Algebraic K-theory of generalized schemes

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Till mina föräldrar.

Abstract: Nikolai Durov has developed a generalization of conventional scheme theory in which commutative algebraic monads replace commutative unital rings as the basic algebraic objects. The resulting geometry is expressive enough to encompass conventional scheme theory, tropical algebraic geometry and geometry over the field with one element. It also permits the construction of important Arakelov theoretical objects, such as the completion $\widehat{\operatorname{Spec}\mathbb{Z}}$ of $\operatorname{Spec}\mathbb{Z}$. In this thesis, we prove a projective bundle theorem for the field with one element and compute the Chow rings of the generalized schemes $\widehat{\operatorname{Spec}\mathbb{Z}}^N$, appearing in the construction of $\widehat{\operatorname{Spec}\mathbb{Z}}$.

Resumé: Nikolai Durov har udviklet en generalisering af konventionel skemateori, hvori kommutative algebraiske monader erstatter kommutative enhedsbærende ringe som de grundlæggende algebraiske objekter. Den resulterende geometri er bred nok til at omfatte konventionel skemateori, tropisk algebraisk geometri og geometri over legemet med et element. Den giver også mulighed for at konstruere objekter fra Arakelovgeometri, f eks fulstændiggørelsen $\widehat{\operatorname{Spec}\mathbb{Z}}$ af $\operatorname{Spec}\mathbb{Z}$. I denne afhandling viser vi en projective bundle-sætning for legemet med et element. Vi undersøger også de generaliserede skemaer $\widehat{\operatorname{Spec}\mathbb{Z}}^N$, som optræder i konstruktionen av $\widehat{\operatorname{Spec}\mathbb{Z}}$, og udregner deres Chow-ringe.

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Contents

Introduction

"Are you sure these specks aren't supposed to be here?" JD tentatively touches the panel. "I mean, maybe it's supposed to be, oh, I don't know, in or something?"

Bret Easton Ellis
Glamorama

When studying the arithmetic of the ring of integers $\mathbb{Z}$, one can sometimes benefit from comparing $\mathbb{Z}$ with the ring $k[x]$ of polynomials in one variable over a field k. The arithmetic of $k[x]$ can be studied by considering projective schemes over k and using results such as the Riemann-Roch and Riemann-Hurwitz theorems. Indeed, when applied to the projective line $\mathbb{P}^1_k$ these theorems yield a proof of an analogue of the *abc*-conjecture for $k[x]$ (cf. e.g. [**19**]). Moreover, as was proved by Pierre Deligne, an analogue of the Riemann hypothesis also holds true for $k[x]$ (cf. [**6**]). These circumstances have left mathematicians searching for a number theoretical analogue of $\mathbb{P}^1_k$, i.e. an algebro-geometric object $\widehat{\text{Spec } \mathbb{Z}}$ that relates to the affine scheme Spec $\mathbb{Z}$ like $\mathbb{P}^1_k$ relates to the affine scheme Spec $k[x]$.

The projective line $\mathbb{P}^1_k$ can be described as an algebraic one-point compactification of Spec $k[x]$ in the following way. For any element $a \in k$, there is an associated *absolute value* on the quotient field $k(x) = \text{Quot } k[x]$:

$$v_a(\cdot) = e^{-\text{ord}_{x-a}(\cdot)}.$$

In fact, any equivalence class of non-trivial absolute values on $k(x)$ has a representative of this form, with the exception of the equivalence class containing the absolute value

$$v_\infty = e^{-\deg(\cdot)}.$$

We make two observations:

- Assuming that the field k is algebraically closed, there is a bijection between k and the set of non-zero prime ideals of $k[x]$.
- The set $\mathcal{O}_v = \{\alpha \in k(x);\ v(\alpha) \le 1\}$ is a local ring, for any absolute value v on $k(x)$.

These facts allow for the construction of the projective scheme $\mathbb{P}^1_k$, by completing the affine scheme Spec $k[x]$ with an additional point corresponding to the absolute value v_∞.

Now one may try to construct the object $\widehat{\text{Spec } \mathbb{Z}}$ by completing the affine scheme Spec $\mathbb{Z}$ in a similar way, using absolute values on the quotient field $\mathbb{Q} = \text{Quot } \mathbb{Z}$. By Ostrowski's theorem, any equivalence class of non-trivial absolute values on $\mathbb{Q}$ is represented either by $p^{-ord_p(\cdot)}$, for some prime number p, or by the unique archimedean absolute value $|\cdot|$. Pursuing the analogy with $\mathbb{P}^1_k$, the extra point of $\widehat{\text{Spec }\mathbb{Z}}$ should be furnished by $|\cdot|$. However, there is a serious issue in this situation: The set

$$\mathcal{O}_\infty = \{\alpha \in \mathbb{Q}; |\alpha| \leq 1\}$$

does not admit a ring structure. Consequently, any attempt to construct $\widehat{\text{Spec }\mathbb{Z}}$ within the conventional theory is bound to fail.

A framework for completing an arithmetic scheme along the lines described above is provided by *Arakelov geometry*. Here the extra data of the "compactification" consist of hermitian metrics on the arithmetic vector bundles over the scheme. This technique was conceived in the 1970's by Suren Arakelov, inspired by Igor Shafarevich's approach to the Mordell conjecture (cf. [**1**]). A proof of the conjecture based on Arakelov geometric methods would be found about 20 years later by Paul Vojta (cf. [**26**]). Today, Arakelov geometry comprises an important branch of diophantine geometry, featuring for instance its own version of the Grothendieck-Riemann-Roch theorem (cf. [**11**]).

A key ingredient in the proof of the analogue of the Riemann hypothesis for $k[x]$ is the fiber product Spec $k[x] \times_{\text{Spec }k}$ Spec $k[x]$. In order to be able to mimic this proof for the ring $\mathbb{Z}$, one would have to be in possession of an affine scheme playing the role of Spec k. More precisely, one would need an object $\mathbb{F}_1$ and a morphism $\mathbb{F}_1 \to \mathbb{Z}$, which render a non-trivial fiber product Spec $\mathbb{Z} \times_{\text{Spec }\mathbb{F}_1}$ Spec $\mathbb{Z}$. Since $\mathbb{Z}$ is the initial object in the category of commutative unital rings, this means that once again one is forced to step outside of the conventional theory.

The object $\mathbb{F}_1$, commonly referred to as the *field with one element*, was first envisioned in 1957 by Jacques Tits in the context of his theory of buildings (cf. [**21**]). After that, the concept was mused upon by Mikhail Kapranov, Alexander Smirnov and Yuri I. Manin (cf. [**13**] and [**17**]). A notion of varieties over $\mathbb{F}_1$ was proposed by Christophe Soulé in 2004 (cf. [**20**]), and his theory was soon to be adopted and developed further by Alain Connes and Caterina Consani. They were motivated by joint work with Matilde Marcolli, in which they had discovered connections between $\mathbb{F}_1$ and non-commutative geometry (cf. [**2**] and [**3**]). In Connes-Consani's theory, an $\mathbb{F}_1$-scheme is a hybrid of a conventional scheme and a *monoid scheme* in the sense defined by Anton Deitmar in 2005 (cf. [**4**]). Monoid schemes were originally defined by Anton Deitmar, who has also done illuminating work on their zeta functions and algebraic K-theory (cf. [**4**] and [**5**]).

A conventional scheme can be studied through its *functor of points*. This is a rule which assigns a set to any commutative unital ring and which should be thought of as a parametrization of zero-sets of algebraic equations (cf. e.g. [**9**]). Bertrand Toën and Michel Vaquié describe a version of algebraic geometry over $\mathbb{F}_1$ using this perspective. They begin more generally by associating a theory of algebraic geometry to any symmetric monoidal category $(\mathcal{C}, \otimes)$, satisfying some mild conditions (cf. [**22**]). When $(\mathcal{C}, \otimes)$ is the category of abelian groups with the usual tensor product, one recovers conventional scheme theory from their construction, and algebraic geometry over $\mathbb{F}_1$ is defined to be the case in which $(\mathcal{C}, \otimes)$ is the category of sets with the cartesian product. Alberto Vezzani later showed that this theory is equivalent to Deitmar's theory of monoid schemes (cf. [**25**]). A further survey over the relations between different $\mathbb{F}_1$-geometries can be found in [**15**].

The framework of Toën and Vaquié permits a theory of *homotopical algebraic geometry* which they develop in [**23**] and [**24**]. Here rings are replaced by ring spectra, according to Friedhelm Waldhausen's vision of a *brave new algebra*. Another materialization of this vision is the *derived algebraic geometry* of Jacob Lurie (cf. [**16**]). These two homotopy theoretical approaches are not directly related to algebraic geometry over $\mathbb{F}_1$, but are in themselves highly interesting as attempts of redeveloping the foundations of algebraic geometry.

Nikolai Durov's generalized schemes

A pervading philosophy in extensions of conventional scheme theory is to use basic algebraic objects which carry an inherent monoid structure. This is due to the fact that when defining prime spectra of commutative unital rings, one is most crucially dependent on the existence of *multiplicative systems*. The theory of *generalized schemes*, as developed by Nikolai Durov, makes use of this observation as well. Durov suggests that the basic objects of a generalization of conventional scheme theory should be certain *algebraic monads* which he calls *generalized rings* (cf. [**8**]). The resulting geometry then becomes expressive enough to encompass conventional scheme theory, tropical algebraic geometry and a version of algebraic geometry over $\mathbb{F}_1$. It also permits the construction of important objects in Arakelov geometry, such as the "one-point compactification" $\widehat{\mathrm{Spec}\,\mathbb{Z}}$.

An *algebraic monad* is the data of an endofunctor Σ on the category of sets, together with natural transformations $\mu : \Sigma \circ \Sigma \to \Sigma$ and $\epsilon : \mathrm{id} \to \Sigma$, which implement multiplication and unit respectively (cf. definitions 1.1 and 1.4 in chapter 1). The set $\Sigma(\mathbf{1})$ is canonically equipped with a monoid structure: An element $x \in \Sigma(\mathbf{1})$ can be viewed as a map $\tilde{x} : \mathbf{1} \to \Sigma(\mathbf{1})$, and so by functoriality gives rise to a map $\Sigma(\tilde{x}) : \Sigma(\mathbf{1}) \to \Sigma \circ \Sigma(\mathbf{1})$. Composition with μ yields an endomorphism of $\Sigma(\mathbf{1})$.

If the algebraic monad Σ is a generalized ring, then the monoid $\Sigma(\mathbf{1})$ is commutative. This provides a notion of multiplicative systems in $\Sigma(\mathbf{1})$ which can be used to define *ideals* and *localizations* (cf. definitions 2.14 and 2.20 in chapter 1). Following the conventional theory, one defines the *prime spectrum* $\mathrm{Spec}\,\Sigma$ and declares that a *generalized scheme* is a *generalized locally ringed space* which admits an open cover by prime spectra of generalized rings (cf. definitions 1.3 and 1.4 in chapter 2).

Let us now briefly summarize how Durov's theory can be used in order to construct the object $\widehat{\mathrm{Spec}\,\mathbb{Z}}$. First, we remark that any commutative unital ring R defines a generalized ring Σ_R, such that the underlying topological space of the prime spectrum $\mathrm{Spec}\,\Sigma_R$ coincides with the underlying topological space of $\mathrm{Spec}\,R$. Given a natural number $N \geq 2$, one defines a generalized ring A_N with the property that

$$A_N(\mathbf{n}) = \left\{ (\lambda_1, ..., \lambda_n) \in \mathbb{Z}[N^{-1}]^n;\ \sum_{i=1}^{n} |\lambda_i| \leq 1 \right\},\ \text{for } n \in \mathbb{N}.$$

Stated differently, $A_N(\mathbf{n})$ is the intersection of $\mathbb{Z}[N^{-1}]^n$ and the n-octahedron.

The set underlying the prime spectrum Spec A_N consists of the following elements.

- The zero ideal of A_N.
- Ideals of A_N which are generated by prime numbers p, such that $p \nmid N$.
- The maximal ideal $\mathfrak{p}_{N\infty} = \{\alpha \in \mathbb{Z}[N^{-1}];\ |\alpha| < 1\}$.

The localization of A_N with respect to the multiplicative subset generated by N^{-1} coincides with the generalized ring defined by the ring $\mathbb{Z}[N^{-1}]$. This permits the glueing of Spec A_N and Spec $\Sigma_{\mathbb{Z}}$ along Spec $\Sigma_{\mathbb{Z}[N^{-1}]}$:

$$\widehat{\text{Spec } \mathbb{Z}}^N = \text{Spec } A_N \sqcup_{\text{Spec } \Sigma_{\mathbb{Z}[N^{-1}]}} \text{Spec } \Sigma_{\mathbb{Z}}.$$

The object $\widehat{\text{Spec } \mathbb{Z}}^N$ is a generalized scheme with the "correct" underlying set in the sense that there is a one-to-one correspondence between its points and the valuations on Quot $\mathbb{Z}$. Passing to the limit of a projective system, one obtains an object with the desired topological properties:

$$\widehat{\text{Spec } \mathbb{Z}} = \varprojlim_{N \geq 2} \widehat{\text{Spec } \mathbb{Z}}^N.$$

(See definition 1.8 in chapter 2.)

The object $\widehat{\text{Spec } \mathbb{Z}}$ is a generalized locally ringed space, such that any algebraic variety X over $\mathbb{Q}$ admits a finitely presented model $\mathcal{X}$ over $\widehat{\text{Spec } \mathbb{Z}}$. Furthermore, if X and $\mathcal{X}$ are projective, then any rational point P of X extends to a uniquely determined section σ_P of such a model, and the logarithmic height of P can be related to the arithmetic degree of $\sigma_P^* \mathcal{O}_{\mathcal{X}}(1)$. This makes Durov's theory suitable for applications in Arakelov geometry.

Let us briefly mention how the field with one element enters in the new theory. The generalized ring $\mathbb{F}_1$ is defined via

$$\mathbb{F}_1(X) = X \sqcup \{*\}.$$

While this object admits a morphism to the generalized ring defined by $\mathbb{Z}$ in the category of generalized rings, it turns out that the corresponding fiber product Spec $\mathbb{Z} \times_{\text{Spec } \mathbb{F}_1}$ Spec $\mathbb{Z}$ is isomorphic to Spec $\mathbb{Z}$. In other words, generalized rings do not provide a setting in which one can approach the Riemann hypothesis according to the ideas described earlier.

The present text

The main results in this thesis are the *projective bundle theorem for* $\mathbb{F}_1$ and a computation of the Chow rings of the generalized shemes $\widehat{\text{Spec } \mathbb{Z}}^N$, for $N \geq 2$ (cf. theorem 1.6 and corollary 2.11 in chapter 3).

The proof of the projective bundle theorem starts out in chapter 1 with the following characterization of finitely generated projective modules over generalized polynomial rings (cf. theorem 3.2 in chapter 1).

Theorem 1. *For $n \in \mathbb{N}$, any finitely generated projective module over the generalized ring $\mathbb{F}_1[x_1, ..., x_n]$ is stably free.*

In chapter 3, theorem 1 is first used in order to compute the Grothendieck group of $\mathbb{P}^1_{\mathbb{F}_1}$, along with its additional ring structure. More precisely, in theorem 1.4 it is established that there is an isomorphism of abelian groups

$$K^0(\mathbb{P}^1_{\mathbb{F}_1}) \simeq \mathbb{Z} \times \mathbb{Z}.$$

Using induction on n, theorem 1 then allows for a proof of the following.

Theorem 2. (Projective bundle theorem for $\mathbb{F}_1$)
For $n \in \mathbb{N}$, there is an isomorphism of rings

$$K^0(\mathbb{P}^n_{\mathbb{F}_1}) \simeq \mathbb{Z}[x]/x^{n+1}.$$

The computation of the Chow rings of the generalized schemes $\widehat{\operatorname{Spec}\mathbb{Z}}^N$ is called for by Durov at the very end of his thesis (cf. (**10.7.16**) in [**8**]). A first step towards such a computation is the following characterization of finitely generated projective modules over the generalized ring A_N (cf. theorem 2.1 of chapter 3).

Theorem 3. *For a natural number $N \geq 2$, any finitely generated projective module over A_N is free.*

The strategy in the proof of this theorem is to show that any finitely generated projective module of rank strictly larger than one is isomorphic to a direct sum of finitely generated projective modules of lower rank. This reduces the situation to rank one finitely generated projective modules, for which the desired characterization is achieved by Durov (cf. (**7.1.33**) in [**8**]). (An alternative proof for the rank one-case is also obtained as a porism of lemma 2.2 in chapter 3.)

A consequence of theorem 3 is that isomorphism classes of vector bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}^N$ can be identified with certain double cosets of matrices (cf. proposition 2.6 in chapter 3). This alternative description reveals that the Grothendieck group $K^0(\widehat{\operatorname{Spec}\mathbb{Z}}^N)$ is generated by line bundles and that the map

$$\begin{aligned} \operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}^N) &\to K^0(\widehat{\operatorname{Spec}\mathbb{Z}}^N), \\ [\mathcal{L}] &\mapsto [\mathcal{L}] - 1, \end{aligned}$$

is an injective homomorphism of abelian groups. In this situation, a result by Durov applies to compute the Grothendieck group.

Theorem 4. *For a natural number $N \geq 2$, the map*

$$\begin{array}{rcl} \mathbb{Z} \times \operatorname{Pic}(\widehat{\operatorname{Spec}\mathbb{Z}}^N) & \to & K^0(\widehat{\operatorname{Spec}\mathbb{Z}}^N), \\ (n, [\mathcal{L}]) & \mapsto & n + [\mathcal{L}] - 1, \end{array}$$

is an isomorphism of rings.

The computations of the Chow rings announced above is obtained as a corollary of theorem 4 (cf. corollary 2.11 in chapter 3).

Corollary 5. *For a natural number $N \geq 2$, the Chow ring of $\widehat{\operatorname{Spec}\mathbb{Z}}^N$ is*

$$\operatorname{CH}(\widehat{\operatorname{Spec}\mathbb{Z}}^N) \simeq \mathbb{Q} \oplus \log \mathbb{Q}[N^{-1}]^*_+.$$

This computation underlines the similarity between $\widehat{\operatorname{Spec}\mathbb{Z}}^N$ and one-dimensional projective spaces, since it reveals that the intersection theory of $\widehat{\operatorname{Spec}\mathbb{Z}}^N$ is one-dimensional. In fact, theorems 2 and 4 show that the rings $K^0(\mathbb{P}^1_{\mathbb{F}_1})$ and $K^0(\widehat{\operatorname{Spec}\mathbb{Z}}^N)$ are isomorphic whenever N is a prime number. In these cases, the twisting bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(-1)$ corresponds to a unique line bundle on $\widehat{\operatorname{Spec}\mathbb{Z}}^N$ defined by N. Generally, each prime divisor of N gives rise to an incarnation of this twisting bundle. Hence the Chow rings reflect at once the “curve-like” nature of $\widehat{\operatorname{Spec}\mathbb{Z}}^N$ and the arithmetic complexity of the natural number N.

Organization of the material

The thesis consists of four chapters and one appendix. Chapters 1 and 2 give an introduction to Durov’s theory of generalized schemes and their Grothendieck groups. The exposition of these chapters follow [**8**] and [**10**] closely and, apart from the results in sections 2 and 3 of chapter 1, they contain no original material. In chapter 3, the projective bundle theorem is proved, and the computations of the rings $K^0(\widehat{\operatorname{Spec}\mathbb{Z}}^N)$ and $\operatorname{CH}(\widehat{\operatorname{Spec}\mathbb{Z}}^N)$ are carried out. Chapter 4 contains two Grothendieck-Riemann-Roch results for generalized schemes. The first one concerns projections to $\operatorname{Spec}\mathbb{F}_1$ and the second one concerns *zero-sections* between generalized schemes with finite intersection theory. Appendix A gives a short summary of sheaf theory and conventional algebraic geometry.

CHAPTER 1

Generalized rings

This chapter begins with a brief summary of the generalization of commutative algebra proposed by Nikolai Durov in his thesis. The new theory allows for a version of algebraic geometry which makes it possible to construct a one-point compactification of Spec $\mathbb{Z}$ and schemes over the field with one element. Moreover, since both commutative unital rings and semi-rings are *generalized rings* in Durov's sense, the new algebraic geometry subsumes both conventional algebraic geometry and tropical algebraic geometry. After introducing the necessary terminology, we proceed to study free and stably free resolutions in the new context. Our observations are then used in order to conclude that any *finitely generated projective module* over a generalized rings of polynomials over the field with one element is *stably free*.

1. Algebraic monads

The algebraic structures which replace commutative unital rings in Durov's theory will be certain *algebraic monads*. These are devices which encode n-ary operations, for $n \in \mathbb{N}$, in the sense of the following definitions.

