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A Colourful Approach to String Topology

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Abstract

For M a compact, orientable manifold and $N \subseteq \mathbb{R}^{n+1}$ a submanifold, we construct the cleavage operad that acts on M^N through correspondences, analogous to the Cacti Operad acting on M^{S^1}, formulating String Topology.

For the unit sphere, $N := S^n \subseteq \mathbb{R}^{n+1}$ we compute the cleavage operad to be a coloured E_{n+1}-operad. We twist this structure by the orthogonal group $\mathrm{SO}(n+1)$ to obtain an operad whose actions prescribe non-unital $(n+1)$-Batalin-Vilkovisky algebras. We show that the action through correspondences transfers to a spectral action on $M^{S^n} \wedge S^{-\dim(M)}$. This action is obtained through an extension of the Cleavage Operad. Homotopically the extension is a simplification, and it adjoins a unit to the action on M^{S^n}.

We finally give advantages of our geometric stance on generalizing String Topology even when $N = S^1$: We improve on equivariance of group actions on M^{S^n}, and provide apparent links between Knot Theory and String Topology.

Resumé

For M en kompakt, orienterbar mangfoldighed og $N \subseteq \mathbb{R}^{n+1}$ en delmangfoldighed konstrueres kløvningsoperaden der ved korrespondancer virker på M^N på samme vis som kaktusoperaden virker på M^{S^1} og giver strengtopologi.

For enhedssfæren, $N := S^n \subseteq \mathbb{R}^{n+1}$ viser vi at kløvningsoperaden er en farvet E_{n+1}-operad. Vi vridder denne med den ortogonale gruppe $\mathrm{SO}(n+1)$ og får en operad hvis virkninger er ikke-unitale $(n+1)$-Batalin-Vilkovisky algebraer. Vi viser hvordan korrespondancevirkningen overføres til en spektral virkning på $M^{S^n} \wedge S^{-\dim(M)}$. Denne virkning opnås gennem en udvidelse af kløvningsoperaden. Homotopisk set er udvidelsen en simplificering, og effekten af udvidelsen på algebrastrukturen af M^{S^n} er en adjungering af enhed.

Yderligere fordele af vores geometriske synspunkt til at generalisere strengtopologi gives, selv når $N = S^1$: Ækvivarians af gruppevirkninger på M^{S^n} forbedres, og vi viser sammenhænge mellem knudeteori og strengtopologi.

To the future – and all its children

Overview

As spaces go, the mapping spaces M^N of contionous maps between manifolds N and M are quite bulky. This thesis is concerned with grasping them better by finding algebraic structures, specified by actions of operads, on the mapping spaces. Our strategy will extend an operadic structure on M^{S^1}, known as String Topology through the Cacti Operad, given in for instance [Vor05] or [Kau05], and reproducing the Chas-Sullivan product [CS99]. We shall provide operads that similarly act on M^N for M a compact manifold, and N a submanifold of $\mathbb{R}^{n+1}$.

The thesis is structured with an introduction for each of the three chapters. Here, we shall here give a brief overview of what one should expect to find in each chapter.

Chapter 1 is a revision of our preprint [Bar10], and here we construct the basic Cleavage Operad that prescribes certain 'cleaving' structures on the mapping space M^N through so-called correspondences. That is, for each $N \subseteq \mathbb{R}^{n+1}$ we give a cleavage operad $\mathcal{C}\ell eav_N$ that act on M^N through correspondences. We identify the unit-sphere $N = S^n \subseteq \mathbb{R}^{n+1}$ as a space where the operad is especially well-behaved, and we show that the operad associated to S^n is an E_{n+1}-operad. We further twist it by $\mathrm{SO}(n+1)$ to an operad whose homology gives $(n+1)$-Batalin-Vilkovisky algebras. In [Hu06] a claim is given about an algebraic method for providing an E_{n+1}-structure on X^{S^n} for X any finite complex. Compared to Hu's claims, we give a full-blown geometric constuction of the operadic structure as well as the statement on

$(n+1)$-Batalin-Vilkovisky Algebras, superceding E_{n+1} algebras.

Although our methods build on a theory of coloured operads – and therefore take a completely different form – it bears more resemblance to the attempts made by Sullivan and Voronov to directly generalize the so-called Cacti Operad to higher dimensions. These ideas, and their drawbacks, are outlined in [CV06, Ch. 5]. In the case of $n = 1$, the construction we give can easily be seen to agree with the action of the Cacti Operad up to homotopy. However, basically since there is a vast complexity of mappings between higher-dimensional spheres, it seems apparent to us that such higher-dimensional cacti will never attain the topological control we are able to obtain in this thesis.

Our explicit E_{n+1}-structure could also be seen as an extension of [GS08], where they obtain a spectral E_n-structure on M^{S^n} based on the action on based loop spaces $\Omega^n M$ of the little disk operad on the fibers of the evaluation map $M^{S^n} \to M$.

Chapter 2 constitutes more recent research whose exposition is not yet in a state ready for publication. In this chapter we present a method for obtaining the actual homological algebra structure

$$\mathrm{H}_*\left(\mathcal{C}\mathit{leav}_{S^n}(-;k)\right) \otimes \mathbb{H}_*\left(M^{S^n}\right)^{\otimes k} \to \mathbb{H}_*\left(M^{S^n}\right), \tag{1}$$

which is given in a stable category of spectra, prior to taking homology.

Reading the literature on String Topology, one might perceive that methods for obtaining such a structures are well-understood. I disagree; there are several beautiful methods available, from which one easily can conclude that there are maps $\mathbb{H}_*(M^{S^n})^{\otimes k} \to \mathbb{H}_*(M^{S^n})$ pointwise, for a fixed element of $\mathcal{C}\mathit{leav}_{S^n}(-;k)$. However, the nature of these methods also tell us that attempting to extend such maps trivially along $\mathcal{C}\mathit{leav}_{S^n}$ is too naïve to hope for. This leaves a gap in our common knowledge on how to obtain the global map (1) from the pointwise constituents. As is the case for the methods with pointwise actions, we as well apply umkehr maps, and fix this gap in our specific case. We only seem to be able to go through with our argument for the global action, based on good topological behaviour of $\mathcal{C}leav_{S^n}$. Understood in the way that it decomposes suitably as manifolds. In fact, the topology of the acting operad seem to play such a crucial role that we haven't been able to go through with the construction for $\mathcal{C}leav_{S^n}$ directly, but need to expand the operad to an even better-behaved 'punctured cleavage operad' $\overrightarrow{\mathcal{C}leav}_{S^n}$.

Our arguments are based on homotopical considerations of Poincaré duality as presented in [Kle01].

We should again warn that attempting to construct the action has been an ongoing theme in the authors years as a PhD-student, and the research we present in chapter 2 is recent progress. One should not expect more from it than one would from a draft.

Finally, chapter 3 contains some selected short observations on further perspectives of the constructions related to $\mathcal{C}leav_{S^n}$. We feel that these should be given special attention in the near future. We show that the action of $\mathcal{C}leav_{S^n}$ is well-behaved with respect to $\mathrm{SO}(n+1)$-equivariance of M^{S^n}. We also show how subtleties on the dependancy of the embedding $N \subseteq \mathbb{R}^{n+1}$, gives some potential links between String Topology and knot-theory. We sketch how Khovanov Homology appears to fit into this picture. Overall, the intention of this chapter is not to give strict mathematical results but more to give a feel of the potential for extending the operations of String Topology, initiating from the constructions of the two previous chapters.

Aknowledgements

My journey as a PhD-student have taught me a lot about myself and math. Lending ears and brains from fellow mathematicians are amongst my treasured memories. Having been included in the highly active Centre for Symmetry and Deformation, generously sponsored by the Danish National Research Foundation, I have been gifted with potential for a vast amount of such interactions. For the equally important financial support of my stipend, I wish to thank both the FNU grant 09-064254 dubbed 'Symmetry and Moduli Problems in Topology' awarded to Ib Madsen, as well as the Faculty of Science at University of Copenhagen.

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Chapter 1

The Cleavage Operad and Its Algebras

1.1 Introduction

In [Vor05], Voronov gave life to the Cacti operad: An operad whose homology acts on the shifted homology of the free loop space over a compact, smooth, orientable manifold M – $\mathbb{H}_*(M^{S^1})$ – giving it the structure of a Batalin-Vilkovisky algebra; hereby recovering the Chas-Sullivan product of [CS99]. As an intermediate operad, Kaufmann in [Kau05] gave an E_2-operad – the spineless cacti operad – whose homology acts to give a Gerstenhaber structure, underlying the Batalin-Vilkovisky structure of Chas and Sullivans String Topology; all reflected in the fact that taking the semi-direct product of the spineless cacti with SO(2) yields the Cacti operad.

We follow the same general string of ideas, but generalize them by replacing S^1 with a manifold $N \subseteq \mathbb{R}^{n+1}$ – of arbitrary dimension – embedded in euclidean space. Our methods will involve certain decompositions of N, and for convexly embedded spheres these decompositions are simple enough to obtain results within topology, our focus will thus take a shift towards $N := S^n \subset \mathbb{R}^{n+1}$ the unit-sphere.

What we construct is a coloured operad that acts on M^N – the space of maps from N to M – in a related manner to how the Cacti operad acts on M^{S^1}. As revealed by the previous sentence, we found it necessary to broaden the scope of the use of the word 'operad' – and enter the realm of coloured operads; coloured over topological spaces. As we describe in section 1.2, this colouring is similar to picking a category internal to topological spaces, with traditional operads being one-object gadgets. We show in 1.5.21 that for $N = S^n$, the homotopy type of this operad is computable, using combinatorial methods of [Ber97], as a coloured E_{n+1}-operad. We then show how to form a semidirect product of this operad with $\mathrm{SO}(n+1)$, providing a $(n+1)$-Batalin-Vilkovisky structure on the homology of M^{S^n}.

In [CV06][Ch. 5], an outline is given for a generalisation of the Cacti operad to the n-dimensional Cacti operad, by replacing lobes with copies of S^n floating in $\mathbb{R}^{n+1}$. Our original motivation was to explicitly compute the structure of this operad; attempting to construct homotopies equivalating the little $(n+1)$-disks operad and the n-dimensional cacti operad, as was done in [Bar08] for the 1-dimensional case. However, such attempts did not seem to have a shortcut, bypassing the structure of the diffeomorphism group $\mathrm{Diff}(S^n)$.

Although we are working with coloured operads, and so give an operad different from the Cacti operad, morally we take the stance of [Kau05] – starting from a more rigid structure, where diffeomorphisms have no influence until the twisting of section 1.6 can be inferred. The coloured operad we define have an operadic structure that is basically given by cleaving N into smaller submanifolds – timber – and therefore we dub the operad, the *Cleavage Operad over* N, $\mathcal{C}\ell eav_N$.

An example of such a cleavage is given in Figure 1.1, starting in picture A, by cleaving N by a single hyperplane chopping N into two pieces of timber N_1 and N_2 in picture B; and hence succesively cleaving the timber produced into smaller subsets of N. Note that the non-linear ordering in which the

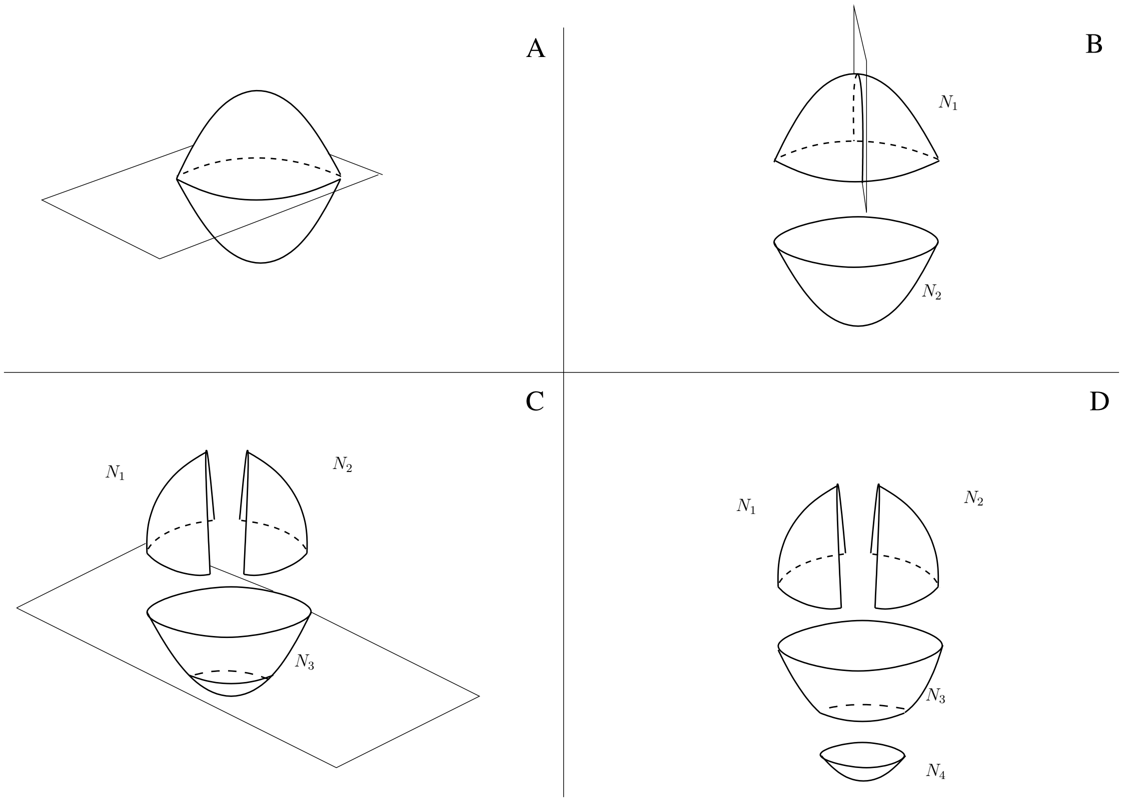

Figure 1.1: Recursive procedure cleaving a sphere into four pieces of timber

cleaving is done constitutes part of the data; for instance, interchanging the cuts made in A and B would yield a different cleavage.

However, in order to have an interesting topology, we need to forget as much of the ordering dictated by the indexing trees as possible. In 1.3.8, we forget what needs to be forgotten by defining $\mathcal{Cleav}_N$ as a quotient of an operad with k-ary structure given as pairs $(T, \underline{P})$ where T is a k-ary binary tree expressing the ordering with which N should be cleaved, and $\underline{P} = \{P_1, \ldots, P_{k-1}\}$ a tuppel of affine hyperplanes – conveying information on where to cleave by decorating the internal vertices of T.

In terms of Figure 1.1, this quotient would allow for an interchange of the ordering of hyperplanes, such that the hyperplane of C cleaves before A and B, as well as allowing an interchange of the hyperplanes of B and C.

Summary of Results

As usual in String Topology, fix a dimension d of the manifold M, and let $\mathbb{H}_*(-) := \mathrm{H}_{*+d}(-)$.

Recall that the correspondence category $\mathrm{Corr}(\mathcal{C})$ over a co-complete category $\mathcal{C}$ is given by letting $\mathrm{Ob}(\mathrm{Corr}(\mathcal{C})) = \mathrm{Ob}(\mathcal{C})$, and the set of morphisms from objects X to Y, $\mathrm{Corr}(\mathcal{C})(X,Y)$ be given as diagrams $X \longleftarrow Z \longrightarrow Y$ where Z is an arbitrary object, and the arrows are morphisms in $\mathcal{C}$. Composition is given by taking pull-backs.

Theorem A The operad $\mathcal{C}\mathit{leav}_N$ acts on M^N in the category of correspondences over topological space.

We furthermore indicate how this gives rise to the statement that letting $N := S^n \subseteq \mathbb{R}^{n+1}$ the unit sphere, and M a compact, smooth, orientable d-manifold, $\mathrm{H}_*(\mathcal{C}\mathit{leav}_{S^n})$ acts on $\mathbb{H}_*(M^{S^n})$.

Producing this homological action requires new techniques, and these are the focus of the upcoming [Bar11]

To examplify the action on correspondences, made precise in section 1.4, take the cleavage in picture D of Figure 1.1, note first that the collective boundary of the submanifolds N_1, N_2, N_3 and N_4 of N has two components C_1 and C_2. Consider the space

$$M^N_{C_1,C_2} := \{f \in M^N \mid f(C_1) = \{k_1\} \subseteq M, f(C_2) = \{k_2\} \subseteq M\}$$

– i.e. the space of maps from N to M that are constant on all of C_1 and all of C_2. The correspondence $M^N \xleftarrow{\iota} M^N_{C_1,C_2} \xrightarrow{\varphi} \left(M^N\right)^k$ is given with ι the inclusion. The other map φ maps into the i'th factor of $\left(M^N\right)^k$ by taking

$\varphi(f)(m) = f|_{N_i}(m)$ for $m \in N_i$ and for $m \notin N_i$ letting $\varphi(f)(m)$ be the same constant as $f(C_i)$ where C_i is the component separating m from N_i.

In [CJ02], it was discovered that for the Cacti operad, a passage to the latter part of 1.1 can be done via spectra to obtain so-called umkehr maps, homologically reversing one of the arrows in the correspondences. We follow this idea, and as in [CJ02] work with something stronger than stated in 1.1, namely that the action is realized by taking homotopy groups of a stable map between spectra. where an extended and punctured version of the Cleavage Operad is introduced as technical assistance for this purpose. This paper will only indicate how the action works pointwise in $\mathcal{Cleav}_{S^n}$, as well as the 2-ary term of the operad – providing reason to the specifics of the definitions of $\mathcal{Cleav}_{S^n}$.

It turns out that working with the punctured cleavage operad corresponds to adjoining a unit to the algebra of $\mathcal{Cleav}_{S^n}$. We therefore stress that the algebra described in Theorem A is a non-untial algebra, and that the coloured operad $\mathcal{Cleav}_{S^n}$ does not have 0-ary terms.

Section 1.5 and beyond are used to show the following:

Theorem B The coloured operad $\mathcal{Cleav}_{S^n}$ is a coloured E_{n+1}-operad. Taking a semidirect product of $\mathcal{Cleav}_{S^n}$ by $\mathrm{SO}(n+1)$ provides a coloured operad $\mathcal{Cleav}_{S^n} \rtimes \mathrm{SO}(n+1)$ whose homologous actions provides $(n+1)$-Batalin-Vilkovisky algebras.

The main technicality in proving the above theorem is the first statement of a coloured E_{n+1}-operad. We apply combinatorial methods of [Ber97] to show this theorem in section 1.5. The final part of the statement follows in section 1.6 by the construction of semidirects products as given in [SW03]. Briefly, since $\mathcal{Cleav}_{S^n}$ and its action on M^{S^n} is well-behaved with respect to an action of $\mathrm{SO}(n+1)$, we can apply a coloured construction of a semidirect product.

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1.2 Operadic and Categorical Concepts

In this section we shall introduce language from the world of operads, used troughout. For a general overview, we refer to the category theoretic [Lei04, 2.1], where they are called *multicategories*.

Definition 1.2.1 A *coloured operad* $\mathcal{C}$ consist of

- A class $\mathrm{Ob}(\mathcal{C})$ of *objects* or *colours*.
- For each $k \in \mathbb{N}$ and $a, a_1, \ldots, a_k \in \mathrm{Ob}(\mathcal{C})$, a class of k-ary morphisms denoted $\mathcal{C}\,(a; a_1, \ldots, a_k)$.
- To $i \in \{1, \ldots, k\}$, $f \in \mathcal{C}(a; a_1, \ldots, a_k)$ k-ary and $g \in \mathcal{C}(a_i; b_1, \ldots, b_n)$ n-ary an operadic composition $f \circ_i g \in \mathcal{C}(a; a_1, \ldots, a_{i-1}, b_1, \ldots, b_n, a_{i+1}, \ldots, a_k)$ of arity $k + n - 1$.

- Units $\mathbb{1}_a \in \mathcal{C}(a; a)$ for any object a.

- An action of Σ_k, that is, given $\sigma \in \Sigma_k$ a map $\sigma.\colon \mathcal{C}(a; a_1, \ldots, a_k) \to \mathcal{C}(a; a_{\sigma(1)}, \ldots, a_{\sigma(k)})$ for all $k \in \mathbb{N}$ and $a_i \in \mathrm{Ob}(\mathcal{C})$.

These are subject to the following conditions, where we to a $H \subseteq \{1, \ldots, m\}$, denote by Σ_H the permutation group of the elements of H. As by convention let $\Sigma_{|H|}$ denote the permutation group on the first $|H|$ natural numbers. The unique monotone map $H \to \{1, \ldots, |H|\}$ defines an isomorphism $\rho_H \colon \Sigma_H \to \Sigma_{|H|}$:

- Associativity: For $f \in \mathcal{C}\left(a; a_1, \ldots, a_k\right), g \in \mathcal{C}\left(a_i; b_1, \ldots, b_m\right), h \in \mathcal{C}\left(b_j; c_1, \ldots, c_l\right)$ the identity
$$f \circ_i (g \circ_j h) = (f \circ_i g) \circ_{j+i-1} h$$
holds for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, m\}$. For further $s \in \mathcal{C}(a_r; d_1, \ldots, d_u)$ where $i < r$ and $u \in \mathbb{N}$ we require that
$$(f \circ_i g) \circ_r s = (f \circ_r s) \circ_{i+u-1} g.$$

- Σ_k-equivariance: For $\sigma \in \Sigma_{k+m}$ and $f \in \mathcal{C}(a; a_1, \ldots, a_k), g \in \mathcal{C}(a_i; b_1, \ldots, b_m)$ the identity
$$\sigma.(f \circ_i g) = \sigma|_I.f \circ_{\sigma|_I(i)} \sigma|_J.g$$
holds where $I := \{1, \ldots, i, i+m+1, \ldots, k+m\}$ is the set of integers from 1 to $k+m$ excluding the set $J := \{i+1, \ldots, i+m\}$. For $H \subseteq \{1, \ldots, k+m\}$ the permutation $\sigma|_H \in \Sigma_{|H|}$ is given, using ρ_H from the top of this definition, as $\sigma|_H := \rho_H\left(\widetilde{\sigma|_H}\right)$ where in turn $\widetilde{\sigma|_H} \in \Sigma|_H$ is defined from σ by requiring that to $r, p \in H$ we have $\widetilde{\sigma|_H}(r) < \widetilde{\sigma|_H}(p)$ whenever $\sigma(r) < \sigma(p)$ so $\sigma|_H$ permutes the ordered symbols of H in the same way that σ permutes $\{1, \ldots, |H|\}$.

- Unit-identity: For $f \in \mathcal{C}(a; a_1, \ldots, a_k)$ we have

$$f \circ_i \mathbb{1}_{a_i} = f \text{ and } \mathbb{1}_a \circ_1 f = f$$

  for all $i \in \{1, \ldots, k\}$.

Indeed, a classical *operad* is simply a coloured operad with a single object. We shall refer to such gadgets as *monochrome operads*. On the other hand, a category $\mathcal{C}$ is the same as a coloured operad with $\mathcal{C}(a, a_1, \ldots, a_k) = \varnothing$ for $k > 1$.

Familiar concepts like functors, hom-sets and adjoints are extended in the obvious ways to this multi-arity setting.

Definition 1.2.2 Let $(\mathcal{A}, \boxtimes)$ be a symmetric monoidal category. The *underlying coloured operad* $\mathcal{U}nd_{\mathcal{A}}$ is given by letting $\mathrm{Ob}(\mathcal{U}nd_{\mathcal{A}}) = \mathrm{Ob}(\mathcal{A})$ and

$$\mathcal{U}nd_{\mathcal{A}}(a; a_1, \ldots, a_n) := \mathrm{Hom}_{\mathcal{A}}(a_1 \boxtimes \cdots \boxtimes a_n, a).$$

The usual (monochrome) endomorphism operad $\mathcal{E}nd_A$ of an object $A \in \mathcal{A}$ is given by considering the full subcategory of $\mathcal{U}nd_{\mathcal{A}}$ generated by $\{A\} \subseteq \mathrm{Ob}(\mathcal{U}nd_{\mathcal{A}})$.

Definition 1.2.3 An *action* of a coloured operad $\mathcal{C}$ on $\mathcal{A}$ is a functor $\alpha\colon \mathcal{C} \to \mathcal{U}nd_{\mathcal{A}}$. A *monochrome action* of $\mathcal{C}$ is a functor $\alpha\colon \mathcal{C} \to \mathcal{E}nd_A$ for an object $A \in \mathcal{A}$.

In (string) topology, operads are sought after for their actions on topological entities. As stated in the introduction, we venture on the same basic safari, but seek monochrome actions of coloured operads. As long as we seek monochrome actions, the extra colours on the operad become somewhat opaque – and we get actions similar to that of monochrome operads. To have topological actions of course requires topology to enter the game:

To $\mathcal{O}$ a coloured operad, denote by $\mathcal{O}(-;k)$ the set of all k-ary morphisms of $\mathcal{O}$. Let $\mathcal{O}(A;k)$ be the restriction of $\mathcal{O}(-;k)$ with $A \in \mathrm{Ob}(\mathcal{O})$ incoming.

Definition 1.2.4 Let $\mathcal{O}$ be a coloured operad. We say that $\mathcal{O}$ is a coloured topological operad if both $\mathrm{Ob}(\mathcal{O})$ and $\mathcal{O}(-;k)$ are topological spaces, along with the data of the following commutative diagram involving a pullback for $m, k \in \mathbb{N}$ and $i \in \{1, \ldots, k\}$:

$$\begin{tikzcd}
\mathcal{O}(-;k+m-1) & \mathcal{O}(-;k) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-;m) \arrow[l, "\circ_i"'] \arrow[r] \arrow[d] & \mathcal{O}(-;m) \arrow[d, "\mathrm{ev}_{\mathrm{in}}"] \\
 & \mathcal{O}(-;k) \arrow[r, "\mathrm{ev}_i"] & \mathrm{Ob}(\mathcal{O})
\end{tikzcd}$$

where ev_i evaluates at the i'th outgoing colour and $\mathrm{ev}_{\mathrm{in}}$ evaluates at the incoming colour

The structure should naturally adhere to the associativity, unit and Σ_k-equivariance conditions as specified in 1.2.1.

Note that homology does not in general preserve direct limits such as a push-out, so applying the homology functor to the diagram in 1.2.4 will not yield another push-out diagram. And in effect not lead to a similar structure in graded modules. One can however define the homology of a coloured topological operad as the coloured operad defined partially by the induced diagram

$$\begin{tikzcd}
\mathrm{H}_*\left(\mathcal{O}(-;k+m-1)\right) & & \\
\mathrm{H}_*\left(\mathcal{O}(-;k) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-;m)\right) \arrow[u, "\circ_i"] \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & \mathrm{H}_*(\mathcal{O}(-;k)) \square_{\mathrm{Ob}(\mathcal{O})} \mathrm{H}_*(\mathcal{O}(-;m)) \arrow[r] \arrow[d] & \mathrm{H}_*(\mathcal{O}(-;m)) \arrow[d, "(\mathrm{ev}_{\mathrm{in}})_*"] \\
 & \mathrm{H}_*\left(\mathcal{O}(-;k)\right) \arrow[r, "(\mathrm{ev}_i)_*"] & \mathrm{H}_*\left(\mathrm{Ob}(\mathcal{O})\right)
\end{tikzcd}$$

where $A\square_C B$ denotes the pullback of maps $A \to C$ and $B \to C$ in graded modules.

This diagram can in turn be taken to lead to a notion of partially defined coloured operads. Partial in the sense that the dotted arrow to the pullback-space is not always invertible. For the purpose of actions of operads, such a slightly more technical notion of partially defined operads would generally suffice.

However, the operads we shall define in this thesis will all have contractible colours, and we can instead of introducing partial operads use the following proposition to see that in our case, applying the homology functor to our operads will result in classical operads.

Proposition 1.2.5 Assume that $\mathcal{O}$ is a coloured topological operad with $\mathrm{Ob}(\mathcal{O}) \simeq *$, and with evalution maps ev_i fibrations for all $i \in \{1, \dots, k\}$, or with $\mathrm{ev}_{\mathrm{in}}$ a fibration. Then applying homology to $\mathcal{O}$ defines $\mathrm{H}_*(\mathcal{O})$ as a classical monochrome operad in the category of graded modules.

