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Approximation properties for Lie groups and noncommutative L^p-spaces

PhD thesis by

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Abstract

The main topic of this thesis is approximation properties for Lie groups and noncommutative L^p-spaces. In the setting of Lie groups, we consider the Approximation Property of Haagerup and Kraus (AP). It is well-known that every connected simple Lie group with real rank at most 1 satisfies the stronger property weak amenability. In 2010, Lafforgue and de la Salle gave the first example of a Lie group without the AP, namely $\mathrm{SL}(3,\mathbb{R})$. In a joint work with Haagerup, we extend this result to connected simple Lie groups with real rank at least 2 and finite center. To this end, it is sufficient to prove that the group $\mathrm{Sp}(2,\mathbb{R})$ does not have the AP. By looking at the universal covering group $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ of $\mathrm{Sp}(2,\mathbb{R})$, we are, in a subsequent work with Haagerup, able to remove the condition of finite center. Hence, a connected simple Lie group has the AP if and only if its real rank is at most 1.

Let $L(G)$ denote the group von Neumann algebra of a group G. In the work of Lafforgue and de la Salle mentioned above, it was also proved that for a lattice Γ in $\mathrm{SL}(3,\mathbb{R})$ and $p \in [1, \frac{4}{3}) \cup (4, \infty]$, the noncommutative L^p-space $L^p(L(\Gamma))$ does not have the completely bounded approximation property (CBAP) or the operator space approximation property (OAP). We show that for a lattice Γ in a connected simple Lie group with real rank at least 2 and finite center, the noncommutative L^p-space $L^p(L(\Gamma))$ does not have the CBAP or OAP for $p \in [1, \frac{12}{11}) \cup (12, \infty]$. In the second work with Haagerup mentioned above, we are able to remove the condition of finite center.

Finally, we study the Grothendieck Theorem for jointly completely bounded (jcb) bilinear forms. This was proved in full generality for jcb bilinear forms on C^*-algebras by Haagerup and Musat. Their method of proof makes use of essentially von Neumann algebraic techniques, although the problem itself is purely C^*-algebraic. We give a modified proof of the theorem that only makes use of C^*-algebraic techniques. In addition, we prove that the best constant in Blecher's inequality is stricly larger than 1.

Resumé

Denne afhandling omhandler approksimationsegenskaber for Lie-grupper og ikke-kommutative L^p-rum. For Lie-grupper ser vi på approximationsegenskaben for grupper defineret af Haagerup og Kraus (AP). Det er velkendt, at enhver sammenhængende simpel Lie-gruppe med reel rang højst 1 opfylder den stærkere egenskab svag amenabilitet. I 2010 gav Lafforgue og de la Salle det første eksempel på en Lie-gruppe uden AP, nemlig $\mathrm{SL}(3,\mathbb{R})$. I et samarbejde med Haagerup udvides dette resultat til sammenhængende simple Lie-grupper med reel rang mindst 2 og endeligt centrum. Til dette formål er det tilstrækkeligt at bevise, at gruppen $\mathrm{Sp}(2,\mathbb{R})$ ikke har AP. Ved at se på den universelle overlejringsgruppe $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ for $\mathrm{Sp}(2,\mathbb{R})$ er vi, i et efterfølgende arbejde med Haagerup, i stand til at fjerne betingelsen om endeligt centrum. Derfor vil en sammenhængende simpel Lie-gruppe have AP, hvis og kun hvis dens reelle rang er højst 1.

Lad $L(G)$ være gruppe von Neumann algebraen af en gruppe G. I arbejdet af Lafforgue og de la Salle nævnt ovenfor blev det også bevist, at for et gitter Γ i $\mathrm{SL}(3,\mathbb{R})$ og $p \in [1, \frac{4}{3}) \cup (4, \infty]$ vil det ikke-kommutative L^p-rum $L^p(L(\Gamma))$ ikke have den såkaldte *completely bounded approximation property* (CBAP) eller *operator space approximation property* (OAP). Vi viser, at for et gitter Γ i en sammenhængende simpel Lie-gruppe med reel rang mindst 2 og endeligt centrum vil det ikke-kommutative L^p-rum $L^p(L(\Gamma))$ have hverken CBAP eller OAP for $p \in [1, \frac{12}{11}) \cup (12, \infty]$. I det andet arbejde med Haagerup nævnt ovenfor, er vi i stand til at fjerne betingelsen om endeligt centrum.

Endelig studerer vi Grothendieck Sætningen for simultant fuldstændigt begrænsede (sfb) bilineære former. Dette blev bevist i fuldt generalitet af Haagerup og Musat for sfb bilineære former på C^*-algebraer. Deres metode gør primært brug af von Neumann algebraiske teknikker, selv om problemet i sig selv er rent C^*-algebraisk. Vi giver et modificeret bevis for sætningen, der kun gør brug af C^*-algebraiske teknikker. Derudover viser vi, at den bedste konstant i Blechers ulighed er større end 1.

Preface

This thesis contains the results of my PhD research carried out at the Department of Mathematical Sciences of the University of Copenhagen from September 2010 until August 2013. The main topic of my research has been approximation properties for Lie groups and noncommutative L^p-spaces. In the setting of Lie groups, I have mainly considered the Approximation Property of Haagerup and Kraus (AP), and in the setting of noncommutative L^p-spaces, I have mainly been interested in the completely bounded approximation property (CBAP) and the operator space approximation property (OAP). In addition, I have studied the Grothendieck Theorem for jointly completely bounded bilinear forms. The main part of this thesis consists of four articles I have written or co-authored. These are included as appendices with permission of the corresponding journals (if applicable).

The AP has been less studied in the literature than other approximation properties for groups, such as amenability, weak amenability and the Haagerup property. This might be due to the fact that until 2010, there were no examples known of exact (countable discrete) groups without the AP. Weak amenability is stronger than the AP. Hence, it follows from the work of Cowling and Haagerup and from the work of Hansen that connected simple Lie groups with real rank at most 1 have the AP. In the article of Haagerup and Kraus, it was conjectured that the group $\mathrm{SL}(3,\mathbb{R})$, which is a Lie group of real rank 2, is a group without it. In 2010, Lafforgue and de la Salle proved this conjecture. Their approach inherently gave information about approximation properties for noncommutative L^p-spaces associated with lattices in $\mathrm{SL}(3,\mathbb{R})$. More precisely, they proved that for a lattice Γ in $\mathrm{SL}(3,\mathbb{R})$ and $p \in [1, \frac{4}{3}) \cup (4, \infty]$, the noncommutative L^p-space $L^p(L(\Gamma))$ does not have the CBAP or OAP, where $L(\Gamma)$ denotes the group von Neumann algebra of Γ.

I started my PhD research not long after the appearance of the article of Lafforgue and de la Salle, and the main goal of my research was to extend their result to a larger class of Lie groups. An important group to look at was the group $\mathrm{Sp}(2,\mathbb{R})$, since any connected simple Lie group of real at least 2 contains a closed subgroup locally isomorphic to $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$. In a joint work with Haagerup, we proved that this group does not have the AP, implying that if G is a connected simple Lie group with real rank at least 2 and finite center, then G does not have the AP. The article on this work was published in Duke Mathematical Journal [**HdL13a**].

As pointed out earlier, the method of Lafforgue and de la Salle (for $\mathrm{SL}(3,\mathbb{R})$) gives information on approximation properties for certain noncommutative L^p-spaces. Indeed, they introduced the property of completely bounded approximation by Schur multipliers on S^p, denoted $\mathrm{AP}_{p,\mathrm{cb}}^{\mathrm{Schur}}$, which is weaker than the AP for $p \in (1,\infty)$. Also, if a noncommutative L^p-space $L^p(L(\Gamma))$ associated with a countable discrete group Γ has the OAP, then Γ has the $\mathrm{AP}_{p,\mathrm{cb}}^{\mathrm{Schur}}$. The method used by Haagerup and myself is more direct than the method of Lafforgue and de la Salle, since we are able to work with completely bounded Fourier multipliers rather than completely bounded multipliers on Schatten classes. In this way, we can treat the AP directly, but we do not obtain results on approximation properties for noncommutative L^p-spaces. Moreover, we systematically make use of the theory of Gelfand pairs and spherical functions.

In a subsequent work, I proved that for a lattice Γ in any connected simple Lie group with real rank at least 2 and finite center, the noncommutative L^p-space $L^p(L(\Gamma))$ does not have the CBAP or OAP for $p \in [1, \frac{12}{11}) \cup (12, \infty]$. The key idea of the approach is similar to the approach of Lafforgue and de la Salle, but again, it proves beneficial to systematically use the theory of Gelfand pairs and spherical functions This work was published in the Journal of Functional Analysis [**dL13a**].

It was clear that in order to remove the finite center condition in the results mentioned above, it would be sufficient to consider one more group, namely, the universal covering group $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ of $\mathrm{Sp}(2,\mathbb{R})$. Recently, in a subsequent joint work, Haagerup and I were able to remove the condition of finite center in the results mentioned earlier. The article on this work was submitted recently [**HdL13b**].

At the beginning of my PhD, I also studied the Grothendieck Theorem for jointly completely bounded (jcb) bilinear forms, also known as the Effros-Ruan Conjecture. Pisier and Shlyakhtenko proved a version of this conjecture for exact operator spaces, as well as a version for C^*-algebras, assuming that at least one of them is exact. It was proved in full generality for jcb bilinear forms on C^*-algebras by Haagerup and Musat. Their method of proof makes use of essentially von Neumann algebraic techniques, although the problem itself is purely C^*-algebraic. I gave a modified proof of the theorem that only makes use of C^*-algebraic techniques. This will appear in Operator Algebra and Dynamics, Proceedings of the Nordforsk Network Closing Conference [**dL13b**]. Recently, Regev and Vidick gave a more elementary proof of both the JCB Grothendieck Theorem for C^*-algebras and its version for exact operator spaces.

The first chapter of this thesis gives an introduction to the problems adressed in the articles, and it states the main results. Moreover, the results are put into a broader context. At the end of the chapter, some related open problems are discussed.

Tim de Laat
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Contents

CHAPTER 1

Introduction

The main topic of this thesis is approximation properties for Lie groups and noncommutative L^p-spaces. In addition, we consider the Grothendieck Theorem for jointly completely bounded bilinear forms (JCB Grothendieck Theorem). In this chapter, we give an introduction to the topics of the author's articles in the appendices, including a brief historical account and some related results, and we state the main theorems. At the end of the chapter, some open problems are discussed.

1.1. On approximation properties for Lie groups

The main approximation properties of interest in this thesis are the Approximation Property of Haagerup and Kraus (AP) for groups, and the completely bounded approximation property (CBAP) and the operator space approximation property (OAP) for noncommutative L^p-spaces. Firstly, we focus on Lie groups, and we describe a more abstract and general framework in which the AP fits. In the next section, we consider the approximation properties for noncommutative L^p-spaces associated with lattices in Lie groups. It turns out that certain important information on these properties can be obtained by looking at the approximation properties of the underlying group, which is our approach.

Approximation properties can be formulated for several classes of objects. They give information about the analytical and topological structure of such objects. For the scope of this text, the most notable examples of such classes are topological groups, operator spaces, C*-algebras and von Neumann algebras. For a thorough account on approximation properties for groups and operator algebras, we refer to [**BO00**]. More specifically, in the methods we use, the most important class of objects for which we consider approximation properties is the class of second countable locally compact Hausdorff groups. Indeed, Lie groups are contained in this class. Also, the noncommutative L^p-spaces we are interested in are the ones associated with countable discrete groups, and as pointed out above, the approximation properties we are interested in for such spaces can, to a certain extent, be studied by means of the approximation properties of the underlying groups.

Heuristically speaking, an approximation property P for groups is a property that a group may have that describes whether or not certain functions on the group can be approximated by a class of other functions in a certain topology. Usually, the approximating functions are "easier to handle" than the functions to be approximated, e.g., they might have a certain decay property. The approximation properties for groups that are of importance in this thesis have analogues for C^*-algebras and von Neumann algebras and are characterized by two conditions, as is made precise below.

Assumption: We assume P to be an approximation property for the class of second countable locally compact Hausdorff groups with analogues C*P for C^*-algebras and W*P for von Neumann algebras such that the following two conditions are satisfied:

C1: if G is a second countable locally compact Hausdorff group and Γ is a lattice in G, then G has P if and only if Γ has P;

C2: if Γ is a countable discrete group, the following are equivalent:
   (1) the group Γ has P,
   (2) the reduced C^*-algebra $C^*_\lambda(\Gamma)$ of Γ has C*P,
   (3) the group von Neumann algebra $L(\Gamma)$ of Γ has W*P.

As mentioned earlier, the AP fits in this framework, but amenability and weak amenability do as well. Also, these three approximation properties pass to closed subgroups and to finite products. Moreover, they can be formulated in such a way that a group has the property if the constant function 1 on the group can be approximated by elements of the Fourier algebra in a certain topology.

We now briefly discuss the Fourier algebra of a group. Let G be a (second countable) locally compact (Hausdorff) group, and let $\lambda : G \longrightarrow \mathcal{B}(L^2(G))$ denote the left-regular representation, which is given by $(\lambda(x)\xi)(y) = \xi(x^{-1}y)$, where $x, y \in G$ and $\xi \in L^2(G)$. The Fourier algebra $A(G)$ of G is defined as the space consisting of the coefficients of λ, as introduced by Eymard [**Eym64**] (see also [**Eym95**]). More precisely, φ belongs to $A(G)$ if and only if there exist $\xi, \eta \in L^2(G)$ such that for all $x \in G$ we have

$$\varphi(x) = \langle \lambda(x)\xi, \eta \rangle.$$

The norm on $A(G)$ is defined by

$$\|\varphi\|_{A(G)} = \min\{\|\xi\|\|\eta\| \mid \varphi(x) = \langle \lambda(x)\xi, \eta\rangle \; \forall x \in G\}.$$

With this norm, $A(G)$ is a Banach space. We have $\|\varphi\|_\infty \leq \|\varphi\|_{A(G)}$ for all $\varphi \in A(G)$, and $A(G)$ is $\|.\|_\infty$-dense in $C_0(G)$. The Fourier algebra can be identified isometrically with the predual of the group von Neumann algebra $L(G)$ of G. The identification is given by the pairing $\langle T, \varphi\rangle = \langle Tf, g\rangle_{L^2(G)}$, where $T \in L(G)$ and $\varphi = \overline{g} * \check{f}$ for certain $f, g \in L^2(G)$. Here, $\overline{f}(x) = \overline{f(x)}$ and $\check{f}(x) = f(x^{-1})$ for all $x \in G$.

A complex-valued function φ is said to be a (Fourier) multiplier if and only if $\varphi\psi \in A(G)$ for all $\psi \in A(G)$. Let $MA(G)$ denote the Banach space of multipliers of $A(G)$ equipped with the norm given by $\|\varphi\|_{MA(G)} = \|m_\varphi\|$, where $m_\varphi : A(G) \longrightarrow A(G)$ denotes the associated multiplication operator. A multiplier φ is called completely bounded if the operator $M_\varphi : L(G) \longrightarrow L(G)$ induced by m_φ is completely bounded. The space of completely bounded multipliers is denoted by $M_0A(G)$, and with the norm $\|\varphi\|_{M_0A(G)} = \|M_\varphi\|_{\mathrm{cb}}$, it is a Banach space. It is known that $A(G) \subset M_0A(G) \subset MA(G)$.

Arguably, the theory of approximation properties for groups goes back to von Neumann, who introduced the notion of amenability in [**vN29**] in order to solve the famous

Banach–Tarski Paradox. Recall that a second countable locally compact Hausdorff group G is amenable if there exists a left-invariant mean on $L^\infty(G)$. It was proven by Leptin [**Lep68**] that G is amenable if and only if $A(G)$ has a bounded approximate unit, i.e., there exists a net (φ_α) in $A(G)$ with $\sup_\alpha \|\varphi_\alpha\|_{A(G)} \leq 1$ such that for all $\psi \in A(G)$ we have $\lim_\alpha \|\varphi_\alpha\psi - \psi\|_{A(G)} = 0$. Amenability satisfies the two conditions **C1** and **C2** above. Indeed, its C^*-analogue is nuclearity and its W^*-analogue is hyperfiniteness (see, e.g., [**BO00**, Theorem 2.6.8]).

For Lie groups, amenability has been studied thoroughly. It is precisely known which connected semisimple Lie groups are amenable, as follows from the following theorem.

THEOREM 1.1.1. (See [**Pie84**, Proposition 3.14.10].) *Let G be a connected semisimple Lie group. Then G is amenable if and only if G is compact.*

Let us point out that much is known about amenability for Lie groups that are not necessarily connected and semisimple (see, e.g., [**Pie84**, Section 3.14]), but in this thesis we focus on connected semisimple Lie groups.

A locally compact group G is called weakly amenable if there exists a net (φ_α) in $A(G)$ with $\sup_\alpha \|\varphi_\alpha\|_{M_0A(G)} \leq C$ for some $C > 0$ such that $\varphi_\alpha \to 1$ uniformly on compact subsets of G. The infimum of these constants C is called the Cowling-Haagerup constant of G and is denoted by $\Lambda(G)$. Weak amenability satisfies the two conditions **C1** and **C2** above. Indeed, its C^*-analogue is the completely bounded approximation property (CBAP) and its W^*-analogue is the weak* completely bounded approximation property (w*CBAP) (see Section 1.2 and [**BO00**]).

Amenability of a group G implies weak amenability with $\Lambda(G) = 1$. Weak amenability was first studied in [**dCH85**], in which de Cannière and Haagerup proved that the free group $\mathbb{F}_n$ on n generators with $n \geq 2$ is weakly amenable with $\Lambda(\mathbb{F}_n) = 1$. This also implied that weak amenability is strictly weaker than amenability, since $\mathbb{F}_n$ is not amenable.

Weak amenability has been studied much in the literature, and the constant $\Lambda(G)$ is known for every connected simple Lie group G. Recall that every connected simple Lie group G (with Lie algebra $\mathfrak{g}$) can be decomposed as a set product $G = KAK$, called a polar decomposition of G, where K arises from a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (the group K has Lie algebra $\mathfrak{k}$), and A is an abelian Lie group such that its Lie algebra $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. If G has finite center, then K is a maximal compact subgroup. The dimension of the Lie algebra $\mathfrak{a}$ of A is called the real rank of G. In general, given a polar decomposition $G = KAK$, it is not the case that for $g \in G$ there exist unique $k_1, k_2 \in K$ and $a \in A$ such that $g = k_1ak_2$. However, after choosing a set of positive roots and restricting to the closure $\overline{A^+}$ of the positive Weyl chamber A^+, we still have $G = K\overline{A^+}K$. Moreover, if $g = k_1ak_2$, where $k_1, k_2 \in K$ and $a \in \overline{A^+}$, then a is unique. Note that we can choose any Weyl chamber to be the positive one by choosing the set of positive roots correspondingly. For details, see [**Hel78**, Section IX.1].

Let G be a connected simple Lie group. The constant $\Lambda(G)$ of G depends on the real rank of G. It is known that if G has real rank 0, then G is compact. A proof of this fact can be given by using a theorem of Weyl (see, e.g., [**HN12**, Theorem 12.1.17]). If

G has real rank 1, then it is locally isomorphic to one of the groups $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$, with $n \geq 2$, or to $F_{4(-20)}$. It is known that

$$\Lambda(G) = \begin{cases} 1 & \text{if } G \text{ is locally isomorphic to } \mathrm{SO}(n,1) \text{ or } \mathrm{SU}(n,1), \\ 2n-1 & \text{if } G \text{ is locally isomorphic to } \mathrm{Sp}(n,1), \\ 21 & \text{if } G \text{ is locally isomorphic to } F_{4(-20)}. \end{cases}$$

This was proved by Cowling and Haagerup for groups with finite center [**CH89**]. The finite center condition was removed by Hansen [**Han90**].

Haagerup proved that all connected simple Lie groups with real rank greater than or equal to two and finite center are not weakly amenable by using the fact that any such group contains a closed subgroup locally isomorphic to $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$, neither of which is weakly amenable [**Haa86**]. Later, Dorofaeff proved that this result also holds for such Lie groups with infinite center [**Dor96**]. Recently, an analogue of this result was proved by Lafforgue for algebraic Lie groups over non-archimedean fields [**Laf10**]. In 2005, Cowling, Dorofaeff, Seeger and Wright gave a characterization of weak amenability for a very large class of connected Lie groups [**CDSW05**].

A weaker approximation approximation property was introduced by Haagerup and Kraus [**HK94**]. Let X denote the completion of $L^1(G)$ with respect to the norm given by

$$\|f\|_X = \sup\left\{ \left| \int_G f(x)\varphi(x)dx \right| \mid \varphi \in M_0A(G), \|\varphi\|_{M_0A(G)} \leq 1 \right\}.$$

Then $X^* = M_0A(G)$. The space X is considered as the natural predual $M_0A(G)_*$ of $M_0A(G)$, and it was first considered by de Cannière and Haagerup [**dCH85**].

Definition 1.1.2. A (second countable) locally compact (Hausdorff) group G is said to have the Approximation Property (AP) if there exists a net (φ_α) in $A(G)$ such that $\varphi_\alpha \to 1$ in the $\sigma(M_0A(G), M_0A(G)_*)$-topology, where $M_0A(G)_*$ denotes the natural predual of $M_0A(G)$.

It was proved by Haagerup and Kraus that if G is a locally compact group and Γ is a lattice in G, then G has the AP if and only if Γ has the AP, so the AP satisfies condition **C1** above. They also proved that it satisfies condition **C2**. Indeed, the C^*-analogue of the AP is given by the operator space approximation property (OAP), and the W^*-analogue of the AP is given by the weak* operator space approximation property (w*OAP) (see Section 1.2 and [**BO00**]). The AP has a nice stability property that weak amenability does not have, namely, if H is a closed normal subgroup of a locally compact group G such that both H and G/H have the AP, then G has the AP. This implies that the group $\mathrm{SL}(2,\mathbb{Z}) \rtimes \mathbb{Z}^2$ has the AP, but it was proved in [**Haa86**] that this group is not weakly amenable, so the AP is strictly weaker than weak amenability. It was proved by Haagerup and Kraus that countable discrete groups satisfying the AP are exact.

The AP has been less studied in the literature than amenability and weak amenability (and the Haagerup property). This might be due to the fact that until 2010, there were no examples known of exact (countable discrete) groups without the AP. Since weak amenability is a stronger property, it followed from the work of Cowling and Haagerup and from the work of Hansen that connected simple Lie groups with real rank smaller than or equal to one have the AP. In the article of Haagerup and Kraus, it was conjectured that the group $\mathrm{SL}(3,\mathbb{R})$, which has real rank two, is a group without it. In 2010, Lafforgue and de la Salle proved this conjecture.

THEOREM 1.1.3. (Lafforgue - de la Salle [**LdlS11**, Theorem C]) The group $\mathrm{SL}(3,\mathbb{R})$ does not have the AP.

In fact, Lafforgue and de la Salle proved that $\mathrm{SL}(3,\mathbb{R})$ fails to have certain weaker approximation properties than the AP, namely, they proved that $\mathrm{SL}(3,\mathbb{R})$ does not have the property of completely bounded approximation by Schur multipliers on S^p (denoted $\mathrm{AP}_{p,\mathrm{cb}}^{\mathrm{Schur}}$) for $p \in [1,\frac{4}{3}) \cup (4,\infty]$ (see Section 1.2). The $\mathrm{AP}_{p,\mathrm{cb}}^{\mathrm{Schur}}$ is also closely related to the OAP for noncommutative L^p-spaces associated with countable discrete groups. In this way, the method of Lafforgue and de la Salle inherently gave information on approximation properties of certain noncommutative L^p-spaces.

As pointed out earlier, the PhD research of the author was mainly aimed at generalizing the result on the AP by Lafforgue and de la Salle to other Lie groups of higher real rank. In the article in Appendix A, we consider the symplectic group $\mathrm{Sp}(2,\mathbb{R})$. Recall that $\mathrm{Sp}(2,\mathbb{R})$ is defined as the Lie group

$$\mathrm{Sp}(2,\mathbb{R}) := \{g \in \mathrm{GL}(4,\mathbb{R}) \mid g^T J g = J\},$$

where g^T denotes the transpose of g, the matrix I_2 denotes the 2×2 identity matrix and the matrix J is defined by

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Let K denote a maximal compact subgroup of $\mathrm{Sp}(2,\mathbb{R})$ given by

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathrm{M}_4(\mathbb{R}) \;\middle|\; A + iB \in \mathrm{U}(2) \right\}.$$

This group is isomorphic to $\mathrm{U}(2)$. A polar decomposition of $\mathrm{Sp}(2,\mathbb{R})$ is given by $\mathrm{Sp}(2,\mathbb{R}) = K\overline{A^+}K$, where

$$\overline{A^+} = \left\{ D(\beta,\gamma) = \begin{pmatrix} e^{\beta} & 0 & 0 & 0 \\ 0 & e^{\gamma} & 0 & 0 \\ 0 & 0 & e^{-\beta} & 0 \\ 0 & 0 & 0 & e^{-\gamma} \end{pmatrix} \;\middle|\; \beta \geq \gamma \geq 0 \right\}.$$

The first main result is the following theorem.

THEOREM 1.1.4. (see Appendix A, Theorem 3.1) The group $\mathrm{Sp}(2,\mathbb{R})$ does not have the Approximation Property.

Together, the groups $\mathrm{SL}(3,\mathbb{R})$ and $\mathrm{Sp}(2,\mathbb{R})$ form a powerful couple. Indeed, every connected simple Lie group with real rank greater than or equal to two has a closed subgroup that is locally isomorphic to either $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$. The failure of the AP for these two groups implies the following theorem.

THEOREM 1.1.5. (see Appendix A, Theorem 5.1) Let G be a connected simple Lie group with real rank greater than or equal to two and finite center. Then G does not have the Approximation Property.

As indicated earlier, there are important differences between the method of proof of Haagerup and the author and the method used by Lafforgue and de la Salle. For the first, it is sufficient to look at completely bounded multipliers, whereas for the second, one essentially needs the technically more involved completely bounded Schur multipliers on Schatten classes. Our approach is more direct, since we can treat the AP at once without considering the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$, but we do not get any information on approximation properties of noncommutative L^p-spaces. In our work, we made systematic use of the theory of Gelfand pairs and spherical functions. For an overview of this theory, see [**vD09**] or [**Far82**]. In a separate work, the author considers approximation properties for noncommutative L^p-spaces associated with lattices in simple higher rank Lie groups with finite center (see Section 1.2 and Appendix B).

In a subsequent work with Haagerup, we show that the finite center condition in Theorem 1.1.5 can be removed. Hereto, it is sufficient to consider the universal covering group $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ of $\mathrm{Sp}(2,\mathbb{R})$. A covering group of a connected Lie group G is a Lie group $\widetilde{G}$ with a surjective Lie group homomorphism $\sigma : \widetilde{G} \to G$, in such a way that $(\widetilde{G}, \sigma)$ is a covering space of G (in the topological sense). A simply connected covering space is called a universal covering space. Every connected Lie group G has a universal covering space $\widetilde{G}$. Let $\sigma : \widetilde{G} \to G$ be the corresponding covering map, and let $\tilde{1} \in \sigma^{-1}(1)$. Then there exists a unique multiplication on $\widetilde{G}$ that makes $\widetilde{G}$ into a Lie group in such a way that σ is a surjective Lie group homomorphism. The group $\widetilde{G}$ is called a universal covering group of the Lie group G. Universal covering groups of connected Lie groups are unique up to isomorphism. They also satisfy the exact sequence $1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$, where $\pi_1(G)$ denotes the fundamental group of G. For details on universal covering groups, see [**Kna96**, Section I.11].

THEOREM 1.1.6. (see Appendix C, Theorem 3.2) The universal covering group $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ of $\mathrm{Sp}(2,\mathbb{R})$ does not have the Approximation Property.

This finishes the description of the AP for connected simple Lie groups, as such groups with real rank zero and one are known to be weakly amenable. This is summarized by the following theorem.

THEOREM 1.1.7. (see Appendix C, Theorem 5.1) Let G be a connected simple Lie group. Then G has the Approximation Property if and only if its real rank is 0 or 1.

COROLLARY 1.1.8. Let $G = S_1 \times \ldots \times S_n$ be a connected semisimple Lie group, where the S_i's denote the simple factors. Then G has the AP if and only if the real rank of all S_i's is smaller than or equal to one.

Recall that a lattice in a Lie group G is a discrete subgroup Γ of G such that G/Γ has finite invariant measure.

COROLLARY 1.1.9. Let Γ be a lattice in a connected simple Lie group G. Then $C^*_\lambda(\Gamma)$ has the OAP, and, equivalently, $L(\Gamma)$ has the w*OAP, if and only if all simple factors of G have real rank smaller than or equal to 1.

1.2. On approximation properties for noncommutative L^p-spaces

Recall that an operator space E is a closed subspace of $\mathcal{B}(H)$ for some Hilbert space H. This gives rise to a norm on $M_n(E)$ for every $n \geq 1$ by the embeddings $M_n(E) \subset M_n(\mathcal{B}(H)) \cong \mathcal{B}(H^n)$. A linear map $T : E \longrightarrow F$ between operator spaces induces linear maps $T_n : M_n(E) \longrightarrow M_n(F)$ defined by $T_n([v_{ij}]) = [T(v_{ij})]$ for all $v = [v_{ij}] \in M_n(E)$. The map T is called completely bounded if the completely bounded norm $\|T\|_{cb} := \sup_{n\geq 1} \|T_n\|$ is finite. For an introduction to operator spaces, we refer to [**ER00**] and [**Pis03**].

Noncommutative L^p-spaces are important examples of operator spaces. Let M be a finite von Neumann algebra with normal faithful trace τ. For $1 \leq p < \infty$, the noncommutative L^p-space $L^p(M,\tau)$ is defined as the completion of M with respect to the norm $\|x\|_p = \tau((x^*x)^{\frac{p}{2}})^{\frac{1}{p}}$, and for $p = \infty$, we put $L^\infty(M,\tau) = M$ equipped with operator norm. In [**Kos84**], Kosaki showed that noncommutative L^p-spaces can be realized by interpolating between M and $L^1(M,\tau)$. This leads to an operator space structure on them, as described by Pisier [**Pis96**] (see also [**JR03**]). Indeed, the pair of spaces $(M, L^1(M,\tau))$ becomes a compatible couple of operator spaces, and for $1 < p < \infty$ we have the isometry $L^p(M,\tau) \cong [M, L^1(M,\tau)]_{\frac{1}{p}}$. By [**Pis98**, Lemma 1.7], we know that for a linear map $T : L^p(M,\tau) \longrightarrow L^p(M,\tau)$, its completely bounded norm $\|T\|_{cb}$ corresponds to $\sup_{n\in\mathbb{N}} \| \mathrm{id}_{S_n^p} \otimes T : S_n^p[L^p(M)] \to S_n^p[L^p(M)]\|$. Using [**Pis98**, Corollary 1.4] and the fact that $S_n^1 \otimes L^1(M) = L^1(M \otimes M_n)$, we obtain that $S_n^p[L^p(M)] = L^p(M \otimes M_n)$, which implies that $\|T\|_{cb} = \sup_{n\in\mathbb{N}} \|T \otimes \mathrm{id} : L^p(M \otimes M_n) \longrightarrow L^p(M \otimes M_n)\|$.

Definition 1.2.1. An operator space E is said to have the completely bounded approximation property (CBAP) if there exists a net F_α of finite-rank maps on E such that $\sup_\alpha \|F_\alpha\|_{cb} < C$ for some $C > 0$ and $\lim_\alpha \|F_\alpha x - x\| = 0$ for every $x \in E$.

The infimum of all possible constants C's is denoted by $\Lambda(E)$. If $\Lambda(E) = 1$, we say that E has the completely contractive approximation property (CCAP).

Let $\mathcal{K}(\ell^2)$ denote the space of compact operators on the Hilbert space ℓ^2.

Definition 1.2.2. An operator space E is said to have the operator space approximation property (OAP) if there exists a net F_α of finite-rank maps on E such that $\lim_\alpha \|(\mathrm{id}_{\mathcal{K}(\ell^2)} \otimes F_\alpha)x - x\| = 0$ for all $x \in \mathcal{K}(\ell^2) \otimes_{\min} E$.

The CBAP goes back to de Cannière and Haagerup [**dCH85**], and the OAP was defined by Effros and Ruan [**ER90**]. By definition, the CCAP implies the CBAP, which in turn implies the OAP.

It was pointed out that the CBAP and the OAP are the C^*-analogues of weak amenability and the AP, respectively. These properties are most generally formulated in the setting of operator spaces (note that C^*-algebras are operator spaces). In this way, they are natural approximation properties for noncommutative L^p-spaces.

In this thesis, the noncommutative L^p-spaces we are interested in are the ones of the form $L^p(L(\Gamma))$, where $L(\Gamma)$ is the group von Neumann algebra of a lattice Γ in a connected simple Lie group G. Such a von Neumann algebra $L(\Gamma)$ is finite and has canonical trace $\tau : x \mapsto \langle x\delta_1, \delta_1\rangle$, where $\delta_1 \in \ell^2(\Gamma)$ is the characteristic function of the unit element $1 \in \Gamma$.

It was proved by Junge and Ruan [**JR03**, Proposition 3.5] that if Γ is a weakly amenable (countable) discrete group, then for $p \in (1,\infty)$, the noncommutative L^p-space $L^p(L(\Gamma))$ has the CBAP. This implies that for every $p \in (1,\infty)$ and every lattice Γ in a connected simple Lie group G of real rank zero or one, the noncommutative L^p-space $L^p(L(\Gamma))$ has the CBAP (see Section 1.1). It was also shown by Junge and Ruan [**JR03**] that if Γ is a discrete group with the AP, and if $p \in (1,\infty)$, then $L^p(L(\Gamma))$ has the OAP.

The first examples of noncommutative L^p-spaces without the CBAP were given by Szankowski for $p > 80$ [**Sza84**]. More examples were given by Lafforgue and de la Salle [**LdlS11**]. They proved that for all $p \in [1,\frac{4}{3}) \cup (4,\infty]$ and all lattices Γ in $\mathrm{SL}(n,\mathbb{R})$, where $n \geq 3$, the space $L^p(L(\Gamma))$ does not have the CBAP or OAP. They also proved analogous results for lattices in Lie groups over non-archimedean fields. As mentioned before, in their work, the failure of the OAP for the aforementioned noncommutative L^p-spaces follows from the failure of a certain approximation property for the groups $\mathrm{SL}(3,\mathbb{R})$. This property, called the property of completely bounded approximation by Schur multipliers on S^p, denoted $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$, was introduced by Lafforgue and de la Salle exactly to this purpose.

For $p \in [1,\infty]$ and a (separable) Hilbert space $\mathcal{H}$, let $S^p(\mathcal{H})$ denote the p^{th} Schatten class on $\mathcal{H}$. Recall that for $p \in [1,\infty]$ and $\psi \in L^\infty(X \times X, \mu \otimes \mu)$, the Schur multiplier with symbol ψ is said to be bounded (resp. completely bounded) on $S^p(L^2(X,\mu))$ if it maps $S^p(L^2(X,\mu)) \cap S^2(L^2(X,\mu))$ into $S^p(L^2(X,\mu))$ (by $T_k \mapsto T_{\psi k}$), and if this map extends (necessarily uniquely) to a bounded (resp. completely bounded) map M_ψ on $S^p(L^2(X,\mu))$. The norm of such a bounded multiplier ψ is defined by $\|\psi\|_{MS^p(L^2(X,\mu))} = \|M_\psi\|$, and its completely bounded norm by $\|\psi\|_{cbMS^p(L^2(X,\mu))} = \|M_\psi\|_{cb}$. The spaces of multipliers and completely bounded multipliers are denoted by $MS^p(L^2(X,\mu))$ and $cbMS^p(L^2(X,\mu))$, respectively. It follows that for every $p \in [1,\infty]$ and $\psi \in L^\infty(X \times X, \mu \otimes \mu)$, we have $\|\psi\|_\infty \leq \|\psi\|_{MS^p(L^2(X,\mu))} \leq \|\psi\|_{cbMS^p(L^2(X,\mu))}$.

Definition 1.2.3. (see [**LdlS11**, Definition 2.2]) Let G be a (second countable) locally compact (Hausdorff) group, and let $1 \leq p \leq \infty$. The group G is said to have the property of completely bounded approximation by Schur multipliers on S^p, denoted $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$, if there exists a $C > 0$ and a net $\varphi_\alpha \in A(G)$ with $\sup_\alpha \|\check{\varphi}_\alpha\|_{cbMS^p(L^2(G))} \leq C$ and $\varphi_\alpha \to 1$ uniformly on compact subsets of G. The infimum of these C's is denoted by $\Lambda^{\mathrm{Schur}}_{p,\mathrm{cb}}(G)$.

It was proved by Lafforgue and de la Salle that if G is a locally compact group and Γ is a lattice in G, then for $1 \leq p \leq \infty$, we have $\Lambda^{\mathrm{Schur}}_{p,\mathrm{cb}}(\Gamma) = \Lambda^{\mathrm{Schur}}_{p,\mathrm{cb}}(G)$, i.e., the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ satisfies condition **C1** of Section 1.1. They also proved that for a discrete group Γ and $p \in (1,\infty)$, it follows that $\Lambda^{\mathrm{Schur}}_{p,\mathrm{cb}}(\Gamma) \in \{1,\infty\}$. Since a semisimple Lie group G has lattices (see [**BH62**]), we conclude that for such a group, it also follows that $\Lambda^{\mathrm{Schur}}_{p,\mathrm{cb}}(G) \in \{1,\infty\}$ for $p \in (1,\infty)$.

Lafforgue and de la Salle related the AP for groups and the OAP for noncommutative L^p-spaces to the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$.

Lemma 1.2.4. (see [**LdlS11**, Corollary 3.12]) If Γ is a countable discrete group with the AP and if $p \in (1,\infty)$, then $\Lambda^{\mathrm{Schur}}_{p,\mathrm{cb}}(\Gamma) = 1$.

LEMMA 1.2.5. (see [**LdlS11**, Corollary 3.13]) If $p \in (1,\infty)$ and Γ is a countable discrete group such that $L^p(L(\Gamma))$ has the OAP, then $\Lambda^{\mathrm{Schur}}_{p,\mathrm{cb}}(\Gamma) = 1$.

One of the main results of Lafforgue and de la Salle is the following.

THEOREM 1.2.6. (see [**LdlS11**, Theorem E]) Let $n \geq 3$. For $p \in [1, \frac{4}{3}) \cup (4,\infty]$, the (exact) group $\mathrm{SL}(n,\mathbb{Z})$ does not have the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$.

As a consequence, the group $\mathrm{SL}(n,\mathbb{R})$ does not have the AP, and for $p \in [1,\frac{4}{3}) \cup (4,\infty]$ and a lattice Γ in $\mathrm{SL}(n,\mathbb{R})$, the noncommutative L^p-space $L^p(L(\Gamma))$ does not have the CBAP or OAP.

Using the property of completely bounded approximation by Schur multipliers on S^p, the author obtains results on approximation properties for noncommutative L^p-spaces. These results only give sufficient conditions for a noncommutative L^p-space to not have the OAP. The following are the main results of Appendix B.

THEOREM 1.2.7. (see Appendix B, Theorem 3.1) For $p \in [1, \frac{12}{11}) \cup (12,\infty]$, the group $\mathrm{Sp}(2,\mathbb{R})$ does not have the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$.

THEOREM 1.2.8. (see Appendix B, Theorem 4.3) Let $p \in [1, \frac{12}{11}) \cup (12,\infty]$, and let Γ be a lattice in a connected simple Lie group with finite center and real rank greater than or equal to two. Then $L^p(L(\Gamma))$ does not have OAP or CBAP.

Similarly to the case of the AP, Haagerup and the author are able to remove the finite center condition, leading to the following general result.

THEOREM 1.2.9. (see Appendix C, Theorem 5.3) Let Γ be a lattice in a connected simple Lie group G. If G has real rank greater than or equal to two, then for $p \in [1, \frac{12}{11}) \cup (12,\infty]$, the noncommutative L^p-space $L^p(L(\Gamma))$ does not have the OAP.

1.3. On the JCB Grothendieck Theorem

In the "Résumé de la théorie métrique des produits tensorielles topologiques" [**Gro53**], Grothendieck proved the following result.

THEOREM 1.3.1 (Fundamental theorem on the metric theory of tensor products). Let K_1 and K_2 be compact spaces, and let $u : C(K_1) \times C(K_2) \longrightarrow \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be a bounded bilinear form. Then there exist probability measures μ_1 and μ_2 on K_1 and K_2 respectively, such that

$$|u(f_1,f_2)| \leq K_G^{\mathbb{K}} \|u\| \left(\int_{K_1} |f_1(t)|^2 d\mu_1(t) \right)^{\frac{1}{2}} \left(\int_{K_2} |f_2(t)|^2 d\mu_2(t) \right)^{\frac{1}{2}}$$

for all $f_1 \in C(K_1)$ and $f_2 \in C(K_2)$, where $K_G^{\mathbb{K}}$ is a universal constant only depending on the field $\mathbb{K}$.

The exact values of $K_G^{\mathbb{R}}$ and $K_G^{\mathbb{C}}$ are still unknown, but several bounds (from above and below) have been discovered by now. We refer to [**Pis12**, Section 4] for details.

Grothendieck also conjectured a noncommutative analogue of Theorem 1.3.1.

Theorem 1.3.2 (Noncommutative Grothendieck Inequality). Let A and B be C*-algebras, and let $u : A \times B \longrightarrow \mathbb{C}$ be a bounded bilinear form. Then there exist states f_1, f_2 on A and g_1, g_2 on B, such that for all $a \in A$ and $b \in B$,

$$|u(a,b)| \leq \|u\| \left(f_1(a^*a) + f_2(aa^*)\right)^{\frac{1}{2}} \left(g_1(b^*b) + g_2(bb^*)\right)^{\frac{1}{2}} .$$

This result was proved by Pisier assuming a certain approximability condition on the bilinear form [**Pis78**]. The general case was proved by Haagerup [**Haa85**].

Effros and Ruan conjectured that an analogue of the (noncommutative) Grothendieck Inequality holds for jointly completely bounded forms on C*-algebras [**ER91**]. For details on the notions of complete boundedness for bilinear forms, we refer to Appendix D.

Theorem 1.3.3 (JCB Grothendieck Theorem). Let A, B be C*-algebras, and let $u : A \times B \longrightarrow \mathbb{C}$ be a jointly completely bounded bilinear form. There exist states f_1, f_2 on A and g_1, g_2 on B, such that

$$|u(a,b)| \leq K\|u\|_{\mathrm{jcb}} \left(f_1(aa^*)^{\frac{1}{2}} g_1(b^*b)^{\frac{1}{2}} + f_2(a^*a)^{\frac{1}{2}} g_2(bb^*)^{\frac{1}{2}}\right),$$

for all $a \in A$ and $b \in B$, where K is a universal constant.

Effros and Ruan also conjectured that $K = 1$.

In [**PS02**], Pisier and Shlyakhtenko proved a version of the conjecture for exact operator spaces, in which the constant K depends on the exactness constants of the operator spaces. They also proved the conjecture for C^*-algebras, assuming that at least one of them is exact, with universal constant $K = 2^{\frac{3}{2}}$.

Haagerup and Musat proved the general conjecture for C^*-algebras, with universal constant $K = 1$ [**HM08**]. They used certain type III factors for the proof. Since the conjecture itself is purely C^*-algebraic, it would be more satisfying to have a proof that relies on C^*-algebras. In the article in Appendix D, we show that the proof of Haagerup and Musat can be modified in such a way that essentially only C^*-algebraic arguments are used. Indeed, in their proof, one tensors the C^*-algebras on which the bilinear form is defined with certain type III factors, whereas we show that it also works to use certain simple nuclear C^*-algebras admitting KMS states instead. We then transform the problem back to the (classical) Noncommutative Grothendieck Inequality, as was also done by Haagerup and Musat.

Recently, Regev and Vidick gave a more elementary proof of both the JCB Grothendieck Theorem for C^*-algebras and its version for exact operator spaces [**RV12**]. Their proof makes use of methods from quantum information theory and has the advantage that the transformation of the problem to the (classical) noncommutative Grothendieck Inequality is explicit and based on finite-dimensional techniques. Moreover, they obtain certain new quantitative estimates.

In [**Ble92**], Blecher stated a conjecture about the norm of elements in the algebraic tensor product of two C^*-algebras. Equivalently, the conjecture can be formulated as follows (see Conjecture 0.2′ of [**PS02**]). For a bilinear form $u : A \times B \to \mathbb{C}$, put $u^t(b,a) = u(a,b)$.

THEOREM 1.3.4 (Blecher's inequality). There is a constant K such that any jointly completely bounded bilinear form $u : A \times B \to \mathbb{C}$ on C^*-algebras A and B decomposes as a sum $u = u_1 + u_2$ of completely bounded bilinear forms on $A \times B$, and $\|u_1\|_{cb} + \|u_2^t\|_{cb} \leq K\|u\|_{jcb}$.

A version of this theorem for exact operator spaces and a version for pairs of C^*-algebras, one of which is assumed to be exact, were proved by Pisier and Shlyakhtenko [**PS02**]. They also showed that the best constant in the inequality is greater than or equal to 1. Haagerup and Musat proved that the inequality holds in general with $K = 2$ [**HM08**, Section 3]. In the article in Appendix D, we prove that the best constant is actually strictly greater than 1.

1.4. Further research

The research described above has answered certain questions and raised new ones. Most importantly, the results above give an answer to the question which connected semisimple Lie groups have the AP. The connectedness assumption is essential, as is usually the case for approximation properties. However, the condition of semisimplicity can be weakened. It is straightforward to formulate necessary and sufficient conditions for the AP to hold for certain larger or different classes of connected Lie groups, e.g., connected reductive Lie groups or connected Lie groups with a sufficiently nice global Levi decomposition. Such similar results and slight generalizations are not included in this thesis. Instead, in a third joint work with Haagerup, we intend to cover the Approximation Property for non-simple Lie groups as generally as possible.

Other important open problems lie in the direction of approximation properties for noncommutative L^p-spaces. The approach in this thesis gives sufficient conditions on p, namely, $p \in [1, \frac{12}{11}) \cup (12, \infty]$, for the failure of the OAP for $L^p(L(\Gamma))$, where Γ is a lattice in a connected simple Lie group with real rank greater than or equal to two. A logical question to ask is whether or not these values of p are optimal. A priori, there is no reason to think so, since the "cut at $p = 12$" just follows from estimates of certain functions that turn up in the problem. Also, we know that when the Lie group under consideration contains $\mathrm{SL}(3, \mathbb{R})$ as a closed subgroup, the set of p-values is larger, namely, "at least" $p \in [1, \frac{4}{3}) \cup (4, \infty]$. It might very well be the case that the only value of p for which $L^p(L(\Gamma))$ has the CBAP and OAP is $p = 2$. A hint in this direction is that whenever F is a non-archimedean local field, the results of Lafforgue and de la Salle (see [**LdlS11**, Theorem A]) imply that for every $p \neq 2$, there exists a noncommutative L^p-space associated with a lattice in $\mathrm{SL}(n, F)$ (for some $n \geq 3$) without the OAP. The idea is that the known "set of p-values for which the OAP fails" becomes larger for increasing n. It is not known whether a similar result holds in the archimedean setting. We point out that there are important differences between the proofs for the archimedean and the non-archimedean cases.

A sublety that arises when considering the exact set of p-values for which the OAP holds or does not hold for $L^p(L(\Gamma))$ is that the approach followed in our work makes use of the earlier mentioned $\mathrm{AP}_{p,\mathrm{cb}}^{\mathrm{Schur}}$, which is an approximation property of Γ. More precisely, the fact we use is that if $L^p(L(\Gamma))$ has the OAP for $p \in (1, \infty)$, then Γ has the

$\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ for that value of p. However, it is not clear whether the converse is true, so in order to prove that the noncommutative L^p-spaces do have the OAP for certain p, it is not known whether it is sufficient to consider the analogous question for the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ of the underlying group. This question was mentioned explicitly, but left unanswered, in the recent work of Caspers and de la Salle [**CdlS13**].

An important motivation for continuing the study of the problems mentioned above is given by the following theoretical fact. If it is true that the exact p-values for which noncommutative L^p-space associated with lattices in $\mathrm{PSL}(n,\mathbb{R})$ have the OAP are actually different for different values of $n \geq 3$, then this would imply that $L(\mathrm{PSL}(m,\mathbb{Z}))$ is not isomorphic to $L(\mathrm{PSL}(n,\mathbb{Z}))$ for $m,n \geq 3$ and $m \neq n$, which is a very famous open problem in von Neumann algebras, posed by Connes.

Another direction that can be investigated is whether the methods of this thesis can be used to prove that every connected simple Lie group with real rank greater than or equal to two has strong property (T) of Lafforgue (see [**Laf08**]). In [**Laf08**], Lafforgue proved that $\mathrm{SL}(3,F)$ has this property for every local field F. By the work of Liao [**Lia12**], it is known that $\mathrm{Sp}(2,F)$ has strong property (T) for local non-archimedean fields F. In an ongoing work with de la Salle, we consider a generalization of strong property (T) to connected simple Lie group with real rank greater than or equal to two.

There are also several open questions related to the jointly completely bounded Grothendieck inequality. As was already indicated in Section 1.3, the constants $K_G^{\mathbb{R}}$ and $K_G^{\mathbb{C}}$ are unknown. Several bounds from above and below have been established (see [**Pis12**, Section 4] for an overview), but it turns out to be highly nontrivial to determine the constants exactly.

Other applications of Grothendieck Theorems were found by studying the deep connection between the Grothendieck Theorem and Bell's inequality in quantum theory, as established by Tsirelson in 1980 [**Tsi80**]. Recently, such applications have been studied by researchers in the field of quantum information theory. We will not go into details here, but let us mention that Regev and Vidick, who, as mentioned earlier, gave the most elementary proof of the Grothendieck Theorem for jointly completely bounded bilinear forms so far, established a connection between quantum XOR games and Grothendieck Theorems in [**RV13**].

APPENDIX A

Simple Lie groups without the Approximation Property

This chapter contains the published version of the following article:

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SIMPLE LIE GROUPS WITHOUT THE APPROXIMATION PROPERTY

UFFE HAAGERUP and TIM DE LAAT

Abstract
For a locally compact group G, let $A(G)$ denote its Fourier algebra, and let $M_0A(G)$ denote the space of completely bounded Fourier multipliers on G. The group G is said to have the Approximation Property (AP) if the constant function 1 *can be approximated by a net in $A(G)$ in the weak-* topology on the space $M_0A(G)$. Recently, Lafforgue and de la Salle proved that* $\mathrm{SL}(3,\mathbb{R})$ *does not have the AP, implying the first example of an exact discrete group without it, namely,* $\mathrm{SL}(3,\mathbb{Z})$*. In this paper we prove that* $\mathrm{Sp}(2,\mathbb{R})$ *does not have the AP. It follows that all connected simple Lie groups with finite center and real rank greater than or equal to two do not have the AP. This naturally gives rise to many examples of exact discrete groups without the AP.*

1. Introduction

Let G be a (second countable) locally compact group, and let $\lambda : G \longrightarrow \mathcal{B}(L^2(G))$ denote the left-regular representation, which is given by $(\lambda(x)\xi)(y) = \xi(x^{-1}y)$, where $x, y \in G$ and $\xi \in L^2(G)$. Let the Fourier algebra $A(G)$ be the space consisting of the coefficients of λ, as introduced by Eymard [12], [13]. More precisely, $\varphi \in A(G)$ if and only if there exist $\xi, \eta \in L^2(G)$ such that for all $x \in G$ we have

$$\varphi(x) = \langle \lambda(x)\xi, \eta \rangle.$$

The norm on $A(G)$ is defined by

$$\|\varphi\|_{A(G)} = \min\big\{\|\xi\|\|\eta\| \;\big|\; \forall x \in G \;\varphi(x) = \langle \lambda(x)\xi, \eta\rangle\big\}.$$

With this norm, $A(G)$ is a Banach space. We have $\|\varphi\|_{A(G)} \geq \|\varphi\|_\infty$ for all $\varphi \in A(G)$, and $A(G)$ is $\|\cdot\|_\infty$-dense in $C_0(G)$.

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In Eymard's work, the following characterization of $A(G)$ is given. For two functions $f, g \in L^2(G)$, the function $\varphi = f * \tilde{g}$, where $\tilde{g}(x) = \overline{\check{g}(x)} = \overline{g(x^{-1})}$ for $x \in G$, belongs to $A(G)$. Conversely, if $\varphi \in A(G)$, then we can find such a decomposition $\varphi = f * \tilde{g}$ so that $\|f\|_2 \|g\|_2 = \|\varphi\|_{A(G)}$.

