

PhD thesis

Jens Kaad

A Calculation of the Multiplicative Character on Higher Algebraic K -theory

Academic advisor: Ryszard Nest

Submitted: 22/09/2009

Assessment committee:

Alan Carey, Australian National University, Australia

Magdalena Musat (chair), University of Copenhagen, Denmark

Matthias Lesch, University of Bonn, Germany

Jens Kaad
Institut for Matematiske Fag
Københavns Universitet
Universitetsparken 5
2100 København Ø
Danmark

`kaad@math.ku.dk`

© Jens Kaad

ISBN 987-87-91927-47-8

Abstract

In this thesis we study the multiplicative character of A. Connes and M. Karoubi. Our first achievement is a comparison result. We show that the multiplicative character on the second algebraic K -group agrees with a determinant invariant defined by L. G. Brown. As a byproduct we get a K -theoretic proof of the classical Helton-Howe identity

$$(1) \quad \det(e^A e^B e^{-A} e^{-B}) = \operatorname{Tr}[A, B] \in \mathbb{C}/(2\pi i)\mathbb{Z} \quad [A, B] \in \mathcal{L}^1(H)$$

Our second result can be interpreted as a multivariate version of (1). We show that the evaluation of the odd multiplicative character on higher Loday symbols is given in terms of the higher Helton-Howe trace form. To be precise, we get that

$$\mathcal{M}_F(e^{a_0} * \dots * e^{a_{2p-1}}) = -\frac{1}{p!} \operatorname{Tr} \left(\sum_{s \in \Sigma_{2p}} \operatorname{sgn}(s) P a_{s(0)} P \cdot \dots \cdot P a_{s(2p-1)} P \right) \in \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

On our way we investigate the multiplicative structure on relative K -theory and the multiplicative properties of the relative Chern character.

Resumé

I denne afhandling studerer vi A. Connes og M. Karoubis multiplikative karakter. Vores første resultat er et sammenlignings resultat. Vi viser at den multiplikative karakter på den anden algebraiske K -gruppe stemmer overens med en determinant invariant defineret af L. G. Brown. Som et biprodukt får vi et K -teoretisk bevis for den klassiske Helton-Howe identitet

$$(2) \quad \det(e^A e^B e^{-A} e^{-B}) = \operatorname{Tr}[A, B] \in \mathbb{C}/(2\pi i)\mathbb{Z} \quad [A, B] \in \mathcal{L}^1(H)$$

Vores andet resultat kan fortolkes som en multivariabel udgave af (2). Vi viser at evalueringen af den ulige multiplikative karakter på højere Loday symboler er givet ved den højere Helton-Howe sporform. Mere præcist får vi at

$$\mathcal{M}_F(e^{a_0} * \dots * e^{a_{2p-1}}) = -\frac{1}{p!} \operatorname{Tr} \left(\sum_{s \in \Sigma_{2p}} \operatorname{sgn}(s) P a_{s(0)} P \cdot \dots \cdot P a_{s(2p-1)} P \right) \in \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

På vores vej undersøger vi den multiplikative struktur i relativ K -teori og den relative Chern karakters multiplikative egenskaber.

CONTENTS

1. Introduction	5
2. The Fredholm determinant	17
2.1. The Schatten ideals and the operator trace	17
2.2. The first relative K -group of Banach algebras and their ideals	18
2.3. The first relative Chern character of an ideal in a Banach algebra	23
2.4. A K -theoretic definition of the Fredholm determinant	26
3. Determinants and traces of multiplicative and additive commutators	28
3.1. Noncommutative and commutative C^2 -functions in two variables	29
3.2. The determinant invariant and determinants of multiplicative commutators	31
3.3. The trace invariant and traces of additive commutators	35
3.4. The relation between the trace invariant and the determinant invariant	42
4. Determinant invariants and multiplicative characters of Fredholm modules	44
4.1. The multiplicative character on the second algebraic K -group	45
4.2. Comparison of the multiplicative character and the determinant invariant	47
5. Relative K -theory and products of contractions	51
5.1. Preliminaries on simplicial sets	51
5.2. The plus construction and higher K -theory	54
5.3. The Loday product in algebraic and relative K -theory	56
5.4. The relative Loday product and the relative Steinberg symbol	62
6. The higher determinant of a finitely summable Fredholm module	65
6.1. The multiplicative character of a finitely summable Fredholm module	66
6.2. The multiplicative properties of the relative Chern character	70
6.3. A calculation of the multiplicative character	72
7. The determinant of an odd dimensional torus	77
7.1. Preliminaries from noncommutative geometry	78
7.2. An integral formula for the higher determinant of a torus	82
References	86