Proof. Since $\mathrm{Ob}(\mathcal{O}) \simeq *$, and the evaluation maps are fibrations, the long exact sequence of homotopy groups along with the 5-lemma tells us that the pullback spaces $\mathcal{O}(-;k)\times_{\mathrm{Ob}(\mathcal{O})}\mathcal{O}(-;m)$ and $\mathcal{O}(-;k)\times\mathcal{O}(-;m)$ are homotopy equivalent for all $k, m \in \mathbb{N}$ so we have that

$$\mathrm{H}_*(\mathcal{O}(-;k) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-;m)) \cong \mathrm{H}_*(\mathcal{O}(-;k) \times \mathcal{O}(-;m)).$$

The Künneth formula now gives the map

$$\circ_i \colon \mathrm{H}_*(\mathcal{O}(-;k)) \otimes \mathrm{H}_*(\mathcal{O}(-;m)) \to \mathrm{H}_*(\mathcal{O}(-;k+m-1))$$

used to define classical operads. By definition of coloured topological operads, this map satisfies the needed associativity, unity and Σ_k-invariance conditions. □

Definition 1.2.6 A *morphism* $F\colon \mathcal{O} \to \mathcal{P}$ of topological operads internal to

$(\mathcal{A}, \boxtimes)$ is given by morphisms

$$F_{\mathrm{Ob}}\colon \mathrm{Ob}(\mathcal{O}) \to \mathrm{Ob}(\mathcal{P}), F_k\colon \mathcal{O}(-;k) \to \mathcal{P}(-;k) \tag{1.1}$$

for all $k, m \in \mathbb{N}$, such that these morphisms provide a natural transformation of the diagrams of the type in 1.2.4 defining the structure for $\mathcal{O}$ and $\mathcal{P}$.

A *weak equivalence* of topological operads is given by a zig-zag of morphisms, where all continous maps of (1.1) are weak homotopy equivalences.

In particular, a weak equivalence $\mathcal{P} \simeq \mathcal{O}$ of topological coloured operads induces an isomorphism $\mathrm{H}_*(\mathcal{P}) \cong \mathrm{H}_*(\mathcal{O})$.

Example 1.2.7 Let $\mathcal{P}$ be a monochrome operad, and X a topological space. We form the trivial X-coloured operad over $\mathcal{P}$, $\mathcal{P} \times X$, by setting

- $\mathrm{Ob}(\mathcal{P} \times X) := X$
- $\mathcal{P} \times X(-;k) := \mathcal{P}(k) \times X$

evaluation maps $\mathcal{P}(k) \times X \to X$ are given by the projection map, $\circ_i$-composition, pointwise in X, the same as $\circ_i$-composition in $\mathcal{P}$.

Definition 1.2.8 We say that a coloured operad $\mathcal{O}$ is a *coloured E_n-operad* if there is a weak equivalences of operads between $\mathcal{O}$ and $\mathcal{P} \times \mathrm{Ob}(\mathcal{O})$, where $\mathcal{P}$ is a monochrome E_n-operad.

In 1.5.5, we use methods of [Ber97] to give a combinatorial way of detecting a coloured E_n-operad. We then use this to show that the operad we define in the next section is a coloured E_{n+1}-operad.

1.3 The Cleavage Operad

1.3.1 Definition of the Cleavage Operad

The operadic structure we shall define will be induced from the operadic structure of trees. The trees we consider will all, without further specification, be:

- Binary and finite, in the sense that all vertices are univalent or trivalent, and there are only finitely many vertices.
- Rooted, in the sense that there is a distinguished univalent vertex called the root.
- Labelled, in the sense that for a k-ary tree, the k remaining univalent vertices are numbered from $1, \ldots, k$.
- Planar, specifying edges out of a trivalent vertex as left- right- or down-going.

$\mathcal{Tree}$ and $\mathcal{Tree}(k)$ denotes the set of isomorphism classes of trees, respectively k-ary trees. Grafting of trees defines $\mathcal{Tree}$ as a (monochrome) operad.

Let $\mathrm{Gr}_n(\mathbb{R}^{n+1})$ be the oriented Grassmanian of codimension 1 subvectorspaces of $\mathbb{R}^{n+1}$.

Definition 1.3.1 Let the space of *affine, oriented hyperplanes* be given as

$$\mathrm{Hyp}^{n+1} := \mathrm{Gr}_n(\mathbb{R}^{n+1}) \times \mathbb{R}$$

where a pair $(H, p) \in \mathrm{Hyp}^{n+1}$ defines an affine hyperplane $P \subset \mathbb{R}^{n+1}$ by translating H along p. The hyperplane $H \in \mathrm{Gr}_n(\mathbb{R}^{n+1})$ is oriented by a choice of normal-vector, we let this choice of normal-vector denote the orientation of any associated affine hyperplane defined by (H, p).

Definition 1.3.2 Let

$$\mathcal{T}ree_{\mathrm{Hyp}^{n+1}}(k) := \mathcal{T}ree(k) \times \left(\mathrm{Hyp}^{n+1}\right)^{k-1}$$

We call $(T, \underline{P}) \in \mathcal{T}ree_{\mathrm{Hyp}^{n+1}}(k)$ a $(n+1)$-decorated k-ary tree. Here $\underline{P} = (P_1, \ldots, P_{k-1})$ denotes the tuple of elements of Hyp^{n+1}. Specifying the trivalent vertices of T as $v_1, \ldots, v_{k-1}$, these are matched to the trivalent vertices of T – and we encourage the reader to pretend that P_i is dangling from v_i.

Denote by $\mathcal{T}ree_{\mathrm{Hyp}^{n+1}}$ the operad constituted from the pieces above.

Convention 1.3.3 Throughout this text, we denote by $N \subseteq \mathbb{R}^{n+1}$ an embedded, smooth manifold We assume further that N has a *recording area*, $\mathrm{Rec}(N) \subseteq \mathbb{R}^{n+1}$ where we have the requirement that N is the boundary of $\mathrm{Rec}(N)$; $N = \partial \, \mathrm{Rec}(N)$.

We shall allow for the recording area of N to be unspecified from the notation, as it will often be the obvious choice associated with it. However, as will become clear in the definition of $\mathcal{C}leav_N$, the choice of recording area is indeed a part of the data of the resulting operad, and a priori two different choices of recording area will result in two different operads.

Example 1.3.4 The main example in this paper is $N := S^n$ the unit-sphere inside $\mathbb{R}^{n+1}$. The associated recording area will always be $\mathrm{Rec}(S^n) := D^{n+1}$, the closed unit-disk inside $\mathbb{R}^{n+1}$.

For $P \in \mathrm{Hyp}^{n+1}$, $\mathbb{R}^{n+1} \backslash P$ consist of the two components $(\mathbb{R}^{n+1})^P_+$ and $(\mathbb{R}^{n+1})^P_-$, where $(\mathbb{R}^{n+1})^P_+$ is the space in the direction of the normal-vector of P. We say that P *bisects* $\mathbb{R}^{n+1}$ into these two open subsets of $\mathbb{R}^{n+1}$.

Let $(T, \underline{P}) \in \mathcal{T}ree_{Hyp^{n+1}}(k)$, and designate by V_T the set of vertices of T that are not the root. To our given manifold N and an open submanifold $U \subseteq N$, we associate for each internal vertex $v \in V_T$ a subspace $U_v \subseteq N$:

If v is the vertex attached through only a single edge to the root, we let $U_v = U$. Since T is binary, for $v \in V_T$ a trivalent vertex the left-going and right-going edge connect v to v_- and v_+, respectively. Let $P_v \in \mathrm{Hyp}^{n+1}$ be the decoration at v. We let $U_{v_-} = (U_v) \cap \left(\mathbb{R}^{n+1}_-\right)^{P_v}$ and $U_{v_+} = U_v \cap \left(\mathbb{R}^{n+1}_+\right)^{P_v}$. This determines U_v for all $v \in V_T$.

This timbering process is illustrated in figure 1.2 for the case $N = \mathbb{R}^2$, and three hyperplanes inside $\mathbb{R}^2$, it gives three different examples where trees are decorated by the hyperplanes in some way.

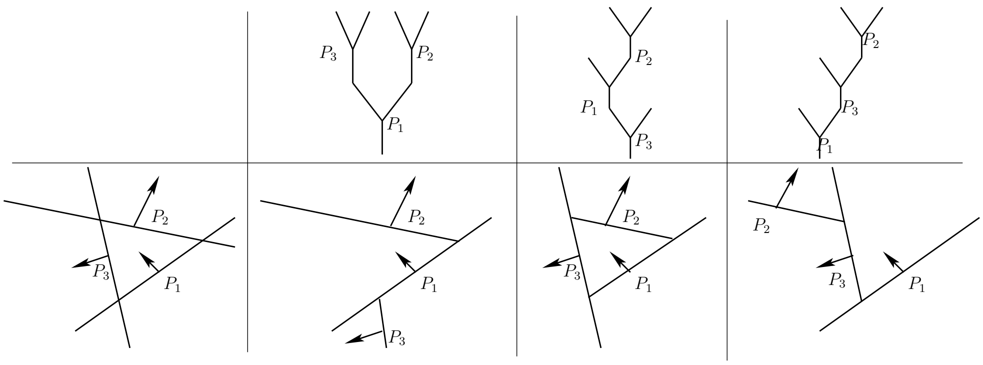

Figure 1.2: Three hyperplanes P_1, P_2, P_3 in $\mathbb{R}^2$ decorating three different trees. The four leafs of the individual trees will be decorated by the four subsets of $\mathbb{R}^2$ indicated by the associated picture below. Removing subtrees including and above a trivalent vertex, and the associated hyperplanes decorating the subtree in the picture below specifies the decoration of the other vertices.

Definition 1.3.5 Let $U \subseteq N$ be an open submanifold. A tree $(T, \underline{P}) \in \mathcal{T}ree_{Hyp^{n+1}}$ is *U-cleaving* if we to each trivalent $v \in V_T$ decorated by P_v and associated with U_v in the recursive process above have that P_v intersects U_v non-trivially and transversally.

We let Timber_N be the set of subsets of N, called *timber*, where $U \in \mathrm{Timber}_N$ if there is an N-cleaving tree, T, with U associated to a leaf of T.

Hereby Timber_N consist of a subset of certain particular open submanifolds, since at each vertex of a N-cleaving tree, the submanifolds associated to the two vertices above the vertex are again open submanifolds. Taking the closure inside N of these submanifolds will yield a codimension 0 submanifold potentially with boundary and corners.

Summarizing the above, we have specified a procedure that to a N-cleaving tree $(T, \underline{P})$ associates at each vertex v of T an open submanifold U_v. The exact same procedure can be extended to the recording area $\mathrm{Rec}(N)$, so that every vertex v of T has a subset $\mathrm{Rec}(U_v)$ associated to it, where the boundary of $\mathrm{Rec}(U_v)$ will be the space U_v.

Definition 1.3.6 The space $\mathrm{Rec}(U_v)$ given above will be called the *associated recording area* of U_v. In case $\mathrm{Rec}(N)$ is a manifold, $\mathrm{Rec}(U_v)$ will in turn be a submanifold of $\mathrm{Rec}(N)$.

There is a natural topology on Timber_N – described by the space of hyperplanes giving rise to each timber, we assume this is given and wait until the next section with describing it explicitly to give the definition of the operad as fast as possible.

Definition 1.3.7 By the pre-N-cleavage operad, we shall understand the coloured operad $\widehat{\mathcal{Cleav}}_N$, given by

- $\mathrm{Ob}(\widehat{\mathcal{Cleav}}_N) = \mathrm{Timber}_N$
- $\widehat{\mathcal{Cleav}}_N(U;k) := \big\{(T,\underline{P}) \in \mathcal{Tree}_{\mathcal{Hyp}^{n+1}}(k) \mid (T,\underline{P}) \text{ is } U\text{–cleaving}\big\}$

Granted the topology on Timber_N, we let

$$\widehat{\mathcal{Cleav}}_N(-;k) = \coprod_{U \in \mathrm{Ob}(\widehat{\mathcal{Cleav}}_N)} \widehat{\mathcal{Cleav}}_N(U;k)$$

and endow this with a topology as a subset of $\mathcal{Tree}_{\mathrm{Hyp}^n} \times \mathrm{Timber}_N$. The operadic composition

$$\circ_i \colon \widehat{\mathcal{Cleav}}_N(U;k) \times_{\mathrm{Ob}(\widehat{\mathcal{Cleav}}_N)} \widehat{\mathcal{Cleav}}_N(-;m) \to \widehat{\mathcal{Cleav}}_N(U;k+m-1)$$

is given by grafting indexing trees, and retaining all decorations of the result.

Definition 1.3.8 We let the N-cleavage operad, $\mathcal{C}\mathit{leav}_N$ be given by letting

$$\mathrm{Ob}(\mathcal{C}\mathit{leav}_N) = \{U \in \mathrm{Timber}_N \mid \complement U \simeq \coprod_{\text{finite}} *\}.$$

Here $\complement U$ denotes the complement of U as a subspace of N.

For the k-ary morphisms, we take the full suboperad of $\widehat{\mathcal{C}\mathit{leav}}_N$ on the objects $\mathrm{Ob}(\mathcal{C}\mathit{leav}_N)$ specified above, and apply a quotient: $\mathcal{C}\mathit{leav}_N(-;k) := \widehat{\mathcal{C}\mathit{leav}}_N(-;k)/\sim$, where $\sim$ is the equivalence relation given by letting $(T,\underline{P}) \sim (T',\underline{P}')$ if for all $i \in \{1,\dots,k\}$ the i'th timber N_i associated to $(T,\underline{P})$ agrees with the i'th timber N_i' of $(T',\underline{P}')$. If $(T,\underline{P})$ and $(T',\underline{P}')$ are equivalent under $\sim$, we say they are *chop-equivalent*

Since the colours are left unchanged under $\sim$, taking operadic composition induced by $\widehat{\mathcal{C}\mathit{leav}}_N$ is well-defined.

We denote an element of $\mathcal{C}\mathit{leav}_N(-;k)$ by $[T,\underline{P}]$, where $(T,\underline{P})$ is a representative of the element.

Remark 1.3.9 A priori it would suffice to be given the set Timber_N of subsets beforehand, and from this define the operadic structure through these subsets, with k open substes whose closure cover U determining the k-ary information operations of $\mathcal{C}\mathit{leav}_N(U;k)$ – avoiding the introduction of trees and hyperplanes.

However, in our forthcoming computations we shall see that it is important that we have this very strict relationship between the trees decorated by hyperplanes and the associated timber. If one had given a more arbitrary space of subsets of N instead of Timber_N, the same combinatorial benefits would not be available for computations.

Remark 1.3.10 Note that for $N = S^n$, the complement inside S^n of $U \in \mathrm{Ob}(\mathcal{C}\mathit{leav}_{S^n})$ is by the generalized Schönflies theorem [Bro60] always given by a disjoint union of wedges of disks – the wedging occurring when hyperplanes intersect directly at S^n.

Remark 1.3.11 That we for $\mathcal{Cleav}_N$ have taken a subspace of objects; i.e. $\mathrm{Ob}(\mathcal{Cleav}_N) \subset \mathrm{Timber}_N$ is necessary in order to obtain homological actions via umkehr maps, as we shall see in the next section. Do however note that it is only necessary for the homological actions, meaning that this contractibility assumption could be skipped if one is interested in a weaker notion of actions through correspondances.

For $N = S^1$ we have $\mathrm{Ob}(\mathcal{Cleav}_{S^1}) = \mathrm{Timber}_{S^1}$; the complements $\complement U$ for $U \in \mathrm{Ob}(\mathcal{Cleav}_{S^1})$ are always intervals.

Letting $N = S^n$ and $n > 1$, it follows by a simple Mayer-Vietoris argument of the H_0-groups along the closure of the timber $\overline{U}$ and $\complement U$ associated to $U \in \mathrm{Ob}(\mathcal{Cleav}_{S^n})$, that we in taking the subspace $\mathrm{Ob}(\mathcal{Cleav}_{S^n}) \subset \mathrm{Timber}_{S^n}$ are excluding the U that are disconnected[1]. Using the generalized Schöenflies theorem as stated in 1.3.10 we get that all $[T, \underline{P}]$ with in- and out-put connected are indeed in $\mathcal{Cleav}_{S^n}$.

If we take $N = S^1 \times S^1$, forming the subset $\mathrm{Ob}(\mathcal{Cleav}_N) \subset \mathrm{Timber}_N$ makes $\mathcal{Cleav}_N(-;k) = \emptyset$ for $k > 1$. This can be seen by noting that if we assume that $[T, \underline{P}] \in \mathcal{Cleav}_N(-;2)$ then one of the outgoing timber will either be a cylinder or have a genus. For higher k, the complement of outgoing timber will contain outgoing timber of $\mathcal{Cleav}_N(-;2)$ as a subset. And since they are subsets of $S^1 \times S^1$, there will not all be contractible.

The following two definitions give relations among trees in $\mathcal{Cleav}_N(-;k)$. These two types of relations – it turns out – are essential in the forthcoming material. One should however note that these two types of relations are only examples, and do not generate all relations imposed on $\mathcal{Cleav}_N(-;k)$.

To $P \in \mathrm{Hyp}^{n+1}$, let $-P \in \mathrm{Hyp}^{n+1}$ denote the hyperplane given by reversing orientation of P

Definition 1.3.12 Assume that $(T, \underline{P})$ is an N-cleaving tree. Let v be an internal vertex of T, decorated by P. Let T' be the tree obtained from T by

[1] such disconnected timber do exist; attempt for instance eating an apple conventionally to disconnect the peel in the last bite

interchanging the branches above v, and let $\underline{P}'$ denote the set of hyperplanes with P interchanged with $-P$.

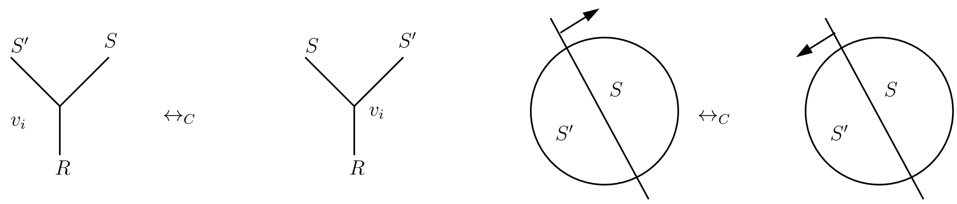

Figure 1.3: The $\leftrightarrow_C$-relation

Alternatively, the local picture 1.3 defines an equivalence relation $(T, \underline{P}) \leftrightarrow_C (T', \underline{P}')$. In $\mathcal{C}\mathit{leav}_N(-;k)$, we have that $[T, \underline{P}] = [T', \underline{P}']$

We say that two hyperplanes $P, Q \in \mathrm{Hyp}^{n+1}$ are *antipodally parallel* if $-Q$ can be obtained from P by translating P via its normal vector.

Definition 1.3.13 The local picture between $(T, \underline{P})$ and $(T', \underline{P})$ N-cleaving in 1.4, describes when two N-cleaving trees are $\leftrightarrow_B$-related.

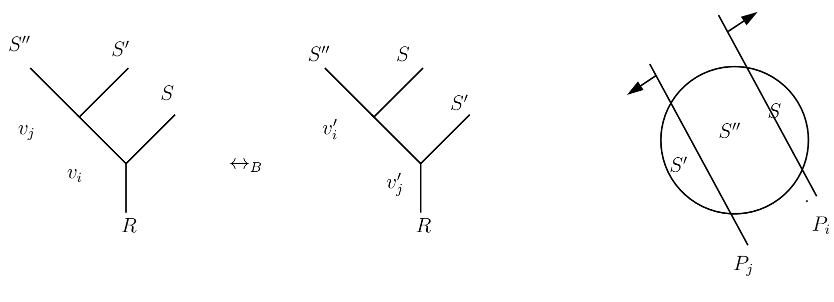

Figure 1.4: The $\leftrightarrow_B$-relation

Here the internal vertices v_j directly above v_i in T, decorated by P_i and P_j of $\underline{P}$ have swapped position – along with the branches specified in the picture – in T' compared to T. We let $(T, \underline{P}) \leftrightarrow_B (T', \underline{P})$ if P_i and P_j are antipodally parallel.

In this case we have $[T, \underline{P}] = [T', \underline{P}]$ in $\mathcal{C}\!\mathit{leav}_N(-; k)$.

We say that P and Q are parallel if either P, Q or $P, -Q$ are antipodally parallel.

Observation 1.3.14 Assume that we are given $(T, \underline{P}) \in \mathcal{C}\!\mathit{leav}_N(-; k)$, where all hyperplanes of $\underline{P}$ are pairwise parallel. Using the $\leftrightarrow_C$ and $\leftrightarrow_B$-relations of 1.3.12 and 1.3.13, we obtain that $[T, \underline{P}] = [L_k, \underline{P}']$, where L_k is a leftblown tree as in 1.3.15, and $\underline{P}'$ is obtained from $\underline{P}$ by reversing the orientations along some hyperplanes.

1.3.2 Topology on the Timber

Definition 1.3.15 We let the *arity k left-blown tree* be the tree $L_k \in \mathcal{T}\!\mathit{ree}(k)$, with the right-going edges all ending at leaves, let the only leaf on a left-going edge be labelled by k – and for the other leaves, if there are i internal vertices between the leaf and the root, we label the leaf by i.

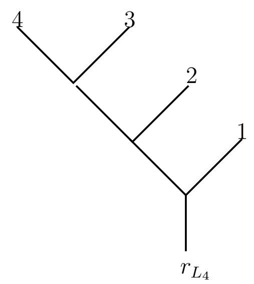

Figure 1.5: The left-blown tree L_4

Let L_1 denote the tree with $V_{L_1} = \emptyset$, and a single leaf and root.

Cultivating the cleaving tree appropriately, that is by reversing orientations of hyperplanes – swapping branches around as in 1.3.14 – and cutting away unnecessary branches, we can assume $U \in \mathrm{Timber}_N$ to be on the leaf labelled $k+1$ of L_k for some $k \in \mathbb{N}$.

Construction 1.3.16 We specify a topology on $\mathrm{Timber}_{\mathbb{R}^{n+1}}$ as a subspace by seeing that there is an injection $\psi\colon \mathrm{Timber}_{\mathbb{R}^{n+1}} \to \coprod_{i \in \mathbb{N}} \left(\left(\mathrm{Hyp}^{n+1}\right)^i / \Sigma_i \right)$, where the permutation group Σ_i permutes the factors of the product.

To specify the injection ψ, note that for $U \in \mathrm{Timber}_{\mathbb{R}^{n+1}}$, the boundary of the closure in N of U, $\partial \overline{U}$ contain the information needed to reconstruct $(L_k, \underline{P})$ having U as the decoration on the top-leaf. Such hyperplanes are given by taking least affine subsets containing certain parts of $\partial \overline{U}$; either distinguished by different components of $\partial \overline{U}$ – and otherwise a corner of $\partial \overline{U}$ will be the distinguishing feature for $(L_k, \underline{P})$. The function ψ now maps $U \in \mathrm{Timber}_{\mathbb{R}^{n+1}}$ to the corresponding hyperplanes, $(P_1, \dots, P_k) \in \left(\mathrm{Hyp}^{n+1}\right)^k$ decorating L_k and determined by $\partial \overline{U}$. This thus hits the component in the image of ψ indexed by k.

There is ambiguity in the above definition of ψ; any reordering of the hyperplanes $(P_1, \dots, P_k)$ will give rise to the same top-level timber. We therefore quotient by Σ_i in the image of ψ.

Let $\mathrm{Timber}_N^{\varnothing} := \mathrm{Timber}_N \cup \{\varnothing\}$.

Construction 1.3.17 For $N \subset \mathbb{R}^{n+1}$, we have a surjection $\mu_N\colon \mathrm{Timber}_{\mathbb{R}^{n+1}} \to \mathrm{Timber}_N^{\varnothing}$, given by $\mu_N(U) = \overline{N \cap U}^{\circ}$, whenever $\overline{N \cap U}^{\circ} \in \mathrm{Timber}_N$, where the $\overline{}$ followed by $^{\circ}$ denotes the opening of the closure inside N; and if $\overline{N \cap U}^{\circ} \notin \mathrm{Timber}_N$, we let $\mu_N(U) = \varnothing$. We specify a topology on $\mathrm{Timber}_N^{\varnothing}$ by letting μ_N be a quotient map.

Note that μ_N is well-defined since hyperplanes resulting in transversal intersections of N gives rise to μ_N mapping to the empty set. We need to take the closure of the opening of $N \cap U$ to ensure that hyperplanes cleaving N tangentially has the same effect as cleaving $\mathbb{R}^{n+1}$ away from N.

We let $\mathrm{Timber}_N \subset \mathrm{Timber}_N^{\varnothing}$ be endowed with the subspace topology

Remark 1.3.18 Specifying μ_N as a quotient map means that certain elements $[L_k, \underline{P}] \in \mathcal{C}\mathit{leav}_{\mathbb{R}^{n+1}}$ will give rise to the same $\overline{U \cap N}^{\circ}$ at the top-leaf, under μ_N. In particular, if $[L_k, \underline{P}]$ giving rise to $\overline{U \cap N}^{\circ}$ has some leaf decorated by $\varnothing$, we can instead consider $[L_{k-1}, \hat{\underline{P}}]$ as giving rise to $\overline{U \cap N}^{\circ}$, where $\hat{\underline{P}}$ is given by removing the hyperplane from $\underline{P}$ decorating the vertex below said leaf, since the hyperplane in question will not be cleaving N.

Proposition 1.3.19 Let N be a compact submanifold of $\mathbb{R}^{n+1}$. Timber_N is contractible. Similarly, $\mathrm{Ob}(\mathcal{C}\mathit{leav}_{S^n})$ is contractible for $S^n \subseteq \mathbb{R}^{n+1}$ the unit-sphere

Proof. Given a point $U \in \mathrm{Timber}_N$ will have $\mathbb{C}U$ consist of a disjoint union of submanifolds of N that has boundary at the points where U has been cleaved from N by hyperplanes $P_1, \ldots, P_k$. Each P_i has a normal-vector in the direction towards U, and one in the direction away from U. The topology on Timber_N, precisely determined by these hyperplanes makes it continuous in Timber_N to translate $P_1, \ldots, P_k$ in the direction away from U. Since N is compact, this translation will in finite time take each hyperplane past tangential hyperplanes of N. Hence each hyperplane eventually disappears from the cleaving data and by 1.3.18, eventually this translation provides an element of $\mathcal{C}\mathit{leav}_N$ given by the 1-ary undecorated tree L_1 as an operation from N to N . This hence defines a homotopy $\Phi_t\colon \mathrm{Timber}_N \to \mathrm{Timber}_N$ with $\Phi_0(U) = U$ and $\Phi_1(U) = N$, and hence the desired null-homotopy onto $N \in \mathrm{Timber}_N$.

For the statement on $\mathrm{Ob}(\mathcal{C}\mathit{leav}_{S^n})$, note that from the definition 1.3.8 and 1.3.10, that the submanifolds of $\mathbb{C}U$ will consist of a disjoint union of disks. The null-homotopy Φ_t above will in this case result in smaller and smaller disks as t increases, and so $\Phi_t(U)$ remains within $\mathrm{Ob}(\mathcal{C}\mathit{leav}_{S^n})$, and the null-homotopy is given as above.

□

The following proposition tells us in conjunction with 1.3.19 that 1.2.5 applies to $\mathcal{C}\!\ell eav_N$.