Another characterization of the Fourier algebra is given by the fact that $A(G)$ can be identified isometrically with the predual of the group von Neumann algebra $L(G)$ of G. The identification is given by the pairing $\langle T, \varphi \rangle = \langle Tf, g \rangle_{L^2(G)}$, where $T \in L(G)$ and $\varphi = \overline{g} * \check{f}$ for certain $f, g \in L^2(G)$.

A complex-valued function φ is said to be a (Fourier) multiplier if and only if $\varphi\psi \in A(G)$ for all $\psi \in A(G)$. Note that a multiplier is a bounded and continuous function. Let $MA(G)$ denote the Banach space of multipliers of $A(G)$ equipped with the norm given by $\|\varphi\|_{MA(G)} = \|m_\varphi\|$, where $m_\varphi : A(G) \longrightarrow A(G)$ denotes the multiplication operator on $A(G)$ associated with φ. A multiplier φ is called completely bounded if the operator $M_\varphi : L(G) \longrightarrow L(G)$ induced by m_φ is completely bounded. The space of completely bounded multipliers is denoted by $M_0A(G)$, and with the norm $\|\varphi\|_{M_0A(G)} = \|M_\varphi\|_{\mathrm{cb}}$, it forms a Banach space. It is known that $A(G) \subset M_0A(G) \subset MA(G)$.

Completely bounded Fourier multipliers were first studied by Herz [22], although he defined them in a different way. Hence, they are also called Herz–Schur multipliers. The equivalence of both notions was proved by Bożejko and Fendler [2]. They also gave an important characterization of completely bounded Fourier multipliers; namely, $\varphi \in M_0A(G)$ if and only if there exist bounded continuous maps $P, Q : G \longrightarrow \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, such that

$$\varphi(y^{-1}x) = \langle P(x), Q(y) \rangle \tag{1}$$

for all $x, y \in G$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$. In this characterization, $\|\varphi\|_{M_0A(G)} = \min\{\|P\|_\infty \|Q\|_\infty\}$, where the minimum is taken over all possible pairs (P, Q) for which (1) holds.

Completely bounded Fourier multipliers naturally give rise to the formulation of a certain approximation property, namely weak amenability, which was studied extensively for Lie groups in [5], [7], [8], [10], [17], and [20]. Other approximation properties can be formulated in terms of multipliers as well (see [3, Chapter 12]).

Recall that a locally compact group G is amenable if there exists a left-invariant mean on $L^\infty(G)$. It was proven by Leptin [29] that G is amenable if and only if $A(G)$ has a bounded approximate unit; that is, there is a net (φ_α) in $A(G)$ with $\sup_\alpha \|\varphi_\alpha\|_{A(G)} \leq 1$ such that for all $\psi \in A(G)$ we have $\lim_\alpha \|\varphi_\alpha \psi - \psi\|_{A(G)} = 0$.

A locally compact group G is called weakly amenable if and only if there is a net (φ_α) in $A(G)$ with $\sup_\alpha \|\varphi_\alpha\|_{M_0A(G)} \leq C$ for some $C > 0$, such that $\varphi_\alpha \to 1$

uniformly on compact subsets of G. The infimum of these constants C is denoted by $\Lambda(G)$, and we put $\Lambda(G) = \infty$ if G is not weakly amenable.

Amenability of a group G implies weak amenability with $\Lambda(G) = 1$. Weak amenability was first studied in [5], in which de Cannière and the first author proved that the free group $\mathbb{F}_n$ on n generators with $n \geq 2$ is weakly amenable with $\Lambda(\mathbb{F}_n) = 1$. This also implied that weak amenability is strictly weaker than amenability, since $\mathbb{F}_n$ is not amenable.

The constant $\Lambda(G)$ is known for every connected simple Lie group G and depends on the real rank of G. First, note that if G has a real rank zero G is amenable. A connected simple Lie group G with real rank one is locally isomorphic to one of the groups $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ with $n \geq 2$, or to $F_{4(-20)}$. It is known that

$$\Lambda(G) = \begin{cases} 1 & \text{if } G \text{ is locally isomorphic to } \mathrm{SO}(n,1) \text{ or } \mathrm{SU}(n,1), \\ 2n-1 & \text{if } G \text{ is locally isomorphic to } \mathrm{Sp}(n,1), \\ 21 & \text{if } G \text{ is locally isomorphic to } F_{4(-20)}. \end{cases}$$

This was proved by Cowling and the first author for groups with finite center [8]. The finite center condition was removed by Hansen [20].

The first author proved that all connected simple Lie groups with finite center and real rank greater than or equal to two are not weakly amenable by using the fact that any such group contains a subgroup locally isomorphic to $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$, neither of which is weakly amenable [17]. Later, Dorofaeff proved that this result also holds for such Lie groups with infinite center [10]. Recently, an analogue of this result was proved by Lafforgue for algebraic Lie groups over non-Archimedean fields [27]. Cowling, Dorofaeff, Seeger, and Wright gave a characterization of weak amenability for almost all connected Lie groups [7].

A weaker approximation property defined in terms of completely bounded Fourier multipliers was introduced by the first author and Kraus [18].

Definition 1.1
A locally compact group G is said to have the Approximation Property for groups (AP) if there is a net (φ_α) in $A(G)$ such that $\varphi_\alpha \to 1$ in the $\sigma(M_0A(G), M_0A(G)_*)$-topology, where $M_0A(G)_*$ denotes the natural predual of $M_0A(G)$, as introduced in [5].

It was proved by the first author and Kraus that if G is a locally compact group and Γ is a lattice in G, then G has the AP if and only if Γ has the AP. The AP has some nice stability properties that weak amenability does not have; for example, if H is a closed normal subgroup of a locally compact group G such that both H and G/H have the AP, then G has the AP. This implies that the group $\mathrm{SL}(2,\mathbb{Z}) \rtimes \mathbb{Z}^2$ has the AP, but it was proven in [17] that this group is not weakly amenable, so the AP is strictly weaker than weak amenability.

A natural question to ask is which groups do have the AP. When this property was introduced, it was not clear that there even exist groups without it, but it was conjectured by the first author and Kraus that $\mathrm{SL}(3,\mathbb{Z})$ would be such a group. This conjecture was recently proved by Lafforgue and de la Salle [28].

Recall that a countable discrete group Γ is exact if and only if its reduced group C*-algebra is exact. For discrete groups it is known that the AP implies exactness (see [3, Section 12.4]). Note that the result of Lafforgue and de la Salle also gives the first example of an exact group without the AP. In their paper the property of completely bounded approximation by Schur multipliers on $S^p(L^2(G))$, denoted by $\mathrm{AP}_{\mathrm{pcb}}^{\mathrm{Schur}}$, was introduced. For discrete groups, this property is weaker than the AP for all $p \in (1,\infty)$. Lafforgue and de la Salle proved that $\mathrm{SL}(3,\mathbb{R})$ does not satisfy the $\mathrm{AP}_{\mathrm{pcb}}^{\mathrm{Schur}}$ for certain values of p in this interval, implying that the exact group $\mathrm{SL}(3,\mathbb{Z})$ indeed fails to have the AP, since both the AP and the $\mathrm{AP}_{\mathrm{pcb}}^{\mathrm{Schur}}$ pass from the group to its lattices and from its lattices to the group.

The main part of this paper concerns the proof of the following result.

THEOREM
The group $\mathrm{Sp}(2,\mathbb{R})$ *does not have the AP.*

Together with the fact that $\mathrm{SL}(3,\mathbb{R})$ does not have the AP, the above result gives rise to the following theorem.

THEOREM
Let G be a connected simple Lie group with finite center and real rank greater than or equal to two. Then G does not have the AP.

In [11], Effros and Ruan introduced the operator approximation property (OAP) for C^*-algebras and the weak-* operator approximation property (w*OAP) for von Neumann algebras. By the results of [18, Section 2], it follows that for every lattice Γ in a connected simple Lie group with finite center and real rank greater than or equal to two, the reduced group C^*-algebra $C_\lambda^*(\Gamma)$ does not have the OAP and the group von Neumann algebra $L(\Gamma)$ does not have the w*OAP.

A natural question is whether all connected simple Lie groups with real rank greater than or equal to two fail to have the AP, that is, if the last mentioned theorem also holds for groups with infinite center. As of now, we do not know the answer to this question (see the comments in Section 4).

This paper is organized as follows. In Section 2 we recall and prove some results about Lie groups, Gelfand pairs, and the AP. Some of these may be of independent interest.

In Section 3 we give a proof of the fact that $\mathrm{Sp}(2,\mathbb{R})$ does not have the AP. It turns out to be sufficient to consider completely bounded Fourier multipliers on $\mathrm{Sp}(2,\mathbb{R})$, rather than multipliers on Schatten classes, so we do not use the $\mathrm{AP}^{\mathrm{Schur}}_{\mathrm{pcb}}$.

In Section 4 we prove the earlier mentioned theorem that all connected simple Lie groups with finite center and real rank greater than or equal to two do not have the AP.

In Section 5 we give a new proof of the result of Lafforgue and de la Salle that $\mathrm{SL}(3,\mathbb{R})$ does not have the AP based on the method of Section 3.

2. Lie groups and the Approximation Property

In this section we recall some results about Lie groups, Gelfand pairs, and the AP, and we prove some technical results.

2.1. Polar decomposition

For the details and proofs of the unproved results in this section, we refer the reader to [21] and [23].

Recall that every connected semisimple Lie group G with finite center can be decomposed as $G = KAK$, where K is a maximal compact subgroup (unique up to conjugation) and A is an abelian Lie group such that its Lie algebra $\mathfrak{a}$ is a Cartan subspace of the Lie algebra $\mathfrak{g}$ of G. The dimension of $\mathfrak{a}$ is called the real rank of G and is denoted by $\mathrm{Rank}_{\mathbb{R}}(G)$. The real rank of a Lie group is an important concept for us, since the main result is formulated for Lie groups with certain real ranks. The KAK decomposition, also called the polar decomposition, is in general not unique. After choosing a set of positive roots and restricting to the closure $\overline{A^+}$ of the positive Weyl chamber A^+, we still have $G = K\overline{A^+}K$. Moreover, if $g = k_1 a k_2$, where $k_1, k_2 \in K$ and $a \in \overline{A^+}$, then a is unique. Note that we can choose any Weyl chamber to be the positive one by choosing the correct polarization. For the purposes of this paper, the existence and the explicit form of the polar decomposition for two certain groups is important.

Example 2.1 (The symplectic groups)

Let the symplectic group be defined as the Lie group

$$\mathrm{Sp}(n,\mathbb{R}) := \left\{ g \in \mathrm{GL}(2n,\mathbb{R}) \;\middle|\; g^t J g = J \right\},$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Here I_n denotes the $(n \times n)$-identity matrix. We will only consider the case $n = 2$ from now on.

The maximal compact subgroup K of $\mathrm{Sp}(2, \mathbb{R})$ is given by

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathrm{M}_4(\mathbb{R}) \;\middle|\; A + iB \in \mathrm{U}(2) \right\}.$$

This group is isomorphic to $\mathrm{U}(2)$. The embedding of an arbitrary element of $\mathrm{U}(2)$ into $\mathrm{Sp}(2, \mathbb{R})$ under this isomorphism is given by

$$\begin{pmatrix} a+ib & e+if \\ c+id & g+ih \end{pmatrix} \mapsto \begin{pmatrix} a & e & -b & -f \\ c & g & -d & -h \\ b & f & a & e \\ d & h & c & g \end{pmatrix},$$

where $a, b, c, d, e, f, g, h \in \mathbb{R}$.

A polar decomposition of $\mathrm{Sp}(2, \mathbb{R})$ is given by $\mathrm{Sp}(2, \mathbb{R}) = K\overline{A^+}K$, where

$$\overline{A^+} = \left\{ D(\alpha_1, \alpha_2) = \begin{pmatrix} e^{\alpha_1} & 0 & 0 & 0 \\ 0 & e^{\alpha_2} & 0 & 0 \\ 0 & 0 & e^{-\alpha_1} & 0 \\ 0 & 0 & 0 & e^{-\alpha_2} \end{pmatrix} \;\middle|\; \alpha_1 \geq \alpha_2 \geq 0 \right\}.$$

Example 2.2 (The special linear group)
Consider the special linear group $\mathrm{SL}(3, \mathbb{R})$. Its maximal compact subgroup is $K = \mathrm{SO}(3)$, sitting naturally inside $\mathrm{SL}(3, \mathbb{R})$. A polar decomposition is given by $\mathrm{SL}(3, \mathbb{R}) = K\overline{A^+}K$, where

$$\overline{A^+} = \left\{ \begin{pmatrix} e^{\alpha_1} & 0 & 0 \\ 0 & e^{\alpha_2} & 0 \\ 0 & 0 & e^{\alpha_3} \end{pmatrix} \;\middle|\; \alpha_1 \geq \alpha_2 \geq \alpha_3,\ \alpha_1 + \alpha_2 + \alpha_3 = 0 \right\}.$$

2.2. Gelfand pairs and spherical functions
Let G be a locally compact group, and let K be a compact subgroup. We denote the (left) Haar measure on G by dx and the normalized Haar measure on K by dk. A function $\varphi : G \longrightarrow \mathbb{C}$ is said to be K-biinvariant if for all $g \in G$ and $k_1, k_2 \in K$, then we have $\varphi(k_1 g k_2) = \varphi(g)$. We identify the space of continuous K-biinvariant functions with the space $C(K \backslash G / K)$. If the subalgebra $C_c(K \backslash G / K)$ of the convolution algebra $C_c(G)$ is commutative, then the pair (G, K) is said to be a Gelfand pair, and K is said to be a Gelfand subgroup of G. Equivalently, the pair (G, K) is a Gelfand pair if and only if for every irreducible representation π on a Hilbert space $\mathcal{H}$ the space

$$\mathcal{H}_e = \{\xi \in \mathcal{H} \mid \forall k \in K : \pi(k)\xi = \xi\}$$

is at most one-dimensional.

For $\varphi \in C(G)$, define $\varphi^K \in C(K\backslash G/K)$ by

$$\varphi^K(g) = \int_{K\times K} \varphi(kgk')\,dk\,dk'.$$

A continuous K-biinvariant function $h : G \longrightarrow \mathbb{C}$ is called a spherical function if the functional χ on $C_c(K\backslash G/K)$ given by

$$\chi(\varphi) = \int_G \varphi(x)h(x^{-1})\,dx, \quad \varphi \in C_c(K\backslash G/K)$$

defines a nontrivial character, that is, $\chi(\varphi * \psi) = \chi(\varphi)\chi(\psi)$ for all $\varphi, \psi \in C_c(K\backslash G/K)$. The following characterization of spherical functions is used later: a continuous K-biinvariant function $h : G \longrightarrow \mathbb{C}$ not identical to zero is a spherical function if and only if for all $x, y \in G$

$$\int_K h(xky)\,dk = h(x)h(y).$$

In particular, $h(e) = 1$.

Spherical functions arise as the matrix coefficients of K-invariant vectors in irreducible representations of G. Hence, they give rise to interesting decompositions of functions on G.

For an overview of the theory of Gelfand pairs and spherical functions, we refer the reader to [14] and [9].

2.3. *Multipliers on compact Gelfand pairs*

For the study of completely bounded Fourier multipliers on a Gelfand pair it is natural to look at multipliers that are biinvariant with respect to the Gelfand subgroup. In the case of a compact Gelfand pair (G, K), that is, G is a compact group and K a closed subgroup such that (G, K) is a Gelfand pair, we get a useful decomposition of completely bounded Fourier multipliers in terms of spherical functions.

Suppose in this section that (G, K) is a compact Gelfand pair. Recall that for compact groups every representation on a Hilbert space is equivalent to a unitary representation, that every irreducible representation is finite-dimensional, and that every unitary representation is the direct sum of irreducible ones. Denote by dx and dk the normalized Haar measures on G and K, respectively. Recall as well that for a Gelfand pair every irreducible representation π on $\mathcal{H}$ the space $\mathcal{H}_e$ as defined in Section 2.2 is at most one-dimensional. Let $P_\pi = \int_K \pi(k)\,dk$ denote the projection onto $\mathcal{H}_e$, and set $\hat{G}_K = \{\pi \in \hat{G} \mid P_\pi \neq 0\}$, where $\hat{G}$ denotes the unitary dual of G, that is, the set of equivalence classes of unitary irreducible representations of G.

PROPOSITION 2.3
Let (G, K) be a compact Gelfand pair, and let φ be a K-biinvariant completely bounded Fourier multiplier. Then φ has a unique decomposition

$$\varphi(x) = \sum_{\pi \in \hat{G}_K} c_\pi h_\pi(x), \quad x \in G,$$

where $h_\pi(x) = \langle \pi(x)\xi_\pi, \xi_\pi \rangle$ is the positive definite spherical function associated with the representation π with K-invariant cyclic vector ξ_π, and $\sum_{\pi \in \hat{G}_K} |c_\pi| = \|\varphi\|_{M_0A(G)}$.

Proof
Note that for a compact group G, we have $A(G) = M_0A(G) = MA(G)$. By definition of $A(G)$, there exist $\xi, \eta \in L^2(G)$ such that for all $x \in G$,

$$\varphi(x) = \langle \lambda(x)\xi, \eta \rangle,$$

and $\|\varphi\|_{A(G)} = \|\xi\|\|\eta\|$. Note that since G is compact, we have

$$L(G) \cong \bigoplus_{\pi \in \hat{G}} B(\mathcal{H}_\pi)$$

as an l^∞ direct sum, and

$$A(G) \cong \bigoplus_{\pi \in \hat{G}} S_1(\mathcal{H}_\pi)$$

as an l^1 direct sum, where $S_1(\mathcal{H}_\pi)$ denotes the space of trace class operators on $\mathcal{H}_\pi$. Hence, we can write

$$\varphi(x) = \sum_{\pi \in \hat{G}} \mathrm{Tr}\big(S_\pi \pi(x)\big), \quad x \in G,$$

where S_π is a trace class operator acting on $\mathcal{H}_\pi$, and it follows that

$$\|\varphi\|_{A(G)} = \sum_{\pi \in \hat{G}} \|S_\pi\|_1,$$

where $\| \cdot \|_1$ denotes the trace class norm.

Since φ is K-biinvariant, S_π can be replaced by $P_\pi S_\pi P_\pi$, which vanishes whenever $\pi \notin \hat{G}_K$, and which equals $c_\pi P_\pi$ for some constant c_π whenever $\pi \in \hat{G}_K$. We have $|c_\pi| = \|c_\pi P_\pi\|_1$, since the dimension of P_π is one. Hence,

$$\varphi(x) = \sum_{\pi \in \hat{G}_K} c_\pi \, \mathrm{Tr}\big(P_\pi \pi(x)\big),$$

and therefore,

$$\|\varphi\|_{A(G)} = \sum_{\pi \in \hat{G}_K} \|P_\pi S_\pi P_\pi\|_1 = \sum_{\pi \in \hat{G}_K} |c_\pi|.$$

For each $\pi \in \hat{G}_K$, choose a unit vector $\xi_\pi \in P_\pi \mathcal{H}_\pi$. Then

$$\varphi(x) = \sum_{\pi \in \hat{G}_K} c_\pi h_\pi(x),$$

where $h_\pi(x) = \langle \pi(x)\xi_\pi, \xi_\pi \rangle$ is the positive definite spherical function associated with $(\pi, \mathcal{H}_\pi, \xi_\pi)$. $\square$

2.4. The Approximation Property

Recall from Section 1 that a locally compact group G has the AP if there is a net (φ_α) in $A(G)$ such that $\varphi_\alpha \to 1$ in the $\sigma(M_0A(G), M_0A(G)_*)$-topology, where $M_0A(G)_*$ denotes the natural predual of $M_0A(G)$.

The natural predual can be described as follows (see [5]). Let X denote the completion of $L^1(G)$ with respect to the norm given by

$$\|f\|_X = \sup\left\{\left|\int_G f(x)\varphi(x)\,dx\right| \;\middle|\; \varphi \in M_0A(G), \|\varphi\|_{M_0A(G)} \leq 1\right\}.$$

Then $X^* = M_0A(G)$. On bounded sets, the $\sigma(M_0A(G), M_0A(G)_*)$-topology coincides with the $\sigma(L^\infty(G), L^1(G))$-topology.

The AP passes to closed subgroups, as is proved in [18, Proposition 1.14]. Also, as was mentioned in Section 1, if H is a closed normal subgroup of a locally compact group G such that both H and G/H have the AP, then G has the AP (see [18, Theorem 1.15]). A related result is the following proposition. First we recall some facts about groups.

For a group G we denote its center by $Z(G)$, and (if G is finite) we denote its order by $|G|$. Recall that the adjoint representation $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is given by $\mathrm{ad}(X)(Y) = [X, Y]$. The image $\mathrm{ad}(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$. Let $\mathrm{Ad}(\mathfrak{g})$ denote the analytic subgroup of $\mathrm{GL}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$. The Lie group $\mathrm{Ad}(\mathfrak{g})$ is called the adjoint group. For a connected Lie group G with Lie algebra $\mathfrak{g}$ we also write the adjoint group as $\mathrm{Ad}(G)$. Note that Lie groups with the same Lie algebra have isomorphic adjoint groups. The adjoint group of a connected Lie group G is isomorphic to $G/Z(G)$. For more details, we refer the reader to [21].

PROPOSITION 2.4

If G_1 and G_2 are two locally isomorphic connected simple Lie groups with finite center such that G_1 has the AP, then G_2 has the AP.

Proof
Let G_1 and G_2 be two locally isomorphic connected simple Lie groups with finite center, and suppose that G_1 satisfies the AP. The two groups have the same Lie algebra and hence, their adjoint groups, which are isomorphic to $G_1/Z(G_1)$ and $G_2/Z(G_2)$, respectively, are also isomorphic.

Let (φ_α^1) be a net of functions in $A(G_1)$ converging to the constant function 1 in the weak-* topology on $M_0A(G_1)$. Define

$$\tilde{\varphi}_\alpha^1\big(xZ(G_1)\big) := \frac{1}{|Z(G_1)|} \sum_{z \in Z(G_1)} \varphi_\alpha^1(xz).$$

The summands are elements of the Fourier algebra of G_1, and $\tilde{\varphi}_\alpha^1$ is independent of the representative of the coset. By [12, Proposition 3.25], the space $A(G_1/Z(G_1))$ can be identified isometrically with the subspace of $A(G_1)$ consisting of the elements of $A(G_1)$ that are constant on the cosets of $Z(G_1)$, and hence $\tilde{\varphi}_\alpha^1$ is in $A(G_1/Z(G_1))$.

From the characterization of $A(G_1/Z(G_1))$ we can also conclude that $\tilde{\varphi}_\alpha^1 \to 1$ in the weak-* topology on $M_0A(G_1/Z(G_1))$. The latter can also be identified with the subspace of $M_0A(G_1)$ consisting of the elements of $M_0A(G_1)$ that are constant on the cosets of $Z(G_1)$. Indeed, the approximating net consists of functions that are finite convex combinations of left translates of functions approximating 1 in the weak-* topology on $M_0A(G_1)$.

Hence $G_1/Z(G_1)$ has the AP, so $G_2/Z(G_2)$ has it, as well. From the fact mentioned above, namely that whenever H is a closed normal subgroup of a locally compact group G such that both H and G/H have the AP, then G has the AP, it follows that G_2 has the AP. □

LEMMA 2.5
Let G be a locally compact group with a compact subgroup K. If G has the AP, then the net approximating the constant function 1 in the weak- topology on $M_0A(G)$ can be chosen to consist of K-biinvariant functions.*

Proof
For $f \in C(G)$ or $f \in L^1(G)$ we put

$$f^K(g) = \int_K \int_K f(kgk')\,dk\,dk', \quad g \in G,$$

where dk is the normalized Haar measure on K. Since the norm $\|\cdot\|_{M_0A(G)}$ is invariant under left and right translation by elements of K, we have $\|\varphi^K\|_{M_0A(G)} \leq \|\varphi\|_{M_0A(G)}$ for all $\varphi \in M_0A(G)$. Moreover, for $\varphi \in M_0A(G)$ and $f \in L^1(G)$, we have

$$\langle \varphi^K, f\rangle = \langle \varphi, f^K\rangle,$$

where $L^1(G)$ is considered as a dense subspace of $M_0A(G)$ and the bracket $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $M_0A(G)$ and $M_0A(G)_*$. Hence, $\|f^K\|_{M_0A(G)_*} \leq \|f\|_{M_0A(G)_*}$ for all $f \in L^1(G)$. Therefore, the map on $L^1(G)$ defined by $f \mapsto f^K$ extends uniquely to a linear contraction R on $M_0A(G)_*$, and $R^*\varphi = \varphi^K$ for all $\varphi \in M_0A(G)$, where $R^* \in \mathcal{B}(M_0A(G))$ is the dual operator of R.

Assume now that G has the AP. Then there exists a net φ_α in $A(G)$ such that $\varphi_\alpha \to 1$ in the $\sigma(M_0A(G), M_0A(G)_*)$-topology. Hence, $\varphi_\alpha^K = R^*\varphi_\alpha \to R^*1 = 1$ in the $\sigma(M_0A(G), M_0A(G)_*)$-topology. Moreover, $\varphi_\alpha^K \in A(G) \cap C(K\backslash G/K)$ for all α. This proves the lemma. $\square$

The following lemma is used to conclude that a certain subspace of $M_0A(G)$ is $\sigma(M_0A(G), M_0A(G)_*)$-closed.

LEMMA 2.6
Let (X,μ) be a σ-finite measure space, and let $v : X \longrightarrow \mathbb{R}$ be a strictly positive measurable function on X. Then the set

$$S := \big\{f \in L^\infty(X) \;\big|\; |f(x)| \leq v(x) \text{ a.e.}\big\}$$

is $\sigma(L^\infty(X), L^1(X))$-closed.

Proof
Let (f_α) be a net in S converging to $f \in L^\infty(X)$ in the $\sigma(L^\infty(X), L^1(X))$-topology. Define $E_n = \big\{x \in X \;\big|\; |f(x)| > \big(1 + \frac{1}{n}\big)v(x)\big\}$. We will prove that $\mu(E_n) = 0$ for all $n \in \mathbb{N}$. Suppose that for some $n \in \mathbb{N}$ we have $\mu(E_n) > 0$. Put $E_{n,k} = \big\{x \in E_n \mid v(x) \geq \frac{1}{k}\big\}$. Then $E_{n,k} \nearrow E_n$ for $k \to \infty$. In particular, $\mu(E_{n,k_n}) > 0$ for some $k_n \in \mathbb{N}$. By σ-finiteness of μ, we can choose $F_n \subset E_{n,k_n}$ such that $0 < \mu(F_n) < \infty$. Note that $F_n \subset E_n$ and $v(x) \geq \frac{1}{k_n}$ for all $x \in F_n$. Define the measurable function $g : X \longrightarrow \mathbb{C}$ by

$$g(x) = \frac{1}{\mu(F_n)}\mathbf{1}_{F_n}(x)\frac{1}{v(x)}\frac{\overline{f(x)}}{|f(x)|}, \quad x \in X.$$

Then $g \in L^1(X)$. It follows that $\mathrm{Re}\big(\int_X f_\alpha g\, d\mu\big) \leq 1$, since $|f_\alpha(x)g(x)| \leq 1$ almost everywhere on F_n. Hence, $\mathrm{Re}\big(\int_X fg\, d\mu\big) \leq 1$. Since this integral is real and $fg \geq 0$, it follows that $\int_X |fg|\, d\mu \leq 1$. On the other hand,

$$\int_X |fg|\, d\mu = \frac{1}{\mu(F_n)}\int_{F_n} \frac{|f(x)|}{v(x)}\, d\mu(x) \geq 1 + \frac{1}{n}.$$

This gives a contradiction, so $\mu(E_n) = 0$ for all $n \in \mathbb{N}$. This implies that the set $E = \bigcup_{n=1}^\infty E_n = \{x \in X \mid |f(x)| > v(x)\}$ has measure 0, so $|f(x)| \leq v(x)$ almost everywhere. $\square$

Let G be a locally compact group with compact subgroup K. Because left and right translations of a function $\varphi \in M_0A(G)$ are continuous with respect to the $\sigma(M_0A(G), M_0A(G)_*)$-topology, the space $M_0A(G) \cap C(K\backslash G/K)$ consisting of K-biinvariant completely bounded Fourier multipliers is $\sigma(M_0A(G), M_0A(G)_*)$-closed. Together with Lemma 2.6 and the fact that $L^1(G) \subset M_0A(G)$, this implies the following.

LEMMA 2.7
Let G be a locally compact group with a compact subgroup K, and let $v : G \longrightarrow \mathbb{R}$ be a strictly positive measurable function. Define

$$S_v(G) = \{ f \in L^\infty(G) \mid |f(x)| \leq v(x) \text{ a.e.} \}.$$

Then the space $M_0A(G) \cap S_v(G) \cap C(K\backslash G/K)$ is $\sigma(M_0A(G), M_0A(G)_*)$-closed.

3. The group $\mathrm{Sp}(2,\mathbb{R})$ does not have the Approximation Property

In this section, let $G = \mathrm{Sp}(2,\mathbb{R})$, and let K, A and $\overline{A^+}$ be as described in Example 2.1. The fact that G does not have the AP follows from the behavior of completely bounded Fourier multipliers that are biinvariant with respect to the maximal compact subgroup of $\mathrm{Sp}(2,\mathbb{R})$. Note that the elements of the Fourier algebra, that is, the possible approximating functions, are themselves completely bounded Fourier multipliers. Moreover, they vanish at infinity. We identify two compact Gelfand pairs sitting inside $\mathrm{Sp}(2,\mathbb{R})$ and relate the values of biinvariant completely bounded Fourier multipliers to the values of certain different multipliers on these compact Gelfand pairs. The spherical functions of these Gelfand pairs satisfy certain Hölder continuity conditions, which give rise to the key idea of the proof: an explicit description of the asymptotic behavior of completely bounded Fourier multipliers that are biinvariant with respect to the maximal compact subgroup. In the proof of Lafforgue and de la Salle for the case $\mathrm{SL}(3,\mathbb{R})$ [28], such an estimate is also one of the important ideas.

THEOREM 3.1
The group $G = \mathrm{Sp}(2,\mathbb{R})$ does not have the AP.

The elements of $M_0A(G) \cap C(K\backslash G/K)$ are constant on the double cosets of K in G, so to describe their asymptotic behavior we only need to consider their restriction to $\overline{A^+}$. Note that by Example 2.1 a general element of $\overline{A^+}$ can be written as $D(\alpha_1,\alpha_2) = \mathrm{diag}(e^{\alpha_1}, e^{\alpha_2}, e^{-\alpha_1}, e^{-\alpha_2})$, where $\alpha_1 \geq \alpha_2 \geq 0$.

PROPOSITION 3.2
There exist constants $C_1, C_2 > 0$ such that for all K-biinvariant completely bounded

Fourier multipliers $\varphi : G \longrightarrow \mathbb{C}$*, the limit* $\lim_{g\to\infty}\varphi(g) = \varphi_\infty$ *exists and for all* $\alpha_1 \geq \alpha_2 \geq 0$ *we have*

$$\left|\varphi\big(D(\alpha_1,\alpha_2)\big) - \varphi_\infty\right| \leq C_1 e^{-C_2\|\alpha\|_2}\|\varphi\|_{M_0A(G)}, \tag{2}$$

where $\|\alpha\|_2 = \sqrt{\alpha_1^2+\alpha_2^2}$.

Let us first state an interesting corollary of Proposition 3.2.

COROLLARY 3.3
Every K*-biinvariant completely bounded Fourier multiplier can be written as the sum of a* K*-biinvariant completely bounded Fourier multiplier vanishing at infinity and an element of* $\mathbb{C}$*. More precisely, if* φ *is a* K*-biinvariant completely bounded Fourier multiplier on* G*, then* $\varphi = \varphi_0 + \varphi_\infty$*, where* $\varphi_0 \in M_0A(G) \cap C_0(K\backslash G/K)$ *and* $\varphi_\infty = \lim_{g\to\infty}\varphi(g) \in \mathbb{C}$.

Proof of Theorem 3.1 using Proposition 3.2.
Recall that the elements of $A(G)$ vanish at infinity. By Lemma 2.7, it follows that the unit ball of the space $M_0A(G) \cap C_0(K\backslash G/K)$, which by Proposition 3.2 satisfies the asymptotic behavior of (2) (with $\varphi_\infty = 0$ and $\|\varphi\|_{M_0A(G)} \leq 1$), is closed in the $\sigma(M_0A(G), M_0A(G)_*)$-topology. Recall the Krein–Smulian theorem, which asserts that whenever X is a Banach space and A is a convex subset of the dual space X^* such that $A \cap \{x^* \in X^* \mid \|x^*\| \leq r\}$ is weak-* closed for every $r > 0$, then A is weak-* closed (see [6, Theorem V.12.1]). In the case where A is a vector space, which is the case here, it suffices to check the case $r = 1$, that is, the weak-* closedness of the unit ball. It follows that the space $M_0A(G) \cap C_0(K\backslash G/K)$ is weak-* closed. Since $A(G) \cap C(K\backslash G/K) \subset M_0A(G) \cap C_0(K\backslash G/K)$, it follows that the constant function 1 is not contained in the $\sigma(M_0A(G), M_0A(G)_*)$-closure of $A(G) \cap C(K\backslash G/K)$. Hence, by Lemma 2.5, $\mathrm{Sp}(2,\mathbb{R})$ does not have the AP. $\square$

The proof of Proposition 3.2 will be given after we prove some preliminary results. First we identify two Gelfand pairs sitting inside G. We describe them, the way they are embedded into G, and their spherical functions, and we characterize the completely bounded Fourier multipliers on them that are biinvariant with respect to the corresponding Gelfand subgroup.

Consider the group $\mathrm{U}(2)$, which contains the circle group $\mathrm{U}(1)$ as a subgroup via the embedding

$$\mathrm{U}(1) \hookrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{U}(1) \end{pmatrix} \subset \mathrm{U}(2).$$

Under the identification $K \cong \mathrm{U}(2)$, the embedded copy of $\mathrm{U}(1)$ has the following form:

$$\mathrm{U}(1) \cong K_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \middle| \; \theta \in [0, 2\pi) \right\},$$

which can be interpreted as the group of rotations in the plane parameterized by the second and the fourth coordinate. The group K_1 commutes with the group generated by the elements $D_\alpha = \mathrm{diag}(e^\alpha, 1, e^{-\alpha}, 1)$, where $\alpha \in \mathbb{R}$. This group is a subgroup of $A \subset G$, where A is as in Example 2.1.

It goes back to Weyl [33] that $(\mathrm{U}(2), \mathrm{U}(1))$ is a Gelfand pair (see, e.g., [23, Theorem IX.9.14]). The homogeneous space $\mathrm{U}(2)/\mathrm{U}(1)$ is homeomorphic to the complex 1-sphere $S^1_{\mathbb{C}} \subset \mathbb{C}^2$ and the space $\mathrm{U}(1)\backslash\mathrm{U}(2)/\mathrm{U}(1)$ of double cosets is homeomorphic to the closed unit disc $\overline{\mathbb{D}} \subset \mathbb{C}$ by the map

$$K_1 \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} K_1 \mapsto u_{11}.$$

The spherical functions for $(\mathrm{U}(2), \mathrm{U}(1))$ can be found in [24]. By the homeomorphism $\mathrm{U}(1)\backslash\mathrm{U}(2)/\mathrm{U}(1) \cong \overline{\mathbb{D}}$, they are functions of one complex variable in the closed unit disc. They are indexed by the integers $p, q \geq 0$ and explicitly given by

$$h_{p,q} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = h^0_{p,q}(u_{11}),$$

where in the point $z \in \overline{\mathbb{D}}$ the function $h^0_{p,q}$ is explicitly given by

$$h^0_{p,q}(z) = \begin{cases} z^{p-q} P_q^{(0,p-q)}(2|z|^2 - 1) & p \geq q, \\ \overline{z}^{q-p} P_p^{(0,q-p)}(2|z|^2 - 1) & p < q. \end{cases}$$

Here $P_n^{(\alpha,\beta)}$ denotes the nth Jacobi polynomial. The following is a special case of a result obtained by the first author and Schlichtkrull [19].

THEOREM 3.4
There exists a constant $C > 0$ such that for all nonnegative integers n, β we have

$$(\sin\theta)^{\frac{1}{2}} (\cos\theta)^{\beta + \frac{1}{2}} |P_n^{(0,\beta)}(\cos 2\theta)| \leq \frac{C}{\sqrt{2}} (2n + \beta + 1)^{-\frac{1}{4}}, \quad \theta \in [0, \pi).$$

In particular, for $\theta = \frac{\pi}{4}$ we get

$$2^{-\frac{\beta+1}{2}}|P_n^{(0,\beta)}(0)| \leq \frac{C}{\sqrt{2}}(2n+\beta+1)^{-\frac{1}{4}}.$$

For the special point $z = \frac{1}{\sqrt{2}}$, it follows that

$$\Big|h^0_{p,q}\Big(\frac{1}{\sqrt{2}}\Big)\Big| \leq C(p+q+1)^{-\frac{1}{4}},$$

where C is a constant independent of p and q.

Recall that a function $f : X \longrightarrow Y$ from a metric space X to a metric space Y is Hölder continuous with exponent $\alpha > 0$ if there exists a constant $C > 0$ such that $d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)^\alpha$, for all $x_1, x_2 \in X$. The following result gives Hölder continuity with exponent $\frac{1}{4}$ of the spherical functions on the circle in $\mathbb{D}$ with radius $\frac{1}{\sqrt{2}}$, centered at the origin, with a constant independent of p and q.

COROLLARY 3.5
For all $p, q \geq 0$, we have

$$\Big|h^0_{p,q}\Big(\frac{e^{i\theta_1}}{\sqrt{2}}\Big) - h^0_{p,q}\Big(\frac{e^{i\theta_2}}{\sqrt{2}}\Big)\Big| \leq \tilde{C}|\theta_1 - \theta_2|^{\frac{1}{4}}$$

for all $\theta_1, \theta_2 \in [0, 2\pi)$, where $\tilde{C}$ is a constant independent of p and q.

Proof
From the explicit form of $h^0_{p,q}$ it follows that for all $\theta \in [0, 2\pi)$,

$$h^0_{p,q}\Big(\frac{e^{i\theta}}{\sqrt{2}}\Big) = e^{i(p-q)\theta} h^0_{p,q}\Big(\frac{1}{\sqrt{2}}\Big).$$

This implies that

$$\begin{aligned}\Big|h^0_{p,q}\Big(\frac{e^{i\theta_1}}{\sqrt{2}}\Big) - h^0_{p,q}\Big(\frac{e^{i\theta_2}}{\sqrt{2}}\Big)\Big| &= |e^{i(p-q)\theta_1} - e^{i(p-q)\theta_2}|\Big|h^0_{p,q}\Big(\frac{1}{\sqrt{2}}\Big)\Big| \\ &\leq |p-q||\theta_1 - \theta_2| C(p+q+1)^{-\frac{1}{4}} \\ &\leq C(p+q+1)^{\frac{3}{4}}|\theta_1 - \theta_2|\end{aligned}$$

for all $\theta_1, \theta_2 \in [0, 2\pi)$. We also have the estimate

$$\Big|h^0_{p,q}\Big(\frac{e^{i\theta_1}}{\sqrt{2}}\Big) - h^0_{p,q}\Big(\frac{e^{i\theta_2}}{\sqrt{2}}\Big)\Big| \leq 2\Big|h^0_{p,q}\Big(\frac{1}{\sqrt{2}}\Big)\Big| \leq 2C(p+q+1)^{-\frac{1}{4}}$$

for all $\theta_1, \theta_2 \in [0, 2\pi)$. Combining the two, we get

$$\begin{aligned}
&\Big|h^0_{p,q}\Big(\frac{e^{i\theta_1}}{\sqrt{2}}\Big) - h^0_{p,q}\Big(\frac{e^{i\theta_2}}{\sqrt{2}}\Big)\Big| \\
&\leq \big(C(p+q+1)^{\frac{3}{4}}|\theta_1-\theta_2|\big)^{\frac{1}{4}}\big(2C(p+q+1)^{-\frac{1}{4}}\big)^{\frac{3}{4}} \\
&= \tilde{C}|\theta_1-\theta_2|^{\frac{1}{4}}
\end{aligned}$$

for all $\theta_1, \theta_2 \in [0, 2\pi)$, where $\tilde{C} = 2^{\frac{3}{4}}C$. □

By Proposition 2.3, a U(1)-biinvariant completely bounded Fourier multiplier $\varphi : \mathrm{U}(2) \longrightarrow \mathbb{C}$ can be decomposed as

$$\varphi = \sum_{p,q=0}^{\infty} c_{p,q} h_{p,q},$$

where $c_{p,q} \in \mathbb{C}$ and $\sum_{p,q=0}^{\infty} |c_{p,q}| = \|\varphi\|_{M_0A(\mathrm{U}(2))}$. It follows that

$$\varphi(u) = \varphi\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22}\end{pmatrix} = \varphi^0(u_{11}), \quad u \in \mathrm{U}(2)$$

for some continuous function $\varphi^0 : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$.

COROLLARY 3.6

Let $\varphi : \mathrm{U}(2) \longrightarrow \mathbb{C}$ be a U(1)*-biinvariant completely bounded Fourier multiplier. Then $\varphi(u) = \varphi^0(u_{11})$, and for all $\theta_1, \theta_2 \in [0, 2\pi)$ we have*

$$\Big|\varphi^0\Big(\frac{e^{i\theta_1}}{\sqrt{2}}\Big) - \varphi^0\Big(\frac{e^{i\theta_2}}{\sqrt{2}}\Big)\Big| \leq \tilde{C}|\theta_1-\theta_2|^{\frac{1}{4}}\|\varphi\|_{M_0A(\mathrm{U}(2))}.$$

Proof

Let $\theta \in [0, 2\pi)$, and let $u_{11,\theta} = \frac{e^{i\theta}}{\sqrt{2}}$. Then the matrix

$$u_\theta = \begin{pmatrix} \frac{e^{i\theta}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{e^{-i\theta}}{\sqrt{2}} \end{pmatrix}$$

is an element of U(2). In this way we get

$$\begin{aligned}
\Big|\varphi^0\Big(\frac{e^{i\theta_1}}{\sqrt{2}}\Big) - \varphi^0\Big(\frac{e^{i\theta_2}}{\sqrt{2}}\Big)\Big| &= |\varphi(u_{\theta_1}) - \varphi(u_{\theta_2})| \\
&\leq \sum_{p,q=0}^{\infty} |c_{p,q}|\Big|h^0_{p,q}\Big(\frac{e^{i\theta_1}}{\sqrt{2}}\Big) - h^0_{p,q}\Big(\frac{e^{i\theta_2}}{\sqrt{2}}\Big)\Big| \\
&= \tilde{C}\|\varphi\|_{M_0A(\mathrm{U}(2))}|\theta_1-\theta_2|^{\frac{1}{4}}.
\end{aligned}$$

□

For $\alpha \in \mathbb{R}$ consider the map $K \longrightarrow G$ defined by $k \mapsto D_\alpha k D_\alpha$, where $D_\alpha = \operatorname{diag}(e^\alpha, 1, e^{-\alpha}, 1)$. Given a K-biinvariant completely bounded Fourier multiplier on G, this map gives rise to a K_1-biinvariant completely bounded Fourier multiplier on K.

LEMMA 3.7
Let $\varphi : G \longrightarrow \mathbb{C}$ be a K-biinvariant completely bounded Fourier multiplier, and for $\alpha \in \mathbb{R}$ let $\psi_\alpha : K \longrightarrow \mathbb{C}$ be defined by $\psi_\alpha(k) = \varphi(D_\alpha k D_\alpha)$. Then ψ_α is K_1-biinvariant and satisfies

$$\|\psi_\alpha\|_{M_0A(K)} \leq \|\varphi\|_{M_0A(G)}.$$

Proof
Using the fact that the group elements D_α commute with K_1, it follows that for all $k \in K$ and $k_1, k_2 \in K_1 \subset K_2$,

$$\psi_\alpha(k_1 k k_2) = \varphi(D_\alpha k_1 k k_2 D_\alpha) = \varphi(k_1 D_\alpha k D_\alpha k_2) = \varphi(D_\alpha k D_\alpha) = \psi_\alpha(k),$$

so ψ_α is K_1-biinvariant.

By the characterization of completely bounded Fourier multipliers due to Bożejko and Fendler (see Section 1), we know that there exist bounded continuous maps $P, Q : G \longrightarrow \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, such that $\varphi(y^{-1}x) = \langle P(x), Q(y)\rangle$ for all $x, y \in G$, and, moreover, $\|\varphi\|_{M_0A(G)} = \|P\|_\infty \|Q\|_\infty$.

For all $k_1, k_2 \in K$ we have

$$\begin{aligned}\psi_\alpha(k_2^{-1}k_1) &= \varphi(D_\alpha k_2^{-1} k_1 D_\alpha) = \varphi\big((k_2 D_\alpha^{-1})^{-1} k_1 D_\alpha\big) \\ &= \langle P(k_1 D_\alpha), Q(k_2 D_\alpha^{-1})\rangle = \langle P_\alpha(k_1), Q_\alpha(k_2)\rangle,\end{aligned}$$

where P_α, Q_α are the bounded continuous maps from K to $\mathcal{H}$ defined by $P_\alpha(k) = P(kD_\alpha)$ and $Q_\alpha(k) = Q(kD_\alpha^{-1})$. Because KD_α and KD_α^{-1} are subsets of G, we get $\|P_\alpha\|_\infty \leq \|P\|_\infty$ and $\|Q_\alpha\|_\infty \leq \|Q\|_\infty$, and hence $\|\psi_\alpha\|_{M_0A(K)} \leq \|\varphi\|_{M_0A(G)}$. ☐

From the fact that ψ_α is K_1-biinvariant, it follows that $\psi_\alpha(u) = \psi_\alpha^0(u_{11})$, where $\psi_\alpha^0 : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ is a continuous function.

Suppose now that $\alpha_1 \geq \alpha_2 \geq 0$, and let $D(\alpha_1, \alpha_2)$ be as defined in Example 2.1; that is, $D(\alpha_1, \alpha_2) = \operatorname{diag}(e^{\alpha_1}, e^{\alpha_2}, e^{-\alpha_1}, e^{-\alpha_2})$. If we find an element of the form $D_\alpha k D_\alpha$ in $KD(\alpha_1, \alpha_2)K$, we can relate the value of a K-biinvariant completely bounded Fourier multiplier φ to the value of the multiplier ψ_α that was defined in Lemma 3.7. This only works for certain $\alpha_1, \alpha_2 \geq 0$. We specify which possibilities of α_1 and α_2 we consider, and it will become clear from our proofs that in these cases such α and k exist. It turns out to be sufficient to consider certain candidates for k, namely, the matrices that in the U(2)-representation of K have the form

$$u = \begin{pmatrix} a+ib & -\sqrt{1-a^2-b^2} \\ \sqrt{1-a^2-b^2} & a-ib \end{pmatrix} \tag{3}$$

with $a^2+b^2 \leq 1$. In particular, $u \in \mathrm{SU}(2)$.

In the following lemmas we let $\|h\|_{HS} = \mathrm{Tr}(h^t h)^{\frac{1}{2}}$ and $\det(h)$ denote the Hilbert–Schmidt norm and the determinant of a matrix in $M_4(\mathbb{R})$, respectively. Note that $\det(k) = 1$ for all $k \in K$, because K is a connected subgroup of the orthogonal group O(4).

LEMMA 3.8

Let $g \in G = \mathrm{Sp}(2,\mathbb{R})$. Then $g \in KD(\beta,\gamma)K$, where $\beta,\gamma \in \mathbb{R}$ are uniquely determined by the condition $\beta \geq \gamma \geq 0$ together with the two equations

$$\begin{cases} \sinh^2\beta + \sinh^2\gamma = \frac{1}{8}\|g-(g^t)^{-1}\|_{HS}^2, \\ \sinh^2\beta\sinh^2\gamma = \frac{1}{16}\det(g-(g^t)^{-1}). \end{cases} \tag{4}$$

Proof

Let $g \in G$. By the $K\overline{A^+}K$-decomposition, we have $g = k_1 D(\beta,\gamma)k_2$ for some $k_1,k_2 \in K$ and some $\beta,\gamma \in \mathbb{R}$ satisfying $\beta \geq \gamma \geq 0$. Since $k_i = (k_i^t)^{-1}$, $i = 1,2$, and $D(\beta,\gamma) = D(\beta,\gamma)^t$, we have $(g^t)^{-1} = k_1 D(\beta,\gamma)^{-1}k_2$. Hence, $g-(g^t)^{-1} = k_1(D(\beta,\gamma)-D(\beta,\gamma)^{-1})k_2$, which implies that

$$\|g-(g^t)^{-1}\|_{HS}^2 = \|D(\beta,\gamma)-D(\beta,\gamma)^{-1}\|_{HS}^2 = 8(\sinh^2\beta+\sinh^2\gamma)$$

and

$$\det\big(g-(g^t)^{-1}\big) = \det\big(D(\beta,\gamma)-D(\beta,\gamma)^{-1}\big) = 16\sinh^2\beta\sinh^2\gamma;$$

that is, (β,γ) satisfies (4).

Put $c_1(g) = \frac{1}{8}\|g-(g^t)^{-1}\|_{HS}^2$ and $c_2(g) = \frac{1}{16}\det(g-(g^t)^{-1})$. Then $\sinh^2\beta$ and $\sinh^2\gamma$ are the two solutions of the second order equation $x^2 - c_1(g)x + c_2(g) = 0$, and since $\beta \geq \gamma \geq 0$, the numbers $\sinh^2\beta$ and $\sinh^2\gamma$ are uniquely determined by (4). This also determines $(\beta,\gamma) \in \mathbb{R}^2$ uniquely under the condition $\beta \geq \gamma \geq 0$. $\square$

LEMMA 3.9

Let $\alpha \geq 0$ and $\beta \geq \gamma \geq 0$. If $u \in K$ is of the form (3) *with respect to the identification of K with* U(2)*, then $D_\alpha u D_\alpha \in KD(\beta,\gamma)K$ if and only if*

$$\begin{cases} \sinh\beta\sinh\gamma = \sinh^2\alpha(1-a^2-b^2), \\ \sinh\beta - \sinh\gamma = \sinh(2\alpha)|a|. \end{cases} \tag{5}$$

Proof
Let $\alpha \geq 0$, and let $\beta \geq \gamma \geq 0$. By Lemma 3.8, $D_\alpha u D_\alpha \in KD(\beta,\gamma)K$ if and only if

$$\begin{aligned}\sinh^2\beta + \sinh^2\gamma &= \frac{1}{8}\|D_\alpha u D_\alpha - D_\alpha^{-1} u D_\alpha^{-1}\|_{HS}^2\\ &= \sinh^2(2\alpha)a^2 + 2\sinh^2\alpha(1-a^2-b^2),\end{aligned} \tag{6}$$

and

$$\begin{aligned}\sinh^2\beta\sinh^2\gamma &= \frac{1}{16}\det(D_\alpha u D_\alpha - D_\alpha^{-1} u D_\alpha^{-1})\\ &= \sinh^4\alpha(1-a^2-b^2)^2.\end{aligned} \tag{7}$$

Note that (7) implies the first equation of the statement. Moreover, by (6) and the first equation of the statement, we have $(\sinh\beta - \sinh\gamma)^2 = \sinh^2(2\alpha)a^2$, which implies the second equation of the statement. Hence, (6) and (7) imply (5). Clearly, (5) also implies equations (6) and (7). This proves the lemma. $\square$

Consider now the second Gelfand pair sitting inside $\mathrm{Sp}(2,\mathbb{R})$, namely, the pair of groups $(\mathrm{SU}(2),\mathrm{SO}(2))$. Both groups are naturally subgroups of $\mathrm{U}(2)$, so under the embedding into G, they give rise to compact Lie subgroups of G. The subgroup corresponding to $\mathrm{SU}(2)$ will be called K_2, and the one corresponding to $\mathrm{SO}(2)$ will be called K_3. The group K_3 commutes with the group generated by the elements $D'_\alpha = \mathrm{diag}(e^\alpha, e^\alpha, e^{-\alpha}, e^{-\alpha})$, where $\alpha \in \mathbb{R}$.