Definition 1.1. Let $\mathcal{C}$ be a category. A *monad* on $\mathcal{C}$ is a triple $\Sigma = (\Sigma, \mu, \epsilon)$, consisting of an endofunctor $\Sigma : \mathcal{C} \to \mathcal{C}$, and two natural transformations

$$\begin{aligned} \mu &: \Sigma \circ \Sigma \to \Sigma, \\ \epsilon &: \mathrm{id}_{\mathcal{C}} \to \Sigma, \end{aligned}$$

which make the following diagrams commute.

$$\begin{array}{ccc} \Sigma \circ \Sigma \circ \Sigma(X) & \xrightarrow{\Sigma(\mu_X)} & \Sigma \circ \Sigma(X) \\ \downarrow{\scriptstyle \mu_{\Sigma(X)}} & & \downarrow{\scriptstyle \mu_X} \\ \Sigma \circ \Sigma(X) & \xrightarrow{\mu_X} & \Sigma(X) \end{array} \qquad \begin{array}{ccccc} \mathrm{id}_{\mathcal{C}} \circ \Sigma(X) & \xrightarrow{\epsilon_{\Sigma(X)}} & \Sigma \circ \Sigma(X) & \xleftarrow{\Sigma(\epsilon_X)} & \Sigma \circ \mathrm{id}_{\mathcal{C}}(X) \\ & {\scriptstyle \mathrm{id}_{\Sigma(X)}} \searrow & \downarrow{\scriptstyle \mu_X} & \swarrow {\scriptstyle \Sigma(\mathrm{id}_X)} & \\ & & \Sigma(X) & & \end{array}$$

A *homomorphism* of monads $\phi : \Sigma_1 \to \Sigma_2$ is a natural transformation of the underlying endofunctors, compatible with the four structural transformations in the sense that for any morphism $f : X \to Y$ in $\mathcal{C}$

$$\begin{aligned} \phi \circ \epsilon_{\Sigma_1} &= \epsilon_{\Sigma_2}, \\ \phi \circ \mu_{\Sigma_1} &= \mu_{\Sigma_2} \circ (\phi \cdot \phi). \end{aligned}$$

DEFINITION 1.2. Let Σ be a monad on a category $\mathcal{C}$. A *module over* Σ is a pair (M, α), consisting of an object M of $\mathcal{C}$, and a morphism $\alpha : \Sigma(M) \to M$ in $\mathcal{C}$, satisfying

$$\alpha \circ \mu_M = \alpha \circ \Sigma(\alpha),$$
$$\alpha \circ \epsilon_M = \mathrm{id}_M.$$

A *homomorphism* $f : (M, \alpha_M) \to (N, \alpha_N)$ of modules over Σ is an element $f \in \mathrm{Hom}_{\mathcal{C}}(M, N)$, such that $f \circ \alpha_M = \alpha_N \circ \Sigma(f)$.

REMARK 1.3. Given an object X of a category $\mathcal{C}$ and a monad $\Sigma = (\Sigma, \mu, \epsilon)$ on $\mathcal{C}$, the natural transformation μ furnishes the object $\Sigma(X)$ with a structure of module over Σ:

$$\mu_{\Sigma(X)} : \Sigma(\Sigma(X)) \to \Sigma(X).$$

DEFINITION 1.4. An endofunctor Σ on the category of sets is *algebraic* if, for any filtered family of sets $\{X_i\}_{i \in I}$, one has

$$\Sigma(\varinjlim_i X_i) \simeq \varinjlim_i \Sigma(X_i).$$

A monad is *algebraic* if its underlying endofunctor is algebraic.

Let us define a few algebraic monads.

DEFINITION 1.5. Given a commutative unital ring R, we define the algebraic monad $\Sigma_R = (\Sigma_R, \mu, \epsilon)$ as follows. For a set X, let

$$\Sigma_R(X) = \mathrm{Hom}^{\mathrm{fin}}_{\mathcal{S}ets}(X, R),$$

be the set of finitely supported maps from X to R. Elements of $\Sigma_R(X)$ can then be identified with finite formal linear combinations

$$\lambda_1\{x_1\} + \dots + \lambda_n\{x_n\},$$

where $\lambda_i \in R$ and $x_i \in X$, for $1 \leq i \leq n$. The natural transformation μ is then defined via

$$\mu_X\left(\sum_i \lambda_i \left\{\sum_j \nu_{ij}\{x_j\}\right\}\right) = \sum_{ij} \lambda_i \nu_{ij}\{x_j\},$$

and the natural transformation ϵ is defined via $\epsilon_X(x) = \{x\}$.

REMARK 1.6. The notion of a module over Σ_R coincides with the conventional notion of a module over R. Indeed, let M be a set, and for each $x \in M$, consider the characteristic function of x:

$$\chi_x : X \to \{0, 1\} \subset R.$$

Now assume that (M, α) is a module over Σ_R. Putting

$$x + y = \alpha(\chi_x + \chi_y),$$
$$\lambda x = \alpha(\lambda \chi_x),$$

one obtains a module over R. Conversely, given a module M over R, the map

$$\lambda_1\{x_1\} + \dots + \lambda_n\{x_n\} \overset{\alpha}{\mapsto} \lambda_1 x_1 + \dots + \lambda_n x_n,$$

equips M with the structure of module over Σ_R.

DEFINITION 1.7. ($\mathbb{F}_\emptyset$ and $\mathbb{F}_1$)
We define the algebraic monad $\mathbb{F}_\emptyset$ by

$$\mathbb{F}_\emptyset(X) = X,$$

with both $\mu_{\mathbb{F}_\emptyset}$ and $\epsilon_{\mathbb{F}_\emptyset}$ given by the identity maps of sets. The *field with one element* is the algebraic monad $\mathbb{F}_1$ defined by

$$\mathbb{F}_1(X) = X \sqcup \{*\}.$$

The natural transformation $\mu_{\mathbb{F}_1}$ is given by collapsing two copies of $*$ to one, and the natural transformation $\epsilon_{\mathbb{F}_1}$ is given by the inclusion of sets.

DEFINITION 1.8. ($\mathbb{Z}_\infty$ and A_N)
The algebraic monad $\mathbb{Z}_\infty$ is defined as a submonad of the algebraic monad $\Sigma_\mathbb{R}$ (cf. definition 1.5):

$$\mathbb{Z}_\infty(X) = \left\{ \sum_i \lambda_i \{x_i\} \in \Sigma_\mathbb{R}(X);\ \sum_i |\lambda_i| \leq 1 \right\}.$$

Further, for any natural number $N \geq 2$, the algebraic monad A_N is defined as the intersection of $\mathbb{Z}_\infty$ with the algebraic monad $\Sigma_{\mathbb{Z}[N^{-1}]}$:

$$A_N(X) = \left\{ \sum_i \lambda_i \{x_i\} \in \mathbb{Z}_\infty(X);\ \lambda_i \in \mathbb{Z}[N^{-1}] \right\}.$$

DEFINITION 1.9. (Endomorphism monads)
The *endomorphism monad* of a set Y is defined by

$$\mathrm{End}(Y)(X) = \mathrm{Hom}(Y^X, Y),$$

with $\mu_{\mathrm{End}(Y)}$ given by the composition of maps and $\epsilon_{\mathrm{End}(Y)}$ given by evaluation. To give a homomorphism $\Sigma \to \mathrm{End}(Y)$ of algebraic monads is equivalent with specifying a module structure $\Sigma(Y) \to Y$ (cf. (**4.3.8**) in [**8**]).

There is an alternative approach to the theory of algebraic monads which sheds light on its connection to universal algebra. First note that given a module $\alpha : \Sigma(M) \to M$, an element $(x_1, ..., x_n) \in M^n$ may be viewed as a map $\tilde{x} : \mathbf{n} \to M$. Hence one may apply the composite $\alpha \circ \Sigma(\tilde{x})$ to an element $u \in \Sigma(\mathbf{n})$ and get an *n-ary operation*

$$\Sigma(\mathbf{n}) \times M^n \to M.$$

Conversely, given sets $\Sigma(\mathbf{n})$ of n-ary operations, for $n \in \mathbb{N}$, which satisfy certain compatibility conditions, one can construct a corresponding algebraic monad (cf. (**4.5.11**) in [**8**]).

The above shows in particular that the set $\Sigma(\mathbf{1})$ carries a monoid structure, so it makes sense to talk about *multiplicative subsets* of $\Sigma(\mathbf{1})$. It also makes sense to define ideals and localizations. However, we will postpone precise definitions until section 2. Instead, we continue by describing two algebraic monads in terms of their operations.

EXAMPLE 1.10. The commutative semi-ring $\mathbb{T}$ of *tropical numbers* is the set $\mathbb{R} \cup \{\infty\}$, equipped with two binary operations $[+]$ and $[\times]$, defined by

$$\begin{aligned}[+](a,b) &= \min(a,b),\\ [\times](a,b) &= a+b.\end{aligned}$$

This data gives rise to an algebraic monad $\Sigma_{\mathbb{T}}$, which makes it possible to formulate tropical algebraic geometry in the new framework.

EXAMPLE 1.11. By adjoining a unary operation $[x] : \mathbb{F}_1(\mathbf{1}) \to \mathbb{F}_1(\mathbf{1})$ to $\mathbb{F}_1$, one obtains the algebraic monad $\mathbb{F}_1[x]$. This object is initial in the category of triples (Σ, ρ, f), where Σ is an algebraic monad, $\rho : \mathbb{F}_1 \to \Sigma$ is a homomorphism of monads, and $f : \{x\} \to \Sigma(\mathbf{1})$ is a map. Explicitly, it is given by

$$\mathbb{F}_1[x](\mathbf{n}) = \left\{ *, \begin{array}{ccc} {}_1x^0, & \cdots & {}_nx^0 \\ {}_1x^1, & \cdots & {}_nx^1 \\ {}_1x^2, & \cdots & {}_nx^2 \\ \vdots & & \vdots \end{array} \right\},$$

with action

$$\begin{array}{rcl} \mathbb{F}_1[x](\mathbf{1}) \times \mathbb{F}_1[x](\mathbf{n}) & \to & \mathbb{F}_1[x](\mathbf{n}), \\ ({}_1x^i, {}_kx^j) & \mapsto & {}_kx^{i+j}, \end{array}$$

for $1 \leq k \leq n$, and $i, j \in \mathbb{N}$.

The algebraic monad $\mathbb{F}_1[x]$ provides an example of a *free algebra* over $\mathbb{F}_1$, in the sense of the following definitions.

DEFINITION 1.12. Let Σ be an algebraic monad. An *algebra* over Σ is a homomorphism of algebraic monads

$$\rho : \Sigma \to T.$$

We will abuse notation and write just T in cases where no confusion can arise.

DEFINITION 1.13. Let $U = \bigsqcup_{n \in \mathbb{N}} U_n$ be a graded set, and consider the category of pairs (Σ, f), which consist of an algebraic monad Σ, and a map of graded sets

$$f : U \to \bigsqcup_{n \in \mathbb{N}} \Sigma(\mathbf{n}).$$

The *free algebra generated by* U is the initial object $\mathbb{F}_\emptyset\langle U \rangle$ in this category. Similarly, for an algebraic monad T, the *free algebra over* T *generated by* U is the initial object $T\langle U \rangle$ in the category whose objects are algebras over T equipped with graded maps as above.

For a proof of the existence of free algebras, see (**4.5.2-4.5.8**) in [**8**].

REMARK 1.14. If Σ is an algebraic monad and U is a graded set, then the data of a module over $\Sigma\langle U \rangle$ consists of the data of a module M over Σ, together with arbitrarily chosen maps

$$[u] : M^n \to M, \text{ for } u \in U_n, \text{ with } n \in \mathbb{N}.$$

(See (**4.5.13**) in [**8**].)

DEFINITION 1.15. Let Σ be an algebraic monad. For $m \in \mathbb{N}$, the *non-commutative polynomial algebra in m variables over* Σ is the algebraic monad $\Sigma\langle x_1, ..., x_m\rangle$, whose set of n-ary operations is built up according to the following rules.

(i) $1, ..., n \in \Sigma\langle x_1, ..., x_m\rangle(\mathbf{n})$.

(ii) If $t \in \Sigma\langle x_1, ..., x_m\rangle(\mathbf{n})$, then $x_i t \in \Sigma\langle x_1, ..., x_m\rangle(\mathbf{n})$, for $0 \leq i \leq m$.

(iii) If $u \in \Sigma(\mathbf{k})$ is a k-ary operation of Σ, and $t_1, ..., t_k \in \Sigma\langle x_1, ..., x_m\rangle(\mathbf{n})$, then $ut_1 \cdots t_k \in \Sigma\langle x_1, ..., x_m\rangle(\mathbf{n})$.

Stated differently, $\Sigma\langle x_1, ..., x_m\rangle(\mathbf{n})$ is the set of "valid expressions" in $x_1, ..., x_m$ and operations from Σ (cf. (**4.5.2**)-(**4.5.9**) in [**8**]).

REMARK 1.16. While it makes sense to talk about the *degree* of an element $t \in \Sigma\langle x_1, ..., x_m\rangle(\mathbf{1})$, it is not always possible to extract coefficients from t (cf. (**5.3.22**) in [**8**]). This leads to substantial technical complications, since it makes it difficult to use properties of Σ to deduce properties of $\Sigma\langle x_1, ..., x_m\rangle$. This issue will be discussed in section 2.1.

In line with conventional algebraic geometry, we shall work with *commutative* polynomial algebras, meaning that relations will be imposed between the operations $[x_1], ..., [x_m]$ of $\Sigma\langle x_1, ..., x_m\rangle$.

DEFINITION 1.17. Let Σ be a generalized ring. We say that a set of equivalence relations

$$R = \{R(\mathbf{n}) \subset \Sigma(\mathbf{n}) \times \Sigma(\mathbf{n})\}_{n \in \mathbf{N}}$$

is *compatible* with Σ if $t \equiv_{R(\mathbf{k})} s$ and $t_i \equiv_{R(\mathbf{n})} s_i$, for $1 \leq i \leq k$, implies that

$$[t]_{\Sigma(\mathbf{n})}(t_1, ..., t_k) \equiv_{R(\mathbf{n})} [s]_{\Sigma(\mathbf{n})}(s_1, ..., s_k).$$

The *quotient of* Σ *with respect to a set* R *of compatible equivalence relations* is the generalized ring T, defined via

$$T(\mathbf{n}) = \Sigma(\mathbf{n})/R(\mathbf{n}).$$

(See (**4.4.8**) in [**8**].)

DEFINITION 1.18. The *polynomial algebra in m variables over an algebraic monad* Σ is the quotient $\Sigma[x_1, ..., x_m]$ of $\Sigma\langle x_1, ..., x_m\rangle$ with respect to the set of equivalence relations generated by relations which force the operations $[x_i]$, for $1 \leq i \leq m$, to commute.

2. Generalized rings

A ring is a set equipped with two binary operations, addition and multiplication, which are subject to certain conditions. In particular, these operations commute with one another in the sense that multiplication distributes over addition. This is an instance of the notion of *commutativity* as defined below.

DEFINITION 2.1. Let Σ be an algebraic monad and let $m, n \in \mathbb{N}$. Two operations $t \in \Sigma(\mathbf{n})$ and $s \in \Sigma(\mathbf{m})$ *commute* if for any module M over Σ, and any family $\{x_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq m}$ of elements of M, one has

$$t(s(x_{11}, ..., x_{1m}), ..., s(x_{n1}, ..., x_{nm})) = s(t(x_{11}, ..., x_{1n}), ..., t(x_{m1}, ..., x_{mn})).$$

We say that Σ is *commutative* if all elements of $\bigsqcup_{n \geq 0} \Sigma(\mathbf{n})$ commute.

For any two modules M and N over an algebraic monad Σ, the set N^M of maps from M to N becomes a module over Σ by virtue of the module structure on N. Requiring Σ to be commutative turns out to be equivalent with the requirement that the subset of homomorphisms of modules $\mathrm{Hom}_\Sigma(M,N)$ is a sub*module* of N^M for any choices of M and N (cf. (**0.5.2**) in [**8**]).

Definition 2.2. A *generalized ring* is a commutative algebraic monad Σ on the category of sets. An element $* \in \Sigma(\mathbf{0})$ is a *zero* of Σ if it is fixed by any operation of Σ. A *homomorphism* of generalized rings is a homomorphism of the underlying monads. Similarly, a *module* over a generalized ring is a module over the underlying monad.

We write $\mathcal{GR}ings$ for the category of generalized rings and $\mathcal{GR}ings_*$ for the category of generalized rings with zeros. The category of modules over a generalized ring has a coproduct, which we denote by $\oplus$.

Examples of algebraic monads with zeros are given by $\mathbb{F}_1$, $\mathbb{Z}_\infty$, A_N and Σ_R for a commutative unital ring R. The algebraic monad $\mathbb{F}_\emptyset$ is a generalized ring that does not have a zero.

Examples 2.3. (Initial and final objects)
The initial object of the category $\mathcal{GR}ings$ is the monad $\mathbb{F}_\emptyset$ (cf. definition 1.7). The final object is the monad $\mathbf{1}$, defined via

$$\mathbf{1}(\mathbf{n}) = \mathbf{1}, \quad \text{for } n \in \mathbb{N}.$$

The initial and final object of $\mathcal{GR}ings_*$ is the field with one element $\mathbb{F}_1$ (cf. definition 1.7).

Definition 2.4. (Free modules)
Let Σ be a generalized ring. A module $\alpha : \Sigma(F) \to F$ is *free* if there exists a set X, such that

$$\mathrm{Hom}_\Sigma(F,M) \simeq M^X,$$

for any module M over Σ. This entails $F \simeq \Sigma(X)$ (cf. (**0.4.10**) in [**8**]). A module over Σ is *finitely generated* if it admits a surjection from a free module of the form $\Sigma(\mathbf{n})$, for some $n \in \mathbb{N}$. If Σ has a zero, then a *finite free resolution* of a module $\beta : \Sigma(M) \to M$ is a diagram of module homomorphisms

$$* \to \Sigma(\mathbf{n}_m) \stackrel{d_m}{\to} \cdots \Sigma(\mathbf{n}_0) \stackrel{d_0}{\to} M \to *,$$

such that $d_{i-1} \circ d_i = *$, for $1 \leq i \leq m$.

Example 2.5. If M is a finitely generated module over the generalized ring $\mathbb{F}_1$, then $M = \{*, 1, ..., n\}$, for some $n \in \mathbb{N}$ (cf. (**10.3.25**) in [**8**]). Hence, M admits the finite free resolution

$$* \to \mathbb{F}_1(\mathbf{n}) \to M \to *.$$

Definition 2.6. (Stably free and projective modules)
Let Σ be a generalized ring. A module $\alpha : \Sigma(E) \to E$ is *stably free* if there exist free modules F and F', such that

$$E \oplus F' \simeq F.$$

Two modules M and M' are *stably isomorphic* if there exist free modules F and F' such that $M \oplus F \simeq M' \oplus F'$. If the generalized ring Σ has a zero, then a *stably free resolution* of a module $\beta : \Sigma(M) \to M$ is a diagram of module homomorphisms

$$* \to E_m \overset{d_m}{\to} \cdots E_0 \overset{d_0}{\to} M \to *,$$

with E_i stably free for $0 \leq i \leq m$ and $d_{i-1} \circ d_i = *$, for $1 \leq i \leq m$. The *stably free dimension* of M is the minimal length of such a resolution. A module $\beta : \Sigma(P) \to P$ is *projective* if any surjection $\pi : M \to P$ of modules over Σ admits a *section*, i.e. a homomorphism of modules $\sigma : P \to M$, such that $\pi \circ \sigma = \mathrm{id}_P$.

Note that for modules over commutative unital rings, the usual notions of *free*, *finitely generated*, *stably free* and *projective* coincide with the ones in definitions 2.4 and 2.6 under the correspondence described in remark 1.6.

Example 2.7. (A non-free projective module)
Consider the set

$$\mathfrak{p}_\infty = \{\lambda \in \mathbb{R};\ |\lambda| < 1\}.$$

In the spirit of valuation theory, the residue field of the generalized ring $\mathbb{Z}_\infty$ should be the quotient

$$Q = \mathbb{Z}_\infty(\mathbf{1})/\mathfrak{p}_\infty \simeq \{-1, 0, 1\}.$$

The set Q inherits the structure of module over $\mathbb{Z}_\infty$, so there is a homomorphism of monads $\mathbb{Z}_\infty \to \mathrm{End}(Q)$. We define the algebraic monad $\mathbb{F}_\infty$ as the image of this homomorphism. Then, for $n \in \mathbb{N}$, an element of the free module $\mathbb{F}_\infty(\mathbf{n})$ corresponds to a face of the octahedron $\mathbb{Z}_\infty(\mathbf{n})$. In particular, $\mathbb{F}_\infty(\mathbf{n})$ has 3^n elements. Durov shows that the submodule $M \subset \mathbb{F}_\infty(\mathbf{2})$ generated by the elements $\{1\}$ and $\frac{1}{2}\{1\} + \frac{1}{2}\{2\}$ is projective (cf. [**8**] **10.4.20**). Since M has precisely five elements, it is not free.