Proposition 1.3.20 The evaluation map $\mathrm{ev}_{\mathrm{in}}\colon \mathcal{C}\!\ell eav_N(-;k) \to \mathrm{Ob}(\mathcal{C}\!\ell eav_N)$ is a fibration

Proof. To $[T,\underline{P}] \in \mathcal{C}\!\ell eav_N(-;k)$ we shall first of all for each of the hyperplanes P_i of $\underline{P}$ prescribe the following transformation:

Under the relation 1.3.12, we have to make a choice of normal-vector ν_i of P_i, this defines an interval $J_i =]j_-, j_+[$ given by the maximal interval such that translating P_i along ν_i with $r \in J_i$ as a scalar the hyperplane still participates in a cleaving configuration as a decoration of T. Note that since $[T,\underline{P}]$ is cleaving we have $0 \in J_i$ and denote by $c(J_i)$ the center-point of the interval. Note that the other choice of normalvector $-\nu_i$ will give rise to the interval $-J_i$ which will leave the following invariant:

Fix $\varepsilon > 0$, if $j^i_{\min} := \min\{|j_-|, j_+\} < \varepsilon$, translate P_i by $\mathrm{sgn}(c(J_i)) \cdot \min\{\varepsilon - j^i_{\min}, c(J_i)\}$ where $\mathrm{sgn}(c(J_i))$ is the sign of $c(J_i)$.

This translation can naturally be done to all the decorations of a decorated tree $[T,\underline{P}] \in \mathcal{C}\!\ell eav_N(-;k)$ simultaneously. Call this transformation $\Gamma_\varepsilon(T,\underline{P})$, note that since we are moving all hyperplanes at once, dependent on how large ε is chosen, $\Gamma_\varepsilon([T,\underline{P}])$ does not a priori result in a cleaving tree.

We seek to find a lift in the diagram

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d] & \mathcal{C}\!\ell eav_N(-;k) \arrow[d, "\mathrm{ev}_{\mathrm{in}}"] \\ Y \times I \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & \mathrm{Ob}(\mathcal{C}\!\ell eav_N) \end{tikzcd}$$

where we can assume that Y is compact, and therefore pick

$$0 < \varepsilon < \inf_{y \in Y}(\min\{j^i_{\min} \mid P_i \text{ decorates } \varphi(y)\})$$

where $j^i_{\min}$ is the minimal value where P_i can be translated in order to have it still participate in a cleavage as defined above.

The lift $\tilde{h}(y,t)$ is now given as $\Gamma_\varepsilon(\varphi(y))$ considered as a cleaving tree of the timber $h(y,t)$. Note that our choice of ε makes $\Gamma_\varepsilon(\varphi(y))$ result in an element of $\mathcal{Cleav}_N(-;k)$, basically since along t, the timber $h(y,t)$ will change continuously and therefore by definition of Γ_ε will for a small neighborhood of $t \in I$ only give rise to a small change in how the configurations of hyperplanes change, guaranteeing their continual cleaving attributes. □

1.4 Action of Cleavages Through Correspondances

Let M be a compact manifold. We set $M^N := \{f \colon N \to M\}$ – i.e. the space of unbased, continous maps from N to M, endowed with the compact-open topology.

Above 1.3.5, we have specified a procedure that to an N-cleaving tree $(T, \underline{P})$ associates at each vertex v of T an open submanifold U_v. As noted in 1.3.6, this procedure can be extended to the recording area $\mathrm{Rec}(N)$, so that every vertex v of T has a subset $\mathrm{Rec}(U)_v$ associated to it, where the boundary of $\mathrm{Rec}(U)_v$ will be the space U_v. In case $\mathrm{Rec}(N)$ is a manifold, $\mathrm{Rec}(U)_v$ will in turn be a submanifold of $\mathrm{Rec}(N)$.

Definition 1.4.1 Let $\mathrm{Rec}(U) \subseteq \mathrm{Rec}(N)$ denote the recording area of $U \in \mathrm{Timber}_N$ as given above. To a U-cleaving tree $(T, \underline{P})$, we associate the *blueprint* of $(T, \underline{P})$ to be the following subset of $\mathrm{Rec}(U)$:

$$\beta_{(T,\underline{P})} := \mathrm{Rec}(U) \backslash \bigcup_{i=1}^{k} \mathrm{Rec}(U)_i$$

where $\mathrm{Rec}(U)_i$ is the subset of $\mathrm{Rec}(U)$ associated to the i'th leaf of $(T, \underline{P})$. By definition of the recursive procedure above 1.3.5, $\beta_{(T,\underline{P})}$ will be contained in the collective union of all the hyperplanes of $\underline{P}$, loosely it will consist of all points of hyperplanes in $\underline{P}$ that have been involved in the recursive bisection process of $\mathrm{Rec}(U)$ described above 1.3.5.

In figure 1.1 of the introduction, the boundary of $\beta_{(T,\underline{P})}$ will be the collective boundary of the closure within S^n of the submanifolds in picture D.

The notion of a blueprint is invariant under the chop-equivalence of 1.3.8, so we can make sense of the blueprint for $[T,\underline{P}] \in \mathcal{C}\mathit{leav}_N(-;k)$ and shall denote this by $\beta_{[T,\underline{P}]}$.

Definition 1.4.2 We let $\pi_0\widehat{(\beta_{[T,\underline{P}]}})$ denote the quotient of $\pi_0(\beta_{[T,\underline{P}]})$, where two pathcomponents are considered equivalent if the same hyperplane in $\underline{P}$ has given rise to these components of $\beta_{[T,\underline{P}]}$.

Example 1.4.3 If $N = S^n \subseteq \mathbb{R}^{n+1}$ as the unit-sphere, bounding the unit disk D^{n+1}, convexity of D^{n+1} entails that $\pi_0(\beta_{[T,\underline{P}]}) = \pi_0\widehat{(\beta_{[T,\underline{P}]}})$.

As an example of where the quotient matters, take N a standard-embedding of $S^1 \times S^1$ in $\mathbb{R}^3$, with recording area $D^1 \times S^1$, then cleaving $S^1 \times S^1$ with a single hyperplane into two annuli would have $\beta_{[T,\underline{P}]}$ consist of two components, whereas $\pi_0\widehat{(\beta_{[T,\underline{P}]}})$ would still be trivial.

Let $\mathrm{Corr}(\mathcal{C})$ denote the correspondance category over $\mathcal{C}$ a co-complete category, as described in the introduction.

Construction 1.4.4 We construct a functor $\Phi_N\colon \mathcal{C}\mathit{leav}_N \to \mathcal{E}\mathit{nd}_{M^N}^{\mathrm{Corr(Top)}}$. That is, an action of $\mathcal{C}\mathit{leav}_N$ on M^N as an object of the category of natural transformations of correspondances over Top.

Let $[T,\underline{P}] \in \mathcal{C}\mathit{leav}_N(U;k)$. Let $\complement N_1, \ldots, \complement N_k$ denote the complement, inside of N, of the timbers associated to the leafs of T.

The action is given through the following pullback-diagram

$$\begin{tikzcd}
M^N_{[T,\underline{P}]} \arrow[r, "\varphi^*"] \arrow[d] & \left(M^N\right)^k \arrow[d, "\mathrm{res}"] \\
M^{\pi_0\widehat{(\beta_{[T,\underline{P}]})}} \arrow[r, "\varphi"] & \prod_{i=1}^k M^{(\complement N_i)}
\end{tikzcd} \tag{1.2}$$

where res is the restriction map onto each complement.

We define φ as the induced of a map $c\colon \coprod_{i=1}^{k}(\complement N_i) \to \pi_0(\widehat{\beta_{[T,\underline{P}]}})$. To define c, note that a component C of $\complement N_i$, will have $\partial C \backslash \partial \overline{U}$ be the result of some cuts of hyperplanes decorating T. By definition of $\pi_0(\widehat{\beta_{[T,\underline{P}]}})$, and since C is a connected component, the cuts will all constitute the same element, $c(C)$ – of $\pi_0(\widehat{\beta_{[T,\underline{P}]}})$ making φ well-defined as the map constant map along the timber intersecting represenatives of $\pi_0(\widehat{\beta_{[T,\underline{P}]}})$ nontrivially.

By glueing the functions in the pullback space, we can identify $M^N_{[T,\underline{P}]}$ as the space of $f \in M^N$ such that f is constant along each subspace of the blueprint that is a representative of $\pi_0(\widehat{\beta_{[T,\underline{P}]}})$. We hence have a canonical inclusion $\iota_{[T,\underline{P}]}\colon M^N_{[T,\underline{P}]} \to M^N$, and in turn a correspondance

$$M^N \xleftarrow{\iota_{[T,\underline{P}]}} M^N_{[T,\underline{P}]} \xrightarrow{\varphi^*} \left(M^N\right)^k$$

Example 1.4.5 For the cleavage $[T, P_1, P_2, P_3] \in \mathcal{Cleav}_{S^n}(S^n; 4)$ in picture D of figure 1.1 in the introduction, we can consider the morphisms defining diagram (1.2) through figure 1.6, using the mapping functor $M^{(-)}$ to dualize the morphisms indicated in the figure, we get the maps that define the pullback diagram (1.2).

Functoriality of the above pullback-construction in 1.4.4 gives us

Proposition 1.4.6 The construction of 1.4.4 defines an action of $\mathcal{Cleav}_N$ on M^N as an element of the symmetric monoidal category $(\mathrm{Corr}(\mathrm{Top}), \times)$.

Proposition 1.4.7 The map res is a fibration.

Proof. Follows directly since the inclusions $\complement N_i \to N$ are closed cofibarations, and res is the dualization under the mapping space functor M^-. □

The rest of this section is devoted to give a hint at why the action through correspondances of $\mathcal{Cleav}_{S^n}$ gives rise to a homological action of $\mathrm{H}_*(\mathcal{Cleav}_{S^n})$ on $\mathbb{H}_*(M^{S^n})$. As mentioned in the introducation, we give the full action in

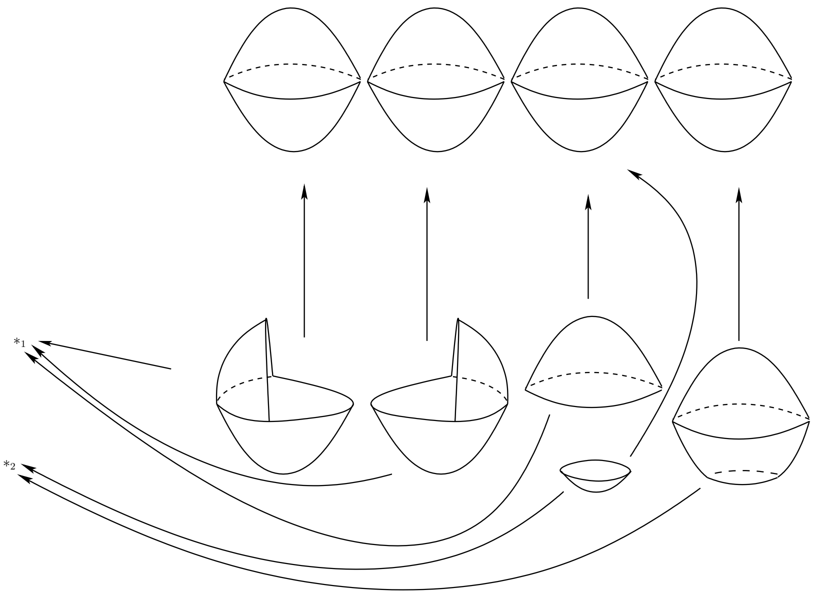

Figure 1.6: The 4-ary operation of the introduction has the complement of its timber $\complement N_1, \complement N_2, \complement N_3, \complement N_4$ drawn as the five disks on the bottom right corner. The upwards arrow are the inclusion maps, so that dualizing them provides the restriction map. The leftwards maps have as target two points, and these should be considered as collapsing the components of the blueprint $\beta_{[T,P_1,P_2,P_3]}$. These maps are given as the ones where the boundary of each disk is contained in a component of the blueprint.

chapter 2, in this section we only give examples and observations that can be seen as a justification of our definition of $\mathcal{C}leav_{S^n}$ and a warm-up to the homological action.

In obtaining homological actions of $\mathcal{C}leav_{S^n}$ on M^{S^n}, there is a shift of degrees by the dimension of M, the following proposition gives the insight along spaces that this holds.

Proposition 1.4.8 Consider the case of $N = S^n$. For any $[T, \underline{P}] \in \mathcal{C}leav_{S^n}(-; k)$, with associated timber $N_1, \ldots, N_k$ the number

$$\left|\pi_0\left(\coprod_{i=1}^{k} \complement N_i\right)\right| - \left|\pi_0\left(\beta_{[T,\underline{P}]}\right)\right|$$

will be constantly $k-1$ for all $[T,\underline{P}] \in \mathcal{C}leav_{S^n}(-;k)$.

Proof. Note that $\beta_{[T,\underline{P}]}$ constitute the boundary of the disjoint union of wedges of disks $\coprod_{i=1}^{k} \complement N_i$, so the number $\left|\pi_0\left(\coprod_{i=1}^{k} \complement N_i\right)\right|$ will be constant as long as $\left|\pi_0\left(\beta_{[T,\underline{P}]}\right)\right|$ is constant.

Consider a path $\gamma\colon [0,1] \to \mathcal{C}leav_{S^n}(-;k)$ with $\left|\pi_0(\beta_{\gamma(0)})\right| = \left|\pi_0(\beta_{\gamma(1)})\right| + 1$ and such that for a specific $t_0 \in [0,1]$, we have for all $t \leqslant t_0$, $\left|\pi_0(\beta_{\gamma(t)})\right| = \left|\pi_0(\beta_{\gamma(0)})\right|$ and for all $t > t_0$, $\left|\pi_0(\beta_{\gamma(t)})\right| = \left|\pi_0(\beta_{\gamma(1)})\right|$.

By definition of the cleaving proces, $\gamma(t_0)$ will have two hyperplanes P_l, P_r such that $P_l \cap P_r \cap \operatorname{Rec}(S^n)$ is a nontrivial subspace of $\partial \operatorname{Rec}(S^n) = S^n$, and for any $\varepsilon > 0$, the same intersection for the hyperplanes of $\gamma(t_0+\varepsilon)$ will be trivial; meaning that for sufficiently small ε, such that the hyperplanes do not become parallel, $P_l \cap P_r \subseteq \mathbb{R}^{n+1}$ will be contained in $\mathbb{R}^{n+1} \setminus \operatorname{Rec}(N)$. This has the effect that there is precisely one $j \in \{1, \ldots, k\}$ such that the complement of the timber indexed by j, $\complement N_j$, for $\gamma(t_0)$ has a connected component containing $P_l \cap P_r \cap \operatorname{Rec}(N)$, whereas for $\gamma(t_0+\varepsilon)$ this becomes disconnected with different boundary components of $\complement N_i$ being formed using intersections with P_l and P_r respectively. Hence for these basic types of paths, an increase in $\left|\pi_0(\beta_{[T,\underline{P}]})\right|$ leads to an equal increase in $\left|\pi_0(\coprod_{i=1}^{k} \complement N_i)\right|$.

One can use these paths to parametrize a single cleaving hyperplane moving within $\mathcal{C}leav_{S^n}(-;k)$, while the other $k-2$ hyperplanes remain fixed. From such parametrizations, one puts together a general path moving all hyperplanes of $\mathcal{C}leav_{S^n}(-;k)$ and the result follows.

To compute the constant, take a configuration of hyperplanes where all hyperplanes are parallel, so that $\beta_{[T,\underline{P}]}$ has $k-1$ components. In this case the space of complements of the associated timber, $\coprod_{i=1}^{k} \complement N_i$ will have $(k-2)$ spaces $\complement N_i$ that consist of two disjoint spaces, and the two extremal complements that consist of a single subdisks of S^n. Hence in total $2(k-2)+2 = 2(k-1)$ components. In effect, the constant will be $2(k-1)-(k-1) = k-1$. $\square$

Remark 1.4.9 The above 1.4.8 is only stated for the particular case $N := S^n$. However the only place in the proof where we use the nature of S^n is to identify that the number $\left|\pi_0\left(\coprod_{i=1}^k \mathtt{C}N_i\right)\right|$ will be constant as long as $\left|\pi_0\left(\widehat{\beta_{[T,\underline{P}]}}\right)\right|$ is constant.

Again, we need to work with $\left|\pi_0\left(\widehat{\beta_{[T,\underline{P}]}}\right)\right|$ as defined in 1.4.2 for more general N, however this also fits directly into the proof.

Since the hyperplanes of $\mathcal{C}leav_N(-;k)$ are required to cleave N transversally, one can indeed use Morse theory to obtain this initial statement of the proof for a general embedded manifold N. However, we shall not go into detail with this, as we shall only apply 1.4.8 in the case where $N = S^n$.

Of course in the more general case, this constant will no longer be $k-1$, and indeed the constant will potentially vary along components of $\mathcal{C}leav_N(-;k)$. With $\mathcal{C}leav_{S^n}(-;k)$ connected, basically since all hyperplanes will cleave S^n transversally, this variance along components is not part of the statement of 1.4.8.

Remark 1.4.10 Note that as we don't use it in the proof, 1.4.8 holds even if we drop the assumption in 1.3.8 that for all elements of $\mathcal{C}leav_{S^n}(-;k)$ the associated timber $N_1,\dots,N_k$ should satisfy $\mathtt{C}N_i \simeq \coprod_{\text{finite}} *$. In fact, this assumption is not needed for this paper, but will only be applied in [Bar11] to define homological actions from the correspondances of 1.4.4.

This is also the case for an extended version of 1.4.8 to $\mathcal{C}leav_N$ as indicated in 1.4.9

Remark 1.4.11 For the case $N = S^n$, as mentioned in the remarks above, the correspondance diagram (1.2) will by certain umkehr maps for the map φ^* eventually lead to a homological action in [Bar11]. What we in fact will show is something stronger, namely that there is a stable action map, residing in the category of spectra:

$$\mathcal{C}\!\mathit{leav}_{S^n}(-;k) \times \left(M^{S^n}\right)^k \to M^{S^n} \wedge S^{\dim(M)\cdot(k-1)} \tag{1.3}$$

Smashing the above map with the Eilenberg-Maclane spectrum, and taking homotopy groups yields the action mentioned below Theorem 1.1 of the introduction.

We shall in 1.4.12 below give an example of the action to illustrate the reasoning behind the definition of $\mathcal{C}\!\mathit{leav}_{S^n}$.

We shall first make some remarks on how the constructions of this section provides the foundation for how this map takes form.

First of all, note that since we in 1.3.8 have assumed that to $[T, \underline{P}] \in \mathcal{C}\!\mathit{leav}_N(-;k)$ the associated timber $N_1, \ldots, N_k$ will have $\complement N_i$ consist of a finite disjoint union of contractible spaces. Therefore, in the diagram (1.2), the space $\prod_{i=1}^k M^{\complement N_i}$ is a Poincaré duality space. That is, up to homotopy it is equivalent to a product of copies of M.

In the case $N = S^n$, taking such a specific $[T, \underline{P}]$ as a pointwise version of (1.3), we should be able to obtain a map $\left(M^{S^n}\right)^k \to M^{S^n} \wedge S^{\dim(M)\times(k-1)}$. Indeed, for fixed $[T, \underline{P}]$ there are methods available in the litterature to do this. For instance adapting the methods of [CK09, p.14/'Umkehr maps in String Topology'] provides such a map. Here it is crucial to note that the map φ is up to homotopy an embedding of codimension $\dim(M) \times (k-1)$ where the $(k-1)$-factor comes from 1.4.7, which provides the dimension-shift in (1.3).

Yet another method, formulated on the chain-level instead of spectra would be [FT09, Theorem A].

However, getting from these pointwise umkehr maps to a map such as (1.3) is a somewhat strenous hike. This is the focus of [Bar11]. Note that while the crucial 1.4.8 hints that it is possible to provide the maps for other N

than the euclidean embedded unit-spheres S^n, the geometry involved makes us look only at the interesting case $\mathcal{C}leav_{S^n}$.

The complexity of the construction of the umkehr map rises with the arity of the involved maps. As the arity rises, the map φ of (1.2) will by 1.4.8 have constant codimension, whereas the actual dimension will be exposed to sudden jumps in the actual dimensions of the spaces φ is mapping between, as is indicated in the proof of 1.4.8. The coherence issues of higher arity lies in patching the instances of such jumps together. The following example illustrates the case of arity 2 operations where there are no such jumps:

Example 1.4.12 The 2-ary portion $\mathcal{C}leav_{S^n}(-;2)$ is a manifold, specified by a single cleaving hyperplane, it deformation retracts onto S^n, which is determined by the direction of the normal-vector of the cleaving hyperplane.

Consider the pull-back diagram

$$\begin{tikzcd} M^{S^n}_{\mathcal{C}leav_{S^n}(-;2)} \arrow[r] \arrow[d] & \left(M^{S^n}\right)^2 \times \mathcal{C}leav_{S^n}(-;2) \arrow[d, "\mathrm{res}"] \\ M \times \mathcal{C}leav_{S^n}(-;2) \arrow[r] & M^{\coprod_{i=1}^2 \complement N_i}_{\mathcal{C}leav_{S^n}(-;2)} \end{tikzcd} \tag{1.4}$$

Where $\complement N_1$ and $\complement N_2$ denotes the complement inside S^n of the timber associated to $[T, P] \in \mathcal{C}leav_{S^n}(-;2)$. Considering it as a set, $M^{\coprod_{i=1}^2 \complement N_i}_{\mathcal{C}leav_{S^n}(-;2)}$ is given by the disjoint union $\coprod_{[T,P]\in\mathcal{C}leav_{S^n}(-;2)} M^{\complement N_i}$. The map res in the diagram is given by letting $\mathrm{res}(f_1, f_2, [T, P])$ be given as the the restriction of f_1 to $\complement N_1$ and f_2 to $\complement N_2$ along the component indexed by $[T, P]$. We topologize $M^{\coprod_{i=1}^2 \complement N_i}_{\mathcal{C}leav_{S^n}(-;2)}$ by making res a quotient map.

Note that by 1.4.7, pointwise in $\mathcal{C}leav_{S^n}(-;2)$, res is a fibration. One sees that the lifts of this global map can be constructed to be continous in $\mathcal{C}leav_{S^n}(-;2)$ as well.

We hereby have homotopy-equivalences

$$M^{\coprod_{i=1}^{2} \complement N_i}_{\mathcal{C}leav_{S^n}(-;2)} \simeq M^2 \times \mathcal{C}leav_{S^n}(-;2) \simeq M^2 \times S^n$$

and

$$M \times \mathcal{C}leav_{S^n}(-;2) \simeq M \times S^n.$$

Stating that the lower portion of the diagram is an embedding of Poincaré duality spaces. For instance applying one of the methods mentioned in 1.4.11, this hereby provides the first sign of an action of $\mathcal{C}leav_{S^n}$ on M^{S^n}, and taking homotopy groups of this map, we get a map in homology

$$\mathrm{H}_*\left(\mathcal{C}leav_{S^n}(-;2)\right) \otimes \mathrm{H}_*(M^{S^n})^{\otimes 2} \to \mathrm{H}_{*+\dim(M)}(M^{S^n})$$

1.5 Spherical Cleavages are E_{n+1}-operads

We now devote energy to prove that $\mathcal{C}leav_{S^n}$ is a coloured E_{n+1}-operad. A concept defined in 1.2.8

1.5.1 Combinatorics of Coloured E_n-operads

The following combinatorial data of full level graphs is the main tool we use to describe a detection principle for E_n-operads.

Definition 1.5.1 By a *full level* (n,k)*-graph*, we shall understand a graph G with k vertices $v_1, \dots, v_k$ such that all pairs (v_i, v_j) are connected by precisely one edge e_{ij}. Let $\mathbf{n} := \{0, \dots, n-1\}$. We let each of the $\binom{k}{2}$ edges of G be labelled by elements of $\mathbf{n}$.

We say that a full level (n,k)-graph G is oriented if there to each edge G e_{ij} is designated a direction; either from v_i to v_j or from v_j to v_i.

To $\sigma \in \Sigma_k$, there is a unique orientation of a full level (n,k)-graph, G. Namely by letting e_{ij} point from v_i to v_j if $\sigma(i) < \sigma(j)$, and point from v_j to

v_i if $\sigma(j) < \sigma(i)$. We call σ the permutation associated to G.

Indeed, assuming that this orientation of G is oriented with no cycles and comes equipped with a sink and a source, one can reconstruct the permutation σ_G associated to G from the orientation of G: Let the index of the sink of G be mapped to k under σ_G, and successively remove the sink of an oriented full (n, i)-graph with sink, source and no cycles to obtain a full $(n, i-1)$-graph with induced orientation being guaranteed a sink by the pigeonhole principle. The index of this sink is mapped to i under σ_G. Continuing this process until only the source of G is left makes σ_G a well-defined permutation since G has no cycles.

Definition 1.5.2 Let $\mathcal{K}^n(k)$ denote the set of full level (n, k)-graphs, oriented via Σ_k as above. This gives a bijection $\mathbf{n}^{\binom{k}{2}} \times \Sigma_k \leftrightarrow \mathcal{K}^n(k)$.

We have that $\mathcal{K}^n$ is an operad: To $G_k \in \mathcal{K}^n(k)$ and $G_m \in \mathcal{K}^n(m)$, inserting the m vertices of G_m instead of i'th vertex of G_k, we obtain a full level $(n, m+k-1)$ graph $G_k \circ_i G_m$, by labelling and orienting the edges of $G_k \circ_i G_m$ in the following way:

Since we have replaced the i'th vertex of G_k with k vertices from $\mathcal{K}^n(m)$, G_k lives as a subgraph of $G_k \circ_i G_m$ in k different ways – one for each choice of replacement vertex for i in G_m. The graph G_m has all its vertices retained in $G_k \circ_i G_m$, so there is only one choice of subgraph for G_m.

We label and orient all edges of $G_k \circ_i G_m$ via the labellings and orientations of the possibilities of subgraphs G_k and G_m.

An example operadic composition is given in figure 1.7

Observation 1.5.3 Using 1.5.2, we see how $\mathcal{K}^n$ is an operad of posets. First of all, letting $\mathcal{K}^n(2) = \mathbf{n} \times \Sigma_2$ be given by setting $\Sigma_2 = \{\mathrm{id}, \tau\}$ and partially ordering through the arrows of the diagram

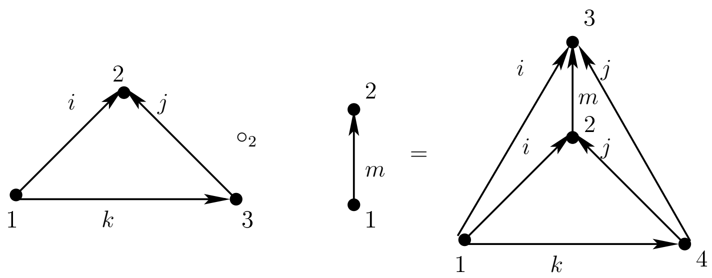

Figure 1.7: An operadic composition of a 3-ary with a 2-ary operation in the full-graph operad. The labellings i, j, k, m are elements of $\mathbf{n}$. Note that in the 4-ary operation both graphs are contained in the final result, and we copy the labelling of the edges that are going to the vertex the operadic composition is happening at

$$\begin{tikzcd}
(0,\mathrm{id}) \arrow[r] \arrow[dr] & (1,\mathrm{id}) \arrow[r] \arrow[dr] & \cdots \arrow[r] \arrow[dr] & (n,\mathrm{id}) \\
(0,\tau) \arrow[r] \arrow[ur] & (1,\tau) \arrow[r] \arrow[ur] & \cdots \arrow[r] \arrow[ur] & (n,\tau)
\end{tikzcd} \tag{1.5}$$

Consider the maps $\gamma_{ij}\colon \mathcal{K}^n(k) \to \mathcal{K}^n(2)$, sending $G_k \in \mathcal{K}^n(k)$ to its subgraph, with one edge, spanned by the vertices v_i and v_j. Following [Ber97, 1.5], we let the partial ordering be given by the coarsest ordering such that γ_{ij} is order preserving for all $i < j \in \{0, \ldots, k\}$.

Definition 1.5.4 Given $\mathcal{O}$ a coloured operad, let $S(\mathcal{O})$ denote the coloured operad in posets, with objects the subsets of $\mathrm{Ob}(\mathcal{O})$, and k-ary morphisms subsets of $\mathcal{O}(-;k)$. The operad is an operad in posets through inclusions of subspaces.