The subgroup $\mathrm{SU}(2) \subset \mathrm{U}(2)$ consisting of matrices of the form

$$u = \begin{pmatrix} a+ib & -c+id \\ c+id & a-ib \end{pmatrix} \tag{8}$$

with $a,b,c,d \in \mathbb{R}$ such that $a^2+b^2+c^2+d^2 = 1$ is after embedding into G identified with

$$\begin{aligned}K_2 &= \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \,\middle|\, u = A+iB \in \mathrm{SU}(2) \right\}\\ &= \begin{pmatrix} a & -c & -b & -d \\ c & a & -d & b \\ b & d & a & -c \\ d & -b & c & a \end{pmatrix},\end{aligned}$$

as follows directly from the considerations in Example 2.1.

Recall from Section 2 that a continuous function h not identical to 0 on G that is biinvariant with respect to a Gelfand subgroup K is a spherical function if and only if for all x and y we have $\int_K h(xky)\,dk = h(x)h(y)$. From this, it follows that if K and K' are two unitarily equivalent Gelfand subgroups such that $K = uK'u^*$ and such that h is a spherical function of the pair (G, K), we have that $\tilde{h}(x) = h(uxu^*)$ defines a spherical function for the pair (G, K'). Indeed,

$$\begin{aligned}\tilde{h}(x)\tilde{h}(y) = h(uxu^*)h(uyu^*) &= \int_K h(uxu^*kuyu^*)\,dk \\ &= \int_{K'} h(uxu^*uk'u^*uyu^*)\,d(uk'u^*) = \int_{K'} \tilde{h}(xk'y)\,dk'.\end{aligned}$$

By a symmetry argument, we find a one-to-one correspondence between the spherical functions for both pairs.

By [4, Theorem 47.6], the pair $(\mathrm{SU}(2), \mathrm{SO}(2))$ is a Gelfand pair. This also follows from [15, Chapter 9]. Indeed, it is explained there that the pair $(\mathrm{SU}(2), K')$, where K' is the subgroup isomorphic to SO(2) consisting of elements of the form $\mathrm{diag}(e^{is}, e^{-is})$ for real numbers s, is a Gelfand pair, and the spherical functions are indexed by the integers $n \geq 0$, and for an element $u \in \mathrm{SU}(2)$, as given in (8), they are given by

$$P_n(2|u_{11}|^2 - 1) = P_n\big(2(a^2 + b^2) - 1\big),$$

where $P_n : [-1, 1] \longrightarrow \mathbb{R}$ is the nth Legendre polynomial. However, the two embeddings of SO(2), that is, the natural one and the one given by K', are unitarily equivalent by the following relation:

$$u \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} u^* = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

where u is the unitary matrix given by

$$u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

More generally, for an element in SU(2) we get

$$u \begin{pmatrix} a + ib & -c + id \\ c + id & a - ib \end{pmatrix} u^* = \begin{pmatrix} a + ic & b + id \\ -b + id & a - ic \end{pmatrix},$$

from which it follows that $(\mathrm{SU}(2), \mathrm{SO}(2))$ is a Gelfand pair, and the spherical functions for this pair are indexed by $n \geq 0$ and are given by

$$P_n\big(2(a^2 + c^2) - 1\big) = P_n(a^2 - b^2 + c^2 - d^2),$$

where the last equality follows from the relation $a^2 + b^2 + c^2 + d^2 = 1$.

Note also that the double cosets of K' in SU(2) are labeled by $a^2+b^2-c^2-d^2$, and therefore the double cosets of SO(2) in SU(2) are labeled by $a^2-b^2+c^2-d^2$. Hence, every SO(2)-biinvariant function $\chi : \mathrm{SU}(2) \longrightarrow \mathbb{C}$ is of the form $\chi(u) = \chi^0(a^2-b^2+c^2-d^2)$ for a certain function $\chi^0 : [-1,1] \longrightarrow \mathbb{C}$.

Remark 3.10

The Legendre polynomials $P_n(\cos\theta)$, without the doubled angle, are the spherical functions for the Gelfand pair $(\mathrm{SO}(3), \mathrm{SO}(2))$ (see [9], [14]).

In what follows, we need the following estimates for the Legendre polynomials and their derivatives. Analogous results were obtained by Lafforgue [25] and used by Lafforgue and de la Salle [28]. Our estimates are slightly different. Therefore, we include a proof.

LEMMA 3.11

For all nonnegative integers n,

$$|P_n(x) - P_n(y)| \leq 4|x-y|^{\frac{1}{2}}$$

for $x, y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$; that is, the Legendre polynomials are uniformly Hölder continuous on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ with exponent $\frac{1}{2}$.

Proof

Since $P_0(x) = 1$ and $P_1(x) = x$ for $x \in [-1,1]$, the statement is clearly satisfied for $n = 0$ and $n = 1$. For $n \geq 2$ we use the same integral representation for Legendre polynomials as in [25, Lemma 2.2]; namely, for all $x \in [-1,1]$ we have

$$P_n(x) = \frac{1}{\pi} \int_0^\pi (x + i\sqrt{1-x^2}\cos\theta)^n \, d\theta.$$

Suppose that $n \geq 1$. Differentiation under the integral sign gives

$$P_n'(x) = \frac{n}{\pi} \int_0^\pi (x + i\sqrt{1-x^2}\cos\theta)^{n-1} \Big(1 - i\frac{x}{\sqrt{1-x^2}}\cos\theta\Big) \, d\theta.$$

We have $\left|1 - i\frac{x}{\sqrt{1-x^2}}\cos\theta\right|^2 \leq \frac{1}{1-x^2}$. For $x \in [-1,1]$ set

$$I_n(x) = \frac{1}{\pi} \int_0^\pi |x + i\sqrt{1-x^2}\cos\theta|^n \, d\theta.$$

It follows that for $n \geq 1$ we have $|P_n(x)| \leq I_n(x)$ and $|P_n'(x)| \leq \frac{n}{1-x^2} I_{n-1}(x)$. Moreover, $|x + i\sqrt{1-x^2}\cos\theta|^2 = 1 - (1-x^2)\sin^2\theta \leq e^{-(1-x^2)\sin^2\theta}$. It follows that

$$\begin{aligned} I_n(x) &\leq \frac{1}{\pi}\int_0^{\pi} e^{-\frac{n}{2}(1-x^2)\sin^2\theta}\,d\theta \\ &\leq \frac{2}{\pi}\int_0^{\pi/2} e^{-\frac{n}{2}(1-x^2)(\frac{2\theta}{\pi})^2}\,d\theta \\ &\leq \frac{2}{\pi}\frac{\pi}{\sqrt{2n(1-x^2)}}\int_0^{\infty} e^{-u^2}\,du. \end{aligned}$$

The last integral is equal to $\frac{\sqrt{\pi}}{2}$. Hence, for $x \in [-\frac{1}{2}, \frac{1}{2}]$, we get $I_n(x) \leq \sqrt{\frac{2\pi}{3n}} \leq \frac{2}{\sqrt{n}}$. Thus, for $n \geq 2$ and $x \in [-\frac{1}{2}, \frac{1}{2}]$, we get $|P_n(x)| \leq \frac{2}{\sqrt{n}}$, and we get $|P_n'(x)| \leq \frac{n}{1-x^2} I_{n-1}(x) \leq \frac{8n}{3\sqrt{n-1}} \leq 4\sqrt{n}$. Let now $n \geq 2$ and $x, y \in [-\frac{1}{2}, \frac{1}{2}]$. From the above inequalities it follows that

$$|P_n(x) - P_n(y)| \leq |P_n(x)| + |P_n(y)| \leq \frac{4}{\sqrt{n}},$$

$$|P_n(x) - P_n(y)| \leq \Big|\int_x^y P_n'(t)\,dt\Big| \leq 4\sqrt{n}|x-y|.$$

Combining the two, we get

$$|P_n(x) - P_n(y)| \leq \Big(\frac{4}{\sqrt{n}}\Big)^{\frac{1}{2}}(4\sqrt{n}|x-y|)^{\frac{1}{2}} = 4|x-y|^{\frac{1}{2}},$$

which proves the statement for $n \geq 2$. □

Remark 3.12
The same result can be obtained from [31] (see Theorem 7.3.3, equation (7.33.9), and Theorem 7.33.3 therein).

For $\alpha \in \mathbb{R}$ consider the map $K \longrightarrow G$ defined by $k \mapsto D_\alpha' k v D_\alpha'$, where $D_\alpha' = \mathrm{diag}(e^\alpha, e^\alpha, e^{-\alpha}, e^{-\alpha})$ and $v \in Z(K)$ is chosen to be the matrix in K that in the U(2)-representation of K is given by

$$v = \begin{pmatrix} \frac{1}{\sqrt{2}}(1+i) & 0 \\ 0 & \frac{1}{\sqrt{2}}(1+i) \end{pmatrix}. \tag{9}$$

Given a K-biinvariant completely bounded Fourier multiplier on G, this map gives rise to a K_3-biinvariant completely bounded Fourier multiplier on K. We state the following result, but omit its proof, as it is similar to the one of Lemma 3.7.

LEMMA 3.13
Let $\varphi : G \longrightarrow \mathbb{C}$ be a K-biinvariant completely bounded Fourier multiplier, and let

for $\alpha \in \mathbb{R}$ the function $\tilde{\chi}_\alpha : K \longrightarrow \mathbb{C}$ be defined by $\tilde{\chi}_\alpha(k) = \varphi(D'_\alpha k v D'_\alpha)$. Then $\tilde{\chi}_\alpha$ is K_3-biinvariant and satisfies

$$\|\tilde{\chi}_\alpha\|_{M_0A(K)} \leq \|\varphi\|_{M_0A(G)}.$$

Consider the restriction $\chi_\alpha = \tilde{\chi}_\alpha|_{K_2}$, which is a K_3-biinvariant completely bounded Fourier multiplier on K_2. It follows that $\chi_\alpha(u) = \chi_\alpha^0(a^2 - b^2 + c^2 - d^2)$ for $u \in K_2$, where a, b, c, d are as before, and $\|\chi_\alpha\|_{M_0A(K_2)} \leq \|\varphi\|_{M_0A(G)}$.

COROLLARY 3.14
Let $\varphi \in M_0A(G) \cap C(K\backslash G/K)$, and let $\chi_\alpha : K_2 \longrightarrow \mathbb{C}$ be as in Lemma 3.13. Then $\chi_\alpha(u) = \chi_\alpha^0(a^2 - b^2 + c^2 - d^2)$ for $u \in K_2$, and $\chi_\alpha^0 : [-1, 1] \longrightarrow \mathbb{C}$ satisfies

$$|\chi_\alpha^0(r_1) - \chi_\alpha^0(r_2)| \leq 4|r_1 - r_2|^{\frac{1}{2}} \|\varphi\|_{M_0A(G)}$$

for $r_1, r_2 \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Proof
By applying Proposition 2.3 to the Gelfand pair $(\mathrm{SU}(2), \mathrm{SO}(2))$, we get $\chi_\alpha(u) = \sum_{n=0}^{\infty} c_n P_n(a^2 - b^2 + c^2 - d^2)$, where $\sum_{n=0}^{\infty} |c_n| = \|\chi_\alpha\|_{M_0A(K_2)} \leq \|\varphi\|_{M_0A(G)}$. Hence, the corollary follows from Lemma 3.11. □

Suppose now that $\alpha_1 \geq \alpha_2 \geq 0$, and let $D(\alpha_1, \alpha_2)$ be as defined in Example 2.1. Again, if we find an element of the form $D'_\alpha u v D'_\alpha$ in $KD(\alpha_1, \alpha_2)K$, where u now has to be an element of $\mathrm{SU}(2)$, we can relate the value of a K-biinvariant completely bounded Fourier multiplier φ to the value of the multiplier χ_α. This again only works for certain $\alpha_1, \alpha_2 \geq 0$. Consider a general element of $\mathrm{SU}(2)$:

$$u = \begin{pmatrix} a + ib & -c + id \\ c + id & a - ib \end{pmatrix} \tag{10}$$

with $a^2 + b^2 + c^2 + d^2 = 1$.

LEMMA 3.15
Let $\alpha \geq 0$ and $\beta \geq \gamma \geq 0$, and let $u, v \in K$ be of the form as in (9) *and* (10) *with respect to the identification of K with* $\mathrm{U}(2)$. *Then $D'_\alpha u v D'_\alpha \in KD(\beta, \gamma)K$ if and only if*

$$\begin{cases} \sinh^2\beta + \sinh^2\gamma = \sinh^2(2\alpha), \\ \sinh\beta \sinh\gamma = \frac{1}{2}\sinh^2(2\alpha)|r|, \end{cases}$$

where $r = a^2 - b^2 + c^2 - d^2$.

Proof
The lemma follows from Lemma 3.8. Since for $g = D'_\alpha uvD'_\alpha$ we have $(g^t)^{-1} = (D'_\alpha)^{-1}uv(D'_\alpha)^{-1}$, it follows by direct computation that

$$\|g - (g^t)^{-1}\|_{HS}^2 = 8\sinh^2(2\alpha),$$
$$\det\big(g - (g^t)^{-1}\big) = 4\sinh^4(2\alpha)r^2. \qquad \square$$

LEMMA 3.16
Let $\beta \geq \gamma \geq 0$. Then the equations

$$\begin{aligned} \sinh^2(2s) + \sinh^2 s &= \sinh^2\beta + \sinh^2\gamma, \\ \sinh(2t)\sinh t &= \sinh\beta\sinh\gamma \end{aligned} \tag{11}$$

have unique solutions $s = s(\beta,\gamma)$, $t = t(\beta,\gamma)$ in the interval $[0,\infty)$. *Moreover,*

$$s \geq \frac{\beta}{4}, \qquad t \geq \frac{\gamma}{2}. \tag{12}$$

Proof
The existence and uniqueness of $s,t \geq 0$ is obvious, since $x \mapsto \sinh x$ is a continuous and strictly increasing function mapping $[0,\infty)$ onto $[0,\infty)$. From (11), it follows that for $\beta \geq \gamma \geq 0$ and $s = s(\beta,\gamma)$,

$$\begin{aligned} 2\sinh^2(2s) &\geq \sinh^2(2s) + \sinh^2(s) \geq \sinh^2(\beta) \\ &= 4\sinh^2\Big(\frac{\beta}{2}\Big)\cosh^2\Big(\frac{\beta}{2}\Big) \geq 2\sinh^2\Big(\frac{\beta}{2}\Big). \end{aligned}$$

Hence, $2s \geq \frac{\beta}{2}$. To prove the second inequality in (12), we use that for $t = t(\beta,\gamma)$, we have

$$\sinh^2(2t) \geq \sinh(2t)\sinh(t) = \sinh(\beta)\sinh(\gamma) \geq \sinh^2(\gamma),$$

from which it follows that $2t \geq \gamma$. $\square$

LEMMA 3.17
There exists a constant $C_3 > 0$ such that whenever $\beta \geq \gamma \geq 0$ and $s = s(\beta,\gamma)$ is chosen as in Lemma 3.16, then for all $\varphi \in M_0A(G) \cap C(K\backslash G/K)$,

$$\big|\varphi\big(D(\beta,\gamma)\big) - \varphi\big(D(2s,s)\big)\big| \leq C_3 e^{-\frac{\beta-\gamma}{8}}\|\varphi\|_{M_0A(G)}.$$

Proof
Assume first that $\beta - \gamma \geq 8$. Let $\alpha \in [0,\infty)$ be the unique solution to $\sinh^2\beta + \sinh^2\gamma = \sinh^2(2\alpha)$, and observe that $2\alpha \geq \beta \geq 2$, so in particular $\alpha > 0$. Define

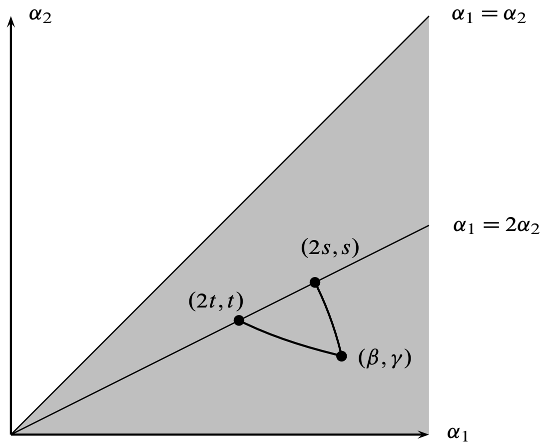

Figure 1. The figure shows the relative position of (β,γ), $(2s,s)$, and $(2t,t)$ as in Lemma 3.17 and Lemma 3.18. Note that (β,γ) and $(2s,s)$ lie on a path in the (α_1,α_2)-plane of the form $\sinh^2\alpha_1 + \sinh^2\alpha_2 =$ constant, and (β,γ) and $(2t,t)$ lie on a path of the form $\sinh\alpha_1 \sinh\alpha_2 =$ constant.

$$r_1 = \frac{2\sinh\beta\sinh\gamma}{\sinh^2\beta + \sinh^2\gamma} \in [0,1],$$

and $a_1 = \left(\frac{1+r_1}{2}\right)^{\frac{1}{2}}$ and $b_1 = \left(\frac{1-r_1}{2}\right)^{\frac{1}{2}}$. Furthermore, put

$$u_1 = \begin{pmatrix} a_1 + ib_1 & 0 \\ 0 & a_1 - ib_1 \end{pmatrix} \in \mathrm{SU}(2),$$

and let

$$v = \begin{pmatrix} \frac{1}{\sqrt{2}}(1+i) & 0 \\ 0 & \frac{1}{\sqrt{2}}(1+i) \end{pmatrix},$$

as previously defined. We now have $2\sinh\beta\sinh\gamma = \sinh^2(2\alpha)r_1$, and $a_1^2 - b_1^2 = r_1$, so by Lemma 3.15, we have $D'_\alpha u_1 v D'_\alpha \in KD(\beta,\gamma)K$. Let $s = s(\beta,\gamma)$ be as in Lemma 3.16. Then $s \geq 0$ and $\sinh^2(2s) + \sinh^2 s = \sinh^2\beta + \sinh^2\gamma = \sinh^2(2\alpha)$. Put

$$r_2 = \frac{2\sinh(2s)\sinh s}{\sinh^2(2s) + \sinh^2 s} \in [0,1],$$

and

$$u_2 = \begin{pmatrix} a_2 + ib_2 & 0 \\ 0 & a_2 - ib_2 \end{pmatrix} \in \mathrm{SU}(2),$$

where $a_2 = \left(\frac{1+r_2}{2}\right)^{\frac{1}{2}}$ and $b_2 = \left(\frac{1-r_2}{2}\right)^{\frac{1}{2}}$. Since $a_2^2 - b_2^2 = r_2$, it follows again by Lemma 3.15 that $D'_\alpha u_2 v D'_\alpha \in KD(2s,s)K$. Now, let $\chi_\alpha(u) = \varphi(D'_\alpha u v D'_\alpha)$ for $u \in K_2 \cong \mathrm{SU}(2)$. Then by Lemma 3.13 and Corollary 3.14, it follows that

$$|\chi_\alpha(u_1) - \chi_\alpha(u_2)| = |\chi_\alpha^0(r_1) - \chi_\alpha^0(r_2)| \leq 4|r_1 - r_2|^{\frac{1}{2}} \|\varphi\|_{M_0A(G)},$$

provided that $r_1, r_2 \leq \frac{1}{2}$. Hence, under this assumption, using the K-biinvariance of φ, we get

$$\left|\varphi\big(D(\beta,\gamma)\big) - \varphi\big(D(2s,s)\big)\right| \leq 4|r_1 - r_2|^{\frac{1}{2}} \|\varphi\|_{M_0A(G)}. \tag{13}$$

Note that $r_1 \leq \frac{2\sinh\beta\sinh\gamma}{\sinh^2\beta} = 2\frac{\sinh\gamma}{\sinh\beta}$. Hence, using $\beta \geq \gamma + 8 \geq \gamma$, we get $r_1 \leq 2\frac{e^\gamma(1-e^{-2\gamma})}{e^\beta(1-e^{-2\beta})} \leq 2e^{\gamma-\beta}$. In particular, $r_1 \leq 2e^{-8} \leq \frac{1}{2}$. Similarly, $r_2 \leq 2\frac{\sinh s}{\sinh 2s} = \frac{1}{\cosh s} \leq 2e^{-s}$. By Lemma 3.16, equation (12), we obtain that $r_2 \leq 2e^{-\frac{\beta}{4}} \leq 2e^{\frac{\gamma-\beta}{4}} \leq 2e^{-2} \leq \frac{1}{2}$. In particular, (13) holds, and since $|r_1 - r_2| \leq \max\{r_1, r_2\} \leq 2e^{\frac{\gamma-\beta}{4}}$, we have proved that

$$\left|\varphi\big(D(\beta,\gamma)\big) - \varphi\big(D(2s,s)\big)\right| \leq 4\sqrt{2}e^{\frac{\gamma-\beta}{8}} \|\varphi\|_{M_0A(G)} \tag{14}$$

under the assumption that $\beta \geq \gamma + 8$. If $\gamma \leq \beta < \gamma + 8$, we get from $\|\varphi\|_\infty \leq \|\varphi\|_{M_0A(G)}$ that $|\varphi(D(\beta,\gamma)) - \varphi(D(2s,s))| \leq 2\|\varphi\|_{M_0A(G)}$. Since $2e \leq 4\sqrt{2}$, it follows that equation (14) holds for all (β,γ) with $\beta \geq \gamma \geq 0$ and $C_3 = 4\sqrt{2}$. □

LEMMA 3.18

There exists a constant $C_4 > 0$ such that whenever $\beta \geq \gamma \geq 0$ and $t = t(\beta,\gamma)$ is chosen as in Lemma 3.16, then for all $\varphi \in M_0A(G) \cap C(K\backslash G/K)$,

$$\left|\varphi\big(D(\beta,\gamma)\big) - \varphi\big(D(2t,t)\big)\right| \leq C_4 e^{-\frac{\gamma}{8}} \|\varphi\|_{M_0A(G)}.$$

Proof

Let $\beta \geq \gamma \geq 0$. Assume first that $\gamma \geq 2$, and let $\alpha \geq 0$ be the unique solution in $[0,\infty)$ to the equation $\sinh\beta\sinh\gamma = \frac{1}{2}\sinh^2\alpha$, and observe that $\alpha > 0$, because $\beta \geq \gamma \geq 2$. Put

$$a_1 = \frac{\sinh\beta - \sinh\gamma}{\sinh(2\alpha)} \geq 0.$$

Since $\sinh(2\alpha) = 2\sinh\alpha\cosh\alpha \geq 2\sinh^2\alpha$, we have

$$a_1 \leq \frac{\sinh\beta}{\sinh(2\alpha)} \leq \frac{\sinh\beta}{2\sinh^2\alpha} = \frac{1}{4\sinh\gamma}.$$

In particular, $a_1 \leq \frac{1}{4\gamma} \leq \frac{1}{8}$. Put now $b_1 = \sqrt{\frac{1}{2} - a_1^2}$. Then $1 - a_1^2 - b_1^2 = \frac{1}{2}$. Hence, $\sinh\beta \sinh\gamma = \sinh^2\alpha(1 - a_1^2 - b_1^2)$ and $\sinh\beta - \sinh\gamma = \sinh(2\alpha)a_1$. Let

$$u_1 = \begin{pmatrix} a_1 + ib_1 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & a_1 - ib_1 \end{pmatrix} \in \mathrm{SU}(2).$$

By Lemma 3.9, we have $D_\alpha u_1 D_\alpha \in KD(\beta, \gamma)K$.

By Lemma 3.16, we have $\sinh(2t)\sinh t = \sinh\beta\sinh\gamma = \frac{1}{2}\sinh^2\alpha$. Moreover, by (12), we have $t \geq \frac{\gamma}{2} \geq 1$. By replacing (β, γ) in the above calculation with $(2t, t)$, we get that the number

$$a_2 = \frac{\sinh(2t) - \sinh t}{\sinh(2\alpha)} \geq 0,$$

satisfies

$$a_2 \leq \frac{1}{4\sinh t} \leq \frac{1}{4\sinh 1} \leq \frac{1}{4}.$$

Hence, we can put $b_2 = \sqrt{\frac{1}{2} - a_2^2}$ and

$$u_2 = \begin{pmatrix} a_2 + ib_2 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & a_2 - ib_2 \end{pmatrix}.$$

Then

$$\begin{aligned} \sinh(2t)\sinh t &= \sinh^2\alpha(1 - a_2^2 - b_2^2), \\ \sinh(2t) - \sinh t &= \sinh(2\alpha)a_2, \end{aligned}$$

and $u_2 \in \mathrm{SU}(2)$. Hence, by Lemma 3.9, $D_\alpha u_2 D_\alpha \in KD(2t, t)K$. Put now $\theta_j = \arg(a_j + ib_j) = \frac{\pi}{2} - \sin^{-1}(\frac{a_j}{\sqrt{2}})$ for $j = 1, 2$. Since $0 \leq a_j \leq \frac{1}{2}$ for $j = 1, 2$, and since $\frac{d}{dt}\sin^{-1} t = \frac{1}{\sqrt{1-t^2}} \leq \sqrt{2}$ for $t \in [0, \frac{1}{\sqrt{2}}]$, it follows that

$$\begin{aligned} |\theta_1 - \theta_2| &\leq \left|\sin^{-1}\Big(\frac{a_1}{\sqrt{2}}\Big) - \sin^{-1}\Big(\frac{a_2}{\sqrt{2}}\Big)\right| \\ &\leq |a_1 - a_2| \\ &\leq \max\{a_1, a_2\} \\ &\leq \max\Big\{\frac{1}{4\sinh\gamma}, \frac{1}{4\sinh t}\Big\} \\ &\leq \frac{1}{4\sinh\frac{\gamma}{2}}, \end{aligned}$$

because $t \geq \frac{\gamma}{2}$. Since $\gamma \geq 2$, we have $\sinh \frac{\gamma}{2} = \frac{1}{2} e^{\frac{\gamma}{2}}(1 - e^{-\gamma}) \geq \frac{1}{4} e^{\frac{\gamma}{2}}$. Hence, we have $|\theta_1 - \theta_2| \leq e^{-\frac{\gamma}{2}}$. Note that $a_j = \frac{1}{\sqrt{2}} e^{i\theta_j}$ for $j = 1, 2$, so by Corollary 3.6 and Lemma 3.7, the function $\psi_\alpha(u) = \varphi(D_\alpha u D_\alpha)$, $u \in \mathrm{U}(2) \cong K$, satisfies

$$\begin{aligned} |\psi_\alpha(u_1) - \psi_\alpha(u_2)| &\leq \tilde{C} |\theta_1 - \theta_2|^{\frac{1}{4}} \|\psi_\alpha\|_{M_0A(K)} \\ &\leq \tilde{C} e^{-\frac{\gamma}{8}} \|\varphi\|_{M_0A(G)}. \end{aligned} \tag{15}$$

Since $D_\alpha u_1 D_\alpha \in KD(\beta,\gamma)K$ and $D_\alpha u_2 D_\alpha \in KD(2t,t)K$, it follows that

$$\bigl|\varphi\bigl(D(\beta,\gamma)\bigr) - \varphi\bigl(D(2t,t)\bigr)\bigr| \leq \tilde{C} e^{-\frac{\gamma}{8}} \|\varphi\|_{M_0A(G)}$$

for all $\gamma \geq 2$. For γ satisfying $0 < \gamma \leq 2$, we can instead use that $\|\varphi\|_\infty \leq \|\varphi\|_{M_0A(G)}$. Hence, with $C_4 = \max\{\tilde{C}, 2e^{\frac{1}{4}}\}$, we obtain

$$\bigl|\varphi\bigl(D(\beta,\gamma)\bigr) - \varphi\bigl(D(2t,t)\bigr)\bigr| \leq C_4 e^{-\frac{\gamma}{8}} \|\varphi\|_{M_0A(G)}$$

for all $\beta \geq \gamma \geq 0$. □

LEMMA 3.19
Let $s \geq t \geq 0$. Then the equations

$$\begin{aligned} \sinh^2 \beta + \sinh^2 \gamma &= \sinh^2(2s) + \sinh^2 s, \\ \sinh \beta \sinh \gamma &= \sinh(2t) \sinh t, \end{aligned} \tag{16}$$

have a unique solution $(\beta,\gamma) \in \mathbb{R}^2$ for which $\beta \geq \gamma \geq 0$. Moreover, if $1 \leq t \leq s \leq \frac{3t}{2}$, then

$$\begin{aligned} |\beta - 2s| &\leq 1, \\ |\gamma + 2s - 3t| &\leq 1. \end{aligned} \tag{17}$$

Proof
Put $\rho(s) = \sinh^2(2s) + \sinh^2 s$ for $s \geq 0$, and $\sigma(t) = 2\sinh(2t)\sinh t$ for $t \geq 0$. Then ρ and σ are strictly increasing functions on $[0,\infty)$, and for all $s \geq 0$, we have $\rho(s) = \sigma(s) + (\sinh(2s) - \sinh s)^2 \geq 0$. Hence, for all $s \geq t \geq 0$, we have $\rho(s) - \sigma(t) \geq \sigma(s) - \sigma(t) \geq 0$. If $(\beta,\gamma) \in \mathbb{R}^2$ is a solution of (16) and $\beta \geq \gamma \geq 0$, then the pair $(x,y) = (\sinh\beta, \sinh\gamma)$ satisfies $x \geq y \geq 0$, and

$$(x \pm y)^2 = \rho(s) \pm \sigma(t).$$

Hence,

$$\begin{aligned} x &= \frac{1}{2}\bigl(\sqrt{\rho(s) + \sigma(t)} + \sqrt{\rho(s) - \sigma(t)}\bigr), \\ y &= \frac{1}{2}\bigl(\sqrt{\rho(s) + \sigma(t)} - \sqrt{\rho(s) - \sigma(t)}\bigr), \end{aligned}$$

and thus $(\beta,\gamma)=(\sinh^{-1}x,\sinh^{-1}y)$ is the unique solution to (16) satisfying $\beta\geq\gamma\geq 0$. To prove (17), first observe that since $\sinh\beta\geq\sinh\gamma$, we obtain from (16) that $\frac{1}{2}\rho(s)\leq\sinh^2\beta\leq\rho(s)$ and $\sinh\beta\sinh\gamma=\frac{1}{2}\sigma(t)$. Hence, $\sqrt{\frac{\rho(s)}{2}}\leq\sinh\beta\leq\sqrt{\rho(s)}$ and $\frac{\sigma(t)}{\sqrt{4\rho(s)}}\leq\sinh\gamma\leq\frac{\sigma(t)}{\sqrt{2\rho(s)}}$. Using $s\geq t\geq 1$, we obtain

$$\begin{aligned}
\rho(s)&\leq\frac{1}{4}(e^{4s}+e^{2s})\leq\frac{e^{4s}}{4}(1+e^{-2})\leq\frac{1}{3}e^{4s},\\
\rho(s)&\geq\frac{1}{4}(1-e^{-4s})^2e^{4s}\geq\frac{e^{4s}}{4}(1-e^{-4})^2\geq\frac{1}{5}e^{4s},\\
\sigma(t)&\leq\frac{1}{2}e^{3t},\\
\sigma(t)&\geq\frac{1}{2}e^{3t}(1-e^{-4})(1-e^{-2})\geq\frac{1}{3}e^{3t}.
\end{aligned}$$

Altogether, we have proved that

$$\frac{e^{2s}}{\sqrt{10}}\leq\sinh\beta\leq\frac{e^{2s}}{\sqrt{3}},$$
$$\frac{1}{2\sqrt{3}}e^{3t-2s}\leq\sinh\gamma\leq\sqrt{\frac{5}{8}}e^{3t-2s}.$$

From the first inequality we have $e^{\beta}\geq\frac{2}{\sqrt{10}}e^2$. Hence, $1-e^{-2\beta}\geq 1-\frac{5}{2}e^{-2}\geq\frac{1}{2}$, which implies that $e^{\beta}\leq 4\sinh\beta\leq\frac{4}{\sqrt{3}}e^{2s}$ and $e^{\beta}\geq 2\sinh\beta\geq\frac{2}{\sqrt{10}}e^{2s}$. Therefore, $|\beta-2s|\leq\max\{\log\frac{4}{\sqrt{3}},\log\frac{\sqrt{10}}{2}\}\leq 1$.

Under the extra assumption $s\leq\frac{3t}{2}$, we have $3t-2s\geq 0$. Hence, $\cosh^2\gamma=\sinh^2\gamma+1\leq\frac{5}{8}e^{6t-4s}+1\leq\frac{13}{18}e^{6t-4s}$, which implies that $e^{\gamma}=\sinh\gamma+\cosh\gamma\leq\big(\sqrt{\frac{5}{8}}+\sqrt{\frac{13}{8}}\big)e^{3t-2s}\leq 3\sqrt{\frac{5}{8}}e^{3t-2s}$. Moreover, $e^{\gamma}\geq 2\sinh\gamma\geq\frac{1}{\sqrt{3}}e^{3t-2s}$. Hence,

$$|\gamma+2s-3t|\leq\max\Big\{\log\Big(3\sqrt{\frac{5}{8}}\Big),\log\sqrt{3}\Big\}\leq 1. \qquad\square$$

LEMMA 3.20

There exists a constant $C_5>0$ such that whenever $s,t\geq 0$ satisfy $2\leq t\leq s\leq\frac{6}{5}t$, then for all $\varphi\in M_0A(G)\cap C(K\backslash G/K)$,

$$\big|\varphi\big(D(2s,s)\big)-\varphi\big(D(2t,t)\big)\big|\leq C_5e^{-\frac{s}{16}}\|\varphi\|_{M_0A(G)}.$$

Proof

Choose $\beta\geq\gamma\geq 0$ as in Lemma 3.19. Then by Lemma 3.17 and Lemma 3.18, we have

$$\begin{aligned}\big|\varphi\big(D(2s,s)\big)-\varphi\big(D(\beta,\gamma)\big)\big| &\leq C_3 e^{-\frac{\beta-\gamma}{8}}\|\varphi\|_{M_0A(G)},\\ \big|\varphi\big(D(2t,t)\big)-\varphi\big(D(\beta,\gamma)\big)\big| &\leq C_4 e^{-\frac{\gamma}{8}}\|\varphi\|_{M_0A(G)}.\end{aligned}$$

Moreover, by (17),

$$\begin{aligned}\beta-\gamma &\geq (2s-1)-(3t-2s+1)=4s-3t-2\geq s-2,\\ \gamma &\geq 3t-2s-1\geq \frac{5}{2}s-2s-1=\frac{s-2}{2}.\end{aligned}$$

Hence, since $s\geq 2$, we have $\min\{e^{-\gamma},e^{-(\beta-\gamma)}\}\leq e^{-\frac{s-2}{2}}$. Thus, the lemma follows from Lemma 3.17 and Lemma 3.18 with $C_5=e^{\frac{1}{8}}(C_3+C_4)$. □

LEMMA 3.21
There exists a constant $C_6>0$ such that for all $\varphi\in M_0A(G)\cap C(K\backslash G/K)$ the limit $c_\infty(\varphi)=\lim_{t\to\infty}\varphi(D(2t,t))$ exists, and for all $t\geq 0$,

$$\big|\varphi\big(D(2t,t)\big)-c_\infty(\varphi)\big|\leq C_6 e^{-\frac{t}{16}}\|\varphi\|_{M_0A(G)}.$$

Proof
By Lemma 3.20, we have for $u\geq 5$ and $\gamma\in[0,1]$ that

$$\big|\varphi\big(D(2u,u)\big)-\varphi\big(D(2u+2\gamma,u+\gamma)\big)\big|\leq C_5 e^{-\frac{u}{16}}\|\varphi\|_{M_0A(G)}. \tag{18}$$

Let $s\geq t\geq 5$. Then $s=t+n+\delta$, where $n\geq 0$ is an integer and $\delta\in[0,1)$. Applying equation (18) to $(u,\gamma)=(t+j,1)$, $j=0,1,\dots,n-1$ and $(u,\gamma)=(t+n,\delta)$, we obtain

$$\begin{aligned}\big|\varphi\big(D(2t,t)\big)-\varphi\big(D(2s,s)\big)\big| &\leq C_5\Big(\sum_{j=0}^{n}e^{-\frac{t+j}{16}}\Big)\|\varphi\|_{M_0A(G)}\\ &\leq C_5' e^{-\frac{t}{16}}\|\varphi\|_{M_0A(G)},\end{aligned}$$

where $C_5'=(1-e^{-\frac{1}{16}})^{-1}C_5$. Hence $\big(\varphi(D(2t,t))\big)_{t\geq 5}$ is a Cauchy net. Therefore, $c_\infty(\varphi)=\lim_{t\to\infty}\varphi(D(2t,t))$ exists, and

$$\big|\varphi\big(D(2t,t)\big)-c_\infty(\varphi)\big|=\lim_{s\to\infty}\big|\varphi\big(D(2t,t)\big)-\varphi\big(D(2s,s)\big)\big|\leq C_5' e^{-\frac{t}{16}}\|\varphi\|_{M_0A(G)}$$

for all $t\geq 5$. Since $\|\varphi\|_\infty\leq\|\varphi\|_{M_0A(G)}$, we have for all $0\leq t<5$,

$$\big|\varphi\big(D(2t,t)\big)-c_\infty(\varphi)\big|\leq 2\|\varphi\|_{M_0A(G)}.$$

Hence, the lemma follows with $C_6=\max\{C_5',2e^{\frac{5}{16}}\}$. □

Proof of Proposition 3.2
Let $\varphi \in M_0A(G) \cap C(K\backslash G/K)$, and let $(\alpha_1, \alpha_2) = (\beta, \gamma)$, where $\beta \geq \gamma \geq 0$. Assume first $\beta \geq 2\gamma$. Then $\beta - \gamma \geq \frac{\beta}{2}$, so by Lemma 3.16 and Lemma 3.17, there exists an $s \geq \frac{\beta}{4}$ such that

$$\left|\varphi\big(D(\beta,\gamma)\big) - \varphi\big(D(2s,s)\big)\right| \leq C_3 e^{-\frac{\beta}{16}} \|\varphi\|_{M_0A(G)}.$$

By Lemma 3.21,

$$\left|\varphi\big(D(2s,s)\big) - c_\infty(\varphi)\right| \leq C_6 e^{-\frac{s}{16}} \|\varphi\|_{M_0A(G)} \leq C_6 e^{-\frac{\beta}{64}} \|\varphi\|_{M_0A(G)}.$$

Hence,

$$\left|\varphi\big(D(\beta,\gamma)\big) - c_\infty(\varphi)\right| \leq (C_3 + C_6) e^{-\frac{\beta}{64}} \|\varphi\|_{M_0A(G)}.$$

Assume now that $\beta < 2\gamma$. Then, by Lemma 3.16 and Lemma 3.18, we obtain that there exists a $t \geq \frac{\gamma}{2} > \frac{\beta}{4}$ such that

$$\left|\varphi\big(D(\beta,\gamma)\big) - \varphi(D(2t,t))\right| \leq C_4 e^{-\frac{\beta}{16}} \|\varphi\|_{M_0A(G)},$$

and by Lemma 3.21,

$$\left|\varphi\big(D(2t,t)\big) - c_\infty(\varphi)\right| \leq C_6 e^{-\frac{t}{16}} \|\varphi\|_{M_0A(G)} \leq C_6 e^{-\frac{\beta}{64}} \|\varphi\|_{M_0A(G)}.$$

Hence,

$$\left|\varphi\big(D(\beta,\gamma)\big) - c_\infty(\varphi)\right| \leq (C_4 + C_6) e^{-\frac{\beta}{64}} \|\varphi\|_{M_0A(G)}.$$

Therefore, for all $\beta \geq \gamma \geq 0$, we have

$$\left|\varphi\big(D(\beta,\gamma)\big) - c_\infty(\varphi)\right| \leq C_1 e^{-\frac{\beta}{64}} \|\varphi\|_{M_0A(G)},$$

where $C_1 = \max\{C_3 + C_6, C_4 + C_6\}$. This proves the proposition, because $\|\alpha\|_2 = \sqrt{\beta^2 + \gamma^2} \leq \sqrt{2}\beta$. □

Remark 3.22
In [26, Definition 4.1], Lafforgue introduces the property (T_{Schur}) for a locally compact group G relative to a specified compact subgroup K of G. It is not hard to see that our Proposition 3.2 implies the degenerate case ($s = 0$) of the property (T_{Schur}) for $G = \mathrm{Sp}(2,\mathbb{R})$ relative to its maximal compact subgroup $K \cong \mathrm{U}(2)$. In the same way, Proposition 5.2 implies the degenerate case of the property (T_{Schur}) for $G = \mathrm{SL}(3,\mathbb{R})$ relative to $K = \mathrm{SO}(3)$.

4. Simple Lie groups with finite center and real rank greater than or equal to two do not have the Approximation Property

In the previous section we proved that $\mathrm{Sp}(2,\mathbb{R})$ does not have the AP. Together with the fact that $\mathrm{SL}(3,\mathbb{R})$ does not have the AP, this implies the following theorem.

THEOREM 4.1
Let G be a connected simple Lie group with finite center and real rank greater than or equal to two. Then G does not have the AP.

Proof
Let G be a connected simple Lie group with finite center and real rank greater than or equal to two. By Wang's method [32], we may assume that G is the adjoint group, so that G has a connected splitting semisimple subgroup H with real rank 2. Such a subgroup is closed, as was proved in [10]. It is known that H has finite center and is locally isomorphic to either $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$ (see [1], [30]). Since the AP is passed to closed subgroups and as it is preserved under local isomorphisms (see Proposition 2.4), we conclude that G does not have the AP, since $\mathrm{SL}(3,\mathbb{R})$ and $\mathrm{Sp}(2,\mathbb{R})$ do not have the AP. $\square$

Remark 4.2
Note that we could as well have stated the theorem for connected semisimple Lie groups with finite center such that at least one simple factor has real rank greater than or equal to two, since this factor would then contain a subgroup that is locally isomorphic to either $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$.

Let $n \geq 1$, and let $\mathbb{K}$ be field. Countable discrete subgroups of $\mathrm{GL}(n,\mathbb{K})$ are exact, as was proven in [16]. Recall that a lattice in a second countable locally compact group is a closed discrete subgroup Γ such that G/Γ has bounded G-invariant measure. As mentioned in Section 1, if Γ is a lattice in a second countable locally compact group G, then G has the AP if and only if Γ has the AP. These observations imply the following result.

THEOREM 4.3
Let Γ be a lattice in a connected simple linear Lie group with finite center and real rank greater than or equal to two. Then Γ is an exact group and does not satisfy the AP.

COROLLARY 4.4
For every lattice in a connected simple Lie group with finite center and real rank

*greater than or equal to two, the reduced group C^*-algebra $C^*_\lambda(\Gamma)$ does not have the OAP and the group von Neumann algebra $L(\Gamma)$ does not have the w*OAP.*

Remark 4.5

We do not know yet if the finite center condition in Theorem 4.1 can be omitted. If G is a connected simple Lie group with real rank greater than or equal to two (and maybe infinite center), it contains a connected splitting semisimple subgroup H locally isomorphic to either $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$. This implies that H is a group isomorphic to a quotient of the universal cover of either $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$ by a discrete subgroup of the center of the universal cover. If H is locally isomorphic to $\mathrm{SL}(3,\mathbb{R})$, our arguments still hold, since the universal cover is finite. However, the universal cover of $\mathrm{Sp}(2,\mathbb{R})$ is infinite, so our arguments do not work any longer. If the universal cover of $\mathrm{Sp}(2,\mathbb{R})$ does not have the AP, then this would imply that the finite center condition in the theorem can be omitted.

5. The group $\mathrm{SL}(3,\mathbb{R})$

In this section we consider the group $G = \mathrm{SL}(3,\mathbb{R})$ with maximal compact subgroup $K = \mathrm{SO}(3)$. Recall that Lafforgue and de la Salle proved the following theorem.

THEOREM 5.1 (Lafforgue and de la Salle [28, Theorem C])
The group $\mathrm{SL}(3,\mathbb{R})$ *does not have the AP.*

We give a proof of this theorem along the same lines as our proof for the group $\mathrm{Sp}(2,\mathbb{R})$. In particular, we do not make use of the $\mathrm{AP}^{\mathrm{Schur}}_{\mathrm{pcb}}$ for $1 < p < \infty$. It is clear that Theorem 5.1 is implied by Proposition 5.2 below in exactly the same way that Theorem 3.1 is implied by Proposition 3.2, namely, by applying the Krein–Smulian theorem to show that the space $M_0A(G) \cap C_0(K\backslash G/K)$ is closed in $M_0A(G)$ in the $\sigma(M_0A(G), M_0A(G)_*)$-topology.

Let G, K, A, $\overline{A^+}$ be as defined in Example 2.2. Then $G = K\overline{A^+}K$. Following the notation of [25, Section 2] and [28, Section 5], put $D(s,t) = e^{-\frac{s+2t}{3}}\,\mathrm{diag}(e^{s+t}, e^t, 1)$, where $s,t \in \mathbb{R}$. Then $A = \{D(s,t) \mid s,t \in \mathbb{R}\}$ and $\overline{A^+} = \{D(s,t) \mid s \geq 0,\ t \geq 0\}$.

PROPOSITION 5.2
Let $G = \mathrm{SL}(3,\mathbb{R})$ and $K = \mathrm{SO}(3)$, and let $M_0A(G) \cap C(K\backslash G/K)$ denote the set of K-biinvariant completely bounded Fourier multipliers on G. Then there exist constants $C_1, C_2 > 0$ such that for all $\varphi \in M_0A(G) \cap C(K\backslash G/K)$ the limit $\varphi_\infty := \lim_{g\to\infty} \varphi(g)$ exists, and for all $s,t \geq 0$,

$$\bigl|\varphi\bigl(D(s,t)\bigr) - \varphi_\infty\bigr| \leq C_1 \|\varphi\|_{M_0A(G)} e^{-C_2(s+t)}.$$

In [25, Proposition 2.3] Lafforgue proved a similar result for coefficients of certain nonunitary representations of $G = \mathrm{SL}(3,\mathbb{R})$. Below we outline a proof of Proposition 5.2 that relies on the methods of [25, Section 2] and on the previous sections of this paper.

Consider the pair of compact groups (K, K_0), where K is as above and K_0 is the subgroup of K isomorphic to SO(2) given by the embedding

$$\mathrm{SO}(2) \hookrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(2) \end{pmatrix}.$$

It is easy to see that if φ is a K_0-biinvariant function on K, then φ depends only on the first matrix element g_{11}; that is, $\varphi(g) = \varphi^0(g_{11})$ for a certain function $\varphi^0 : [-1,1] \longrightarrow \mathbb{C}$.

LEMMA 5.3

Let $\varphi : K \longrightarrow \mathbb{C}$ be a K_0-biinvariant completely bounded Fourier multiplier. Then $\varphi(g) = \varphi^0(g_{11})$ and for all $x \in [-1,1]$,

$$|\varphi^0(x) - \varphi^0(0)| \leq 4\|\varphi\|_{M_0A(K)}|x|^{\frac{1}{2}}.$$

Proof

By [14] and [9], the pair (SO(3), SO(2)) is a compact Gelfand pair, and the spherical functions are indexed by $n \geq 0$ and given by $\varphi_n(g) = P_n(g_{11})$, where P_n again denotes the nth Legendre polynomial. By Proposition 2.3 the function φ^0 can be written as $\varphi^0 = \sum_{n\geq 0} c_n P_n$, where $c_n \in \mathbb{C}$ and $\sum_{n\geq 0} |c_n| = \|\varphi\|_{M_0A(K)}$. Moreover, by Lemma 3.11 we know that

$$|P_n(x) - P_n(0)| \leq 4|x|^{\frac{1}{2}} \tag{19}$$

for $n \in \mathbb{N}_0$ and $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Since $|P_n(x)| \leq 1$ for all $n \in \mathbb{N}_0$ and $x \in [-1,1]$, the inequality given by (19) holds for $\frac{1}{2} < |x| \leq 1$ as well. The result now follows. □

LEMMA 5.4

Let $\varphi \in M_0A(G) \cap C(K\backslash G/K)$, and let $r \geq 0$. Then the function $\psi_r : K \longrightarrow \mathbb{C}$ defined by $\psi_r(k) = \varphi(D(r,0)kD(r,0))$ is K_0-biinvariant and $\|\psi_r\|_{M_0A(K)} \leq \|\varphi\|_{M_0A(G)}$.

Proof

The matrix $D(r,0) = e^{-\frac{r}{3}}\,\mathrm{diag}(e^r,1,1)$ commutes with K_0. Therefore the lemma follows from the proof of Lemma 3.7. □

LEMMA 5.5
Let $\varphi \in M_0A(G) \cap C(K\backslash G/K)$, and let $q, r \in \mathbb{R}$ such that $r \geq q \geq 0$. Then

$$\left|\varphi\big(D(2q, r-q)\big) - \varphi\big(D(0,r)\big)\right| \leq 4e^{-\frac{r-q}{2}} \|\varphi\|_{M_0A(G)}. \tag{20}$$

Proof
If $r = q = 0$, then (20) is trivial, so we can assume that $r > 0$. Let $\psi_r(g) = \psi_r^0(g_{11})$ be the map defined in Lemma 5.4. It follows that

$$\begin{aligned}\psi_r^0(\cos\theta) &= \varphi\left(D(r,0)\begin{pmatrix}\cos\theta & \sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}D(r,0)\right)\\ &= \varphi\left(e^{-\frac{2r}{3}}\begin{pmatrix}e^{2r}\cos\theta & -e^r\sin\theta & 0\\ e^r\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\right).\end{aligned}$$

By the polar decomposition of $\mathrm{SL}(2,\mathbb{R})$, there exist $k_1, k_2 \in \mathrm{SO}(2)$, and a $q \geq 0$ such that

$$\begin{pmatrix}e^r\cos\theta & -\sin\theta\\ \sin\theta & e^{-r}\cos\theta\end{pmatrix} = k_1\begin{pmatrix}e^q & 0\\ 0 & e^{-q}\end{pmatrix}k_2.$$

Comparing the Hilbert–Schmidt norms (similar to the method we applied for the case $\mathrm{Sp}(2,\mathbb{R})$) and subtracting $2 = 2(\sin^2\theta + \cos^2\theta)$ on both sides, we obtain $(e^r - e^{-r})^2\cos^2\theta = (e^q - e^{-q})^2$. It follows that

$$\sinh q = |\cos\theta|\sinh r, \tag{21}$$

and all values of $q \in [0, r]$ occur for some $\theta \in [0, \frac{\pi}{2}]$. By defining $\tilde{k_i} = \left(\begin{smallmatrix}k_i & 0\\ 0 & 1\end{smallmatrix}\right)$ for $i = 1, 2$, we get

$$D(r,0)\begin{pmatrix}\cos\theta & \sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}D(r,0) = \tilde{k_1}D(2q, r-q)\tilde{k_2},$$

and hence, by the SO(3)-biinvariance of φ, we get $\psi_r^0(\cos\theta) = \varphi(D(2q, r-q))$. For $\theta = \frac{\pi}{2}$, we have $q = 0$. Therefore $\psi_r^0(0) = \varphi(D(0,r))$. Hence, for $r > 0$ and $r \geq q \geq 0$, we have $\psi_r^0(\cos\theta) - \psi_r^0(0) = \varphi(D(2q, r-q)) - \varphi(D(0,r))$ if (21) holds. Hence, by Lemma 5.3 we have

$$\begin{aligned}\left|\varphi\big(D(2q, r-q)\big) - \varphi\big(D(0,r)\big)\right| &\leq 4\|\varphi\|_{M_0A(G)}\left(\frac{\sinh q}{\sinh r}\right)^{\frac{1}{2}}\\ &\leq 4\|\varphi\|_{M_0A(G)}e^{-\frac{r-q}{2}},\end{aligned}$$

where we have used that for $r \geq q \geq 0$ and $r > 0$ the following holds:

$$\frac{\sinh q}{\sinh r} = e^{q-r}\Big(\frac{1-e^{-2q}}{1-e^{-2r}}\Big) \leq e^{q-r}.$$

This proves the lemma. □

LEMMA 5.6
Let $\varphi \in M_0A(G) \cap C(K\backslash G/K)$. For $s,t \geq 0$,

$$\Big|\varphi\big(D(s,t)\big) - \varphi\Big(D\Big(\frac{s+2t}{3}, \frac{s+2t}{3}\Big)\Big)\Big| \leq 8\|\varphi\|_{M_0A(G)}e^{-\frac{t}{3}},$$

$$\Big|\varphi\big(D(s,t)\big) - \varphi\Big(D\Big(\frac{2s+t}{3}, \frac{2s+t}{3}\Big)\Big)\Big| \leq 8\|\varphi\|_{M_0A(G)}e^{-\frac{s}{3}}.$$

Proof
From Lemma 5.5, it follows that in the special case $q = \frac{r}{3}$ we have

$$\Big|\varphi\Big(D\Big(\frac{2r}{3}, \frac{2r}{3}\Big)\Big) - \varphi\big(D(0,r)\big)\Big| \leq 4\|\varphi\|_{M_0A(G)}e^{-\frac{r}{3}}.$$

Combined with the estimate of Lemma 5.5 it follows that in the general case we have $\big|\varphi(D(2q,r-q)) - \varphi\big(D\big(\frac{2r}{3},\frac{2r}{3}\big)\big)\big| \leq A_1\|\varphi\|_{M_0A(G)}$, where $A_1 = 4(e^{-\frac{r-q}{2}} + e^{-\frac{r}{3}})$. Substituting $(s,t) = (2q, r-q)$, we get for all $s,t \geq 0$ that

$$\Big|\varphi\big(D(s,t)\big) - \varphi\Big(D\Big(\frac{s+2t}{3}, \frac{s+2t}{3}\Big)\Big)\Big| \leq A_2\|\varphi\|_{M_0A(G)},$$

where $A_2 = 4(e^{-\frac{t}{2}} + e^{-\frac{s+2t}{6}}) \leq 8e^{-\frac{t}{3}}$, which proves the first inequality of the lemma.