Proposition 2.8. *A projective module over a generalized ring is stably free if and only if it admits a finite free resolution.*

Proof. It is obvious that stably free modules admit finite free resolutions. Conversely, let P be a projective module over a generalized ring Σ, and assume that P admits a finite free resolution

$$* \to \Sigma(\mathbf{n}_m) \overset{d_m}{\to} \cdots \Sigma(\mathbf{n}_0) \overset{d_0}{\to} P \to *.$$

Then $\Sigma(\mathbf{n}_0) \simeq \ker d_0 \oplus P$, since P is projective. Since $\ker d_0$ has a finite free resolution whose length is smaller than that of P, induction on m shows that $\ker d_0$ is stably free, i.e. $\ker d_0 \oplus \Sigma(\mathbf{k})$ is free, for some $k \in \mathbb{N}$. Since $\Sigma(\mathbf{n}_0) \oplus \Sigma(\mathbf{k})$ is free, this concludes the proof. □

The incarnations of short exact sequences in the theory of generalized rings are the *cofibration sequences* in the sense of the next definition.

Definition 2.9. Let Σ be a generalized ring with zero and let $M' \overset{j}{\rightarrowtail} M$ be a monomorphism of modules over Σ. The *cokernel* of j is the pushout along $*$:

$$M/M' = M \sqcup_{M'} *.$$

A *cofibration sequence* is a diagram of modules of the form

$$M' \rightarrowtail M \twoheadrightarrow M/M'.$$

LEMMA 2.10. *Consider two cofibration sequences*

$$\begin{aligned} K \rightarrowtail \quad & E \quad \twoheadrightarrow M, \\ K' \rightarrowtail \quad & E' \quad \twoheadrightarrow M', \end{aligned}$$

such that E and E' are stably free. If M and M' are stably isomorphic, then K and K' are stably isomorphic.

PROOF. Let F and F' be free modules such that $M \oplus F \simeq M' \oplus F'$. Then one has the two cofibration sequences

$$\begin{aligned} K \rightarrowtail \quad & E \oplus F \quad \stackrel{\phi}{\twoheadrightarrow} M \oplus F, \\ K' \rightarrowtail \quad & E' \oplus F' \quad \stackrel{\phi'}{\twoheadrightarrow} M' \oplus F', \end{aligned}$$

Set $X = (E \oplus F) \times_{M \oplus F} (E' \oplus F')$ and consider the projection onto the first factor

$$X \to E \oplus F.$$

The kernel of this map coincides with the kernel of ϕ' and since $E \oplus F$ is free, there is a split cofibration sequence

$$K' \rightarrowtail X \twoheadrightarrow E \oplus F.$$

Hence $X \simeq K' \oplus E \oplus F$. A similar argument shows that $X \simeq K \oplus E' \oplus F'$, so K and K' are stably isomorphic. □

PROPOSITION 2.11. *Let $* \to E_n \to \cdots \to E_0 \to M \to *$ be a stably free resolution and assume that $E'_0, ..., E'_m$ are stably free modules such that any composition of two maps in the sequence*

$$E'_m \to \cdots \to E'_0 \to M \to *$$

is zero. If $m < n-1$, then there exists a stably free module E'_{m+1} extending this sequence. If $m = n-1$, then $\ker(E'_{n-1} \to E'_{n-2})$ is stably free. In particular, given a stably free module E and a cofibration sequence

$$N_1 \rightarrowtail E \twoheadrightarrow N,$$

the stably free dimension of N_1 is strictly smaller than the stably free dimension of N.

PROOF. Let

$$K_m = \begin{cases} \ker(E_m \to E_{m-1}) & \text{for } m \neq 0, \\ \ker(E_0 \to M) & \text{for } m = 0, \end{cases}$$

and define K'_m similarly. By lemma 2.11, there exist free modules F and F', such that $K_m \oplus F \simeq K'_m \oplus F'$. For $m < n-1$, we can choose E'_{m+1} to be the stably free module $E_{m+1} \oplus F$. In the case when $m = n-1$, $K_m \oplus F$ is stably free, since $K_n \simeq E_n$. But then K'_m is stably free as well, since $K_m \oplus F \simeq K'_m \oplus F'$. This concludes the proof. □

PROPOSITION 2.12. *Let Σ be a generalized ring with zero and let $M' \rightarrowtail M \twoheadrightarrow M''$ be a cofibration sequence of finitely generated modules over Σ. If M' and M admit finite free resolutions, then so does M''.*

PROOF. It is possible to construct a commutative diagram of cofibration sequences of modules

$$\begin{array}{ccccc} M_1' & \rightarrowtail & M_1 & \twoheadrightarrow & M_1'' \\ \downarrow & & \downarrow & & \downarrow \\ E' & \rightarrowtail & E & \twoheadrightarrow & E'' \\ \downarrow & & \downarrow & & \downarrow \\ M' & \rightarrowtail & M & \twoheadrightarrow & M'' \end{array}$$

with E', E and E'' stably free. Indeed, since M'' and M' are finitely generated, it is possible to choose epimorphisms $E'' \twoheadrightarrow M''$, $E_0 \twoheadrightarrow E'' \times_{M''} M$ and $E_1 \twoheadrightarrow M'$. Then we put $E = E_0 \oplus E_1$ and $E' = \ker(E \twoheadrightarrow E'')$, and let M_1', M_1 and M_1'' be the kernels of the homomorphisms $E' \twoheadrightarrow M'$, $E \twoheadrightarrow M$ and $E'' \twoheadrightarrow M''$, respectively. By proposition 2.11, M_1 has stably free dimension smaller than that of M, and M_1' has finite stably free dimension. By induction, we can reduce to the case when M is stably free. In this case, the existence of a finite free resolution of M'' follows from the existence of a finite free resolution of M'. □

PROPOSITION 2.13. *Let $M' \rightarrowtail M \twoheadrightarrow M''$ be a cofibration sequence of modules. If M' and M'' admit finite free resolutions, then so does M.*

PROOF. If M' and M'' are stably free, then $M \simeq M' \oplus M''$, since M'' is projective, and hence M admits a finite free resolution since it is stably free. Generally, we can use the same argument as in the proof of proposition 2.12 and construct a square like the one above with E', E and E'' finitely generated free. It then follows by proposition 2.11 and induction on the maximum of the stably free dimensions of M' and M'' that M_1 has finite stably free dimension. Hence M admits a finite free resolution. □

2.1. Ideals and localizations. This section is in part a preparation for the construction of *prime spectra* of generalized rings, carried out in chapter 2. It also introduces some terminology which will be used in section 3 in order to conclude that any finitely generated projective module over the polynomial algebra $\mathbb{F}_1[x_1, ..., x_m]$ is stably free.

DEFINITION 2.14. An *ideal* of a generalized ring Σ is a module $\mathfrak{a}$ over Σ, such that $\mathfrak{a} \subset \Sigma(\mathbf{1})$. If this containment is strict, then $\mathfrak{a}$ is said to be *proper*. Further, $\mathfrak{a}$ is *prime* if $\Sigma(\mathbf{1}) \setminus \mathfrak{a}$ is a multiplicative system and $\mathfrak{a}$ is *maximal* if it is maximal with respect to the partial order defined by inclusion of proper ideals of Σ. The generalized ring Σ is *local* if it has a unique maximal ideal.

In particular, if R is a commutative unital ring, then the set of prime ideals of Σ_R coincides with the set of prime ideals of R.

EXAMPLE 2.15. Non-zero ideals of the generalized ring $\mathbb{F}_1[x]$ are of the form

$$\langle x^n \rangle, \text{ for } n \in \mathbb{N}.$$

More generally, any non-zero ideal of $\mathbb{F}_1[x_1, ..., x_m]$ can be generated by a monomial of the form $x_1^{j_1} \cdots x_m^{j_m}$, for $j_1, ..., j_m \in \mathbb{N}$. In particular, the proper prime ideals of $\mathbb{F}_1[x_1, ..., x_m]$ are

$$\{*\}, \ \langle x_1 \rangle, \ ... \ \langle x_m \rangle.$$

Hilbert's basis theorem states that the commutative ring of polynomials over a Noetherian commutative unital ring is itself Noetherian, and its proof relies on the extraction of highest degree coefficients (cf. e.g. IV:4.1 in [**14**]). Since it is not always possible to extract coefficients when working with generalized polynomial algebras (see remark 1.16), it is not obvious that a generalization of this result holds for generalized rings with the following definition of the Noetherian property.

DEFINITION 2.16. A generalized ring Σ is *Noetherian* if every ascending chain of ideals

$$\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots$$

is stationary, i.e. if there exists an $n \in \mathbb{N}$ such that $\mathfrak{a}_n = \mathfrak{a}_N$, whenever $N \geq n$.

The description of the ideal structure of polynomial algebras over $\mathbb{F}_1$ in example 2.15 shows that the generalized ring $\mathbb{F}_1$ satisfies a Hilbert basis theorem. More generally, it allows us to conclude the following.

PROPOSITION 2.17. *For any $m \in \mathbb{N}$, the polynomial algebra $\mathbb{F}_1[x_0, ..., x_m]$ is Noetherian.*

DEFINITION 2.18. Let Σ be a generalized ring with zero and let M be a module over Σ. An ideal $\mathfrak{a} \subset \Sigma(\mathbf{1})$ is the *annihilator* of an element $x \in M$, if

$$\mathfrak{a} = \{a \in \Sigma(\mathbf{1}); \quad ax = *\}.$$

A prime ideal $\mathfrak{p} \subset \Sigma(\mathbf{1})$ is an *associated prime* of M if there exists an $x \in M$ such that $\mathfrak{p}$ is the annihilator of x.

PROPOSITION 2.19. *Let Σ be a generalized ring with zero and let M be a non-zero module over Σ. If Σ is Noetherian, then M has an associated prime.*

PROOF. Since M is non-zero, the set of proper ideals which are annihilators is non-empty. Since Σ is Noetherian, any ascending chain of annihilators has a maximal element with respect to inclusion. Let $\mathfrak{a}$ be such a maximal element and suppose that $\mathfrak{a}$ is the annihilator of $x \in M$. If $a, b \in \Sigma(\mathbf{1})$ are such that $ab \in \mathfrak{a}$ and $a \notin \mathfrak{a}$, then $ax \neq *$. The ideal generated by b and $\mathfrak{a}$ is the annihilator of ax, so the maximality of $\mathfrak{a}$ implies that $b \in \mathfrak{a}$. Hence $\mathfrak{a}$ is an associated prime of M. □

In line with the conventional theory, the definition of the *structure sheaf* of a prime spectrum will use a notion of *localizations* of the corresponding generalized ring.

DEFINITION 2.20. Let Σ be a generalized ring and let $S \subset \Sigma(\mathbf{1})$ be a multiplicative system. The *localization* of Σ with respect to S is the quotient

$$\Sigma[S^{-1}](\mathbf{n}) = \{(a, s) \in \Sigma(\mathbf{n}) \times S\} / \sim,$$

where $(a, s) \sim (b, t)$ if and only if there exists a $u \in S$, such that $uta = usb$. If M is a module over Σ, the *localization* of M with respect to S is constructed in the analogous fashion.

REMARK 2.21. Let us consider the category of pairs (T, ρ), where $\rho : \Sigma \to T$ is an algebra such that all elements of $\rho_{\Sigma(\mathbf{1})}(S) \subset T(\mathbf{1})$ are invertible in $T(\mathbf{1})$. By construction, the localization $\Sigma[S^{-1}]$ is the initial object of this category. Given a module $\Sigma(M) \to M$, one may consider the case when $T = \mathrm{End}(M)$, and use this universal property to conclude that a module over $\Sigma[S^{-1}]$ is the datum of a module over Σ, such that all elements $s \in S$ act as bijections (cf. (**6.1.6**) in [**8**]).

We consider three instances of localization.

EXAMPLE 2.22. Let R be a commutative unital ring, and let $S \subset R$ be a multiplicative system. It is immediate from the construction that the generalized ring $\Sigma_{R[S^{-1}]}$ coincides with the localization of the generalized ring Σ_R in the sense of definition 2.20:

$$\Sigma_R[S^{-1}] = \Sigma_{R[S^{-1}]}.$$

We also have an analogous correspondence between the two notions of localization of modules for this case.

EXAMPLE 2.23. Localizing the polynomial algebra $\mathbb{F}_1[x]$ with respect to the multiplicative system $\langle x \rangle$ amounts to formally inverting the map $[x] : \mathbb{F}_1[x](\mathbf{1}) \to \mathbb{F}_1[x](\mathbf{1})$. In particular, for $n \in \mathbb{N}$, the set of n-ary operations of the localization is given by

$$\mathbb{F}_1[x][x^{-1}](\mathbf{n}) = \mathbb{F}_1[x, x^{-1}](\mathbf{n}) = \left\{ \begin{matrix} & \vdots & & \vdots \\ & {}_1x^{-1}, & \dots & {}_nx^{-1} \\ *, & {}_1x^{0}, & \dots & {}_nx^{0} \\ & {}_1x^{1}, & \dots & {}_nx^{1} \\ & \vdots & & \vdots \end{matrix} \right\}.$$

EXAMPLE 2.24. The localization of the generalized ring A_N with respect to the multiplicative system $\langle N^{-1} \rangle$ is $\Sigma_{\mathbb{Z}[N^{-1}]}$. Indeed, the monoid $A_N(\mathbf{1})$ consists of the elements of $\mathbb{Z}[N^{-1}]$ whose L^1-norm is smaller than or equal to 1. The canonical embedding $A_N \hookrightarrow \Sigma_{\mathbb{Z}[N^{-1}]}$ induces an injective homomorphism of the localizations

$$A_N[\langle N^{-1} \rangle^{-1}] \hookrightarrow \Sigma_{\mathbb{Z}[N^{-1}]}[\langle N^{-1} \rangle^{-1}] = \Sigma_{\mathbb{Z}[N^{-1}]}.$$

Now, whenever $\lambda = (\lambda_1, ..., \lambda_n) \in \Sigma_{\mathbb{Z}[N^{-1}]}(\mathbf{n})$, one can find an integer $k \geq 0$, such that

$$\sum_{i=1}^{n} |\lambda_i| \leq N^k.$$

Since $N^k \lambda \in A_N(\mathbf{n})$, one has $\lambda \in A_N[\langle N^{-1} \rangle^{-1}](\mathbf{n})$. Hence $\Sigma_{\mathbb{Z}[N^{-1}]}$ injects into $A_N[\langle N^{-1} \rangle^{-1}]$, which proves the claim.

3. Finitely generated projective modules over $\mathbb{F}_1[x_0, ..., x_m]$

Let R be a Noetherian commutative unital ring. If any finitely projective module over R is stably free, then the same property holds for modules over the polynomial ring $R[x_1, ..., x_m]$ (cf. e.g. XXI theorem 2.8 in [**14**]). The aim of this section is to establish the analogous result for the generalized ring $\mathbb{F}_1$. While the proof is just an adaptation of the one in [**14**], we choose to include it for instructional purposes.

LEMMA 3.1. *If M is a finitely generated module over $\mathbb{F}_1[x_1,...,x_m]$, then there exist prime ideals $\mathfrak{p}_i \subset \mathbb{F}_1[x_1,...,x_m](\mathbf{1})$ and a filtration of M by submodules*

$$M = M_1 \supset M_2 \supset \cdots \supset M_k = *,$$

such that $M_i/M_{i+1} \simeq \mathbb{F}_1[x_1,...,x_m](\mathbf{1})/\mathfrak{p}_i$, for $1 \leq i \leq k$.

PROOF. By propositions 2.17 and 2.19, M has an associated prime $\mathfrak{p}$. Let $x \in M$ be such that $\mathfrak{p}$ is the annihilator of x. Then the submodule of $\mathbb{F}_1[x_1,...,x_m](\mathbf{1})$ generated by x is isomorphic to $\mathbb{F}_1[x_1,...,x_m](\mathbf{1})/\mathfrak{p}$, so the set of submodules of M admitting a filtration as above is non-empty. Let N be a maximal element in this set and assume that $N \neq M$. Then the above argument shows that there is a submodule N' of M/N such that $N'/N \simeq \mathbb{F}_1[x_1,...,x_m](\mathbf{1})/\mathfrak{q}$, for some prime ideal $\mathfrak{q}$ of $\mathbb{F}_1[x_1,...,x_m]$. But N' is a submodule of M containing N, which contradicts the maximality of N. □

THEOREM 3.2. *If P is a finitely generated projective module over $\mathbb{F}_1[x_1,...,x_m]$, then P is stably free.*

In view of proposition 2.8, this is a consequence of the following.

THEOREM 3.3. *If M is a finitely generated module over $\mathbb{F}_1[x_1,...,x_m]$, then M admits a finite free resolution.*

PROOF. Let

$$M = M_1 \supset M_2 \supset \cdots \supset M_k = *,$$

be a filtration of M by submodules, as in lemma 3.1. By proposition 2.12, proposition 2.13 and induction, it suffices to prove that associated primes admit finite free resolutions. In order to get a contradiction, let us assume that $\mathfrak{p} \subset \mathbb{F}_1[x_1,...,x_m](\mathbf{1})$ is an associated prime which does not admit such a resolution.

Consider the prime ideal

$$\mathfrak{q} = \mathfrak{p} \cap \mathbb{F}_1[x_1,...,x_{m-1}](\mathbf{1}),$$

and assume that $\mathfrak{p}$ has been chosen so that $\mathfrak{q}$ is maximal with respect to inclusion of ideals of $\mathbb{F}_1[x_1,...,x_{m-1}]$ obtained in this way. By example 2.5 and induction on m, $\mathfrak{q}$ admits a finite free resolution. Hence the module

$$\widetilde{\mathfrak{p}} = \mathfrak{q} \otimes_{\mathbb{F}_1[x_1,...,x_{m-1}]} \mathbb{F}_1[x_1,...,x_m](\mathbf{1})$$

admits a finite free resolution as well. By proposition 2.13, it remains to prove that $\mathfrak{p}/\widetilde{\mathfrak{p}}$ admits a finite free resolution.

Now let $p \in \mathfrak{p}/\widetilde{\mathfrak{p}}$ be a polynomial of minimal degree in x_m. Then the coefficient $d \in \mathbb{F}_1[x_1,...,x_{m-1}](\mathbf{1})$ of p is such that

$$d \cdot \mathfrak{p}/\widetilde{\mathfrak{p}} \subset \langle p \rangle.$$

Since $\mathbb{F}_1[x_1,...,x_m](\mathbf{1})$ does not have any divisors of zero, the principal ideal $\langle p \rangle$ is (stably) free as a module over $\mathbb{F}_1[x_1,...,x_m]$. In particular, it admits a finite free resolution, so by proposition 2.13, is enough to show that $\mathfrak{r} = (\mathfrak{p}/\widetilde{\mathfrak{p}})/\langle p \rangle$ admits a finite free resolution.

Using lemma 3.1 again, one obtains a filtration of $\mathfrak{r}$ by submodules, such that each factor module is of the form $\mathbb{F}_1[x_1, ..., x_m](\mathbf{1})/\mathfrak{r}_i$ for some associated prime $\mathfrak{r}_i$ of $\mathfrak{r}$. Since $d \cdot \mathfrak{r} = *$, the polynomial d lies in every associated prime of $\mathfrak{r}$. The maximality of $\mathfrak{q}$ implies that each factor module in the filtration of $\mathfrak{r}$ admits a finite free resolution, so proposition 2.13 shows that $\mathfrak{r}$ itself admits a finite free resolution. This concludes the proof. $\square$

CHAPTER 2

Generalized schemes

Dual to the theory of generalized rings is one of *affine generalized schemes*. In this chapter, we introduce the corresponding version of algebraic geometry and use it to construct objects such as a one-point compactification of Spec $\mathbb{Z}$ and generalized schemes over the field with one element. We also study *Grothendieck groups* in the new context and equip them with the additional structures of λ-*rings* in order to define Chern and Todd classes of vector bundles.

1. Definitions and examples

We begin by defining the *prime spectrum* of a generalized ring. The reader is encouraged to compare this process with the one described in section 2 of appendix A.

Definition 1.1. Let Σ be a generalized ring and let Spec Σ be the set of prime ideals of Σ. The *Zariski topology* on Spec Σ is generated by the basis which consists of the sets

$$D_a = \{\mathfrak{p} \in \text{Spec } \Sigma;\ a \notin \mathfrak{p}\},\ \text{for } a \in \Sigma(\mathbf{1}).$$

We define a sheaf of generalized rings $\mathcal{O}_{\text{Spec } \Sigma}$ on the topological space Spec Σ by declaring that

$$\mathcal{O}_{\text{Spec } \Sigma}(D_a) = \Sigma[a^{-1}],\ \text{for } a \in \Sigma(\mathbf{1}).$$

The pair

$$\text{Spec } \Sigma = (\text{Spec } \Sigma, \mathcal{O}_{\text{Spec } \Sigma})$$

is the *prime spectrum* of Σ.