That is, we let $\mathrm{Ob}(S(\mathcal{O})) = \{0,1\}^{\mathrm{Ob}(\mathcal{O})}$. Let $S(\mathcal{O})(-;k) := \{0,1\}^{\mathcal{O}(-;k)}$.

These fit into the diagram

$$\begin{tikzcd} S(\mathcal{O})(-;k+m-1) & S(\mathcal{O})(-;m) \times_{\mathrm{Ob}(S(\mathcal{O}))} S(\mathcal{O})(-;k) \arrow[l, "\circ_i"'] \arrow[r] \arrow[d] & S(\mathcal{O})(-;k) \arrow[d, "\mathrm{ev}_i"] \\ & S(\mathcal{O})(-;m) \arrow[r, "\mathrm{ev}_{\mathrm{in}}"] & \mathrm{Ob}(S(\mathcal{O})) \end{tikzcd}$$

Where as usual ev_i evaluates at the i'th colour of $S(\mathcal{O})(-;k)$, given as a subset of $\mathrm{Ob}(\mathcal{O})$ and $\mathrm{ev}_{\mathrm{in}}$ evaluates the incoming colour of $S(\mathcal{O})(-;m)$.

Operadic composition is induced from composition in $\mathcal{O}$, pointwise. In the sense that $\circ_i\colon S(\mathcal{O})(-;m)\times_{\mathrm{Ob}(S(\mathcal{O}))}S(\mathcal{O})(-;k) \to S(\mathcal{O})(-;k+m-1)$ is given by the subset of $\mathcal{O}(-;k+m-1)$ obtained by to any point of $\mathcal{O}(-;m)\times_{\mathrm{Ob}(\mathcal{O})}$ $\mathcal{O}(-;k)$ as an element of $S(\mathcal{O})(-;m) \times_{\mathrm{Ob}(S(\mathcal{O}))} S(\mathcal{O})(-;k)$ applying the $\circ_i$-operation from $\mathcal{O}(-;m) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-;k)$ to $\mathcal{O}(-;k+m-1)$, and taking the union over the specific subset of these compositions.

Recall from 1.2.7 that to a monochrome operad $\mathcal{P}$ and a space X, the coloured topological operad $\mathcal{P} \times X$ is coloured over X and the k-ary morphisms are formed by taking the cartesian product of $\mathcal{P}(k)$ with X.

The Berger Cellularization Theorem, written in a monochrome fashion in [Ber97, Th. 1.16] hereby transfers to our coloured setting:

Theorem 1.5.5 Let $\mathcal{O}$ be a topological coloured operad. The operad $\mathcal{O}$ is a coloured E_n-operad, 1.2.7 if there is a functor $F_k\colon \mathcal{K}^n(k) \times S(\mathrm{Ob}(\mathcal{O})) \to S(\mathcal{O})(-;k)$ that is, both a functor with respect to the poset structure as well as a morphism of coloured operads, satisfying the following:

(A) Let $C_0 \in S(\mathrm{Ob}(\mathcal{O})$. The *latching space of* $\alpha \in \mathcal{K}^n(k)$ is given by $L_{(\alpha,C_0)} := \bigcup_{\beta<\alpha} F_k(\beta, C_0)$. We require that the morphism

$$L_{(\alpha,C_0)} \hookrightarrow F_k(\alpha, C_0)$$

is a cofibration.

(B) For all $\alpha \in \mathcal{K}^n(k)$, we require maps $F_k(\alpha, B) \to B$ where $B \in S(\mathrm{Ob}(\mathcal{O}))$ such that these are weak equivalences, and natural with respect to morphisms $B \hookrightarrow C \in S(\mathrm{Ob}(\mathcal{O}))$. between the functor $F_k(\alpha, -)$.

(C) $\mathrm{colim}_{(\alpha,C_0)\in\mathcal{K}^n(k)\times\mathrm{Ob}(S(\mathcal{O}))} F_k(\alpha, C_0) = \mathcal{O}(-;k)$, where the colimit is using the poset-structure on $\mathcal{K}^n(k)$ – given in 1.5.3 – and inclusions of subsets in $S(\mathrm{Ob}(\mathcal{O}))$. These inclusions should be compatible with the equation, in the sense that $\mathcal{O}(U;k)$ should be given by restricting the colimit to the $C_0 \in \mathrm{Ob}(S(\mathcal{O}))$ satisfying $C_0 \subseteq U$ in the indexing category.

Proof. First of all, note that $\mathcal{K}^n(k)$ as a finite poset is a Reedy Category in the sense of [Dug08, 13.1]; explicitly a degree function $\deg\colon \mathcal{K}^n(k) \to \mathbb{Z}$ can be given by letting $\deg(\alpha)$ be determined by the sum of the $\binom{k}{2}$ labels in $\{0, \ldots, n-1\}$ of the edges of the graph α.

From the assumptions (A)-(C) we get the following homotopy equivalence:

$$\begin{aligned}\mathcal{O}(-;k) = \mathrm{colim}_{(\alpha,C_0)\in\mathcal{K}^n(k)\times S(\mathrm{Ob}(\mathcal{O}))} F_k(\alpha, C_0) &\cong \\ \mathrm{colim}_{C_0\in S(\mathrm{Ob}(\mathcal{O}))}\,\mathrm{colim}_{\alpha\in\mathcal{K}^n(k)} F_k(\alpha, C_0) &\simeq \\ \mathrm{colim}_{C_0\in S(\mathrm{Ob}(\mathcal{O}))}(\mathrm{hocolim}_{\alpha\in\mathcal{K}^n(k)} * \times C_0) &\cong \\ \mathrm{colim}_{C_0\in S(\mathrm{Ob}(\mathcal{O}))}(|\mathcal{N}(\mathcal{K}^n)| \times C_0)(k) &\cong \\ (|\mathcal{N}(\mathcal{K}^n)| \times \mathrm{Ob}(\mathcal{O}))(k)&\end{aligned}$$

The first identification is given in assumption (C), which in turn splits out into a double colimit. To obtain the homotopy equivalence: Since $\mathcal{K}^n(k)$ is a Reedy Category, the assumption (A) allows us to apply [Dug08, 13.4] giving a homotopy equivalence $\mathrm{hocolim}_{\alpha\in\mathcal{K}^n(k)} F_k(\alpha, C_0) \to \mathrm{colim}_{\alpha\in\mathcal{K}^n(k)} F_k(\alpha, C_0)$ for fixed incoming colours $C_0 \in S(\mathrm{Ob}(\mathcal{O}))$. From (B) we get a homotopical identification of $F_k(\alpha, C_0)$ with C_0 and this computes the homotopy colimit, geometrically realizing the nerve of the full graph operad, along with a cartesian product of the colours $C_0 \in S(\mathrm{Ob}(\mathcal{O}))$ – independent of $\alpha \in \mathcal{K}^n(k)$. The final identification follows since the naturality of (B) supplies us with a cartesian product of the nerve along with an actual direct limit of all inclusions of $S(\mathrm{Ob}(\mathcal{O}))$ which can be identified with the final target of the inclusions, $\mathrm{Ob}(\mathcal{O})$.

One now utilizes F as a morphism between coloured operads to check that this gives an operadic weak equivalence $\mathcal{O} \simeq |\mathcal{N}(\mathcal{K}^n)| \times \mathrm{Ob}(\mathcal{O})$

$\square$

Definition 1.5.6 Let $\mathcal{O}$ be a coloured topological operad. We call $F\colon \mathcal{K}^n \times S(\mathrm{Ob}(\mathcal{O})) \to S(\mathcal{O})$ an *E_n-functor* if it satisifes the conditions of 1.5.5

Remark 1.5.7 In order to get an equivalence back to something known, let $\mathcal{D}isk_{n+1}$ denote the little disk operad. We hereby have that the coloured operad $\mathcal{D}isk_{n+1} \times \mathrm{Ob}(\mathcal{C}leav_{S^n})$ is a coloured E_{n+1}-operad in the sense of 1.2.8, coloured over the same objects as $\mathcal{C}leav_{S^n}$.

1.5.2 An E_{n+1}-functor for $\mathcal{C}leav_{S^n}$

Construction 1.5.8 We shall provide the combinatorial data, giving the link between the full graph operad, and the spherical cleavage operads.

For each $n \in \mathbb{N}$, let $I = [-1, 1] \hookrightarrow \mathbb{R}^{n+1}$ denote the interval as sitting inside the first coordinate axis of $\mathbb{R}^{n+1}$. Choosing $k-1$ distinct points $x_1, \ldots, x_{k-1} \in I$ specifies a partition of I into k intervals $X_1 = [-1, x_1], X_2 = [x_1, x_2], \ldots, X_k = [x_{k-1}, 1]$. We endow the collection of these intervals with an ordering, determined by $\sigma \in \Sigma_k$, ordering them as $X_{\sigma(1)}, \ldots, X_{\sigma(k)}$.

Parametrize $S^n := \{\underline{s} = (s_1, \ldots, s_{n+1}) \in \mathbb{R}^{n+1} \mid \|\underline{s}\| = 1\}$ to consider the map $\eta\colon S^n \to I$ given by $\eta(s_1, \ldots, s_{n+1}) = s_1$. Any subinterval $X_i \subseteq I$ defines timber $\tilde{X}_i$ of S^n, by $\eta^{-1}(X_i)$.

Positioning hyperplanes $P_1, \ldots, P_{k-1}$ orthogonal to I – such that P_i contains the point $(x_i, 0, \ldots, 0)$, as decorations on a cleaving tree we can choose their normal-vector of P_i to point towards $(1, 0, \ldots, 0)$ and get colours that are labelled by 1 to k from left to right along the first coordinate axis.

Under the chop-equivalence of 1.3.8, we can always choose a representing cleaving tree with two leaves labelled i and $i+1$ directly above an internal vertex for any $i \in \{1, \ldots, k-1\}$. For this particular representative of a cleaving tree with the particular orientations of $P_1, \ldots, P_{k-1}$, applying the transposition between i and $i+1$ corresponds exactly to inverting the orientation of the decoration at the vertex below the two leafs.

Since Σ_k is generated by these transpositions, we can permute the labelleing of the colours by $\sigma \in \Sigma_k$, and hereby obtain $[T_\sigma, \underline{P_\sigma}] \in \mathcal{C}\mathit{leav}_{S^n}(Uk)$, with $U \in \mathrm{Ob}(\mathcal{C}\mathit{leav}_{S^n})$ and outgoing colours decorated by $\sigma(1), \ldots, \sigma(k)$ from left to right along the first coordinate axis of $\mathbb{R}^{n+1}$.

Definition 1.5.9 For a given $\sigma \in \Sigma_k$, the collection of all $[T_\sigma, \underline{P}_\sigma]$ as given above specifies an element of the subsets of $\mathcal{C}\mathit{leav}_{S^n}(-;k)$, where for $U \in \mathrm{Ob}(\mathcal{C}\mathit{leav}_{S^n})$ this involves a restriction to the U-cleaving trees $(T_\sigma, \underline{P}_\sigma)$.

Note that to $U \in \mathrm{Ob}(\mathcal{C}\mathit{leav}_{S^n})$ choosing the hyperplanes $\underline{P}_\sigma$ to be in equidistant position from each other, and the closest hyperplanes that no longer cleave U, defines an embedding of $U \hookrightarrow \mathcal{C}\mathit{leav}_{S^n}(U;k)$ for each $\sigma \in \Sigma_k$.

Observation 1.5.10 We define $J_U \subseteq [-1, 1]$ as sitting inside the first coordinate axis of $\mathbb{R}^{n+1}$ as the subspace where any U-cleaving $(T_\sigma, \underline{P}_\sigma)$ have decorating hyperplanes contain points of J_U.

We shall for the sake of this section allow ourselves to assume that $J_U \subseteq [-1, 1]$ is a non-empty subinterval; formally, this can be done by redefining $\mathcal{C}\mathit{leav}_{S^n}$ as a full suboperad of $\mathcal{C}\mathit{leav}_{S^n}$ given by restricting $\mathrm{Ob}(\mathcal{C}\mathit{leav}_{S^n})$ to the timber U for which J_U is U-cleaving.

Similar to 1.3.19, one can define a homotopy that pushes the hyperplanes defining U towards tangenthyperplanes of S^n to show that this restriction defines a deformation retraction of the objects, and hence makes the inclusion a weak equivalence of operads.

Remark 1.5.11 We find it enlightening to note that we can form a suboperad, the *caterpillar operad* [2] $\mathcal{C}ater_{S^n}$, of $\mathcal{C}leav_{S^n}$ by taking the full suboperad under the condition that $[T, \underline{P}] \in \mathcal{C}ater_{S^n}(-;k)$ if $[T, \underline{P}]$ is of the form $[T_\sigma, \underline{P}_\sigma]$ as in 1.5.9 for some $\sigma \in \Sigma_k$.

The caterpillar operad will control the product structure on $\mathbb{H}_*(M^{S^n})$, however in order to obtain the higher bracket in the Gerstenhaber Algebra, we shall need more than just parallel hyperplanes – and engage all the ways hyperplanes can rotate in $\mathcal{C}leav_{S^n}$, in contrast to the sole translational data of $\mathcal{C}ater_{S^n}$.

Said in a different way, there is an obvious operadic map from $\mathcal{C}ater_{S^n}$ to the little intervals operad, see e.g. [MS04, ch. 2], determined by how the hyperplanes of $\mathcal{C}ater_{S^n}$ partition the x-axis into intervals. The little intervals operad has the k-ary space given as the space of embeddings of k intervals inside $[0, 1]$. We can expand these little intervals linearly until they touch each other, and hereby similarly to $\mathcal{C}ater_{S^n}(-;k)$ partitioning $[0, 1]$ into k smaller intervals. This provides a map that is a weak equivalence of coloured operads from $\mathcal{C}ater_{S^n}$ to the little intervals operad, considered as a coloured operad with trivial colours. This hence provide the A_∞- or E_1-structure on the String Product associated to M^{S^n} for the action in spectra that is constructed in [Bar11]. The rest of this section is hence devoted to determining the rest of the E_{n+1}-structure of $\mathcal{C}leav_{S^n}$.

In order to engage the combinatorics of this rotational data, i.e. define a E_{n+1}-functor, we shall prescribe explicit transformations of the hemispheres parametrizing the hyperplanes involved in the cleavages.

Definition 1.5.12 We prescribe a function $\kappa\colon S^n \times \mathbb{R}^{n+1} \to \mathrm{Hyp}^{n+1}$ by letting $\kappa(s, t)$ be given as the oriented hyperplane that contains the point $s + t$ and has s as a normal vector. In comparison to the tangent plane at the unit-sphere at $s \in S^n \subseteq \mathbb{R}^{n+1}$, $\kappa(s, t)$ has been translated by t.

[2] collapse the components of the blueprint to a single point, to make a visual link to this name

Definition 1.5.13 In the following, when referring to a sphere S^i, we shall generally consider it as sitting inside a string of inclusions

$$S^0 \overset{\iota^0}{\hookrightarrow} S^1 \overset{\iota^1}{\hookrightarrow} \cdots \overset{\iota^{n-1}}{\hookrightarrow} S^{n-1} \overset{\iota^{n-1}}{\hookrightarrow} S^n \tag{1.6}$$

where all are subsets of $\mathbb{R}^{n+1}$, where S^{i-1} is embedded equatorially into the first i coordinates of $S^i \subset \mathbb{R}^{i+1}$.

Let $S^i_+ = \{(x_0, \ldots, x_n) \in S^n \mid x_i \geqslant 0, x_{i+1} = \cdots = x_{n-1} = 0\}$ and $S^i_- = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \leqslant 0, x_{i+1} = \cdots = x_{n-1} = 0\}$. Restrictions of ι_i in (1.6), yields the partially ordered set of inclusions:

$$\begin{tikzcd}
S^0_+ \arrow[r, hook] \arrow[dr, hook] & S^1_+ \arrow[r, hook] \arrow[dr, hook] & \cdots \arrow[r, hook] \arrow[dr, hook] & S^{n-1}_+ \arrow[r, hook] \arrow[dr, hook] & S^n_+ \\
S^0_- \arrow[r, hook] \arrow[ur, hook] & S^1_- \arrow[r, hook] \arrow[ur, hook] & \cdots \arrow[r, hook] \arrow[ur, hook] & S^{n-1}_- \arrow[r, hook] \arrow[ur, hook] & S^n_-
\end{tikzcd} \tag{1.7}$$

Since these are all inclusions of closed lower-dimensional submanifolds, it is a partially ordered set of cofibrations.

Definition 1.5.14 Consider the partially ordered set I_k with objects $\alpha_j \subseteq \{1, \ldots, k\}$, where j indicates that α_j is of cardinality j, and morphisms generated by the opposite arrows of *simple inclusions* $\alpha_{j-1} \to \alpha_j$.

To $f \in I_k$, where $f\colon \alpha_j \to \alpha_p$ let the domain be denoted by $D(f) := \alpha_j$ denote the domain of f, and the target $T(f) := \alpha_p$. A simle inclusion ι_l defines a *lost number* $j_{\iota_l} := T(\iota_l) \backslash D(\iota_l) \in \{1, \ldots, k\}$.

An *i-string* of morphisms in I_k is given by a sequence $\underline{\iota} = (\iota_1, \ldots, \iota_{k-1})$ of opposite arrows of simple inclusions such that $D(\iota_r) = T(\iota_{r+1})$ – and with $D(\iota_1) = \{1, \ldots, k\}$ and $T(\iota_{k-1}) = \{i\}$.

Let $I_k|_i$ denote the set of i-strings of I_k.

Remark 1.5.15 While we shall mainly find it convenient to use the notation of 1.5.14, note that the data of $\underline{\iota} \in I_k|_i$ exactly corresponds to a permutation of $\{1, \ldots, k\}$, where we to i assign the lost number of ι_i.

We shall thus allow ourselves to consider $\underline{\iota}$ as an element of Σ_k where we here use the notation $\underline{\iota}(j) \in \{1, \ldots, k\}$ to indicate the value of j under the permutation.

Construction 1.5.16 We shall conglomerate the above constructions into a specific recursively defined function that provide the technical core in the definition of the E_n-functor.

For each $i \in \{1, \ldots, k-1\}$, we want to define a function

$$\Theta_U \colon \Sigma_k \times I_{k-1}|_i \times \left(R_U \cap \left(S^n \times \mathbb{R}^{n+1}\right)^{k-1}\right) \to \mathcal{C}\mathit{leav}_{S^n}(U; k).$$

Where $R_U \subseteq (S^n \times \mathbb{R}^{n+1})^{k-1}$, amounts to a corestriction of each $(S^n \times \mathbb{R}^{n+1})$-factor that will be specified below.

Each ι_i will specify a hyperplane via its lost number, 1.5.14, as $P_{j_{\iota_i}}$ of $\underline{P}_\sigma$.

We shall produce a U-cleaving tree that is decorated by the hyperplanes

$$\kappa(s_1, r_1), \ldots, \kappa(s_{k-1}, r_{k-1}).$$

In order to specify the U-cleaving tree that these hyperplanes decorate, we utilize the ordering from left to right along the x-axis of the hyperplanes of $\underline{P}_\sigma$, specified in 1.5.8, as well as the i-string $\underline{\iota}$.

We build this tree recursively, and start by positioning $\kappa(s_1, r_1)$ as the decoration of a 2-ary tree T_2. We hereby restrict Θ_U by letting the first $(S^n \times \mathbb{R}^{n+1})$-factor of R_U be such that $(T_2, \kappa(s_1, r_1))$ is U-cleaving.

In the recursive step, assume that we have defined the first $l-1$ factors of R_U and that we are given an l-ary U-cleaving tree $(T_l, \kappa(s_1, r_1), \ldots, \kappa(s_{l-1}, r_{l-1}))$ such that taking $l-1$ hyperplanes of $\underline{P}_\sigma$, $P_{j_{\iota_1}}, \ldots, P_{j_{\iota_{l-1}}}$, and assigning $P_{j_{\iota_i}}$ to replace the decoaration $\kappa(s_i, r_i)$, also yields a U-cleaving tree.

The hyperplane $P_{j_{\iota_l}}$ cleaves timber associated to a specific leaf of the decorated tree $(T_l, P_{j_{\iota_1}}, \ldots, P_{j_{\iota_{l-1}}})$. We graft a 2-ary tree onto T_l at this leaf, to obtain the $(l+1)$-ary T_{l+1}. We let $\kappa(s_l, r_l)$ be the decoration at the new internal vertex of T_{l+1}, where we define the l'th factor of R_U by requiring that $(s_l, r_l) \in S^n \times \mathbb{R}^{n+1}$ makes the decorated $(T_{l+1}, \kappa(s_1, r_1), \ldots, \kappa(s_l, r_l))$ U-cleaving. The timber at the leaves of the decorated tree $(T_{l+1}, \kappa(s_1, r_1), \ldots, \kappa(s_l, r_l))$ are induced by the timber at the same leafs of $(T_{l+1}, P_{j_{\iota_1}}, \ldots, P_{j_{\iota_l}})$.

We make the following restriction on R_U that will be a technical condition for the proof of 1.5.20:

(†) We intersect R_U by $\prod_{l=1}^{k-1}(S^n \times \mathrm{Rec}(U_{\kappa(s_l,r_l)}))$, where the cleaved recording area $\mathrm{Rec}(U_{\kappa(s_l,r_l)}) \subset \mathbb{R}^{n+1}$ associated to the vertex decorated by $\kappa(s_l, r_l)$ is given in 1.3.6.

Any edge e_{ij} of a graph $G \in \mathcal{K}^n(k)$ is uniquely determined as the edge attached to the vertices labelled by some ordered pair $i < j \in \{1, \ldots, k\}$. Let $\omega_G(i,j) \in \mathbb{Z}_2 =: \{'+', '-'\}$ be given by $\omega_G(i,j) =' +'$ if e_{ij} points from i to j and $\omega_G(i,j) =' -'$ if e_{ij} points from j to i.

Denote by $\lambda_G(i,j) \in \{0, \ldots, n-1\}$ the labelling of the edge e_{ij}.

Construction 1.5.17 We shall construct an E_{n+1}-functor for $\mathcal{C}leav_{S^n}$. That is, we are after a functor $D\colon \mathcal{K}^{n+1} \times S(\mathrm{Ob}(\mathcal{C}leav_{S^n})) \to S(\mathcal{C}leav_{S^n})$.

Let $G \in \mathcal{K}^{n+1}(k)$ be a graph with underlying permutation given by $\sigma_G \in \Sigma_k$

Let a i-string $\underline{\iota} \in I_{k-1}|_i$ be given, and let $l \in \{1, \ldots, k-1\}$. We consider the two lost numbers j_{ι_l} and $j_{\iota_{l-1}}$ in the sense of 1.5.14. To account for the case $l = 1$, we let $j_{\iota_0} := k$. Denote by $\iota_{l_{\max}} := \max\{j_{\iota_l}, j_{\iota_{l-1}}\}$ and $\iota_{l_{\min}} := \min\{j_{\iota_l}, j_{\iota_{l-1}}\}$.

We define the k'th operadic constituent of D as:

$$D_k(G, A_0) =$$

$$\bigcup_{a_0 \in A_0} \bigcup_{i \in \{1,\ldots,k-1\}} \bigcup_{\underline{\iota} \in I_{k-1}|_i} \Theta_{a_0}(\sigma_G, \underline{\iota}, (R_{a_0}(\iota_1), \ldots, R_{a_0}(\iota_{k-1}))), \tag{1.8}$$

where we to $\iota \in I_{k-1}|_i$ let the spaces $R_U(\iota_l)$ be a further restriction of the spaces $S^{\lambda_G(\iota_{l_{\min}},\iota_{l_{\max}})}_{\omega_G(\iota_{l_{\min}},\iota_{l_{\max}})} \times \mathbb{R}^{n+1}$ considered as the i'th input to Θ_U, that is a restriction of the space R_U of 1.5.16.

The further restriction is given by restricting to the pathcomponent of $[T_{\sigma_G}, P_{\sigma_G}]$

Here, the hemispheres $S^{\lambda_G(\iota_{l_{\min}},\iota_{l_{\max}})}_{\omega_G(\iota_{l_{\min}},\iota_{l_{\max}})}$ are given in the diagram (1.7).

1.5.3 Proof that Cleavages are E_{n+1}

To state the first lemma, note that maps $G \to G' \in \mathcal{K}^n(k)$ given by raising the index of an edge of G from i to l where $i < l$, will induce injective maps $D(G, A_0) \hookrightarrow D(G', \underline{A})$ given by a restriction of the inclusion $S^i \hookrightarrow S^l$ as one of the coordinates of $R_U(\iota_j)$ under $D_k(G, A_0) \to D_k(G', A_0)$. This describes how D is a functor of posets, and we use the following three lemmas to check the conditions (A)-(C) of 1.5.5 to prove that $\mathcal{C}leav_{S^n}$ is a coloured E_{n+1}-operad.

Lemma 1.5.18 As in (A) of 1.5.5, consider the latching space $L_{(G,A_0)} = \bigcup_{G'<G} D_k(G', A_0)$. The induced map $L_{(G,A_0)} \hookrightarrow D_k(G', A_0)$ is a cofibration

Proof. The inclusions of submanifolds in (1.7) are of codimension strictly larger than 0, so these are automatically cofibrations. The maps out of $D_k(G', A_0)$ are built out of these maps by restricting to $a_0 \in A_0$-cleaving trees, along with pushouts and factors of cartesian products. Since whether a decoration of a a_0-cleaving tree is cleaving or not is an open condition (that is, if it holds for the hyperplane it holds for a small neighborhood of the hyperplane), the associated restriction of inclusions induced from (1.7) will again be an inclusion of submanifolds that are of codimension greater than 0. Each $D_k(G', A_0) \hookrightarrow D_k(G, A_0)$ will hence result in a cofibration, that can be obtained as a lower-dimensional skeleton of a CW-structure on $D_k(G, A_0)$. Since the latching space is given by a finite union of these lower-dimensional spaces, the map from the latching space is again a cofibration. $\square$

Lemma 1.5.19 $\operatorname{colim}_{(G,A_0)\in(\mathcal{K}^{n+1}\times \mathrm{Ob}(\mathcal{C}leav_{S^n}))(k)} D_k(G, A_0) = (\mathcal{C}leav_{S^n})(-; k)$

Proof. Any $[T, \underline{P}] \in \mathcal{C}\!\mathit{leav}_{S^n}(-;k)$ can be obtained as an element of (1.8) for some choice of hemispheres determined by $G \in \mathcal{K}^n(k)$. Note namely that the definition of Θ_U is a function that exactly mimics the cleaving procedure above 1.3.5, and therefore will$[T, \underline{P}]$ as obtained by this cleaving procedure be obtained since the hemispheres involved in the image of $D_k(G, A_0)$ cover S^n, and since we in 1.5.17 are taking of path-components of $[T_\sigma, P_\sigma]$ for all $\sigma \in \Sigma_k$

□

We say that for $(T, \underline{P})$ an S^n-cleaving tree that the hyperplane of the decoration P_j *dominates* another hyperplane P_l of the decoration of T if P_j and P_l intersect within D^{n+1} and there are points of P_l that lie on $\beta_{[T,\underline{P}]}$ and on one of the subspaces of $\mathbb{R}^{n+1}$ that has been bisected by P_j, but none on the other side.

Lemma 1.5.20 Given $A_0 \in S(\mathrm{Ob}(\mathcal{C}\!\mathit{leav}_{S^n}))$ and $G \in \mathcal{K}^{n+1}(k)$, there is a homotopy equivalence $D_k(G, A_0) \simeq A_0$, where A_0 is considered as a subspace of $\mathcal{C}\!\mathit{leav}_{S^n}(-;k)$ by 1.5.9.

Another usage of terminology would be that we supply a deformation retraction onto A_0, that is not a strong deformation retraction, in that we supply a homotopy $F\colon \mathcal{C}\!\mathit{leav}_{S^n}(-;k) \times [0,1] \to \mathcal{C}\!\mathit{leav}_{S^n}(-;k)$, where we only for $t \in \{0,1\}$ guarantee $F(a_0, t) = a_0$ for $a_0 \in A_0 \subset \mathcal{C}\!\mathit{leav}_{S^n}(-;k)$ included as above.