By the SO(3)-biinvariance of φ, it follows that

$$\varphi\big(\mathrm{diag}(e^{\alpha_1}, e^{\alpha_2}, e^{\alpha_3})\big) = \varphi\big(\mathrm{diag}(e^{\alpha_3}, e^{\alpha_2}, e^{\alpha_1})\big)$$

whenever $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Hence $\varphi(D(s,t)) = \varphi(D(-t,-s)) = \check{\varphi}(D(t,s))$, where $\check{\varphi}(g) = \varphi(g^{-1})$ for all $g \in G$. Since $\|\check{\varphi}\|_{M_0A(G)} = \|\varphi\|_{M_0A(G)}$, we obtain the second inequality of the lemma by applying the first inequality to $\check{\varphi}$ with s and t interchanged. □

LEMMA 5.7
Let $\varphi \in M_0A(G) \cap C(K\backslash G/K)$, and let $u,v \geq 0$ such that $\frac{2}{3}u \leq v \leq \frac{3}{2}u$. Then

$$\big|\varphi\big(D(u,u)\big) - \varphi\big(D(v,v)\big)\big| \leq 16\|\varphi\|_{M_0A(G)}e^{-w/6},$$

where $w = \min\{u,v\}$.

Proof
Put $s = 2v - u$ and $t = 2u - v$. Then $s, t \geq 0$, and $u = \frac{s+2t}{3}$ and $v = \frac{2s+t}{3}$. Hence, by Lemma 5.6, we get $|\varphi(D(s,t)) - \varphi(D(u,u))| \leq 8\|\varphi\|_{M_0A(G)}e^{-\frac{t}{3}}$, and $|\varphi(D(s,t)) - \varphi(D(v,v))| \leq 8\|\varphi\|_{M_0A(G)}e^{-\frac{s}{3}}$. Hence,

$$\left|\varphi\big(D(u,u)\big) - \varphi\big(D(v,v)\big)\right| \leq A_3\|\varphi\|_{M_0A(G)},$$

where $A_3 = 8(e^{-\frac{s}{3}} + e^{-\frac{t}{3}}) = 8(e^{-\frac{2u-v}{3}} + e^{-\frac{2v-u}{3}})$. By the assumptions on u and v, we obtain $\frac{2u-v}{3} \geq \frac{u}{6}$ and $\frac{2v-u}{3} \geq \frac{v}{6}$. Hence, $A_3 \leq 8(e^{-\frac{u}{6}} + e^{-\frac{v}{6}}) \leq 16e^{-\frac{w}{6}}$, where $w = \min\{u, v\}$. This proves the lemma. □

Proof of Proposition 5.2.
Applying the method of the proof of the case $\mathrm{Sp}(2,\mathbb{R})$, it is clear that Lemma 5.7 implies that $c := \lim_{u\to\infty} \varphi(D(u,u))$ exists. Moreover, for $u \geq 2$,

$$\begin{aligned}\left|\varphi\big(D(u,u)\big) - c\right| &\leq \sum_{n=0}^{\infty}\left|\varphi\big(D(u+n+1, u+n+1)\big) - \varphi\big(D(u+n, u+n)\big)\right| \\ &\leq 16e^{-\frac{u}{6}}\|\varphi\|_{M_0A(G)}\sum_{n=0}^{\infty} e^{-\frac{n}{6}} \\ &\leq 112e^{-\frac{u}{6}}\|\varphi\|_{M_0A(G)},\end{aligned}$$

since $\sum_{n=0}^{\infty} e^{-\frac{n}{6}} \leq 7$. Since $|\varphi(D(u,u)) - c| \leq 2\|\varphi\|_{M_0A(G)}$ for $0 \leq u \leq 2$, we have for all $u \geq 0$ that $|\varphi(D(u,u)) - c| \leq 112e^{-\frac{u}{6}}\|\varphi\|_{M_0A(G)}$. Let now $s, t \geq 0$. If $s \leq t$, then this implies that

$$\begin{aligned}\left|\varphi\big(D(s,t)\big) - c\right| &\leq (8e^{-\frac{t}{3}} + 112e^{-\frac{s+2t}{18}})\|\varphi\|_{M_0A(G)} \\ &\leq (8e^{-\frac{s+t}{6}} + 112e^{-\frac{s+t}{12}})\|\varphi\|_{M_0A(G)}.\end{aligned}$$

If $s \geq t$, then we get the same inequality. Hence the proposition holds with $\varphi_\infty = c$, $C_1 = 120$ and $C_2 = \frac{1}{12}$. □

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APPENDIX B

Approximation properties for noncommutative L^p-spaces associated with lattices in Lie groups

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Approximation properties for noncommutative L^p-spaces associated with lattices in Lie groups

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Abstract

In 2010, Lafforgue and de la Salle gave examples of noncommutative L^p-spaces without the operator space approximation property (OAP) and, hence, without the completely bounded approximation property (CBAP). To this purpose, they introduced the property of completely bounded approximation by Schur multipliers on S^p, denoted $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$, and proved that for $p \in [1, \frac{4}{3}) \cup (4, \infty]$ the groups $\mathrm{SL}(n, \mathbb{Z})$, with $n \geqslant 3$, do not have the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$. Since for $p \in (1, \infty)$ the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ is weaker than the approximation property of Haagerup and Kraus (AP), these groups were also the first examples of exact groups without the AP. Recently, Haagerup and the author proved that also the group $\mathrm{Sp}(2, \mathbb{R})$ does not have the AP, without using the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$. In this paper, we prove that $\mathrm{Sp}(2, \mathbb{R})$ does not have the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ for $p \in [1, \frac{12}{11}) \cup (12, \infty]$. It follows that a large class of noncommutative L^p-spaces does not have the OAP or CBAP.

Keywords: Approximation properties; Noncommutative L^p-spaces; Lie groups; Schur multipliers

1. Introduction

Let M be a finite von Neumann algebra with normal faithful trace τ. For $1 \leqslant p < \infty$, the noncommutative L^p-space $L^p(M, \tau)$ is defined as the completion of M with respect to the norm

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$\|x\|_p = \tau((x^*x)^{\frac{p}{2}})^{\frac{1}{p}}$, and for $p = \infty$, we put $L^\infty(M, \tau) = M$ with operator norm. In [23], Kosaki showed that noncommutative L^p-spaces can be realized by interpolating between M and $L^1(M, \tau)$. This leads to an operator space structure on them, as described by Pisier [27] (see also [20]).

An operator space E is said to have the completely bounded approximation property (CBAP) if there exists a net F_α of finite-rank maps on E such that $\sup_\alpha \|F_\alpha\|_{cb} < C$ for some $C > 0$, and $\lim_\alpha \|F_\alpha x - x\| = 0$ for every $x \in E$. The infimum of all possible C's is denoted by $\Lambda(E)$. If $\Lambda(E) = 1$, we say that E has the completely contractive approximation property (CCAP). An operator space E is said to have the operator space approximation property (OAP) if there exists a net F_α of finite-rank maps on E such that $\lim_\alpha \|(\mathrm{id}_{\mathcal{K}(\ell^2)} \otimes F_\alpha)x - x\| = 0$ for all $x \in \mathcal{K}(\ell^2) \otimes_{\min} E$. Here $\mathcal{K}(\ell^2)$ denotes the space of compact operators on the Hilbert space ℓ^2. The CBAP goes back to De Cannière and Haagerup [5], and the OAP was defined by Effros and Ruan [9]. By definition, the CCAP implies the CBAP, which in turn implies the OAP.

Recall that a lattice in a Lie group G is a discrete subgroup Γ of G such that G/Γ has finite invariant measure. In this paper, we consider noncommutative L^p-spaces of the form $L^p(L(\Gamma))$, where $L(\Gamma)$ is the group von Neumann algebra of a lattice Γ in a connected simple Lie group G. Such a von Neumann algebra $L(\Gamma)$ is finite and has canonical trace $\tau : x \mapsto \langle x\delta_1, \delta_1\rangle$, where $\delta_1 \in \ell^2(\Gamma)$ is the characteristic function of the unit element $1 \in \Gamma$.

It was proved by Junge and Ruan [20, Proposition 3.5] that if Γ is a weakly amenable (countable) discrete group, then for $p \in (1, \infty)$, the noncommutative L^p-space $L^p(L(\Gamma))$ has the CBAP. Recall that connected simple Lie groups of real rank zero are amenable. By the work of Cowling and Haagerup [6] and Hansen [17], all connected simple Lie groups of real rank one are weakly amenable. This implies that for every $p \in (1, \infty)$ and every lattice Γ in a connected simple Lie group G of real rank zero or one, the noncommutative L^p-space $L^p(L(\Gamma))$ has the CBAP.

The existence of noncommutative L^p-spaces without the CBAP follows from the work of Szankowski [29]. The first concrete examples were given recently by Lafforgue and de la Salle [24]. They proved that for all $p \in [1, \frac{4}{3}) \cup (4, \infty]$ and all lattices Γ in $\mathrm{SL}(n, \mathbb{R})$, where $n \geqslant 3$, the space $L^p(L(\Gamma))$ does not have the OAP (or CBAP). They also proved analogous results for lattices in Lie groups over nonarchimedean fields. In their work, the failure of the OAP for the aforementioned noncommutative L^p-spaces follows from the failure of a certain approximation property for the groups $\mathrm{SL}(n, \mathbb{R})$. This property, called the property of completely bounded approximation by Schur multipliers on S^p (see Section 2.6), denoted $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$, was introduced by Lafforgue and de la Salle exactly to this purpose.

Other approximation properties for groups (see [3]), e.g., amenability, weak amenability, and the approximation property of Haagerup and Kraus (AP) (see [14]), are related to the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$. It is well-known that amenability of a group G (strictly) implies weak amenability, which in turn (strictly) implies the AP. For $p \in (1, \infty)$, the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ is weaker than the AP. In this way, the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ gave rise to the first example of an exact group without the AP, namely $\mathrm{SL}(3, \mathbb{Z})$. Recently, Haagerup and the author proved that also $\mathrm{Sp}(2, \mathbb{R})$ does not have the AP [15], in a more direct way than Lafforgue and de la Salle did for $\mathrm{SL}(3, \mathbb{R})$. Indeed, the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ was not used in the proof. On the other hand, as was mentioned earlier, the method of Lafforgue and de la Salle also gives information about approximation properties of certain noncommutative L^p-spaces. For this, it is actually crucial to use the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$. Haagerup and the author also proved that all connected simple Lie groups with finite center and real rank greater than or equal to two do not have the AP, building on the failure of the AP for both $\mathrm{SL}(3, \mathbb{R})$ and $\mathrm{Sp}(2, \mathbb{R})$.

The following are the main results of this article.

Theorem 3.1. *For $p \in [1, \frac{12}{11}) \cup (12, \infty]$, the group $\mathrm{Sp}(2, \mathbb{R})$ does not have the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$.*

Theorem 4.3. *Let $p \in [1, \frac{12}{11}) \cup (12, \infty]$, and let Γ be a lattice in a connected simple Lie group with finite center and real rank greater than or equal to two. Then $L^p(L(\Gamma))$ does not have OAP (or CBAP).*

The paper is organized as follows. In Section 2, we recall some preliminary results, and we make a study of Schur multipliers on Schatten classes corresponding to (compact) Gelfand pairs, which provides us with suitable tools for our proof. In Section 3, we prove Theorem 3.1, and in Section 4, we prove Theorem 4.3.

2. Preliminaries

2.1. Schur multipliers on Schatten classes

This section partly follows the exposition of [24, Section 1]. More details can be found there.

For $p \in [1, \infty]$ and a (separable) Hilbert space $\mathcal{H}$, let $S^p(\mathcal{H})$ denote the pth Schatten class on $\mathcal{H}$. Recall that $S^\infty(\mathcal{H})$ is the Banach space $\mathcal{K}(\mathcal{H})$ of compact operators (with operator norm) on $\mathcal{H}$, and for $p \in [1, \infty)$, the space $S^p(\mathcal{H})$ consists of the operators T on $\mathcal{H}$ such that $\|T\|_p = \mathrm{Tr}((T^*T)^{\frac{p}{2}})^{\frac{1}{p}} < \infty$, where Tr denotes the (semifinite) trace on $\mathcal{B}(\mathcal{H})$. In this way, $S^p(\mathcal{H})$ is a Banach space for all $p \in [1, \infty]$. We use the notation $S^p_n = S^p(\ell^2_n)$ and $S^p = S^p(\ell^2)$. Note that the space $S^2(\mathcal{H})$ corresponds to the Hilbert–Schmidt operators on $\mathcal{H}$.

Schatten classes can be realized by interpolating between certain noncommutative L^p-spaces in the semifinite setting. Indeed, we have $S^p(\mathcal{H}) = L^p(\mathcal{B}(\mathcal{H}), \mathrm{Tr})$. Noncommutative L^p-spaces in the semifinite setting can be defined analogously to the finite case, which was described in Section 1. For details, see [28]. The natural operator space structure on $S^p(\mathcal{H})$ follows from [27]. For our purposes, the following characterization of the completely bounded norm of a linear map between Schatten classes is important. Recall that $S^p(\mathcal{H}) \otimes S^p(\mathcal{K})$ (algebraic tensor product) embeds naturally into $S^p(\mathcal{H} \otimes \mathcal{K})$ (Hilbert space tensor product). Let $T : S^p(\mathcal{H}) \to S^p(\mathcal{H})$ be a bounded linear map, and let $\mathcal{K} = \ell^2$. Then T is completely bounded if the map $T \otimes \mathrm{id}_{S^p}$ extends to a bounded linear map on $S^p(\mathcal{H} \otimes \ell^2)$, and we have $\|T\|_{cb} = \|T \otimes \mathrm{id}_{S^p}\| = \sup_{n \in \mathbb{N}} \|T \otimes \mathrm{id}_{S^p_n}\|$ (see [28, Lemma 1.7]).

A linear map $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ of the form $[x_{ij}] \mapsto [\psi_{ij} x_{ij}]$ for some matrix $\psi \in M_n(\mathbb{C})$ is called a Schur multiplier on $M_n(\mathbb{C})$. More precisely, the operator T is called the Schur multiplier on $M_n(\mathbb{C})$ with symbol ψ, and it is also denoted by M_ψ. In what follows, we need more general notions of Schur multipliers.

Let (X, μ) be a σ-finite measure space. Let $k \in L^2(X \times X, \mu \otimes \mu)$. It is well-known that the map $T_k : L^2(X, \mu) \to L^2(X, \mu)$ defined by $(T_k f)(x) = \int_X k(x, y) f(y)\, d\mu(y)$, is a Hilbert–Schmidt operator on $L^2(X, \mu)$. Conversely, if $T \in S^2(L^2(X, \mu))$, then $T = T_k$ for some $k \in L^2(X \times X, \mu \otimes \mu)$. In this way, we can identify $S^2(L^2(X, \mu))$ with $L^2(X \times X, \mu \otimes \mu)$, and we see that every Schur multiplier on $S^2(L^2(X, \mu))$ comes from a function $\psi \in L^\infty(X \times X, \mu \otimes \mu)$ acting by multiplication on $L^2(X \times X, \mu \otimes \mu)$.

Definition 2.1. Let $p \in [1, \infty]$, and let $\psi \in L^\infty(X \times X, \mu \otimes \mu)$. The Schur multiplier with symbol ψ is said to be bounded (resp. completely bounded) on $S^p(L^2(X, \mu))$ if it maps

$S^p(L^2(X,\mu)) \cap S^2(L^2(X,\mu))$ into $S^p(L^2(X,\mu))$ (by $T_k \mapsto T_{\psi k}$), and if this map extends (necessarily uniquely) to a bounded (resp. completely bounded) map M_ψ on $S^p(L^2(X,\mu))$.

The norm of such a bounded multiplier ψ is defined by $\|\psi\|_{MS^p(L^2(X,\mu))} = \|M_\psi\|$, and its completely bounded norm by $\|\psi\|_{cbMS^p(L^2(X,\mu))} = \|M_\psi\|_{cb}$. The spaces of multipliers and completely bounded multipliers are denoted by $MS^p(L^2(X,\mu))$ and $cbMS^p(L^2(X,\mu))$, respectively. It follows that for every $p \in [1,\infty]$ and $\psi \in L^\infty(X \times X, \mu \otimes \mu)$, we have $\|\psi\|_\infty \leqslant \|\psi\|_{MS^p(L^2(X,\mu))} \leqslant \|\psi\|_{cbMS^p(L^2(X,\mu))}$.

If $\frac{1}{p} + \frac{1}{q} = 1$, we have $\|\psi\|_{MS^p(L^2(X,\mu))} = \|\psi\|_{MS^q(L^2(X,\mu))}$. By interpolation and duality we have that whenever $2 \leqslant p \leqslant q \leqslant \infty$, then $\|\psi\|_{MS^p(L^2(X,\mu))} \leqslant \|\psi\|_{MS^q(L^2(X,\mu))}$. These results also hold for the completely bounded norm.

Lemma 2.2. *(See [24, Lemma 1.5 and Remark 1.6].) The Schur multiplier corresponding to $\psi \in L^\infty(X \times X, \mu \otimes \mu)$ is completely bounded on $S^p(L^2(X,\mu))$ if and only if the Schur multiplier corresponding to $\tilde{\psi}(x,\xi,y,\eta) = \psi(x,y)$ is completely bounded on $S^p(L^2(X \times \Omega, \mu \otimes \nu))$, where (Ω,ν) is a σ-finite measure space, and*

$$\|\psi\|_{cbMS^p(L^2(X,\mu))} = \|\tilde{\psi}\|_{cbMS^p(L^2(X\times\Omega,\mu\otimes\nu))}.$$

If $L^2(\Omega,\nu)$ is infinite-dimensional, these norms equal $\|\tilde{\psi}\|_{MS^p(L^2(X\times\Omega,\mu\otimes\nu))}$.

Lemma 2.3. *(See [24, Theorem 1.19].) Let (X,μ) be a locally compact space with a σ-finite Radon measure μ, and let $\psi : X \times X \to \mathbb{C}$ be a bounded continuous function. Let $1 \leqslant p \leqslant \infty$. The following are equivalent*:

(1) *we have $\psi \in MS^p(L^2(X,\mu))$ with $\|\psi\|_{MS^p(L^2(X,\mu))} \leqslant C$,*
(2) *for every finite set $F = \{x_1,\ldots,x_n\} \subset X$ such that $F \subset \mathrm{supp}(\mu)$, the Schur multiplier given by $(\psi(x_i,x_j))_{i,j}$ is bounded on $S^p(\ell^2(F))$ with norm smaller than or equal to C.*

The analogous statement holds in the completely bounded case. In particular, the norm and the completely bounded norm of the multiplier only depend on the support of μ, and if this support does not have any isolated points, then the norm and the completely bounded norm coincide.

2.2. Schur multipliers on locally compact groups

For a locally compact group G and a function $\varphi \in L^\infty(G)$, we define the function $\check{\varphi} \in L^\infty(G \times G)$ by $\check{\varphi}(g,h) = \varphi(g^{-1}h)$. The notation $\check{\varphi}$ will be used without further mentioning. In what follows, we will consider continuous functions $\varphi : G \to \mathbb{C}$ such that $\check{\varphi}$ is a (completely bounded) Schur multiplier on $S^p(L^2(G))$.

2.3. KAK decomposition for Lie groups

Recall that every connected semisimple Lie group G with finite center can be decomposed as $G = KAK$, where K is a maximal compact subgroup (unique up to conjugation) and A is an abelian Lie group such that its Lie algebra $\mathfrak{a}$ is a Cartan subspace of the Lie algebra $\mathfrak{g}$ of G. The dimension of $\mathfrak{a}$ is called the real rank of G and is denoted by $\mathrm{Rank}_{\mathbb{R}}(G)$. The KAK decomposition is in general not unique. However, after choosing a set of positive roots and restricting to the closure $\overline{A^+}$ of the positive Weyl chamber A^+, we still have $G = K\overline{A^+}K$. Moreover, if $g = k_1 a k_2$, where $k_1, k_2 \in K$ and $a \in \overline{A^+}$, then a is unique. For more details, see [18,21].

2.4. Gelfand pairs and spherical functions

Let G be a Lie group with compact subgroup K. We denote the (left) Haar measure on G by dx and the normalized Haar measure on K by dk. A function $\varphi : G \to \mathbb{C}$ is said to be K-bi-invariant if $\varphi(k_1 g k_2) = \varphi(g)$ for all $g \in G$ and $k_1, k_2 \in K$. Note that for $\varphi \in C(G)$, the continuous function defined by $\varphi^K(g) = \int_K \int_K \varphi(kgk')\,dk\,dk'$ is K-bi-invariant. By abuse of notation, we denote the space of K-bi-invariant compactly supported continuous functions on G by $C_c(K \setminus G/K)$. This space can be considered as a subalgebra of the convolution algebra $C_c(G)$. If this subalgebra is commutative, then the pair (G, K) is said to be a Gelfand pair. Equivalently, if G is a Lie group with compact subgroup K, then (G, K) is a Gelfand pair if and only if for every irreducible unitary representation π of G on a Hilbert space $\mathcal{H}_\pi$, the space $\mathcal{H}_{\pi_e}$ consisting of K-invariant vectors, i.e., $\mathcal{H}_{\pi_e} = \{\xi \in \mathcal{H} \mid \forall k \in K\colon\ \pi(k)\xi = \xi\}$, is at most one-dimensional. Also, the pair (G, K) is a Gelfand pair if and only if the representation $L^2(G/K)$ is multiplicity free.

Let (G, K) be a Gelfand pair. A function $h \in C(K \setminus G/K)$ is called a spherical function if the functional χ on $C_c(K \setminus G/K)$ given by $\chi(\varphi) = \int_G \varphi(x) h(x^{-1})\,dx$ defines a nontrivial character, i.e., $\chi(\varphi * \psi) = \chi(\varphi)\chi(\psi)$ for all $\varphi, \psi \in C_c(K \setminus G/K)$. Spherical functions arise as the matrix coefficients of K-invariant vectors in irreducible representations of G.

It is possible to consider Gelfand pairs in more general settings than Lie groups, e.g., in the setting of locally compact groups (see [7,11]).

2.5. Schur multipliers on compact Gelfand pairs

Let G and K be Lie groups such that (G, K) is a Gelfand pair, and let $X = G/K$ denote the homogeneous space (with quotient topology) corresponding with the canonical (transitive) action of G. It follows that K is the stabilizer subgroup of a certain element $e_0 \in X$. In this section we consider Schur multipliers on the Schatten classes $S^p(\mathcal{H})$, where $\mathcal{H} = L^2(G)$ or $L^2(X)$. To this end, it is natural to look at multipliers on G that are K-bi-invariant. Denote by D the space $K \setminus G/K$ as a topological space, and denote by $f : K \setminus G/K \to D$, $KgK \mapsto \xi$ the corresponding homeomorphism. It follows that every function φ in $C(K \setminus G/K)$ induces a continuous function φ^0 on D such that $\varphi(g) = \varphi^0(\xi)$ for all $g \in G$, where ξ is the image under the homeomorphism f.

A Gelfand pair (G, K) is called compact if G is a compact group. In this section, all Gelfand pairs are assumed to be compact, unless explicitly stated otherwise. For compact groups every representation on a Hilbert space is equivalent to a unitary representation, every irreducible representation is finite-dimensional, and every unitary representation is the direct sum of irreducible ones. For an irreducible unitary representation π of G on a Hilbert space $\mathcal{H}_\pi$, let $P_\pi = \int_K \pi(k)\,dk$ denote the projection onto $\mathcal{H}_{\pi_e}$ (see Section 2.4), and let $\hat{G}_K$ denote the space of equivalence classes of the irreducible unitary representations π of G such that $P_\pi \neq 0$.

Lemma 2.4. *Let (G, K) be a compact Gelfand pair, and let $X = G/K$ be the corresponding (compact) homogeneous space. Then*

$$L^2(X) = \bigoplus_{\pi \in \hat{G}_K} \mathcal{H}_\pi.$$

Let h_π denote the spherical function corresponding to the equivalence class π of representations. Then for every $\varphi \in L^2(K \setminus G/K)$ we have

$$\varphi = \sum_{\pi \in \hat{G}_K} c_\pi \dim \mathcal{H}_\pi h_\pi,$$

where $c_\pi = \langle \varphi, h_\pi \rangle$. Moreover, denoting by h_π^0 the (spherical) function on D corresponding to h_π, we have $\varphi^0 = \sum_{\pi \in \hat{G}_K} c_\pi (\dim \mathcal{H}_\pi) h_\pi^0$.

This lemma follows from the Peter–Weyl theorem applied to a compact homogeneous space (see, e.g., [19, Section V.4]). The decomposition of φ (and hence φ^0) is stated explicitly in [32, Proposition 9.10.4].

Lemma 2.5. *Let (G, K) be a (not necessarily compact) Gelfand pair, and let $X = G/K$ denote the corresponding homogeneous space. Choose $e_0 \in X$ so that K is its stabilizer subgroup. Let $\varphi \in C(K \setminus G/K)$. Then there exists a continuous function $\psi : X \times X \to \mathbb{C}$ such that for all $g, h \in G$,*

$$\varphi\bigl(g^{-1}h\bigr) = \psi(ge_0, he_0).$$

Proof. If $ge_0 = g'e_0$ for $g, g' \in G$, then $g^{-1}g' \in K$, and hence $g' = gk$ for some $k \in K$. Hence, by the K-bi-invariance of φ, we know that $\varphi(g^{-1}h)$ depends only on the pair $(ge_0, he_0) \in X \times X$, so there exists a function $\psi : X \times X \to \mathbb{C}$ such that $\varphi(g^{-1}h) = \psi(ge_0, he_0)$. Since $X = G/K$ is equipped with the quotient topology, this function is continuous. $\square$

Lemma 2.6. *Let (G, K) be a compact Gelfand pair. If $\varphi : G \to \mathbb{C}$ is a continuous K-bi-invariant function such that $\check{\varphi} \in cbMS^p(L^2(G))$ (see Section 2.2) for some $p \in [1, \infty]$, then $\|\psi\|_{cbMS^p(L^2(X))} = \|\check{\varphi}\|_{cbMS^p(L^2(G))}$, where $\psi : X \times X \to \mathbb{C}$ is as defined in Lemma 2.5. If K is an infinite group, then these norms are equal to $\|\check{\varphi}\|_{MS^p(L^2(G))}$.*

Proof. By [25, Lemma 1.1], the quotient map $G \to G/K$ has a Borel cross section. Let Y denote the image of this cross section. The result now follows directly from Lemma 2.2 by putting $\Omega = K$, so that $G = Y \times K$ as a measure space by the map $(y, k) \mapsto yk$ for $y \in Y$ and $k \in K$. $\square$

We can now prove a decomposition result for Schur multipliers on $S^p(L^2(G))$ coming from K-bi-invariant functions.

Proposition 2.7. *Let (G, K) be a compact Gelfand pair, suppose that K has infinitely many elements, and let $p \in [1, \infty)$. Let $\varphi : G \to \mathbb{C}$ be a continuous K-bi-invariant function such that $\check{\varphi} \in MS^p(L^2(G))$. Then*

$$\left(\sum_{\pi \in \hat{G}_K} |c_\pi|^p (\dim \mathcal{H}_\pi) \right)^{\frac{1}{p}} \leqslant \|\check{\varphi}\|_{MS^p(L^2(G))},$$

where c_π and $\mathcal{H}_\pi$ are as in Lemma 2.4.

Proof. As before, let $(T_k f)(x) = \int_G k(x,y)f(y)\,dy$. Then T_1 is the projection on $\mathbb{C}1 \in L^2(X)$. It follows that $\|T_1\|_{S^p(L^2(X))} = 1$. It is sufficient to prove that $(\sum_{\pi \in \hat{G}_K} |c_\pi|^p (\dim \mathcal{H}_\pi))^{\frac{1}{p}} \leqslant \|T_\psi\|_{S^p(L^2(X))}$, where ψ is as before. Indeed, we have $\|T_\psi\|_{S^p(L^2(X))} = \frac{\|T_\psi\|_{S^p(L^2(X))}}{\|T_1\|_{S^p(L^2(X))}} \leqslant \|\psi\|_{MS^p(L^2(X))}$, which is smaller than or equal to $\|\psi\|_{cbMS^p(L^2(X))} = \|\check{\varphi}\|_{MS^p(L^2(G))}$ by Lemma 2.6 under the assumption that K is an infinite group.

By Lemma 2.4, we have $\varphi = \sum_{\pi \in \hat{G}_K} c_\pi \dim \mathcal{H}_\pi h_\pi$. By [19, Theorem V.4.3], it follows that the operator $P_{\mathcal{H}_\pi} = \dim \mathcal{H}_\pi T_{h'_\pi}$ is the projection onto $\mathcal{H}_\pi$, where $h'_\pi : X \times X \to \mathbb{C}$ denotes the function induced by h_π (see Lemma 2.5). Since $L^2(X)$ decomposes as a direct sum of Hilbert spaces, we have

$$\begin{aligned}\|T_\psi\|^p_{S^p(L^2(X))} &= \Big\| \sum_{\pi \in \hat{G}_K} c_\pi \dim \mathcal{H}_\pi T_{h'_\pi} \Big\|^p_{S^p(L^2(X))} \\ &= \sum_{\pi \in \hat{G}_K} |c_\pi|^p \operatorname{Tr}\big(|P_{\mathcal{H}_\pi}|^p\big) = \sum_{\pi \in \hat{G}_K} |c_\pi|^p \dim \mathcal{H}_\pi. \qquad \square\end{aligned}$$

Lemma 2.8. *Let G be a locally compact group with compact subgroup K. For $p \in [1,\infty]$, let $\varphi \in C(G)$ be such that $\check{\varphi} \in MS^p(L^2(G))$. Then the continuous function φ^K defined by $\varphi^K(g) = \int_K \int_K \varphi(kgk')\,dk\,dk'$ induces an element $\check{\varphi}^K$ of $MS^p(L^2(G))$, and $\|\check{\varphi}^K\|_{MS^p(L^2(G))} \leqslant \|\check{\varphi}\|_{MS^p(L^2(G))}$. The analogous statement holds in the completely bounded case.*

Proof. Let ν_n be a sequence of finitely supported probability measures on K pointwise converging to the Haar measure μ. Let $\varphi_n : G \to \mathbb{C}$ be defined by $\varphi_n(g) = \int_K \int_K \varphi(kgk')\,d\nu_n(k)\,d\nu_n(k')$. Each φ_n is a convex combination of functions ${}_k\varphi_{k'}$ of the form ${}_k\varphi_{k'}(g) = \varphi(kgk')$, where $k, k' \in K$ are fixed. Hence, φ^K is an element of the pointwise closure of $\operatorname{conv}\{{}_k\varphi_{k'} \mid k,k' \in K\}$. One easily checks that for all $k,k' \in K$, we have $\|{}_k\check{\varphi}_{k'}\|_{MS^p(L^2(G))} = \|\check{\varphi}\|_{MS^p(L^2(G))}$. Hence, by Lemma 2.3, we have $\check{\varphi}^K \in MS^p(L^2(G))$, and $\|\check{\varphi}^K\|_{MS^p(L^2(G))} \leqslant \|\check{\varphi}\|_{MS^p(L^2(G))}$. The result for the completely bounded case follows in an analogous way. $\square$

2.6. The property $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$

In this section we recall the definition of the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$, as given by Lafforgue and de la Salle in [24]. First, recall that the Fourier algebra $A(G)$ (see [10]) consists of the coefficients of the left-regular representation of G. More precisely, $\varphi \in A(G)$ if and only if there exist $\xi, \eta \in L^2(G)$ such that for all $x \in G$ we have $\varphi(x) = \langle \lambda(x)\xi, \eta \rangle$. With the norm $\|\varphi\|_{A(G)} = \min\{\|\xi\|\|\eta\| \mid \forall x \in G\ \varphi(x) = \langle \lambda(x)\xi, \eta\rangle\}$, it is a Banach space.

Definition 2.9. (See [24, Definition 2.2].) Let G be a locally compact Hausdorff second countable group, and let $1 \leqslant p \leqslant \infty$. The group G is said to have the property of completely bounded approximation by Schur multipliers on S^p, denoted $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$, if there exists a constant $C > 0$ and a net $\varphi_\alpha \in A(G)$ such that $\varphi_\alpha \to 1$ uniformly on compacta and $\sup_\alpha \|\check{\varphi}_\alpha\|_{cbMS^p(L^2(G))} \leqslant C$. The infimum of these C's is denoted by $\Lambda^{\mathrm{Schur}}_{p,\mathrm{cb}}(G)$.

The following result is a key property of the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ (see [24, Theorem 2.5]).

Theorem 2.10. *Let G be a locally compact Hausdorff group, and let Γ be a lattice in G. Then for $1 \leqslant p \leqslant \infty$, we have $\Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(\Gamma) = \Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(G)$.*

Lafforgue and de la Salle also proved that for a discrete group Γ and $p \in (1,\infty)$, it follows that $\Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(\Gamma) \in \{1,\infty\}$. Since a semisimple Lie group G has lattices [1], we conclude by the above proposition that for such a group, it also follows that $\Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(G) \in \{1,\infty\}$ for $p \in (1,\infty)$.

Proposition 2.11. *Let G be a locally compact Hausdorff group. The $\mathrm{AP}_{p,\mathrm{cb}}^{\mathrm{Schur}}$ satisfies the following properties*:

(1) *for $p = \infty$ (or $p = 1$, by the third statement of this proposition), the group G has the $\mathrm{AP}_{p,\mathrm{cb}}^{\mathrm{Schur}}$ if and only if it is weakly amenable, and $\Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(G) = \Lambda(G)$, where $\Lambda(G)$ denotes the Cowling–Haagerup constant of G*;
(2) *for every locally compact group, $\Lambda_{2,\mathrm{cb}}^{\mathrm{Schur}}(G) = 1$*;
(3) *if $p,q \in [1,\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then $\Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(G) = \Lambda_{q,\mathrm{cb}}^{\mathrm{Schur}}(G)$*;
(4) *if $2 \leqslant p \leqslant q \leqslant \infty$, then $\Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(G) \leqslant \Lambda_{q,\mathrm{cb}}^{\mathrm{Schur}}(G)$*;
(5) *if H is a closed subgroup of G and $1 \leqslant p \leqslant \infty$, then $\Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(H) \leqslant \Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(G)$*;
(6) *if G has a compact subgroup K, and if φ_α is a net in $A(G)$ converging to 1 uniformly on compacta such that $\sup_\alpha \|\check{\varphi}_\alpha\|_{cbMS^p(L^2(G))} \leqslant C$, then there exists a net $\tilde{\varphi}_\alpha$ in $A(G) \cap C(K \setminus G/K)$ such that $\sup_\alpha \|\check{\tilde{\varphi}}_\alpha\|_{cbMS^p(L^2(G))} \leqslant C$ that converges to 1 uniformly on compacta*;
(7) *if K is a compact normal subgroup of G and $1 \leqslant p \leqslant \infty$, then $\Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(G) = \Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(G/K)$*;
(8) *if G_1 and G_2 are locally isomorphic connected (semi)simple Lie groups with finite centers, then for $p \in [1,\infty]$, we have $\Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(G_1) = \Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(G_2)$.*

Proof. The first statement is clear. The second through the fifth statements are covered in [24, Section 2]. The sixth statement follows from Lemma 2.8. By combining the sixth statement and Lemma 2.6, the seventh statement follows. The fact that the net on the group converges uniformly on compacta if and only if the net on the quotient does, is straightforward (see [6]). For the eighth statement, note that the center is a normal subgroup of a group. Using the seventh statement and the fact that the adjoint groups $G_1/Z(G_1)$ and $G_2/Z(G_2)$, where $Z(G_i)$ denotes the center of G_i, are isomorphic, we obtain the result. □

2.7. Approximation properties for noncommutative L^p-spaces

The operator space structure on a noncommutative L^p-space $L^p(M,\tau)$ can be obtained by considering this space as a certain interpolation space (see [23]). Indeed, the pair of spaces $(M, L^1(M,\tau))$ becomes a compatible couple of operator spaces, and for $1 < p < \infty$ we have the isometry $L^p(M,\tau) \cong [M, L^1(M,\tau)]_{\frac{1}{p}}$. By [28, Lemma 1.7], we know that for a linear map $T : L^p(M,\tau) \to L^p(M,\tau)$, its completely bounded norm $\|T\|_{cb}$ corresponds to $\sup_{n\in\mathbb{N}} \|\mathrm{id}_{S_n^p} \otimes T : S_n^p[L^p(M)] \to S_n^p[L^p(M)]\|$. Using [28, Corollary 1.4] and the fact that $S_n^1 \otimes L^1(M) = L^1(M \otimes M_n)$, we obtain that $S_n^p[L^p(M)] = L^p(M \otimes M_n)$, which implies that $\|T\|_{cb} = \sup_{n\in\mathbb{N}} \|T \otimes \mathrm{id} : L^p(M \otimes M_n) \to L^p(M \otimes M_n)\|$.

In Section 1 of this article, we recalled the definition of the CBAP, CCAP and OAP. It was shown by Junge and Ruan [20] that if Γ is a discrete group with the AP (of Haagerup and Kraus), and if $p \in (1, \infty)$, then $L^p(L(\Gamma))$ has the OAP, where $L(\Gamma)$ denotes the group von Neumann algebra of Γ. Lafforgue and de la Salle related the AP for groups and the OAP for noncommutative L^p-spaces to the $\mathrm{AP}_{p,\mathrm{cb}}^{\mathrm{Schur}}$.

Lemma 2.12. *(See [24, Corollary 3.12].) If Γ is a countable discrete group with the AP, and if $p \in (1, \infty)$, then $\Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(\Gamma) = 1$.*

Lemma 2.13. *(See [24, Corollary 3.13].) If $p \in (1, \infty)$ and Γ is a countable discrete group such that $L^p(L(\Gamma))$ has the OAP, then $\Lambda_{p,\mathrm{cb}}^{\mathrm{Schur}}(\Gamma) = 1$.*

One of the main results of Lafforgue and de la Salle is the following.

Theorem 2.14. *(See [24, Theorem E].) Let $n \geqslant 3$. For $p \in [1, \frac{4}{3}) \cup (4, \infty]$, the group $\mathrm{SL}(n, \mathbb{R})$ does not have the $\mathrm{AP}_{p,\mathrm{cb}}^{\mathrm{Schur}}$.*

As a consequence, the group $\mathrm{SL}(n, \mathbb{R})$ does not have the AP, and for $p \in [1, \frac{4}{3}) \cup (4, \infty]$ and a lattice Γ in $\mathrm{SL}(n, \mathbb{R})$, the noncommutative L^p-space $L^p(L(\Gamma))$ does not have the OAP or CBAP.

3. The group $\mathrm{Sp}(2, \mathbb{R})$

In this section, we prove the following theorem. The proof is along the same lines as the proof of the failure of the AP for $\mathrm{Sp}(2, \mathbb{R})$ in [15] (and for some details we will refer to that article), but obtaining sufficiently sharp estimates for Schur multipliers on Schatten classes is technically more involved.

Theorem 3.1. *For $p \in [1, \frac{12}{11}) \cup (12, \infty]$, the group $\mathrm{Sp}(2, \mathbb{R})$ does not have the $\mathrm{AP}_{p,\mathrm{cb}}^{\mathrm{Schur}}$.*

In this section, we write $G = \mathrm{Sp}(2, \mathbb{R})$. Recall that G is defined as the Lie group

$$G := \{g \in \mathrm{GL}(4, \mathbb{R}) \mid g^t J g = J\},$$

where

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Here I_2 denotes the 2×2 identity matrix. The maximal compact subgroup K of G is isomorphic to $\mathrm{U}(2)$ and explicitly given by

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathrm{M}_4(\mathbb{R}) \,\middle|\, A + iB \in \mathrm{U}(2) \right\}.$$

Let $\overline{A^+} = \{D(\alpha_1, \alpha_2) = \mathrm{diag}(e^{\alpha_1}, e^{\alpha_2}, e^{-\alpha_1}, e^{-\alpha_2}) \mid \alpha_1 \geqslant \alpha_2 \geqslant 0\}$. It follows that $G = K\overline{A^+}K$.

For $p = 1$ and ∞, the $\mathrm{AP}_{p,\mathrm{cb}}^{\mathrm{Schur}}$ is equivalent to weak amenability (as mentioned in Proposition 2.11), and the failure of weak amenability for G was proved in [13]. Therefore, we can restrict ourselves to the case $p \in (1, \infty)$. As follows from Proposition 2.11, it suffices to consider approximating nets consisting of K-bi-invariant functions. The following result gives a certain asymptotic behaviour of continuous K-bi-invariant functions φ for which the induced function $\check{\varphi}$ is a Schur multiplier on $S^p(L^2(G))$. From this, it follows that the constant function 1 cannot be approximated pointwise (and hence not uniformly on compacta) by a K-bi-invariant net in $A(G)$ in such a way that the net of associated multipliers is uniformly bounded in the $MS^p(L^2(G))$-norm. This implies Theorem 3.1.

Proposition 3.2. *Let $p > 12$. There exist constants $C_1(p), C_2(p)$ (depending on p only) such that for all $\varphi \in C(K \setminus G/K)$ for which $\check{\varphi} \in MS^p(L^2(G))$, the limit $\varphi_\infty = \lim_{\|\alpha\| \to \infty} \varphi(D(\alpha_1, \alpha_2))$ exists, and for all $\alpha_1 \geqslant \alpha_2 \geqslant 0$,*

$$\left|\varphi\big(D(\alpha_1, \alpha_2)\big) - \varphi_\infty\right| \leqslant C_1(p) \|\check{\varphi}\|_{MS^p(L^2(G))} e^{-C_2(p)\|\alpha\|_2},$$

where $\|\alpha\|_2 = \sqrt{\alpha_1^2 + \alpha_2^2}$.

Remark 3.3. Note that Proposition 3.2 is stated in terms of the $MS^p(L^2(G))$-norm rather than the $cbMS^p(L^2(G))$-norm. However, we have $\|.\|_{MS^p(L^2(G))} \leqslant \|.\|_{cbMS^p(L^2(G))}$, which shows that Proposition 3.2 is indeed sufficient to prove Theorem 3.1. Moreover, by [24, Theorem 1.18], the claims are equivalent for non-discrete groups.

For the proof of Proposition 3.2, we will identify two Gelfand pairs in G and describe certain properties of their spherical functions.

Consider the group $\mathrm{U}(2)$, which contains the circle group $\mathrm{U}(1)$ as a subgroup via the embedding

$$\mathrm{U}(1) \hookrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{U}(1) \end{pmatrix} \subset \mathrm{U}(2).$$

Let K_1 denote the copy of $\mathrm{U}(1)$ in G under the identification of $\mathrm{U}(2)$ with K. It goes back to Weyl [31] that $(\mathrm{U}(2), \mathrm{U}(1))$ is a Gelfand pair (see, e.g., [21, Theorem IX.9.14]). The homogeneous space $\mathrm{U}(2)/\mathrm{U}(1)$ is homeomorphic to the complex 1-sphere $S^1_{\mathbb{C}} \subset \mathbb{C}^2$ and the double coset space $\mathrm{U}(1) \setminus \mathrm{U}(2)/\mathrm{U}(1)$ is homeomorphic to the closed unit disc $\overline{\mathbb{D}} \subset \mathbb{C}$ by the map

$$\mathrm{U}(1) \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \mathrm{U}(1) \mapsto u_{11}.$$

The spherical functions for $(\mathrm{U}(2), \mathrm{U}(1))$ can be found in [22]. By the homeomorphism $\mathrm{U}(1) \setminus \mathrm{U}(2)/\mathrm{U}(1) \cong \overline{\mathbb{D}}$, they can be considered as functions of one complex variable in the closed unit disc. They are indexed by the integers $l, m \geqslant 0$ and explicitly given by

$$h_{l,m} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = h^0_{l,m}(u_{11}),$$

where in the point $z \in \overline{\mathbb{D}}$, the function $h^0_{l,m}$ is explicitly given by

$$h^0_{l,m}(z) = \begin{cases} z^{l-m} P_m^{(0,l-m)}(2|z|^2 - 1), & l \geqslant m, \\ \bar{z}^{m-l} P_l^{(0,m-l)}(2|z|^2 - 1), & l < m. \end{cases}$$

Here $P_n^{(\alpha,\beta)}$ denotes the nth Jacobi polynomial. These spherical functions satisfy a certain Hölder continuity condition, as is stated in the following lemma (see [15, Corollary 3.5]). The proof of this lemma makes use of recent results by Haagerup and Schlichtkrull [16].

Lemma 3.4. *For all $l,m \geqslant 0$, and for $\theta_1,\theta_2 \in [0,2\pi)$, we have*

$$\left|h^0_{l,m}\left(\frac{e^{i\theta_1}}{\sqrt{2}}\right) - h^0_{l,m}\left(\frac{e^{i\theta_2}}{\sqrt{2}}\right)\right| \leqslant C(l+m+1)^{\frac{3}{4}}|\theta_1-\theta_2|,$$

$$\left|h^0_{l,m}\left(\frac{e^{i\theta_1}}{\sqrt{2}}\right) - h^0_{l,m}\left(\frac{e^{i\theta_2}}{\sqrt{2}}\right)\right| \leqslant 2C(l+m+1)^{-\frac{1}{4}}.$$

Here $C>0$ is a uniform constant. Combining the two, we get

$$\left|h^0_{l,m}\left(\frac{e^{i\theta_1}}{\sqrt{2}}\right) - h^0_{l,m}\left(\frac{e^{i\theta_2}}{\sqrt{2}}\right)\right| \leqslant 2^{\frac{3}{4}}C|\theta_1-\theta_2|^{\frac{1}{4}}.$$

Let $\varphi : \mathrm{U}(2) \to \mathbb{C}$ be a $\mathrm{U}(1)$-bi-invariant continuous function. Then

$$\varphi(u) = \varphi\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22}\end{pmatrix} = \varphi^0(u_{11}), \quad u \in \mathrm{U}(2),\ u_{11} \in \overline{\mathbb{D}},$$

for some continuous function $\varphi^0 : \overline{\mathbb{D}} \to \mathbb{C}$. By Lemma 2.4, we know that $L^2(X) = \bigoplus_{l,m\geqslant 0}\mathcal{H}_{l,m}$, where $X = \mathrm{U}(2)/\mathrm{U}(1) \cong S_1^{\mathbb{C}}$. It is known that $\dim \mathcal{H}_{l,m} = l+m+1$, so, by Proposition 2.7, we get

$$\varphi^0 = \sum_{l,m=0}^{\infty} c_{l,m}(l+m+1)h^0_{l,m},$$

for certain $c_{l,m} \in \mathbb{C}$. Moreover, by the same proposition, we obtain that if $p \in (1,\infty)$, then $(\sum_{l,m\geqslant 0}|c_{l,m}|^p(l+m+1))^{\frac{1}{p}} \leqslant \|\check{\varphi}\|_{MS^p(L^2(\mathrm{U}(2)))}$, where $\check{\varphi}$ is defined as above by $\check{\varphi}(g,h) = \varphi(g^{-1}h)$.

Lemma 3.5. *Let $p>12$, and let $\varphi : \mathrm{U}(2) \to \mathbb{C}$ be a continuous $\mathrm{U}(1)$-bi-invariant function such that $\check{\varphi}$ is an element of $MS^p(L^2(\mathrm{U}(2)))$. Then φ^0 satisfies*

$$\left|\varphi^0\left(\frac{e^{i\theta_1}}{\sqrt{2}}\right) - \varphi^0\left(\frac{e^{i\theta_2}}{\sqrt{2}}\right)\right| \leqslant \tilde{C}(p)\|\check{\varphi}\|_{MS^p(L^2(\mathrm{U}(2)))}|\theta_1-\theta_2|^{\frac{1}{8}-\frac{3}{2p}}$$

for $\theta_1,\theta_2 \in [0,2\pi)$. Here, $\tilde{C}(p)$ is a constant depending only on p.

Proof. Let $p,q \in (1,\infty)$ be such that $\frac{1}{p}+\frac{1}{q}=1$. Then for $\theta_1,\theta_2 \in [0,2\pi)$,

$$\begin{aligned}&\left|\varphi^0\left(\frac{e^{i\theta_1}}{\sqrt{2}}\right) - \varphi^0\left(\frac{e^{i\theta_2}}{\sqrt{2}}\right)\right| \\ &\quad = \sum_{l,m\geqslant 0}|c_{l,m}|(l+m+1)\left|h^0_{l,m}\left(\frac{e^{i\theta_1}}{\sqrt{2}}\right) - h^0_{l,m}\left(\frac{e^{i\theta_2}}{\sqrt{2}}\right)\right|\end{aligned}$$

$$\begin{aligned}&\leqslant \left(\sum_{l,m\geqslant 0}|c_{l,m}|^q(l+m+1)\right)^{\frac{1}{q}}\left(\sum_{l,m\geqslant 0}(l+m+1)\left|h_{l,m}^0\left(\frac{e^{i\theta_1}}{\sqrt{2}}\right)-h_{l,m}^0\left(\frac{e^{i\theta_2}}{\sqrt{2}}\right)\right|^p\right)^{\frac{1}{p}}\\&\leqslant \|\check{\varphi}\|_{MS^q(L^2(\mathrm{U}(2)))}\left(\sum_{l,m\geqslant 0}(l+m+1)\left|h_{l,m}^0\left(\frac{e^{i\theta_1}}{\sqrt{2}}\right)-h_{l,m}^0\left(\frac{e^{i\theta_2}}{\sqrt{2}}\right)\right|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Note that $\|\check{\varphi}\|_{MS^q(L^2(\mathrm{U}(2)))}=\|\check{\varphi}\|_{MS^p(L^2(\mathrm{U}(2)))}$. If we look at the terms of the last sum, we get, using Lemma 3.4 and the fact that $\min\{x,y\}\leqslant x^{\varepsilon}y^{1-\varepsilon}$ for $x,y>0$ and $\varepsilon\in(0,1)$, that

$$\begin{aligned}&(l+m+1)\left|h_{l,m}^0\left(\frac{e^{i\theta_1}}{\sqrt{2}}\right)-h_{l,m}^0\left(\frac{e^{i\theta_2}}{\sqrt{2}}\right)\right|^p\\&\quad\leqslant \min\left\{C^p(l+m+1)^{1+\frac{3}{4}p}|\theta_1-\theta_2|^p,2^pC^p(l+m+n)^{1-\frac{1}{4}p}\right\}\\&\quad\leqslant 2^{p(1-\varepsilon)}C^p|\theta_1-\theta_2|^{p\varepsilon}(l+m+1)^{1+p\varepsilon-\frac{1}{4}p}\end{aligned}$$

for $\varepsilon\in(0,1)$. Hence, the sum converges for $0<\varepsilon<\frac{1}{4}-\frac{3}{p}$. Such an ε only exists for $p>12$. Hence, if $p>12$, and putting $\varepsilon=\frac{1}{2}(\frac{1}{4}-\frac{3}{p})=\frac{1}{8}-\frac{3}{2p}$, then

$$\left|\varphi^0\left(\frac{e^{i\theta_1}}{\sqrt{2}}\right)-\varphi^0\left(\frac{e^{i\theta_2}}{\sqrt{2}}\right)\right|\leqslant \tilde{C}(p)\|\check{\varphi}\|_{MS^p(L^2(\mathrm{U}(2)))}|\theta_1-\theta_2|^{\frac{1}{8}-\frac{3}{2p}}$$

for some constant $\tilde{C}(p)$ depending only on p. □

For $\alpha\in\mathbb{R}$ consider the map $K\to G$ defined by $k\mapsto D_\alpha kD_\alpha$, where $D_\alpha=\mathrm{diag}(e^{\alpha},1,e^{-\alpha},1)$.