Let R be a commutative unital ring, and consider the generalized ring Σ_R (cf. definition 1.5 in chapter 1). Since the ideal structure of Σ_R coincides with the ideal structure of R, the underlying topological spaces of Spec Σ_R and Spec R are the same. Moreover, example 2.22 in chapter 1 shows that $\mathcal{O}_{\text{Spec } \Sigma_R}(U)$ is the generalized ring defined by the commutative unital ring $\mathcal{O}_{\text{Spec } R}(U)$, for any open subset $U \subset \text{Spec } R$. Hence, the prime spectrum of Σ_R in the sense of definition 1.1 may be identified with the prime spectrum of R.

Example 1.2. Recall the generalized ring A_N from example 1.8 in chapter 1. Let us now show that the underlying set of Spec A_N is

$$\{\langle 0\rangle, \langle p\rangle, ..., \mathfrak{p}_{N\infty}\}_{p\nmid N},$$

where $\mathfrak{p}_{N\infty} = \{\alpha \in \mathbb{Z}[N^{-1}];\ |\alpha| < 1\}$ is the maximal ideal of A_N. Any prime number p, such that $p \nmid N$, generates a prime ideal $\langle p\rangle$ of A_N. The inverse image of this prime ideal under the localization homomorphism $A_N \to \mathbb{Z}[N^{-1}]$ is a prime ideal of A_N. Since prime ideals of A_N which do not contain N^{-1} are in bijection with prime ideals of $\mathbb{Z}[N^{-1}]$, it then suffices to show that the only prime ideal $\mathfrak{p}$ of A_N containing N^{-1} is the maximal ideal $\mathfrak{p}_{N\infty}$. But for any $\lambda \in \mathfrak{p}_{N\infty}$, there exists a $k \in \mathbb{N}$, such that $|\lambda|^k < N^{-1}$. Hence $N\lambda^k \in A_N(\mathbf{1})$, and $\lambda^k \in \mathfrak{p}$. Being a prime ideal, $\mathfrak{p}$ has to contain $\mathfrak{p}_{N\infty}$.

In order to generalize the notion of a locally ringed space, consider a morphism in the category of locally ringed spaces

$$(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y).$$

(See section 2 of appendix A.) The requirement that for any $x \in X$, the induced homomorphism of stalks $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of local rings is equivalent with the requirement that for any affine open subschemes $U \subset X$ and $V \subset Y$, such that $f(U) \subset V$, the restricted morphism

$$f_{U,V} : U \to V,$$

is induced by a homomorphism of rings $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$. This leads to the following definition of a category of *generalized locally ringed spaces.*

Definition 1.3. A *generalized locally ringed space* is a pair $(X, \mathcal{O}_X)$, consisting of a topological space X and a sheaf of generalized rings $\mathcal{O}_X$ on X. The stalk $\mathcal{O}_{X,x}$ of X at any given point $x \in X$ is required to be a local generalized ring. A *morphism* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of generalized locally ringed spaces is a pair $(f, f^\sharp)$, consisting of a continuous map $f : X \to Y$ and a homomorphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves on Y. Moreover, it is required that for any open prime spectra Spec $\Sigma_1 \subset X$ and Spec $\Sigma_2 \subset Y$, such that $f(\text{Spec } \Sigma_1) \subset \text{Spec } \Sigma_2$, the restricted morphism $f : \text{Spec } \Sigma_1 \to \text{Spec } \Sigma_2$ is induced by a homomorphism $\Sigma_2 \to \Sigma_1$ of generalized rings. We write $\mathcal{GLRS}$ for the category of generalized locally ringed spaces.

Definition 1.4. A *generalized scheme* is a generalized locally ringed space which admits an open cover by prime spectra of generalized rings. A generalized scheme is *quasi-compact* if this cover can be chosen to be finite. A generalized scheme is *affine* if it is isomorphic to the prime spectrum of some generalized ring in the category $\mathcal{GLRS}$.

An immediate consequence of this definition is that morphisms of affine generalized schemes are in one-to-one correspondence with homomorphisms of generalized rings:

$$\text{Hom}_{\mathcal{GLRS}}(\text{Spec } \Sigma_1, \text{Spec } \Sigma_2) \simeq \text{Hom}_{\mathcal{GR}ings}(\Sigma_2, \Sigma_1).$$

Furthermore, if we let $\Gamma : \mathcal{GLRS} \to \mathcal{GR}ings$ be the functor of global sections, then the functor $\text{Spec} : \mathcal{GR}ings \to \mathcal{GLRS}$, satisfies

$$\text{Hom}_{\mathcal{GLRS}}(X, \text{Spec } R) \simeq \text{Hom}_{\mathcal{GR}ings}(R, \Gamma(X, \mathcal{O}_X)).$$

(See (**6.5.2**) in [**8**].) This is in complete analogy with the classical situation (cf. section 2 of appendix A).

Example 1.5. (The affine spaces $\mathbb{A}^n_\Sigma$)
For $n \in \mathbb{N}$, the *affine n-space* over a generalized ring Σ is the prime spectrum of the polynomial algebra $\Sigma[x_1, ..., x_n]$:

$$\mathbb{A}^n_\Sigma = \text{Spec } \Sigma[x_1, ..., x_n].$$

Example 1.6. (The projective one-space $\mathbb{P}^1_\Sigma$)
Our first example of a non-affine generalized scheme is obtained by glueing two affine lines together. More precisely, let Σ be a generalized ring and consider the multiplicative system

generated by ${}_1x^1 \in \Sigma[x](\mathbf{1})$ (cf. example 1.11 in chapter 1). The localization $\Sigma[x] \to \Sigma[x, x^{-1}]$ defines an open embedding Spec $\Sigma[x, x^{-1}] \hookrightarrow$ Spec $\Sigma[x]$, so it is possible to glue two copies of Spec $\Sigma[x]$ along the open subsets Spec $\Sigma[x, x^{-1}]$. The resulting generalized scheme is the *projective one-space* over Σ, and will be denoted by $\mathbb{P}^1_\Sigma$.

Example 1.7. (The generalized schemes $\widehat{\text{Spec } \mathbb{Z}}^N$)
For a natural number $N \geq 2$, we consider the generalized ring A_N, and its localization $\Sigma_{\mathbb{Z}[N^{-1}]} = A_N[\langle N^{-1}\rangle^{-1}]$ (cf. example 2.24 in chapter 1). Since $\Sigma_{\mathbb{Z}[N^{-1}]} = \Sigma_{\mathbb{Z}}[N^{-1}]$ is also a localization of $\Sigma_{\mathbb{Z}}$, we can construct a generalized scheme $\widehat{\text{Spec } \mathbb{Z}}^N$ by glueing the affine schemes Spec $\Sigma_{\mathbb{Z}}$ and Spec A_N along the open subsets Spec $\Sigma_{\mathbb{Z}[N^{-1}]}$:

$$\widehat{\text{Spec } \mathbb{Z}}^N = \text{Spec } A_N \sqcup_{\text{Spec } \Sigma_{\mathbb{Z}[N^{-1}]}} \text{Spec } \Sigma_{\mathbb{Z}}.$$

The underlying set of the resulting generalized scheme is Spec $\mathbb{Z} \cup \{\mathfrak{p}_{N\infty}\}$, and is hence in bijection with what *should be* the underlying set of a one-point compactification of Spec $\mathbb{Z}$. However, $\widehat{\text{Spec } \mathbb{Z}}^N$ has a topological property which we do not expect from such a compactification: For any prime number p such that $p \nmid N$, the point $\langle p\rangle \in \widehat{\text{Spec } \mathbb{Z}}^N$ is tangled with the point $\mathfrak{p}_{N\infty}$, in the sense that

$$\overline{\{\langle p\rangle\}} = \{\langle p\rangle, \mathfrak{p}_{N\infty}\}.$$

(See (**7.1.6**) in [**8**]).

Given natural numbers $M, N \geq 2$, consider the inclusion homomorphism $\phi_{MN} : A_N \hookrightarrow A_{MN}$. By the one-to-one correspondence between morphisms of affine generalized schemes and homomorphisms of generalized rings, there is an induced morphism

$$\phi^{MN} : \text{Spec } A_{NM} \to \text{Spec } A_N.$$

On the open subset Spec $\mathbb{Z} \cap$ Spec A_{MN}, this morphism is the identity, so it is compatible with the identity morphism Spec $\mathbb{Z} \to$ Spec $\mathbb{Z}$. Hence, one obtains a morphism of generalized schemes

$$f^{MN} : \widehat{\text{Spec } \mathbb{Z}}^{MN} \to \widehat{\text{Spec } \mathbb{Z}}^{N},$$

which is a bijection of the underlying sets. This morphism has the effect of moving points $\langle p\rangle$, for which $p|M$ and $p \nmid N$, to the cluster around $\mathfrak{p}_{N\infty}$.

Definition 1.8. (The algebraic one-point compactification $\widehat{\text{Spec } \mathbb{Z}}$)
With notation as above, the generalized locally ringed space $\widehat{\text{Spec } \mathbb{Z}}$ is defined as the limit of the projective system formed by the set $\{f^{MN}\}_{M,N\geq 2}$:

$$\widehat{\text{Spec } \mathbb{Z}} = \varprojlim_{N\geq 2} \widehat{\text{Spec } \mathbb{Z}}^N.$$

Next, we shall generalize example 1.6 and construct the *projective spaces* $\mathbb{P}^n_\Sigma$ over a generalized ring Σ, for $n \in \mathbb{N}$. Let $\Sigma[x_0, ..., x_n](\mathbf{1})^+$ be the subset of $\Sigma[x_0, ..., x_n](\mathbf{1})$ consisting of elements of strictly positive degree. Given $a \in \Sigma[x_0, ..., x_n](\mathbf{1})^+$, we let $\Sigma[x_0, ..., x_n]_{(a)}$ be the degree-zero part of the localization $\Sigma[x_0, ..., x_n][a^{-1}]$, and put

$$D_+(a) = \text{Spec } \Sigma[x_0, ..., x_n]_{(a)}.$$

Then for $0 \leq i \leq n$, there are isomorphisms

$$D_+(x_i) \simeq \mathbb{A}^n_\Sigma = \text{Spec } \Sigma[x_1, ..., x_n].$$

Also, for $0 \leq i, j \leq n$, we have

$$D_+(x_i x_j) \simeq \text{Spec } \Sigma[x_0, ..., x_n]_{(x_i)}[(x_j/x_i)^{-1}],$$

which shows that the collection of the $n+1$ affine spaces $D_+(x_i)$, for $0 \leq i \leq n$, can be glued together.

Definition 1.9. (The projective spaces $\mathbb{P}^n_\Sigma$)
With notation as above, the generalized scheme obtained by glueing the $D_+(x_i)$'s along the $D_+(x_i x_j)$'s, for $0 \leq i, j \leq n$, is the *projective n-space over* Σ. We denote it by $\mathbb{P}^n_\Sigma$.

1.1. A short note on pseudolocalizations. For a commutative unital ring R, inclusions of basic open subsets $D_a \subset \text{Spec } R$ are in one-to-one correspondence with finitely presented flat epimorphisms with domain R (cf. remark 1.5 in appendix A). This is no longer true for generalized rings, as we shall see now.

To illustrate the situation, we consider a corresponding generalized ring Aff_R, defined via

$$\text{Aff}_R(\mathbf{n}) = \{(\lambda_1, ..., \lambda_n) \in R^n;\ \sum_{i=1}^n \lambda_i = 1\}.$$

In some sense, Aff_R mirrors affine spaces over R, so we would expect that the underlying topological space of its spectrum coincides with the underlying topological space of Spec R. However, this fails to be the case unless Spec R has precisely one point. The reason for this is that the ideal structure of a generalized ring is determined by its monoid of unary operations. Since Aff_R has only one unary operation, the underlying topological space of its prime spectrum reveals very little about the structure of Aff_R itself.

A related phenomenon is that a homomorphism $\text{Aff}_R \to \text{Aff}_{R[S^{-1}]}$, induced by a conventional localization $R \to R[S^{-1}]$, need not be a localization of Aff_R in the sense of definition 2.20 in chapter 1 (cf. (**6.1.27**) in [**8**]). This is due to the fact that we only defined localizations with respect to sets of unary operations. In view of the characterization of basic open subsets in terms of finitely presented flat morphisms (cf. remark 1.5 in appendix A), one may approach localization in a different way, which accounts for the existence of operations of higher arity.

Definition 1.10. (See (**6.1.24**) in [**8**])
A homomorphism of generalized rings $\rho : \Sigma \to T$ is a *pseudolocalization* if the induced functor between module categories is fully faithful, and if T is a flat algebra over Σ.

Every localization is a pseudolocalization, but the converse is not true. Too see this, we need only consider the homomorphism $\text{Aff}_R \to \text{Aff}_{R[S^{-1}]}$ above, which is a pseudolocalization, but not a localization. In particular, this implies that Durov's theory can not be exhibited as a a special case of the theory suggested by Toën-Vaquié (cf. [**22**] and remark 2.3 in appendix A).

2. Vector bundles

Given a generalized locally ringed space $(X, \mathcal{O}_X)$, we consider the category of *sheaves of modules over* $\mathcal{O}_X$. An object of this category is a sheaf $\mathcal{E}$ on X, such that $\mathcal{E}(U)$ is a module over $\mathcal{O}_X(U)$, for any open set $U \subset X$. Furthermore, for each inclusion of open sets $V \subset U$, the restriction homomorphism $\mathcal{E}(U) \to \mathcal{E}(V)$ is required to be compatible with the module structures via the homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ of generalized rings. A *homomorphism* of sheaves of modules over $\mathcal{O}_X$ is a homomorphism of sheaves $\mathcal{E} \to \mathcal{F}$, such that for each open set $U \subset X$, the map $\mathcal{E}(U) \to \mathcal{F}(U)$ is a homomorphism of modules over $\mathcal{O}_X(U)$.

Given a module M over a generalized ring Σ, there is a canonical sheaf of modules $\widetilde{M}$ over $\mathcal{O}_{\text{Spec }\Sigma}$. It is defined by its values on the basic open sets $D_a \subset \text{Spec }\Sigma$:

$$\widetilde{M}(D_a) = M[a^{-1}].$$

The assignment $M \mapsto \widetilde{M}$ extends to a functor which provides a left-adjoint to the global sections functor.

Definition 2.1. A *quasi-coherent sheaf* on a generalized locally ringed space $(X, \mathcal{O}_X)$ is a sheaf of modules $\mathcal{E}$ over $\mathcal{O}_X$, such that $\mathcal{E}_{|U_i} \simeq \widetilde{M_i}$ as a sheaf of modules over $\mathcal{O}_{X|U_i}$, for some open cover $\{U_i\}_{i\in I}$ of X and some modules M_i over $\mathcal{O}_X(U_i)$. A quasi-coherent sheaf is a *vector bundle* if the cover $\{U_i\}_{i\in I}$ can be chosen such that $\widetilde{M_i} \simeq \mathcal{O}_{X|U_i}(\mathbf{n}_i)$, for some integers $n_i \geq 1$. If all n_i can be chosen to have the same value, the vector bundle $\mathcal{E}$ has *rank* n_i. A *line bundle* is a vector bundle of rank one.

Remark 2.2. If $\mathcal{E}$ is a vector bundle on a generalized scheme X, then for any affine open set $U \subset X$, the set $\mathcal{E}(U)$ is a finitely generated projective module. Contrary to the situation in the conventional theory, the converse is not always true. This is illustrated by the non-free projective module Q over $\mathbb{F}_\infty$ (cf. example 2.7 in chapter 1). Since Spec $\mathbb{F}_\infty$ is a one-point space, the corresponding sheaf of modules $\widetilde{Q}$ over $\mathcal{O}_{\mathbb{F}_\infty}$ cannot be a vector bundle.

Important examples of vector bundles on projective spaces are given by the *Serre twists*. In describing them, we first observe that for any polynomial algebra $\Sigma[x_0, ..., x_n]$, the free module $\Sigma[x_0, ..., x_n](\mathbf{1})$ admits a canonical $\mathbb{Z}$-grading: For $j \in \mathbb{Z}$, the degree-j component is the subset consisting of the elements of degree j (cf. (**5.3.22**) in [**8**]). Let us write L_1 for the resulting $\mathbb{Z}$-graded set. Further, for $k \in \mathbb{Z}$, let us write $L_1[k]$ for the $\mathbb{Z}$-graded set obtained from L_1 after shifting the degree by k, i.e. the degree-j component of $L_1[k]$ is the subset of L_1 consisting of the elements of degree $j + k$.

For $0 \leq i \leq n$, denote by $L_1[k]_{(x_i)}$ the degree-zero part of the localization $L_1[k][x_i^{-1}]$. Since $L_1[k]_{(x_i)}$ is a module over $\Sigma[x_0, ..., x_n][x_i^{-1}]$, there is an associated quasi-coherent sheaf $\widetilde{L_1[k]_{(x_i)}}$ on $D_+(x_i)$, which is in fact isomorphic to the trivial line bundle. Moreover, since

$$L_1[k]_{(x_i)}[(x_j/x_i)^{-1}] \simeq L_1[k]_{(x_ix_j)}, \quad \text{for } 0 \leq i, j \leq n,$$

the collection $\{\widetilde{L_1[k]_{(x_i)}}; 0 \leq i \leq n\}$ can be glued together to form a line bundle on $\mathbb{P}^n_\Sigma$.

Definition 2.3. (Serre twists on $\mathbb{P}^n_\Sigma$)
With notation as above, the line bundle obtained by glueing the $\widetilde{L_1[k]_{(x_i)}}$'s is the *k:th Serre twist* on $\mathbb{P}^n_\Sigma$. We denote it by $\mathcal{O}_{\mathbb{P}^n_\Sigma}(k)$.

3. Grothendieck groups

The *Grothendieck group* of a conventional scheme is by definition the abelian group generated by isomorphism classes of vector bundles, with relations arising from short exact sequences (cf. e.g. §8 in chapter II of [**27**]). Given the notion of vector bundles on a generalized scheme, it is almost evident how to extend this definition. The only difference is that relations have to be imposed by means of *cofibrations* in the following sense, since a category of vector bundles may fail to be pointed in the generalized context.

Definition 3.1. A *homomorphism* $u : \mathcal{E}' \to \mathcal{E}$ of vector bundles on a generalized scheme $(X, \mathcal{O}_X)$ is a *cofibration* if it can be locally presented as a retract of a standard embedding $e : \mathcal{E}'_{|U_i} \to \mathcal{E}'_{|U_i} \oplus \mathcal{O}_{X|U_i}(\mathbf{n})$, i.e. if there exist homomorphisms of modules over $\mathcal{O}_{X|U_i}$

$$j : \mathcal{E}_{|U_i} \to \mathcal{E}'_{|U_i} \oplus \mathcal{O}_{X|U_i}(\mathbf{n}),$$
$$q : \mathcal{E}'_{|U_i} \oplus \mathcal{O}_{X|U_i}(\mathbf{n}) \to \mathcal{E}_{|U_i},$$

such that $q \circ j = \mathrm{id}$, $e = j \circ u$, and $u = q \circ e$.

Definition 3.2. Let $(X, \mathcal{O}_X)$ be a generalized locally ringed space. The *Grothendieck group* $K^0(X)$ is generated by isomorphism classes $[\mathcal{E}]$ of vector bundles on X, with relations $[\emptyset] = 0$ and

$$[\mathcal{E}] - [\mathcal{F}] = [\mathcal{E}'] - [\mathcal{F}'],$$

whenever there exist cofibrations u and u' and a pushout square

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\text{u}} & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{F}' & \xrightarrow{\text{u'}} & \mathcal{E}' \end{array}$$

in the category of vector bundles on X.

Remark 3.3. When X is a conventional scheme, a cofibration is precisely an injective homomorphism with cokernel a vector bundle, and definition 3.2 is consistent with the classical definition of the Grothendieck group of X (cf. (**10.3.9**) in [**8**]).

Example 3.4. If Σ is a generalized ring such that every finitely generated projective module over Σ is stably free, one has

$$K^0(\mathrm{Spec}\ \Sigma) \simeq \mathbb{Z}.$$

In particular, corollary 3.2 in chapter 1 implies that $K^0(\mathbb{A}^n_{\mathbb{F}_1}) \simeq \mathbb{Z}$, for all $n \in \mathbb{N}$.

Whenever M and N are two modules over a generalized ring Σ, there exists a module $M \otimes_\Sigma N$ over Σ, such that

$$\mathrm{Hom}_\Sigma(M \otimes_\Sigma N, P) \simeq \mathrm{Hom}_\Sigma(M, \mathrm{Hom}_\Sigma(N, P)),$$

for any module P over Σ (cf. (**0.5.3**) in [**8**]). Moreover, this *tensor product* of M and N over Σ satisfies a universal property with respect to so called *bilinear maps*, and can be constructed in the expected way (cf. (**5.3.5**) in [**8**]).

For two vector bundles $\mathcal{E}$ and $\mathcal{F}$ on a generalized scheme X, the *tensor product* $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ is defined pointwise by the tensor products of the corresponding modules. This operation induces the structure of commutative unital ring on the Grothendieck group $K^0(X)$:

$$[\mathcal{E}] \cdot [\mathcal{F}] = [\mathcal{E} \otimes \mathcal{F}].$$

Furthermore, it equips the set of isomorphism classes of line bundles over X with the structure of multiplicative abelian group.