Proof. We break the proof, specifying the homotopy into three steps. Our overall strategy will be that we in the first step show that the effect of different elements a_0 of $\mathrm{Ob}(\mathcal{C}\!\mathit{leav}_{S^n})$ as input to can be neglected, as we similar to the proof of 1.3.19 can push the defining hyperplanes towards tangent-hyperplanes of S^n.

The second step will show that for a given $\iota \in I_{k-1}|_i$, the space

$$\Theta_{a_0}(\sigma_G, \underline{\iota}, (R_{a_0}(\iota_1), \ldots, R_{a_0}(\iota_{k-1}))$$

given as one of the constiuents in the union of (1.8) is contractible. Again, the overall idea will be similar to 1.3.19, in that we push hyperplanes close enough to be tangential to S^n that all points of the hemisphere gives rise to a cleaving element. However, in pushing these one needs to take extra care since the pushing needs to go along cleaving elements of a_0 for more than one output timber.

In the final step, we show that the contractible spaces of step 2 are all glued together along a contractible space containing the element $[T_{\sigma_G}, \underline{P_{\sigma_G}}]$, hence making the entire $D_k(G, A)$ contractible.

Step 1: Assume that the labelling of the edges of $G \in \mathcal{K}^{n+1}(k)$ satisfy $\lambda_G(i,j) = 0$ for all $i < j \in \{1, \ldots, k\}$, that is all edges of G are labelled by 0. In this case, we have that $D_k(G, A_0)$ will pointwise in A_0 be the space of all sets of $k-1$ hyperplanes $P_1, \ldots, P_{k-1}$ orthogonal to the first coordinate axis of $\mathbb{R}^{n+1}$ that cleaves $a_0 \in A_0$, where the parallel hyperplanes are ordered as P_{σ_G} of 1.5.8, and $\sigma_G \in \Sigma_k$ is the permutation associated to G.

The space $D_k(G, A_0)$ is homotoped onto A_0 by considering the interval $J_{a_0} \subseteq [-1, 1]$ of the first coordinate axis of $\mathbb{R}^{n+1}$ as given in 1.5.10. Homotoping $P_1, \ldots, P_k$ to be equidistant within J_{a_0} for each $a_0 \in A_0$ yields a deformation retraction onto A_0 in the sense of 1.5.9, since the topology of $\mathrm{Ob}(\mathcal{C}leav_{S^n})$ as determined by hyperplanes forming the timber lets the end-points of J_{a_0} – as points in $\mathbb{R}$ vary continuously as functions of $\mathrm{Ob}(\mathcal{C}leav_{S^n})$.

Step 2: For a general $G \in \mathcal{K}^n(k)$ and a fixed i-string $\underline{\iota}$, and $a_0 \in A_0$, we see that the space $\Theta_{a_0}(\sigma_G, \underline{\iota}, (R_{a_0}(\iota_1), \ldots, R_{a_0}(\iota_{k-1})))$ is weakly equivalent to a product of hemispheres, considered as a subspace of $\mathcal{C}leav_{S^n}(-; k)$.

We homotope Θ in $k-1$ steps according to its recursive definition of 1.5.16. For the first step, the points $(s_1, r_1) \in R_{a_0}(\iota_1)$ defines a hyperplane by $\kappa(s_1, r_1)$ that cleaves a_0. We define a function $\mu\colon \mathrm{Ob}(\mathcal{C}leav_{S^n}) \times R_{a_0}(\iota_1) \times [0,1] \to \mathrm{Ob}(\mathcal{C}leav_{S^n})$, such that for any $s' \in S^n$, $\kappa(s', r_1)$ will cleave $\mu(a_0, (s_1, r_1), 1) \in \mathrm{Ob}(\mathcal{C}leav_{S^n})$.

In order to do define this function, note that a_0 as an element of $\mathrm{Ob}(\mathcal{C}leav_{S^n})$

is defined by choosing some hyperplanes $H_1, \ldots, H_r$ that cleave S^n. For any one of these r hyperplanes, H_i, there is a well-defined normal-vector ν_i that points away from the hyperplane $\kappa(s_1, r_1)$. Understood in the sense that translating H_i in the direction of ν_i will leave $\kappa(s_1, r_1)$ cleaving. Similar to the proof of 1.3.19, we can therefore define the function μ by translating the hyperplanes H_i simultaneously in the direction of ν_i, we do this until the hyperplanes are close enough to tangent-hyperplanes of S^n, formally requiring that the minimal distance between $H_i \cap S^n$ and $H_j \cap S^n$ is larger than some given $\varepsilon > 0$ for all $i < j \in \{1, \ldots, r\}$. That any $s' \in S^n$ hereby will have $\kappa(s', r_1)$ cleave $\mu(a_0, (r_1, r_1), 1)$ can be seen by 1.3.11 since the complement of this subspace of S^n will consist of disjoint disks, and by (†) of 1.5.16, $r_1 \in \mathrm{Rec}(a_0) \subseteq \mathrm{Rec}(a_1)$ so $\kappa(s', r_1)$ will always intersect a_1 non-trivially.

This hence defines $a_1 := \mu_1(a_0, (s_1, r_1), 1) \in \mathrm{Ob}(\mathcal{Cleav}_{S^n})$. In a very similar fashion, we wish to further deform the $k-1$ hyperplanes involved in the recursively defined Θ_{a_0}. To this end, assume we are given the hyperplanes $H_1, \ldots, H_{r+l}$ where for all $i, j < r + l$ where $i \neq j$ $H_i \cap S^n$ and $H_j \cap S^n$ are of distance at least ε to $H_i \cap S^n$. We can take these $r + l$ as data for the l'th step of a deformation of $\Theta_{a_0}(R_{a_0}(\iota_1), \ldots, R_{a_0}(\iota_{k-1}))$, where the first r hyperplanes defines timber as defined under μ above, and the remaining l are hyperplanes that cleave this timber. By assumption in the definition of Θ_{a_0}, we are given a hyperplane $\kappa(s_{l+1}, r_{l+1})$ that cleave some $a_l \in \mathrm{Ob}(\mathcal{Cleav}_{S^n})$ as defined through a portion of the hyperplanes $H_1, \ldots, H_{l+r}$.

Only choosing $j := l + r$ will potentially satisfy $H_j \cap H_i \cap S^n \neq \varnothing$, and we therefore wish to push H_{r+l} in the direction of the normal-vector ν_{r+l}, away from $\kappa(s_{l+1}, r_{l+1})$. Blindly pushing $\kappa(s_{l+1}, r_{l+1})$ in the direction of ν_{r_l} will deform a_l to incorporate more hyperplanes than the ones used to defining a_l, and we need to ensure continuity with respect to these hyperplanes. Let therefore again ν_i denote the normal-vector of H_i in the direction away from $\kappa(s_{l+1}, r_{l+1})$. Let similarly $\mathrm{dist}(H_i) \in \mathbb{R}_+$ denote the distance by the hyperplane H_i must be translated along ν_i to become a hyperplane, let $A(U) \in \mathbb{R}_+$

denote the area of an $n-1$-dimensional subspace inside S^n. Assume that $H_{r+l} \cap S^n$ and $H_i \cap S^n$ are within ε distance of each other; potentially intersecting, for $i \in K_{H_{r+l}} \subseteq \{1, \ldots, l+r-1\}$. We push the hyperplanes indexed by $K_{H_{r+l}}$ and H_{r+l} simultaneously to obtain hyperplanes $H_{r+l}(t)$ and $H_i(t)$ where $i \in K_{H_{r+l}(t)}$ at time t, hereby forming the timber $a_l(t)$. Let as usual $\partial\overline{a_l(t)}$ denote the boundary of the closure of a_l within S^n. We push H_i in the direction of ν_i with with speed $\operatorname{dist}(H_i(t)) \cdot (1 - \frac{A(\partial\overline{a_l(t)} \cap H_i(t)}{A(S^n \cap H_i(t))})$ while H_{r+l} is pushed with speed $\operatorname{dist}(H_{r+l}(t)) \cdot \max_{i \in K_{H_{r+l}}}(\frac{A(\partial\overline{a_l(t)} \cap H_i(t)}{A(S^n \cap H_i(t))})$, and we stop once $H_{r+l}(t)$ is at distance at least $\varepsilon \cdot \min_{i \in \{1,\ldots,r+l-1\}} \operatorname{dist}(H_i(t))$ from $H_i(t)$ for all $i \in \{1, \ldots, r_{r+l-1}\}$.

These formula ensure that H_{r+l} is only pushed past the subspace $H_i \cap S^n$ if there is a significant portion of this subspace forming part of $\partial\overline{a_l}$. Hereby, we hence obtain the timber a_l that $\kappa(s', r_{l+1})$ cleaves for all $s' \in S^n$, since the complement $\complement a_l$ will consist of a disjoint union of subdisks of S^n.

For $l = k-1$, this hence gives a subspace of $\mathcal{Cleav}_{S^n}(-;k)$ parametrized by the space $\prod_{l=1}^{k-1}(S_{\omega_{\sigma_G}(\iota_{l_{\min}},\iota_{l_{\max}})}^{\lambda_{\sigma_G}(\iota_{l_{\min}},\iota_{l_{\max}})} \times \operatorname{Rec}(a_l))$, where we again use (†) of 1.5.16, to identify the parametrizing $\mathbb{R}^{n+1}$-factor. Since $\operatorname{Rec}(U)$ can be seen to be contractible, in fact convex, for any $U \in \operatorname{Ob}(\mathcal{Cleav}_{S^n})$, the subspace to our given i-string is weakly equivalent to a product of hemispheres. I.e. it is weakly contractible.

Step 3: Given $\underline{\iota}$ a i-string, and $\underline{\lambda}$ a l-string, we show that the associated spaces of $\underline{\iota}$ and $\underline{\lambda}$,

$$\Theta_{a_0}(\sigma_G, \underline{\iota}, (R_{a_0}(\iota_1), \ldots, R_{a_0}(\iota_{k-1}))) \text{ and } \Theta_{a_0}(\sigma_G, \underline{\lambda}, (R_{a_0}(\lambda_1), \ldots, R_{a_0}(\lambda_{k-1})))$$

respectively, are glued together along contractible subsets, where we consider these as subsets of $\mathcal{Cleav}_{S^n}(-;k)$. Since step 2 tells us that the space associated to $\underline{\iota}$ and the one associated to $\underline{\lambda}$ are contractible, we have that the glued spaces are contractible. This follows combining the Mayer-Vietoris sequence, Hurewicz map and the Van Kampen theorem so that if A and B are two contractible spaces and $A \cap B$ is contractible then $A \cup B$ is again a contractible space. As we shall see, all spaces associated to elements of $\bigcup_{i=1}^{k-1} I_i$ will be glued together along the common basepoint determined by the hyperplanes $\underline{P}_{\sigma_G}$ orthogonal to the first coordinate axis of $\mathbb{R}^{n+1}$ as determined by the orientation of G. In effect, the glueing of all these spaces will result in a contractible space.

In order to see how the spaces are glued together, we see in 1.5.16 that the spaces associated to $\underline{\iota}$ and $\underline{\lambda}$ are given by choosing new normal-vectors for the hyperplanes $P_1, \dots, P_k$ of $\underline{P}_{\sigma_G}$ and decorating them on different trees.

Considering $\underline{\iota}$ and $\underline{\lambda}$ as permutations in Σ_k as given in 1.5.15, it a necessary but not sufficient condition that points in the spaces associated to $\underline{\iota}$ and $\underline{\lambda}$ determined by the tuppels $((s_1, r_1), \dots, (s_{k-1}, r_{k-1})$ and $((s'_1, r'_1), \dots, (s'_{k-1}, r'_{k-1}))$ satisfy

$$(s_{\underline{\iota}^{-1}(i)}, r_{\underline{\iota}^{-1}(i)}) = (s'_{\underline{\lambda}^{-1}(i)}, r'_{\underline{\lambda}^{-1}(i)}) \tag{1.9}$$

for all $i \in \{1, \dots, k-1\}$, since the associated timber of the cleaving trees have to agree.

The condition (1.9) is not sufficient since the way the hyperplanes dominate each other might be different according to the two trees that the hyperplanes are decorating. Note first of all that if (1.9) is satisfied, and $s_{\underline{\iota}^{-1}(i)} = s_{\underline{\lambda}^{-1}(i)}$ is the point $S^0_{\omega(\iota(i)}$ all the hyperplanes are parallel and orthogonal to the first coordinate axis, where there is no dominance amongst hyperplanes, identifying the points.

We have an ordering on the hyperplanes of the spaces associated to $\underline{\iota}$ and $\underline{\lambda}$, given by $P_{\underline{\iota}(1)}, \dots, P_{\underline{\iota}(k-1)}$ and $P_{\underline{\lambda}(1)}, \dots, P_{\underline{\lambda}(k-1)}$. Assume we are given $[T, P_{\underline{\iota}}]$ and $[T', P_{\underline{\lambda}}]$ satisfying (1.9) such that they agree as elements of $D_k(G, A_0)$. We want to show that there is a unique path homotoping them onto trees decorated by hyperplanes orthogonal to the first coordinate axis, in which case step 1 applies to provide the deformation retraction.

For the hyperplane $P_{\underline{\iota}(1)}$, no other hyperplanes $P_{\underline{\iota}(2)}, \dots, P_{\underline{\iota}(k-1)}$ will dominate $P_{\underline{\iota}(1)}$.

We can use step 2 to assume that for the 2-ary tree decorated by only $P_{\underline{\iota_1}}$, the reparametization of the hyperplane along the geodesic path along the hemisphere $S^{\lambda_G(\iota_{1_{\min}},\iota_{1_{\max}})}_{\omega_G(\iota_{1_{\min}},\iota_{1_{\max}})}$ will always be contained in $\mathcal{C}leav_{S^n}(-;2)$.

However, we need to consider these paths for our trees decorated by multiple hyperplanes. That we in 1.5.17 are taking path-components tells us that the geodesic path along the hemisphere $S^{\lambda_G(\iota_{1_{\min}},\iota_{1_{\max}})}_{\omega_G(\iota_{1_{\min}},\iota_{1_{\max}})}$ parametrizing $P_{\underline{\iota}(1)}$ will be contained in the space associated to $\underline{\iota}$. Since the elements $[T, P_{\underline{\iota}}]$ and $[T', P_{\underline{\lambda}}]$ agree, the hyperplane $P_{\underline{\lambda}^{-1}(\underline{\iota}(1)}$ as a decoration of $[T', \underline{P_{\lambda}}]$ will be bound to dominate the same hyperplanes as $P_{\underline{\iota}(1)}$ as an element of $[T, \underline{P_{\iota}}]$, and this will remain true along the reparametization along the geodesic, until $P_{\underline{\iota}(1)}$ no longer dominates any other hyperplanes, since otherwise either $[T, \underline{P_{\iota}}]$ or $[T, \underline{P_{\lambda}}]$ would be reparametrized to non-cleaving trees. To ensure that no new dominance occurs after the potential step where $P_{\underline{\iota}(1)}$ no longer dominates any other hyperplanes, one translates along the first coordinate of $\mathbb{R}^{n+1}$ in $S^{\lambda_G(\iota_{1_{\min}},\iota_{1_{\max}})}_{\omega_G(\iota_{1_{\min}},\iota_{1_{\max}})} \times \mathbb{R}^{n+1}$ at the same speed in the opposite direction as $P_{\underline{\iota}(1)}$ is moving away from the hyperplanes it used to dominate.

This eventually brings the hyperplanes $P_{\underline{\iota}(1)}$ parallel with the first coordinate axis of $\mathbb{R}^{n+1}$, and one iterates this construction for the remaining $P_{\underline{\iota}(2)}, \dots, P_{\underline{\iota}(k-1)}$ in that order, to – along a geodesic in the parametrizing hemispheres – bring them all parallel to the first coordinate axis. Finally, having used step 2, one applies the inverse homotopy of μ in step 2, to make these hyperplanes cleave a_0.

As noted previously, step 1 now applies to finish the proof. □

Theorem 1.5.21 $\mathcal{C}leav_{S^n}$ is a coloured E_{n+1}-operad.

Proof. The lemmas 1.5.18, 1.5.19, 1.5.20 check (A)-(C) in 1.5.5 the functor D of 1.5.17 should satisfy in order for $\mathcal{C}leav_{S^n}$ to be E_{n+1}. Note that 1.5.20 indeed gives the full naturality as stated in (B), since we give an explicit homotopy equivalence onto $\mathrm{Ob}(\mathcal{C}leav_{S^n}) \subset \mathcal{C}leav_{S^n}(-;k)$ that respects morphisms of $S(\mathrm{Ob}(\mathcal{C}leav_{S^n}))$. □

Corollary 1.5.22 There is an equivalence of operads

$$\mathrm{H}_*(\mathcal{C}\mathit{leav}_{S^n}) \cong \mathrm{H}_*(\mathcal{D}\mathit{isk}_{n+1})$$

Proof. By 1.3.19 the colours of $\mathcal{C}\mathit{leav}_{S^n}$ are contractible, and by further 1.3.20, we can apply 1.2.5, so $\mathrm{H}_*(\mathcal{C}\mathit{leav}_{S^n})$ is a monochrome operad and 1.5.21 together with 1.5.7 gives the corollary. □

1.6 Deforming Cleavages Symmetrically

Given a topological space, X, we let $\mathrm{Homeo}(X)$ denote the group of self-homeomorphisms of X.

Similar to having a monochrome G-operad in the category of G-spaces, for coloured topological operads we give the following definition

Definition 1.6.1 A topological group G *acts* on a coloured topological operad, $\mathcal{O}$ if we are given

- $.\colon G \to \mathrm{Homeo}(\mathrm{Ob}(\mathcal{O}))$ a continuous map. Given $g \in G$ and $U \in \mathrm{Ob}(\mathcal{O})$, under adjunction, we denote the corresponding acted upon element as $g.U$.

- $\alpha_i\colon G \to \mathrm{Homeo}(\mathcal{O}(-;m))$ continous maps for all $i \in \{1,\ldots,m\}$ and $m \in \mathbb{N}$.

and these respect the topological structure of operads; i.e. letting $g \in G$ the following diagram should be commutative for all $i \in \{1,\ldots,k\}$ and $j \in$

$\{1, \ldots, m\}$:

$$
\begin{tikzcd}
 & & \mathcal{O}(A; k+m-1) \arrow[ddll, "\alpha_{i+j-1}(g)"'] & \\
 & & \mathcal{O}(A; k) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-; m) \arrow[u, "\circ_i"'] \arrow[r] \arrow[d] \arrow[ddll] & \mathcal{O}(-; m) \arrow[d, "\mathrm{ev_{in}}"] \arrow[ddl, "\alpha_j(g)"] \\
\mathcal{O}(A; k+m-1) & & \mathcal{O}(A; k) \arrow[r, "\mathrm{ev}_i"] \arrow[ddll, "\alpha_i(g)"'] & \mathrm{Ob}(\mathcal{O}) \arrow[ddl, ".g"] \\
\mathcal{O}(A; k) \times_{\mathrm{Ob}(\mathcal{O})} \mathcal{O}(-; m) \arrow[u, "\circ_i"'] \arrow[rr] \arrow[d] & & \mathcal{O}(-; m) \arrow[d, "\mathrm{ev_{in}}"] & \\
\mathcal{O}(A; k) \arrow[rr, "\mathrm{ev}_i"] & & \mathrm{Ob}(\mathcal{O}) &
\end{tikzcd}
\tag{1.10}
$$

In the classical setting, when $\mathcal{O}$ is a monochrome operad, this recovers the notion of a G-operad. In this spirit we call a coloured operad satisfying the above a *coloured G-operad*.

In [SW03, 2.1], semidirect products of monochrome operads are introduced, and we can expand the notion to the coloured setting by only expanding a little on the operadic evaluation maps:

Definition 1.6.2 For a coloured topological operad $\mathcal{O}$ with an action of a group G, we can form the *semidirect product* of $\mathcal{O}$ by G, as for the monochrome setting denoted $\mathcal{O} \rtimes G$, by letting

- $\mathrm{Ob}(\mathcal{O} \rtimes G) = \mathrm{Ob}(\mathcal{O})$
- $\mathcal{O} \rtimes G = \mathcal{O}(-; k) \times G^k$

Letting ev_i and $\mathrm{ev_{in}}$ denote the operadic evaluation maps of $\mathcal{O}$. The operadic evaluation maps ev_i^G and $\mathrm{ev}_{\mathrm{in}}^G$ for the semidirect product are given by $\mathrm{ev}_i^G(\omega, \rho_1, \ldots, \rho_k) = \rho_i . \mathrm{ev}_i(\omega)$ and $\mathrm{ev}_{\mathrm{in}}^G = \mathrm{ev_{in}}$. As for the monochrome case, the operadic composition $\circ_i \colon (\mathcal{O} \rtimes G)(-; k) \times_{\mathrm{Ob}(\mathcal{O})} (\mathcal{O} \rtimes G)(-; m) \to$

$(\mathcal{O} \rtimes G)(-; k+m-1)$ is given by twisting the composition of $\mathcal{O}$ through the action of G in the following sense:

$$(\omega; \rho_1, \ldots, \rho_k) \circ_i (\omega'; \eta_1, \ldots, \eta_m) = (\omega \circ_i \rho_i . \omega'; \rho_1, \ldots, \rho_{i-1}, \rho_i . \eta_1, \ldots, \rho_i . \eta_m, \rho_{i+1}, \ldots, \rho_k)$$

Observation 1.6.3 As a gadget constructed from $\mathbb{R}^{n+1}$, there is an induced action of $\mathrm{SO}(n+1)$ on $\mathcal{C}\mathit{leav}_{S^n}$, precisely:

To an affine oriented hyperplane $P \in \mathrm{Hyp}^{n+1}$ and $\rho \in \mathrm{SO}(n+1)$, letting ρ act on $\mathbb{R}^{n+1}$ by rotation leads P to the affine oriented hyperplane $\rho.P$. Hereby $\mathrm{SO}(n+1)$ acts on the space Hyp^{n+1}.

There is also an action of $\mathrm{SO}(n+1)$ on $\mathrm{Ob}(\mathcal{C}\mathit{leav}_{S^n})$ rotating timber $U \in \mathrm{Ob}(\mathcal{C}\mathit{leav}_{S^n})$ obtained by cleaving S^n along some hyperplanes to the timber $\rho.U$ obtained by cleaving S^n along the hyperplanes rotated along ρ.

Let $[T, \underline{P}] \in \mathcal{C}\mathit{leav}_{S^n}(U; k)$, for some $U \in \mathrm{Ob}(\mathcal{C}\mathit{leav}_{S^n})$. Let $\rho.[T, \underline{P}] = [T, \rho.\underline{P}]$, understood in the sense that ρ rotates the decorations of T simultaneously. Having $[T, \underline{P}]$ an element of $\mathcal{C}\mathit{leav}_{S^n}(U; k)$ this should be interpreted in the sense that $\rho.[T, \underline{P}]$ is an element of $\mathcal{C}\mathit{leav}_{S^n}(\rho.U; k)$, ensuring that the decorated tree cleaving $\rho.U$ and in turn making this action of $\mathrm{SO}(n+1)$ on $\mathcal{C}\mathit{leav}_{S^n}$ satisfy 1.6.1.

The action of 1.6.3 defines the semidirect product of $\mathcal{C}\mathit{leav}_{S^n}$ by $\mathrm{SO}(n+1)$, in the sense of 1.6.2.

Observation 1.6.4 We can extend the action of $\mathcal{C}\mathit{leav}_{S^n}$ along correspondances as given in 1.4.4 to an action of $\mathcal{C}\mathit{leav}_{S^n} \rtimes \mathrm{SO}(n+1)$ on M^{S^n}. We do this by considering $([T, \underline{P}], \rho_1, \ldots, \rho_k)$ an element of $(\mathcal{C}\mathit{leav}_{S^n} \rtimes \mathrm{SO}(n+

1))(−; k) and take the following pullback-diagram:

$$\begin{tikzcd} M^{S^n}_{[T,\underline{P}]} \arrow[r, "\varphi^*_{\underline{\rho}}"] \arrow[d] & \left(M^{S^n}\right)^k \arrow[d, dashed, "\mathrm{res}_{\underline{\rho}}"] \\ M^{\pi_0(\widehat{\beta_{[T,\underline{P}]}})} \arrow[r, "\varphi"] & \prod_{i=1}^k M^{(\complement N_i)} \end{tikzcd}$$

Where all spaces are as in 1.4.4, as is the map φ. However, the *twisted restriction map* $\mathrm{res}_{\underline{\rho}}$ is given at the i'th factor of $\left(M^N\right)^k$ as $\mathrm{res}_{\complement N_i} .\rho_i^{-1}$. Here the map $\mathrm{res}_X : M^{S^n} \to M^X$ denotes the restriction map to the space $X \subseteq S^n$. The element $\rho_i \in \mathrm{SO}(n+1)$ is considered as a diffeomorphism of S^n, and $\mathrm{res}_X .\rho_i^{-1}$ denotes the precomposition of ρ_i^{-1} of the domain of $f \in M^{S^n}$ prior to applying the restriction map. This preapplication of ρ_i^{-1} allows us to consider $\mathrm{res}_{\complement N_i} .\rho_i^{-1}$ as a map that takes points of $\rho_i(\complement N_i) \subseteq S^n$ and brings these to $\complement N_i$ where a restriction map is subsequently applied.

Note that whereas this allows us to let φ be given as the same morphism as in 1.4.4, which in turn allows us to again identify the pullback space $M^{S^n}_{[T,\underline{P}]}$ as maps from S^n to M that are constant along the blueprint of $[T,\underline{P}]$. However, the fact that we are applying a twisted restriction map means that the associated map $\varphi^*_{\underline{\rho}}$ in the pullback will be different from 1.4.4. Concretely, $\varphi^*_{\underline{\rho}}$ is given by having the i'th image maps from S^n to M that are constant along $\rho_i(\complement N_i)$.

Note that this description of $\varphi^*_{\underline{\rho}}$ makes this action of correspondanes respect the operadic composition, in the sense that for $([T,\underline{P}],\rho_1,\ldots,\rho_k)$ and $([T',\underline{P}'],\eta_1,\ldots,\eta_m)$ $\circ_i$-composable as elements of $\mathcal{C}\!\mathit{leav}_{S^n} \rtimes \mathrm{SO}(n+1)$, the change of colours by $\rho_i : S^n \to S^n$ in the operadic composition of the semidirect product makes it necessary for commutativity of operadic associatativity diagrams to map $M^{S^n}_{[T,\underline{P}]\circ_i\rho_i.[T',\underline{P}']}$ to the $i+j-1$'th factor of $\left(M^{S^n}\right)^{k+m-1}$ that are constant along the subspace $\eta_j(\rho_i(\complement N_{i+j-1})) \subseteq S^n$, where $j \in \{1,\ldots,m\}$ and $i \in \{1,\ldots,k\}$.

Proposition 1.6.5 Actions of $\mathrm{H}_*(\mathcal{C}\!\mathit{leav}_{S^n} \rtimes \mathrm{SO}(n+1))$ are BV_{n+1}-algebras, when $\mathrm{H}_*(-)$ denotes homology with coefficients in a field.

Proof. Since $\mathcal{Cleav}_{S^n}$ is a coloured E_{n+1}-operad; 1.5.21, with contractible colours as; 1.3.19, we have that $\mathrm{H}_*(\mathcal{Cleav}_{S^n})$ is in particular a quadratic operad, with 3-ary operations relations determined by the Gerstenhaber relations.

Hereby, the statement follows directly by applying [SW03, 4.4], to see that operadic actions of $\mathrm{H}_*(\mathcal{Cleav}_{S^n} \rtimes \mathrm{SO}(n+1))$ agrees with operadic actions of $\mathrm{H}_*(\mathcal{Disk}_{n+1} \rtimes \mathrm{SO}(n+1))$.