Lemma 3.6. *Let $\varphi:G\to\mathbb{C}$ be a continuous K-bi-invariant function such that $\check{\varphi}\in MS^p(L^2(G))$ for some $p\in(1,\infty)$, and for $\alpha\in\mathbb{R}$, let $\psi_\alpha:K\to\mathbb{C}$ be defined by $\psi_\alpha(k)=\varphi(D_\alpha kD_\alpha)$. Then ψ_α is K_1-bi-invariant and satisfies*

$$\|\check{\psi}_\alpha\|_{MS^p(L^2(\mathrm{U}(2)))}\leqslant\|\check{\varphi}\|_{MS^p(L^2(G))}.$$

Proof. Using the fact that the group elements D_α commute with K_1, it follows that for all $k\in K$ and $k_1,k_2\in K_1\subset K_2$,

$$\psi_\alpha(k_1kk_2)=\varphi(D_\alpha k_1kk_2D_\alpha)=\varphi(k_1D_\alpha kD_\alpha k_2)=\varphi(D_\alpha kD_\alpha)=\psi_\alpha(k),$$

so ψ_α is K_1-bi-invariant.

The second part follows by the fact that $D_\alpha KD_\alpha$ is a subset of G and by applying Lemma 2.3. □

From the fact that ψ_α is K_1-bi-invariant, it follows that $\psi_\alpha(u)=\psi_\alpha^0(u_{11})$, where $\psi_\alpha^0:\overline{\mathbb{D}}\to\mathbb{C}$ is a continuous function.

Suppose that $\alpha_1 \geqslant \alpha_2 \geqslant 0$, and let $D(\alpha_1, \alpha_2)$ be as defined above. If we find an element of the form $D_\alpha k D_\alpha$ in $KD(\alpha_1, \alpha_2)K$, we can relate the value of a K-bi-invariant multiplier φ to the value of the multiplier ψ_α that was just defined. This only works for certain $\alpha_1, \alpha_2 \geqslant 0$. It turns out to be sufficient to consider certain candidates for k, namely the ones of the form

$$u = \begin{pmatrix} a+ib & -\sqrt{1-a^2-b^2} \\ \sqrt{1-a^2-b^2} & a-ib \end{pmatrix} \tag{1}$$

with $a^2+b^2 \leqslant 1$. For a proof of the following result, see [15, Lemma 3.9].

Lemma 3.7. *Let $\alpha \geqslant 0$ and $\beta \geqslant \gamma \geqslant 0$. If $u \in K$ is of the form (1) with respect to the identification of K with* U(2)*, then $D_\alpha u D_\alpha \in KD(\beta, \gamma)K$ if and only if*

$$\begin{cases} \sinh\beta \sinh\gamma = \sinh^2\alpha\left(1-a^2-b^2\right), \\ \sinh\beta - \sinh\gamma = \sinh(2\alpha)|a|. \end{cases} \tag{2}$$

Consider the second Gelfand pair sitting inside G, namely the pair of groups (SU(2), SO(2)). Both groups are subgroups of U(2), so under the embedding into G, they give rise to compact Lie subgroups of G. The subgroup corresponding to SU(2) will be called K_2, and the one corresponding to SO(2) will be called K_3. The group K_3 commutes with the group generated by the elements $D'_\alpha = \operatorname{diag}(e^\alpha, e^\alpha, e^{-\alpha}, e^{-\alpha})$, where $\alpha \in \mathbb{R}$. The subgroup $\mathrm{SU}(2) \subset \mathrm{U}(2)$ consists of matrices of the form

$$u = \begin{pmatrix} a+ib & -c+id \\ c+id & a-ib \end{pmatrix},$$

with $a, b, c, d \in \mathbb{R}$ such that $a^2+b^2+c^2+d^2 = 1$.

By [4, Theorem 47.6], the pair (SU(2), SO(2)) is a Gelfand pair. This also follows from [12, Chapter 9]. The homogeneous space SU(2)/SO(2) is the sphere S^2, and the spherical functions on the double coset space $[-1, 1]$ are indexed by $n \geqslant 0$, and given by the Legendre polynomials

$$P_n\left(2\left(a^2+c^2\right)-1\right) = P_n\left(a^2-b^2+c^2-d^2\right).$$

Note that the double cosets of SO(2) in SU(2) are labeled by $a^2-b^2+c^2-d^2$. We use the following estimate (see [15, Lemma 3.11]).

Lemma 3.8. *For all non-negative integers n, and $x, y \in [-\frac{1}{2}, \frac{1}{2}]$,*

$$\left|P_n(x) - P_n(y)\right| \leqslant \left|P_n(x)\right| + \left|P_n(y)\right| \leqslant \frac{4}{\sqrt{n}},$$

$$\left|P_n(x) - P_n(y)\right| \leqslant \left|\int_x^y P'_n(t)\,dt\right| \leqslant 4\sqrt{n}|x-y|.$$

Combining the two, we get

$$\left|P_n(x) - P_n(y)\right| \leqslant 4|x-y|^{\frac{1}{2}}$$

for $x, y \in [-\frac{1}{2}, \frac{1}{2}]$*, i.e., the Legendre polynomials are uniformly Hölder continuous on* $[-\frac{1}{2}, \frac{1}{2}]$ *with exponent* $\frac{1}{2}$.

Let $\varphi : \mathrm{SU}(2) \to \mathbb{C}$ be an SO(2)-bi-invariant continuous function. Then

$$\varphi(u) = \varphi\begin{pmatrix} a+ib & -c+id \\ c+id & a-ib \end{pmatrix} = \varphi^0\bigl(2\bigl(a^2+c^2\bigr)-1\bigr) = \varphi^0\bigl(a^2-b^2+c^2-d^2\bigr),$$

where $u \in \mathrm{U}(2)$, $u_{11} \in \overline{\mathbb{D}}$, and where $\varphi^0 : \overline{\mathbb{D}} \to \mathbb{C}$ is some continuous function. By Lemma 2.4, we know that $L^2(X) = \bigoplus_{n\geqslant 0} \mathcal{H}_n$, where $X = \mathrm{SU}(2)/\mathrm{SO}(2) \cong S^2$. It is known that $\dim \mathcal{H}_n = 2n+1$, so, by Proposition 2.7, we get

$$\varphi^0 = \sum_{n=0}^{\infty} c_n(2n+1)P_n,$$

for certain $c_n \in \mathbb{C}$. Moreover, by the same proposition, we obtain that if $p \in (1,\infty)$, then $(\sum_{n\geqslant 0} |c_n|^p(2n+1))^{\frac{1}{p}} \leqslant \|\check{\varphi}\|_{MS^p(L^2(\mathrm{SU}(2)))}$, where $\check{\varphi}$ is defined as above by $\check{\varphi}(g,h) = \varphi(g^{-1}h)$.

Lemma 3.9. *Let* $p > 4$*, and let* $\varphi : \mathrm{SU}(2) \to \mathbb{C}$ *be a continuous* SO(2)*-bi-invariant function such that* $\check{\varphi} \in MS^p(L^2(\mathrm{SU}(2)))$*. Then* φ^0 *satisfies*

$$\bigl|\varphi^0(\delta_1) - \varphi^0(\delta_2)\bigr| \leqslant \hat{C}(p)\|\varphi\|_{MS^p(L^2(\mathrm{SU}(2)))}|\delta_1-\delta_2|^{\frac{1}{4}-\frac{1}{p}}$$

for $\delta_1, \delta_2 \in [-\frac{1}{2}, \frac{1}{2}]$*. Here* $\hat{C}(p)$ *is a constant depending only on* p.

Proof. Let $p, q \in (1,\infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, and let $\delta_1, \delta_2 \in [-\frac{1}{2}, \frac{1}{2}]$. Then

$$\begin{aligned}
\bigl|\varphi^0(\delta_1) - \varphi^0(\delta_2)\bigr| &= \sum_{n\geqslant 0} |c_n|(2n+1)\bigl|P_n(\delta_1) - P_n(\delta_2)\bigr| \\
&\leqslant \Bigl(\sum_{n\geqslant 0} |c_n|^q(2n+1)\Bigr)^{\frac{1}{q}} \Bigl(\sum_{n\geqslant 0}(2n+1)\bigl|P_n(\delta_1) - P_n(\delta_2)\bigr|^p\Bigr)^{\frac{1}{p}} \\
&\leqslant \|\check{\varphi}\|_{MS^q(L^2(\mathrm{SU}(2)))}\Bigl(\sum_{n\geqslant 0}(2n+1)\bigl|P_n(\delta_1) - P_n(\delta_2)\bigr|^p\Bigr)^{\frac{1}{p}}.
\end{aligned}$$

Note that $\|\check{\varphi}\|_{MS^q(L^2(\mathrm{SU}(2)))} = \|\check{\varphi}\|_{MS^p(L^2(\mathrm{SU}(2)))}$. If we look at the terms of the last sum, we get, using Lemma 3.8 and the fact that $\min\{x,y\} \leqslant x^{\varepsilon}y^{1-\varepsilon}$ for $x, y > 0$ and $\varepsilon \in (0,1)$, that

$$\begin{aligned}
(2n+1)\bigl|P_n(\delta_1) - P_n(\delta_2)\bigr|^p &\leqslant \min\bigl\{4^p(2n+1)n^{-\frac{p}{2}}, 4^p(2n+1)n^{\frac{p}{2}}|\delta_1-\delta_2|^p\bigr\} \\
&\leqslant 4^p(3n)^{1+p\varepsilon-\frac{p}{2}}|\delta_1-\delta_2|^{p\varepsilon}
\end{aligned}$$

for $\varepsilon \in (0,1)$. Hence, the sum converges for $\varepsilon \in (0, \frac{1}{2} - \frac{2}{p})$. Such an ε only exists for $p > 4$. Hence, if $p > 4$, and putting $\varepsilon = \frac{1}{2}(\frac{1}{2} - \frac{2}{p}) = \frac{1}{4} - \frac{1}{p}$, we have

$$\left|\varphi^0(\delta_1) - \varphi^0(\delta_2)\right| \leqslant \hat{C}(p)\|\check{\varphi}\|_{MS^p(L^2(\mathrm{U}(2)))}|\delta_1 - \delta_2|^{\frac{1}{4}-\frac{1}{p}},$$

where $\hat{C}(p)$ is a constant depending only on p. □

For $\alpha \in \mathbb{R}$ consider the map $K \to G$ defined by $k \mapsto D'_\alpha k v D'_\alpha$, where $D'_\alpha = \mathrm{diag}(e^\alpha, e^\alpha, e^{-\alpha}, e^{-\alpha})$ and $v \in Z(K)$ is chosen to be the matrix in K that in the U(2)-representation of K is given by

$$v = \begin{pmatrix} \frac{1}{\sqrt{2}}(1+i) & 0 \\ 0 & \frac{1}{\sqrt{2}}(1+i) \end{pmatrix}. \tag{3}$$

Given a K-bi-invariant multiplier on G, this map gives rise to a K_3-bi-invariant multiplier on K. We state the following result, but omit its proof, as it is similar to the one of Lemma 3.6.

Lemma 3.10. *Let $\varphi : G \to \mathbb{C}$ be a continuous K-bi-invariant function such that $\check{\varphi} \in MS^p(L^2(G))$ for some $p \in (1,\infty)$, and for $\alpha \in \mathbb{R}$ let $\tilde{\chi}_\alpha : K \to \mathbb{C}$ be defined by $\tilde{\chi}_\alpha(k) = \varphi(D'_\alpha k v D'_\alpha)$. Then $\tilde{\chi}_\alpha$ is K_3-bi-invariant and satisfies*

$$\|\check{\tilde{\chi}}_\alpha\|_{MS^p(L^2(K))} \leqslant \|\check{\varphi}\|_{MS^p(L^2(G))}.$$

Consider the restriction $\chi_\alpha = \tilde{\chi}_\alpha|_{K_2}$, which is a K_3-bi-invariant multiplier on K_2. It follows that $\chi_\alpha(u) = \chi_\alpha^0(a^2 - b^2 + c^2 - d^2)$, where $u \in K_2$, and where a, b, c, d are as before, and $\|\check{\chi}_\alpha\|_{MS^p(L^2(K_2))} \leqslant \|\check{\varphi}\|_{MS^p(L^2(G))}$.

Suppose that $\alpha_1 \geqslant \alpha_2 \geqslant 0$ and let $D(\alpha_1, \alpha_2)$ be as defined above. Again, if we find an element of the form $D'_\alpha u v D'_\alpha$ in $KD(\alpha_1, \alpha_2)K$, where now u has to be an element of SU(2), we can relate the value of a K-bi-invariant multiplier φ to the value of the multiplier χ_α. This again only works for certain $\alpha_1, \alpha_2 \geqslant 0$. Consider a general element of SU(2),

$$u = \begin{pmatrix} a+ib & -c+id \\ c+id & a-ib \end{pmatrix},$$

with $a^2 + b^2 + c^2 + d^2 = 1$. For a proof of the following, see [15, Lemma 3.15].

Lemma 3.11. *Let $\alpha \geqslant 0$ and $\beta \geqslant \gamma \geqslant 0$, and let $u, v \in K$ be of the form as in (1) and (3) with respect to the identification of K with* U(2). *Then $D'_\alpha u v D'_\alpha \in KD(\beta,\gamma)K$ if and only if*

$$\begin{cases} \sinh^2\beta + \sinh^2\gamma = \sinh^2(2\alpha), \\ \sinh\beta \sinh\gamma = \frac{1}{2}\sinh^2(2\alpha)|r|, \end{cases}$$

where $r = a^2 - b^2 + c^2 - d^2$.

Now we can combine the results that we obtained for both Gelfand pairs.

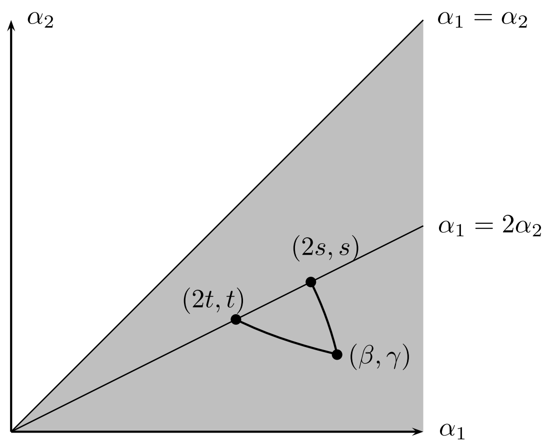

Fig. 1. The figure shows the relative position of (β, γ), $(2s, s)$ and $(2t, t)$ as in Lemmas 3.13 and 3.14.

Lemma 3.12. *Let $\beta \geqslant \gamma \geqslant 0$. Then the equations*

$$\sinh^2(2s) + \sinh^2 s = \sinh^2 \beta + \sinh^2 \gamma,$$
$$\sinh(2t)\sinh t = \sinh\beta \sinh\gamma \tag{4}$$

have unique solutions $s = s(\beta, \gamma)$, $t = t(\beta, \gamma)$ in the interval $[0, \infty)$. *Moreover,*

$$s \geqslant \frac{\beta}{4}, \qquad t \geqslant \frac{\gamma}{2}. \tag{5}$$

A proof of this lemma can be found in [15, Lemma 3.16].

Fig. 1 shows the relative position of (β, γ), $(2s, s)$ and $(2t, t)$ as in Lemmas 3.13 and 3.14 below. Note that (β, γ) and $(2s, s)$ lie on a path in the (α_1, α_2)-plane of the form $\sinh^2 \alpha_1 + \sinh^2 \alpha_2 = \text{constant}$, and (β, γ) and $(2t, t)$ lie on a path of the form $\sinh \alpha_1 \sinh \alpha_2 = \text{constant}$.

Lemma 3.13. *For $p > 4$, there exists a constant $C_3(p) > 0$ (depending only on p) such that whenever $\beta \geqslant \gamma \geqslant 0$ and $s = s(\beta, \gamma)$ is chosen as in Lemma 3.12, then for all $\varphi \in C(K \setminus G/K)$ for which $\check{\varphi} \in MS^p(L^2(G))$,*

$$\left|\varphi\big(D(\beta, \gamma)\big) - \varphi\big(D(2s, s)\big)\right| \leqslant C_3(p) e^{-\frac{\beta-\gamma}{4}(\frac{1}{4}-\frac{1}{p})} \|\check{\varphi}\|_{MS^p(L^2(G))}.$$

Proof. Assume first that $\beta - \gamma \geqslant 8$. Let $\alpha \in [0, \infty)$ be the unique solution to $\sinh^2 \beta + \sinh^2 \gamma = \sinh^2(2\alpha)$, and observe that $2\alpha \geqslant \beta \geqslant 2$, so in particular $\alpha > 0$. Define

$$r_1 = \frac{2\sinh\beta \sinh\gamma}{\sinh^2\beta + \sinh^2\gamma} \in [0, 1],$$

and $a_1 = (\frac{1+r_1}{2})^{\frac{1}{2}}$ and $b_1 = (\frac{1-r_1}{2})^{\frac{1}{2}}$. Furthermore, put

$$u_1 = \begin{pmatrix} a_1 + ib_1 & 0 \\ 0 & a_1 - ib_1 \end{pmatrix} \in \mathrm{SU}(2),$$

and let

$$v = \begin{pmatrix} \frac{1}{\sqrt{2}}(1+i) & 0 \\ 0 & \frac{1}{\sqrt{2}}(1+i) \end{pmatrix},$$

as previously defined. We now have $2\sinh\beta\sinh\gamma = \sinh^2(2\alpha)r_1$, and $a_1^2 - b_1^2 = r_1$, so by Lemma 3.11, we have $D'_\alpha u_1 v D'_\alpha \in KD(\beta,\gamma)K$. Let $s = s(\beta,\gamma)$ be as in Lemma 3.12. Then $s \geqslant 0$ and $\sinh^2(2s) + \sinh^2 s = \sinh^2\beta + \sinh^2\gamma = \sinh^2(2\alpha)$. Put

$$r_2 = \frac{2\sinh(2s)\sinh s}{\sinh^2(2s) + \sinh^2 s} \in [0,1],$$

and

$$u_2 = \begin{pmatrix} a_2 + ib_2 & 0 \\ 0 & a_2 - ib_2 \end{pmatrix} \in \mathrm{SU}(2),$$

where $a_2 = (\frac{1+r_2}{2})^{\frac{1}{2}}$ and $b_2 = (\frac{1-r_2}{2})^{\frac{1}{2}}$. Since $a_2^2 - b_2^2 = r_2$, it follows again by Lemma 3.11 that $D'_\alpha u_2 v D'_\alpha \in KD(2s,s)K$. Now, let $\chi_\alpha(u) = \varphi(D'_\alpha u v D'_\alpha)$ for $u \in K_2 \cong \mathrm{SU}(2)$. Then by Lemmas 3.9 and 3.10, it follows that

$$\left|\chi_\alpha(u_1) - \chi_\alpha(u_2)\right| = \left|\chi_\alpha^0(r_1) - \chi_\alpha^0(r_2)\right| \leqslant \hat{C}(p)|r_1 - r_2|^{\frac{1}{4}-\frac{1}{p}}\|\check{\varphi}\|_{MS^p(L^2(G))},$$

provided that $r_1, r_2 \leqslant \frac{1}{2}$. Hence, under this assumption, using the K-bi-invariance of φ, we get

$$\left|\varphi\big(D(\beta,\gamma)\big) - \varphi\big(D(2s,s)\big)\right| \leqslant \hat{C}(p)|r_1 - r_2|^{\frac{1}{4}-\frac{1}{p}}\|\check{\varphi}\|_{MS^p(L^2(G))}. \tag{6}$$

Note that $r_1 \leqslant \frac{2\sinh\beta\sinh\gamma}{\sinh^2\beta} = 2\frac{\sinh\gamma}{\sinh\beta}$. Hence, using $\beta \geqslant \gamma + 8 \geqslant \gamma$, we get $r_1 \leqslant 2\frac{e^\gamma(1-e^{-2\gamma})}{e^\beta(1-e^{-2\beta})} \leqslant 2e^{\gamma-\beta}$. In particular, $r_1 \leqslant 2e^{-8} \leqslant \frac{1}{2}$. Similarly, $r_2 \leqslant 2\frac{\sinh s}{\sinh 2s} = \frac{1}{\cosh s} \leqslant 2e^{-s}$. By Lemma 3.12, Eq. (5), we obtain that $r_2 \leqslant 2e^{-\frac{\beta}{4}} \leqslant 2e^{\frac{\gamma-\beta}{4}} \leqslant 2e^{-2} \leqslant \frac{1}{2}$. In particular, (6) holds, and since $|r_1 - r_2| \leqslant \max\{r_1, r_2\} \leqslant 2e^{\frac{\gamma-\beta}{4}}$, we have proved that

$$\left|\varphi\big(D(\beta,\gamma)\big) - \varphi\big(D(2s,s)\big)\right| \leqslant \hat{C}(p)2^{\frac{1}{4}-\frac{1}{p}}e^{\frac{\gamma-\beta}{4}(\frac{1}{4}-\frac{1}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))} \tag{7}$$

under the assumption that $\beta \geqslant \gamma + 8$. If $\gamma \leqslant \beta < \gamma + 8$, we get from $\|\varphi\|_\infty \leqslant \|\check{\varphi}\|_{MS^p(L^2(G))}$ that $|\varphi(D(\beta,\gamma)) - \varphi(D(2s,s))| \leqslant 2\|\check{\varphi}\|_{MS^p(L^2(G))}$. It follows that

$$\left|\varphi\big(D(\beta,\gamma)\big) - \varphi\big(D(2s,s)\big)\right| \leqslant C_3(p)e^{\frac{\gamma-\beta}{4}(\frac{1}{4}-\frac{1}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))}$$

for all (β,γ) with $\beta \geqslant \gamma \geqslant 0$, if for all $p \in (1,\infty)$, we put $C_3(p) = \max\{\hat{C}(p)2^{\frac{1}{4}-\frac{1}{p}}, 2e^{\frac{1}{2}}\}$. □

Lemma 3.14. *For $p > 12$, there exists a constant $C_4(p) > 0$ (depending only on p) such that whenever $\beta \geqslant \gamma \geqslant 0$ and $t = t(\beta, \gamma)$ is chosen as in Lemma 3.12, then for all $\varphi \in C(K \setminus G/K)$ for which $\check{\varphi} \in MS^p(L^2(G))$,*

$$\left|\varphi\bigl(D(\beta,\gamma)\bigr) - \varphi\bigl(D(2t,t)\bigr)\right| \leqslant C_4(p) e^{-\frac{\gamma}{4}(\frac{1}{4}-\frac{3}{p})} \|\check{\varphi}\|_{MS^p(L^2(G))}.$$

Proof. Let $\beta \geqslant \gamma \geqslant 0$. Assume first that $\gamma \geqslant 2$, and let $\alpha \geqslant 0$ be the unique solution in $[0,\infty)$ to the equation $\sinh\beta \sinh\gamma = \frac{1}{2}\sinh^2\alpha$, and observe that $\alpha > 0$, because $\beta \geqslant \gamma \geqslant 2$. Put

$$a_1 = \frac{\sinh\beta - \sinh\gamma}{\sinh(2\alpha)} \geqslant 0.$$

Since $\sinh(2\alpha) = 2\sinh\alpha\cosh\alpha \geqslant 2\sinh^2\alpha$, we have

$$a_1 \leqslant \frac{\sinh\beta}{\sinh(2\alpha)} \leqslant \frac{\sinh\beta}{2\sinh^2\alpha} = \frac{1}{4\sinh\gamma}.$$

In particular, $a_1 \leqslant \frac{1}{4\gamma} \leqslant \frac{1}{8}$. Put now $b_1 = \sqrt{\frac{1}{2} - a_1^2}$. Then $1 - a_1^2 - b_1^2 = \frac{1}{2}$. Hence, $\sinh\beta\sinh\gamma = \sinh^2\alpha(1 - a_1^2 - b_1^2)$ and $\sinh\beta - \sinh\gamma = \sinh(2\alpha)a_1$. Let

$$u_1 = \begin{pmatrix} a_1 + ib_1 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & a_1 - ib_1 \end{pmatrix} \in \mathrm{SU}(2).$$

By Lemma 3.7, we have $D_\alpha u_1 D_\alpha \in K D(\beta,\gamma) K$.

By Lemma 3.12, we have $\sinh(2t)\sinh t = \sinh\beta\sinh\gamma = \frac{1}{2}\sinh^2\alpha$. Moreover, by (5), we have $t \geqslant \frac{\gamma}{2} \geqslant 1$. By replacing (β,γ) in the above calculation with $(2t, t)$, we get that the number

$$a_2 = \frac{\sinh(2t) - \sinh t}{\sinh(2\alpha)} \geqslant 0,$$

satisfies

$$a_2 \leqslant \frac{1}{4\sinh t} \leqslant \frac{1}{4\sinh 1} \leqslant \frac{1}{4}.$$

Hence, we can put $b_2 = \sqrt{\frac{1}{2} - a_2^2}$ and

$$u_2 = \begin{pmatrix} a_2 + ib_2 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & a_2 - ib_2 \end{pmatrix}.$$

Then

$$\sinh(2t)\sinh t = \sinh^2\alpha\bigl(1 - a_2^2 - b_2^2\bigr),$$
$$\sinh(2t) - \sinh t = \sinh(2\alpha)a_2,$$

and $u_2 \in \mathrm{SU}(2)$. Hence, by Lemma 3.7, $D_\alpha u_2 D_\alpha \in KD(2t,t)K$. Put now $\theta_j = \arg(a_j + ib_j) = \frac{\pi}{2} - \sin^{-1}(\frac{a_j}{\sqrt{2}})$ for $j=1,2$. Since $0 \leqslant a_j \leqslant \frac{1}{2}$ for $j=1,2$, and since $\frac{d}{dt}\sin^{-1} t = \frac{1}{\sqrt{1-t^2}} \leqslant \sqrt{2}$ for $t \in [0, \frac{1}{\sqrt{2}}]$, it follows that

$$\begin{aligned}
|\theta_1 - \theta_2| &\leqslant \left|\sin^{-1}\left(\frac{a_1}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{a_2}{\sqrt{2}}\right)\right| \\
&\leqslant |a_1 - a_2| \\
&\leqslant \max\{a_1, a_2\} \\
&\leqslant \max\left\{\frac{1}{4\sinh\gamma}, \frac{1}{4\sinh t}\right\} \\
&\leqslant \frac{1}{4\sinh\frac{\gamma}{2}},
\end{aligned}$$

because $t \geqslant \frac{\gamma}{2}$. Since $\gamma \geqslant 2$, we have $\sinh\frac{\gamma}{2} = \frac{1}{2}e^{\frac{\gamma}{2}}(1 - e^{-\gamma}) \geqslant \frac{1}{4}e^{\frac{\gamma}{2}}$. Hence, $|\theta_1 - \theta_2| \leqslant e^{-\frac{\gamma}{2}}$. Note that $a_j = \frac{1}{\sqrt{2}}e^{i\theta_j}$ for $j=1,2$, so by Lemmas 3.5 and 3.6, the function $\psi_\alpha(u) = \varphi(D_\alpha u D_\alpha)$, $u \in \mathrm{U}(2) \cong K$ satisfies

$$\begin{aligned}
\left|\psi_\alpha(u_1) - \psi_\alpha(u_2)\right| &\leqslant \tilde{C}(p)|\theta_1 - \theta_2|^{\frac{1}{8} - \frac{3}{2p}} \|\check{\psi}_\alpha\|_{MS^p(L^2(K))} \\
&\leqslant \tilde{C}(p)e^{-\frac{\gamma}{4}(\frac{1}{4} - \frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))}.
\end{aligned} \tag{8}$$

Since $D_\alpha u_1 D_\alpha \in KD(\beta,\gamma)K$ and $D_\alpha u_2 D_\alpha \in KD(2t,t)K$, it follows that

$$\left|\varphi\big(D(\beta,\gamma)\big) - \varphi\big(D(2t,t)\big)\right| \leqslant \tilde{C}(p)e^{-\frac{\gamma}{4}(\frac{1}{4} - \frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))}$$

for all $\gamma \geqslant 2$. For γ satisfying $0 < \gamma \leqslant 2$, we can instead use that $\|\varphi\|_\infty \leqslant \|\check{\varphi}\|_{MS^p(L^2(G))}$. Hence, for all $p \in (1,\infty)$ putting $C_4(p) = \max\{\tilde{C}(p), 2e^{\frac{1}{8}}\}$, we obtain

$$\left|\varphi\big(D(\beta,\gamma)\big) - \varphi\big(D(2t,t)\big)\right| \leqslant C_4(p)e^{-\frac{\gamma}{4}(\frac{1}{4} - \frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))}$$

for all $\beta \geqslant \gamma \geqslant 0$. $\square$

For a proof of the following lemma, see [15, Lemma 3.19].

Lemma 3.15. *Let $s \geqslant t \geqslant 0$. Then the equations*

$$\begin{aligned}
\sinh^2\beta + \sinh^2\gamma &= \sinh^2(2s) + \sinh^2 s, \\
\sinh\beta\sinh\gamma &= \sinh(2t)\sinh t,
\end{aligned} \tag{9}$$

have a unique solution $(\beta,\gamma) \in \mathbb{R}^2$ for which $\beta \geqslant \gamma \geqslant 0$. Moreover, if $1 \leqslant t \leqslant s \leqslant \frac{3t}{2}$, then

$$|\beta - 2s| \leqslant 1,$$
$$|\gamma + 2s - 3t| \leqslant 1. \tag{10}$$

Lemma 3.16. *For all $p > 12$, there exists a constant $C_5(p) > 0$ such that whenever $s, t \geqslant 0$ satisfy $2 \leqslant t \leqslant s \leqslant \frac{6}{5}t$, then for all $\varphi \in C(K \setminus G/K)$ for which $\check{\varphi} \in MS^p(L^2(G))$,*

$$\left|\varphi\big(D(2s,s)\big) - \varphi\big(D(2t,t)\big)\right| \leqslant C_5(p)e^{-\frac{s}{8}(\frac{1}{4}-\frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))}.$$

Proof. Choose $\beta \geqslant \gamma \geqslant 0$ as in Lemma 3.15. Then by Lemmas 3.13 and 3.14, we have for $p > 12$,

$$\left|\varphi\big(D(2s,s)\big) - \varphi\big(D(\beta,\gamma)\big)\right| \leqslant C_3(p)e^{-\frac{\beta-\gamma}{4}(\frac{1}{4}-\frac{1}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))},$$
$$\left|\varphi\big(D(2t,t)\big) - \varphi\big(D(\beta,\gamma)\big)\right| \leqslant C_4(p)e^{-\frac{\gamma}{4}(\frac{1}{4}-\frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))}.$$

Moreover, by (10),

$$\beta - \gamma \geqslant (2s-1) - (3t - 2s + 1) = 4s - 3t - 2 \geqslant s - 2,$$
$$\gamma \geqslant 3t - 2s - 1 \geqslant \frac{5}{2}s - 2s - 1 = \frac{s-2}{2}.$$

Hence, since $s \geqslant 2$, we have $\min\{e^{-\gamma}, e^{-(\beta-\gamma)}\} \leqslant e^{-\frac{s-2}{2}}$. Thus, the lemma follows from Lemmas 3.13 and 3.14 with $C_5(p) = e^{\frac{1}{16}}(C_3(p) + C_4(p))$. $\square$

Lemma 3.17. *For $p > 12$, there exists a constant $C_6(p) > 0$ such that for all $\varphi \in C(K \setminus G/K)$ for which $\check{\varphi} \in MS^p(L^2(G))$, the limit $c_\infty(\varphi) = \lim_{t\to\infty} \varphi(D(2t,t))$ exists, and for all $t \geqslant 0$,*

$$\left|\varphi\big(D(2t,t)\big) - c_\infty(\varphi)\right| \leqslant C_6(p)e^{-\frac{t}{8}(\frac{1}{4}-\frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))}.$$

Proof. By Lemma 3.16, we have for $u \geqslant 5$ and $\gamma \in [0,1]$, that

$$\left|\varphi\big(D(2u,u)\big) - \varphi\big(D(2u+2\gamma, u+\gamma)\big)\right| \leqslant C_5(p)e^{-\frac{u}{8}(\frac{1}{4}-\frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))}, \tag{11}$$

since $u \leqslant u + \gamma$. Let $s \geqslant t \geqslant 5$. Then $s = t + n + \delta$, where $n \geqslant 0$ is an integer and $\delta \in [0,1)$. Applying Eq. (11) to $(u,\gamma) = (t+j, 1)$, $j = 0, 1, \ldots, n-1$ and $(u,\gamma) = (t+n, \delta)$, we obtain

$$\begin{aligned}\left|\varphi\big(D(2t,t)\big) - \varphi\big(D(2s,s)\big)\right| &\leqslant C_5(p)\left(\sum_{j=0}^{n} e^{-\frac{t+j}{8}(\frac{1}{4}-\frac{3}{p})}\right)\|\check{\varphi}\|_{MS^p(L^2(G))} \\ &\leqslant C_5(p)' e^{-\frac{t}{8}(\frac{1}{4}-\frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))},\end{aligned}$$

where $C_5'(p) = C_5(p)\sum_{j=0}^{\infty} e^{-\frac{j}{8}(\frac{1}{4}-\frac{3}{p})}$. Hence, $(\varphi(D(2t,t)))_{t\geqslant 5}$ is a Cauchy net. Therefore, $c_\infty(\varphi) = \lim_{t\to\infty}\varphi(D(2t,t))$ exists, and

$$\left|\varphi\big(D(2t,t)\big) - c_\infty(\varphi)\right| = \lim_{s\to\infty}\left|\varphi\big(D(2t,t)\big) - \varphi\big(D(2s,s)\big)\right| \leqslant C_5'(p)e^{-\frac{t}{8}(\frac{1}{4}-\frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))}$$

for all $t \geqslant 5$. Since $\|\varphi\|_\infty \leqslant \|\check{\varphi}\|_{MS^p(L^2(G))}$, we have for all $0 \leqslant t < 5$,

$$\left|\varphi\big(D(2t,t)\big) - c_\infty(\varphi)\right| \leqslant 2\|\check{\varphi}\|_{MS^p(L^2(G))}.$$

Hence, the lemma follows with $C_6(p) = \max\{C_5'(p), 2e^{\frac{5}{32}}\}$. $\square$

Proof of Proposition 3.2. Let $\varphi \in C(K \setminus G/K)$ be such that $\check{\varphi} \in MS^p(L^2(G))$, and let $(\alpha_1, \alpha_2) = (\beta, \gamma)$, where $\beta \geqslant \gamma \geqslant 0$. Assume first $\beta \geqslant 2\gamma$. Then $\beta - \gamma \geqslant \frac{\beta}{2}$, so by Lemmas 3.12 and 3.13, there exists an $s \geqslant \frac{\beta}{4}$ such that

$$\left|\varphi\big(D(\beta,\gamma)\big) - \varphi\big(D(2s,s)\big)\right| \leqslant C_3(p)e^{-\frac{\beta}{8}(\frac{1}{4}-\frac{1}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))}.$$

By Lemma 3.17,

$$\begin{aligned}\left|\varphi\big(D(2s,s)\big) - c_\infty(\varphi)\right| &\leqslant C_6(p)e^{-\frac{s}{8}(\frac{1}{4}-\frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))} \\ &\leqslant C_6(p)e^{-\frac{\beta}{32}(\frac{1}{4}-\frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))}.\end{aligned}$$

Hence,

$$\left|\varphi\big(D(\beta,\gamma)\big) - c_\infty(\varphi)\right| \leqslant \big(C_3(p) + C_6(p)\big)e^{-\frac{\beta}{32}(\frac{1}{4}-\frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))}.$$

Assume now that $\beta < 2\gamma$. Then, by Lemmas 3.12 and 3.14, we obtain that there exists a $t \geqslant \frac{\gamma}{2} > \frac{\beta}{4}$ such that

$$\left|\varphi\big(D(\beta,\gamma)\big) - \varphi\big(D(2t,t)\big)\right| \leqslant C_4(p)e^{-\frac{\beta}{8}(\frac{1}{4}-\frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))},$$

and again by Lemma 3.17,

$$\begin{aligned}\left|\varphi\big(D(2t,t)\big) - c_\infty(\varphi)\right| &\leqslant C_6(p)e^{-\frac{t}{8}(\frac{1}{4}-\frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))} \\ &\leqslant C_6(p)e^{-\frac{\beta}{32}(\frac{1}{4}-\frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))}.\end{aligned}$$

Hence,

$$\left|\varphi\big(D(\beta,\gamma)\big) - c_\infty(\varphi)\right| \leqslant \big(C_4(p) + C_6(p)\big)e^{-\frac{\beta}{32}(\frac{1}{4}-\frac{3}{p})}\|\check{\varphi}\|_{MS^p(L^2(G))}.$$

Combining these results, and using that $\|\alpha\|_2 = \sqrt{\beta^2+\gamma^2} \leqslant \sqrt{2}\beta$, it follows that for all $\beta \geqslant \gamma \geqslant 0$,

$$\left|\varphi\big(D(\beta,\gamma)\big) - c_\infty(\varphi)\right| \leqslant C_1(p)e^{-C_2(p)\|\alpha\|_2}\|\check{\varphi}\|_{MS^p(L^2(G))},$$

where $C_1(p) = \max\{C_3(p) + C_6(p), C_4(p) + C_6(p)\}$ and $C_2(p) = \frac{1}{32\sqrt{2}}(\frac{1}{4} - \frac{3}{p})$. This proves the proposition. $\square$

The values $p \in [1, \frac{12}{11}) \cup (12, \infty]$ give sufficient conditions for $\mathrm{Sp}(2, \mathbb{R})$ to fail the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$. We would like to point out that the set of these values might be bigger.

4. Noncommutative L^p-spaces without the OAP

In the previous section we proved that $\mathrm{Sp}(2, \mathbb{R})$ does not have the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ for $p \in [1, \frac{12}{11}) \cup (12, \infty]$. By Lemma 2.13, this directly implies the following theorem.

Theorem 4.1. *Let $p \in [1, \frac{12}{11}) \cup (12, \infty]$, and let Γ be a lattice in $\mathrm{Sp}(2, \mathbb{R})$. Then the noncommutative L^p-space $L^p(L(\Gamma))$ does not have the OAP (or CBAP).*

Combining Theorem 3.1 and Theorem 2.14, this implies the following result.

Theorem 4.2. *Let $p \in [1, \frac{12}{11}) \cup (12, \infty]$, and let G be a connected simple Lie group with finite center and real rank greater than or equal to two. Then G does not have the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$.*

Proof. Let G be a connected simple Lie group with finite center and real rank greater than or equal to two. By Wang's method [30], we may assume that G is the adjoint group, so that G has a connected semisimple subgroup H with real rank 2. Such a subgroup is closed, as was proved in [8]. It is known that H has finite center and is locally isomorphic to either $\mathrm{SL}(3, \mathbb{R})$ or $\mathrm{Sp}(2, \mathbb{R})$ [2,26]. Since the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ passes to closed subgroups and is preserved under local isomorphisms (see Proposition 2.11), we conclude that G does not have the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ for $p \in [1, \frac{12}{11}) \cup (12, \infty]$, since both $\mathrm{SL}(3, \mathbb{R})$ and $\mathrm{Sp}(2, \mathbb{R})$ do not have the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ for such p. $\square$

Combining this result with Proposition 2.10 and Lemma 2.13, we obtain the main theorem of this article.

Theorem 4.3. *Let $p \in [1, \frac{12}{11}) \cup (12, \infty]$, and let Γ be a lattice in a connected simple Lie group with finite center and real rank greater than or equal to two. Then $L^p(L(\Gamma))$ does not have OAP (or CBAP).*

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APPENDIX C

Simple Lie groups without the Approximation Property II

This chapter contains the preprint version of the following article:

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SIMPLE LIE GROUPS WITHOUT THE APPROXIMATION PROPERTY II

UFFE HAAGERUP AND TIM DE LAAT

Abstract. We prove that the universal covering group $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ of $\mathrm{Sp}(2,\mathbb{R})$ does not have the Approximation Property (AP). Together with the fact that $\mathrm{SL}(3,\mathbb{R})$ does not have the AP, which was proved by Lafforgue and de la Salle, and the fact that $\mathrm{Sp}(2,\mathbb{R})$ does not have the AP, which was proved by the authors of this article, this finishes the description of the AP for connected simple Lie groups. Indeed, it follows that a connected simple Lie group has the AP if and only if its real rank is zero or one. By an adaptation of the methods we use to study the AP, we obtain results on approximation properties for noncommutative L^p-spaces associated with lattices in $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$. Combining this with earlier results of Lafforgue and de la Salle and results of the second named author of this article, this gives rise to results on approximation properties of noncommutative L^p-spaces associated with lattices in any connected simple Lie group.

1. Introduction

This is the second article of the authors on the Approximation Property (AP) for Lie groups. In the first article on this topic, the authors proved that $\mathrm{Sp}(2,\mathbb{R})$ does not satisfy the AP [21]. Together with the earlier established fact that $\mathrm{SL}(3,\mathbb{R})$ does not have the AP, which was proved by Lafforgue and de la Salle in [31], this implied that if G is a connected simple Lie group with finite center and real rank greater than or equal to two, then G does not satisfy the AP. In [21], it was pointed out that in order to extend this result to the class of connected simple Lie groups with real rank greater than or equal to two, i.e., not necessarily with finite center, it would be sufficient to prove that the universal covering group $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ of $\mathrm{Sp}(2,\mathbb{R})$ does not satisfy the AP. The main goal of this article is to prove this. This finishes the description of the AP for connected simple Lie groups. Indeed, it follows that a connected simple Lie group has the AP if and only if its real rank is zero or one.

In this article we are mainly interested in Lie groups, but many definitions are given in the setting of locally compact groups. We always assume locally compact groups to be second countable and Hausdorff. Before we state the main results of this article, we give some background (see Section 1 of [21] for a more extensive account of the background).

Let G be a locally compact group. Denote by $A(G)$ its Fourier algebra and by $M_0A(G)$ the space of completely bounded Fourier multipliers on G. Recall that G

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is said to have the Approximation Property for groups (AP) if there is a net (φ_α) in the Fourier algebra $A(G)$ such that $\varphi_\alpha \to 1$ in the $\sigma(M_0A(G), M_0A(G)_*)$-topology, where $M_0A(G)_*$ denotes the natural predual of $M_0A(G)$, as introduced in [4].

The AP was defined by the first named author and Kraus in [20] as a version for groups of the Banach space approximation property (BSAP) of Grothendieck. To see the connection, recall first that Banach spaces have a natural noncommutative analogue, namely, operator spaces. Recall that an operator space E is a closed linear subspace of the bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. Operator spaces have a remarkably rich structure (see [11], [36]). For the class of operator spaces, which contains the class of C^*-algebras, a well-known version of the BSAP is known, namely, the operator space approximation property (OAP). The first named author and Kraus proved that a discrete group Γ has the AP if and only if its reduced C^*-algebra $C^*_\lambda(\Gamma)$ has the OAP.

The AP also relates to other approximation properties for groups (see [3] for an extensive text on approximation properties for groups and operator algebras). It is known that weak amenability (which is strictly weaker than amenability) strictly implies the AP. Amenability and weak amenability have been studied thoroughly for Lie groups. Indeed, a connected simple Lie group with real rank zero is amenable and a connected simple Lie group with real rank one is weakly amenable (see [7] and [23]). Also, it has been known for some time that connected simple Lie groups with real rank greater than or equal to two are not weakly amenable (see [19] and [9]). In addition, weak amenability was studied for a larger class of connected Lie groups in [6]. The AP has been less studied than weak amenability. In particular, until the work of Lafforgue and de la Salle, no example of an exact group without the AP was known.

The key theorem of this article is as follows.

Theorem 3.2. The universal covering group $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ of the symplectic group $\mathrm{Sp}(2,\mathbb{R})$ does not have the Approximation Property.

Combining this with the fact that $\mathrm{SL}(3,\mathbb{R})$ does not have the AP, as established by Lafforgue and de la Salle, and the fact that $\mathrm{Sp}(2,\mathbb{R})$ does not have the AP, as proved by the authors, the following main result follows.

Theorem 5.1. Let G be a connected simple Lie group. Then G has the Approximation Property if and only if G has real rank zero or one.

There are important differences between the approach of Lafforgue and de la Salle for the proof of the fact that $\mathrm{SL}(3,\mathbb{R})$ does not have the AP in [31] and the approach of the authors for proving the failure of the AP for $\mathrm{Sp}(2,\mathbb{R})$ in [21] and for its universal covering group in this article. Indeed, the method of Lafforgue and de la Salle gives information about approximation properties for certain noncommutative L^p-spaces associated with lattices in $\mathrm{SL}(3,\mathbb{R})$, which the method of the authors does not. However, the latter is more direct, since it suffices to consider completely bounded Fourier multipliers rather than completely bounded multipliers on Schatten classes.

Noncommutative L^p-spaces are important examples of the earlier mentioned operator spaces. Let M be a finite von Neumann algebra with normal faithful trace τ. For $1 \leq p < \infty$, the noncommutative L^p-space $L^p(M,\tau)$ is defined as the completion of M with respect to the norm $\|x\|_p = \tau((x^*x)^{\frac{p}{2}})^{\frac{1}{p}}$, and for $p = \infty$, we put $L^\infty(M,\tau) = M$ (with operator norm). Noncommutative L^p-spaces can be realized by interpolating between M and $L^1(M,\tau)$ (see [28]). This leads to an operator space structure on them (see [34],[25]).

An operator space E is said to have the completely bounded approximation property (CBAP) if there exists a net F_α of finite-rank maps on E with $\sup_\alpha \|F_\alpha\|_{cb} < C$ for some $C > 0$ such that $\lim_\alpha \|F_\alpha x - x\| = 0$ for every $x \in E$. The infimum of all possible C's is denoted by $\Lambda(E)$. If $\Lambda(E) = 1$, then E has the completely contractive approximation property (CCAP). An operator space E is said to have the operator space approximation property (OAP) if there exists a net F_α of finite-rank maps on E such that $\lim_\alpha \|(\mathrm{id}_{\mathcal{K}(\ell^2)} \otimes F_\alpha)x - x\| = 0$ for all $x \in \mathcal{K}(\ell^2) \otimes_{\min} E$. Here, $\mathcal{K}(\ell^2)$ denotes the space of compact operators on ℓ^2. The CBAP goes back to [4], and the OAP was defined in [10]. By definition, the CCAP implies the CBAP, which in turn implies the OAP.

It was shown by Junge and Ruan [25] that if Γ is a weakly amenable countable discrete group (resp. a countable discrete group with the AP), and if $p \in (1,\infty)$, then $L^p(L(\Gamma))$ has the CBAP (resp. the OAP), where $L(\Gamma)$ denotes the group von Neumann algebra of Γ. The method of Lafforgue and de la Salle can be used to prove the failure of the CBAP and OAP for noncommutative L^p-spaces. The key ingredient of their method is the property of completely bounded approximation by Schur multipliers on S^p, denoted $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$, which is weaker than the AP for $p \in (1,\infty)$. Indeed, they prove that if $p \in (1,\infty)$ and Γ is a countable discrete group with the AP, then $\Lambda^{\mathrm{Schur}}_{p,\mathrm{cb}}(\Gamma) = 1$ (see [31, Corollary 3.12]). Also, they prove that if $p \in (1,\infty)$ and Γ is a countable discrete group such that $L^p(L(\Gamma))$ has the OAP, then $\Lambda^{\mathrm{Schur}}_{p,\mathrm{cb}}(\Gamma) = 1$ (see [31, Corollary 3.13]). Using this, they prove that for $p \in [1, \frac{4}{3}) \cup (4,\infty]$ and a lattice Γ in $\mathrm{SL}(3,\mathbb{R})$, the noncommutative L^p-space $L^p(L(\Gamma))$ does not have the OAP or CBAP.

In [29], the second named author generalized the results of Lafforgue and de la Salle on approximation properties for noncommutative L^p-spaces associated with lattices in $\mathrm{SL}(3,\mathbb{R})$ to noncommutative L^p-spaces associated with lattices in connected simple Lie groups with finite center and real rank greater than or equal to two. In this article, we will in turn generalize these results to connected simple Lie groups with real rank greater than or equal to two that do not necessarily have finite center, as is illustrated by our main result on noncommutative L^p-spaces.

Theorem 5.3. Let Γ be a lattice in a connected simple Lie group with real rank greater than or equal to two. For $p \in [1, \frac{12}{11}) \cup (12,\infty]$, the noncommutative L^p-space $L^p(L(\Gamma))$ does not have the OAP or CBAP.

It may very well be possible that the range of p-values for which the CBAP and OAP fail is larger than $[1, \frac{12}{11}) \cup (12,\infty]$. We will comment on this in further detail in Section 5.

This article is organized as follows. In Section 2, we recall some preliminaries. In Section 3, we prove that $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ does not have the AP. We prove the results on noncommutative L^p-spaces in Section 4. The results will be summarized and combined to our general results in Section 5. Appendix A gives a connection between spherical functions for Gelfand pairs and their analogues for strong Gelfand pairs that might give a deeper understanding of certain results that are proved in Section 3. The material in that appendix follows from discussions of the second named author with Thomas Danielsen. This material might be known to experts, but we could not find an explicit reference.

2. Preliminaries

2.1. **Universal covering groups.** Let G be a connected Lie group. A covering group of G is a Lie group $\widetilde{G}$ with a surjective Lie group homomorphism $\sigma : \widetilde{G} \to G$, in such a way that $(\widetilde{G}, \sigma)$ is a covering space of G (in the topological sense). A simply connected covering space is called a universal covering space. Every connected Lie group G has a universal covering space $\widetilde{G}$. Let $\sigma : \widetilde{G} \to G$ be the corresponding covering map, and let $\tilde{1} \in \sigma^{-1}(1)$. Then there exists a unique multiplication on $\widetilde{G}$ that makes $\widetilde{G}$ into a Lie group in such a way that σ is a surjective Lie group homomorphism. The group $\widetilde{G}$ is called a universal covering group of the Lie group G. Universal covering groups of connected Lie groups are unique up to isomorphism. They also satisfy the exact sequence $1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$, where $\pi_1(G)$ denotes the fundamental group of G. For details on universal covering groups, see [26, Section I.11].

2.2. **Polar decomposition of Lie groups.** Every connected semisimple Lie group G has a polar decomposition $G = KAK$, where K arises from a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (the group K has Lie algebra $\mathfrak{k}$), and A is an abelian Lie group such that its Lie algebra $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. If G has finite center, then K is a maximal compact subgroup. The dimension of the Lie algebra $\mathfrak{a}$ of A is called the real rank of G and is denoted by $\mathrm{rank}_{\mathbb{R}}(G)$. In general, given a polar decomposition $G = KAK$, it is not the case that for $g \in G$ there exist unique $k_1, k_2 \in K$ and $a \in A$ such that $g = k_1 a k_2$. However, after choosing a set of positive roots and restricting to the closure $\overline{A^+}$ of the positive Weyl chamber A^+, we still have $G = K\overline{A^+}K$. Moreover, if $g = k_1 a k_2$, where $k_1, k_2 \in K$ and $a \in \overline{A^+}$, then a is unique. Note that we can choose any Weyl chamber to be the positive one by choosing the set of positive roots correspondingly. We also use the terminology polar decomposition for such a $K\overline{A^+}K$ decomposition. For details, see [24, Section IX.1].

2.3. **Gelfand pairs and spherical functions.** Let G be a locally compact group (with Haar measure dg) with a compact subgroup K (with normalized Haar measure dk). A function $\varphi : G \to \mathbb{C}$ is said to be K-bi-invariant if for all $g \in G$ and $k_1, k_2 \in K$, we have $\varphi(k_1 g k_2) = \varphi(g)$. We denote the space of continuous K-bi-invariant compactly supported functions by $C_c(K\backslash G/K)$. If the subalgebra $C_c(K\backslash G/K)$ of the (convolution) algebra $C_c(G)$ is commutative, then the pair (G, K) is called a Gelfand pair. Equivalently, the pair (G, K) is a Gelfand pair if and only if for every irreducible unitary representation π on a Hilbert space $\mathcal{H}$, the space $\mathcal{H}_e = \{\xi \in \mathcal{H} \mid \forall k \in K : \pi(k)\xi = \xi\}$ consisting of K- invariant vectors is at most one-dimensional. For a Gelfand pair (G, K), a function $h \in C(K\backslash G/K)$ is called spherical if the functional χ on $C_c(K\backslash G/K)$ given by $\chi(\varphi) = \int_G \varphi(g) h(g^{-1}) dg$ for $\varphi \in C_c(K\backslash G/K)$ defines a nontrivial character. The theory of Gelfand pairs and spherical functions is well-established and goes back to Gelfand [16]. For more recent accounts of the theory, we refer the reader to [8], [14], [39].