Definition 3.5. The *Picard group* of a generalized scheme X is the abelian group $\mathrm{Pic}(X)$, whose underlying set consists of the isomorphism classes of line bundles on X, and whose operation is induced by the tensor product of vector bundles.

Example 3.6. For any generalized ring Σ, the Serre twists $\mathcal{O}_{\mathbb{P}^n_\Sigma}(k)$ satisfy

$$\mathcal{O}_{\mathbb{P}^n_\Sigma}(k_1) \otimes_{\mathcal{O}_{\mathbb{P}^n_\Sigma}} \mathcal{O}_{\mathbb{P}^n_\Sigma}(k_2) \simeq \mathcal{O}_{\mathbb{P}^n_\Sigma}(k_1 + k_2), \quad \text{for } k_1, k_2 \in \mathbb{Z}.$$

When k is a field, the Picard group of $\mathbb{P}^n_{\Sigma_k}$ consists precisely of classes of Serre twists, and thus

$$\mathrm{Pic}(\mathbb{P}^n_{\Sigma_k}) \simeq \mathbb{Z}.$$

Remark 3.7. Isomorphism classes of line bundles over the algebraic one-point compactification $\widehat{\mathrm{Spec}\,\mathbb{Z}}$ are in one-to-one correspondence with strictly positive rational numbers (cf. (**10.5.8**) in [**8**]). With the intended analogy between $\mathbb{P}^1_k$ and $\widehat{\mathrm{Spec}\,\mathbb{Z}}$ in mind, one could say that the positive rational numbers serve as arithmetic Serre twists.

4. λ-ring structures

The aim of the following two sections is to introduce the terminology needed to formulate the Grothendieck-Riemann-Roch theorem. In particular, the Grothendieck groups of definition 3.2 will be equipped with so called λ-*operations*. These operations generalize the notion of binomial coefficients, and serve to capture combinatorial properties of generalized schemes.

Definition 4.1. A λ-*ring structure* on a commutative unital ring R consists of a set of ring homomorphisms

$$\lambda^i : R \to R, \text{ for } i \in \mathbb{N},$$

such that $\lambda^0(x) = 1$, $\lambda^1(x) = x$, and

$$\lambda^k(x + y) = \sum_{i=0}^{k} \lambda^i(x)\lambda^{k-i}(y), \text{ for } k > 0.$$

This kind of homomorphisms will be called λ-*operations*, and a ring equipped with λ-operations will be called a λ-*ring*.

A set of ring homomorphisms $\{\lambda^i\}_{i\in\mathbb{N}}$ induces a λ-ring structure on a ring R if and only if the λ-*polynomial*

$$\lambda_t(x) = \sum_{i\in\mathbb{N}} \lambda^i(x)t^i$$

induces a ring homomorphism

$$\begin{array}{rcl} R & \to & 1 + tR[[t]], \\ x & \mapsto & \lambda_t(x). \end{array}$$

(See chapter 1 of [**10**].)

DEFINITION 4.2. An *augmentation* of a commutative unital ring R is a surjective ring homomorphism

$$\epsilon : R \to \mathbb{Z}.$$

EXAMPLE 4.3. The (unique) augmentation of the ring $\mathbb{Z}$ is given by the identity map. The (unique) λ-ring structure on $\mathbb{Z}$ is defined by the binomial coefficients:

$$\lambda^i(n) = \binom{n}{i}, \text{ for } n \in \mathbb{Z}.$$

For a given λ-ring R, there are associated σ-*operations* $\sigma^i : R \to R$, defined via the σ-*polynomial*:

$$\sigma_t(x) = \lambda_{-t}(x)^{-1} = \sum_{i\in\mathbb{N}} \sigma^i(x)t^i.$$

The λ-operations can be recovered recursively from the σ-operations using the relation

$$\sum_{i=0}^{k} (-1)^i \sigma^i(x)\lambda^{k-i}(x) = 0.$$

This makes it possible to use σ-operations for an indirect definition of λ-operations. In defining a λ-ring structure on the Grothendieck groups of generalized schemes, we shall adopt this approach.

DEFINITION 4.4. Let $\mathcal{E}$ be a vector bundle on a generalized scheme X. For $i \in \mathbb{N}$, the *i:th symmetric power* of $\mathcal{E}$ is the quotient

$$\mathrm{Sym}^i(\mathcal{E}) = \mathcal{E}^{\otimes i}/\mathfrak{S}_i,$$

with respect to the natural action of the symmetric group $\mathfrak{S}_i$ on i letters. The Grothendieck group $K^0(X)$ is equipped with the structure of λ-ring by inducing λ-operations from the σ-operations

$$\sigma^i([\mathcal{E}]) = [\mathrm{Sym}^i(\mathcal{E})], \quad \text{for } i \in \mathbb{Z}.$$

An augmentation of $K^0(X)$ is given by the rank map $[\mathcal{E}] \mapsto \mathrm{rank}(\mathcal{E})$. (See (**10.3.18**) in [**8**].)

Remark 4.5. In the classical theory, λ-operations on Grothendieck groups are defined directly, by means of exterior powers of vector bundles. The reason why we need to proceed as we do is that it is impossible to define exterior powers of modules over a generalized ring Σ which fails to be *alternating* (cf. (**5.5.11**) in [**8**]). In particular, it is impossible to define exterior powers of modules over $\mathbb{F}_\emptyset$ and $\mathbb{F}_1$.

We close this section by recording a lemma which will be used in chapter 3 to compute Grothendieck groups.

Definition 4.6. (The λ-ring $\widetilde{\mathrm{Pic}(X)}$)
Given a generalized scheme X, consider the abelian group

$$\widetilde{\mathrm{Pic}(X)} = \mathbb{Z} \times \mathrm{Pic}(X).$$

For $i, j \in \mathbb{Z}$ and $l_1, l_2 \in \mathrm{Pic}(X)$, a structure of commutative unital ring is defined via the multiplication

$$(i, l_1) \cdot (j, l_2) = (ij, jl_1 + il_2),$$

and λ-operations are defined by

$$\lambda^i(j, l_1) = \left(\binom{j}{i}, \binom{j-1}{i-1} \cdot l_1 \right).$$

The map $(i, l_1) \mapsto i$ provides an augmentation of the resulting ring.

Lemma 4.7. *Let X be a generalized scheme, such that the map*

$$\begin{aligned} c_1 : \mathrm{Pic}(X) &\rightarrow K^0(X), \\ [\mathcal{L}] &\mapsto [\mathcal{L}] - 1, \end{aligned}$$

is a homomorphism of abelian groups. Then the map

$$\begin{aligned} \varphi : \widetilde{\mathrm{Pic}(X)} &\rightarrow K^0(X), \\ (i, [\mathcal{L}]) &\mapsto i - 1 + [\mathcal{L}], \end{aligned}$$

is a homomorphism of λ-rings. If $K^0(X)$ is generated by line bundles, then φ is surjective. If both $\mathbb{Z} \rightarrow K^0(X)$ and c_1 are injective, then φ is injective.

Proof. See (**10.5.21**) in [**8**]. □

5. Characteristic classes of vector bundles

Given λ-operations on a commutative unital ring R, one defines the associated *γ-operations* $\gamma^i : R \rightarrow R$ via the *γ-polynomial*

$$\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x) = \sum_{i \in \mathbb{N}} \gamma^i(x) t^i.$$

Since the γ-polynomial is obtained from the λ-polynomial through an invertible change of variable, the γ-operations define a new λ-ring structure on R. If R admits an augmentation ϵ, one can define a corresponding *γ-filtration* by putting

$$\begin{aligned} F^0 R &= R, \\ F^1 R &= \ker \epsilon, \end{aligned}$$

and for $m \geq 2$, letting $F^m R$ be the abelian group generated by elements of the form

$$\gamma^{i_1}(x_1) \cdots \gamma^{i_k}(x_k), \text{ with } x_1, ..., x_k \in F^1 R \text{ and } \sum_{j=1}^{k} i_j \geq m.$$

We write $\mathrm{Gr}(R)$ for the graded ring associated with the γ-filtration:

$$\mathrm{Gr}(R) = \bigoplus_{m=0}^{\infty} F^m R / F^{m+1} R.$$

Definition 5.1. The *Chow ring* of a generalized scheme X is the rationalization of the graded ring associated with the γ-filtration of $K^0(X)$:

$$\mathrm{CH}(X) = \mathrm{Gr}(K^0(X)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Any morphism $f : X \to Y$ of generalized schemes gives rise to a homomorphism of λ-rings

$$f^K : K^0(Y) \to K^0(X),$$

via pullback of vector bundles. Moreover, since this homomorphism maps $F^m K^0(Y)$ into $F^m K^0(X)$, for $m \in \mathbb{N}$, it induces a homomorphism of Chow rings

$$f^{\mathrm{CH}} : \mathrm{CH}(Y) \to \mathrm{CH}(X).$$

(See (**10.6.2**) in [**8**].) Defining maps in the opposite direction is a more complicated task, which we shall return to when formulating the Grothendieck-Riemann-Roch results of chapter 4.

Next, we define certain subsets of λ-rings which serve as domains of functions giving characteristic classes. This will give us an abstract way of referring to the elements of a Grothendieck group which correspond to vector bundles.

Definition 5.2. A *positive structure* on an augmented λ-ring R is a subset $\mathbb{E}$ of the additive group of R, such that

$$\begin{array}{ll} (i) & \mathbb{Z}^+ \subset \mathbb{E},\ \mathbb{E} \cdot \mathbb{E} = \mathbb{E}, \text{ and } R = \mathbb{E} - \mathbb{E}. \\ (ii) & \epsilon(e) > 0, \text{ for all } e \in \mathbb{E}. \\ (iii) & \lambda^i(e) = 0, \text{ for } i > \epsilon(e). \\ (iv) & \lambda^{\epsilon(e)}(e) \text{ is a unit in } R. \end{array}$$

A *line element* is a positive element u, such that $\epsilon(u) = 1$.

Example 5.3. For a generalized scheme X, a canonical positive structure on the λ-ring $K^0(X)$ is given by the set of equivalence classes of vector bundles. A line element for this structure is then an equivalence class of a line bundle.

Given a positive element e of a λ-ring, it is possible to construct a finite extension in which e splits as a sum of line elements:

$$e = u_1 + ... + u_{\epsilon(e)}.$$

(See §1 of chapter I in [**10**].) This allows us to make the following definitions.

DEFINITION 5.4. Let e be a positive element of a λ-ring R and let $e = u_1 + ... + u_{\epsilon(e)}$ be a splitting of e in an extension $R' \supset R$. The *Chern character* of e is

$$\mathrm{ch}(e) = \sum_{i=1}^{\epsilon(e)} \exp(u_i - 1) = \sum_{i=1}^{\epsilon(e)} \sum_{j=0}^{\infty} \frac{(u_i-1)^j}{j!} \in \mathrm{Gr}(R') \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The *Todd class* of e is

$$\mathrm{td}(e) = \prod_{i=1}^{\epsilon(e)} \frac{(u_i - 1)\cdot \exp(u_i - 1)}{\exp(u_i-1) - 1} \in \mathrm{Gr}(R') \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The Chern character and the Todd class are *multiplicative* in the sense that

$$\mathrm{ch}(ee') = \mathrm{ch}(e)\mathrm{ch}(e'),$$
$$\mathrm{td}(e+e') = \mathrm{td}(e)\mathrm{td}(e'),$$

for any two positive elements e and e'.

Further, one has

$$\begin{aligned} \mathrm{ch} \circ \phi &= \mathrm{Gr}(\phi) \otimes_{\mathbb{Z}} \mathbb{Q} \circ \mathrm{ch}, \\ \mathrm{td} \circ \phi &= \mathrm{Gr}(\phi) \otimes_{\mathbb{Z}} \mathbb{Q} \circ \mathrm{td}, \end{aligned}$$

for any homomorphism ϕ of λ-rings.

REMARK 5.5. If all elements of $F^mR/F^{m+1}R$ are nilpotent for $m \geq 1$, then the Chern character and the Todd class take values in the ring $\mathrm{Gr}(R) \otimes_{\mathbb{Z}} \mathbb{Q}$ (cf. III:2 in [**10**]). The projective bundle theorem in chapter 3 will imply that $R = K^0(\mathbb{P}^n_{\mathbb{F}_1})$ satisfies this nilpotency condition for $n \in \mathbb{N}$, and hence that both ch and td define homomorphisms

$$K^0(\mathbb{P}^n_{\mathbb{F}_1}) \to \mathrm{CH}(\mathbb{P}^n_{\mathbb{F}_1}).$$

We have the following explicit calculation of the σ-operations of $\mathbb{Z}$.

LEMMA 5.6. *Let e be a positive element of an augmented λ-ring. Then*

$$\epsilon(\sigma^i(e)) = \binom{\epsilon(e) - 1 + i}{i}.$$

PROOF. Using a splitting $e = u_1 + ... + u_{\epsilon(e)}$, we can write $\lambda_t(e) = \prod_{j=1}^{\epsilon(e)}(1 + u_j t)$. Hence

$$\begin{aligned} \epsilon(\sigma_t(e)) &= \epsilon(\lambda_{-t}(e))^{-1} = \prod_{j=1}^{\epsilon(e)} \epsilon(1 - u_j t)^{-1} \\ &= (1-t)^{-\epsilon(e)} = \sum_j \binom{\epsilon(e) - 1 + j}{j} t^j. \end{aligned}$$

□

CHAPTER 3

Computations

1. The projective bundle theorem

Our aim in this section is to prove a *projective bundle theorem* for the generalized ring $\mathbb{F}_1$, i.e. to carry out the computations

$$K^0(\mathbb{P}^n_{\mathbb{F}_1}) \simeq \mathbb{Z}[x]/x^{n+1}, \quad \text{for } n \in \mathbb{N}.$$

The proof will be loosely based on an argument due to A. Suslin (cf. [**18**]), and goes by induction on $n \geq 1$. In order to compute $K^0(\mathbb{P}^1_{\mathbb{F}_1})$, we intend to use lemma 4.7 in chapter 2. Hence we begin by computing the abelian group $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{F}_1})$.

LEMMA 1.1. *The abelian group $K^0(\mathbb{P}^1_{\mathbb{F}_1})$ is generated by the isomorphism classes of the line bundles $\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(k)$, for $k \in \mathbb{Z}$. In particular*

$$\mathrm{Pic}(\mathbb{P}^1_{\mathbb{F}_1}) \simeq \mathbb{Z}.$$

PROOF. In section 3 of chapter 1, it was proved that finitely generated projective modules over $\mathbb{F}_1[x]$ are stably free. Hence a vector bundle $\mathcal{E}$ over $\mathbb{P}^1_{\mathbb{F}_1}$ is determined by two stably free modules E_1 and E_2, and an isomorphism of localizations $\tilde{f} : E_1[x^{-1}] \xrightarrow{\sim} E_2[x^{-1}]$. Let $m, n \in \mathbb{N}$ be such that $E_1 \oplus \mathbb{F}_1[x](\mathbf{m}) \simeq E_2 \oplus \mathbb{F}_1[x](\mathbf{m}) \simeq \mathbb{F}_1[x](\mathbf{n})$, and let f be the canonical extension of $\tilde{f}$ to an automorphism of $\mathbb{F}_1[x, x^{-1}](\mathbf{n})$. Then one obtains a permutation of the generators ${}_1x^0, ..., {}_n x^0$ by precomposing f with an automorphism of the form

$$\begin{array}{rl} {}_1x^0 & \mapsto {}_1\, x^{k_1} \\ \vdots & \\ {}_nx^0 & \mapsto {}_n\, x^{k_n} \end{array}, \quad \text{for some } k_1, ..., k_n \in \mathbb{Z}.$$

This shows that the vector bundle over $\mathbb{P}^1_{\mathbb{F}_1}$ determined by f is isomorphic to $\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(k_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(k_n)$. Hence its direct summand $\mathcal{E}$ is also isomorphic to a direct sum of line bundles. □

REMARK 1.2. Using a similar argument and the fact that finitely generated projective modules over $\mathbb{F}_1[x_0, ..., x_n]$ are stably free, one can show that $K^0(\mathbb{P}^n_{\mathbb{F}_1})$ is generated by line bundles and that

$$\mathrm{Pic}(\mathbb{P}^n_{\mathbb{F}_1}) \simeq \mathbb{Z}, \text{ for } n \in \mathbb{N}.$$

In order to show that the map φ of lemma 4.7 in chapter 2 is surjective, we need to show that the map

$$\begin{array}{rcl} c_1 : \mathrm{Pic}(\mathbb{P}^1_{\mathbb{F}_1}) & \to & K^0(\mathbb{P}^1_{\mathbb{F}_1}), \\ {[\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(k)]} & \mapsto & [\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(k)] - 1, \end{array}$$

is a homomorphism of abelian groups. This is achieved through the following lemma.

LEMMA 1.3. *For $k_1, k_2 \in \mathbb{Z}$, one has*

$$\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(k_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(k_2) \simeq \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(k_1+k_2).$$

PROOF. Consider the following two automorphisms of $\mathbb{F}_1[x, x^{-1}](\mathbf{2})$.

$$f: \begin{array}{ll} {}_1x^0 & \mapsto_1 x^{k_1} \\ {}_2x^0 & \mapsto_2 x^0 \end{array}$$

$$g: \begin{array}{ll} {}_1x^0 & \mapsto_1 x^0 \\ {}_2x^0 & \mapsto_2 x^{k_2} \end{array}$$

The vector bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(k_1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(k_2)$ is determined by the composite automorphism $g \circ f$. Precomposing f and g with the automorphism of $\mathbb{F}_1[x](\mathbf{2})$ which swaps the generators defines an isomorphic vector bundle. But the latter composite is given by

$$\begin{array}{ll} {}_1x^0 & \mapsto_1 x^0 \\ {}_2x^0 & \mapsto_2 x^{k_1+k_2} \end{array}$$

and hence the vector bundle it defines is $\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(k_1+k_2)$. □

THEOREM 1.4. *There is an isomorphism of λ-rings*

$$K^0(\mathbb{P}^1_{\mathbb{F}_1}) \simeq \mathbb{Z} \times \mathrm{Pic}(\mathbb{P}^1_{\mathbb{F}_1}).$$

PROOF. Combining lemmas 1.1 and 1.3 with lemma 4.7 in chapter 2, it is sufficient to establish injectivity of the homomorphisms $\mathbb{Z} \to K^0(\mathbb{P}^1_{\mathbb{F}_1})$ and

$$\begin{array}{rcl} c_1 : \mathrm{Pic}(\mathbb{P}^1_{\mathbb{F}_1}) & \to & K^0(\mathbb{P}^1_{\mathbb{F}_1}), \\ [\mathcal{L}] & \mapsto & [\mathcal{L}] - 1. \end{array}$$

To prove injectivity of the first one, consider the open inclusion $\mathbb{A}^1_{\mathbb{F}_1} \hookrightarrow \mathbb{P}^1_{\mathbb{F}_1}$. This map induces a homomorphism $K^0(\mathbb{P}^1_{\mathbb{F}_1}) \to K^0(\mathbb{A}^1_{\mathbb{F}_1}) \simeq K^0(\mathrm{Spec}\ \mathbb{F}_1) \simeq \mathbb{Z}$, which must be a left-inverse, since $\mathbb{Z}$ is the initial object in the category of commutative unital rings. To prove injectivity of c_1, consider the homomorphism

$$\begin{array}{rcl} \deg : K^0(\mathbb{P}^1_{\mathbb{F}_1}) & \to & \mathrm{Pic}(\mathbb{P}^1_{\mathbb{F}_1}), \\ [\mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(k_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(k_n)] & \mapsto & \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}_1}}(k_1 + \ldots k_n). \end{array}$$

It provides a left-inverse of c_1. □

DEFINITION 1.5. Let X be a generalized scheme and let $Z \subset X$ be a closed set. The *Grothendieck group with support on Z* is the subgroup $K^0_Z(X)$ of $K^0(X)$, consisting of the elements which vanish when restricted to any open set U, such that $U \cap Z = \emptyset$.

THEOREM 1.6. (Projective bundle theorem for $\mathbb{F}_1$)
For $n \in \mathbb{N}$, let $\xi_n = [\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-1)] - 1$ be the first Chern class of the line bundle $\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-1)$. Then the map

$$\begin{array}{rcl} \mathbb{Z}[x]/x^{n+1} & \to & K^0(\mathbb{P}^n_{\mathbb{F}_1}), \\ x & \mapsto & \xi_n, \end{array}$$

is an isomorphism of rings.