□

It is natural to conjecture that there is a weak equivalence of operads $\mathcal{Cleav}_{S^n} \rtimes \mathrm{SO}(n+1) \simeq \mathcal{Disk}_{n+1} \rtimes \mathrm{SO}(n+1)$. However the string of equivalences of 1.5.5 does not produce an equivalence extending 1.5.21 to the setting of BV_{n+1}-operads. One notices that the intermediate terms given in 1.5.5 involves the nerve of the full graph operad. Heurestically, this nerve construction does not see the action of $\mathrm{SO}(n+1)$ on $\mathbb{R}^{n+1}$. One option for proving such an extension of 1.5.21 could be to attempt at making $\mathrm{SO}(n+1)$-equivariant versions of the nerve of the full graph operad. We shall however not go into these considerations in this thesis.

Chapter 2

Punctured Cleavages and Umkehr Maps

2.1 Introduction

The exposition of this chapter is still work in progress, however we go into detail with the action of $\mathcal{Cleav}_{S^n}$ on M^{S^n}. That is, we shall give a first sketch of a proof of the following theorem

Theorem C For M a compact orientable manifold, there are (stable) maps in the category of spectra

$$\Big(\mathcal{Cleav}_{S^n}(-;k)\Big) \times \left(M^{S^n}\right)^k \to \left(M^{S^n}\right) \wedge S^{\dim(M)\cdot(k-1)} \tag{2.1}$$

such that taking homology, we get an action of $\mathrm{H}_*(\mathcal{Cleav}_{S^n})$ on $\mathbb{H}_*(M^{S^n})$.

which is done in 2.4.9 of this section.

Naïvely, we seek – for any $[T,\underline{P}] \in \mathcal{Cleav}_{S^n}(-;k)$ – to provide umkehr maps $\varphi^!$ for the correspondance $M^{S^n} \longleftarrow M_{[T,\underline{P}]} \xrightarrow{\varphi} \left(M^{S^n}\right)^k$ described in Theorem A of section 1.4 in chapter 1. For a fixed $[T,\underline{P}] \in \mathcal{Cleav}_{S^n}(-;k)$, there are several methods available in the litterature for obtaining such an umkehr map for a fixed $[T, \underline{P}]$, and as a homotopical application of [Kle01] we give one such method in section 2.3.

However, being able to construct umkehr maps pointwise for a single element $[T, \underline{P}]$ only hints at the potential for the map (2.1), it does not automatically imply that these umkehr map are parametrized by $\mathcal{C}\!\mathit{leav}_{S^n}(-;k)$ as a topological entity. The bulk of this chapter is aimed at patching such umkehr maps into a global umkehr map.

The main idea will be to find manifolds for which the desired umkehr maps exist, and then glue the associated umkehr maps together to an action. It turns out that working directly with $\mathcal{C}\!\mathit{leav}_{S^n}$ has technical disadvantages in that the associated spaces does not break nicely into acting compact manifolds. Therefore we are lead to consider the larger punctured cleavage operad $\overrightarrow{\mathcal{C}\!\mathit{leav}}_{S^n}$, which essentially allows for the cleaving hyperplanes to be tangential to S^n.

Having cleaving hyperplanes become tangential to S^n means that we allow a point in the k-ary spaces of the coloured operad $\overrightarrow{\mathcal{C}\!\mathit{leav}}_{S^n}(-;k)$ to be a set consisting of cleaving hyperplanes, along with some points, or punctures, decorated with elements of the commutative operad; where the arity of the commutative operad counts the amount of tangential hyperplanes present in the point. The topology of $\overrightarrow{\mathcal{C}\!\mathit{leav}}_{S^n}(-;k)$ is rigged such that the punctures can transfer between being points and being hyperplanes cleaving S^n. This transfer is illustrated in picture 2.1, where we have shown four steps in a path of $\overrightarrow{\mathcal{C}\!\mathit{leav}}_{S^n}(S^n;7)$.

As indicated in picture 2.1, punctures can slide past cleaving hyperplanes and other punctures with no complication. This simplifies the homotopical type compared to $\mathcal{C}\!\mathit{leav}_{S^n}$, in that we have the following:

Theorem D There is a homotopy equivalence

$$\overrightarrow{\mathcal{C}\!\mathit{leav}}_{S^n}(-;k) \simeq (S^n)^k \,.$$

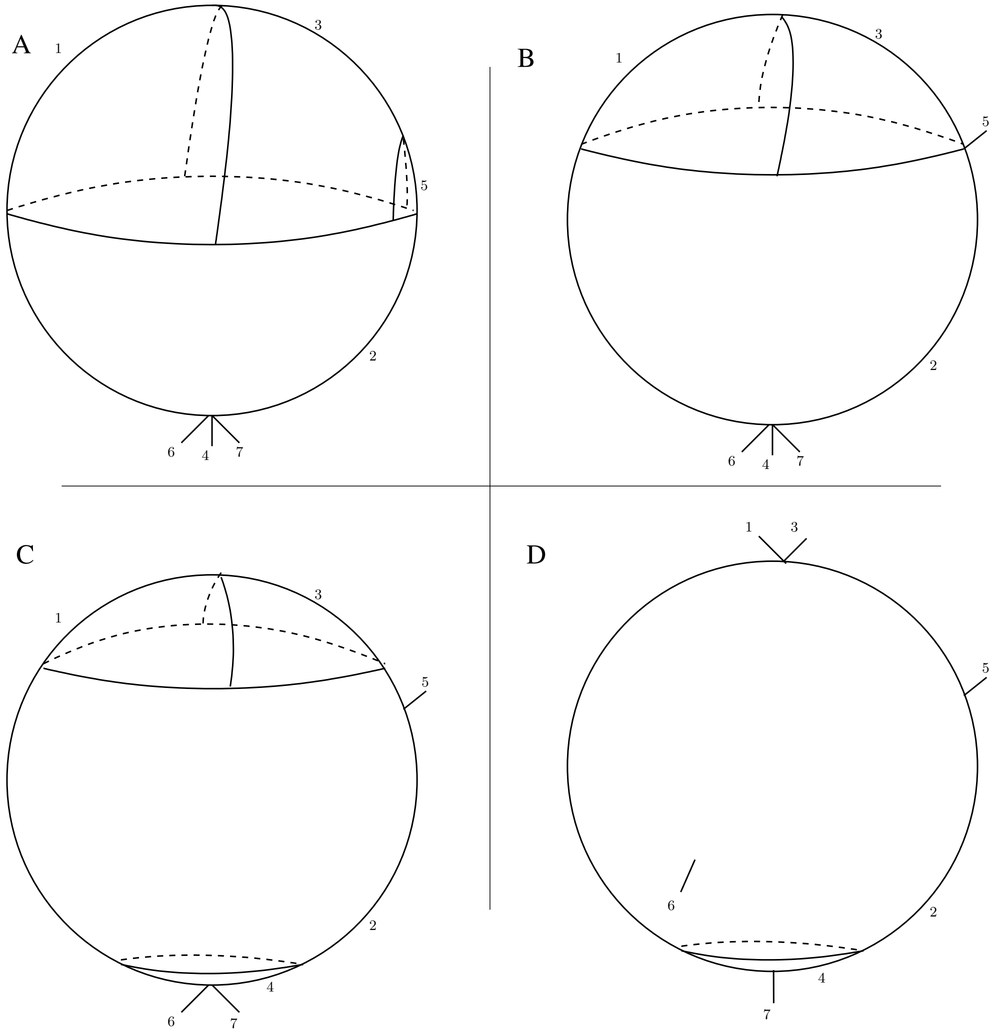

Figure 2.1: Starting from (A) the central cleaving hyperplane moves towards the top, transferring cleaving hyperplanes to punctures along the way. From (C) the multiplicity of punctures is indicated by a decrease in the arity of the associated commutative operad

To state the advantages briefly, we show in 2.4.2 how the homotopically simpler $\overrightarrow{\mathcal{C}leav}_{S^n}(-;k)$ break up into products of spheres that have the desired umkehr map. And since it breaks up into acting Poincaré duality spaces, the homotopy theory explained in 2.3 can be used to show that the umkehr maps patch together in a suitable homotopy colimit. Restricting this umkehr map to $\mathcal{C}\mathit{leav}_{S^n}$ we can compose the umkehr map with a continous map to M^{S^n} – providing the promised action of chapter 1.

As mentioned, the point of this chapter is to introduce $\overrightarrow{\mathcal{C}\mathit{leav}}_{S^n}$ as a gadget for obtaining the action of $\mathcal{C}\mathit{leav}_{S^n}$. However as a byproduct of this construction, we in 2.4.9 also get an action of the larger $\overrightarrow{\mathcal{C}\mathit{leav}}_{S^n}$.

However, it should be emphasized that the construction of $\mathcal{C}\mathit{leav}_{S^n}$ we gave in the previous chapter will have its actions by non-unital algebra, and there are no 0-ary operations of $\mathcal{C}\mathit{leav}_{S^n}$. There appears to be no way to append unit elements of the algebra of $\mathcal{C}\mathit{leav}_{S^n}$, without destroying the operadic composition inside $\mathcal{C}\mathit{leav}_{S^n}$.

In $\overrightarrow{\mathcal{C}\mathit{leav}}_{S^n}$, there are 0-ary operations forgetting the punctures, and it turns out that the enlargement of the action of $\mathcal{C}\mathit{leav}_{S^n}$ to an action of $\overrightarrow{\mathcal{C}\mathit{leav}}_{S^n}$ is exactly given by adjoining units to the Batalin-Vilkovisky algebra. This appears to emphasize the importance of the Batalin-Vilkovisky structure of String Topology as a non-unital algebra.

This chapter is structured with section 2.2 introducing the punctured cleavage operad $\overrightarrow{\mathcal{C}\mathit{leav}}_{S^n}$, section 2.3 introduces the homotopy theory needed for providing umkehr maps over compact manifolds, and section 2.4 proves Theorems C and D, and goes through the argument of how we in the case of cleavage operads are capable of forming umkehr maps over the entire operad, not just a k-ary element hereof.

Acknowledgement

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2.2 The Punctured Cleavage Operad

A point $[T, \underline{P}] \in \mathcal{Cleav}_{S^n}(-;k)$ is determined by the set of hyperplanes $\{P_1, \ldots, P_{k-1}\}$ decorated on the tree T, subject to an equivalence relation. As mentioned in the introduction, we extend $\mathcal{Cleav}_{S^n}$ through a larger coloured operad $\overrightarrow{\mathcal{Cleav}}_{S^n}$ with marked points mimicking the cleaving hyperplanes $\underline{P}$ vanishing towards tangentplanes of S^n.

Definition 2.2.1 Given $U \in \text{Timber}_{S^n}$, we let the *open puncture U-operad* with k-ary term be given by the space

$$\mathcal{Pun}_U^\circ(k) := \coprod_{k=k_1+\cdots+k_j} \text{Conf}_j(U) \times \mathcal{Comm}(k_1) \times \cdots \times \mathcal{Comm}(k_j)$$

Here, the coproduct is taken over all partitions of k, $\text{Conf}_j(U)$ denotes the ordered configuration space on j points and $\mathcal{Comm}(k_i)$ is the commutative operad with k_i outputs. We relabel the outputs of each $\mathcal{Comm}(k_i)$ monotonely by the elements $\left(\sum_{m=1}^{i-1} k_m\right) + 1, \ldots, \left(\sum_{m=1}^{i-1} k_m\right) + k_i$, and let $\circ_m$-composition induced by the operadic composition of the component $\mathcal{Comm}(k_i)$ having an output labelled by m.

We define the *compactified puncture U-operad* of $\mathcal{Pun}_U^\circ$, by allowing points x_i and x_h in $\text{Conf}_U(j)$ to collide. When $x_i = x_h$, we identifthe factors $\mathcal{Comm}(k_i)$ and $\mathcal{Comm}(k_h)$ of $\mathcal{Pun}_U^\circ(k)$ with $\mathcal{Comm}(k_i + k_h)$.

As in picture 2.1 of the introduction, a point in $\mathcal{Pun}_U(k)$ is an ordered configuration of j points in U decorated with j trees, with the tree associated to x_i having k_i leafs, and representing an element of the commutative operad $\mathcal{Comm}(k_i)$. We call a point labelled by $\mathcal{Comm}(k)$ a *k-clustering* of punctures.

Observation 2.2.2 We have that $\mathcal{Pun}_U(k) \cong U^k$, since in the compactification, the labellings of the commutative operad count the multiplicity of the points collided – so we have just reconstructed U^k allowing collisions of configurations of $\text{Conf}_U(k) := \{(x_1, \ldots, x_k) \in U^k | x_i \neq x_j \forall 1 \leqslant i < j \leqslant k\}$

We have forgetfull maps $\pi_1, \ldots \pi_k \colon \mathcal{P}un_U(k) \to \mathcal{P}un_U(k-1)$. Assume i is labelling an element of $\mathcal{C}omm(k_j)$ as a factor of $\mathcal{P}un_U(k)$. When $k_j > 1$, we let π_i be given as the 0-ary operadic operation $\circ_i \colon \mathcal{C}omm(k_j) \to \mathcal{C}omm(k_j - 1)$ removing the element labelled i, and let π_i be the identity along the rest of the factors of $\mathcal{P}un_U(k)$.

When $k_j = 1$, we let π_i be induced by the forgetful map $\rho_j \colon \mathrm{Conf}_j(U) \to \mathrm{Conf}_{j-1}(U)$ along the configuration factor of $\mathcal{P}un_U(k)$, and let it be the identity along the rest of the factors.

Construction 2.2.3 Let Σ_k be the permutation group of the letters $\{1, \ldots, k\}$. As a set, we let

$$\overrightarrow{\mathcal{C}leav_{S^n}}(U; k) := \coprod_{i=1}^{k} \mathcal{C}leav_{S^n}(U; i) \times \mathcal{P}un_U(k-i) \times \Sigma_k \tag{2.2}$$

We make this into an operad coloured over spaces by extending Timber_{S^n} to $\overrightarrow{\mathrm{Timber}_{S^n}} := \mathrm{Timber}_{S^n} \cup S^n$ and topologizing this as points in $z \in S^n$ given as limit points of timber $U_n \in \mathrm{Timber}_{S^n}$ converging towards z.

With the extended timber, relabel the elements of $\mathcal{P}un_U(k-i)$ monotonely to start from $k - i + 1$ and end at k. We let the output of the $k - i$ final elements of $\mathcal{C}leav_{S^n}(U_i) \times \mathcal{P}un_U(k-i)$ be given by the point $z \in S^n$ that the associated tree labelled by the given point is decorating.

Finally, we topologize $\coprod_{i=1}^{k} \mathcal{C}leav_{S^n}(U; i) \times \mathcal{P}un_U(k-i)$ by letting an open set be generated from a set, where a subset is described by following data:

- An open set $W \subseteq \mathcal{C}leav_{S^n} \times \mathcal{P}un_U(k-i)$
- for each $k - i \leqslant j \leqslant k$, an open set

  $$V \subseteq \mathcal{C}leav_{S^n}(U; i+1) \times \mathcal{P}un_U(k-i-1),$$

  where we require that applying the map $\pi_j \colon \mathcal{P}un_U(k-i) \to \mathcal{P}un_U(k-i-1)$ to the second factor of W yields the second factor of V. Further, for all $[T, \underline{P}] \in V$, there is a fixed hyperplane P_j, such that forgetting this hyperplane makes the first factor of V agree with the first factor of V. Finally, for every configuration of P_j in V, the timber labelled by $i+1$ should be entirely contained in the neighborhood W_x of the puncture$x$, specified by W as the factor affected by π_j.

Such pairs (W, V) forms a basis for the topology of $\coprod_{i=1}^{k} \mathcal{Cleav}_{S^n}(U; i) \times \mathcal{Pun}_U(k-i)$.

A topology for $\overrightarrow{\mathcal{Cleav}_{S^n}}(-; k)$ is given by letting the Σ_k-factor in (2.2) act by permuting the operadic outputs of $\mathcal{Cleav}_{S^n}(U; i) \times \mathcal{Pun}_U(k-i)$ and equivalating elements of $\mathcal{Cleav}_{S^n}(U; i) \times \mathcal{Pun}_U(k-i)$ that agree after being acted upon. We finally topologize $\overrightarrow{\mathcal{Cleav}_{S^n}}(-; k) := \coprod_{U \in \overrightarrow{\mathrm{Timber}_{S^n}}} \overrightarrow{\mathcal{Cleav}_{S^n}}(U; k)$ as a subspace of $\overrightarrow{\mathcal{Cleav}_{S^n}}(\{S^n\}, k) \times \overrightarrow{\mathrm{Timber}_{S^n}}$.

Remark 2.2.4 An immediate consequence of the construction of the coloured operad $\overrightarrow{\mathcal{Cleav}_{S^n}}$, is that there is an inclusion of coloured operads $\mathcal{Cleav}_{S^n} \hookrightarrow \overrightarrow{\mathcal{Cleav}_{S^n}}$, induced by the inclusion of $\mathcal{Cleav}_{S^n}(U; k)$ into the $i = k$ part of (2.2). As explained in the introduction, this inclusion will be a cornerstone in providing an action of $\mathrm{H}_*(\mathcal{Cleav}_{S^n})$ on $\mathbb{H}_*(M^{S^n})$.

To an element $\chi \in \overrightarrow{\mathcal{Cleav}_{S^n}}(U; k)$, let $[T, \underline{P}]_\chi \in \mathcal{Cleav}_{S^n}(U; i)$ denote the element in the first factor of (2.2) describing cleaving hyperplanes, and $\underline{p}_\chi \in \mathcal{Pun}_U(k-i)$ the factor of the second, describing punctures. Let $(p_{i_1}, \ldots, p_{i_{k-i}})_\chi$ denote the $k-i$ points in the configuration space part of $\mathcal{Pun}_U(k-i)$, counted with clustering multiplicity. We let p_{i_m} be the point labelled by i_m. We call these points *punctures* of $\overrightarrow{\mathcal{Cleav}_{S^n}}(U; k)$.

Construction 2.2.5 To $\chi \in \overrightarrow{\mathcal{Cleav}_{S^n}}(-; k)$ we can form the diagram akin to the action diagram in 1.4.4 of $\mathcal{Cleav}_{S^n}$:

$$
\begin{tikzcd}
M_{\chi}^{S^n} \arrow[r] \arrow[d] & \left(M^{S^n}\right)^k \arrow[d, "N(\mathrm{res}_{\chi})"] \\
M^{\pi_0(\beta_{\chi})} \arrow[r, "\varphi"] & \widehat{M^{\coprod_{i=1}^k \complement N_i}}
\end{tikzcd}
\tag{2.3}
$$

However, we impose the following modifications compared to 1.4.4 for an element $\chi \in \overrightarrow{\mathcal{C}leav}_{S^n}(-;k)$:

- We let the *blueprint* of χ be given as the following subset of S^n:

$$\beta_{\chi} := \beta_{[T,\underline{P}]_{\chi}} \cup \{p_1\} \cup \cdots \cup \{p_m\}$$

  where $\beta_{[T,\underline{P}]_{\chi}}$ is the blueprint associated to $[T,\underline{P}]_{\chi}$, and p_i the punctures of χ.

- When i is a label of $[T,\underline{P}]_{\chi}$, we as here let $\complement N_i$ denote the closure of the complement of the associated timber N_i as a subset of S^n, and the only modification on $\widehat{M^{\coprod_{i=1}^k \complement N_i}}$ compared to $M^{\coprod_{i=1}^k \complement N_i}$ is that when the ouput labelled by i a puncture we want the i'th factor in the mapping space to be a copy of M. Since $\complement N_i = S^n$ as the closure of the associated timber, we shall in this case realize M as M^{S^n}/Ω_{-p_i}; that is we quotient the factor associated to $N_i = \{p_i\}$ out by the based loop space, based opposite the puncture.

- The notation $M_{\chi}^{S^n}$ is misleading in the sense that when $\chi \notin \mathcal{C}leav_{S^n}(-;k)$, we do not have a natural way of identifying of $M_{\chi}^{S^n}$ as a subspace of M^{S^n}. In picture 2.2 we show the effect of $M_{\chi}^{S^n}$ when there is a puncture. As before, each cleaving hyperplane gives rise to a codimension one subspace where the mapping is constant. However each puncture gives rise to an 'implosion' where the i'th mapping space from $\left(M^{S^n}\right)^k$ 'bubbles out' and survives as attached at $-p_i$ to the mappings that are constant at the blueprint.

Note that when $\chi \in \mathcal{C}leav_{S^n}$, the above diagram is precisely the same diagram as described in 1.4.4.

Remark 2.2.6 As indicated in 2.2, there is a map $M_\chi^{S^n} \to M^{S^n}$, namely the map that excludes f_i in the pullback for all punctures i and include the remaining as maps $S^n \to M$, patched together from the result of hyperplanes that are actually cleaving, opposed to being punctures.

Each puncture hereby acts as a unit for its associated algebra. In the sense that the operation for $\chi \in \overrightarrow{\mathcal{C}leav_{S^n}}(-;k)$ factors from $\left(M^{S^n}\right)^k$ through $\left(M^{S^n}\right)^{k-l}$ where l is the amount of punctures of χ, induced by the l 0-ary operations $\overrightarrow{\mathcal{C}leav_{S^n}}(-;k) \to \overrightarrow{\mathcal{C}leav_{S^n}}(-;k-l)$ forgetting all the punctures.

2.3 Umkehr Maps Along Manifolds

Throughout this chapter, let $F \longrightarrow X \xrightarrow{p} Y$ denote a fibration sequence, with F the fiber. Extending the fibration via the homotopy fiber $F_p := \{(x, f) \in X \times B^I \mid f(0) = *, f(1) = p(x)\}$, leads to the usual fibration sequence $\Omega Y \longrightarrow F_p \xrightarrow{pr} X$

Note by the above that the monoid ΩY acts on F_p by concatenation of based loops, and the associated Borel construction $E\Omega Y \times_{\Omega Y} F_p$ fits into a morphism of fibrations

$$\begin{array}{ccccc} F_p & \longrightarrow & E\Omega Y \times_{\Omega Y} F_p & \longrightarrow & B(\Omega Y) \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & X & \longrightarrow & Y \end{array}$$

Assuming that Y is path connected, the two outmost morphisms are the standard homotopy equivalences, and the middle morphism is the projection onto X from F_p. We shall use the notation $F_{h\Omega Y} := E\Omega Y \times_{\Omega Y} F_p$. The

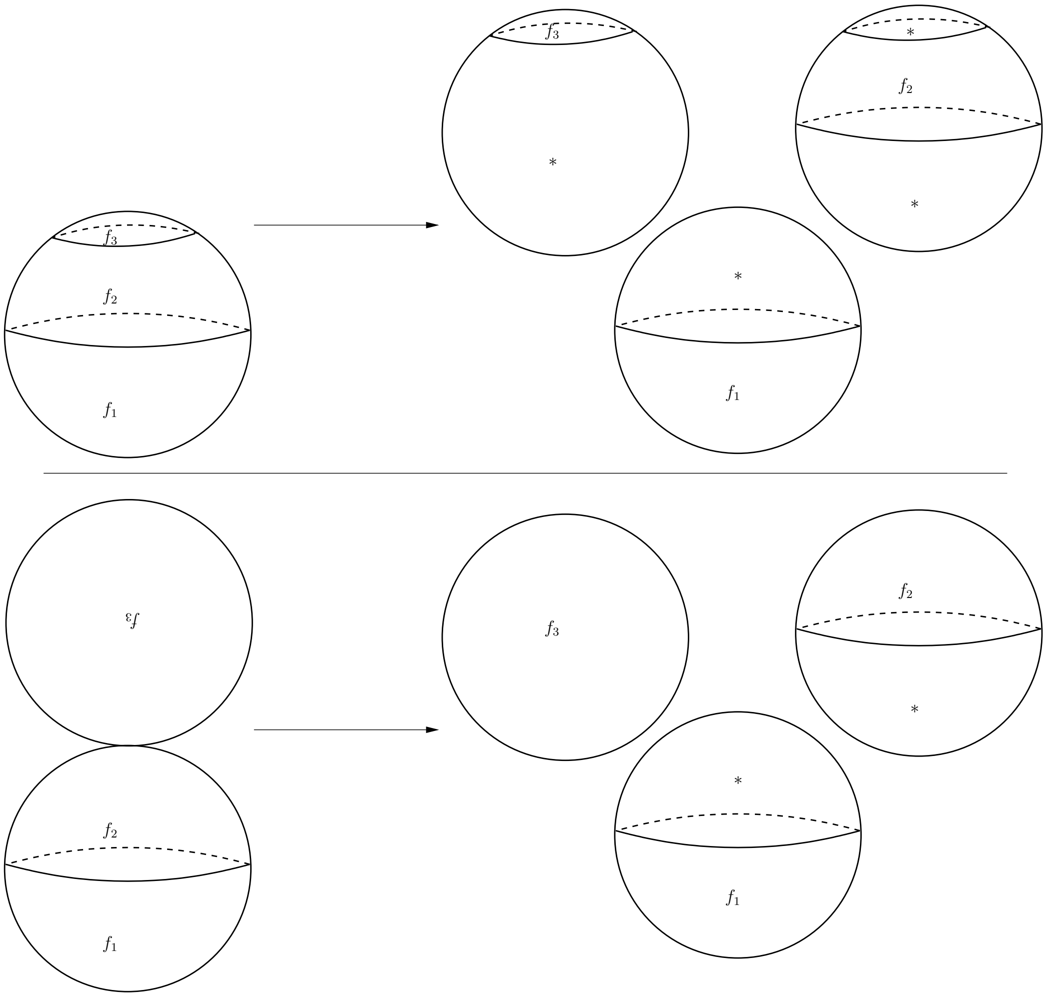

Figure 2.2: The top picture shows an operation of an element in $\mathcal{Cleav}_{S^n}(-;3)$ with all cleaving hyperplane. Here, the result is a map $S^n \to M$ that is constant along the blueprint. The bottom shows the result of sliding the top hyperplane to a puncture at the top of S^n. Here, suddenly the function f_3 is wedged onto the function patched together from the cleavage. Note that Disregarding f_3, we still have a map to S^n that is constant along the blueprint.

long exact sequence of homotopy groups now tells us that we have a weak equivalence:

Lemma 2.3.1 $E\Omega Y \times_{\Omega Y} F_p \simeq X$

This has the effect that in a pullback diagram

$$\begin{tikzcd} f^*(X) \arrow[r, "\varphi^*"] \arrow[d] & X \arrow[d, "f"] \\ A \arrow[r, "\varphi"] & B \end{tikzcd} \tag{2.4}$$

with f a fibration, we can identify the map φ^* as a map $\varphi^*\colon F_{h\Omega A} \to F_{h\Omega B}$.

In [Kle01], the dualizing spectrum of a group G is given as the G-equviariant function spectrum $\mathrm{Map}_G(EG_+, G)$, with both source and target being suspension spectra. From theorem [Kle01, Th. D], there is a norm map $\eta_B\colon F_{h\Omega B} \wedge D_{\Omega B} \to F^{h\Omega B}$ as well as a norm map $\eta_A\colon F_{h\omega A} \wedge D_{\Omega A} \to F^{h\Omega A}$.

Here, X^{hG} denotes the homotopy fixed-point spectrum $\mathrm{Map}(EG_+, X)^G$, that is the fixed point of the associated mapping space, where X is a G-spectrum.

Taking the induced map of $\varphi\colon A \to B$ produces an umkehr map $\varphi^\dagger = \mathrm{Map}(E(\varphi)_+, X)\colon F^{h\Omega B} \to F^{h\Omega A}$. This map is what leads to the following:

Lemma 2.3.2 If A and B in (2.4) are Poincaré duality spaces, and f is a fibration, then the above defines an umkehr map in spectra $\varphi^\dagger\colon X \wedge D_{\Omega B} \to f^*(X) \wedge D_{\Omega A}$.

Proof. Since A and B are Poincaré duality spaces, [Kle01, Th. D] gives us that the norm-maps involved in the following string of morphisms are homotopy-equivalences of spectra:

$$\begin{tikzcd} X \wedge D_{\Omega B} \arrow[r] & F_{h\Omega B} \wedge D_{\Omega B} \arrow[r, "\eta_B"] & F^{h\Omega B} \arrow[d, "\varphi^\dagger"] \\ f^*(X) \wedge D_{(\Omega A} & F_{h(\Omega A)} \wedge D_{\Omega A} \arrow[l] & F^{h\Omega A} \arrow[l] \end{tikzcd} \tag{2.5}$$

The two extremal morphisms are also homotopy equivalences from calculations above. Therefore, an umkehr map $X \wedge D_{\Omega B} \to f^*(X) \wedge D_{\Omega A}$ is given by choosing homotopy inverses in the string of morphisms above. $\square$

From [Kle01, Th. A], we furthermore have that $D_{\Omega A}$ is a sphere spectrum of dimension $-\dim(A)$.

