(\frac{n}{N},e^t\right)+O\left(\frac{e^t\epsilon}{t}\right).
\end{aligned}$$

We now see that

$$\begin{aligned}
\frac{1}{\pi(t)}\sum_{\substack{[\gamma]\in\Gamma'\\ l(\gamma)\le t\\ N\Phi(\gamma)\in A\\ M_1<|N\Phi(\gamma)|\le Kt}} 1 &= \frac{1}{\pi(t)}\left(\frac{4d(A)}{Nt}\sum_{M_1<|n|\le Kt} li\left(\frac{n}{N},e^t\right)+O\left(\frac{e^t}{t^2}\right)\right) \\
&= \frac{d(A)}{\pi(t)}\sum_{\substack{[\gamma]\in\Gamma'\\ l(\gamma)\le t\\ M_1<|N\Phi(\gamma)|\le Kt}} 1+O\left(\frac{1}{t}\right).
\end{aligned}$$

So (5.24) and (5.26) implies

$$\begin{aligned}
\left|\lim_{t\to\infty}\frac{\pi_A(t)}{\pi(t)} - d(A)\right| &\leq \left|\lim_{t\to\infty}\frac{d(A)}{\pi(t)}\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq t \\ M_1<|N\Phi(\gamma)|\leq Kt}} 1 - d(A)\right| + \epsilon \\
&\leq \left|\lim_{t\to\infty}\frac{d(A)}{\pi(t)}\sum_{\substack{[\gamma]\in\Gamma' \\ l(\gamma)\leq t}} 1 - d(A)\right| + \epsilon(1+d(A)) \\
&= \epsilon(1+d(A)).
\end{aligned}$$

Since this is true for arbitrary $\epsilon > 0$, we have proved the theorem. $\square$

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