PROOF. For $n = 0$, the statement of the theorem is that $K^0(\text{Spec }\mathbb{F}_1) \simeq \mathbb{Z}$, which is true by (**10.3.25**) in [**8**]. The case $n = 1$ was proved in theorem 1.4. Now assume that n is such that the statement of the theorem holds for $n-1$. If $\{0\} \in \mathbb{A}^n_{\mathbb{F}_1} \subset \mathbb{P}^n_{\mathbb{F}_1}$ is a closed point, then since $K^0(\mathbb{A}^1_{\mathbb{F}_1}) \simeq K^0(\text{Spec }\mathbb{F}_1)$, the construction of projective spaces produce isomorphisms

$$K^0(\mathbb{P}^n_{\mathbb{F}_1} \setminus \{0\}) \xrightarrow{\sim} K^0(\mathbb{P}^{n-1}_{\mathbb{F}_1} \times_{\text{Spec }\mathbb{F}_1} \mathbb{A}^1_{\mathbb{F}_1}) \xrightarrow{\sim} K^0(\mathbb{P}^{n-1}_{\mathbb{F}_1}),$$

such that $\xi_n \mapsto \xi_{n-1}$ under the composite map. By the inductive assumption, $K^0(\mathbb{P}^{n-1}_{\mathbb{F}_1})$ is freely generated by $1, \xi_{n-1}, ..., \xi^{n-1}_{n-1}$, so the restriction map $K^0(\mathbb{P}^n_{\mathbb{F}_1}) \to K^0(\mathbb{P}^n_{\mathbb{F}_1} \setminus \{0\})$ is a split surjection. Hence, one has a short exact sequence of abelian groups

$$0 \to K_{\{0\}}(\mathbb{P}^n_{\mathbb{F}_1}) \to K^0(\mathbb{P}^n_{\mathbb{F}_1}) \to K^0(\mathbb{P}^n_{\mathbb{F}_1} \setminus \{0\}) \to 0,$$

and it suffices to show that $K_{\{0\}}(\mathbb{P}^n_{\mathbb{F}_1})$ is freely generated by ξ^n_n. In order to do this, consider the double restriction homomorphism

$$K^0(\mathbb{P}^n_{\mathbb{F}_1}) \to K^0(\mathbb{P}^{n-1}_{\mathbb{F}_1}) \oplus K^0(\mathbb{A}^n_{\mathbb{F}_1} \setminus \{0\}).$$

By the inductive assumption, an element of its kernel has to restrict to a multiple of ξ^n_{n-1} on $\mathbb{P}^{n-1}_{\mathbb{F}_1}$. Since $K^0(\mathbb{A}^n_{\mathbb{F}_1}) \simeq \mathbb{Z}$, we know that this element has to be the difference of equivalence classes of two bundles which have the same rank. Hence it has to be a multiple of ξ^n_n, since $\text{Pic}(\mathbb{P}^n_{\mathbb{F}_1}) \simeq \mathbb{Z}$. □

2. Intersection theory of $\widehat{\text{Spec }\mathbb{Z}}^N$

In this section, we solve the open problem stated in (**10.7.16**) of [**8**] by computing the Chow rings $\text{CH}(\widehat{\text{Spec }\mathbb{Z}}^N)$, for natural numbers $N \geq 2$. We begin by proving that any finitely generated projective module over the generalized ring A_N is free in section 2.1. In section 2.2, we combine this with lemma 4.7 of chapter 2 and conclude that

$$K^0(\widehat{\text{Spec }\mathbb{Z}^N}) \simeq \mathbb{Z} \times \log \mathbb{Z}[N^{-1}]^*_+, \text{ for } N \geq 2.$$

As a corollary, we obtain isomorphisms

$$\text{CH}(\widehat{\text{Spec }\mathbb{Z}}^N) \simeq \mathbb{Q} \oplus \log \mathbb{Q}[N^{-1}]^*_+, \quad \text{for } N \geq 2.$$

2.1. Finitely generated projective modules over A_N are free. For a natural number $N \geq 2$, the generalized ring A_N was defined via

$$A_N(\mathbf{n}) = \left\{ (\lambda_1, ..., \lambda_n); \lambda_i \in \mathbb{Z}[N^{-1}] \quad \sum_{i=1}^n |\lambda_i| \leq 1 \right\}, \text{ for } n \in \mathbb{N}.$$

(See definition 1.8 in chapter 1.)

THEOREM 2.1. *If P is a finitely generated projective module over A_N, then P is free.*

Let us introduce some notation in order to prove this. We let $\pi : A_N(\mathbf{n}) \to P$ be a surjective homomorphism from the free module over A_N on n generators (cf. definition 2.6 in chapter 1), where n is assumed to be minimal. Further, we fix a section $\sigma : P \to A_N(\mathbf{n})$ of π, and put $a = \sigma \circ \pi$. Then $a^2 = a$, and a is given by a matrix $A = (a_{ij})$ with entries $a_{ij} \in A_N(\mathbf{1})$.

For $1 \leq j \leq n$, we write u_j for the image of the standard generator e_j of $A_N(\mathbf{n})$. Then

$$\|u_j\|_1 = \sum_{i=1}^{n} |a_{ij}| \leq 1, \text{ for all } 1 \leq j \leq n,$$

since $u_j = a(e_j) \in A_N(\mathbf{n})$.

LEMMA 2.2. *One has $\|u_i\|_1 = 1$, for all $1 \leq i \leq n$.*

PROOF. Let $I = \{i \in \mathbf{n};\ \|u_i\|_1 = 1\}$. For any vector $x = (x_1, ..., x_n) \in \mathbb{R}^n$, one has

$$\|Ax\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}||x_j| \leq \sum_{j=1}^{n} |x_j| = \|x\|_1.$$

In particular, if $\|Ax\|_1 = \|x\|_1$, then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{n} |x_j|,$$

which is possible only if for all j, either $x_j = 0$ or $\sum_i |a_{ij}| = 1$, i.e. $j \in I$. Since $\|Au_i\|_1 = \|u_i\|_1$, u_i belongs to the $A_N(\mathbf{1})$-span of $\{u_i\}_{i \in I}$, and hence $I = \mathbf{n}$ by the minimality of n. □

Note that if $R = (r_{ij})$ is the matrix with entries $r_{ij} = |a_{ij}|$, then $R^2 = R$. Indeed, since $A^2 = A$, one has $a_{ik} = \sum_j a_{ij} a_{jk}$, so

$$r_{ik} = |a_{ik}| \leq \sum_{j=1}^{n} |a_{ij}||a_{jk}| = \sum_{j=1}^{n} r_{ij} r_{jk}.$$

Furthermore

$$1 = \sum_{i=1}^{n} r_{ik} \leq \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} r_{jk} = \sum_{j=1}^{n} r_{jk} = 1,$$

and hence $r_{ik} = \sum_j r_{ij} r_{jk}$.

PROPOSITION 2.3. *If $r_{ij} > 0$, for all $1 \leq i, j \leq n$, then all columns of R are equal.*

PROOF. Let $x = (x_1, ..., x_n)^t$ be the transpose of the i:th row of R. In showing that all components of x are equal, one may assume that at least one $x_k = 0$. Indeed, one can put $m = \min_i x_i$ and replace x by $x - m(1, ..., 1)^t$. Since $R^2 = R$, one has $R^t x = x$. Hence

$$x_k = \sum_{i=1}^{n} r_{ik} x_i,$$

which implies that $x_i = 0$, for all $1 \leq i \leq n$, since all r_{ik}'s are strictly positive. □

Let us now show that if $r_{ij} > 0$, for all $1 \leq i, j \leq n$, then P is free of rank one. First we note that any equality $|x_1 + ... + x_n| = |x_1| + ... + |x_n|$ implies that $x_1 + ... + x_n$ has the same sign as x_i, for all $1 \leq i \leq n$. In particular

$$\left| \sum_{j=1}^{n} a_{ij} a_{jk} \right| = |a_{ik}| = r_{ik} = \sum_{j=1}^{n} r_{ij} r_{jk} = \sum_{j=1}^{n} |a_{ij} a_{jk}|,$$

so $a_{ij}a_{jk}$ has the same sign as $\sum_j a_{ij}a_{jk} = a_{ik}$. Hence $a_{ii} > 0$, for all $1 \leq i \leq n$. Put

$$\epsilon_i = \frac{a_{1i}}{|a_{1i}|},$$
$$B = (\epsilon_i a_{ij} \epsilon_j^{-1}).$$

Then B defines the same finitely generated projective module over A_N as A. Since $\epsilon_1 = 1$, and $\epsilon_1 a_{1j} \epsilon_j^{-1} = |a_{1j}| = r_{1j}$, it may be assumed that $a_{1i} > 0$, for all $1 \leq i \leq n$. But then $a_{ij} > 0$, for all $1 \leq i, j \leq n$, since $a_{1i}a_{ij}$ has the same sign as a_{1j}. By proposition 2.3, all columns of A are equal, whence the minimality of n and lemma 2.2 implies that $A = (1)$.

THEOREM 2.4. *If $r_{ij} = 0$ for some $i, j \in \mathbf{n}$, then there exist finitely generated projective modules P' and P'' over A_N, such that*

$$P = P' \oplus P''.$$

PROOF. For $j \in \mathbf{n}$, we consider the set

$$S_j = \{i;\ r_{ij} > 0\}.$$

Now choose $i_0 \in \mathbf{n}$, such that the cardinality $|S_{i_0}|$ is minimal, and put $S = S_{i_0}$. One has $i_0 \in S$, since otherwise u_{i_0} would lie in the $A_N(\mathbf{1})$-span of $\{e_j\}_{j \neq i_0}$, which would contradict the minimality of n. Note that $k \in S_j$ implies that $S_k \subset S_j$, since $r_{ij} = \sum_k r_{ik} r_{kj}$. Consequently, $i, j \in S$ implies that $S_i = S_j = S$, by the minimality of $|S|$. Hence

$$i, j \in S \quad \Rightarrow \quad r_{ij} > 0,$$
$$i \notin S, j \in S \quad \Rightarrow \quad r_{ij} = 0.$$

In other words, one may assume that the matrix A is block-triangular

$$A = \begin{pmatrix} A' & C \\ 0 & A'' \end{pmatrix}$$

with $A'^2 = A'$ and $A''^2 = A''$, since it is possible to reorder indices in such a way that $S = \mathbf{k}$, for some $k < n$.

Write P' for the finitely generated projective module over A_N defined by the block A'. By (**10.2.12**) in [**8**], there exist homomorphisms

$$\sigma' : P' \oplus A_N(\mathbf{n} - \mathbf{k}) \to P,$$
$$j : P \to P' \oplus A_N(\mathbf{n} - \mathbf{k}),$$

such that $\sigma' \circ j = \mathrm{id}_P$, $j \circ (P' \hookrightarrow P) = (P' \hookrightarrow P' \oplus A_N(\mathbf{n} - \mathbf{k}))$ and $\sigma' \circ (P' \hookrightarrow P' \oplus A_N(\mathbf{n} - \mathbf{k})) = (P' \hookrightarrow P)$. Put $q = j \circ \sigma'$, and let q'' be the composite

$$A_N(\mathbf{n} - \mathbf{k}) \to P' \oplus A_N(\mathbf{n} - \mathbf{k}) \xrightarrow{q} P' \oplus A_N(\mathbf{n} - \mathbf{k}) \to A_N(\mathbf{n} - \mathbf{k}).$$

Then $q^2 = q$ and $q''^2 = q''$. In order to show that $q = \mathrm{id}_{P'} \oplus q''$, consider the standard basis $\{e_i\}$ of $A_N(\mathbf{n} - \mathbf{k})$, and put

$$m_i = q(e_i), \quad 1 \leq i \leq n - k.$$

Then

$$m_i = \lambda_i v_i + \mu_i w_i,$$

for some $v_i \in P'$, $w_i \in A_N(\mathbf{n}-\mathbf{k})$, and with $|\lambda_i| + |\mu_i| \leq 1$. Write m_i' for the projection of m_i to $A_N(\mathbf{n}-\mathbf{k})$, that is

$$m_i' = q''(e_i) = \mu_i w_i.$$

Since $q''^2 = q''$, one has $\|q''(m_i')\|_1 = \|m_i'\|_1$, and hence

$$\|q''(w_i)\|_1 = \|w_i\|_1.$$

Using the same argument as in the proof of lemma 2.2, one concludes that $(w_i)_j = 0$, unless $\|m_i'\|_1 = 1$. With $I = \{i;\ \|m_i'\|_1 = 1\}$, one has

$$\begin{aligned} w_i &= \sum_{j\in I} (w_i)_j e_j, \\ q(w_i) &= \sum_{j\in I} (w_i)_j m_j. \end{aligned}$$

Since

$$m_i = q(m_i) = \lambda_i v_i + \mu_i q(w_i),$$

it follows that the A_N-submodule of $P' \oplus A_N(\mathbf{n}-\mathbf{k})$ generated by P' and $\{m_j\}_{j\in I}$ contains m_i. Note that

$$|\mu_i|\|w_i\|_1 = \|m_i'\|_1 = 1, \quad \text{for } 1 \leq i \leq n-k.$$

Since $w_i \in A_N(\mathbf{n}-\mathbf{k})$, one has $\|w_i\|_1 \leq 1$. Since also $|\lambda_i| + |\mu_i| \leq 1$, one concludes that $\lambda_i = 0$, i.e. $m_i \in A_N(\mathbf{n}-\mathbf{k})$. Hence $q = \mathrm{id}_{P'} \oplus q''$, so that $P = P' \oplus P''$ for the finitely generated projective module P'' over A_N defined by q''. □

By induction on the rank of projective modules, we can now conclude that P is free, since we have just shown that it is a direct sum of free modules. This concludes the proof of theorem 2.1. An inspection of the argument yields the following result.

Porism 2.5. If Σ is the stalk at a point of the generalized scheme $\widehat{\operatorname{Spec}\mathbb{Z}}^N$, then any finitely projective module over Σ is free.

2.2. Vector bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}^N$**.** The category of vector bundles over $\widehat{\operatorname{Spec}\mathbb{Z}}^N$ is equivalent with the category of triples $(E_{\mathbb{Z}}, E_N, \theta_N)$, where $E_{\mathbb{Z}}$ is a free module over $\mathbb{Z}$, E_N a free module over A_N, and $\theta_N : E_{\mathbb{Z}}[N^{-1}] \xrightarrow{\sim} E_N[N^{-1}]$ an isomorphism of modules over $\mathbb{Z}[N^{-1}]$. This follows by combining (**7.1.19**) in [**8**] with theorem 2.1. In particular, any vector bundle over $\widehat{\operatorname{Spec}\mathbb{Z}}^N$ has a well-defined rank.

Given a positive integer r, the vector bundles of rank r admit an alternative description as a certain double coset of matrices. Namely, let E be a free module of rank r over $\mathbb{Z}[N^{-1}]$ and choose a basis $(f_i)_{1\leq i\leq r}$ of $E_{\mathbb{Z}} \subset E$, and a basis $(e_i)_{1\leq i\leq r}$ of $E_N \subset E$. Both being bases of E, they are related to each other by means of a matrix $A = (a_{ij}) \in \mathrm{GL}_r(\mathbb{Z}[N^{-1}])$:

$$e_i = \sum_{j=1}^{r} a_{ij} f_j, \qquad a_{ij} \in \mathbb{Z}[N^{-1}].$$

If we choose another basis for $E_{\mathbb{Z}}$, it is related to $(f_i)_{1\le i\le r}$ by means of a matrix $B = (b_{ij}) \in \mathrm{GL}_r(\mathbb{Z})$:

$$f_i = \sum_{j=1}^{r} b_{ij} f'_j.$$

Similarly, if we replace $(e_i)_{1\le i\le r}$ by another basis of E_N, the two bases are related to each other by means of a matrix $C = (c_{ij})$ in the group Oct_r of symmetries of the r-octahedron:

$$e'_i = \sum_{j=1}^{r} c_{ij} e_j.$$

Hence, multiplying A from the right by matrices from $\mathrm{GL}_r(\mathbb{Z})$ and from the left by matrices from Oct_r does not change the corresponding vector bundle. Conversely, if A and A' define isomorphic vector bundles, we may assume that they arise from different choices of bases in the same modules $E_{\mathbb{Z}}$ and E_N. Hence $A' = CAB$ for some $C \in \mathrm{Oct}_r$ and some $B \in \mathrm{GL}_r(\mathbb{Z})$. We have just shown:

PROPOSITION 2.6. *Isomorphism classes of vector bundles of rank r over $\widehat{\text{Spec }\mathbb{Z}}^N$ are in one-to-one correspondence with double cosets*

$$\mathrm{Oct}_r \backslash \mathrm{GL}_r(\mathbb{Z}[N^{-1}])/\mathrm{GL}_r(\mathbb{Z}).$$

In particular,

$$\mathrm{Pic}(\widehat{\text{Spec }\mathbb{Z}}^N) \simeq \{\pm 1\}\backslash \mathbb{Z}[N^{-1}]^*/\{\pm 1\} \simeq \log \mathbb{Z}[N^{-1}]^*_+.$$

We denote the line bundle corresponding to an element $\log\lambda \in \log \mathbb{Z}[N^{-1}]^*_+$ by $\mathcal{O}(\log\lambda)$.

REMARK 2.7. The notation $\log \mathbb{Z}[N^{-1}]^*_+$ arises from the convention that $\mathrm{Pic}(\widehat{\text{Spec }\mathbb{Z}}^N)$ should be written additively, while $\mathbb{Z}[N^{-1}]^*_+$ is written multiplicatively.

Now let $u : \mathcal{E}' \to \mathcal{E}$ be a cofibration of vector bundles over $\widehat{\text{Spec }\mathbb{Z}}^N$ with cofiber $\mathcal{E}''$ and put

$$\begin{aligned} E'_{\mathbb{Z}} &= \Gamma(\text{Spec }\mathbb{Z}, \mathcal{E}'),\\ E_{\mathbb{Z}} &= \Gamma(\text{Spec }\mathbb{Z}, \mathcal{E}),\\ E'_N &= \Gamma(\text{Spec }A_N, \mathcal{E}'),\\ E_N &= \Gamma(\text{Spec }A_N, \mathcal{E}). \end{aligned}$$

Let $r = \mathrm{rank}(\mathcal{E}')$ and $s = \mathrm{rank}(\mathcal{E}'')$. Choose a free module E of rank $r+s$ over $\mathbb{Z}[N^{-1}]$ and let $(f'_i)_{1\le ı\le r}$ and $(f''_i)_{1\le i\le s}$ be bases of $E'_{\mathbb{Z}}$ and $E_{\mathbb{Z}}/E'_{\mathbb{Z}}$ in E, respectively. If f_{r+i} are any lifts of f''_i, $1 \le i \le s$, then a basis $\{f_i\}_{1\le i\le r+s}$ of $E_{\mathbb{Z}}$ is given by

$$f_i = \begin{cases} f'_i & \text{for } 1 \le i \le r,\\ f_i & \text{for } r < i \le r+s. \end{cases}$$

By theorem 2.1, $E'_N \to E_N$ is isomorphic to $A_N(\mathbf{r}) \to A_N(\mathbf{r}+\mathbf{s})$, so we may choose a basis $(e_i)_{1\le i\le r+s}$ of E_N, such that its first r elements constitute a basis of E'_N, and such that the images of the remaining s elements constitute a basis of the cofiber E''_N of $E'_N \to E_N$. Let $A \in \mathrm{GL}_{r+s}(\mathbb{Z}[N^{-1}])$ be the matrix relating $(e_i)_{1\le i\le r+s}$ to $(f_i)_{1\le i\le r+s}$, and define $A' \in \mathrm{GL}_r(\mathbb{Z}[N^{-1}])$

and $A'' \in \mathrm{GL}_s(\mathbb{Z}[N^{-1}])$ similarly. Then A, A' and A'' describe $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{E}''$ respectively, and by construction

$$A = \begin{pmatrix} A' & 0 \\ C & A'' \end{pmatrix}.$$

Conversely, if a vector bundle $\mathcal{E}$ admits a description in terms of a block triangular matrix A of the above form, then we obtain a cofibration $\mathcal{E}' \to \mathcal{E}$ of vector bundles over $\widehat{\mathrm{Spec}\ \mathbb{Z}}^N$. In this way, cofibrations of vector bundles over $\widehat{\mathrm{Spec}\ \mathbb{Z}}^N$ correspond to block-triangular decompositions of matrices.

Given any right coset $\mathrm{GL}_r(\mathbb{Z}[N^{-1}])/\mathrm{GL}_r(\mathbb{Z})$, one can construct a matrix $A = (a_{ij}) \in \mathrm{GL}_r(\mathbb{Z}[N^{-1}])$ in the same coset, such that

(i) A is lower-triangular, i.e. $a_{ij} = 0$ for $i < j$.

(ii) A has positive diagonal elements, i.e. $a_{ii} > 0$.

This is done in [**8**] (**10.5.15**).