Consider a diagram of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, bend right, "\iota_X"'] & B \arrow[d, bend right, "\iota_B"'] & A \arrow[l, "\varphi"'] \arrow[d, bend right, "\iota_A"'] \\ X' \arrow[r, "f'"] \arrow[u, bend right, "\rho_X"'] & B' \arrow[u, bend right, "\rho_B"'] & A \arrow[l, "\varphi'"'] \arrow[u, bend right, "\rho_A"'] \end{tikzcd} \tag{2.6}$$

with the circular morphisms being retractions, $\rho_E \circ \iota_E = \mathbb{1}_E$, $\rho_B \circ \iota_B = \mathbb{1}_B$ and $\rho_A \circ \iota_A = \mathbb{1}_A$ – and where these retractive properties are commuting with the morphisms f and φ.

Assume that f and f' are fibrations, with fibers F and F'.

The morphisms ι_X and $\iota_{f^*(X)}$ induced from the diagram 2.6 gives a diagram

$$\begin{tikzcd} E \wedge D_{\Omega B} \arrow[r, "\varphi^!"] \arrow[d, "\iota_E \wedge \mathbb{1}"'] & f^*(E) \wedge D_{\Omega A} \arrow[d, "\iota_{f^*(E)} \wedge \mathbb{1}"] \\ E' \wedge D_{\Omega B} \arrow[r] & f'^*(E') \wedge D_{\Omega A} \end{tikzcd} \tag{2.7}$$

of maps, with the top map being the umkehr map of 2.3.2. The diagram 2.6 also gives group extensions

$$\Omega A \xrightarrow{\iota_A} \Omega A' \longrightarrow \Omega(A'/A)$$

$$\Omega B \xrightarrow{\iota_B} \Omega B' \longrightarrow \Omega(B'/B)$$

By [Kle01, Th. B+C], there is an equivalence of spectra $D_{\Omega A'} \simeq D_{\Omega A} \wedge$

$D_{\Omega A'/A}$ and $D_{\Omega B'} \simeq D_{\Omega B} \wedge D_{\Omega B/B'}$. Taking the morphism into the first factor hence produces morphisms of spectra that extends (2.7) by composition to a morphism

$$\begin{tikzcd} E \wedge D_{\Omega B} \arrow[r, "\varphi^!"] \arrow[d, "\iota_E"] & f^*(E) \wedge D_{\Omega A} \arrow[d, "\iota_{f^*(E)}"] \\ E' \wedge D_{\Omega B'} \arrow[r] & f'^*(E') \wedge D_{\Omega A'} \end{tikzcd} \tag{2.8}$$

By 2.6, there are retraction maps $\hat{\rho_E}$ and $\hat{\rho_{f^*(E)}}$ fitting into the above diagram.

Lemma 2.3.3 Assuming that A, A', B, B' are Poincaré duality spaces, in (2.6)we have the identity

$$\varphi^\dagger = \hat{\rho_E} \circ \varphi'^\dagger \circ \hat{\iota_E}$$

Proof. By [Kle01, Th. A] we have an identification of $D_{\Omega C}$ as a sphere spectrum of dimension $-\dim(C)$ if C is Poincaré duality space. The progression from (2.7) to (2.8) is hence a desuspension map, so $\varphi'^!$ fits into the lower morphism of (2.7) and the identity follows. □

2.4 Patching Actions in Cleavage Operads

We now go into detail with the promised homologous action of the previous chapter; that is we shall provide the action of $\mathrm{H}_*(\mathcal{C}\mathit{leav}_{S^n})$ on $\mathbb{H}_*(M^{S^n})$. The strategy being that we will decompose $\overrightarrow{\mathcal{C}\mathit{leav}}_{S^n}(-;k)$ into smaller manifolds. We then apply the methods of section 2.3 to obtain umkehr maps for the constituents we have decomposed the space into. Via a homotopy colimit, we then show that these patch together to a global umkehr map, for the correspondance diagrams given in 2.2.5. An action of $\mathcal{C}\mathit{leav}_{S^n}$ is then obtained through the embedding of 2.2.4

Obtaining the action through an action of $\overrightarrow{\mathcal{Cleav}}_{S^n}$ gives us the advantage that the larger acting space is homotopically simple, and we avoid having to deal with certain considerations on manifolds with boundary involved in the underlying Poincaré duality arguments, had we attempted to make the construction for $\mathcal{Cleav}_{S^n}$ directly.

2.4.1 Patching Actions In Families

To any timber $U \in \overrightarrow{\text{Timber}_{S^n}}$ there is an embedding $\overrightarrow{\mathcal{Cleav}_{S^n}}(U;k) \hookrightarrow \overrightarrow{\mathcal{Cleav}_{S^n}}(S^n;k)$, extending the cleaving hyperplanes on U to cleaving hyperplanes on S^n. We shall therefore restrict attention to $\overrightarrow{\mathcal{Cleav}_{S^n}}(S^n;k)$ in decomposing the space. The action for other $\overrightarrow{\mathcal{Cleav}_{S^n}}(U;k)$ will be given through this embedding.

We start out by covering $\overrightarrow{\mathcal{Cleav}_{S^n}}(S^n;k)$ by a suitable set of closed spaces. To $\chi \in \overrightarrow{\mathcal{Cleav}_{S^n}}(S^n;k)$, let $|\pi_0(\beta_\chi)|$ denote the amount of components of the blueprint.

Definition 2.4.1 Let the m-evasion space $A_m \subseteq \overrightarrow{\mathcal{Cleav}_{S^n}}(S^n;k)$ be the subspace given by requiring that $\chi \in A_m$ has $|\pi_0(\beta_\chi)| = m+1$

Hereby, A_0 will consist of punctured cleavages where all the hyperplanes and punctures involved intersect non-trivially inside D^{n+1}. On the other hand A_{k-2} consist of cleavages where none of the involved hyperplanes intersect non-trivially inside D^{n+1}, and configuration points disjoint from the hyperplanes.

Let $\overline{A_m}$ denote the componentwise closure of the m-veering space inside $\overrightarrow{\mathcal{Cleav}_{S^n}}(S^n;k)$. Componentwise, in the sense that if two components $C_1 \subseteq A_m$ and $C_2 \subseteq A_m$ have $\overline{C_1} \cap \overline{C_2} \neq \emptyset$, we let $\overline{C_1}$ and $\overline{C_2}$ be disjoint spaces in $\overline{A_m}$

We specify a covering of $\overrightarrow{\mathcal{Cleav}_{S^n}}(S^n;k)$ by taking intersections of all the components of the spaces $\overline{A_m}$ for all $m \in \{0,\dots,k-2\}$.

This covering defines a category, $\mathrm{Pat}_{\{\overline{A_m}\}}$ with objects given as intersections of components of elements of $\{A_m\}$ and morphisms given by inclusions of subspaces.

Lemma 2.4.2 For any $A \in \mathrm{Ob}(\mathrm{Pat}_{\{A_i\}})$, A is homotopy equivalent to a disjoint union of a product of spheres.

Proof. An element $[T, \underline{P}] \in \overrightarrow{\mathcal{C}leav}_{S^n}(-;k)$ is called a *m-slinky* if $\beta_{[T,\underline{P}]}$ is connected and consists of precisely $m-1$ hyperplanes and punctures. We require that each hyperplane intersects nontrivially with at least one other hyperplane or puncture, and that such intersections will always be a single wedge-point.

Note that $\overline{A_0} \cap \overline{A_{k-2}}$ consists precisely of all k-slinkys. As we progress towards larger spaces in $\mathrm{Pat}_{\{A_i\}}$, hyperplanes of the involved spaces will be allowed to form in other patterns than slinkies.

The claim we want to show is that the spaces $\overline{A_i} \cap \overline{A_j}$ where $i < j$ as well as A_j, deformation retract onto the space of punctures on S^n where in the first case at least $w := j - i$, and the second, $w := j$ punctures are part of a cluster of punctures. Meaning that there are in total $k - 1 - w$ punctures counted without multiplicity. Showing this, we will have shown that $A_j \simeq (S^n)^{k-1-w} \simeq A_i \cap A_j$.

Choose $l \in \{1, \dots, k\}$, associated to a point $a \in A$, there is timber $U_l \subseteq S^n$, and timber $\tilde{U}_l$ for the associated Cleavage Operad over D^{n+1}. Being convex, $\tilde{U}_l$ has a well-defined center-of-mass u_l, and we can define the deformation retracting by pushing hyperplanes in the direction away from u_l in the following way:

A direction vector v_p is defined by the shortest line from u_l to a given hyperplane P_p, pointing away from u_l. If P_p is a hyperplane containing u_l, we let $v_p = 0$; note that in this case P_p will always be dominated by some hyperplane closer to $\tilde{U}_l$. In order to ensure continuity with respect to these hyperplanes, normalize v_p and scale it by the norm of the inner product $|\nu_p \cdot v_p|$.

We push the hyperplanes in the following way:

(A) If P_p is not participating in a slinky, we translate it by the direction vector v_p. If there are hyperplanes in the direction of P_p that have not yet been translated into configuration points, we leave P_p stationary.

(B) If P_p is participating in a slinky, and only a single wedge-point a of other hyperplanes, discounting configuration points, is attached to P_p, we specify a rotation of P_p, centered around a, by letting v_p denote the tangential direction of the rotation.

(C) If P_p is bounding $\tilde{U}_l$, but it is participating in a slinky where none of the other hyperplanes bound $\tilde{U}_l$, we in (B) leave P_p stationary until P_p is the only hyperplane left that is not a configuration point in the slinky.

(D) If P_p is participating in a slinky, and has multiple wedge-points arising from intersections with other hyperplanes, we let it be stationary until it's wedge-points have decimated to configuration points enough that it falls under (B) above.

The above procedure will lead all cleaving hyperplanes to eventually become tangent to S^n, and hence become punctures. Hereby providing the desired deformation retraction of A. Continuity of the procedure in the transition of hyperplanes from (A) to (B) can be checked using the fact that we are using what is in (A) translational data to what in (B) is tangential data for the rotation.

(A) ensures that not only the hyperplanes participating in slinkies, but all hyperplanes will eventually turn into configuration points.

(C) and the condition on hyperplanes being stationary ensures that hyperplanes won't be pushed into a position where they are no longer cleaving.

Note that the deformation retratction is well-defined, in the sense that the retraction never goes out of A. Note namely that (B) and (D) ensures that the sum of the amount of hyperplanes and configuration points participating in slinkies never decreases.

□

Although not entirely contained in the lemma above, we note that the same homotopy as specified in the proof above computes the homotopy type of $\overrightarrow{\mathcal{C}\mathit{leav}}_{S^n}$. This thus proves Theorem D stated in the introduction.

Definition 2.4.3 To $\chi \in \overline{A_m}$, there are associated timber $N_1^\chi, \ldots, N_k^\chi$, and the complement of the associated timber $\coprod_{i=1}^k \complement N_i^\chi$ will consist of a disjoint union of wedges of disks and points, as described in 1.3.10.

A point $\chi \in \overline{A_m} \backslash A_m$ will have $|\pi_0(\beta_\chi| < m+1$. Hence, for some $j \in \{1, \ldots, k\}$, the closure inside S^n of the complement of the timber of χ will have $\complement N_j^\chi$ consisting of less components compared to the case of $\chi' \in A_m$ lying in the same path-component of $\overline{A_m}$ as χ. Since a point in $\overline{A_m} \backslash A_m$ can be considered a limiting point of elements of A_m, the lesser components of $\complement N_j^\chi$ will be signified by a wedge $\bigvee_l C_l$ of components C_l that in any path away from $\overline{A_m} \backslash A_m$ becomes disjoint.

We shall make the dogma that whenever $\bigvee_l C_l$ occurs as as above in $\overline{A_m}$, the symbol $\complement N_j^\chi$ shall indicate the space with $\coprod_l C_l$ replacing $\bigvee_l C_l$.

Similarly, if $\complement N_i^\chi$ consist of the clustered punctures $p_{i_1} = \cdots = p_{i_j}$, we let $\complement N_i = \coprod_{l=1}^j \{p_l\}$.

The above dogma ensures that $|\pi_0(\beta_\chi)|$ is constant for any $\chi \in \overline{A_m}$. Hence timber will always 'evade collision' in $\overline{A_m}$.

We shall define a functor F that goes from the category $\mathrm{Pat}_{\{\overline{A_m}\}}$ to the subcategory of the diagram-category of topology spaces, given by pullbacks, I.e. Diagrams of the form

$$\begin{tikzcd} F(A)_{1,1} \arrow[r] \arrow[d] & F(A)_{1,2} \arrow[d] \\ F(A)_{2,1} \arrow[r] & F(A)_{2,2} \end{tikzcd}$$

We shall generally adhere to the above notation, and use the pairs of elements of $\{1,2\}$; $(1,1),(1,2),(2,1),(2,2)$ to refer to the specific spot above in a pullback diagram. Call the category of these diagrams Udiag.

Our first step in defining the umkehr map is an intermediate functor G that will be enough for providing an umkehr for correspondences over the space $\overline{A_m}$.

Construction 2.4.4 Let G be the following functor from the category $\{\overline{A_m}\}$ of the evasion spaces $\overline{A_m}$, into Udiag:

$$\begin{tikzcd} G(\overline{A_m})_{1,1} \arrow[r] \arrow[d, "\hat{\mathrm{res}}"] & \left(M^{S^n}\right)^{k+m} \times \overline{A_m} \arrow[d, "\hat{\mathrm{res}}"] \\ M^{|\pi_0(\beta_{[T,\underline{P}]})|} \times \overline{A_m} \arrow[r, "\varphi_{\overline{A_m}}"] & G(\overline{A_m})_{2,2} \end{tikzcd} \tag{2.9}$$

Since we have a pullback-diagram, $G(\overline{A_m})_{1,1}$ will be specified in the following.

We first define $G(\overline{A_m})_{2,2}$, and the morphism rês. First of all, as a set, we let

$$M^{\complement N_i}_{\overline{A_m}} := \coprod_{\chi \in \overline{A_m}} M^{\complement N_i^\chi}$$

where $\complement N_i^\chi$ is the i'th output of the element $\chi \in \overrightarrow{\mathcal{C}leav}_{S^n}(S^n;k)$ as in 2.4.3.

We specify a topology on this space as the quotient space under the restriction map from $\left(M^{S^n}\right)^{k+m} \times \overline{A_m}$, here as specified in 2.4.3, each component of $\complement N_i^\chi$ is included into one of the $k+m$ copies of S^n involved in the domain. Since we have taken componentwise closure of $\overline{A_m}$, the map is well defined as the $\complement N_i^\chi$ exercising multiple components will be constant throughout each component of $\overline{A_m}$.

Note as in 2.2.5 that since we are taking closures of complements, when $N_i^\chi = \{p_i\}$ is a puncture we have that $\complement N_i^\chi = S^n$. Consider the subspace

$(\Omega^n_{\{-p_i\}}M)_{\overline{A_m}} \subseteq M^{\complement N_i}_{\overline{A_m}}$ given as the mappings where whenever $\complement N_i^\chi = S^n$, and thus $N_i = \{p_i\}$, the associated map $f_i \colon S^n \to M$ will have $f_i(-p_i) = * \in M$ some fixed basepoint. That is, $(\Omega^n_{\{-p_i\}}M)_{\overline{A_m}}$ is the subspace of all based loopspaces occurring in $M^{\complement N_i}_{\overline{A_m}}$. We hereby let $G(\overline{A_m})_{2,2}$ be the quotient space

$$G(\overline{A_m})_{2,2} := M^{\complement N_i}_{\overline{A_m}}/(\Omega_{\{-p_i\}})_{\overline{A_m}}$$

The map rês is given as the restriction map $\mathrm{res}\colon \left(M^{S^n}\right)^{k+m} \times \overline{A_m} \to M^{\complement N_i}_{\overline{A_m}}$, composed with the quotient map to $G(\overline{A_m})_{2,2}$.

Pointwise, the map $\varphi_{\overline{A_m}}$ is given as in 1.2 – as the constant maps along the components that share points with the component of the blueprint.

For a disk of $\complement N_i^\chi \neq S^n$, we can define a point specifying the centre-of mass – for instance by maximizing the time with which any geodesic flow will reach the boundary of the disk. For a sequence in $\overline{A_m}$ with N_i converging to $\{p_i\}$, the centre of masses will converge towards $\{-p_i\}$. Evaluating at centre of mass in a tuppel $(f_1, \dots, f_{k+m})_\chi \in M^{\complement N_i}_{\overline{A_m}}$, and $-p_i$ where $N_i = \{p_i\}$, hereby provides the following homotopy-equivalence:

Lemma 2.4.5 $G(\overline{A_m})_{2,2}$ is homotopy equivalent to $M^{k+m} \times \overline{A_m}$.

Remark 2.4.6 Disregarding the punctures in 2.4.4, the map

$$\mathrm{res}\colon \left(M^{S^n}\right)^{k+m} \times \left(\overline{A_m} \cap \mathcal{C}leav_{S^n}(-;k)\right) \to M^{\complement N_i}_{\overline{A_m} \cap \mathcal{C}leav_{S^n}(-;k)}$$

is a fibration.

This can be seen by restricting to a fixed $\chi \in \overline{A_m} \cap \mathcal{C}leav_{S^n}(-;k)$, the map consist of a dualisation of cofibrations $\complement N_i^\chi \hookrightarrow S^n$, cofibrations taken componentwise if $\complement N_i^\chi$ has multiple components. Since we can choose the maps that make $(S^n, \complement N_i)$ a neighbourhood deformation retraction pair to be depending continuously on $\overline{A_m} \cap \mathcal{C}leav_{S^n}(-;k)$ we can define the associated lift pointwise in $\overline{A_m} \cap \mathcal{C}leav_{S^n}(-;k)$.

Recall that a quasifibration $f\colon E \to B$ is a surjective map such that $(E, f^{-1}(b)) \to (B, b)$ is a weak equivalence.

We doubt that the map res as given in 2.4.4 with punctures included is an actual fibration. However, basically by its definition, the map res – where we have quotiented the 'missing' fibers of rês in the base of $G(\overline{A})_{2,2}$, we have that

Proposition 2.4.7 The maps res and $\widetilde{\text{res}}$ of 2.4.4 are quasifibrations

The statement on $\widetilde{\text{res}}$ can be seen identifying the map as the evaluation along the blueprint of χ where they in $F(\overline{A})_{1,1}$ will be constant.

By 2.4.5 and 2.4.2, we have a pullback-diagram with the lower portion of the spaces being manifolds. Since we have vertical quasifibrations, we can in turn produce an umkehr, for each U-diagram indexed by $\overline{A_m}$ independently. However, we shall need one more step in extending our umkehr-diagrams in order to patch the local umkehr maps into a global umkehr map.

We shall thus replace the functor G by a functor $F\colon \operatorname{Pat}_{\{A_m\}}$:

Construction 2.4.8 Fixing $\chi \in \overline{A_m}$, as seen in 2.4.4, the set of complements of associated timber $\coprod_{i=1}^{k} \complement N_i$ consist of $k+m$ contractible components. Choosing any $k-2-m$ of these components $D_1, \dots, D_{k-2-m}$ gives a restriction map in the same way as in 2.4.4

$$\operatorname{res}\colon M^{S^n} \times \overline{A_m} \to M_{\overline{A_m}}^{\coprod_{i=1}^{k} \complement N_i \sqcup \coprod_{i=1}^{k-2-m} D_i},$$

where the latter is a quotient space under the map.

Again, when χ has a puncture p_i, the associated component $\complement N_i$, as well as any chosen D_j corresponding to p_i will be a copy of S^n. Again, analogous to 2.4.4, we let $(\Omega_{\{-p_i\}})_{\overline{A_m}})$ denote the based loop spaces $-p_i$ for such values of χ. We let

$$F(\overline{A_m})_{2,2} := M_{\overline{A_m}}^{\coprod_{i=1}^{k} \complement N_i \sqcup \complement_{i=1}^{k-2-m} D_i} / (\Omega_{\{-p_i\}})_{\overline{A_m}}$$

and consider the diagram, with the square involving $G(\overline{A_m})_{1,1}$ and $G(\overline{A_m})_{2,2}$ being the diagram (2.9):

$$
\begin{tikzcd}
F(\overline{A_m})_0 & & \left(M^{S^n}\right)^k \times \overline{A_m} \arrow[dr] \arrow[dd] & \\
& G(\overline{A_m})_{1,1} \arrow[ul] \arrow[rr] \arrow[dl] \arrow[dd] & & \left(M^{S^n}\right)^{k+m} \times \overline{A_m} \arrow[dl, "\Delta"'] \arrow[dd, "\mathrm{res}"] \\
F(\overline{A_m})_{1,1} \arrow[uu, "\mathrm{pr}"] \arrow[rr] \arrow[dd] & & \left(M^{S^n}\right)^{2(k-1)} \times \overline{A_m} \arrow[dd, "\varphi_{\overline{A_m}}"] & \\
& M^{|\pi_0(\beta_{[T,\underline{P}]})|} \times \overline{A_m} \arrow[rr, "\mathrm{res}"] \arrow[dl, "\Delta"'] & & G(\overline{A_m})_{2,2} \arrow[dl, "\eta"'] \\
M^{k-1} \times \overline{A_m} \arrow[rr, "\gamma"] & & F(\overline{A_m})_{2,2} &
\end{tikzcd}
\tag{2.10}
$$

The maps labelled Δ are all a suitable iteration of diagonal maps, making the diagram commute. The map η is specified by noting that for $\chi \in \overline{A_m}$ we have $|\pi_0(\beta_\chi| = m+1$. The difference between $k-1$ and $m+1$ is $k-2-m$, and having chosen the components $D_1, \dots, D_{k-2-m}$, we specify η as the map that is given by $\varphi_{\overline{A_m}}$ along $m+1$ of the factors of M^{k-1}, and for each $j \in k-2-m$, we choose one of the other factors in M^{k-1} doubled by the diagonal map, and let η be constant along D_j.

We let $F\colon \mathrm{Pat}_{\{\overline{A_m}\}} \to \mathrm{Udiag}$ be given as the pullback square involving $F(\overline{A_m})_{1,1}$.

We let $F(\overline{A_m})_0$ be given by noting that the extra choices involved in this construction will in the pullback space $F(\overline{A_m})_{1,1}$ give rise to k mappings from $S^n \to M$ that fit together as in 2.2.5, as well $k-2$ mappings given by the choices of components involved in F and G. pr is the projection map that projects these extra $k-2$ mappings away. To any $\chi \in A_m$, the associated space $F(\overline{A_m})_0$ restricted to χ will be given by $M_\chi^{S^n}$ of 2.2.5.

The explicit definition of F is highly dependent on the choices made above, as we shall see, 2.3.3 will make these choices cancel out in taking a pullback along the top-maps in the diagram of 2.4.8.

Note that with the definition of F – which image has no reference to m, F is also defined for $\overline{A_m} \cap \overline{A_n}$, so F can be extended to a functor $F\colon \operatorname{Pat}_{\{A_m\}} \to$ Udiag. This is the functor for which we need to show that the umkehr maps for $\overline{A_m}$ defined by 2.3.2 patch together to a global umkehr map of the entire operad $\mathcal{C}\!leav_{S^n}$:

Theorem 2.4.9 The diagrams (2.10) exhibit an action of $\overrightarrow{\mathcal{C}\!leav_{S^n}}$ in the category of spectra. Meaning that we have a map

$$\left(M^{S^n}\right)^k \times \overrightarrow{\mathcal{C}\!leav_{S^n}}(-;k) \to M^{S^n} \wedge S^{\dim(M)(k-1)}.$$

Restricting this map to $\mathcal{C}\!leav_{S^n}(-;k)$ provides an action of $\mathrm{H}_*(\mathcal{C}\!leav_{S^n})$ on $\mathbb{H}_*(M^{S^n})$.

Homotopically, the result of Theorem D tells us that the larger space $\overrightarrow{\mathcal{C}\!leav_{S^n}}(-;k)$ is much simpler than $\mathcal{C}\!leav_{S^n}(-;k)$, which is the k'th level of an E_{n+1}-operad. However, as in 2.2.6, the algebra we get from acting by $\overrightarrow{\mathcal{C}\!leav_{S^n}}$ is a unital algebra, whereas the algebra we get from $\mathcal{C}\!leav_{S^n}$ is a non-unital one.

Proof. To $A \in \operatorname{Ob}(\operatorname{Pat}_{\{A_i\}})$, we take a choice of functor F. Lemma 2.4.2 tells us that A is a Poincaré duality space, and therefore, using 2.4.5, the spaces $F(A)_{2,2}$ and $F(A)_{2,1}$ of 2.4.8 are Poincaré duality spaces as well.

Since 2.4.7 tells us that the needed maps are quasifibrations, we by 2.3.2 get that there is an umkehr map

$$\varphi_A^!\colon F(A)_{1,2} \wedge D_{\Omega F(A)_{2,2}} \to F(A)_{1,1} \wedge D_{\Omega F(A)_{2,1}}$$

We have stated 2.3.2 with fibrations instead of quasifibrations to ensure that the pullback of fibrations is again a fibration. In 2.4.7, we have however seen that in our case with quasifibrations, this is still the case for all pullback-diagrams we consider. The results of chapter 2.3 hence holds with fibration replaced with quasifibrations.

By [Kle01, Th. A], the dualizing spectra will be given by desuspended sphere spectra. The domain of $\varphi^!_A$ will be a sphere spectrum desuspended by $\dim(M) \cdot (2(k-1)) + \dim(A)$ and the target desuspended by $\dim(M) \cdot (k-1) + \dim(A)$, where $\dim(A)$ denotes the dimension of the product of spheres given by the deformation retraction of 2.4.2.

Smashing $\left(M^{S^n}\right)^k \times A$ and $M_A^{S^n}$ with the same dualising spectra as above, and in that order, we get that composition with the induced map in spectra with the maps Δ and pr above provide an action

$$A \times \left(M^{S^n}\right)^k \to F(A)_0 \wedge S^{\dim(M)\cdot(k-1)}$$

where we have suspended the map suitably to have no desuspensions on the domain of the action.

Let $A' \to A$ be a morphism in $\mathrm{Pat}_{\{A_i\}}$, these are inclusions. Restricting the action of A to A' will by 2.3.3 yield the same umkehr map, as producing the action for A' directly.

Therefore, taking the colimit of the associated umkehr maps provide a morphism in spectra, and using 2.3.2, we get the desired morphism

$$\left(M^{S^n}\right)^k \times \overrightarrow{\mathcal{C}\mathit{leav}_{S^n}} \to \mathrm{hocolim}_{A \in \mathrm{Pat}_{\{A_i\}}} F(A)_0) \wedge S^{\dim(M)\cdot(k-1)}$$

We identify the target of the above as in 2.2.6, where we see that the homotopy colimit $\mathrm{hocolim}_{\mathrm{Pat}_{\{A_i\}}} F(A)_0$ is equivalent to the space of maps $S^n \to M$ constant along the blueprint of $\chi \in \overrightarrow{\mathcal{C}\mathit{leav}_{S^n}}(-;k)$, with extra functions f_i wedged on whenever i is a puncture. Again as in 2.2.6, forgetting the extra f_i defines a map to M^{S^n} which in turn provides the action.

The action of $\mathcal{C}\mathit{leav}_{S^n}(-;k)$ is given by restricting the spectrum in the domain from $\overrightarrow{\mathcal{C}\mathit{leav}_{S^n}}(-;k)$, and in effect also restricting the intermediate spectrum $M^{S^n}_{\overrightarrow{\mathit{Cleav}}_{S^n}(-;k)} \wedge S^{\dim(M)\cdot(k-1)}$ to $M^{S^n}_{\mathit{Cleav}_{S^n}(-;k)} \wedge S^{\dim(M)\cdot(k-1)}$.