Let G be a locally compact group with closed subgroup H. A function $\varphi : G \to \mathbb{C}$ is said to be $\mathrm{Int}(H)$-invariant if $\varphi(hgh^{-1}) = \varphi(g)$ for all $g \in G$ and $h \in H$. The space of continuous $\mathrm{Int}(H)$-invariant functions is denoted by $C(G//H)$.

Let now G be a locally compact group with compact subgroup K. The pair (G,K) is called a strong Gelfand pair if and only if the subalgebra $C_c(G//K)$ of $C_c(G)$ is commutative. In the setting of locally compact groups, the notion of strong Gelfand pair goes back to Goldrich and Wigner [18]. It is well-known that whenever G is a locally compact group with a compact subgroup K, then (G,K) is a strong Gelfand pair if and only if $(G \times K, \Delta K)$ (where ΔK is the diagonal subgroup) is a Gelfand pair.

It turns out that certain results of Section 3 can be understood on a deeper level in the setting of strong Gelfand pairs, in particular when one considers the analogue of spherical functions in this setting. This is discussed in Appendix A. The analogues of spherical functions already occurred in [17].

2.4. **The Fourier algebra.** Let G be a (second countable) locally compact group. The Fourier algebra $A(G)$ is defined as the space consisting of the coefficients of the left-regular representation $\lambda : G \to \mathcal{B}(L^2(G))$. It was introduced by Eymard [12] (see also [13]). More precisely, $\varphi \in A(G)$ if and only if there exist $\xi, \eta \in L^2(G)$ such that for all $g \in G$, we have $\varphi(g) = \langle \lambda(g)\xi, \eta \rangle$. The Fourier algebra $A(G)$ is a Banach space with respect to the norm defined by $\|\varphi\|_{A(G)} = \min\{\|\xi\|\|\eta\| \mid \forall g \in G\ \varphi(g) = \langle \lambda(g)\xi, \eta\rangle\}$. We have $\|\varphi\|_\infty \leq \|\varphi\|_{A(G)}$ for all $\varphi \in A(G)$, and $A(G)$ is $\|.\|_\infty$-dense in $C_0(G)$. Eymard showed that $A(G)$ can be identified isometrically with the predual of the group von Neumann algebra $L(G)$ of G.

2.5. **Completely bounded Fourier multipliers on compact Gelfand pairs.** A function $\varphi : G \to \mathbb{C}$ is said to be a Fourier multiplier if and only if $\varphi\psi \in A(G)$ for all $\psi \in A(G)$. Let $MA(G)$ denote the Banach space of multipliers of $A(G)$ equipped with the norm given by $\|\varphi\|_{MA(G)} = \|m_\varphi\|$, where $m_\varphi : A(G) \to A(G)$ denotes the associated multiplication operator. A multiplier φ is said to be completely bounded if the operator $M_\varphi : L(G) \to L(G)$ induced by m_φ is completely bounded. The space of completely bounded multipliers is denoted by $M_0A(G)$, and with the norm $\|\varphi\|_{M_0A(G)} = \|M_\varphi\|_{\mathrm{cb}}$, it forms a Banach space. It is known that $A(G) \subset M_0A(G) \subset MA(G)$.

It was proved by Bożejko and Fendler in [2] that $\varphi \in M_0A(G)$ if and only if there exist bounded continuous maps $P, Q : G \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, such that $\varphi(g_2^{-1}g_1) = \langle P(g_1), Q(g_2)\rangle$ for all $g_1, g_2 \in G$. Here $\langle .,. \rangle$ denotes the inner product on $\mathcal{H}$. In this characterization, $\|\varphi\|_{M_0A(G)} = \min\{\|P\|_\infty \|Q\|_\infty\}$, where the minimum is taken over all possible pairs (P,Q) for which $\varphi(g_2^{-1}g_1) = \langle P(g_1), Q(g_2)\rangle$ for all $g_1, g_2 \in G$.

Suppose now that (G,K) is a compact Gelfand pair, i.e., the group G is compact and (G,K) is a Gelfand pair. Then for every irreducible representation π on $\mathcal{H}$, the space $\mathcal{H}_e$ as defined in Section 2.3 is at most one-dimensional. Let $P_\pi = \int_K \pi(k)dk$ denote the projection onto $\mathcal{H}_e$, and set $\hat{G}_K = \{\pi \in \hat{G} \mid P_\pi \neq 0\}$, where $\hat{G}$ denotes the unitary dual of G. We proved the following result in [21, Proposition 2.3].

Proposition 2.1. Let (G,K) be a compact Gelfand pair, and let φ be a K-bi-invariant completely bounded Fourier multiplier. Then φ has a unique decomposition $\varphi(g) = \sum_{\pi \in \hat{G}_K} c_\pi h_\pi(g)$ for all $g \in G$, where $h_\pi(g) = \langle \pi(g)\xi_\pi, \xi_\pi\rangle$ is the positive definite spherical function associated with the representation π with K-invariant cyclic vector ξ_π, and $\sum_{\pi \in \hat{G}_K} |c_\pi| = \|\varphi\|_{M_0A(G)}$.

2.6. **The Approximation Property.** We recall the definition and basic properties of the Approximation Property for groups (AP), as introduced by the first named author and Kraus [20].

Definition 2.2. A locally compact group G is said to have the Approximation Property for groups (AP) if there is a net (φ_α) in $A(G)$ such that $\varphi_\alpha \to 1$ in the $\sigma(M_0A(G), M_0A(G)_*)$-topology, where $M_0A(G)_*$ denotes the natural predual of $M_0A(G)$ as introduced in [4] (see also [20] and [21]).

It was proved by the first named author and Kraus that if G is a locally compact group and Γ is a lattice in G, then G has the AP if and only if Γ has the AP [20, Theorem 2.4]. The AP passes to closed subgroups, as is proved in [20, Proposition 1.14]. Also, if H is a closed normal subgroup of a locally compact group G such that both H and G/H have the AP, then G has the AP [20, Theorem 1.15]. Moreover, if G_1 and G_2 are two locally isomorphic connected simple Lie groups with finite center such that G_1 has the AP, then G_2 has the AP [21, Proposition 2.4].

2.7. **Preliminaries for the results on noncommutative L^p-spaces.** These preliminaries are only relevant for Section 4. For a more extensive account, we refer to [31], [29].

2.7.1. *Schur multipliers on Schatten classes.* For $p \in [1,\infty]$ and a Hilbert space $\mathcal{H}$, let $S^p(\mathcal{H})$ denote the p^{th} Schatten class on $\mathcal{H}$. We identify $S^2(\mathcal{H})$ with $\mathcal{H}^* \otimes \mathcal{H}$, and for a σ-finite measure space (X,μ), we identify $L^2(X,\mu)^*$ with $L^2(X,\mu)$ by the duality bracket $\langle f,g\rangle = \int_X fg d\mu$. It follows that $S^2(L^2(X,\mu))$ can be identified with $L^2(X\times X, \mu\otimes\mu)$. Hence, every Schur multiplier on $S^2(L^2(X,\mu))$ comes from a function $\psi \in L^\infty(X\times X,\mu\otimes\mu)$ acting by multiplication on $L^2(X\times X,\mu\otimes\mu)$.

Definition 2.3. Let $p \in [1,\infty]$, and let $\psi \in L^\infty(X\times X, \mu\otimes\mu)$. The Schur multiplier with symbol ψ is said to be bounded (resp. completely bounded) on $S^p(L^2(X,\mu))$ if it maps $S^p(L^2(X,\mu)) \cap S^2(L^2(X,\mu))$ into $S^p(L^2(X,\mu))$ by $T_k \mapsto T_{\psi k}$ (where T_k denotes the integral operator with kernel k), and if this map extends (necessarily uniquely) to a bounded (resp. completely bounded) map M_ψ on $S^p(L^2(X,\mu))$.

The norm of a bounded multiplier ψ is defined by $\|\psi\|_{MS^p(L^2(X,\mu))} = \|M_\psi\|$, and its completely bounded norm by $\|\psi\|_{cbMS^p(L^2(X,\mu))} = \|M_\psi\|_{cb}$. The spaces of multipliers and completely bounded multipliers are denoted by $MS^p(L^2(X,\mu))$ and $cbMS^p(L^2(X,\mu))$, respectively. It follows that for every $p \in [1,\infty]$ and $\psi \in L^\infty(X\times X,\mu\otimes\mu)$, we have $\|\psi\|_\infty \leq \|\psi\|_{MS^p(L^2(X,\mu))} \leq \|\psi\|_{cbMS^p(L^2(X,\mu))}$.

2.7.2. *Schur multipliers on compact Gelfand pairs.* In this section, we recall results from [29, Section 2] that are analogues in the setting of multipliers on Schatten classes of the results of Section 2.5. For proofs, we refer to [29].

For a locally compact group G and a function $\varphi \in L^\infty(G)$, we define the function $\check{\varphi} \in L^\infty(G\times G)$ by $\check{\varphi}(g_1,g_2) = \varphi(g_1^{-1}g_2)$.

In what follows, let G and K be Lie groups such that (G,K) is a compact Gelfand pair. Let $X = G/K$ denote the homogeneous space corresponding with the canonical transitive action of G. The group K is the stabilizer subgroup of a certain element $e_0 \in X$. It follows that $L^2(X) = \oplus_{\pi\in\hat{G}_K}\mathcal{H}_\pi$. Let h_π denote the spherical function corresponding to the equivalence class π of representations. Then for every $\varphi \in L^2(K\backslash G/K)$ we have $\varphi = \sum_{\pi\in\hat{G}_K} c_\pi \dim\mathcal{H}_\pi h_\pi$, where $c_\pi = \langle\varphi, h_\pi\rangle$.

It also follows that for any $\varphi \in C(K\backslash G/K)$, there exists a continuous function $\psi : X \times X \to \mathbb{C}$ such that for all $g_1, g_2 \in G$, we have $\varphi(g_1^{-1}g_2) = \psi(g_1 e_0, g_2 e_0)$. Let $\varphi : G \to \mathbb{C}$ be a continuous K-bi-invariant function such that $\check{\varphi} \in cbMS^p(L^2(G))$ for some $p \in [1,\infty]$. Then $\|\psi\|_{cbMS^p(L^2(X))} = \|\check{\varphi}\|_{cbMS^p(L^2(G))}$, where $\psi : X \times X \to \mathbb{C}$ is as defined above. If K is an infinite group, then these norms are equal to $\|\check{\varphi}\|_{MS^p(L^2(G))}$.

Let (G,K) be a compact Gelfand pair, let $p \in [1,\infty)$, and let $\varphi : G \to \mathbb{C}$ be a continuous K-bi-invariant function such that $\check{\varphi} \in MS^p(L^2(G))$. Then $\left(\sum_{\pi \in \hat{G}_K} |c_\pi|^p (\dim \mathcal{H}_\pi)\right)^{\frac{1}{p}} \leq \|\check{\varphi}\|_{MS^p(L^2(G))}$, where c_π and $\mathcal{H}_\pi$ are as before.

2.7.3. *The* $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$. The $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ was defined in [31]. Its relevance to us, including certain important properties, was described in Section 1.

Definition 2.4. (see [31, Definition 2.2]) Let G be a locally compact Hausdorff second countable group, and let $1 \leq p \leq \infty$. The group G is said to have the property of completely bounded approximation by Schur multipliers on S^p, denoted $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$, if there exists a constant $C > 0$ and a net $\varphi_\alpha \in A(G)$ such that $\varphi_\alpha \to 1$ uniformly on compacta and $\sup_\alpha \|\check{\varphi}_\alpha\|_{cbMS^p(L^2(G))} \leq C$. The infimum of these C's is denoted by $\Lambda^{\mathrm{Schur}}_{p,\mathrm{cb}}(G)$.

It was proved by Lafforgue and de la Salle that if G is a locally compact group and Γ is a lattice in G, then for $1 \leq p \leq \infty$, we have $\Lambda^{\mathrm{Schur}}_{p,\mathrm{cb}}(\Gamma) = \Lambda^{\mathrm{Schur}}_{p,\mathrm{cb}}(G)$ (see [31, Theorem 2.5]). More properties of the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ are discussed in [31] and [29].

3. The group $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ does not have the AP

In this section, we prove that the universal covering group $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ of $\mathrm{Sp}(2,\mathbb{R})$ does not have the AP. Hereto, let us first recall the definition of $\mathrm{Sp}(2,\mathbb{R})$ and describe a realization of $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$.

Let I_2 denote the 2×2 identity matrix, and let the matrix J be defined by

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Recall that the symplectic group $\mathrm{Sp}(2,\mathbb{R})$ is defined as the Lie group

$$\mathrm{Sp}(2,\mathbb{R}) := \{ g \in \mathrm{GL}(4,\mathbb{R}) \mid g^T J g = J \}.$$

Here, g^T denotes the transpose of g. Let K denote the maximal compact subgroup of $\mathrm{Sp}(2,\mathbb{R})$ given by

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathrm{M}_4(\mathbb{R}) \;\middle|\; A + iB \in \mathrm{U}(2) \right\}.$$

This group is isomorphic to $\mathrm{U}(2)$. A polar decomposition of $\mathrm{Sp}(2,\mathbb{R})$ is given by $\mathrm{Sp}(2,\mathbb{R}) = K\overline{A^+}K$, where

$$\overline{A^+} = \left\{ D(\beta,\gamma) = \begin{pmatrix} e^\beta & 0 & 0 & 0 \\ 0 & e^\gamma & 0 & 0 \\ 0 & 0 & e^{-\beta} & 0 \\ 0 & 0 & 0 & e^{-\gamma} \end{pmatrix} \;\middle|\; \beta \geq \gamma \geq 0 \right\}.$$

Different explicit realizations of $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ can be found in the literature. An incomplete list is given by [32], [37], [40]. We use the realization in terms of circle functions, given recently by Rawnsley [37], and in what follows we use of some of his computations. In fact, he describes a method that gives a realization of the universal covering group of any connected Lie group G with fundamental group $\pi_1(G)$ isomorphic to $\mathbb{Z}$ admitting a so-called (normalized) circle function. Firstly, we briefly describe Rawnsley's general construction.

Let G be a connected Lie group with $\pi_1(G) \cong \mathbb{Z}$. A circle function on G is a smooth function $c : G \to \mathbb{T}$, where $\mathbb{T}$ denotes the circle (as a subspace of $\mathbb{C}$), that induces an isomorphism of the fundamental groups of G and $\mathbb{T}$. Such a function is said to be normalized if $c(1) = 1$ and $c(g^{-1}) = c(g)^{-1}$. If G admits a circle function, it admits one and only one normalized circle function.

Let G be a connected Lie group with fundamental group isomorphic to $\mathbb{Z}$ that admits a normalized circle function. Then there exists a unique smooth function $\eta : G \times G \to \mathbb{R}$ such that

$$c(g_1 g_2) = c(g_1)c(g_2)e^{i\eta(g_1,g_2)}$$

for all $g_1, g_2 \in G$ and $\eta(1,1) = 0$. Furthermore, it follows that $\eta(g,1) = \eta(1,g) = \eta(g,g^{-1}) = 0$ and $\eta(g_1,g_2) + \eta(g_1g_2,g_3) = \eta(g_1,g_2g_3) + \eta(g_2,g_3)$ for all $g \in G$ and $g_1, g_2, g_3 \in G$.

Let G be a connected Lie group with normalized circle function c, and let

$$\widetilde{G} = \{(g,t) \in G \times \mathbb{R} \mid c(g) = e^{it}\}. \tag{1}$$

The space $\widetilde{G}$ is a smooth manifold of the same dimension as G. A multiplication on $\widetilde{G}$ is given by

$$(g_1,t_1)(g_2,t_2) = (g_1g_2, t_1 + t_2 + \eta(g_1,g_2)).$$

With this multiplication, $\widetilde{G}$ is a Lie group with identity $\tilde{1} = (1,0)$, where 1 denotes the identity element of G, and inverse given by $(g,t)^{-1} = (g^{-1}, -t)$. The map $\sigma : \widetilde{G} \to G$, $(g,t) \mapsto g$ (with kernel $\{(1, 2\pi k) \in G \times \mathbb{R} \mid k \in \mathbb{Z}\}$) defines a universal covering map from $\widetilde{G}$ onto G.

In the rest of this section, let $G = \mathrm{Sp}(2,\mathbb{R})$ and $\widetilde{G} = \widetilde{\mathrm{Sp}}(2,\mathbb{R})$.

We now give the explicit functions c and η for $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$. Let $M_4(\mathbb{R})_0$ denote the subspace of $M_4(\mathbb{R})$ given by

$$M_4(\mathbb{R})_0 = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \;\middle|\; A, B \in M_2(\mathbb{R}) \right\},$$

and let $\iota : M_4(\mathbb{R})_0 \to M_2(\mathbb{C})$ be given by

$$\iota : \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto A + iB.$$

The map ι is an algebra homomorphism. For an element $g \in G$, let $C_g = \frac{1}{2}(g + (g^T)^{-1})$ and $D_g = \frac{1}{2}(g - (g^T)^{-1})$. Note that $g = C_g + D_g$. As described by Rawnsley, the connected Lie group G admits a normalized circle function; namely, the function $c : G \to \mathbb{T}$ given by

$$c(g) = \frac{\det(\iota(C_g))}{|\det(\iota(C_g))|}. \tag{2}$$

With this circle function, the manifold $\widetilde{G}$ is given through (1). Let $Z_g = C_g^{-1} D_g$. The function η (which is needed to define the multiplication on $\widetilde{G}$) corresponding to the circle function c is given by

$$\eta(g_1,g_2)=\mathrm{Im}(\mathrm{Tr}(\iota(\log(1-Z_{g_1}Z_{g_2^{-1}}))))=\mathrm{Im}(\mathrm{Tr}(\iota(\log(C_{g_1}^{-1}C_{g_1g_2}C_{g_2}^{-1})))).$$

The logarithm is well-defined, since $\|Z_{g_1}Z_{g_2^{-1}}\|<1$ (see [37, Section 4]). It was also proved by Rawnsley that $|\eta(g_1,g_2)|<2\pi$ for all $g_1,g_2\in G$ (see [37, Lemma 14]).

Remark 3.1. Everything that we described so far for G can be generalized to $\mathrm{Sp}(n,\mathbb{R})$ for $n\geq 1$ (see [37]).

The rest of this section is devoted to proving the following theorem.

Theorem 3.2. The group $\widetilde{G}=\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ does not have the AP.

Firstly, we elaborate on the structure of $\widetilde{G}$. Let $\mathfrak{g}$ denote the Lie algebra of G and $\widetilde{G}$, and denote by $\exp:\mathfrak{g}\to G$ and $\widetilde{\exp}:\mathfrak{g}\to\widetilde{G}$ the corresponding exponential maps. These exponential maps have as their image a neighbourhood of the identity. The group $\widetilde{G}$ has a polar decomposition (see Section 2.2) $\widetilde{G}=\widetilde{K}\overline{\widetilde{A}^{+}}\widetilde{K}$ that is strongly related to the polar decomposition $G=KAK$ of G. It is known that the exponential map of a connected simple Lie group is a bijection from the $\mathfrak{a}$-summand of the KAK-decomposition on the Lie algebra level to A. Therefore, it follows that $\widetilde{A}\cong A$. This implies that the "infinite covering" part of G is intrinsic to the K-part of the polar decomposition. It is known that $\exp:\mathfrak{k}\to K$ is surjective, because K is connected and compact. Also, since $\mathfrak{k}=\mathfrak{su}(2)\oplus\mathbb{R}$ (see [37, Lemma 9]), it follows that $\widetilde{\exp}:\mathfrak{k}\to\widetilde{K}$ is surjective. We summarize these facts (based on [24, Section IX.1]) in the following proposition.

Proposition 3.3. We have $G=KAK$ and $\widetilde{G}=\widetilde{K}\widetilde{A}\widetilde{K}$, where K and A are as above, and

$$K=\exp(\mathfrak{k}),\qquad A=\exp(\mathfrak{a}),$$
$$\widetilde{K}=\widetilde{\exp}(\mathfrak{k}),\qquad \widetilde{A}=\widetilde{\exp}(\mathfrak{a}).$$

Here, $\mathfrak{k}$ and $\mathfrak{a}$ denote the Lie algebras of K and A, respectively. The group $\widetilde{A}$ is isomorphic to A. We can restrict to the positive Weyl chamber, and get

$$\overline{\widetilde{A}^{+}}=\widetilde{\exp}(\{\mathrm{diag}(e^{\beta},e^{\gamma},e^{-\beta},e^{-\gamma})\mid\beta\geq\gamma\geq 0\}),$$

which yields the decomposition $\widetilde{G}=\widetilde{K}\overline{\widetilde{A}^{+}}\widetilde{K}$.

Note that the group $\mathrm{SU}(2)$ is a natural subgroup of $\mathrm{U}(2)$. Denote by H the corresponding subgroup of K. We also get a corresponding group $\widetilde{H}$, which is isomorphic to H, since $\mathrm{SU}(2)$ is simply connected.

Definition 3.4. We define $\mathcal{C}$ to be the following class of functions:

$$\mathcal{C}:=\{\varphi\in C(\widetilde{G})\mid\varphi\text{ is }\widetilde{H}\text{-bi-invariant and }\mathrm{Int}(\widetilde{K})\text{-invariant}\}.$$

We refer to Section 2.3 for the notions of $\widetilde{H}$-bi-invariant and $\mathrm{Int}(\widetilde{K})$-invariant functions. In the notation used in that section, we have $\mathcal{C}=C(\widetilde{H}\backslash\widetilde{G}/\widetilde{H})\cap C(\widetilde{G}//\widetilde{K})$.

Consider the generator $\begin{pmatrix} i & 0\\ 0 & i\end{pmatrix}$ of the Lie algebra of the center of $\mathrm{U}(2)$. Let Z denote the corresponding element of $\mathfrak{k}$. The elements $v_t=\exp(tZ)$ and $\widetilde{v}_t=\widetilde{\exp}(tZ)$

for $t \in \mathbb{R}$ are elements of the centers of K and $\widetilde{K}$, respectively. Also, the family v_t is periodic with period 2π. Explicitly, we have

$$v_t = \begin{pmatrix} \cos t & 0 & -\sin t & 0 \\ 0 & \cos t & 0 & -\sin t \\ \sin t & 0 & \cos t & 0 \\ 0 & \sin t & 0 & \cos t \end{pmatrix}.$$

Remark 3.5. Every $k \in K$ can be written as the product $k = v_t h$ for some $t \in \mathbb{R}$ and $h \in H$, and, similarly, every $\widetilde{k} \in \widetilde{K}$ can be written as the product $\widetilde{k} = \widetilde{v}_t \widetilde{h}$ for some $t \in \mathbb{R}$ and $\widetilde{h} \in \widetilde{H}$. Hence, the class $\mathcal{C}$ can also be defined in the following way:

$$\mathcal{C} := \{\varphi \in C(\widetilde{G}) \mid \varphi \text{ is } \widetilde{H}\text{-bi-invariant and } \varphi(\widetilde{v}_t g \widetilde{v}_t^{-1}) = \varphi(g) \, \forall g \in \widetilde{G} \, \forall t \in \mathbb{R}\}.$$

For $\beta \geq \gamma \geq 0$, let $D(\beta, \gamma) = \operatorname{diag}(e^{\beta}, e^{\gamma}, e^{-\beta}, e^{-\gamma}) \in G$, which is, as pointed out before, an element of $\overline{A^+}$. Since $\widetilde{A} \cong A$, there is one and only one element $\widetilde{D}(\beta, \gamma)$ in $\overline{\widetilde{A}^+}$ that surjects onto $D(\beta, \gamma) \in G$. We now show that functions in $\mathcal{C}$ are completely determined by their values at elements of the form $\widetilde{v}_t \widetilde{D}(\beta, \gamma)$. Firstly, let us prove the following lemma.

Lemma 3.6. In the realization of (1), we have $\widetilde{v}_t \widetilde{D}(\beta, \gamma) = (v_t D(\beta, \gamma), 2t)$ for $\beta \geq \gamma \geq 0$ and $t \in \mathbb{R}$.

Proof. By the description of $\widetilde{G}$ and the fact that the covering map is a homomorphism, it follows that $\widetilde{v}_t \widetilde{D}(\beta, \gamma) = (v_t D(\beta, \gamma), s)$ for some $s \in \mathbb{R}$. Using that $(v_t^T)^{-1} = v_t$ and that ι is an algebra homomorphism, it follows that $\iota(C_{v_t D(\beta,\gamma)}) = \iota(\frac{1}{2} v_t (D(\beta, \gamma) + D(-\beta, -\gamma))) = \iota(v_t) \operatorname{diag}(\cosh(\beta), \cosh(\gamma))$. Hence,

$$c(\widetilde{v}_t \widetilde{D}(\beta, \gamma)) = \frac{\det(\iota(v_t))}{|\det(\iota(v_t))|},$$

because $\det(\operatorname{diag}(\cosh(\beta), \cosh(\gamma))) = |\det(\operatorname{diag}(\cosh(\beta), \cosh(\gamma)))|$ and the determinant is multiplicative. Using the fact that $\{\widetilde{v}_\sigma \widetilde{D}(\beta, \gamma) \mid \sigma \in \mathbb{R}\}$ defines a continuous path in $\widetilde{G}$, the value of s is computed by

$$s = \tan^{-1}\left(\frac{2 \sin t \cos t}{\cos^2 t - \sin^2 t}\right) + 2k\pi = 2t + 2k\pi$$

for some $k \in \mathbb{Z}$. Since we can connect every element $\widetilde{v}_\sigma \widetilde{D}(\beta, \gamma)$ continuously to $\widetilde{v}_0 = \widetilde{1} = (1, 0)$ (by varying σ, β and γ), it follows that $k = 0$. Hence, $s = 2t$. □

Lemma 3.7. A function in $\mathcal{C}$ is determined by its values at the elements of the form $\widetilde{v}_t \widetilde{D}(\beta, \gamma)$.

Proof. Let $\varphi \in \mathcal{C}$, and let $g \in \widetilde{G}$. By the polar decomposition of $\widetilde{G}$, we can write $g = \widetilde{k}_1 \widetilde{D}(\beta, \gamma) \widetilde{k}_2$ for some $\beta \geq \gamma \geq 0$ and $\widetilde{k}_1, \widetilde{k}_2 \in \widetilde{K}$. For $i = 1, 2$, let $t_i \in \mathbb{R}$ and $\widetilde{h}_i \in \widetilde{H}$ be so that $\widetilde{k}_i = \widetilde{v}_{t_i} \widetilde{h}_i = \widetilde{h}_i \widetilde{v}_{t_i}$. Using both invariance properties of functions in $\mathcal{C}$, we obtain

$$\varphi(g) = \varphi(\widetilde{h}_1 \widetilde{v}_{t_1} \widetilde{D}(\beta, \gamma) \widetilde{v}_{t_2} \widetilde{h}_2) = \varphi(\widetilde{v}_{t_1 + t_2} \widetilde{D}(\beta, \gamma)).$$

□

Notation 3.8. The value of $\varphi \in \mathcal{C}$ at $g = (g_0, t) \in \widetilde{G}$ does not change if we multiply g from the left or the right with an element of $\widetilde{H}$ or if we conjugate g with an element of $\widetilde{K}$. This induces an equivalence relation on $\widetilde{G}$. Let $S_{\beta,\gamma,t}$ denote the corresponding equivalence class of the element $\widetilde{v}_{\frac{t}{2}} \widetilde{D}(\beta, \gamma)$ (note that the t-parameter of the equivalence class corresponds to the t-parameter coming from the equation $c(g_0) = e^{it}$). Also, for $\varphi \in \mathcal{C}$, we put $\dot{\varphi}(\beta, \gamma, t) = \varphi(\widetilde{v}_{\frac{t}{2}} \widetilde{D}(\beta, \gamma))$.

Lemma 3.9. The class $\mathcal{C}$ is invariant under the action of the one-parameter family $\widetilde{v}_t$. More precisely, if $\varphi \in \mathcal{C}$ and $t \in \mathbb{R}$, then $\varphi_t : \widetilde{G} \to \mathbb{C}$ defined by $\varphi_t(g) = \varphi(\widetilde{v}_t g)$ is also in $\mathcal{C}$. Clearly, for an element $\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C}$, it follows that for all $t \in \mathbb{R}$, we have $\|\varphi_t\|_{M_0A(\widetilde{G})} = \|\varphi\|_{M_0A(\widetilde{G})}$.

Proof. Let $\varphi \in \mathcal{C}$. We have $\varphi_t(\widetilde{h}_1 g \widetilde{h}_2) = \varphi(\widetilde{v}_t \widetilde{h}_1 g \widetilde{h}_2) = \varphi(\widetilde{h}_1 \widetilde{v}_t g \widetilde{h}_2) = \varphi(\widetilde{v}_t g) = \varphi_t(g)$ for all $g \in \widetilde{G}$, $t \in \mathbb{R}$ and $\widetilde{h}_1, \widetilde{h}_2 \in \mathcal{H}$. Moreover, we have $\varphi_t(\widetilde{v}_s g \widetilde{v}_s^{-1}) = \varphi(\widetilde{v}_t \widetilde{v}_s g \widetilde{v}_s^{-1}) = \varphi(\widetilde{v}_s \widetilde{v}_t g \widetilde{v}_s^{-1}) = \varphi(\widetilde{v}_t g) = \varphi_t(g)$ for all $g \in \widetilde{G}$ and $s, t \in \mathbb{R}$. This proves the invariance properties of $\mathcal{C}$ of Remark 3.5 for φ_t. $\square$

Lemma 3.10. If $\widetilde{G}$ has the AP, then the approximating net can be chosen in the set $A(\widetilde{G}) \cap \mathcal{C}$.

Proof. For $f \in C(\widetilde{G})$ or $f \in L^1(\widetilde{G})$, we define

$$f^{\mathcal{C}}(g) = \frac{1}{\pi} \int_{\mathbb{R}/\pi\mathbb{Z}} \int_{\widetilde{H}} \int_{\widetilde{H}} f(\widetilde{h}_1 \widetilde{v}_t g \widetilde{v}_t^{-1} \widetilde{h}_2) d\widetilde{h}_1 d\widetilde{h}_2 dt, \qquad g \in \widetilde{G},$$

where $d\widetilde{h}_1$ and $d\widetilde{h}_2$ both denote the normalized Haar measure on $\widetilde{H}$. The function $f^{\mathcal{C}}$ clearly satisfies the invariance properties of Remark 3.5.

The rest of the proof is similar to the proof of [21, Lemma 2.5]. $\square$

Proposition 3.11. There exist constants $C_1, C_2 > 0$ such that for all functions φ in $M_0A(\widetilde{G}) \cap \mathcal{C}$ and $t \in \mathbb{R}$, the limit $c_\varphi(t) = \lim_{s \to \infty} \dot{\varphi}(2s, s, t)$ exists, and for all $\beta \geq \gamma \geq 0$, we have

$$|\dot{\varphi}(\beta, \gamma, t) - c_\varphi(t)| \leq C_1 e^{-C_2 \sqrt{\beta^2 + \gamma^2}} \|\varphi\|_{M_0A(\widetilde{G})}.$$

The proof of this proposition will be postponed. Using the following lemma, we will explain how the proposition implies Theorem 3.2.

Lemma 3.12. The space consisting of φ in $M_0A(\widetilde{G}) \cap \mathcal{C}$ for which $c_\varphi(t) \equiv 0$ is $\sigma(M_0A(\widetilde{G}), M_0A(\widetilde{G})_*)$-closed.

Proof. Let (φ_α) be a net in $M_0A(\widetilde{G}) \cap \mathcal{C}$ converging to $\varphi \in M_0A(\widetilde{G})$. It follows that for all $f \in L^1(\widetilde{G})$, we have $\langle \varphi, f \rangle = \lim_\alpha \langle \varphi_\alpha, f \rangle = \lim_\alpha \langle \varphi_\alpha^{\mathcal{C}}, f \rangle = \lim_\alpha \langle \varphi_\alpha, f^{\mathcal{C}} \rangle = \langle \varphi, f^{\mathcal{C}} \rangle = \langle \varphi^{\mathcal{C}}, f \rangle$, i.e., the space $M_0A(\widetilde{G}) \cap \mathcal{C}$ is $\sigma(M_0A(G), M_0A(G)_*)$-closed, since $L^1(\widetilde{G})$ is dense in $M_0A(\widetilde{G})_*$.

It was proved in [21, Lemma 2.6] that whenever (X, μ) is a σ-finite measure space and $v : X \to \mathbb{R}$ is a strictly positive measurable function on X, then the set $S := \{f \in L^\infty(X) \mid |f(x)| \leq v(x) \text{ a.e.}\}$ is $\sigma(L^\infty(X), L^1(X))$-closed. We can apply this fact to the unit ball of the space $\{\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C} \mid c_\varphi(t) \equiv 0\}$. Indeed, the conditions are satisfied with v given by Proposition 3.11 (putting $\|\varphi\|_{M_0A(\widetilde{G})} \leq 1$).

Recall the Krein-Smulian Theorem, asserting that whenever X is a Banach space and A is a convex subset of the dual space X^* such that $A \cap \{x^* \in X^* \mid \|x^*\| \leq r\}$ is weak-* closed for every $r > 0$, then A is weak-* closed [5, Theorem V.12.1]. In the case where A is a vector space, which is the case here, it suffices to check the case $r = 1$, i.e., the weak-* closedness of the unit ball. It follows that the space consisting of φ in $M_0A(\widetilde{G}) \cap \mathcal{C}$ for which $c_\varphi(t) \equiv 0$ is $\sigma(M_0A(\widetilde{G}), M_0A(\widetilde{G})_*)$-closed. $\square$

Proof of Theorem 3.2 using Proposition 3.11. By Lemma 3.10, it follows that if there is no net in $A(\widetilde{G}) \cap \mathcal{C}$ that approximates the constant function 1 in the $\sigma(M_0A(\widetilde{G}), M_0A(\widetilde{G})_*)$-topology, then $\widetilde{G}$ does not have the AP. However, since the space $\{\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C} \mid c_\varphi(t) \equiv 0\}$ is $\sigma(M_0A(\widetilde{G}), M_0A(\widetilde{G})_*)$-closed by Lemma 3.12, it follows immediately that the constant function 1 cannot be approximated by such a net. ☐

The rest of this section will be devoted to proving Proposition 3.11. Hereto, we identify certain pairs of groups in $\widetilde{G}$, as was also done for G in [21]. However, since $\widetilde{K}$ is not compact (unlike K in G), one of the pairs we consider here is slightly different.

Firstly, note that U(1) is contained as a subgroup in SU(2) by the embedding

$$\begin{pmatrix} e^{i\nu} & 0 \\ 0 & e^{-i\nu} \end{pmatrix} \hookrightarrow \mathrm{SU}(2), \tag{3}$$

where $\nu \in \mathbb{R}$. We point out that the quotient of SU(2) with respect to the equivalence relation $g \sim kgk^{-1}$ for $k \in \mathrm{U}(1)$ is homeomorphic to the closed unit disc $\overline{\mathbb{D}}$ in the complex plane. This homeomorphism is given by

$$z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \mapsto z_{11}. \tag{4}$$

Let H_0 denote the corresponding subgroup of H. It can be proved that (H, H_0) is a strong Gelfand pair (see Section 2.3). However, because the theory on strong Gelfand pairs is not as well-developed as the theory of Gelfand pairs, we use a more explicit approach, and prove the things we need in a more ad hoc manner.

For $l, m \in \mathbb{Z}_{\geq 0}$, consider the so-called disc polynomials (see [27]) $h^0_{l,m} : \overline{\mathbb{D}} \to \mathbb{C}$ from the closed unit disc $\overline{\mathbb{D}}$ to $\mathbb{C}$, given by

$$h^0_{l,m}(z) = \begin{cases} z^{l-m} P_m^{(0,l-m)}(2|z|^2 - 1) & l \geq m, \\ \overline{z}^{m-l} P_l^{(0,m-l)}(2|z|^2 - 1) & l < m. \end{cases}$$

where $P_n^{(\alpha,\beta)}$ denotes the n^{th} Jacobi polynomial.

Recall that a function $f : X \to Y$ from a metric space X to a metric space Y is Hölder continuous with exponent $\alpha > 0$ if there exists a constant $C > 0$ such that $d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)^\alpha$, for all $x_1, x_2 \in X$. The following result (see [21, Corollary 3.5]) gives Hölder continuity with exponent $\frac{1}{4}$ of the functions $h^0_{l,m}$ on the circle in $\mathbb{D}$ centered at the origin with radius $\frac{1}{\sqrt{2}}$, with a constant independent of l and m. It is a corollary of results of de first named author and Schlichtkrull [22].

Lemma 3.13. For all $l, m \geq 0$, we have

$$\left| h^0_{l,m}\left(\frac{e^{i\theta_1}}{\sqrt{2}}\right) - h^0_{l,m}\left(\frac{e^{i\theta_2}}{\sqrt{2}}\right) \right| \leq \tilde{C} |\theta_1 - \theta_2|^{\frac{1}{4}}$$

for all $\theta_1, \theta_2 \in [0, 2\pi)$, where $\tilde{C}$ is a constant independent of l and m.

We now prove the following decomposition result.

Lemma 3.14. Let $\varphi \in M_0A(\mathrm{SU}(2)/\!/\,\mathrm{U}(1))$ (recall the embedding (3)). Let

$$z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathrm{SU}(2).$$

Then $\varphi(z) = \varphi^0(z_{11})$ for a certain function $\varphi^0 : \mathbb{D} \to \mathbb{C}$, and

$$\varphi^0 = \sum_{l,m \geq 0} c_{l,m} h^0_{l,m}$$

such that $\sum_{l,m\geq 0} |c_{l,m}| = \|\varphi\|_{M_0A(\mathrm{SU}(2))}$. Moreover, φ^0 satisfies

$$\left| \varphi^0 \left(\frac{e^{i\theta_1}}{\sqrt{2}} \right) - \varphi^0 \left(\frac{e^{i\theta_2}}{\sqrt{2}} \right) \right| \leq \tilde{C} |\theta_1 - \theta_2|^{\frac{1}{4}} \|\varphi\|_{M_0A(\mathrm{SU}(2))}$$

for all $\theta_1, \theta_2 \in [0.2\pi)$.

Proof. Let $L \cong \mathrm{U}(1)$ denote the subgroup of $\mathrm{U}(2)$ given by the elements of the form

$$l_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \qquad \theta \in \mathbb{R}.$$

Note that $(\mathrm{U}(2), L)$ is the Gelfand pair that played an important role in [21]. We now prove that there is an isometric isomorphism between $M_0A(\mathrm{SU}(2)//\,\mathrm{U}(1))$ and $M_0A(L\backslash\,\mathrm{U}(2)/L)$.

Let $\Phi : M_0A(L\backslash\,\mathrm{U}(2)/L) \to M_0A(\mathrm{SU}(2)//\,\mathrm{U}(1))$ be the map given by $\varphi \mapsto \tilde{\varphi}$, where $\tilde{\varphi} = \varphi|_{\mathrm{SU}(2)}$. It is clear that $\tilde{\varphi} \in M_0A(\mathrm{SU}(2)//\,\mathrm{U}(1))$ and that Φ is norm-decreasing.

Write $\mathrm{U}(2) = \mathrm{SU}(2) \rtimes L$ by the action given by multiplication from the right, i.e., $g = hl$, where $g \in \mathrm{U}(2)$, $h \in \mathrm{SU}(2)$ and $l \in L$. Consider the map $\Psi : M_0A(\mathrm{SU}(2)//\,\mathrm{U}(1)) \to M_0A(L\backslash\,\mathrm{U}(2)/L)$ given by $\varphi \mapsto \psi$, where $\psi(g) = \varphi(h)$ if $g = hl$ according to the unique factorization that follows from $\mathrm{U}(2) = \mathrm{SU}(2) \rtimes L$. It follows that $\psi(l_1hl_2) = \varphi(h)$ for all $h \in \mathrm{SU}(2)$ and $l_1, l_2 \in L$. Indeed, $\psi(l_1hl_2) = \psi(l_1hl_1^{-1}l_1l_2) = \varphi(l_1hl_1^{-1}) = \varphi(h)$, since $lhl^{-1} \in \mathrm{SU}(2)$ for all $h \in \mathrm{SU}(2)$ and $l \in L$. From this, it follows that $\psi((h_2l_2)^{-1}h_1l_1) = \psi(l_2^{-1}h_2^{-1}h_1l_1) = \varphi(h_2^{-1}h_1)$. Let now $P, Q : \mathrm{SU}(2) \to \mathcal{H}$ be bounded continuous maps such that $\varphi(h_2^{-1}h_1) = \langle P(h_1), Q(h_2) \rangle$ for all $h_1, h_2 \in \mathrm{SU}(2)$ and $\|\varphi\|_{M_0A(\mathrm{SU}(2))} = \|P\|_\infty \|Q\|_\infty$. This is possible by the result of Bożejko and Fendler mentioned in Section 2.5. It follows from this that also the map Ψ is norm-decreasing, since the maps $\tilde{P}(hl) = P(h)$ and $\tilde{Q}(hl) = Q(h)$ give maps such that $\psi((h_2l_2)^{-1}h_1l_1) = \langle \tilde{P}(h_1l_1), \tilde{Q}(h_2l_2) \rangle$ for all $h_1, h_2 \in \mathrm{SU}(2)$ and $l_1, l_2 \in L$. Moreover, it is easy to check that Φ and Ψ are each other's inverses.

From Proposition 2.1 we get a decomposition of elements of $M_0A(L\backslash\,\mathrm{U}(2)/L)$ in terms of the functions $h_{l,m}$, as was also explained in [21, Section 3]. Indeed $(\mathrm{U}(2), L)$ is a compact Gelfand pair. Applying the map Φ to this decomposition, i.e., restricting to $\mathrm{SU}(2)$, and by using the homeomorphism of (4), it follows that for $\varphi \in M_0A(\mathrm{SU}(2)//\,\mathrm{U}(1))$ we have $\varphi(h) = \varphi^0(h_{11})$ for a certain function $\varphi^0 : \mathbb{D} \to \mathbb{C}$, and $\varphi^0 = \sum_{l,m\geq 0} c_{l,m} h^0_{l,m}$ such that $\sum_{l,m\geq 0} |c_{l,m}| = \|\varphi\|_{M_0A(\mathrm{SU}(2))}$. The last assertion of the lemma follows directly from Lemma 3.13. □

Remark 3.15. This lemma shows that the disc polynomials act like analogues of spherical functions for the strong Gelfand pair $(\mathrm{SU}(2), \mathrm{U}(1))$. The disc polynomials also occur as the spherical functions of the Gelfand pair $(\mathrm{U}(2), L)$, where L is as above. It turns out that there is a general connection between the spherical functions for certain Gelfand pairs and their analogues for certain strong Gelfand pairs. A brief account on this connection is given in Appendix A.

Note that we can identify $M_0A(H//H_0)$ with $M_0A(\mathrm{SU}(2)//\,\mathrm{U}(1))$.

Proposition 3.16. Let $\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C}$. For $\alpha \geq 0$, let $\psi_\alpha : H \to \mathbb{C}$ be given by $h \mapsto \varphi(\widetilde{D}(\alpha,0)\widetilde{h}\widetilde{D}(\alpha,0))$. This function is an element of $M_0A(H//H_0)$, and $\|\psi_\alpha\|_{M_0A(H)} \leq \|\varphi\|_{M_0A(\widetilde{G})}$.

Proof. Let L be the subgroup of $\mathrm{U}(2)$ as in Lemma 3.14, and let K_0 (resp. $\widetilde{K}_0$) be the corresponding subgroup of K (resp. $\widetilde{K}$). For $\widetilde{h}_0 \in \widetilde{H}_0$, we can write $\widetilde{h}_0 = \widetilde{k}_0\widetilde{v}_t$ for some $\widetilde{k}_0 \in \widetilde{K}_0$ and $t \in \mathbb{R}$. Since $\widetilde{k}_0$ is the exponential of an element in the Lie algebra (note that this does not hold for every element in $\widetilde{G}$), and since this element of the Lie algebra commutes with the Lie algebra element corresponding to $\widetilde{D}(\alpha,0)$, the elements also commute on the Lie group level. Hence, for all $h \in H$ and $h_0 = k_0v_t \in H_0$,

$$\begin{aligned}\psi_\alpha(h_0hh_0^{-1}) &= \varphi(\widetilde{D}(\alpha,0)\widetilde{k}_0\widetilde{v}_t\widetilde{h}\widetilde{v}_t^{-1}\widetilde{k}_0^{-1}\widetilde{D}(\alpha,0))\\ &= \varphi(\widetilde{k}_0\widetilde{D}(\alpha,0)\widetilde{h}\widetilde{D}(\alpha,0)\widetilde{k}_0^{-1})\\ &= \varphi(\widetilde{D}(\alpha,0)\widetilde{h}\widetilde{D}(\alpha,0))\\ &= \psi_\alpha(h),\end{aligned}$$

so ψ_α is an element of $C(H//H_0)$. The statement on the norms follows in the same way as in [21, Lemma 3.7]. □

Suppose that $\beta \geq \gamma \geq 0$, and let $D(\beta,\gamma)$ and $\widetilde{D}(\beta,\gamma)$ be as before. Let $S_{\beta,\gamma,t}$ be as in Notation 3.8. In what follows, let $\|.\|_{HS}$ denote the Hilbert-Schmidt norm of an operator, and let $h \in H$ be such that

$$\iota(h) = \begin{pmatrix} a+ib & -c+id \\ c+id & a-ib \end{pmatrix}, \tag{5}$$

with $a^2+b^2+c^2+d^2 = 1$. The following is an easy adaptation of [21, Lemma 3.8].

Lemma 3.17. Let $g = (g_0,t) \in \widetilde{G}$. Then $g \in S_{\beta,\gamma,t}$, where $\beta,\gamma \in \mathbb{R}$ are uniquely determined by the condition $\beta \geq \gamma \geq 0$ together with the equations

$$\begin{aligned}\sinh^2\beta + \sinh^2\gamma &= \frac{1}{8}\|g_0 - (g_0^T)^{-1}\|_{HS}^2,\\ \sinh^2\beta\sinh^2\gamma &= \frac{1}{16}\det(g_0 - (g^T)^{-1}).\end{aligned}$$

Lemma 3.18. Let $\alpha > 0$ and $\beta \geq \gamma \geq 0$. If $\widetilde{h} \in \widetilde{H}$ is such that the corresponding $h \in H$ satisfies (5), and $c = \sqrt{1-a^2-b^2} = \frac{1}{\sqrt{2}}$ and $d = 0$, then $\widetilde{D}(\alpha,0)\widetilde{h}\widetilde{D}(\alpha,0) \in S_{\beta,\gamma,t}$ if and only if

$$\begin{aligned}\sinh\beta\sinh\gamma &= \frac{1}{2}\sinh^2\alpha(1-a^2-b^2),\\ \sinh\beta - \sinh\gamma &= \sinh(2\alpha)|a|,\\ t &= -\tan^{-1}\left(\frac{2ab}{\coth^2(\alpha)+a^2-b^2}\right).\end{aligned} \tag{6}$$

Proof. Let $\alpha > 0$ and $\beta \geq \gamma \geq 0$. By Lemma 3.17, $\widetilde{D}(\alpha,0)\widetilde{h}\widetilde{D}(\alpha,0) \in S_{\beta,\gamma,t}$ if and only if

$$\begin{aligned}\sinh^2\beta + \sinh^2\gamma &= \frac{1}{8}\|D(\alpha,0)hD(\alpha,0) - D(\alpha,0)^{-1}hD(\alpha,0)^{-1}\|_{HS}^2\\ &= \sinh^2(2\alpha)a^2 + \sinh^2\alpha,\end{aligned} \tag{7}$$

and

$$\begin{aligned}\sinh^2\beta\sinh^2\gamma &= \frac{1}{16}\det(D(\alpha,0)hD(\alpha,0) - D(\alpha,0)^{-1}hD(\alpha,0)^{-1}) \\ &= \frac{1}{4}\sinh^4\alpha,\end{aligned} \tag{8}$$

and, using the explicit expression of (2),

$$e^{it} = \frac{\det(\iota(C_{D(\alpha,0)hD(\alpha,0)}))}{|\det(\iota(C_{D(\alpha,0)hD(\alpha,0)}))|}. \tag{9}$$

The fact that the first two equations of (6) hold if and only if (7) and (8) hold was proved in [21, Lemma 3.9]. The rest of the proof consists of computing t. From (9), it follows that

$$t = \arg(\det(\iota(C_{D(\alpha,0)hD(\alpha,0)}))) + 2k\pi \tag{10}$$

for some $k \in \mathbb{Z}$. It is elementary to check that

$$\begin{aligned}\iota(C_{D(\alpha,0)hD(\alpha,0)}) &= \iota(\frac{1}{2}\,(D(\alpha,0)hD(\alpha,0) + D(-\alpha,0)hD(-\alpha,0))) \\ &= \begin{pmatrix}\cosh(2\alpha)a + ib & -\frac{\cosh(\alpha)}{\sqrt{2}} \\ \frac{\cosh(\alpha)}{\sqrt{2}} & a - ib\end{pmatrix}.\end{aligned}$$

Computing the determinant of this matrix yields

$$\det(\iota(C_{D(\alpha,0)hD(\alpha,0)})) = \cosh(2\alpha)a^2 + b^2 + iab - iab\cosh(2\alpha) + \frac{1}{2}\cosh^2(\alpha).$$

Determining the argument is done by taking the inverse tangent of the the imaginary part of this determinant divided by its real part, which yields

$$\begin{aligned}&\arg(\det(\iota(C_{D(\alpha,0)hD(\alpha,0)}))) = \tan^{-1}\left(\frac{ab(1-\cosh(2\alpha))}{\cosh(2\alpha)a^2 + b^2 + \frac{1}{2}\cosh^2(\alpha)}\right) \\ &= -\tan^{-1}\left(\frac{2ab\sinh^2(\alpha)}{a^2 + 2a^2\sinh^2(\alpha) + b^2\cosh^2(\alpha) - b^2\sinh^2(\alpha) + \frac{1}{2}\cosh^2(\alpha)}\right) \\ &= -\tan^{-1}\left(\frac{2ab\sinh^2(\alpha)}{(a^2-b^2)\sinh^2(\alpha) + (\frac{1}{2} + a^2 + b^2)\cosh^2(\alpha)}\right) \\ &= -\tan^{-1}\left(\frac{2ab}{\coth^2(\alpha) + a^2 - b^2}\right).\end{aligned}$$

Since $\coth^2(\alpha) \geq 1$ for all $\alpha > 0$, the argument of the inverse tangent is clearly a bounded function. Hence, the value of k in (10) is the same for the whole family of elements of the form $\widetilde{D}(\alpha,0)\widetilde{h}\widetilde{D}(\alpha,0)$. Since there exists a continuous path from any $\widetilde{D}(\alpha,0)\widetilde{h}\widetilde{D}(\alpha,0)$ to the identity element of $\widetilde{G}$, it follows that $k = 0$. □

We now consider a different pair of groups in $\widetilde{G}$. The natural embedding of SO(2) in SU(2) gives rise to a subgroup H_1 of H and to a subgroup $\widetilde{H}_1$ of $\widetilde{H}$. The pair (H, H_1) is a compact Gelfand pair and was used in [21] as well. If $h \in \mathrm{SU}(2)$ satisfies (5), then the double cosets of SO(2) in SU(2) are labeled by $a^2-b^2+c^2-d^2$. Hence, every SO(2)-bi-invariant function $\chi : \mathrm{SU}(2) \to \mathbb{C}$ is of the form $\chi(h) = \chi^0(a^2-b^2+c^2-d^2)$ for a certain function $\chi^0 : [-1,1] \to \mathbb{C}$, since $\mathrm{SO}(2)\backslash\mathrm{SU}(2)/\mathrm{SO}(2) \cong [-1,1]$. The spherical functions for this Gelfand pair are indexed by $n \geq 0$, and given by $P_n(a^2-b^2+c^2-d^2)$, where P_n denotes the n^{th} Legendre polynomial. For details, we refer to [21].

The following estimate was proved (in this explicit form) in [21, Lemma 3.11]. Similar estimates were already proved in [30], and, as was remarked in [21, Remark 3.12], they can also be obtained from Szegö's book [38].

Lemma 3.19. For all non-negative integers n,

$$|P_n(x) - P_n(y)| \leq 4|x-y|^{\frac{1}{2}}$$

for $x, y \in [-\frac{1}{2}, \frac{1}{2}]$, i.e., the Legendre polynomials are uniformly Hölder continuous on $[-\frac{1}{2}, \frac{1}{2}]$ with exponent $\frac{1}{2}$.