Lemma 2.8. *$K^0(\widehat{\mathrm{Spec}\ \mathbb{Z}}^N)$ is generated by line bundles.*

Proof. Let $\mathcal{E}$ be a vector bundle of rank r, and let A be a matrix of the above type describing $\mathcal{E}$. Since A is in particular block triangular, it corresponds to a cofibration of vector bundles. Since A is in fact completely triangular, we have the equality

$$[\mathcal{E}] = \sum_{i=1}^{r} [\mathcal{O}(\log a_{ii})] \quad \text{in } K^0(\widehat{\mathrm{Spec}\ \mathbb{Z}}^N).$$

□

Lemma 2.9. *The map*

$$\begin{aligned} c_1 : \log \mathbb{Z}[N^{-1}]^*_+ &\to K^0(\widehat{\mathrm{Spec}\ \mathbb{Z}}^N), \\ \log \lambda &\mapsto [\mathcal{O}(\log \lambda)] - 1, \end{aligned}$$

is an injective homomorphism of abelian groups.

Proof. First we check that

$$[\mathcal{O}(\log \lambda + \log \mu)] - 1 = [\mathcal{O}(\log \lambda)] - 1 + [\mathcal{O}(\log \mu)] - 1,$$

in $K^0(\widehat{\mathrm{Spec}\ \mathbb{Z}}^N)$. Consider the vector bundle $\mathcal{E}$ over $K^0(\widehat{\mathrm{Spec}\ \mathbb{Z}}^N)$ defined by the matrix

$$A = \begin{pmatrix} \lambda & 0 \\ 1 & \mu \end{pmatrix}.$$

Then $[\mathcal{E}] = [\mathcal{O}(\log \lambda)] + [\mathcal{O}(\log \mu)]$ in $K^0(\widehat{\mathrm{Spec}\ \mathbb{Z}}^N)$. Multiplying A by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Oct}_2$ from the left, we obtain

$$A' = \begin{pmatrix} 1 & \mu \\ \lambda & 0 \end{pmatrix}.$$

Consider the canonical lower-triangular form $A'' = (a''_{ij})$ of A'. Since A' is congruent to A'' modulo $\mathrm{GL}_2(\mathbb{Z})$, the row gcd's of A' and A'' must be equal, so $a''_{11} = gcd(1,\mu) = 1$. Also, since matrices in $\mathrm{GL}_2(\mathbb{Z})$ have determinant ± 1, $a''_{22} = \det A'' = \pm\det A = \pm\lambda\mu$. All numbers involved are positive, so we get $a''_{22} = \lambda\mu$. Since A'' also defines $\mathcal{E}$, we have $[\mathcal{E}] = [\mathcal{O}(\log\lambda + \log\mu)] + 1$ in $K^0(\widehat{\text{Spec}\,\mathbb{Z}^N})$, proving that c_1 is a homomorphism of abelian groups. To prove injectivity, we consider the map

$$\deg : K^0(\widehat{\text{Spec}\,\mathbb{Z}^N}) \;\to\; \log\mathbb{Z}[N^{-1}]^*_+,$$

which sends the class of a vector bundle $\mathcal{E}$ represented by the matrix A to the element $\log|\det A|$. Since $\deg\circ c_1 = \mathrm{id}$, c_1 is injective. □

THEOREM 2.10. *Let $N \geq 2$. Then there is an isomorphism of λ-rings*

$$K^0(\widehat{\text{Spec}\,\mathbb{Z}^N}) \simeq \mathbb{Z}\times\log\mathbb{Z}[N^{-1}]^*_+.$$

PROOF. Recall from proposition 2.6 that $\mathrm{Pic}(\widehat{\text{Spec}\,\mathbb{Z}^N}) \simeq \log\mathbb{Z}[N^{-1}]^*_+$, and consider the map

$$\varphi : \mathbb{Z}\times\log\mathbb{Z}[N^{-1}]^*_+ \;\to\; K^0(\widehat{\text{Spec}\,\mathbb{Z}^N}),$$

from lemma 4.7 in chapter 2. Combining this lemma with lemmas 2.8 and 2.9, we conclude that φ is a surjective homomorphism of λ-rings. To establish injectivity, it remains to show that the homomorphism $\mathbb{Z}\to K^0(\widehat{\text{Spec}\,\mathbb{Z}^N})$ is injective. But this follows from the existence of a morphism $\text{Spec}\,\mathbb{Q}\to\widehat{\text{Spec}\,\mathbb{Z}^N}$, inducing a homomorphism $K^0(\widehat{\text{Spec}\,\mathbb{Z}^N})\to K^0(\text{Spec}\,\mathbb{Q}) = \mathbb{Z}$ of commutative unital rings. □

COROLLARY 2.11. *Let $N \geq 2$. Then*

$$\mathrm{CH}(\widehat{\text{Spec}\,\mathbb{Z}}^N) \simeq \mathbb{Q}\oplus\log\mathbb{Q}[N^{-1}]^*_+.$$

PROOF. One has

$$F^0K^0(\widehat{\text{Spec}\,\mathbb{Z}}^N) = K^0(\widehat{\text{Spec}\,\mathbb{Z}}^N),$$
$$F^1K^0(\widehat{\text{Spec}\,\mathbb{Z}}^N) = \ker\epsilon = 0\times\log\mathbb{Z}[N^{-1}]^*_+.$$

Furthermore, for any $l\in\log\mathbb{Z}[N^{-1}]^*_+$, one has

$$\lambda^i(0,l) = \lambda^i(i-1,l) = (0,\binom{i-2}{i-1}l) = 0,\ \text{for } i\geq 2.$$

This implies that $F^2K^0(\widehat{\text{Spec}\,\mathbb{Z}}^N)$ is generated by elements of the form $\lambda^1(l_1)\lambda^1(l_2)$, for $l_1,l_2\in\ker\epsilon$. But λ^1 is the identity, and $\log\mathbb{Z}[N^{-1}]^{*\,2}_+ = 0$ in $K^0(\widehat{\text{Spec}\,\mathbb{Z}}^N)$, so $F^2K^0(\widehat{\text{Spec}\,\mathbb{Z}}^N) = \{0\}$. This shows that $\mathrm{Gr}(K^0(\widehat{\text{Spec}\,\mathbb{Z}}^N)) \simeq \mathbb{Z}\oplus\log\mathbb{Z}[N^{-1}]^*_+$. Hence

$$\mathrm{CH}(\widehat{\text{Spec}\,\mathbb{Z}}^N) = \mathrm{Gr}(K^0(\widehat{\text{Spec}\,\mathbb{Z}}^N))\otimes_{\mathbb{Z}}\mathbb{Q} \simeq \mathbb{Q}\oplus\log\mathbb{Q}[N^{-1}]^*_+.$$

□

CHAPTER 4

Grothendieck-Riemann-Roch results

One of the most important tools in algebraic geometry is Grothendieck's relativization of the classical Riemann-Roch theorem. Originating in the theory of Riemann surfaces, the Riemann-Roch theorem gives a formula which can be used to calculate the Euler characteristic of a line bundle $\mathcal{L}(D)$ over a non-singular projective curve X of genus g:

$$\dim H^0(X, \mathcal{L}(D)) - \dim H^1(X, \mathcal{L}(D)) = \deg D + 1 - g.$$

(See e.g. IV:1.3 in [**12**]).

Using the formalism of Chern classes, Hirzebruch generalized the Riemann-Roch formula to the following statement about higher dimensional varieties.

THEOREM. (Hirzebruch-Riemann-Roch)
If $\mathcal{E}$ is a locally free sheaf on a non-singular projective variety X of dimension n, then

$$\sum_{i=0}^{n}(-1)^i \dim H^i(X, \mathcal{E}) = \deg(\mathrm{ch}(\mathcal{E})\mathrm{td}_X)_{2n}.$$

Here td_X is the Todd class of X and $(-)_{2n}$ denotes the degree $2n$-component of the cohomology ring (cf. appendix A.4 in [**12**]).

In generalizing Hirzebruch's formula, Grothendieck took as his point of departure certain morphisms $f : X \to Y$ between schemes. In this relative setting, a Riemann-Roch type result concerns the relation between the Chern characters ch_X and ch_Y induced by f. Indeed, given a coherent sheaf $\mathcal{F}$ on X, the relative incarnations of the cohomology groups on the left-hand side of the Hirzebruch-Riemann-Roch formula are the coherent sheaves $R^i f_* \mathcal{F}$, for $0 \le i \le n$. Now let ϕ be any homomorphism from the abelian group generated by the coherent sheaves on X to another abelian group, such that

$$\phi(\mathcal{F}) = \phi(\mathcal{F}') + \phi(\mathcal{F}'')$$

for any short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$. By the construction of Grothendieck groups, the homomorphism ϕ factors through $K^0(Y)$, and hence there is an induced additive map

$$\begin{aligned} \psi : K^0(X) &\to K^0(Y), \\ [\mathcal{F}] &\mapsto \sum_i (-1)^j [R^i f_* \mathcal{F}]. \end{aligned}$$

Using the pushforward in cohomology

$$f_H : H(X) \to H(Y),$$

and the *Todd class* td_f of f, one arrives at the following generalization of the Riemann-Roch theorem (cf. e.g. appendix A:5 in [**12**]).

Theorem. (Grothendieck-Riemann-Roch)
Let $f : X \to Y$ be a projective morphism between connected schemes which are quasi-projective over a Noetherian base. Then the following diagram is commuative.

$$\begin{array}{ccc} K^0(X) & \xrightarrow{f_K} & K^0(Y) \\ {\scriptstyle \mathrm{ch}(-)\cdot \mathrm{td}_f}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}(-)} \\ H(X) & \xrightarrow{f_H} & H(Y) \end{array}$$

In this chapter, we prove Grothendieck-Riemann-Roch type results for certain morphisms between generalized schemes. Section 1 concerns the morphisms of the form

$$\mathbb{P}^n_{\mathbb{F}_1} \to \mathrm{Spec}\ \mathbb{F}_1, \quad \text{for } n \in \mathbb{N}.$$

The key to the Grothendieck-Riemann-Roch theorem in this case turns out to be the projective bundle theorem of the previous chapter. In section 2, we define *zero-sections* of generalized schemes and prove a corresponding Grothendieck-Riemann-Roch theorem under a certain finiteness assumption.

1. Projections to Spec $\mathbb{F}_1$

For $n \in \mathbb{N}$, the structure homomorphism $\mathbb{F}_1 \to \mathbb{F}_1[x_0, ..., x_n]$ of the polynomial algebra in $n+1$ variables over $\mathbb{F}_1$ induces a morphism of generalized schemes

$$\mathbb{P}^n_{\mathbb{F}_1} \xrightarrow{p} \mathrm{Spec}\ \mathbb{F}_1.$$

We will now define the pushforward of this morphism:

$$p_K : K^0(\mathbb{P}^n_{\mathbb{F}_1}) \to K^0(\mathrm{Spec}\ \mathbb{F}_1).$$

Observe that since $K^0(\mathbb{P}^n_{\mathbb{F}_1})$ is generated by the elements $[\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-1)]^k = [\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-k)]$, for $0 \le k \le n$ (cf. theorem 1.6 in chapter 3), it will suffice to know where these elements map under p_K.

We take our cues from the conventional theory and consider a projection $\mathbb{P}^n_X \xrightarrow{\tilde{p}} X$, where X is a Noetherian scheme. For $0 \le k \le n$, one has the equality

$$\sum_i (-1)^i [R^i \tilde{p}_* \mathcal{O}_{\mathbb{P}^n_X}(k)] = [\mathrm{Sym}^k(\mathcal{O}_X^{\oplus(n+1)})] \quad \text{in } K^0(X).$$

(See V:§2 in [**10**].) In cases when $K^0(X) \simeq \mathbb{Z}$, e.g. when X is the prime spectrum of a field, one can combine this with lemma 5.6 in chapter 2 to show that

$$\tilde{p}_K([\mathcal{O}_{\mathbb{P}^n_X}(k)]) = \sigma^k(n+1) = \binom{n+k}{k}, \quad \text{for } 0 \le k \le n.$$

LEMMA 1.1. *Let $n \in \mathbb{N}$. If $\phi : K^0(\mathbb{P}^n_{\mathbb{F}_1}) \to K^0(\text{Spec } \mathbb{F}_1)$ is the $\mathbb{Z}$-linear functional such that*

$$\phi([\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(k)]) = \binom{n+k}{k}, \quad \textit{for } 0 \leq k \leq n,$$

then $\phi([\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-k)]) = 0$, for $0 < k \leq n$.

PROOF. The projective bundle theorem yields the identity $([\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-1)] - 1)^{n+1} = 0$. Multiplying it by $[\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(1)]^n$, we obtain the relation

$$[\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-1)] = \sum_{i=0}^{n} (-1)^{n-i} \binom{n+1}{i} [\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(1)]^{n-i}.$$

The relation between λ- and σ-operations now implies that

$$\phi([\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-1)]) = \sum_{i=0}^{n} (-1)^{n-i} \lambda^i(n+1) \sigma^{n-i}(n+1) = 0.$$

The general case is proved in a similar manner, through multiplication of the above identity with $[\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(1)]^{n+1-k}$ and use of induction on $k \geq 1$. □

DEFINITION 1.2. For $n \in \mathbb{N}$, the homomorphism $p_K : K^0(\mathbb{P}^n_{\mathbb{F}_1}) \to K^0(\text{Spec } \mathbb{F}_1)$ of abelian groups is defined via

$$p_K([\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-k)]) = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } 0 < k \leq n. \end{cases}$$

One checks that p_K is of graded degree $-(n+1)$, which implies that there is an induced graded homomorphism

$$p_{\text{Gr}} : \text{Gr}(K^0(\mathbb{P}^n_{\mathbb{F}_1})) \to \text{Gr}(K^0(\text{Spec } \mathbb{F}_1)).$$

The following proposition gives an explicit description of the latter map.

PROPOSITION 1.3. *Let $\xi = \Big([\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(1)] - 1\Big) \mod F^2(K^0(\mathbb{P}^n_{\mathbb{F}_1}))$. Then*

$$p_{\text{Gr}}(\xi^j) = \begin{cases} 1 & \textit{for } j = n, \\ 0 & \textit{otherwise.} \end{cases}$$

In words, p_{Gr} picks up the n:th coefficient of its argument, expressed as a power series in ξ.

PROOF. For $j \neq n$, the value 0 is obtained, since $p_K(F^i K^0(\mathbb{P}^n_{\mathbb{F}_1})) \subset F^{i-n-1} K^0(\mathbb{P}^n_{\mathbb{F}_1})$. For $j = n$, consider the identities

$$(1-t)^{-n-1} = \sum_{i \in \mathbb{N}} \binom{n+i}{i} t^i,$$

$$(1-t)^n = \sum_{i \in \mathbb{N}} (-1)^{n-i} \binom{n}{i} t^{n-i}.$$

They imply that $\sum_{i=0}^{n}(-1)^{n-i}\binom{n+i}{i}\binom{n}{i}$ is the coefficient of t^n in the power series expansion of $(1-t)^{-1}$. Hence

$$\begin{aligned} 1 = \sum_{i=0}^{n}(-1)^{n-i}\binom{n+i}{i}\binom{n}{i} &= \sum_{i=0}^{n}(-1)^{n-i}p_K([\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(i)])\binom{n}{i} \\ &= p_K(([\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(1)]-1)^n) = p_{\mathrm{Gr}}(\xi^n), \end{aligned}$$

This concludes the proof of the proposition. □

Now we claim that the Chow ring $\mathrm{CH}(\mathbb{P}^n_{\mathbb{F}_1})$ is a receptacle for Chern characters and Todd classes of vector bundles on $\mathbb{P}^n_{\mathbb{F}_1}$. Indeed, by remark 5.5 in chapter 2, this will follow if we can show that any element of $F^iK^0(\mathbb{P}^n_{\mathbb{F}_1})/F^{i+1}K^0(\mathbb{P}^n_{\mathbb{F}_1})$ is nilpotent for $i \geq 1$. But this is clear, since

$$F^1K^0(\mathbb{P}^n_{\mathbb{F}_1}) = \langle [\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-1)] - 1\rangle.$$

We arrive at the following Grothendieck-Riemann-Roch type result.

THEOREM 1.4. (Grothendieck-Riemann-Roch for projections to Spec $\mathbb{F}_1$) *For $n \in \mathbb{N}$, let*

$$\mathrm{td}^n = \mathrm{td}([\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(1)])^{n+1}.$$

If p_{CH} is the $\mathbb{Q}$-linear extension of p_{Gr}, then the following diagram commutes.

$$\begin{array}{ccc} K^0(\mathbb{P}^n_{\mathbb{F}_1}) & \xrightarrow{p_K} & K^0(\mathrm{Spec}\ \mathbb{F}_1) \\ \Big\downarrow {\scriptstyle \mathrm{ch}(-)\cdot \mathrm{td}^n} & & \Big\downarrow {\scriptstyle \mathrm{ch}(-)} \\ \mathrm{CH}(\mathbb{P}^n_{\mathbb{F}_1}) & \xrightarrow{p_{\mathrm{CH}}} & \mathrm{CH}(\mathrm{Spec}\ \mathbb{F}_1) \end{array}$$

PROOF. It suffices to check commutativity for the classes $[\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-1)]^k = [\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-k)]$, for $0 \leq k \leq n$, since they generate $K^0(\mathbb{P}^n_{\mathbb{F}_1})$. The homomorphism $\mathrm{ch} : K^0(\mathrm{Spec}\ \mathbb{F}_1) \to \mathrm{CH}(\mathrm{Spec}\ \mathbb{F}_1)$ is the canonical inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$, so we have

$$\mathrm{ch} \circ p_K([\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-k)]) = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } 0 < k \leq n, \end{cases}$$

by definition 1.2. By proposition 1.3, we need to show that $\mathrm{ch} \circ p_K([\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-k)])$ coincides with the n:th coefficient of $\mathrm{ch}([\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(-k)]) \cdot \mathrm{td}^n$, expressed as a power series in

$$\xi = \Big([\mathcal{O}_{\mathbb{P}^n_{\mathbb{F}_1}}(1)] - 1\Big) \mod F^2(K^0(\mathbb{P}^n_{\mathbb{F}_1})).$$

Multiplicativity of the Chern character implies that the power series in question is

$$\exp(\xi)^{-k} \cdot \left(\frac{\xi \cdot \exp(\xi)}{\exp(\xi) - 1}\right)^{n+1}.$$

The coefficient of ξ^n is the residue of

$$\frac{\exp(-\xi k)d\xi}{(1-\exp(-\xi))^{n+1}}.$$

Changing variables $y = 1 - \exp(-\xi)$, we see that this is the same the residue of

$$\frac{(1-y)^{k-1}dy}{y^{n+1}},$$

which is the coefficient of y^n in the power series $(1-y)^{k-1}$. Since this coefficient is 1 for $k = 0$ and 0 for $0 < k \leq n$, the statement of the theorem follows. □

2. Zero-sections

Let $\phi : \Sigma \to \Sigma'$ be a surjective homomorphism of generalized rings and assume that Σ has a zero $*$. If ϕ is obtained by imposing relations of the form $a_i = *$, then the corresponding morphism Spec $\Sigma' \to$ Spec Σ is topologically closed. However, if ϕ is obtained by imposing relations of the form $a_i = b_i$, then Spec $\Sigma' \to$ Spec Σ is not necessarily topologically closed, since one may not be able to replace those relations by relations of the form $a_i - b_i = *$. Consequently, topologically closed morphisms between generalized schemes are relatively rare. For this reason, Durov defines a morphism of generalized schemes $f : X \to Y$ to be a *closed immersion* if the induced homomorpism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an epimorphism. In particular, it is not required that a closed immersion of generalized schemes is topologically closed.

Given any generalized scheme X and a vector bundle $\mathcal{E}$ over X, it is possible to generalize the construction of projective spaces (cf. definition 1.9 in chapter 2) and obtain a *projective bundle*, denoted $\mathbb{P}(\mathcal{E})$. For a detailed description of the construction, see (**6.6**) in [**8**]. In this section, we shall consider a special case of closed immersions whose targets are projective bundles.

Definition 2.1. A morphism $s : X \to Y$ of generalized schemes is a *zero-section* if there exists a vector bundle $\mathcal{E}$ on X such that $Y = \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)$ and $s : X \to \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)$ is induced by projection to the second factor:

$$\mathcal{E} \oplus \mathcal{O}_X \to \mathcal{O}_X.$$

In particular, a zero-section is a section of the canonical projection $p : \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X) \to X$, so the induced pullback

$$s^K : K^0(\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)) \to K^0(X)$$

is surjective. In order to formulate and prove a Grothendieck-Riemann-Roch result for zero-sections, we need to introduce the concept of an *involution* of a λ-ring.

Definition 2.2. Let R be an augmented λ-ring with positive structure. An *involution* of R is a ring homomorphism

$$\begin{array}{rcl} R & \to & R, \\ x & \mapsto & x^\vee, \end{array}$$

such that $x^{\vee\vee} = x$ and $\epsilon(x^\vee) = \epsilon(x)$, for all $x \in R$. Furthermore, we require that $u^\vee = u^{-1}$ for all line elements $u \in R$.