Restricting $\mathrm{hocolim}_{\overline{A}\in \mathrm{Pat}_{\{A_i\}}} F(\overline{A})_0$ to $\mathit{Cleav}_{S^n}(-;k)$, hereby avoiding the punctures, gives the space $M^{S^n}_{\mathit{Cleav}_{S^n}}$ of maps that are constant along each blueprint of $\chi \in \mathit{Cleav}_{S^n}$. The map $M^{S^n}_{\mathit{Cleav}_{S^n}} \to M^{S^n}$ is now pointwise in $\mathit{Cleav}_{S^n}(-;k)$ an inclusion. Whereas one sees that the punctures are precisely units for the algebra

Smashing with the associated Eilenberg-Maclane spectrum, and taking homotopy groups, we get the statement for homology groups.

□

Remark 2.4.10 The proof of 2.4.9 deals with the operad Cleav_{S^n}. It is however easy to see that the action constructed in this theorem extends to $\mathit{Cleav}_{S^n} \rtimes \mathrm{SO}(n+1)$ as constructed in 1.6.

Note namely that the top-right triangle of (2.10) can be extended to the diagram

$$\begin{tikzcd}
\left(M^{S^n}\right)^k \times (\mathrm{SO}(n+1))^k \times \overline{A_m} \arrow[dd] \arrow[dr] & \\
& \left(M^{S^n}\right)^{k+m} \times (\mathrm{SO}(n+1))^k \times \overline{A_m} \arrow[dl, "\Delta"] \\
\left(M^{S^n}\right)^{2(k-1)} \times (\mathrm{SO}(n+1))^k \times \overline{A_m} &
\end{tikzcd}$$

using 1.6.4, we see that the entire diagram of (2.10) can be extended to include the above diagram, having the exact same entries ats (2.10) in the lower entries.

Finally, the space $\overline{A_m}$ is closed under action by $\mathrm{SO}(n+1)$ and this hence leads to a action of the decomposition of the semidirect product; and the proof of 2.4.9 can be directly extended to provide a stable action map

$$\left(M^{S^n}\right)^k \times \overrightarrow{\mathit{Cleav}_{S^n}} \rtimes \mathrm{SO}(n+1)(-;k) \to M^{S^n} \wedge S^{\dim(M)(k-1)}.$$

In light of 1.6.5, The above remark hence gives us the following corollary:

Corollary 2.4.11 $\mathbb{H}_*(M^{S^n})$ is a $(n+1)$-Batalin-Vilkovisky algebra.

Chapter 3

Further Perspectives

3.1 Summing Up

In the two previous chapters, we have formed an operadic foundation for higher dimensional String Topology, as prescribed by the Cleavage Operads. In the first chapter our most prominent results were the fact that there are coloured E_{n+1}-operads and operads encoding $(n+1)$-Batalin-Vilkovisky algebras acting on M^{S^n}. In the second chapter, we gave the spectral action of $\mathcal{Cleav}_{S^n}$ on M^{S^n}. These results all seem to have been anticipated since the early years of String Topology. However, we do stress that the main novelty we present lies in the explicit geometric construction of the Cleavage Operads.

Compared to the 1-dimensional Cacti Operad as introduced by Voronov in [Vor05], the operad $\mathcal{Cleav}_{S^1}$ act on M^{S^1} similar to the Cacti Operad; but only up to homotopy. Where the Cacti Operad acts by lifting actual embeddings of finite products of M to M^{S^1} via a pullback, our operads specify this lift through certain mapping spaces homotopy equivalent to products of M. In the first section of this chapter, we show how this 'up to homotopy' viewpoint has advantages when considering equivariance of M^{S^1} or generally M^{S^n} compared to the classical case.

Furthermore, we have not only specified acting operads for M^{S^n}, but for any $N \subseteq \mathbb{R}^{n+1}$ an acting operad on M^N. The necessary conditions given in 1.3.8 that the complement of all timber N_i of N is componentwise contractible makes the action less interesting for general manifolds $N \subseteq \mathbb{R}^{n+1}$. That these operad acts to give an actual algebra, in the classical non-operadic sense appears to be a rare phenomenon. However, the choice of embeddings $N \subseteq \mathbb{R}^{n+1}$ still gives leavage for different structure of the acting operads. In the final section of this chapter we indicate how knots $S^1 \hookrightarrow \mathbb{R}^3$, appear to give new structures on free loop spaces. As for the Cleavage Operad, we can produce an operad that cleave knots into smaller components. Inspired by a tale, we call this operad the Gordian Knot Operad. Khovanov Homology can be used to determine the complexity of a knot. Dependent on this complexity, different algebraic structures will – conjecturely – come from the cleaved knot.

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3.2 Operational Equivariance

One problem in other approaches to String Topology, via for instance the Cacti Operad, is that in the pullback 'pinch diagram', describing the basic operation on the free loop-space, we do not have S^1-equivariance:

$$\begin{tikzcd} M^{S^1}_\infty \arrow[r, "\varphi"] \arrow[d] & M^{S^1} \times M^{S^1} \arrow[d, "\mathrm{ev}"] \\ M \arrow[r, "\Delta"] & M \times M \end{tikzcd} \tag{3.1}$$

where the map ev is the evaluation map for some fixed points of S^1. $M_\infty^{S^1}$ is the space of maps from the 'figure eight', given as maps from $S^1 \coprod S^1/\sim$, where $\sim$ is the equivalence relation that wedges the circles together at the evaluation points of ev.

Non-equivariance of the action is understood in the sense that we have an action of S^1 in the domain $M^{S^1} \times M^{S^1}$ of the evaluation map ev, acting by S^1 on M^{S^1} by precomposing on the domain of the maps $S^1 \to M$. However, there is in general no natural way to act by S^1 on $M \times M$. This makes the map φ similarly not S^1-equivariant. This has the implications that if one inspects φ and its associated umkehr, $\varphi^!$, the equivariance is ill-behaved. The picture gets worse in higher arity than two in diagrams similar to 3.1. These are described by cacti with several lobes intersecting each other, in the sense of for instance [Vor05]. Both [Wes08] and [KM07, Prospectus] are examples where this is an outspoken problem.

We shall point out how the operad $\mathcal{C}\!\mathit{leav}_{S^n}$ is much more well-behaved with respect to $\mathrm{SO}(n+1)$-equivariance of the action on M^{S^n}. The basic action diagram related to $\mathcal{C}\!\mathit{leav}_{S^n}$ is given in 1.4.4 in (1.2). Compared to 3.1, the restriction map res does carry $\mathrm{SO}(n+1)$-equivariance. Indeed, the space of objects of $\mathcal{C}\!\mathit{leav}_{S^n}$, $\mathrm{Ob}(\mathcal{C}\!\mathit{leav}_{S^n})$ has an action of $\mathrm{SO}(n+1)$, rotating timber of S^n via the action of $\mathrm{SO}(n+1)$ on S^n. Since the action of $\mathrm{SO}(n+1)$ on $\mathbb{R}^{n+1}$ transports cleaving hyperplanes to cleaving hyperplanes, it also transports timber to timber.

Denoting the action on timber $N_i \in \mathrm{Timber}_{S^n}$ as $\alpha.N_i$ for $\alpha \in \mathrm{SO}(n+1)$, the restriction map is equivariant in the sense that we have the commutative diagram

$$\begin{tikzcd}
\left(M^{S^n}\right)^k \arrow[r, ".\alpha"] \arrow[d, "\mathrm{res}"] & \left(M^{S^n}\right)^k \arrow[d, "\mathrm{res}"] \\
M^{\coprod_{i=1}^k \complement N_i} \arrow[r, ".\alpha"] & M^{\coprod_{i=1}^k \complement \alpha.N_i}
\end{tikzcd}$$

where the topmost map is the diagonal action of $\mathrm{SO}(n+1)$ on the domain along the k factors of M^{S^n}. This $\mathrm{SO}(n+1)$-equivariance has the effect that the associated maps $\varphi\colon M^{S^n}_{[T,\underline{P}]} \to \left(M^{S^n}\right)^k$ and its umkehr are $\mathrm{SO}(n+1)$-equivariant as well. In the sense that the $\mathrm{SO}(n+1)$-action will rotate the blueprint of $[T,\underline{P}]$, along which maps in $M^{S^n}_{[T,\underline{P}]}$ are constant; and this is precisely the action with wich φ is equivariant.

This improved equivariance leads us to the following question, inspired by the work of [Wes08]:

Question 3.2.1 What does an $(\mathrm{SO}(n+1))$-equivariant version of $\mathcal{C}leav_{S^n}$ tell us about the equivariant homology $\mathbb{H}_*^{\mathrm{SO}(n+1)}(M^{S^n})$?

3.3 Dependency of the Embedding

The results we obtain in chapter 1 are all relying on the fact that we are working with $S^n \subseteq \mathbb{R}^{n+1}$ the unit-sphere.We shall in this section employ the fact that any embedding of N inside $\mathbb{R}^{n+1}$ gives a new operad. In particular any knot $S^1 \hookrightarrow \mathbb{R}^3$ gives a new operad. However we shall show indications that this direct approach does not give much new interesting information to String Topology.

Instead we shall in the end of this section aim for hinting towards an extension of String Topology via Khovanov Homology – in the sense that every knot $K \in \mathrm{Emb}(S^1, \mathbb{R}^3)$ gives a different theory, with the un-knot providing standard String Topology.

3.3.1 Change of Embeddings

To an embedding $e\colon N \hookrightarrow \mathbb{R}^{n+1}$, we have in section 2.2 defined an associated Cleavage Operad, and for the sake of this chapter, we shall therefore refer to the Cleavage Operad associated to e as $\mathcal{C}leav_e$.

As a starting observation, the dimension of the ambient space N lives in does not affect the homotopy type of the Cleavage Operad:

Proposition 3.3.1 Let an embedding $e\colon N \hookrightarrow \mathbb{R}^n$ be given, and denote by $\iota\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ the inclusion of the first n coordinates. There is a weak equivalence of operads $\mathcal{C}leav_e \simeq \mathcal{C}leav_{\iota\circ e}$.

Proof. An oriented hyperplane of $\mathbb{R}^{n+1}$, with points entirely contained in $\iota(\mathbb{R}^n)$ does not intersect N transversally. Neither is an oriented hyperplane of $\mathbb{R}^{n+1}$ transversal if they do not does not intersect N. Given a hyperplane P of $\mathbb{R}^n$ intersecting N transversally, there is hence an $S^1\backslash S^0$ worth of choices of hyperplanes of $\mathbb{R}^{n+1}$ whose intersection with $N \subseteq \mathbb{R}^{n+1}$ agrees with $P \cap N$. The two contractible components of $S^1\backslash S^0$ corresponds to a choice of orientation of P. Contraction of these intervals provides the homotopy equivalence. $\square$

Given an embedded knot $K\colon S^1 \to \mathbb{R}^3$, we have an associated operad $\mathcal{C}leav_K$. One could hope that this operad would provide new operations on M^{S^1} – other than the E_2-operations determined in 1.5.21. We take the following example as indication that such hopes are best off shattered.

Example 3.3.2 The left part of picture 3.1 shows an operation associated to an non-convex embedding $S^1 \to \mathbb{R}^2$. The point of the example being that we obtain timber N_1 and N_2 with multiple components. Hereby $M^{\coprod_{i=1}^2 \complement N_i} \simeq M^4$, whereas we still have that the blueprint consists of a single element. Therefore, in the diagram (1.2) of 1.4.4 the map $M^{\pi_0(\beta_{[T,P]})} \to M^{\coprod_{i=1}^2 \complement N_i}$ is up to homotopy an embedding of codimension $3\dim(M)$. This has the effect that the operation associated to this particular cleavage will be a map $\mathrm{H}_*(M^{S^1})\otimes \mathrm{H}_*(M^{S^1}) \to \mathrm{H}_{*-3\dim(M)}(M^{S^1})$, with the degree of the target differing from the 2-ary of operation $\mathcal{C}leav_{S^n}(-;2)$ by $2\dim(M)$. A priori the operations associated to a non-convex embedding is hereby a new 'higher' operation.

In [CG04] a fat-graph model for higher string operations on $\mathrm{H}_*(M^{S^1})$ is given. In the sense that any fat-graph gives rise to a new higher operations as well. In picture 3.1, we have indicated a fat-graph associated to the non-convex-embedding. Briefly, any such fat-graph can be obtained by collapsing the blueprint associated to $[T, \underline{P}]$to a point, and hereby for each timber N_i obtain as many loops as there are components of N_i. We make it into a fat-graph, by making sure that there is a single incoming edge, wrapping the loops by encapsulating the loops by an extra full edge. Finally, we join the timber separated by a hyperplane through a single full edge.

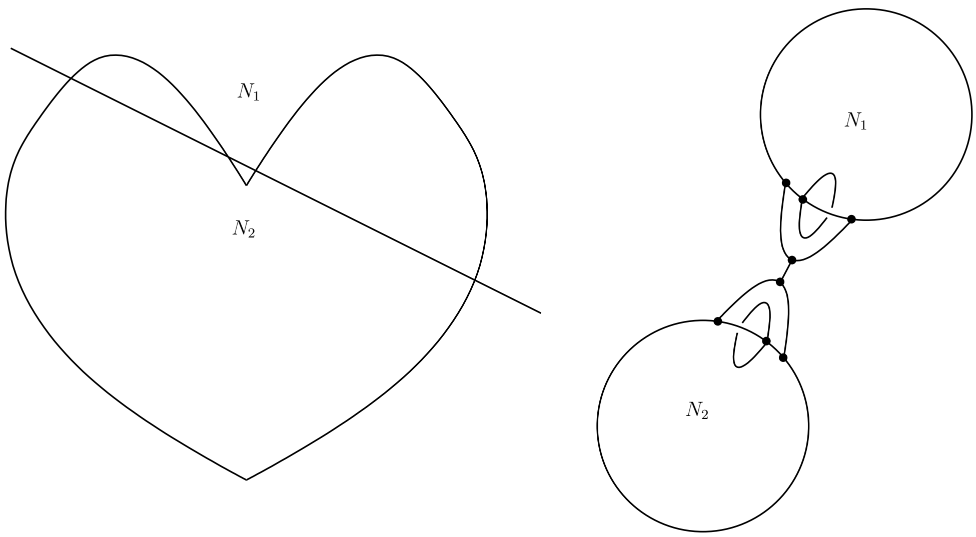

Figure 3.1: An example of a cloven non-convex embedding is displayed to the left. Notice that a single cleavage here makes N_1 consist of two components, which also holds for N_2 and their complements. To the cleavage we can associate a fat-graph with two out-going and one incoming boundary circles. The two components of N_1 and N_2 are signified by the two loops at the outgoing boundary circles

It can be seen that the operation associated to such a non-convex embedding are precisely the same as the operation associated to the fat-graph model. The surface associated to this particular fat-graph will have genus greater than zero, as will all higher operations arising from non-convex embeddings. In [Tam10, Th. A], it is shown that all such higher genus operations are trivial.

As a consequence, all new operations we attempt at constructing for $\mathcal{C}leav_e$

were $e\colon S^1 \to \mathbb{R}^n$ is an embedding will all be trivial. There are no interesting operations asides the ones arising from the unit-circle $S^1 \subset \mathbb{R}^2$.

Morally, and at least for $S^1 \subset \mathbb{R}^2$ this says that isotopies doesn't create new operations. Low-dimensional topologists would therefore consider the apparantly negative result of Tamanoi a positive one instead. And indeed we shall follow this positive spin, and consider it a starting point for making String Topology a knot-invariant. We dedicate the rest of this section to briefly indicate how we envision extending String Topology extended via Khovanov Homology.

In order to work from the perspective of knot theory, we need to turn to knot diagrams rather than actual embeddings $K \in \mathrm{Emb}(S^1, \mathbb{R}^3)$. That is, up to finitely many self-intersections, i.e. points $p \in \mathbb{R}^2$ with $|K^{-1}(p)| = 2$, $K\colon S^1 \to \mathbb{R}^2$ is an embedding. A knot diagram signifies a particular angle to look at a knot, considering $\mathbb{R}^2$ as the first two coordinates of $\mathbb{R}^3$. An actual embedding of S^1 inside $\mathbb{R}^3$ can hereby be reconstructed from the knot diagram. Using 3.3.1 as intuition, this knot reconstruction procedure does not appear to contain any homotopical information – and we shall not pay it any attention, only define the Gordian Knot Operad via knot diagrams.

3.3.2 The Gordian Knot Operad

We shall show how to twist the Cleavage Operad of a knot to obtain a bi-graded homological operad that acts skew-graded on the homology of $\mathrm{H}_*(M^{S^1})$ – skew-graded in the sense that the grading of the action will depend on the knot involved in Khovanov Homology.

To a knot-diagram $K \subseteq \mathbb{R}^2$, let $\gamma_K \subseteq \mathbb{R}^2$ denote the set of crossing points of $\mathbb{R}^2$. Given an oriented hyperplane $P \in \mathrm{Hyp}^2$ we say that P is transverse to K if $P \cap \gamma_K = \varnothing$ and P intersects $K \backslash \gamma_K$ transversally as submanifolds of $\mathbb{R}^2$.

Using this definition of transversality for knot diagrams, we literally define the Cleavage Operad $\mathcal{C}\mathit{leav}_K$ of a knot diagram $K \subseteq \mathbb{R}^2$ as in 1.3.8.

Recall from, say [BN05] that Khovanov Homology is given as the Homology of the Khovanov Complex C_K. The complex C_K is basically defined by smoothing the knot diagram at each of the crossing points γ_K, using one of the four Skein relations depicted in picture 3.2.

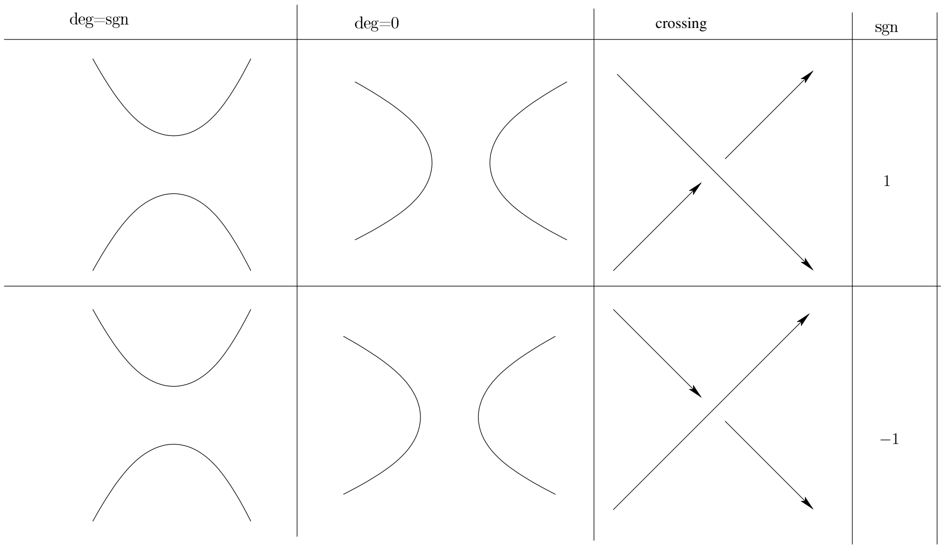

Figure 3.2: The Skein relations giving the Khovanov Complex. The sign of the crossing is listed in the final column, and the degree – that can attain the values $-1, 0, 1$ – are indicated in the top row.

Dividing γ_K into γ_K^+ consisting of positive crossings, and γ_K^- the negative crossings. To both γ_K^+ and γ_K^- there are two Skein relations, we hence arrive at $2^{|\gamma_K^+|} + 2^{|\gamma_K^-|}$ different ways of applying the Skein relations to smoothen the knot. To $i \in \{1, \dots, 2^{|\gamma_K^+|} + 2^{|\gamma_K^-|}\}$, let K_i denote the smoothing associated to a given choice of smoothing, suitably ordered. All smoothings K_i is hence a disjoint union of circles inside $\mathbb{R}^2$. There is a $\mathbb{Z}$-grading associated to a smoothing, given as the sum of the smoothing degrees in 3.2. One builds a complex using this grading, and by generating a module freely from all K_i, and choosing signs of morphisms appropriately as is usually done to obtain complexes. Again, we refer to [BN05] for the details.

The important point to note is that to each K_i, there is an associated Cleavage Operad $\mathcal{Cleav}_{K_i}$, simply because K_i is given as disjoint circles inside of $\mathbb{R}^2$. Hence each of this operads fit into the exact same picture as the Khovanov Complex, and each $\mathcal{Cleav}_{K_i}$ is assigned a degree according to what smoothing has been performed to it. This leads us to the following definition

Definition 3.3.3 The *Gordian Knot Complex* $\mathcal{Gord}_K$ is the complex of coloured operads given with a disjoint union of $\mathcal{Gord}_{K_i}$ of degree j at the j'th spot of the complex. Choosing formal morphisms between the degrees provides the structure of a complex of coloured operads.

Observation 3.3.4 The Gordian Knot Complex acts on $M^{\coprod_{\mathbb{N}} S^1}$ in the following sense:

Each $\mathcal{Cleav}_{K_i}$ acts on M^{K_i} in the sense of 2.4.9. It hence also acts on $M^{\coprod_{\mathbb{N}} S^1}$ by only affecting the $\pi_0(K_i)$ first components of $M^{\coprod_{\mathbb{N}} S^1}$. We can hence build a Khovanov complex along the algebra structure on $M^{\coprod_{\mathbb{N}} S^1}$ induced from the action of $\mathcal{Cleav}_{K_i}$. Again, this complex is built completely in the same way as one builds the Khovanov Complex; each K_i generates an algebra-structure on $M^{\coprod_{\mathbb{N}} S^1}$, and the morphisms of the Khovanov-Complex transfer into morphism $M^{\coprod_{\mathbb{N}} S^1} \to M^{\coprod_{\mathbb{N}} S^1}$ given by diagonal maps, inserting an extra copy of M^{S^1} whenever the Skein Relation gives rise to an extra copy of S^1, and projection maps, forgetting a copy of M^{S^1} whenever the Skein relation makes two copies of S^1 into a single S^1.

We can hence create a Khovanov Complex of $M^{\coprod_{\mathbb{N}} S^1}$, acted upon by $\mathcal{Gord}_K$. We call the homology of this complex the *gordian knot operad*. However the question of what Homology of the complex $\mathcal{Gord}_K$ actually means in an operadic context is more subtle, at least in the sense that in order to obtain the action of the associated Gordian Knot Operad a knot invariant. Analogous to 3.3.2, we can't simply hope that isotoping the knot $S^1 \hookrightarrow \mathbb{R}^2$ and obtain an invariant Khovanov Complex. We shall not go into a concrete description of this issue, but just note that the cleaving hyperplanes separate the knot K into tangles. This is precisely the issue dealt with in [LP09], where they in [LP09, Th. 4.4] show how the Khovanov Homology of a knot decomposed into tangles glue together along a limit-construction to fit into a so-called Knowledgeable Frobenius Algebra. As a cliffhanger, we conjecture that the construction of Lauda and Pfeiffer transfer to the setting of $M^{\coprod_{\mathbb{N}} S^1}$, in the sense that the associated algebraic structure induced from the Gordian Knot operad will be a knot invariant; and furthermore it seems likely that it is possible to formulate how $\mathrm{H}_*(M^{\coprod_{\mathbb{N}} S^1})$ is a knowledgeable Frobenius algebra 'over the Chas-Sullivan Loop product'.

Bibliography

[Bar08] Tarje Bargheer, *Cultivating operads in string topology*, 2008, Master's Thesis, University of Copenhagen, www.math.ku.dk/∼bargheer.

[Bar10] ———, *The cleavage operad and higher dimensional Chas-Sullivan products*, 2010, Arxiv:1012.4839.

[Bar11] ———, *The punctured cleavage operad and patched umkehr maps*, 2011, preprint.

[Ber97] Clemens Berger, *Combinatorial models for real configuration spaces and E_n-operads*, Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995), Contemp. Math., vol. 202, Amer. Math. Soc., Providence, RI, 1997, pp. 37–52. MR MR1436916 (98j:18014)

[BN05] Dror Bar-Natan, *Khovanov's homology for tangles and cobordisms*, Geom. Topol. **9** (2005), 1443–1499 (electronic). MR 2174270 (2006g:57017)

[Bro60] Morton Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76. MR MR0117695 (22 #8470b)

[CG04] Ralph L. Cohen and Véronique Godin, *A polarized view of string topology*, Topology, geometry and quantum field theory, London

Math. Soc. Lecture Note Ser., vol. 308, Cambridge Univ. Press, Cambridge, 2004, pp. 127–154. MR 2079373 (2005m:55014)

[CJ02] Ralph L. Cohen and John D. S. Jones, *A homotopy theoretic realization of string topology*, Math. Ann. **324** (2002), no. 4, 773–798.

[CK09] Ralph L. Cohen and John R. Klein, *Umkehr maps*, Homology, Homotopy Appl. **11** (2009), no. 1, 17–33. MR 2475820 (2010a:55006)

[CS99] Moira Chas and Dennis Sullivan, *String topology*, 1999, arXiv.org:math/9911159.

[CV06] Ralph L. Cohen and Alexander A. Voronov, *Notes on string topology*, String topology and cyclic homology, Adv. Courses Math. CRM Barcelona, Birkhäuser, Basel, 2006, pp. 1–95.

[Dug08] Dan Dugger, *A primer on homotopy colimits*, http://math.uoregon.edu/ ddugger/hocolim.pdf, 2008.

[FT09] Yves Félix and Jean-Claude Thomas, *String topology on Gorenstein spaces*, Math. Ann. **345** (2009), no. 2, 417–452. MR 2529482 (2010k:57050)

[GS08] Kate Gruher and Paolo Salvatore, *Generalized string topology operations*, Proc. Lond. Math. Soc. (3) **96** (2008), no. 1, 78–106. MR 2392316 (2009e:55013)

[Hu06] Po Hu, *Higher string topology on general spaces*, Proc. London Math. Soc. (3) **93** (2006), no. 2, 515–544. MR 2251161 (2007f:55007)

[Kau05] Ralph M. Kaufmann, *On several varieties of cacti and their relations*, Algebr. Geom. Topol. **5** (2005), 237–300 (electronic).

[Kle01] John R. Klein, *The dualizing spectrum of a topological group*, Math. Ann. **319** (2001), no. 3, 421–456. MR 1819876 (2001m:55037)

[KM07] Nitu Kitchloo and Jack Morava, *Thom prospectra for loopgroup representations*, Elliptic cohomology, London Math. Soc. Lecture Note Ser., vol. 342, Cambridge Univ. Press, Cambridge, 2007, pp. 214–238. MR 2330514 (2008f:55007)

[Lei04] Tom Leinster, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004. MR MR2094071 (2005h:18030)

[LP09] Aaron D. Lauda and Hendryk Pfeiffer, *Open-closed TQFTS extend Khovanov homology from links to tangles*, J. Knot Theory Ramifications **18** (2009), no. 1, 87–150. MR 2490019 (2011c:57067)

[MS04] James E. McClure and Jeffrey H. Smith, *Operads and cosimplicial objects: an introduction*, Axiomatic, enriched and motivic homotopy theory, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., Dordrecht, 2004, pp. 133–171. MR 2061854 (2005b:55018)

[SW03] Paolo Salvatore and Nathalie Wahl, *Framed discs operads and Batalin-Vilkovisky algebras*, Q. J. Math. **54** (2003), no. 2, 213–231.

[Tam10] Hirotaka Tamanoi, *Loop coproducts in string topology and triviality of higher genus TQFT operations*, J. Pure Appl. Algebra **214** (2010), no. 5, 605–615. MR 2577666

[Vor05] Alexander A. Voronov, *Notes on universal algebra*, Graphs and patterns in mathematics and theoretical physics, Proc. Sympos. Pure Math., vol. 73, Amer. Math. Soc., Providence, RI, 2005, pp. 81–103.

[Wes08] Craig Westerland, *Equivariant operads, string topology, and Tate cohomology*, Math. Ann. **340** (2008), no. 1, 97–142. MR 2349769 (2008k:55014)