Lemma 3.20. Let $\varphi \in M_0A(\mathrm{SO}(2)\backslash \mathrm{SU}(2)/\mathrm{SO}(2))$. Suppose that $h \in \mathrm{SU}(2)$ is of the form

$$h = \begin{pmatrix} a+ib & -c+id \\ c+id & a-ib \end{pmatrix},$$

where $a, b, c, d \in \mathbb{R}$ such that $a^2+b^2+c^2+d^2 = 1$. Then $\varphi(h) = \varphi^0(r)$, where $r = a^2 - b^2 + c^2 - d^2$, for a certain function $\varphi^0 : [-1,1] \to \mathbb{C}$, and

$$\varphi^0 = \sum_{n \geq 0} c_n P_n$$

such that $\sum_{n\geq 0} |c_n| = \|\varphi\|_{M_0A(\mathrm{SU}(2))}$. Moreover, φ^0 satisfies

$$|\varphi^0(r_1) - \varphi^0(r_2)| \leq 4|r_1 - r_2|^{\frac{1}{2}} \|\varphi\|_{M_0A(\widetilde{G})}$$

for all $r_1, r_2 \in [-\frac{1}{2}, \frac{1}{2}]$.

The above lemma follows directly from Proposition 2.1 and Lemma 3.19. Note that we can identify $M_0A(H_1\backslash H/H_1)$ with $M_0A(\mathrm{SO}(2)\backslash \mathrm{SU}(2)/SO(2))$.

Notation 3.21. In what follows, we use the notation $v = v_{\frac{\pi}{4}}$ and $\widetilde{v} = \widetilde{v}_{\frac{\pi}{4}}$.

The proof of the following proposition is similar to the proof of Proposition 3.16.

Proposition 3.22. Let $\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C}$. For $\alpha \geq 0$, let $\chi'_\alpha : H \to \mathbb{C}$ be given by $h \mapsto \varphi(\widetilde{D}(\alpha,\alpha)\widetilde{v}\widetilde{h}\widetilde{D}(\alpha,\alpha))$, and let $\chi''_\alpha : H \to \mathbb{C}$ be given by $h \mapsto \varphi(\widetilde{D}(\alpha,\alpha)\widetilde{v}^{-1}\widetilde{h}\widetilde{D}(\alpha,\alpha))$. These functions are elements of $M_0A(H_1\backslash H/H_1)$, and $\|\chi'_\alpha\|_{M_0A(H)} \leq \|\varphi\|_{M_0A(\widetilde{G})}$ and $\|\chi''_\alpha\|_{M_0A(H)} \leq \|\varphi\|_{M_0A(\widetilde{G})}$.

Suppose that $\beta \geq \gamma \geq 0$, and let $D(\beta,\gamma)$ and $\widetilde{D}(\beta,\gamma)$ be as before.

Lemma 3.23. Let $\alpha > 0$ and $\beta \geq \gamma \geq 0$. If $\widetilde{h}$ is such that the corresponding h satisfies (5), then $\widetilde{D}(\alpha,\alpha)\widetilde{v}\widetilde{h}\widetilde{D}(\alpha,\alpha) \in S_{\beta,\gamma,t}$ if and only if

$$\begin{cases} \sinh^2\beta + \sinh^2\gamma = \sinh^2(2\alpha), \\ \sinh\beta\sinh\gamma = \frac{1}{2}\sinh^2(2\alpha)|r|, \\ t = \frac{\pi}{2} - \tan^{-1}\left(\frac{\sinh^2(2\alpha)}{2\cosh(2\alpha)}r\right), \end{cases}$$

and $\widetilde{D}(\alpha,\alpha)\widetilde{v}^{-1}\widetilde{h}\widetilde{D}(\alpha,\alpha) \in S_{\beta,\gamma,t}$ if and only if

$$\begin{cases} \sinh^2\beta + \sinh^2\gamma = \sinh^2(2\alpha), \\ \sinh\beta\sinh\gamma = \frac{1}{2}\sinh^2(2\alpha)|r|, \\ t = -\frac{\pi}{2} + \tan^{-1}\left(\frac{\sinh^2(2\alpha)}{2\cosh(2\alpha)}r\right), \end{cases} \tag{11}$$

where $r = a^2 - b^2 + c^2 - d^2$.

Proof. Let $\alpha > 0$ and $\beta \geq \gamma \geq 0$. By Lemma 3.17, $\widetilde{D}(\alpha,\alpha)\widetilde{v}\widetilde{h}\widetilde{D}(\alpha,\alpha) \in S_{\beta,\gamma,t}$ if and only if

$$\begin{aligned}\sinh^2\beta + \sinh^2\gamma &= \frac{1}{8}\|D(\alpha,\alpha)vhD(\alpha,\alpha) - D(\alpha,\alpha)^{-1}vhD(\alpha,\alpha)^{-1}\|_{HS}^2 \\ &= \sinh^2(2\alpha),\end{aligned} \tag{12}$$

and

$$\begin{aligned}\sinh^2\beta\sinh^2\gamma &= \frac{1}{16}\det(D(\alpha,\alpha)vhD(\alpha,0) - D(\alpha,\alpha)^{-1}vhD(\alpha,\alpha)^{-1}) \\ &= \frac{1}{4}\sinh^4(2\alpha)r^2,\end{aligned} \tag{13}$$

and, using the explicit expression of (2),

$$e^{it} = \frac{\det(\iota(C_{D(\alpha,\alpha)vhD(\alpha,\alpha)}))}{|\det(\iota(C_{D(\alpha,\alpha)vhD(\alpha,\alpha)}))|}. \tag{14}$$

The first two equations of (11) are now obvious. The last part of the proof consists of computing t. From (14), it follows that

$$t = \arg(\det(\iota(C_{D(\alpha,\alpha)vhD(\alpha,\alpha)}))) + 2k\pi$$

for some $k \in \mathbb{Z}$. It is elementary to check that

$$\begin{aligned}\iota(C_{D(\alpha,\alpha)vhD(\alpha,\alpha)}) &= \iota(\frac{1}{2}(D(\alpha,\alpha)vhD(\alpha,\alpha) + D(-\alpha,-\alpha)vhD(-\alpha,-\alpha))) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}\cosh(2\alpha)(a-b) + i(a+b) & -\cosh(2\alpha)(c+d) - i(c-d) \\ \cosh(2\alpha)(c-d) + i(c+d) & \cosh(2\alpha)(a+b) + i(a-b)\end{pmatrix}.\end{aligned}$$

Computing the determinant of this matrix yields

$$\begin{aligned}&\det(\iota(C_{D(\alpha,\alpha)vhD(\alpha,\alpha)})) \\ &= \frac{1}{2}(a^2 - b^2 + c^2 - d^2)(\cosh^2(2\alpha) - 1) + i(a^2 + b^2 + c^2 + d^2)\cosh(2\alpha).\end{aligned}$$

Determining the argument is done by taking the inverse tangent of the the imaginary part of this determinant divided by its real part. By $\arg(x+iy) = \tan^{-1}(\frac{y}{x}) = \frac{\pi}{2} - \tan^{-1}(\frac{x}{y})$ for $x \neq 0$ and $y > 0$, we obtain

$$\begin{aligned}\arg(\det(\iota(C_{D(\alpha,\alpha)vhD(\alpha,\alpha)}))) &= \frac{\pi}{2} - \tan^{-1}\left(\frac{(a^2 - b^2 + c^2 - d^2)(2\cosh^2(2\alpha) - 1)}{(a^2 + b^2 + c^2 + d^2)\cosh(2\alpha)}\right) \\ &= \frac{\pi}{2} - \tan^{-1}\left(\frac{\sinh^2(2\alpha)}{2\cosh(2\alpha)}r\right).\end{aligned}$$

The second inclusion we have to consider, i.e., $\widetilde{D}(\alpha,\alpha)\widetilde{v}^{-1}\widetilde{h}\widetilde{D}(\alpha,\alpha) \in S_{\beta,\gamma,t}$ is very similar. It is easy to check that this holds if and only if (12) and (13) hold. As for the value of t, it is very similar to the first case. Indeed, it is again elementary to check that

$$\begin{aligned}\iota(C_{D(\alpha,\alpha)v^{-1}hD(\alpha,\alpha)}) &= \iota(\frac{1}{2}(D(\alpha,\alpha)v^{-1}hD(\alpha,\alpha) + D(-\alpha,-\alpha)v^{-1}hD(-\alpha,-\alpha))) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix}\cosh(2\alpha)(a+b) - i(a-b) & -\cosh(2\alpha)(c-d) + i(c+d) \\ \cosh(2\alpha)(c+d) - i(c-d) & \cosh(2\alpha)(a-b) - i(a+b)\end{pmatrix}.\end{aligned}$$

It follows that

$$\arg(\det(\iota(C_{D(\alpha,\alpha)v^{-1}hD(\alpha,\alpha)}))) = -\frac{\pi}{2} + \tan^{-1}\left(\frac{\sinh^2(2\alpha)}{2\cosh(2\alpha)}r\right).$$

By an argument similar to the one in the proof of Lemma 3.18, it follows that $k = 0$, giving the correct values of t. □

We will now prove that multipliers in $M_0A(\widetilde{G}) \cap \mathcal{C}$ are almost constant on certain paths in the groups.

Proposition 3.24. Let $\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C}$. If $\alpha > 0$ and $|\tau_1 - \tau_2| \leq \frac{\pi}{2}$, then

$$|\dot{\varphi}(2\alpha, 0, \tau_1) - \dot{\varphi}(2\alpha, 0, \tau_2)| \leq 24e^{-\alpha} \|\varphi\|_{M_0A(\widetilde{G})}.$$

In order to prove this result, we need the following lemma.

Lemma 3.25. Let $\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C}$, let $\alpha \geq 2$ and $\tau \in [-\frac{\pi}{4}, \frac{\pi}{4}]$. Let $r = -\frac{2\cosh(2\alpha)}{\sinh^2(2\alpha)} \tan(\tau)$, and let $\beta \geq \gamma \geq 0$ be the unique numbers for which

$$\sinh\beta = \frac{1}{2}\sinh(2\alpha)(\sqrt{1+|r|} + \sqrt{1-|r|}),$$
$$\sinh\gamma = \frac{1}{2}\sinh(2\alpha)(\sqrt{1+|r|} - \sqrt{1-|r|}).$$

Then

$$|\dot{\varphi}(\beta, \gamma, \tau) - \dot{\varphi}(2\alpha, 0, 0)| \leq 12e^{-\alpha} \|\varphi\|_{M_0A(\widetilde{G})}.$$

Proof. One easily checks that $\sinh^2\beta + \sinh^2\gamma = \sinh^2(2\alpha)$ and $2\sinh\beta\sinh\gamma = \sinh^2(2\alpha)|r|$. Put

$$g(r) = \widetilde{D}(\alpha, \alpha)\widetilde{v}\widetilde{h}(r)\widetilde{D}(\alpha, \alpha) \in S_{\beta,\gamma,\tau'}$$

where

$$\tau' = \frac{\pi}{2} - \tan^{-1}\left(\frac{\sinh^2(2\alpha)}{2\cosh(2\alpha)} r\right) = \frac{\pi}{2} + \tau$$

and $\widetilde{h}(r)$ is any element in $\widetilde{H}$ satisfying $a^2 - b^2 + c^2 - d^2 = r$. By Proposition 3.22, we obtain

$$|\varphi(g(r)) - \varphi(g(0))| \leq 4|r|^{\frac{1}{2}} \|\varphi\|_{M_0A(\widetilde{G})},$$

provided that $|r| \leq \frac{1}{2}$. Since $g(0)$ corresponds to $r = 0$, it follows that the corresponding $\tau_0 = \frac{\pi}{2}$. Hence, $g(0) \in S_{2\alpha,0,\frac{\pi}{2}}$.

Hence, by the invariance property of $\mathcal{C}$ of Lemma 3.9,

$$|\dot{\varphi}(\beta, \gamma, \tau) - \dot{\varphi}(2\alpha, 0, 0)| \leq 4|r|^{\frac{1}{2}} \|\varphi\|_{M_0A(\widetilde{G})},$$

provided that $|r| \leq \frac{1}{2}$. However, since $|\tau| \leq \frac{\pi}{4}$, we have $|\tan\tau| \leq 1$. It follows that $|r| \leq \frac{2\cosh(2\alpha)}{\sinh^2(2\alpha)} \leq \frac{4e^{2\alpha}(1+e^{-4\alpha})}{e^{4\alpha}(1-e^{-4\alpha})^2} \leq 4e^{-2\alpha}\left(\frac{1+e^{-8}}{(1-e^{-8})^2}\right) \leq 5e^{-2\alpha}$ for $\alpha \geq 2$. Then $|r| \leq 5e^{-4} < \frac{1}{2}$. This implies that

$$|\dot{\varphi}(\beta, \gamma, \tau) - \dot{\varphi}(2\alpha, 0, 0)| \leq 12e^{-\alpha} \|\varphi\|_{M_0A(\widetilde{G})}.$$

□

Proof of Proposition 3.24. Put $\tau = \frac{\tau_1 - \tau_2}{2}$. It is sufficient to prove that

$$|\dot{\varphi}(2\alpha, 0, \tau) - \dot{\varphi}(2\alpha, 0, -\tau)| \leq 24e^{-\alpha} \|\varphi\|_{M_0A(\widetilde{G})}.$$

Construct $\beta \geq \gamma \geq 0$ as in Lemma 3.25. Observe that this gives the same for τ and $-\tau$. Replacing $g(r) = \widetilde{D}(\alpha, \alpha)\widetilde{v}\widetilde{h}(r)\widetilde{D}(\alpha, \alpha)$ in that lemma by $g(r) = \widetilde{D}(\alpha, \alpha)\widetilde{v}^{-1}\widetilde{h}(r)\widetilde{D}(\alpha, \alpha)$, we obtain

$$|\dot{\varphi}(\beta, \gamma, -\tau) - \dot{\varphi}(2\alpha, 0, 0)| \leq 12e^{-\alpha} \|\varphi\|_{M_0A(\widetilde{G})}$$

for $\alpha \geq 2$. Combining the results, we obtain

$$|\dot{\varphi}(\beta,\gamma,\pm\tau) - \dot{\varphi}(2\alpha,0,0)| \leq 12e^{-\alpha}\|\varphi\|_{M_0A(\widetilde{G})}.$$

Then the invariance property of $\mathcal{C}$ (see Lemma 3.9) implies that

$$|\dot{\varphi}(\beta,\gamma,0) - \dot{\varphi}(2\alpha,0,\mp\tau)| \leq 12e^{-\alpha}\|\varphi\|_{M_0A(\widetilde{G})}$$

for $\alpha \geq 2$. Since $2e^2 \leq 24$, it follows that the desired estimate holds for every $\alpha > 0$. □

Lemma 3.26. Let $\beta \geq \gamma \geq 0$. Then the equations

$$\sinh^2(2s_1) + \sinh^2 s_1 = \sinh^2\beta + \sinh^2\gamma,$$
$$\sinh(2s_2)\sinh s_2 = \sinh\beta\sinh\gamma$$

have unique solutions $s_1 = s_1(\beta,\gamma)$, $s_2 = s_2(\beta,\gamma)$ in the interval $[0,\infty)$. Moreover,

$$s_1 \geq \frac{\beta}{4}, \qquad s_2 \geq \frac{\gamma}{2}. \tag{15}$$

For a proof, see [21, Lemma 3.16]. Note that we have changed notation here.

Lemma 3.27. There exists a constant $\tilde{B} > 0$ such that for $\alpha > 0$, $t \in \mathbb{R}$, $\tau \in [-\frac{\pi}{2},\frac{\pi}{2}]$, $s_1 = s_1(2\alpha,0)$ chosen as in Lemma 3.26, and $\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C}$, we have

$$|\dot{\varphi}(2s_1,s_1,t) - \dot{\varphi}(2s_1,s_1,t+\tau)| \leq \tilde{B}e^{-\frac{\alpha}{4}}\|\varphi\|_{M_0A(\widetilde{G})}.$$

Proof. Let $t \in \mathbb{R}$, and let $\tau \in [-\frac{\pi}{2},\frac{\pi}{2}]$. Suppose first that $\alpha \geq 4$, and let $h_1 \in H$ be such that

$$\iota(h_1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \in \mathrm{SU}(2),$$

i.e., in the parametrization of (5), we have $a = b = \frac{1}{\sqrt{2}}$, $c = d = 0$, and, hence, $r_1 = 0$. By Lemma 3.23, we have $\widetilde{D}(\alpha,\alpha)\widetilde{v}\widetilde{h}_1\widetilde{D}(\alpha,\alpha) \in S_{2\alpha,0,t''}$ for some $t'' \in \mathbb{R}$. Let $s_1 = s_1(2\alpha,0)$ be as in Lemma 3.26. Then $s_1 \geq 0$ and $\sinh^2(2s_1) + \sinh^2 s_1 = \sinh^2(2\alpha)$. Put

$$r_2 = \frac{2\sinh(2s_1)\sinh s_1}{\sinh^2(2s_1) + \sinh^2 s_1} \in [0,1],$$

and let $h_2 \in H$ be such that

$$\iota(h_2) = \begin{pmatrix} a_2 + ib_2 & 0 \\ 0 & a_2 - ib_2 \end{pmatrix} \in \mathrm{SU}(2),$$

where $a_2 = \left(\frac{1+r_2}{2}\right)^{\frac{1}{2}}$ and $b_2 = \left(\frac{1-r_2}{2}\right)^{\frac{1}{2}}$. Since $a_2^2 - b_2^2 = r_2$, it follows again by Lemma 3.23 that $\widetilde{D}(\alpha,\alpha)\widetilde{v}\widetilde{h}_2\widetilde{D}(\alpha,\alpha) \in S_{2s_1,s_1,t'}$ for some $t' \in \mathbb{R}$.

Let $\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C}$, and let $\chi'_\alpha(h) = \varphi(\widetilde{D}(\alpha,\alpha)\widetilde{v}\widetilde{h}\widetilde{D}(\alpha,\alpha))$ for $h \in H$ as in Proposition 3.22. By the same proposition, given the fact that $r_1 = 0$ and provided that $r_2 \leq \frac{1}{2}$, it follows that

$$\begin{aligned} |\dot{\varphi}(2s_1,s_1,t') - \dot{\varphi}(2\alpha,0,t'')| &\leq |\chi'_\alpha(h_1) - \chi'_\alpha(h_2)| \\ &= |\chi'^{,0}_\alpha(r_1) - \chi'^{,0}_\alpha(r_2)| \\ &\leq 4r_2^{\frac{1}{2}}\|\varphi\|_{M_0A(\widetilde{G})}, \end{aligned} \tag{16}$$

where $\chi'^{,0}_\alpha$ is the function on $[-1,1]$ induced by χ'_α. Note that $r_2 \leq 2\frac{\sinh s_1}{\sinh 2s_1} = \frac{1}{\cosh s_1} \leq 2e^{-s_1}$. By Lemma 3.26, equation (15), we obtain that $r_2 \leq 2e^{-\frac{\alpha}{2}} \leq 2e^{-2} \leq \frac{1}{2}$. In particular, (16) holds, and we have $r_2 \leq 2e^{-\frac{\alpha}{2}}$. The estimate above is independent on the choice of $\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C}$, so by the invariance property of Lemma 3.9, it follows that

$$\begin{aligned}&|\dot{\varphi}(2s_1, s_1, t') - \dot{\varphi}(2s_1, s_1, t' + \tau)|\\ &\leq |\dot{\varphi}(2s_1, s_1, t') - \dot{\varphi}(2\alpha, 0, t'')| + |\dot{\varphi}(2\alpha, 0, t'') - \dot{\varphi}(2\alpha, 0, t'' + \tau)|\\ &\quad + |\dot{\varphi}(2\alpha, 0, t'' + \tau) - \dot{\varphi}(2s_1, s_1, t' + \tau)|\\ &\leq (8\sqrt{2}e^{-\frac{\alpha}{4}} + 24e^{-\alpha})\|\varphi\|_{M_0A(\widetilde{G})}.\end{aligned}$$

By the invariance property of Lemma 3.9, the desired estimate follows with $\tilde{B} = 8\sqrt{2} + 24$. □

By the following two lemmas, we can estimate the difference between $\dot{\varphi}(\beta, \gamma, t)$ and the value of φ at a certain point on the line $\{(2s, s, t) \mid s \in \mathbb{R}_+\}$. The method is similar to the one used in [21, Lemma 3.17 and Lemma 3.18], but because of the t-dependence, there is an extra parameter. Lemma 3.27 provides us with the tools to deal with this extra parameter.

Lemma 3.28. There exists a constant $B_1 > 0$ such that whenever $\beta \geq \gamma \geq 0$, $t \in \mathbb{R}$, and $s_1 = s_1(\beta, \gamma)$ is chosen as in Lemma 3.26, then for all $\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C}$,

$$|\dot{\varphi}(\beta, \gamma, t) - \dot{\varphi}(2s_1, s_1, t)| \leq B_1 e^{-\frac{\beta - \gamma}{8}} \|\varphi\|_{M_0A(\widetilde{G})}.$$

Proof. Let $\beta \geq \gamma \geq 0$ and $t \in \mathbb{R}$. Assume first that $\beta - \gamma \geq 8$. Let $\alpha \in [0, \infty)$ be the unique solution to $\sinh^2 \beta + \sinh^2 \gamma = \sinh^2(2\alpha)$, and observe that $2\alpha \geq \beta \geq 2$, so in particular $\alpha > 0$. Define

$$r_1 = \frac{2 \sinh \beta \sinh \gamma}{\sinh^2 \beta + \sinh^2 \gamma} \in [0, 1],$$

and $a_1 = \left(\frac{1+r_1}{2}\right)^{\frac{1}{2}}$ and $b_1 = \left(\frac{1-r_1}{2}\right)^{\frac{1}{2}}$. Furthermore, let $h_1 \in H$ be such that

$$\iota(h_1) = \begin{pmatrix} a_1 + ib_1 & 0 \\ 0 & a_1 - ib_1 \end{pmatrix} \in \mathrm{SU}(2),$$

and let $\widetilde{v}$ be as before. We now have $2 \sinh \beta \sinh \gamma = \sinh^2(2\alpha) r_1$, and $a_1^2 - b_1^2 = r_1$, so by Lemma 3.23, we have $\widetilde{D}(\alpha, \alpha)\widetilde{v}\widetilde{h}_1\widetilde{D}(\alpha, \alpha) \in S_{\beta,\gamma,t'}$ for some $t' \in \mathbb{R}$.

Let now $s_1 = s_1(\beta, \gamma)$ be as in Lemma 3.26. Then $s_1 \geq 0$ and $\sinh^2(2s_1) + \sinh^2 s_1 = \sinh^2 \beta + \sinh^2 \gamma = \sinh^2(2\alpha)$. Similar to the proof of Lemma 3.27, put

$$r_2 = \frac{2 \sinh(2s_1) \sinh s_1}{\sinh^2(2s_1) + \sinh^2 s_1} \in [0, 1]$$

and let $h_2 \in H$ be such that

$$\iota(h_2) = \begin{pmatrix} a_2 + ib_2 & 0 \\ 0 & a_2 - ib_2 \end{pmatrix} \in \mathrm{SU}(2),$$

where $a_2 = \left(\frac{1+r_2}{2}\right)^{\frac{1}{2}}$ and $b_2 = \left(\frac{1-r_2}{2}\right)^{\frac{1}{2}}$. Since $a_2^2 - b_2^2 = r_2$, it follows again by Lemma 3.23 that $\widetilde{D}(\alpha, \alpha)\widetilde{v}\widetilde{h}_2\widetilde{D}(\alpha, \alpha) \in S_{2s_2, s_2, t''}$ for some $t'' \in \mathbb{R}$.

Now, let $\chi'_\alpha(h) = \varphi(\widetilde{D}(\alpha, \alpha)\widetilde{v}\widetilde{h}\widetilde{D}(\alpha, \alpha))$ for $h \in H$ as in Proposition 3.22. By the same proposition, it follows that

$$|\chi'_\alpha(h_1) - \chi'_\alpha(h_2)| = |\chi'^{,0}_\alpha(r_1) - \chi'^{,0}_\alpha(r_2)| \leq 4|r_1 - r_2|^{\frac{1}{2}} \|\varphi\|_{M_0A(\widetilde{G})}, \tag{17}$$

provided that $r_1, r_2 \leq \frac{1}{2}$. Note that $r_1 \leq \frac{2 \sinh \beta \sinh \gamma}{\sinh^2 \beta} = 2\frac{\sinh \gamma}{\sinh \beta}$. Hence, using $\beta \geq \gamma + 8 \geq \gamma$, we get $r_1 \leq 2\frac{e^\gamma(1-e^{2\gamma})}{e^\beta(1-e^{2\beta})} \leq 2e^{\gamma - \beta}$. In particular, $r_1 \leq 2e^{-8} \leq \frac{1}{2}$.

Similarly, $r_2 \leq 2\frac{\sinh s_1}{\sinh 2s_1} = \frac{1}{\cosh s_1} \leq 2e^{-s_1}$. By Lemma 3.26, equation (15), we obtain that $r_2 \leq 2e^{-\frac{\beta}{4}} \leq 2e^{\frac{\gamma-\beta}{4}} \leq 2e^{-2} \leq \frac{1}{2}$. In particular, (17) holds. Moreover, $|r_1 - r_2| \leq \max\{r_1, r_2\} \leq 2e^{\frac{\gamma-\beta}{4}}$.

Because of the explicit form of t' and t'', we have $|t' - t''| \leq \frac{\pi}{2}$. It follows that

$$|\dot{\varphi}(\beta,\gamma,t')-\dot{\varphi}(2s_2,s_2,t')| \leq |\dot{\varphi}(\beta,\gamma,t')-\dot{\varphi}(2s_1,s_1,t'')|+|\dot{\varphi}(2s_1,s_1,t')-\dot{\varphi}(2s_1,s_1,t'')|.$$

The first summand is estimated by $4\sqrt{2}e^{-\frac{\beta-\gamma}{8}}\|\varphi\|_{M_0A(\widetilde{G})}$ by (17), and the second summand is estimated by $\tilde{B}e^{-\frac{\alpha}{4}}$ by Lemma 3.27. It follows that

$$\begin{aligned}|\dot{\varphi}(\beta,\gamma,t') - \dot{\varphi}(2s_2,s_2,t')| &\leq 4\sqrt{2}e^{-\frac{\beta-\gamma}{8}}\|\varphi\|_{M_0A(\widetilde{G})} + \tilde{B}e^{-\frac{\alpha}{4}}\|\varphi\|_{M_0A(\widetilde{G})} \\ &\leq (4\sqrt{2}+\tilde{B})e^{-\frac{\beta-\gamma}{8}}\|\varphi\|_{M_0A(\widetilde{G})}\end{aligned}$$

under the assumption that $\beta \geq \gamma + 8$. By shifting over $t - t'$ (cf. Lemma 3.9), we obtain the estimate of the lemma for $\beta \geq \gamma + 8$. In general, the assertion of the lemma follows with $B_1 = \max\{4\sqrt{2}+\tilde{B}, 2e^2\} = 4\sqrt{2}+\tilde{B}$. □

Lemma 3.29. There exists a constant $B_2 > 0$ such that whenever $\beta \geq \gamma \geq 0$, $t \in \mathbb{R}$, and $s_2 = s_2(\beta,\gamma)$ is chosen as in Lemma 3.26, then for all $\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C}$,

$$|\dot{\varphi}(\beta,\gamma,t) - \dot{\varphi}(2s_2,s_2,t)| \leq B_2 e^{-\frac{\gamma}{8}}\|\varphi\|_{M_0A(G)}.$$

Proof. Let $\beta \geq \gamma \geq 0$ and $t \in \mathbb{R}$. Assume first that $\gamma \geq 2$, and let $\alpha \in [0,\infty)$ be the unique solution in $[0,\infty)$ to the equation $\sinh\beta\sinh\gamma = \frac{1}{2}\sinh^2\alpha$, and observe that $\alpha > 0$, because $\beta \geq \gamma \geq 2$. Put

$$a_1 = \frac{\sinh\beta - \sinh\gamma}{\sinh(2\alpha)} \geq 0.$$

Since $\sinh(2\alpha) = 2\sinh\alpha\cosh\alpha \geq 2\sinh^2\alpha$, we have

$$a_1 \leq \frac{\sinh\beta}{\sinh(2\alpha)} \leq \frac{\sinh\beta}{2\sinh^2\alpha} = \frac{1}{4\sinh\gamma}.$$

In particular, $a_1 \leq \frac{1}{4\gamma} \leq \frac{1}{8}$. Put now $b_1 = \sqrt{\frac{1}{2} - a_1^2}$. Then $1 - a_1^2 - b_1^2 = \frac{1}{2}$. Hence, we have $\sinh\beta - \sinh\gamma = \sinh(2\alpha)a_1$. Let $h_1 \in H$ be such that

$$\iota(h_1) = \begin{pmatrix} a_1 + ib_1 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & a_1 - ib_1 \end{pmatrix} \in \mathrm{SU}(2).$$

By Lemma 3.18, we have $\widetilde{D}(\alpha,0)\widetilde{h_1}\widetilde{D}(\alpha,0) \in S_{\beta,\gamma,t'}$, where t' is determined by the equations in that lemma. By Lemma 3.26, we have $\sinh(2s_2)\sinh s_2 = \sinh\beta\sinh\gamma = \frac{1}{2}\sinh^2\alpha$. Moreover, by (15), we have $s_2 \geq \frac{\gamma}{2} \geq 1$. By replacing (β,γ) in the above calculation with $(2s_2, s_2)$, we get that the number

$$a_2 = \frac{\sinh(2s_2) - \sinh s_2}{\sinh(2\alpha)} \geq 0,$$

satisfies

$$a_2 \leq \frac{1}{4\sinh s_2} \leq \frac{1}{4\sinh 1} \leq \frac{1}{4}.$$

Hence, we can put $b_2 = \sqrt{\frac{1}{2} - a_2^2}$ and let $h_2 \in H$ be such that

$$\iota(h_2) = \begin{pmatrix} a_2 + ib_2 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & a_2 - ib_2 \end{pmatrix}.$$

Then

$$\sinh(2s_2)\sinh s_2 = \sinh^2\alpha(1-a_2^2-b_2^2),$$
$$\sinh(2s_2)-\sinh s_2 = \sinh(2\alpha)a_2,$$

and $\iota(h_2)\in \mathrm{SU}(2)$. Hence, by Lemma 3.18, $\widetilde{D}(\alpha,0)\widetilde{h}_2\widetilde{D}(\alpha,0)\in S_{2s_2,s_2,t''}$, where t'' is determined by the equations in that lemma. It follows from the explicit formula for t, and from the fact that they have the same sign, that $|t'-t''|\leq \frac{\pi}{2}$. Put now $\theta_j = \arg(a_j+ib_j) = \frac{\pi}{2}-\sin^{-1}\left(\frac{a_j}{\sqrt{2}}\right)$ for $j=1,2$. Since $0\leq a_j\leq \frac{1}{2}$ for $j=1,2$, and since $\frac{d}{dy}\sin^{-1}y = \frac{1}{\sqrt{1-y^2}}\leq\sqrt{2}$ for $y\in[0,\frac{1}{\sqrt{2}}]$, it follows that

$$\begin{aligned}|\theta_1-\theta_2| &\leq \left|\sin^{-1}\left(\frac{a_1}{\sqrt{2}}\right)-\sin^{-1}\left(\frac{a_2}{\sqrt{2}}\right)\right|\\ &\leq |a_1-a_2|\\ &\leq \max\{a_1,a_2\}\\ &\leq \max\left\{\frac{1}{4\sinh\gamma},\frac{1}{4\sinh t}\right\}\\ &\leq \frac{1}{4\sinh\frac{\gamma}{2}},\end{aligned}$$

because $y\geq\frac{\gamma}{2}$. Since $\gamma\geq 2$, we have $\sinh\frac{\gamma}{2} = \frac{1}{2}e^{\frac{\gamma}{2}}(1-e^{-\gamma})\geq\frac{1}{4}e^{\frac{\gamma}{2}}$. Hence, $|\theta_1-\theta_2|\leq e^{-\frac{\gamma}{2}}$. Note that $a_j = \frac{1}{\sqrt{2}}e^{i\theta_j}$ for $j=1,2$, so by Proposition 3.16, we have

$$\begin{aligned}|\dot{\varphi}(2s_2,s_2,t'')-\dot{\varphi}(\beta,\gamma,t')| &\leq |\psi_\alpha(h_1)-\psi_\alpha(h_2)|\\ &\leq \tilde{C}|\theta_1-\theta_2|^{\frac{1}{4}}\|\psi_\alpha\|_{M_0A(H)}\\ &\leq \tilde{C}e^{-\frac{\gamma}{8}}\|\varphi\|_{M_0A(\widetilde{G})}.\end{aligned}\tag{18}$$

Since $\widetilde{D}(\alpha,0)\widetilde{h}_1\widetilde{D}(\alpha,0)\in S_{\beta,\gamma,t'}$ and $\widetilde{D}(\alpha,0)\widetilde{h}_2\widetilde{D}(\alpha,0)\in S_{2s_2,s_2,t''}$, it follows that

$$|\dot{\varphi}(\beta,\gamma,t')-\dot{\varphi}(2s_2,s_2,t')|\leq|\dot{\varphi}(\beta,\gamma,t')-\dot{\varphi}(2s_2,s_2,t'')|+|\dot{\varphi}(2s_1,s_1,t')-\dot{\varphi}(2s_2,s_2,t'')|.$$

The first summand is estimated by $\tilde{C}e^{-\frac{\gamma}{8}}\|\varphi\|_{M_0A(\widetilde{G})}$ by (18), and the second summand is estimated by $\tilde{B}e^{-\frac{\alpha}{4}}$. It now follows that

$$|\dot{\varphi}(\beta,\gamma,t')-\dot{\varphi}(2s_1,s_1,t')|\leq\tilde{C}e^{-\frac{\gamma}{8}}\|\varphi\|_{M_0A(\widetilde{G})}+\tilde{B}e^{-\frac{\alpha}{4}}\|\varphi\|_{M_0A(\widetilde{G})}.$$

Using the fact that $e^{-\frac{\alpha}{4}}\leq e^{-\frac{\gamma}{8}}$ and using the invariance property of Lemma 3.9, the desired estimate follows with $B_2=\max\{\tilde{C},2e^{\frac{1}{4}}\}$. □

We state the following lemma. For a proof, see [21, Lemma 3.19].

Lemma 3.30. Let $s_1\geq s_2\geq 0$. Then the equations

$$\sinh^2\beta+\sinh^2\gamma = \sinh^2(2s_1)+\sinh^2 s_1,$$
$$\sinh\beta\sinh\gamma = \sinh(2s_2)\sinh s_2,$$

have a unique solution $(\beta,\gamma)\in\mathbb{R}^2$ for which $\beta\geq\gamma\geq 0$. Moreover, if $1\leq s_2\leq s_1\leq\frac{3}{2}s_2$, then

$$\begin{aligned}|\beta-2s_1|&\leq 1,\\ |\gamma+2s_1-3s_2|&\leq 1.\end{aligned}\tag{19}$$

Lemma 3.31. There exists a constant $B_3 > 0$ such that whenever $s_1, s_2 \geq 0$ satisfy $2 \leq s_2 \leq s_1 \leq \frac{6}{5}s_2$ and $t \in \mathbb{R}$, then for all $\varphi \in M_0A(G) \cap \mathcal{C}$,

$$|\dot{\varphi}(2s_1, s_1, t) - \varphi(2s_2, s_2, t)| \leq B_3 e^{-\frac{s_1}{16}} \|\varphi\|_{M_0A(\widetilde{G})}.$$

Proof. Choose $\beta \geq \gamma \geq 0$ as in Lemma 3.30. Then by Lemma 3.28 and Lemma 3.29, we have

$$|\dot{\varphi}(\beta, \gamma, t) - \dot{\varphi}(2s_1, s_1, t)| \leq B_1 e^{-\frac{\beta-\gamma}{8}} \|\varphi\|_{M_0A(\widetilde{G})}$$
$$|\dot{\varphi}(\beta, \gamma, t) - \dot{\varphi}(2s_2, s_2, t)| \leq B_2 e^{-\frac{\gamma}{8}} \|\varphi\|_{M_0A(\widetilde{G})}.$$

Moreover, by (19), we have

$$\beta - \gamma \geq (2s_1 - 1) - (3s_2 - 2s_1 + 1) = 4s_1 - 3s_2 - 2 \geq s_1 - 2,$$
$$\gamma \geq 3s_2 - 2s_1 - 1 \geq \frac{5}{2}s_1 - 2s_1 - 1 = \frac{s_1 - 2}{2}.$$

Hence, since $s_1 \geq 2$, we have $\min\{e^{-\gamma}, e^{-(\beta-\gamma)}\} \leq e^{-\frac{s_1-2}{2}}$. Thus, the lemma follows from Lemma 3.28 and Lemma 3.29 with $B_3 = e^{\frac{1}{8}}(B_1 + B_2)$. ☐

Lemma 3.32. There exists a constant $B_4 > 0$ such that for all $\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C}$ and $t \in \mathbb{R}$ the limit $c_\varphi(t) = \lim_{s_1 \to \infty} \dot{\varphi}(2s_1, s_1, t)$ exists, and for all $s_2 \geq 0$,

$$|\dot{\varphi}(2s_2, s_2, t) - c_\varphi(t)| \leq B_4 e^{-\frac{s_2}{16}} \|\varphi\|_{M_0A(\widetilde{G})}.$$

Proof. Let $\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C}$, and let $t \in \mathbb{R}$. By Lemma 3.31, we have for $u \geq 5$ and $\kappa \in [0,1]$, that

$$|\dot{\varphi}(2u, u, t) - \dot{\varphi}(2(u+\kappa), u+\kappa, t)| \leq B_3 e^{-\frac{u}{16}} \|\varphi\|_{M_0A(\widetilde{G})}. \tag{20}$$

Let $s_1 \geq s_2 \geq 5$. Then $s_1 = s_2 + n + \delta$, where $n \geq 0$ is an integer and $\delta \in [0,1)$. Applying equation (20) to $(u, \kappa) = (s_2 + j, 1)$, $j = 0, 1, \ldots, n-1$ and $(u, \kappa) = (s_2 + n, \delta)$, we obtain

$$|\dot{\varphi}(2s_1, s_1, t) - \dot{\varphi}(2s_2, s_2, t)| \leq B_3 \left(\sum_{j=0}^{n} e^{-\frac{s_2+j}{16}} \right) \|\varphi\|_{M_0A(\widetilde{G})} \leq B_3' e^{-\frac{s_2}{16}} \|\varphi\|_{M_0A(\widetilde{G})},$$

where $B_3' = (1 - e^{-\frac{1}{16}})^{-1} B_3$. Hence $(\dot{\varphi}(2s_1, s_1, t))_{s_1 \geq 5}$ is a Cauchy net for every $t \in \mathbb{R}$. Therefore, $c_\varphi(t) = \lim_{s_1 \to \infty} \dot{\varphi}(2s_1, s_1, t)$ exists, and

$$|\dot{\varphi}(2s_2, s_2, t) - c_\varphi(t)| = \lim_{s_1 \to \infty} |\dot{\varphi}(2s_1, s_1, t) - \dot{\varphi}(2s_2, s_2, t)| \leq B_3' e^{-\frac{s_2}{16}} \|\varphi\|_{M_0A(\widetilde{G})}$$

for all $s_2 \geq 5$. Since $\|\varphi\|_\infty \leq \|\varphi\|_{M_0A(\widetilde{G})}$, we have for all $0 \leq s_2 < 5$,

$$|\dot{\varphi}(2s_2, s_2, t) - c_\varphi(t)| \leq 2\|\varphi\|_{M_0A(\widetilde{G})}.$$

Hence, the lemma follows with $B_4 = \max\{B_3', 2e^{\frac{5}{16}}\}$. ☐

Proof of Proposition 3.11. Let $\varphi \in M_0A(G) \cap \mathcal{C}$ and let $t \in \mathbb{R}$. Let $\beta \geq \gamma \geq 0$. Suppose first that $\beta \geq 2\gamma$. Then $\beta - \gamma \geq \frac{\beta}{2}$, so by Lemma 3.26 and Lemma 3.28, there exists an $s_1 \geq \frac{\beta}{4}$ such that

$$|\dot{\varphi}(\beta, \gamma, t) - \dot{\varphi}(2s_1, s_1, t)| \leq B_1 e^{-\frac{\beta}{16}} \|\varphi\|_{M_0A(\widetilde{G})}.$$

Suppose now that $\beta < 2\gamma$. Then, by Lemma 3.26 and Lemma 3.29, we obtain that there exists an $s_2 \geq \frac{\gamma}{2} > \frac{\beta}{4}$ such that

$$|\dot{\varphi}(\beta, \gamma, t) - \dot{\varphi}(2s_2, s_2, t)| \leq B_2 e^{-\frac{\beta}{16}} \|\varphi\|_{M_0A(\widetilde{G})}.$$

Combining these estimates with Lemma 3.32, and using again that s_1 and s_2 majorize $\frac{\beta}{4}$, it follows that for all $\beta \geq \gamma \geq 0$, we have

$$|\dot{\varphi}(\beta,\gamma,t) - c_\varphi(t)| \leq C_1 e^{-\frac{\beta}{64}} \|\varphi\|_{M_0A(\widetilde{G})}.$$

where $C_1 = \max\{B_1 + B_4, B_2 + B_4\}$. This proves the proposition, for $\sqrt{\beta^2+\gamma^2} \leq \sqrt{2}\beta$. □

Proposition 3.33. For every $\varphi \in M_0A(\widetilde{G}) \cap \mathcal{C}$, the limit function $c_\varphi(t)$ is a constant function.

Proof. From Proposition 3.11 and its proof, we know that for every $\varphi \in M_0A(\widetilde{G})$ the limit $c_\varphi(t) = \lim_{\beta^2+\gamma^2 \to \infty} \dot{\varphi}(\beta,\gamma,t)$ exists and that φ satisfies a certain asymptotic behaviour. It is clear from this expression that the limit may depend on t, but it does not depend on how $\beta^2+\gamma^2$ goes to infinity. In particular, we have $c_\varphi(t) = \lim_{\alpha\to\infty} \dot{\varphi}(2\alpha, 0, t)$. Let τ_1, τ_2 be such that $|\tau_1 - \tau_2| \leq \frac{\pi}{2}$. By Proposition 3.24, we have

$$|\dot{\varphi}(2\alpha,0,\tau_1) - \dot{\varphi}(2\alpha,0,\tau_2)| \leq 24 e^{-2\alpha} \|\varphi\|_{M_0A(\widetilde{G})}.$$

In the limit $\alpha \to \infty$, this expression gives $c_\varphi(\tau_1) = c_\varphi(\tau_2)$. i.e., the function $c_\varphi(t)$ is constant on any interval of length smaller than or equal to $\frac{\pi}{2}$. Hence, the function c_φ is constant. □

Corollary 3.34. The space $M_0A(\widetilde{G}) \cap \mathcal{C}_0$ of completely bounded Fourier multipliers φ in $\mathcal{C}$ for which $c_\varphi \equiv 0$ is a subspace of $M_0A(\widetilde{G}) \cap \mathcal{C}$ of codimension one.

4. Noncommutative L^p-spaces associated with lattices in $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$

Let again $G = \mathrm{Sp}(2,\mathbb{R})$ and $\widetilde{G} = \widetilde{\mathrm{Sp}}(2,\mathbb{R})$. We use the same realization of $\widetilde{G}$ as in Section 3, and we use the same notation as in that section (e.g., for the subgroups K, A, $\overline{A^+}$, H, H_0, H_1 of G and the corresponding subgroups of $\widetilde{G}$). The main result of this section is a statement about the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ for $\widetilde{G}$. This gives rise to the failure of the OAP for certain noncommutative L^p-spaces, which will be explained in Section 5.

Theorem 4.1. For $p \in [1, \frac{12}{11}) \cup (12, \infty]$, the group $\widetilde{G}$ does not have the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$.

The proof follows by combining the method of proof of the failure of the AP for $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ in Section 3 with the methods that were used in [31], [29] to prove the failure of the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ for $\mathrm{SL}(3,\mathbb{R})$ and $\mathrm{Sp}(2,\mathbb{R})$ for certain values of $p \in (1,\infty)$, respectively.

Note that for $p = 1$ and ∞, the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ is equivalent to weak amenability (see [31, Proposition 2.3]), and the failure of weak amenability for $\widetilde{G}$ was proved in [9], so from now on, it suffices to consider $p \in (1,\infty)$. Using an averaging argument similar to the one in Lemma 3.10 (see [29, Lemma 2.8] for more details on averaging functions in the setting of the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$), it follows that if $\widetilde{G}$ has the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ for some $p \in (1,\infty)$, then the approximating net can be chosen in $A(\widetilde{G}) \cap \mathcal{C}$.

The following result, which is a direct analogue of Proposition 3.11, gives a certain asymptotic behaviour of continuous functions φ in $\mathcal{C}$ for which the induced function $\check{\varphi}$ is a Schur multiplier on $S^p(L^2(\widetilde{G}))$. From this, it follows that the constant function 1 cannot be approximated pointwise (and hence not uniformly on compacta) by a net in $A(\widetilde{G}) \cap \mathcal{C}$ in such a way that the net of associated multipliers is uniformly bounded in the $MS^p(L^2(\widetilde{G}))$-norm. This implies Theorem 4.1.

Proposition 4.2. Let $p > 12$. There exist constants $C_1(p), C_2(p)$ (depending on p only) such that for all $\varphi \in C(\widetilde{G}) \cap \mathcal{C}$ for which $\check{\varphi} \in MS^p(L^2(\widetilde{G}))$, and for all $t \in \mathbb{R}$, the limit $\tilde{c}^p_\varphi(t) = \lim_{s\to\infty} \dot{\varphi}(2s, s, t)$ exists, and for all $\beta \geq \gamma \geq 0$,

$$|\dot{\varphi}(\beta, \gamma, t) - \tilde{c}^p_\varphi(t)| \leq C_1(p) e^{-C_2(p)\sqrt{\beta^2+\gamma^2}} \|\check{\varphi}\|_{MS^p(L^2(\widetilde{G}))}.$$

To prove this, we again use the strong Gelfand pair $(\mathrm{SU}(2), \mathrm{U}(1))$ and the Gelfand pair $(\mathrm{SU}(2), \mathrm{SO}(2))$, which sit inside $\widetilde{G}$. For the disc polynomials $h_{l,m}$, we need better estimates than in Lemma 3.13. These were already given in [21, Corollary 3.5].

Lemma 4.3. For all $l, m \geq 0$, and for $\theta_1, \theta_2 \in [0, 2\pi)$, we have

$$\left| h^0_{l,m}\left(\frac{e^{i\theta_1}}{\sqrt{2}}\right) - h^0_{l,m}\left(\frac{e^{i\theta_2}}{\sqrt{2}}\right) \right| \leq C(l+m+1)^{\frac{3}{4}} |\theta_1 - \theta_2|,$$

$$\left| h^0_{l,m}\left(\frac{e^{i\theta_1}}{\sqrt{2}}\right) - h^0_{l,m}\left(\frac{e^{i\theta_2}}{\sqrt{2}}\right) \right| \leq 2C(l+m+1)^{-\frac{1}{4}}.$$

Here $C > 0$ is a uniform constant.

Combining the above two estimates, we get the estimate of Lemma 3.13. Combining Lemma 3.14 and [29, Lemma 2.4], we obtain that for $\varphi \in L^2(\mathrm{SU}(2)/\!/\,\mathrm{U}(1))$, there is an induced function $\varphi^0 : \mathbb{D} \to \mathbb{C}$, and

$$\varphi^0 = \sum_{l,m=0}^{\infty} c_{l,m}(l+m+1) h^0_{l,m}$$

for certain $c_{l,m} \in \mathbb{C}$. Moreover, by [29, Proposition 2.7], we obtain that if $p \in (1, \infty)$, then $\left(\sum_{l,m\geq 0} |c_{l,m}|^p (l+m+1)\right)^{\frac{1}{p}} \leq \|\check{\varphi}\|_{MS^p(L^2(\mathrm{U}(2)))}$.

Lemma 4.4. Let $p > 12$, and let $\varphi \in \mathrm{SU}(2) \to \mathbb{C}$ be a continuous $\mathrm{Int}(\mathrm{U}(1))$-invariant function such that $\check{\varphi}$ is an element of $MS^p(L^2(\mathrm{SU}(2)))$. Then φ^0 satisfies

$$\left| \varphi^0\left(\frac{e^{i\theta_1}}{\sqrt{2}}\right) - \varphi^0\left(\frac{e^{i\theta_2}}{\sqrt{2}}\right) \right| \leq \tilde{C}(p) \|\check{\varphi}\|_{MS^p(L^2(\mathrm{U}(2)))} |\theta_1 - \theta_2|^{\frac{1}{8} - \frac{3}{2p}}$$

for $\theta_1, \theta_2 \in [0, 2\pi)$. Here, $\tilde{C}(p)$ is a constant depending only on p.

The proof of this lemma is exactly the same as the proof of [29, Lemma 3.5] after identifying the spaces $C(\mathrm{SU}(2)/\!/\,\mathrm{U}(1))$ and $C(L \backslash \mathrm{U}(2)/L)$ and proving an isometric isomorphism in the setting of multipliers on Schatten classes as was done in Lemma 3.14 in the setting of completely bounded Fourier multipliers.

Lemma 4.5. Let $\varphi \in C(\widetilde{G}) \cap \mathcal{C}$ such that $\check{\varphi} \in MS^p(L^2(\widetilde{G}))$ for some $p \in (1, \infty)$, and for $\alpha \in \mathbb{R}$, let $\psi_\alpha : H \to \mathbb{C}$ be defined by $\psi_\alpha(h) = \varphi(\widetilde{D}(\alpha, 0)\widetilde{h}\widetilde{D}(\alpha, 0))$. Then ψ_α is an element of $C(H/\!/H_0)$ and satisfies

$$\|\check{\psi}_\alpha\|_{MS^p(L^2(H))} \leq \|\check{\varphi}\|_{MS^p(L^2(\widetilde{G}))}.$$

Proof. The fact that $\psi_\alpha \in C(H/\!/H_0)$ follows as in Proposition 3.16. The second part follows by the fact that $\widetilde{D}(\alpha, 0)\widetilde{H}\widetilde{D}(\alpha, 0)$ is a subset of $\widetilde{G}$ and by applying [29, Lemma 2.3]. □

We now turn to the second pair of groups (H, H_1). We again need the Legendre polynomials, which act as spherical functions. The following estimate was proved in [29, Lemma 3.8].

Lemma 4.6. For all non-negative integers n, and $x, y \in [-\frac{1}{2}, \frac{1}{2}]$,

$$|P_n(x) - P_n(y)| \leq |P_n(x)| + |P_n(y)| \leq \frac{4}{\sqrt{n}},$$

$$|P_n(x) - P_n(y)| \leq \left| \int_x^y P_n'(t) dt \right| \leq 4\sqrt{n}|x - y|.$$

Combining the two estimates above, yields the estimate of Lemma 3.19. Let $\varphi : \mathrm{SU}(2) \to \mathbb{C}$ be a $\mathrm{SO}(2)$-bi-invariant continuous function. Then $\varphi(h) = \varphi^0(r)$ as in Section 3. It follows that $\varphi^0 = \sum_{n=0}^{\infty} c_n(2n+1)P_n$ for certain $c_n \in \mathbb{C}$. Moreover, as above, we obtain that if $p \in (1, \infty)$, then $(\sum_{n \geq 0} |c_n|^p(2n+1))^{\frac{1}{p}} \leq \|\check{\varphi}\|_{MS^p(L^2(\mathrm{SU}(2)))}$, where $\check{\varphi}$ is defined as above by $\check{\varphi}(g, h) = \varphi(g^{-1}h)$. The following result can be found in [29, Lemma 3.9].

Lemma 4.7. Let $p > 4$, and let $\varphi \in C(\mathrm{SO}(2) \backslash \mathrm{SU}(2) / \mathrm{SO}(2))$ be such that $\check{\varphi} \in MS^p(L^2(\mathrm{SU}(2)))$. Then φ^0 satisfies

$$|\varphi^0(\delta_1) - \varphi^0(\delta_2)| \leq \hat{C}(p) \|\varphi\|_{MS^p(L^2(\mathrm{SU}(2)))} |\delta_1 - \delta_2|^{\frac{1}{4} - \frac{1}{p}}$$

for $\delta_1, \delta_2 \in [-\frac{1}{2}, \frac{1}{2}]$. Here $\hat{C}(p)$ is a constant depending only on p.