Example 2.3. For a generalized locally ringed space $(X, \mathcal{O}_X)$, we define an involution of the λ-ring $K^0(X)$ via

$$[\mathcal{F}]^\vee = [\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)].$$

The circumstance which motivates us to consider involutions in the first place is the description of the K-theoretical pushforward of a zero-section of conventional schemes

$$s : X \to \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X).$$

The pushforward s_K satisfies the *projection formula* with respect to the K-theoretical pullback s^K:

$$s_K(a \cdot s^K(b)) = s_K(a) \cdot b, \text{ for all } a \in K^0(X) \text{ and all } b \in K^0(\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)).$$

Consequently, the map s_K is determined by the element $s_K(1)$. In order to describe this element, we use the *hyperplane sheaf* $\mathcal{H}$ on $\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)$, which fits into the short exact sequence

$$0 \to \mathcal{H} \to p^*(\mathcal{E}) \oplus \mathcal{O}_{\mathbb{P}(\mathcal{E}\oplus\mathcal{O}_X)} \to \mathcal{O}_{\mathbb{P}(\mathcal{E}\oplus\mathcal{O}_X)}(1) \to 0.$$

In view of the Koszul resolution

$$0 \to \Lambda^n\mathcal{H}^\vee \to \Lambda^{n-1}\mathcal{H}^\vee \to \cdots \to \Lambda^1\mathcal{H}^\vee \xrightarrow{s^\vee} \mathcal{O}_{\mathbb{P}(\mathcal{E}\oplus\mathcal{O}_X)} \to s_*\mathcal{O}_X \to 0,$$

the element $s_K(1)$ is given by the sum $\lambda_{-1}([\mathcal{H}]^\vee)$. These observations lead us to the following definition.

Definition 2.4. Let $s : X \to Y = \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)$ be a zero-section of generalized schemes. The associated *hyperplane element* is

$$h_s = [p^*(\mathcal{E})] + 1 - [\mathcal{O}_{\mathbb{P}(\mathcal{E}\oplus\mathcal{O}_X)}(1)] \in K^0(Y).$$

The K-theoretical pushforward of s is defined via

$$s_K([\mathcal{F}]) = p^K([\mathcal{F}]) \cdot \lambda_{-1}(h_s^\vee),$$

where $p : Y \to X$ is the canonical projection.

Lemma 2.5. *For a zero-section of generalized schemes $s : X \to Y$, the pushforward s_K is of graded degree $d = \epsilon(h_s^\vee)$.*

Proof. For $n \in \mathbb{N}$ and $a \in F^nK^0(X)$, let $b \in F^nK^0(Y)$ be such that $a = s^K(b)$. Then

$$s_K(a) = s_K(s^K(b)) = s_K(1) \cdot b = \lambda_{-1}(h_s^\vee) \cdot b \in F^{n+d}K^0(Y).$$

□

Hence, any zero-section $s : X \to Y$ of generalized schemes induces a graded homomorphism $s_{\mathrm{Gr}} : \mathrm{Gr}(K^0(X)) \to \mathrm{Gr}(K^0(Y))$. The next two lemmas will yield a description of the element $s_{\mathrm{Gr}}(1) \in \mathrm{Gr}(K^0(Y))$.

Lemma 2.6. *Let R be an augmented λ-ring with positive structure and involution. Let e be a positive element in R, splitting as a sum $u_1 + ... + u_m$ of line elements in an extension. Then*

$$(i) \qquad \gamma_v(e - m) = \sum_{i=0}^{m} \lambda^i(e)v^i(1 - v)^{m-i},$$

$$(ii) \qquad \lambda^i(e) = \lambda^{m-i}(e^\vee)\lambda^m(e).$$

PROOF. (*i*) Since the γ-operations define a new λ-ring structure on R, the γ-polynomial is a ring homomorphism. Hence

$$\gamma_v(e-m) = \frac{\gamma_v(e)}{\gamma_v(m)}.$$

Since

$$\gamma_v(m) = \lambda_{\frac{v}{1-v}}(m) = \sum_{i=0}^{m} \binom{m}{i} \left(\frac{v}{1-v}\right)^i = \left(1 + \frac{v}{1-v}\right)^m = (1-v)^{-m},$$

we obtain

$$\begin{aligned} \gamma_v(e-m) &= \lambda_{\frac{v}{1-v}}(e) \cdot (1-v)^m \\ &= \sum_{i=0}^{m} \lambda^i(e) v^i (1-v)^{m-i}. \end{aligned}$$

(*ii*) We have

$$\begin{aligned} \lambda_t(e) &= \prod_{i=1}^{m}(1+u_i t) = \prod_{i=1}^{m} u_i t (u_i^{-1} t^{-1} + 1) \\ &= t^m \cdot \prod_{i=1}^{m} u_i \cdot \prod_{i=1}^{m}(1 + u_i^{\vee} t^{-1}) \\ &= t^m \cdot \lambda^m(e) \cdot \lambda_{t^{-1}}(e^{\vee}) \\ &= \lambda^m(e) \cdot \sum_{i=0}^{m} \lambda^{m-i}(e^{\vee}) t^i. \end{aligned}$$

This identity proves the claim. □

LEMMA 2.7. *Let $s : X \to Y$ be a zero-section of generalized schemes, and let $h_s = u_1 + ... + u_m$ be a splitting of the associated hyperplane element into a sum of line elements. Then*

$$s_{\mathrm{Gr}}(1) = \prod_{i=1}^{m}(u_i - 1).$$

PROOF. For a positive element e and $i \in \mathbb{N}$, put $c^i(e) = \gamma^i(e - \epsilon(e)) \mod F^{i+1}$. As a consequence of the splitting, we have

$$\sum_{i=0}^{m} c^i(h_s) t^i = \prod_{i=1}^{m}(1 + c^1(u_i) t),$$

and hence $\gamma^m(h_s - m) = \prod_{i=1}^{m}(u_i - 1)$. Setting $v = t^{-1}$ in lemma 2.6 (*i*) and multiplying by t^m, one obtains

$$\sum_{i=0}^{m} \gamma^i(h_s - m) t^{m-i} = \sum_{i=0}^{m} \lambda^i(h_s)(t-1)^{m-i}.$$

Evaluating this equality in $t = 0$ and using lemma 2.6 (ii), one obtains

$$\begin{aligned}\gamma^m(h_s - m) &= \sum_{i=0}^{m} \lambda^{m-i}(h_s^\vee)\lambda^m(h_s)(-1)^{m-i} \\ &= \lambda_{-1}(h_s^\vee)\lambda^m(h_s).\end{aligned}$$

Hence

$$\prod_{i=1}^{m}(u_i - 1) = \gamma^m(h_s - m) = \lambda_{-1}(h_s^\vee)\lambda^m(h_s) \equiv \lambda_{-1}(h_s^\vee) \mod F^1,$$

since $\lambda^m(h_s)$ is a line element. The assertion follows. □

PROPOSITION 2.8. *With notation as in lemma 2.7*

$$\mathrm{ch}(s_K(1)) = \frac{s_{\mathrm{Gr}}(1)}{\mathrm{td}(h_s)}.$$

PROOF. By definition

$$\mathrm{ch}(s_K(1)) = \mathrm{ch}(\lambda_{-1}(h_s^\vee)) = \prod_{i=1}^{m}\left(1 - \mathrm{ch}(u_i^\vee)\right) = \prod_{i=1}^{m}\left(1 - \exp(1 - u_i)\right).$$

Now since

$$\mathrm{td}(h_s) \cdot \mathrm{ch}(s_K(1)) = \prod_{i=1}^{m} \frac{(u_i - 1)\exp(u_i - 1)}{\exp(u_i - 1) - 1} \cdot \prod_{i=1}^{m}\left(1 - \exp(1 - u_i)\right) = \prod_{i=1}^{m}\left(u_i - 1\right),$$

we are done by lemma 2.7. □

In what remains of this chapter, we shall assume that all involved generalized schemes satisfy the *nilpotency condition* of remark 5.5 in chapter 2. This guarantees that the respective Chern characters and Todd classes define ring homomorphisms

$$\begin{aligned}\mathrm{ch} &: K^0(-) \to \mathrm{CH}(-), \\ \mathrm{td} &: K^0(-) \to \mathrm{CH}(-).\end{aligned}$$

THEOREM 2.9. *(Grothendieck-Riemann-Roch for zero-sections) Let $s : X \to Y$ be a zero-section of generalized schemes which satisfy the nilpotency condition of remark 5.5 in chapter 2. Let*

$$\mathrm{td}_s = \mathrm{td}(-s^K(h_s)) \in \mathrm{CH}(X),$$

and let s_{CH} be the $\mathbb{Q}$-linear extension of s_{Gr}. Then the following diagram commutes.

$$\begin{array}{ccc} K^0(X) & \xrightarrow{s_K} & K^0(Y) \\ \Big\downarrow{\scriptstyle \mathrm{ch}(-)\cdot \mathrm{td}_s} & & \Big\downarrow{\scriptstyle \mathrm{ch}(-)} \\ \mathrm{CH}(X) & \xrightarrow{s_{\mathrm{CH}}} & \mathrm{CH}(Y) \end{array}$$

PROOF. For $a \in K^0(X)$, let $b \in K^0(Y)$ be such that $s^K(b) = a$. Then

$$\mathrm{ch} \circ s_K(a) = \mathrm{ch}(s_K(1) \cdot b) = \mathrm{ch}(s_K(1)) \cdot \mathrm{ch}(b),$$

by the multiplicativity of ch. Now proposition 2.8 gives

$$\mathrm{ch}(s_K(1)) \cdot \mathrm{ch}(b) = \frac{s_{\mathrm{Gr}}(1)}{\mathrm{td}(h_s)} \cdot \mathrm{ch}(b).$$

Since

$$\begin{aligned}
\frac{s_{\mathrm{Gr}}(1)}{\mathrm{td}(h_s)} \cdot \mathrm{ch}(b) &= s_{\mathrm{Gr}}\left(s^{\mathrm{Gr}}\left(\frac{\mathrm{ch}(b)}{\mathrm{td}(h_s)}\right)\right) \\
&= s_{\mathrm{Gr}}\left(s^{\mathrm{Gr}} \circ \mathrm{ch}(b) \cdot s^{\mathrm{Gr}}(\mathrm{td}(h_s)^{-1})\right) \\
&= s_{\mathrm{Gr}}\left(s^{\mathrm{Gr}} \circ \mathrm{ch}(b) \cdot \mathrm{td}(s^K(h_s))^{-1}\right) \\
&= s_{\mathrm{Gr}}(\mathrm{ch} \circ s^K(b) \cdot \mathrm{td}_s) \\
&= s_{\mathrm{Gr}}(\mathrm{ch}(a) \cdot \mathrm{td}_s),
\end{aligned}$$

we are done. □

APPENDIX A

Preliminaries on sheaves, sites and algebraic geometry

1. Sheaves and sites

Consider three open subsets of the real line, U, V and W, such that

$$U \supset W,\ V \supset W, \text{ and } U \cup V = \mathbb{R}.$$

Suppose that we are given a continuous real-valued function f on U and a continuous real-valued function g on V, such that the two restrictions $f|_W$ and $g|_W$ coincide. Then there is a unique real-valued continuous function h on $\mathbb{R}$, which is obtained by "patching" f_U and f_V along W, in the sense that

$$h|_U = f \text{ and } h|_V = g.$$

In contrast, let $\{U_i\}_{i\in\mathbb{N}}$, be the open cover of $\mathbb{R}$ with U_i given by

$$U_i = \{x \in \mathbb{R};\ |x| < i\}.$$

If we are given bounded functions f_i on U_i, for each $i \in \mathbb{N}$, such that

$$f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}, \text{ for all } i, j \in \mathbb{N},$$

it may be the case that the function f on $\mathbb{R}$, obtained by "patching" as above, fails to be bounded on $\mathbb{R}$. Indeed, an instance of this is when $f_i = \mathrm{id}_{U_i}$, for all $i \in \mathbb{N}$. The *sheaf property* provides a rigorous way of expressing that a class of functions is stable under the kind of patching outlined above.

We begin by considering *presheaves*. These are devices which will be used to "bookkeep" collections of local data, such as different sets of functions on the open subsets of a topological space.

Definition 1.1. A *presheaf* on a category $\mathcal{C}$ is a contravariant functor

$$\Gamma : \mathcal{C} \to \mathcal{S}ets.$$

By a *presheaf of groups*, a *presheaf of rings*, or a *presheaf of objects of* $\mathcal{D}$, for a fixed category $\mathcal{D}$, we shall mean a presheaf taking its values in the category of groups, in the category of rings, or in the category $\mathcal{D}$, respectively. A *homomorphism* of presheaves is a natural transformation of functors.

Definition 1.2. Given a topological space X, we let $O(X)$ denote the category whose objects are the open subsets of X and whose morphisms are the inclusions of open subsets. By a slight abuse of terminology, we shall refer to a presheaf on $O(X)$ as a *presheaf on* X. The *stalk* of such a presheaf $\mathcal{F}$ at a point $x \in X$ is the set

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U).$$

DEFINITION 1.3. Let $\mathcal{C}$ be a category admitting pullbacks. A (*basis for a*) *Grothendieck topology* on $\mathcal{C}$ is a rule $\mathbf{T}$, which to each object U of $\mathcal{C}$ assigns a collection $\mathbf{T}(U)$ consisting of families of morphisms with codomain U, such that the following conditions are satisfied.

(**G1**) For any isomorphism $V \to U$, one has $\{V \to U\} \in \mathbf{T}(U)$.

(**G2**) For any $\{U_i \to U;\ i \in I\} \in \mathbf{T}(U)$, and any morphism $V \to U$, the collection of pullbacks $\{U_i \times_U V \to V;\ i \in I\}$ belongs to $\mathbf{T}(V)$.

(**G3**) If $\{U_i \to U;\ i \in I\} \in \mathbf{T}(U)$ and $\{V_{ij} \to U_i;\ j \in J_i\} \in \mathbf{T}(U_i)$, for each $i \in I$, then the collection of composites $\{V_{ij} \to U;\ j \in J_i, i \in I\}$ belongs to $\mathbf{T}(U)$.

A *site* is a pair $(\mathcal{C}, \mathbf{T})$, where $\mathcal{C}$ is a category and $\mathbf{T}$ is a Grothendieck topology on $\mathcal{C}$.

Given a topological space X, we define a Grothendieck topology $\mathbf{T}_X$ on the category $O(X)$ by choosing $\mathbf{T}_X(U)$ as the collection of open covers of the open subset $U \subset X$, with each member contained in U. Condition (**G1**) is then satisfied since any open subset is covered by itself. Condition (**G2**) is satisfied since a family $\{U_i \cap V\}_{i\in I}$ covers V whenever the family $\{U_i\}_{i\in I}$ covers U. Finally, condition (**G3**) is satisfied since whenever one has a family $\{U_i\}_{i\in I}$ which covers U and a family $\{V_{ij}\}_{j\in J_i}$ which covers U_i, for each $i \in I$, then the family $\{V_{ij}\}_{j\in J_i, i\in I}$ covers U. In this way, we associate a site $(O(X), \mathbf{T}_X)$ to any topological space X.

DEFINITION 1.4. Let $\mathcal{R}ings$ be the category of commutative unital rings. The *Zariski site* is the pair $(\mathcal{R}ings^{\mathrm{op}}, \mathbf{T}_{Zar})$, where $\mathbf{T}_{Zar}$ is the Grothendieck topology which to an object R assigns the collection of families of duals of localizations $R \to R[r^{-1}]$, such that the elements $r \in R$ generate the unit ideal:

$$\mathbf{T}_{Zar}(R) = \{\{R[r^{-1}] \to R\}_{r\in S};\ S \subset R, \langle S\rangle = R\}.$$

REMARK 1.5. Localizations of the form $R \to R[r^{-1}]$ are precisely the finitely presented flat epimorphisms with domain R (cf. [**22**]). Since being finitely presented and flat is a property that makes sense for any morphism between monoid objects of a symmetric monoidal category, it is possible to define the *Zariski site* starting with any category of monoid objects in a symmetric monoidal category.

DEFINITION 1.6. Let $(\mathcal{C}, \mathbf{T})$ be a site such that the category $\mathcal{C}$ has all limits. A presheaf $\mathcal{F}$ on $\mathcal{C}$ is a *sheaf* if, for any object U and any family $\{U_i \to U;\ i \in I\} \in \mathbf{T}(U)$, the following diagram is an equalizer.

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)$$

A *homomorphism* of sheaves is a homomorphism of the underlying presheaf. By a *sheaf* on a topological space X, we mean a sheaf on the site $(O(X), \mathbf{T}_X)$.

EXAMPLES 1.7. The presheaf which to an open subset $U \subset \mathbb{R}$ assigns the set of real-valued continuous functions on U is a sheaf on $\mathbb{R}$ by virtue of the fact that a set of such functions which coincide on restrictions may be glued together to a continuous real-valued function. Since bounded functions cannot always be glued together to form a bounded function, the presheaf which to an open subset $U \subset \mathbb{R}$ assigns the set of bounded functions on U fails to be a sheaf on $\mathbb{R}$. For any commutative unital ring R, the functor $\mathrm{Hom}_{\mathcal{R}ings}(-, R)$ is a sheaf on the Zariski site.

2. Algebraic geometry

Consider the category $\mathcal{LRS}$ of *locally ringed spaces*. An object of $\mathcal{LRS}$ is a pair $(X, \mathcal{O}_X)$, consisting of a topological space X and a sheaf $\mathcal{O}_X$ of commutative unital rings on X. It is also required that the stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at any given point $x \in X$ is a local ring. A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in $\mathcal{LRS}$ is a pair $(f, f^\sharp)$, consisting of a continuous map $X \to Y$ and a homomorphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves on Y, such that the induced homomorphism $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of local rings.

The contravariant functor of global sections

$$\begin{aligned} \Gamma : \mathcal{LRS} &\to \mathcal{R}ings, \\ (X, \mathcal{O}_X) &\mapsto \mathcal{O}_X(X), \end{aligned}$$

has a left-adjoint Spec : $\mathcal{R}ings \to \mathcal{LRS}$. This means that there are isomorphisms

$$\mathrm{Hom}_{\mathcal{LRS}}(X, \mathrm{Spec}\ R) \simeq \mathrm{Hom}_{\mathcal{R}ings}(R, \Gamma(X, \mathcal{O}_X)),$$

for any choices of a locally ringed space X and a commutative unital ring R. One can describe the locally ringed space Spec R explicitly as follows. First, the *Zariski topology* is defined on the set Spec R of prime ideals of R by choosing a basis consisting of the sets

$$D_a = \{\mathfrak{p} \in \mathrm{Spec}\ R;\ a \notin \mathfrak{p}\},\ \text{for } a \in R.$$

Then the structure sheaf $\mathcal{O}_{\mathrm{Spec}\ R}$ is defined by declaring that its values on the basis sets are

$$\mathcal{O}_{\mathrm{Spec}\ R}(D_a) = R[a^{-1}].$$

Definition 2.1. An *affine scheme* is a locally ringed space of the form Spec R, for some commutative unital ring R. A *scheme* is a locally ringed space which admits an open cover by affine schemes.

The definition above bears much resemblance to the definition of a manifold. Indeed, both schemes and manifolds are topological spaces which admit covers by specific basic spaces. Nevertheless, the behavior of schemes is usually more complicated than that of manifolds. For instance, a point of a scheme may be non-closed and residue fields may vary between different points of the same scheme.

An instructive way to approach a scheme can be to consider its *functor of points*. This perspective illuminates the transformation of algebraic geometry from the study of zero-sets of algebraic equations to the study of *rules* which parametrize such zero-sets. More precisely, consider affine scheme X, defined by a system of polynomial equations $\{f_i = 0\}_{i \in I}$:

$$X = \mathrm{Spec}\ \mathbb{Z}[x_1, ...]/(f_i)_{i \in I}.$$

The characterizing property of the functor Spec is that a morphism from an affine scheme Spec R to X can be identified with a homomorphism of rings $\mathbb{Z}[x_1, ...]/(f_i)_{i\in I} \to R$. Such a morphism exists if and only if the images of the variables $x_1, ...$ satisfy the equations $f_i = 0$, for $i \in I$.

DEFINITION 2.2. The *functor of points* of a scheme X is the contravariant functor

$$\begin{aligned} \mathcal{F}_X : \mathcal{R}ings &\rightarrow \mathcal{S}ets \\ R &\mapsto \mathrm{Hom}_{\mathcal{LRS}}(\mathrm{Spec}\ R, X). \end{aligned}$$

Using the terminology of section 1, a functor of points $\mathcal{F}_X$ is a presheaf on the category $\mathcal{R}ings$. Furthermore, a comparison between the definition of the Zariski topology and the definition of the Grothendieck topology $\mathbf{T}_{Zar}$ reveals that $\mathcal{F}_X$ is a sheaf on the Zariski site.

REMARK 2.3. It is an interesting problem to characterize the sheaves $\mathcal{R}ings \rightarrow \mathcal{S}ets$ which are represented by schemes. This is done in [**7**], and has inspired alternatives to conventional algebraic geometry which are non-additive and/or homotopical in nature (cf. [**22**], [**23**] and [**24**]).

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