Lemma 4.8. Let $\varphi \in C(\widetilde{G}) \cap \mathcal{C}$ such that $\check{\varphi} \in MS^p(L^2(\widetilde{G}))$ for some $p \in (1, \infty)$. For $\alpha \geq 0$, let $\chi_\alpha' : H \to \mathbb{C}$ be defined by $h \mapsto \varphi(\widetilde{D}(\alpha, \alpha)\widetilde{v}\widetilde{h}\widetilde{D}(\alpha, \alpha))$, and let $\chi_\alpha'' : H \to \mathbb{C}$ bedefined by $h \mapsto \varphi(\widetilde{D}(\alpha, \alpha)\widetilde{v}^{-1}\widetilde{h}\widetilde{D}(\alpha, \alpha))$. These maps are H_1-bi-invariant such that $\check{\chi}_\alpha', \check{\chi}_\alpha'' \in MS^p(L^2(H))$. Moreover, we obtain that $\|\check{\chi}_\alpha'\|_{MS^p(L^2(H))} \leq \|\check{\varphi}\|_{MS^p(L^2(H))}$ and $\|\check{\chi}_\alpha''\|_{MS^p(L^2(H))} \leq \|\check{\varphi}\|_{MS^p(L^2(H))}$.

The fact that the maps are H_1-bi-invariant is similar to the case of completely bounded Fourier multipliers. The second part follows by the fact that the sets $\widetilde{D}(\alpha, \alpha)\widetilde{v}\widetilde{H}\widetilde{D}(\alpha, \alpha)$ and $\widetilde{D}(\alpha, \alpha)\widetilde{v}^{-1}\widetilde{H}\widetilde{D}(\alpha, \alpha)$ are subsets of $\widetilde{G}$ and by applying [29, Lemma 2.3].

Proposition 4.9. Let $p > 4$, and let $\varphi \in \mathcal{C}$ such that $\check{\varphi} \in MS^p(L^2(G))$. If $|\tau_1 - \tau_2| \leq \frac{\pi}{2}$ and $\alpha \geq 0$, then

$$|\dot{\varphi}(2\alpha, 0, \tau_1) - \dot{\varphi}(2\alpha, 0, \tau_2)| \leq D(p) e^{-2\alpha(\frac{1}{4} - \frac{1}{p})} \|\check{\varphi}\|_{MS^p(L^2(\widetilde{G})},$$

where $D(p) > 0$ is a constant depending only on p.

The proof of this proposition is similar to the proof of Proposition 3.24. One uses the Hölder continuity coming from the Legendre polynomials in the p-setting (see Lemma 4.7) rather than the Hölder continuity in the setting of completely bounded Fourier multipliers. In the lemma yielding the above proposition, replace the Hölder continuity accordingly.

Lemma 4.10. There exists a constant $\tilde{B}(p) > 0$ such that for $\alpha > 0$, $t \in \mathbb{R}$, $\tau \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and $s_1 = s_1(2\alpha, 0)$ is chosen as in Lemma 3.26, then for all $\varphi \in \mathcal{C}$ such that $\check{\varphi} \in MS^p(L^2(G))$,

$$|\dot{\varphi}(2s_1, s_1, t) - \dot{\varphi}(2s_1, s_1, t + \tau)| \leq \tilde{B}(p) e^{-\frac{\alpha}{2}(\frac{1}{4} - \frac{1}{p})} \|\check{\varphi}\|_{MS^p(L^2(\widetilde{G})}.$$

The following two lemmas replace Lemmas 3.28 and 3.29.

Lemma 4.11. For $p > 4$, there exists a constant $B_1(p) > 0$ (depending only on p) such that whenever $\beta \geq \gamma \geq 0$ and $s_1 = s_1(\beta, \gamma)$ is chosen as in Lemma 3.26, then for all $\varphi \in \mathcal{C}$ for which $\check{\varphi} \in MS^p(L^2(\widetilde{G}))$,

$$|\dot{\varphi}(\beta, \gamma, t) - \dot{\varphi}(2s_1, s_1, t)| \leq B_1(p) e^{-\frac{\beta - \gamma}{4}(\frac{1}{4} - \frac{1}{p})} \|\check{\varphi}\|_{MS^p(L^2(G))}.$$

Lemma 4.12. For $p > 12$, there exists a constant $B_2(p) > 0$ (depending only on p) such that whenever $\beta \geq \gamma \geq 0$ and $s_2 = s_2(\beta, \gamma)$ is chosen as in Lemma 3.26, then for all $\varphi \in \mathcal{C}$ for which $\check{\varphi} \in MS^p(L^2(\widetilde{G}))$,

$$|\dot{\varphi}(\beta, \gamma, t) - \dot{\varphi}(2s_2, s_2, t)| \leq B_2(p) e^{-\frac{\gamma}{4}(\frac{1}{4} - \frac{3}{p})} \|\check{\varphi}\|_{MS^p(L^2(\widetilde{G}))}.$$

The following lemma follows in a similar way from the previous two lemmas as Lemma 3.31 follows from Lemmas 3.28 and 3.29.

Lemma 4.13. For all $p > 12$, there exists a constant $B_3(p) > 0$ such that whenever $s_1, s_2 \geq 0$ satisfy $2 \leq s_2 \leq s_1 \leq \frac{6}{5} s_2$, then for all $\varphi \in \mathcal{C}$ for which $\check{\varphi} \in MS^p(L^2(\widetilde{G}))$ and for all $t \in \mathbb{R}$,

$$|\dot{\varphi}(2s_1, s_1, t) - \dot{\varphi}(2s_2, s_2, t)| \leq B_3(p) e^{-\frac{s_1}{8}(\frac{1}{4} - \frac{3}{p})} \|\check{\varphi}\|_{MS^p(L^2(\widetilde{G}))}.$$

The following lemma replaces 3.32.

Lemma 4.14. For $p > 12$, there exists a constant $B_4(p) > 0$ such that for all $\varphi \in \mathcal{C}$ for which $\check{\varphi} \in MS^p(L^2(\widetilde{G}))$ and for all $t \in \mathbb{R}$, the limit $\tilde{c}^p_\varphi(t) = \lim_{s_1 \to \infty} \dot{\varphi}(2s_1, s_1, t)$ exists, and for all $s_2 \geq 0$,

$$|\dot{\varphi}(2s_2, s_2, t) - \tilde{c}^p_\varphi(t)| \leq B_4(p) e^{-\frac{s_2}{8}(\frac{1}{4} - \frac{3}{p})} \|\check{\varphi}\|_{MS^p(L^2(\widetilde{G}))}.$$

Proof of Proposition 4.2. Let $\varphi \in \mathcal{C}$ be such that $\check{\varphi} \in MS^p(L^2(\widetilde{G}))$. The proof of the proposition now follows in the same way as the proof of Proposition 3.11. Indeed, assume first $\beta \geq 2\gamma$. Then $\beta - \gamma \geq \frac{\beta}{2}$, and it follows for all $t \in \mathbb{R}$ that

$$|\dot{\varphi}(\beta, \gamma, t) - \tilde{c}^p_\varphi(t)| \leq (B_1(p) + B_4(p)) e^{-\frac{\beta}{32}(\frac{1}{4} - \frac{3}{p})} \|\check{\varphi}\|_{MS^p(L^2(\widetilde{G}))}.$$

Assume now that $\beta < 2\gamma$. Then

$$|\dot{\varphi}(\beta, \gamma, t) - \tilde{c}^p_\varphi(t)| \leq (B_2(p) + B_4(p)) e^{-\frac{\beta}{32}(\frac{1}{4} - \frac{3}{p})} \|\check{\varphi}\|_{MS^p(L^2(\widetilde{G}))}.$$

Combining these results, it follows that for all $\beta \geq \gamma \geq 0$,

$$|\dot{\varphi}(\beta, \gamma, t) - \tilde{c}^p_\varphi(t)| \leq C_1(p) e^{-C_2(p)\sqrt{\beta^2 + \gamma^2}} \|\check{\varphi}\|_{MS^p(L^2(\widetilde{G}))},$$

where $C_1(p) = \max\{B_1(p) + B_4(p), B_2(p) + B_4(p)\}$ and $C_2(p) = \frac{1}{32\sqrt{2}}(\frac{1}{4} - \frac{3}{p})$. This proves the proposition. ☐

The values $p \in [1, \frac{12}{11}) \cup (12, \infty]$ give sufficient conditions for $\widetilde{G}$ to fail the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$. We would like to point out that the set of these values might be bigger, as already mentioned in Section 1.

5. Main results

In this section, we state and prove the main results of this article.

Theorem 5.1. Let G be a connected simple Lie group. Then G has the Approximation Property if and only if it has real rank zero or one.

Proof. Since it is well-known that if a connected simple Lie group G has real rank zero or one, then G has the AP (see Section 1), it suffices to prove that any connected simple Lie group with real rank greater than or equal to two does not have the AP.

Let G be a connected simple Lie group with real rank greater than or equal to two. Then G has a closed connected subgroup H locally isomorphic to $\mathrm{SL}(3, \mathbb{R})$ or $\mathrm{Sp}(2, \mathbb{R})$ (see, e.g., [1],[9],[33]).

Firstly, suppose that H is locally isomorphic to $\mathrm{SL}(3,\mathbb{R})$. Since the universal covering $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ has finite center, it follows that H automatically has finite center. Using the fact that the AP is preserved under local isomorphism of connected simple Lie groups with finite center (see Section 2.6) and the fact that $\mathrm{SL}(3,\mathbb{R})$ does not have the AP, it follows that G does not have the AP, since the AP passes from a group to closed subgroups.

Secondly, suppose that H is locally isomorphic to $\mathrm{Sp}(2,\mathbb{R})$, i.e., H is isomorphic to $\widetilde{\mathrm{Sp}}(2,\mathbb{R})/\Gamma$, where Γ is a discrete subgroup of the center $Z(\widetilde{\mathrm{Sp}}(2,\mathbb{R}))$ of $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$. If H has finite center, then the result follows in the same way as the case $\mathrm{SL}(3,\mathbb{R})$. If H has infinite center, then $H \cong \widetilde{\mathrm{Sp}}(2,\mathbb{R})$, because all nontrivial subgroups of the center of $\widetilde{Sp}(2,\mathbb{R})$ are infinite subgroups of finite index (which make H have finite center). This implies that H does not have the AP, which finishes the proof. □

Note that the proof of this theorem follows from combining the failure of the AP for $\mathrm{SL}(3,\mathbb{R})$, which was proved by Lafforgue and de la Salle and the failure of the AP for $\mathrm{Sp}(2,\mathbb{R})$ and $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$.

Corollary 5.2. Let $G = S_1 \times \ldots \times S_n$ be a connected semisimple Lie group with connected simple factors S_i, $i = 1,\ldots,n$. Then G has the AP if and only if for all $i = 1,\ldots,n$ the real rank of S_i is smaller than or equal to 1.

We now state our results on noncommutative L^p-spaces. Combining [31, Theorem E] by Lafforgue and de la Salle, [29, Theorem 3.1] and Theorem 4.1 of this article, it follows that whenever G is a connected simple Lie group with real rank greater than or equal to two and whenever $p \in [1, \frac{12}{11}) \cup (12,\infty]$, then G does not have the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$. Combining this with the fact that the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$ passes from a group to its lattices and vice versa and the earlier mentioned result of Lafforgue and de la Salle that whenever Γ is a discrete group such that $L^p(L(\Gamma))$ has the OAP for $p \in (1,\infty)$, then Γ has the $\mathrm{AP}^{\mathrm{Schur}}_{p,\mathrm{cb}}$, we obtain the following result.

Theorem 5.3. Let Γ be a lattice in a connected simple Lie group with real rank greater than or equal to two. For $p \in [1, \frac{12}{11}) \cup (12,\infty]$, the noncommutative L^p-space $L^p(L(\Gamma))$ does not have the OAP or CBAP.

Note that this result only gives sufficient conditions on the value of p for the failure of the CBAP and OAP for noncommutative L^p-spaces associated with lattices in connected higher rank simple Lie groups. The set of such p-values might be bigger than $[1, \frac{12}{11}) \cup (12,\infty]$. In particular, if we consider $L^p(L(\Gamma))$, where Γ is a lattice in a connected simple Lie group that contains a closed subgroup locally isomorphic to $\mathrm{SL}(3,\mathbb{R})$, then we know by the results of Lafforgue and de la Salle that the CBAP and OAP for $L^p(L(\Gamma))$ fail for $p \in [1, \frac{4}{3}) \cup (4,\infty]$.

Appendix A. Harmonic analysis on strong Gelfand pairs

This appendix discusses the analogues of spherical functions in the setting of strong Gelfand pairs. In particular, we explain their relation to spherical functions for Gelfand pairs and their meaning in representation theory. The material discussed here is not needed for the rest of this article, but might give a deeper understanding of certain results proved in Sections 3 and 4 (see in particular Lemma 3.14). The main result of this section, Theorem A.2, might be known to experts, and special cases of it were considered in [15], but we could not find a reference for the general statement. The content of this appendix arose from discussions between the second named author and Thomas Danielsen.

The definitions of Gelfand pairs, spherical functions and strong Gelfand pairs were given in Section 2.3. It was pointed out there (and it is elementary to prove) that a pair (G,K) consisting of a locally compact group G and a compact subgroup K is a strong Gelfand pair if and only if $(G\times K,\Delta K)$ (where ΔK is the diagonal subgroup) is a Gelfand pair. We refer to [8] and [14] for a thorough account of the theory of Gelfand pairs.

Suppose that G is a locally compact group with compact subgroup K. An equivalent definition of spherical functions (see [8],[14] for a proof of the equivalence) is that for a Gelfand pair (G,K), a function $h\in C(K\backslash G/K)$ that is not identical to zero is spherical if for all $g_1,g_2\in G$ we have $\int_K h(g_1kg_2)dk=h(g_1)h(g_2)$. We denote the set of spherical functions by $S(G,K)$. Spherical functions parametrize the nontrivial characters (multiplicative linear functionals) of the algebra $C_c(K\backslash G/K)$, since any such character is of the form $\chi(\varphi)=\chi_h(\varphi)=\int_G\varphi(g)h(g^{-1})dg$. Furthermore, if h is a bounded spherical function, the expression above defines a continuous multiplicative functional on the Banach algebra $L^1(K\backslash G/K)$, and the set $BS(G,K)$ of bounded spherical functions parametrizes bijectively the set of continuous characters of $L^1(K\backslash G/K)$.

We can now define the analogues of spherical functions in the setting of strong Gelfand pairs. For a strong Gelfand pair (G,K), we say that a function $h\in C(G//K)$ that is not identical to zero is s-spherical if for all $g_1,g_2\in G$ we have $\int_K h(k^{-1}g_1kg_2)dk=h(g_1)h(g_2)$. The set of s-spherical functions is denoted by $SS(G,K)$. Analogous to the case of spherical functions, the s-spherical functions parametrise the space of nontrivial characters of the convolution algebra $C_c(G//K)$, since an s-spherical function h gives rise to a character by $\chi_h(\varphi):=\int_G\varphi(g)h(g^{-1})dg$.

It is clear that $S(G,K)\subset SS(G,K)$. We can now relate the spaces of s-spherical functions for (G,K) and spherical functions for $(G\times K,\Delta K)$. First, we state a lemma, the proof of which is elementary and left to the reader.

Lemma A.1. The map $\Phi:\Delta K\backslash(G\times K)/\Delta K\to G//K$ given by $\Delta K(g,k)\Delta K\mapsto[k^{-1}g]=[gk^{-1}]$ is a homeomorphism with inverse $\Phi^{-1}([g])=\Delta K(g,e)\Delta K$. Here, $[g]$ denotes the K-conjugation class of g.

The map Φ of Lemma A.1 induces a bijection $\Phi^*:C(G//K)\to C(K\backslash G/K)$ given by $f\mapsto f\circ\Phi$.

Theorem A.2. The map $\Phi^*:C(G//K)\to C(K\backslash G/K)$ given by $f\mapsto f\circ\Phi$ defines a bijection between $SS(G,K)$ and $S(G\times K,\Delta K)$.

Proof. For $h\in SS(G,K)$, we have $(h\circ\Phi)((k_1,k_1)(g,k)(k_2,k_2))=h(k_2^{-1}k^{-1}gk_2)=h(k^{-1}g)=(h\circ\Phi)((g,k))$ for all $g\in G$ and $k,k_1,k_2\in K$, so $h\circ\Phi$ is ΔK-bi-invariant on $G\times K$. Moreover, we check that for $h\circ\Phi$ (which is not identical to the zero function), we have

$$\int_K(h\circ\Phi)((g_1,k_1)(k,k)(g_2,k_2))dk=\int_K h(k_2^{-1}k^{-1}k_1^{-1}g_1kg_2)dk$$
$$=\int_K h(k^{-1}k_1^{-1}g_1kg_2k_2^{-1})dk=(h\circ\Phi)(g_1,k_1)(h\circ\Phi)(g_2,k_2)$$

for all $g_1, g_2 \in G$ and $k, k_1, k_2 \in K$. Let now $h \in S(G,K)$. It follows that $(h \circ \Phi^{-1})(kgk^{-1}) = h(kg,k) = h(g,e) = (h \circ \Phi^{-1})(g)$ for all $g \in G$ and $k \in K$, so $h \circ \Phi^{-1}$ is $\mathrm{Int}(K)$-invariant on G. Moreover, we check that $\varphi \circ \Phi^{-1}$ (which is not identical to the zero function) satisfies

$$\begin{aligned}\int_K (\varphi \circ \Phi^{-1})(k^{-1}g_1 k g_2)dk &= \int_K \varphi((g_1,e)(k,k)(g_2,e))dk \\ &= \varphi((g_1,e))\varphi((g_2,e)) = (\varphi \circ \Phi^{-1})(g_1)(\varphi \circ \Phi^{-1})(g_2)\end{aligned}$$

for all $g_1, g_2 \in G$. □

Remark A.3. For a compact group G with compact subgroup K such that (G,K) is a strong Gelfand pair, the map Φ^* extends to a bijective isometry from $L^2(G//K)$ onto $L^2(\Delta K \backslash G \times K/\Delta K)$. The fact that Φ^* is isometric on $C(G//K)$ follows by elementary computation.

Let (G,K) be a compact strong Gelfand pair, i.e., the group G is compact and (G,K) is a strong Gelfand pair. In particular, (G,K) is a Gelfand pair. Let $X = G/K$ denote the corresponding homogeneous space. For an irreducible unitary representation π of G, let $\mathcal{H}_\pi$, $\mathcal{H}_{\pi_e}$, P_π and $\hat{G}_K$ be as in Section 2.3. Then $L^2(X) = \oplus_{\pi \in \hat{G}_K} \mathcal{H}_\pi$ (see Section 2.7). Let h_π denote the spherical function corresponding to the equivalence class π of representations. Then for every $\varphi \in L^2(K\backslash G/K)$ we have $\varphi = \sum_{\pi \in \hat{G}_K} c_\pi \dim \mathcal{H}_\pi h_\pi$, where $c_\pi = \langle \varphi, h_\pi \rangle$.

Recall that any unitary irreducible representation of a product of compact Lie groups arises as the tensor product of unitary irreducible representations of these groups. Also, it was already known from [18] that a pair (G,K) consisting of a locally compact group and a compact subgroup K of G is a strong Gelfand pair if and only if for every unitary irreducible representation π of G, the space $\mathrm{Hom}_K(\pi,\tau)$ is at most one-dimensional for all unitary irreducible representations τ of K. Combining this with Theorem A.2 and Remark A.3, the following result follows.

Theorem A.4. Let (G,K) be a compact strong Gelfand pair, and let $f \in L^2(G//K)$. Then

$$f = \sum_{\pi \in \widehat{G \times K}_{\Delta K}} c_\pi \dim \mathcal{H}_\pi (h_\pi \circ \Phi^{-1}) = \sum_{\pi \in \hat{G}} c_\pi \dim \mathcal{H}_\pi h_\pi^s,$$

where h_π^s denotes the s-spherical function associated with π.

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APPENDIX D

On the Grothendieck Theorem for jointly completely bounded bilinear forms

This chapter contains the preprint version of the following article:

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ON THE GROTHENDIECK THEOREM FOR JOINTLY COMPLETELY BOUNDED BILINEAR FORMS

TIM DE LAAT

Abstract. We show how the proof of the Grothendieck Theorem for jointly completely bounded bilinear forms on C^*-algebras by Haagerup and Musat can be modified in such a way that the method of proof is essentially C^*-algebraic. To this purpose, we use Cuntz algebras rather than type III factors. Furthermore, we show that the best constant in Blecher's inequality is strictly greater than one.

1. Introduction

In [10], Grothendieck proved his famous *Fundamental Theorem on the metric theory of tensor products*. He also conjectured a noncommutative analogue of this theorem for bounded bilinear forms on C^*-algebras. This *noncommutative Grothendieck Theorem* was proved by Pisier assuming a certain approximability condition on the bilinear form [16]. The general case was proved by Haagerup [11]. Effros and Ruan conjectured a "sharper" analogue of this theorem for bilinear forms on C^*-algebras that are jointly completely bounded (rather than bounded) [9]. More precisely, they conjectured the following result, with universal constant $K = 1$.

Theorem 1.1 (JCB Grothendieck Theorem)**.** Let A, B be C^*-algebras, and let $u : A \times B \to \mathbb{C}$ be a jointly completely bounded bilinear form. Then there exist states f_1, f_2 on A and g_1, g_2 on B such that for all $a \in A$ and $b \in B$,

$$|u(a,b)| \leq K\|u\|_{jcb} \left(f_1(aa^*)^{\frac{1}{2}} g_1(b^*b)^{\frac{1}{2}} + f_2(a^*a)^{\frac{1}{2}} g_2(bb^*)^{\frac{1}{2}} \right),$$

where K is a constant.

We call this *Grothendieck Theorem for jointly completely bounded bilinear forms on C^*-algebras* the *JCB Grothendieck Theorem*. It is often referred to as the *Effros-Ruan conjecture*.

In [18], Pisier and Shlyakhtenko proved a version of Theorem 1.1 for exact operator spaces, in which the constant K depends on the exactness constants of the operator spaces. They also proved the conjecture for C^*-algebras, assuming that at least one of them is exact, with universal constant $K = 2^{\frac{3}{2}}$.

Haagerup and Musat proved the general conjecture (for C^*-algebras), i.e., Theorem 1.1, with universal constant $K = 1$ [12]. They used certain type III factors in the proof. Since the conjecture itself is purely C^*-algebraic, it would be more satisfactory to have a proof that relies on C^*-algebras. In this note, we show how the proof of Haagerup and Musat can be modified in such a way that essentially

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only C^*-algebraic arguments are used. Indeed, in their proof, one tensors the C^*-algebras on which the bilinear form is defined with certain type III factors, whereas we show that it also works to tensor with certain simple nuclear C^*-algebras admitting KMS states instead. We then transform the problem back to the (classical) noncommutative Grothendieck Theorem, as was also done by Haagerup and Musat.

Recently, Regev and Vidick gave a more elementary proof of both the JCB Grothendieck Theorem for C^*-algebras and its version for exact operator spaces [19]. Their proof makes use of methods from quantum information theory and has the advantage that the transformation of the problem to the (classical) noncommutative Grothendieck Theorem is more explicit and based on finite-dimensional techniques. Moreover, they obtain certain new quantitative estimates.

For an extensive overview of the different versions of the Grothendieck Theorem, as well as their proofs and several applications, we refer to [17].

This text is organized as follows. In Section 2, we recall two different notions of complete boundedness for bilinear forms on operator spaces. In Section 3, we recall some facts about Cuntz algebras and their KMS states. This is needed for the proof of the JCB Grothendieck Theorem, which is given in Section 4 (with a constant $K > 1$) by using (single) Cuntz algebras. We explain how to obtain $K = 1$ in Section 5. In Section 6, we show that using a recent result by Haagerup and Musat on the best constant in the noncommutative little Grothendieck Theorem, we are able to improve the best constant in Blecher's inequality.

2. Bilinear forms on operator spaces

Recall that an operator space E is a closed linear subspace of $\mathcal{B}(H)$ for some Hilbert space H. For $n \geq 1$, the embedding $M_n(E) \subset M_n(\mathcal{B}(H)) \cong \mathcal{B}(H^n)$ gives rise to a norm $\|.\|_n$ on $M_n(E)$. In particular, C^*-algebras are operator spaces. A linear map $T : E \to F$ between operator spaces induces a linear map $T_n : M_n(E) \to M_n(F)$ for each $n \in \mathbb{N}$, defined by $T_n([x_{ij}]) = [T(x_{ij})]$ for all $x = [x_{ij}] \in M_n(E)$. The map T is called completely bounded if the completely bounded norm $\|T\|_{cb} := \sup_{n\geq 1} \|T_n\|$ is finite.

There are two common ways to define a notion of complete boundedness for bilinear forms on operator spaces. For the first one, we refer to [5]. Let E and F be operator spaces contained in C^*-algebras A and B, respectively, and let $u : E \times F \to \mathbb{C}$ be a bounded bilinear form. Let $u_{(n)} : M_n(E) \times M_n(F) \to M_n(\mathbb{C})$ be the map defined by $([a_{ij}], [b_{ij}]) \mapsto [\sum_{k=1}^n u(a_{ik}, b_{kj})]$.

Definition 2.1. The bilinear form u is called *completely bounded* if

$$\|u\|_{cb} := \sup_{n\geq 1} \|u_{(n)}\|$$

is finite. We put $\|u\|_{cb} = \infty$ if u is not completely bounded.

Equivalently (see Section 3 of [12] or the Introduction of [18]), u is completely bounded if there exists a constant $C \geq 0$ and states f on A and g on B such that for all $a \in E$ and $b \in F$,

$$|u(a, b)| \leq C f(aa^*)^{\frac{1}{2}} g(b^*b)^{\frac{1}{2}}, \tag{1}$$

and $\|u\|_{cb}$ is the smallest constant C such that (1) holds.

For the second notion, we refer to [3], [9]. Let E and F be operator spaces contained in C^*-algebras A and B, respectively, and let $u : E \times F \to \mathbb{C}$ be a bounded bilinear form. Then there exists a unique bounded linear operator $\tilde{u} : E \to F^*$ such that

$$u(a, b) = \langle \tilde{u}(a), b\rangle$$

for all $a \in E$ and $b \in F$, where $\langle ., .\rangle$ denotes the pairing between F and its dual.

Definition 2.2. The bilinear form u is called *jointly completely bounded* if the map $\tilde{u} : E \to F^*$ is completely bounded, and we set

$$\|u\|_{jcb} := \|\tilde{u}\|_{cb}.$$

We put $\|u\|_{jcb} = \infty$ if u is not jointly completely bounded.

Equivalently, if we define maps $u_n : M_n(E) \otimes M_n(F) \to M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ by

$$u_n\left(\sum_{i=1}^{k} a_i \otimes c_i, \sum_{j=1}^{l} b_j \otimes d_j\right) = \sum_{i=1}^{k}\sum_{j=1}^{l} u(a_i, b_j)c_i \otimes d_j$$

for $a_1, \dots, a_k \in A$, $b_1, \dots, b_l \in B$, and $c_1, \dots, c_k, d_1, \dots, d_l \in M_n(\mathbb{C})$, then we have $\|u\|_{jcb} = \sup_{n\geq 1} \|u_n\|$.

3. KMS states on Cuntz algebras

For $2 \leq n < \infty$, let $\mathcal{O}_n$ denote the Cuntz algebra generated by n isometries, as introduced by Cuntz in [6], in which one of the main results is that the algebras $\mathcal{O}_n$ are simple. We now recall some results by Cuntz. If $\alpha = (\alpha_1, \dots, \alpha_k)$ denotes a multi-index of length $k = l(\alpha)$, where $\alpha_j \in \{1, \dots, n\}$ for all j, we write $S_\alpha = S_{\alpha_1} \dots S_{\alpha_k}$, and we put $S_0 = 1$. It follows that for every nonzero word M in $\{S_i\}_{i=1}^n \bigcup \{S_i^*\}_{i=1}^n$, there are unique multi-indices μ and ν such that $M = S_\mu S_\nu^*$.

For $k \geq 1$, let $\mathcal{F}_n^k$ be the C^*-algebra generated by $\{S_\mu S_\nu^* \mid l(\mu) = l(\nu) = k\}$, and let $\mathcal{F}_n^0 = \mathbb{C}1$. It follows that $\mathcal{F}_n^k$ is $*$-isomorphic to $M_{n^k}(\mathbb{C})$, and, as a consequence, $\mathcal{F}_n^k \subset \mathcal{F}_n^{k+1}$. The C^*-algebra $\mathcal{F}_n$ generated by $\bigcup_{k=0}^\infty \mathcal{F}_n^k$ is a UHF-algebra of type n^∞.

If we write $\mathcal{P}_n$ for the algebra generated algebraically by $S_1, \dots, S_n$, $S_1^*, \dots, S_n^*$, each element A in $\mathcal{P}_n$ has a unique representation

$$A = \sum_{k=1}^{N} (S_1^*)^k A_{-k} + A_0 + \sum_{k=1}^{N} A_k S_1^k,$$

where $N \in \mathbb{N}$ and $A_k \in \mathcal{P}_n \cap \mathcal{F}_n$. The maps $F_{n,k} : \mathcal{P}_n \to \mathcal{F}_n$ $(k \in \mathbb{Z})$ defined by $F_{n,k}(A) = A_k$ extend to norm-decreasing maps $F_{n,k} : \mathcal{O}_n \to \mathcal{F}_n$. It follows that $F_{n,0}$ is a conditional expectation.

The existence of a unique KMS state on each Cuntz algebra was proved by Olesen and Pedersen [15]. Firstly, we give some background on C^*-dynamical systems.

Definition 3.1. A C^*-dynamical system $(A, \mathbb{R}, \rho)$ consists of a C^*-algebra A and a representation $\rho : \mathbb{R} \to \mathrm{Aut}(A)$, such that each map $t \mapsto \rho_t(a)$, $a \in A$, is norm continuous.

C^*-dynamical systems can be defined in more general settings. In particular, one can replace $\mathbb{R}$ with arbitrary locally compact groups.

Let A^a denote the dense $*$-subalgebra of A consisting of analytic elements, i.e., $a \in A^a$ if the function $t \mapsto \rho_t(a)$ has a (necessarily unique) extension to an entire operator-valued function. This extension is implicitly used in the following definition.

Definition 3.2. Let $(A, \mathbb{R}, \rho)$ be a C^*-dynamical system. An invariant state ϕ on A, i.e., a state for which $\phi \circ \rho_t = \phi$ for all $t \in \mathbb{R}$, is a KMS state if

$$\phi(\rho_{t+i}(a)b) = \phi(b\rho_t(a))$$

for all $a \in A^a$, $b \in A$ and $t \in \mathbb{R}$.

This definition is similar to the one introduced by Takesaki (see [20], Definition 13.1). It corresponds to ϕ being a β-KMS state for ρ_{-t} with $\beta = 1$ according to the conventions of [4] and [15]. In the latter, the following two results were proved (see Lemma 1 and Theorem 2 therein). We restate these results slightly according to the conventions of Definition 3.2.

Proposition 3.3. *(Olesen-Pedersen)* For all $t \in \mathbb{R}$ and the generators $\{S_k\}_{k=1}^n$ of $\mathcal{O}_n$, define $\rho_t^n(S_k) = n^{it} S_k$. Then ρ_t^n extends uniquely to a $*$-automorphism of $\mathcal{O}_n$ for every $t \in \mathbb{R}$ in such a way that $(\mathcal{O}_n, \mathbb{R}, \rho^n)$ becomes a C^*-dymamical system. Moreover, $\mathcal{F}_n$ is the fixed-point algebra of ρ^n in $\mathcal{O}_n$, and $\mathcal{P}_n \subset (\mathcal{O}_n)^a$.

Let $\tau_n = \otimes_{k=1}^{\infty} \frac{1}{n}\mathrm{Tr}$ denote the unique tracial state on $\mathcal{F}_n$.

Proposition 3.4. *(Olesen-Pedersen)* For $n \geq 2$, the C^*-dynamical system given by $(\mathcal{O}_n, \mathbb{R}, \rho^n)$ has exactly one KMS state, namely $\phi_n = \tau_n \circ F_{n,0}$.

For a C^*-algebra A, let $\mathcal{U}(A)$ denote its unitary group. The following result was proved by Archbold [1]. It implies the *Dixmier property* for $\mathcal{O}_n$.

Proposition 3.5. *(Archbold)* For all $x \in \mathcal{O}_n$,

$$\phi_n(x)1_{\mathcal{O}_n} \in \overline{\mathrm{conv}\{uxu^* \mid u \in \mathcal{U}(\mathcal{F}_n)\}}^{\|\cdot\|}.$$

As a corollary, we obtain the following (well-known) fact (see also [7]).

Corollary 3.6. The relative commutant of $\mathcal{F}_n$ in $\mathcal{O}_n$ is trivial, i.e.,

$$(\mathcal{F}_n)' \cap \mathcal{O}_n = \mathbb{C}1.$$

Proof. Let $x \in (\mathcal{F}_n)' \cap \mathcal{O}_n$. By Proposition 3.5, we know that for every $\varepsilon > 0$, there exists a finite convex combination $\sum_{i=1}^m \lambda_i u_i x u_i^*$, where $u_i \in \mathcal{U}(\mathcal{F}_n)$, such that $\| \sum_{i=1}^m \lambda_i u_i x u_i^* - \phi_n(x)1_{\mathcal{O}_n} \| < \varepsilon$. Since $x \in (\mathcal{F}_n)' \cap \mathcal{O}_n$, we have $\sum_{i=1}^m \lambda_i u_i x u_i^* = \sum_{i=1}^m \lambda_i x u_i u_i^* = x$. Hence, $\|x - \phi_n(x)1_{\mathcal{O}_n}\| < \varepsilon$. This implies that $x \in \mathbb{C}1$. $\square$

Proposition 3.5 can be extended to finite sets in $\mathcal{O}_n$, as described in the following lemma, by similar methods as in [8], Part III, Chapter 5. For an invertible element v in a C^*-algebra A, we define $\mathrm{ad}(v)(x) = vxv^{-1}$ for all $x \in A$.

Lemma 3.7. Let $\{x_1, \ldots, x_k\}$ be a subset of $\mathcal{O}_n$, and let $\varepsilon > 0$. Then there exists a convex combination α of elements in $\{\mathrm{ad}(u) \mid u \in \mathcal{U}(\mathcal{F}_n)\}$ such that

$$\|\alpha(x_i) - \phi_n(x_i)1_{\mathcal{O}_n}\| < \varepsilon \quad \text{for all } i = 1, \ldots, k.$$

Moreover, there exists a net $\{\alpha_j\}_{j \in J} \subset \mathrm{conv}\{\mathrm{ad}(u) \mid u \in \mathcal{U}(\mathcal{F}_n)\}$ such that

$$\lim_j \|\alpha_j(x) - \phi_n(x)1_{\mathcal{O}_n}\| = 0$$

for all $x \in \mathcal{O}_n$.

Proof. Suppose that $\|\alpha'(x_i) - \phi_n(x_i)1_{\mathcal{O}_n}\| < \varepsilon$ for $i = 1, \dots, k-1$. By Proposition 3.5, we can find a convex combination $\tilde{\alpha}$ such that

$$\|\tilde{\alpha}(\alpha'(x_k)) - \phi_n(\alpha'(x_k))1_{\mathcal{O}_n}\| < \varepsilon.$$

Note that $\phi_n(\alpha'(x_k)) = \phi_n(x_k)$ and $1_{\mathcal{O}_n} = \tilde{\alpha}(1_{\mathcal{O}_n})$. By the fact that $\|\tilde{\alpha}(x)\| \leq \|x\|$ for all $x \in \mathcal{O}_n$, we conclude that $\alpha = \tilde{\alpha} \circ \alpha'$ satisfies $\|\alpha(x_i) - \phi_n(x_i)1_{\mathcal{O}_n}\| < \varepsilon$ for $i = 1, \dots, k$.

Let J denote the directed set consisting of pairs (F, η), where F is a finite subset of $\mathcal{O}_n$ and $\eta \in (0,1)$, with the ordering given by $(F_1, \eta_1) \preceq (F_2, \eta_2)$ if $F_1 \subset F_2$ and $\eta_1 \geq \eta_2$. By the first assertion, this gives rise to a net $\{\alpha_j\}_{j \in J}$ with the desired properties. □

4. Proof of the JCB Grothendieck Theorem

In this section, we explain the proof of the Grothendieck Theorem for jointly completely bounded bilinear forms on C^*-algebras. As mentioned in Section 1, the proof is along the same lines as the proof by Haagerup and Musat, but we tensor with Cuntz algebras instead of type III factors.

Applying the GNS construction to the pair $(\mathcal{O}_n, \phi_n)$, we obtain a $*$-representation π_n of $\mathcal{O}_n$ on the Hilbert space $H_{\pi_n} = L^2(\mathcal{O}_n, \phi_n)$, with cyclic vector ξ_n, such that $\phi_n(x) = \langle \pi_n(x)\xi_n, \xi_n \rangle_{H_{\pi_n}}$. We identify $\mathcal{O}_n$ with its GNS representation. Note that ϕ_n extends in a normal way to the von Neumann algebra $\mathcal{O}_n''$, which also acts on H_{π_n}. This normal extension is a KMS state for a W^*-dynamical system with $\mathcal{O}_n''$ as the underlying von Neumann algebra (see Corollary 5.3.4 of [4]). The commutant $\mathcal{O}_n'$ of $\mathcal{O}_n$ is also a von Neumann algebra, and using Tomita-Takesaki theory (see [4], [20]), we obtain, via the polar decomposition of the closure of the operator $Sx\xi_n = x^*\xi_n$, a conjugate-linear involution $J : H_{\pi_n} \to H_{\pi_n}$ satisfying $J\mathcal{O}_nJ \subset \mathcal{O}_n'$.

Lemma 4.1. For $k \in \mathbb{Z}$, we have

$$\mathcal{O}_n^k := \{x \in \mathcal{O}_n \mid \rho_t^n(x) = n^{-ikt}x \forall t \in \mathbb{R}\} = \{x \in \mathcal{O}_n \mid \phi_n(xy) = n^{-k}\phi_n(yx) \forall y \in \mathcal{O}_n\}.$$

The proof of this lemma is analogous to Lemma 1.6 of [21]. Note that $\mathcal{O}_n^0 = \mathcal{F}_n$, and that for all $k \in \mathbb{Z}$, we have $\mathcal{O}_n^k \neq \{0\}$.

Lemma 4.2. For every $k \in \mathbb{Z}$, there exists a $c_k \in \mathcal{O}_n$ such that

$$\phi_n(c_k^* c_k) = n^{\frac{k}{2}}, \qquad \phi_n(c_k c_k^*) = n^{-\frac{k}{2}},$$

and, moreover, $\langle c_k J c_k J \xi_n, \xi_n \rangle = 1$.

The proof is similar to the proof of Lemma 2.1 of [12].

Proposition 4.3. Let A, B be C^*-algebras, and let $u : A \times B \to \mathbb{C}$ be a jointly completely bounded bilinear form. There exists a bounded bilinear form $\hat{u}$ on $(A \otimes_{\min} \mathcal{O}_n) \times (B \otimes_{\min} J\mathcal{O}_nJ)$ given by

$$\hat{u}(a \otimes c, b \otimes d) = u(a,b)\langle cd\xi_n, \xi_n \rangle$$

for all $a \in A$, $b \in B$, $c \in \mathcal{O}_n$ and $d \in J\mathcal{O}_nJ$. Moreover, $\|\hat{u}\| \leq \|u\|_{jcb}$.

The C^*-algebra $J\mathcal{O}_nJ$ is just a copy of $\mathcal{O}_n$. This result is analogous to Proposition 2.3 of [12], and the proof is the same. Note that in our case, we use $\|\sum_{i=1}^k c_i d_i\|_{\mathcal{B}(L^2(\mathcal{O}_n,\phi_n))} = \|\sum_{i=1}^k c_i \otimes d_i\|_{\mathcal{O}_n \otimes_{min} J\mathcal{O}_nJ}$ for all $c_1, \dots, c_k \in \mathcal{O}_n$ and $d_1, \dots, d_k \in J\mathcal{O}_nJ$. This equality is elementary, since $\mathcal{O}_n$ is simple and nuclear. In the proof of Haagerup and Musat, one takes the tensor product of A and a certain type III factor M and the tensor product of B with the commutant M' of M, respectively. Note that $J\mathcal{O}_nJ \subset \mathcal{O}_n'$.

One can formulate analogues of Lemma 2.4, Lemma 2.5 and Proposition 2.6 of [12]. They can be proved in the same way as there, and one explicitly needs the existence and properties of KMS states on the Cuntz algebras (see Section 3). The analogue of Proposition 2.6 gives the "transformation" of the JCB Grothendieck Theorem to the noncommutative Grothendieck Theorem for bounded bilinear forms.

Using Lemma 2.7 of [12], we arrive at the following conclusion, which is the analogue of [12], Proposition 2.8.

Proposition 4.4. Let $K(n) = \sqrt{(n^{\frac{1}{2}} + n^{-\frac{1}{2}})/2}$, and let $u : A \times B \to \mathbb{C}$ be a jointly completely bounded bilinear form on C^*-algebras A, B. Then there exist states f_1^n, f_2^n on A and g_1^n, g_2^n on B such that for all $a \in A$ and $b \in B$,

$$|u(a,b)| \leq K(n)\|u\|_{jcb}\left(f_1^n(aa^*)^{\frac{1}{2}}g_1^n(b^*b)^{\frac{1}{2}} + f_2^n(a^*a)^{\frac{1}{2}}g_2^n(bb^*)^{\frac{1}{2}}\right).$$

The above proposition is the JCB Grothendieck Theorem. However, the (universal) constant and states depend on n. This is because the noncommutative Grothendieck Theorem gives states on $A \otimes_{min} \mathcal{O}_n$ and $B \otimes_{min} J\mathcal{O}_nJ$, which clearly depend on n, and these states are used to obtain the states on A and B. The best constant we obtain in this way comes from the case $n = 2$, which yields the constant $K(2) = \sqrt{(2^{\frac{1}{2}} + 2^{-\frac{1}{2}})/2} \sim 1.03$.

5. The best constant

In order to get the best constant $K = 1$, we consider the C^*-dynamical system $(A, \mathbb{R}, \rho)$, with $A = \mathcal{O}_2 \otimes \mathcal{O}_3$ and $\rho_t = \rho_t^2 \otimes \rho_t^3$. It is straightforward to check that it has a KMS state, namely $\phi = \phi_2 \otimes \phi_3$. It is easy to see that $\mathcal{F} = \mathcal{F}_2 \otimes \mathcal{F}_3$ is contained in the fixed point algebra. (Actually, it is equal to the fixed point algebra, but we do not need this.) These assertions follow by the fact that the algebraic tensor product of $\mathcal{O}_2$ and $\mathcal{O}_3$ is dense in $\mathcal{O}_2 \otimes \mathcal{O}_3$. Note that ρ is not periodic.

Applying the GNS construction to the pair (A, ϕ), we obtain a $*$-representation π of A on the Hilbert space $H_\pi = L^2(A, \phi)$, with cyclic vector ξ, such that $\phi(x) = \langle \pi(x)\xi, \xi\rangle_{H_\pi}$. We identify A with its GNS representation. Using Tomita-Takesaki theory, we obtain a conjugate-linear involution $J : H_\pi \to H_\pi$ satisfying $JAJ \subset A'$ (see also Section 4).

It follows directly from Proposition 3.5 that $\phi(x)1_A \in \overline{\text{conv}\{uxu^* \mid u \in \mathcal{U}(\mathcal{F})\}}^{\|\cdot\|}$ for all $x \in A$. Also, the analogue of Lemma 3.7 follows in a similar way, as well as the fact that $\mathcal{F}' \cap A = \mathbb{C}1$.

It is elementary to check that

$$A_{\lambda,k} := \{x \in A \mid \rho_t(x) = \lambda^{ikt}x \forall t \in \mathbb{R}\} = \{x \in A \mid \phi(xy) = \lambda^k\phi(yx) \forall y \in \mathcal{O}_n\}.$$

Let $\Lambda := \{2^p3^q \mid p, q \in \mathbb{Z}\} \cap (0, 1)$. For all $\lambda \in \Lambda$ and $k \in \mathbb{Z}$, we have $A_{\lambda,k} \neq \{0\}$. This leads, analogous to Lemma 4.2, to the following result.

Lemma 5.1. Let $\lambda \in \Lambda$. For every $k \in \mathbb{Z}$ there exists a $c_{\lambda,k} \in A$ such that

$$\phi(c_{\lambda,k}^*c_{\lambda,k}) = \lambda^{-\frac{k}{2}}, \qquad \phi(c_{\lambda,k}c_{\lambda,k}^*) = \lambda^{\frac{k}{2}}$$

and

$$\langle c_{\lambda,k}Jc_{\lambda,k}J\xi, \xi\rangle = 1.$$

In this way, by the analogues of Lemma 2.4, Lemma 2.5 and Proposition 2.6 of [12], we obtain the following result, which is the analogue of [12], Proposition 2.8.

Proposition 5.2. Let $\lambda \in \Lambda$, and let $C(\lambda) = \sqrt{(\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}})/2}$. Let $u : A \times B \to \mathbb{C}$ be a jointly completely bounded bilinear form. Then there exist states f_1^λ, f_2^λ on A and g_1^λ, g_2^λ on B such that for all $a \in A$ and $b \in B$,

$$|u(a,b)| \leq C(\lambda)\|u\|_{jcb} \left(f_1^\lambda(aa^*)^{\frac{1}{2}} g_1^\lambda(b^*b)^{\frac{1}{2}} + f_2^\lambda(a^*a)^{\frac{1}{2}} g_2^\lambda(bb^*)^{\frac{1}{2}} \right).$$

Note that $C(\lambda) > 1$ for $\lambda \in \Lambda$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in Λ converging to 1. By the weak*-compactness of the unit balls $(A_+^*)_1$ and $(B_+^*)_1$ of A_+^* and B_+^*, respectively, the Grothendieck Theorem for jointly completely bounded bilinear forms with $K = 1$ follows in the same way as in the "Proof of Theorem 1.1" in [12].

Remark 5.3. By Kirchberg's second "Geneva Theorem" (see [14] for a proof), we know that $\mathcal{O}_2 \otimes \mathcal{O}_3 \cong \mathcal{O}_2$. This implies that the best constant in Theorem 1.1 can also be obtained by tensoring with the single Cuntz algebra $\mathcal{O}_2$, but considered with a different action that defines the C^*-dynamical system. Since the explicit form of the isomorphism is not known, we cannot adjust the action accordingly.

6. A remark on Blecher's inequality

In [2], Blecher stated a conjecture about the norm of elements in the algebraic tensor product of two C^*-algebras. Equivalently, the conjecture can be formulated as follows (see Conjecture 0.2′ of [18]). For a bilinear form $u : A \times B \to \mathbb{C}$, put $u^t(b,a) = u(a,b)$.

Theorem 6.1 (Blecher's inequality)**.** There is a constant K such that any jointly completely bounded bilinear form $u : A \times B \to \mathbb{C}$ on C^*-algebras A and B decomposes as a sum $u = u_1 + u_2$ of completely bounded bilinear forms on $A \times B$, and $\|u_1\|_{cb} + \|u_2^t\|_{cb} \leq K\|u\|_{jcb}$.

A version of this conjecture for exact operator spaces and a version for pairs of C^*-algebras, one of which is assumed to be exact, were proved by Pisier and Shlyakhtenko [18]. They also showed that the best constant in Theorem 6.1 is greater than or equal to 1. Haagerup and Musat proved that Theorem 6.1 holds with $K = 2$ [12, Section 3]. We show that the best constant is actually strictly greater than 1.

In the following, let $\mathrm{OH}(I)$ denote Pisier's operator Hilbert space based on $\ell^2(I)$ for some index set I. Recall the noncommutative little Grothendieck Theorem.

Theorem 6.2 (Noncommutative little Grothendieck Theorem)**.** Let A be a C^*-algebra, and let $T : A \to \mathrm{OH}(I)$ be a completely bounded map. Then there exists a universal constant $C > 0$ and states f_1 and f_2 on A such that for all $a \in A$,

$$\|Ta\| \leq C\|T\|_{cb} f_1(aa^*)^{\frac{1}{4}} f_2(a^*a)^{\frac{1}{4}}.$$

For a completely bounded map $T : A \to \mathrm{OH}(I)$, denote by $C(T)$ the smallest constant $C > 0$ for which there exist states f_1, f_2 on A such that for all $a \in A$, we have $\|Ta\| \leq Cf_1(aa)^{\frac{1}{4}} f_2(a^*a)^{\frac{1}{4}}$. In [12], Haagerup and Musat proved that $C(T) \leq \sqrt{2}\|T\|_{cb}$. Pisier and Shlyakhtenko proved in [18] that $\|T\|_{cb} \leq C(T)$ for all $T : A \to \mathrm{OH}(I)$. Haagerup and Musat proved that for a certain $T : M_3(\mathbb{C}) \to \mathrm{OH}(3)$, the inequality is actually strict, i.e., $\|T\|_{cb} < C(T)$ [13, Section 7]. We can now apply this knowledge to improve the best constant in Theorem 6.1.

Theorem 6.3. The best constant K in Theorem 6.1 is strictly greater than 1.

Proof. Let A be a C^*-algebra, and let $T : A \to \mathrm{OH}(I)$ be a completely bounded map for which $\|T\|_{cb} < C(T)$. Define the map $V = \overline{T^*}JT$ from A to $\overline{A^*} = \overline{A}^*$, where $J : \mathrm{OH}(I) \to \overline{\mathrm{OH}(I)^*}$ is the canonical complete isomorphism and $T^* : \mathrm{OH}(I)^* \to A^*$ is the adjoint of T. Hence, V is completely bounded. It follows that $V = \tilde{u}$ for some jointly completely bounded bilinear form $u : A \times \overline{A} \to \mathbb{C}$. Moreover, $\|u\|_{jcb} = \|V\|_{cb} = \|T\|_{cb}^2$, where the last equality follows from the proof of Corollary 3.4 in [18]. By Blecher's inequality, i.e., Theorem 6.1, we have a decomposition $u = u_1 + u_2$ such that $\|u_1\|_{cb} + \|u_2^t\|_{cb} \leq K\|u\|_{jcb}$.

By the second characterization of completely bounded bilinear forms (in the Christensen-Sinclair sense) in Section 2, we obtain

$$|u_1(a,b)| \leq \|u_1\|_{cb} f_1(aa^*)^{\frac{1}{2}} g_1(b^*b)^{\frac{1}{2}}, \quad |u_2(a,b)| \leq \|u_2^t\|_{cb} f_2(a^*a)^{\frac{1}{2}} g_2(bb^*)^{\frac{1}{2}}.$$

It follows that

$$|u(a,b)| \leq \|u_1\|_{cb} f_1(aa^*)^{\frac{1}{2}} g_1(b^*b)^{\frac{1}{2}} + \|u_2^t\|_{cb} f_2(a^*a)^{\frac{1}{2}} g_2(bb^*)^{\frac{1}{2}}.$$

Let $\overline{g}_i(a) = g_i(\overline{a^*})$ for $i = 1, 2$, and define states

$$\tilde{f} = \frac{\|u_1\|_{cb} f_1 + \|u_2^t\|_{cb} \overline{g}_2}{\|u_1\|_{cb} + \|u_2^t\|_{cb}} \quad \text{and} \quad \tilde{g} = \frac{\|u_1\|_{cb} \overline{g}_1 + \|u_2^t\|_{cb} f_2}{\|u_1\|_{cb} + \|u_2^t\|_{cb}}.$$

We obtain

$$\begin{aligned}
\|T(a)\|^2 = |u(a,\overline{a})| &\leq \|u_1\|_{cb} f_1(aa^*)^{\frac{1}{2}} \overline{g}_1(a^*a)^{\frac{1}{2}} + \|u_2^t\|_{cb} f_2(a^*a)^{\frac{1}{2}} \overline{g}_2(aa^*)^{\frac{1}{2}} \\
&\leq (\|u_1\|_{cb} f_1 + \|u_2^t\|_{cb} \overline{g}_2)(aa^*)^{\frac{1}{2}} (\|u_1\|_{cb} \overline{g}_1 + \|u_2^t\|_{cb} f_2)(a^*a)^{\frac{1}{2}} \\
&\leq (\|u_1\|_{cb} + \|u_2^t\|_{cb}) \tilde{f}(aa^*)^{\frac{1}{2}} \tilde{g}(a^*a)^{\frac{1}{2}}.
\end{aligned}$$

Hence, $\|u_1\|_{cb} + \|u_2^t\|_{cb} \geq C(T)^2 > \|T\|_{cb}^2 = \|u\|_{jcb}$. This proves the theorem. □

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