1. INTRODUCTION

Let us consider a bounded operator $T \in \mathcal{L}(H)$ on a separable Hilbert space of infinite dimension. We would like to extract information from this data. For example, let us suppose that the operator is essentially normal. This means that the commutator between T and its adjoint T^* is a compact operator, $[T, T^*] \in \mathcal{K}(H)$. We let $q : \mathcal{L}(H) \rightarrow \mathcal{L}(H)/\mathcal{K}(H)$ denote the quotient map from the C^* -algebra of bounded operators to the Calkin algebra. Let $E \subseteq \mathcal{L}(H)$ denote the smallest unital C^* -algebra generated by T and the compact operators. We thus have a short exact sequence of C^* -algebras.

$$(3) \quad 0 \longrightarrow \mathcal{K}(H) \xrightarrow{i} E \xrightarrow{q} q(E) \longrightarrow 0$$

The quotient C^* -algebra is commutative and identifies by the Gelfand transform with the *continuous* functions on the essential spectrum of T ,

$$q(E) \cong C(Z) \quad Z := \text{Sp}(q(T))$$

Let us take an invertible continuous function on the essential spectrum, $f : E \rightarrow \mathbb{C}^*$. This data determines an element in the topological K -theory of $C(Z)$, $[f] \in K_1^{\text{top}}(C(Z))$. Furthermore, the short exact sequence (3) provides us with a boundary map

$$\partial : K_1^{\text{top}}(C(Z)) \rightarrow K_0(\mathcal{K}(H)) \cong \mathbb{Z}$$

in topological K -theory. In particular, we can associate the integer $\partial[f] \in \mathbb{Z}$ to the function f . This integer carries a lot of information. For example, to any $\lambda \notin Z = \text{Sp}(q(T))$ we can associate the number $\partial[\lambda - \text{Id}] \in \mathbb{Z}$ which is precisely the index of the Fredholm operator $\lambda - T \in \mathcal{L}(H)$, see [15, Example 4.8.8]. By the work of L. G. Brown, R. G. Douglas and P. A. Fillmore, the essential spectrum together with these indices characterize essentially normal operators up to essential unitary equivalence. See for example, [15, Theorem 2.4.8]. From the definition of the first topological K -group it follows that the function

$$\begin{aligned} g : \mathbb{C} - Z &\rightarrow \mathbb{Z} \\ g : \lambda &\mapsto \partial[\lambda - \text{Id}] = \text{Ind}(\lambda - T) \\ &= \text{Dim}(\text{Ker}(\bar{\lambda} - T)) - \text{Dim}(\text{Ker}(\lambda - T^*)) \end{aligned}$$

is locally constant. In particular, it is constant on the connected components of $\mathbb{C} - Z$. We could now ask the question, if we can extend this invariant to the whole complex plane extracting even more information about the operator $T \in \mathcal{L}(H)$? Let us assume that the commutator $[T, T^*]$ satisfies a more restrictive growth condition than merely compactness. Namely, we assume that the singular values are 1-summable, that is to say, the additive commutator is of trace class, $[T, T^*] \in \mathcal{L}^1(H) \subseteq \mathcal{K}(H)$. The trace ideal $\mathcal{L}^1(H)$ comes equipped with a tracial continuous linear functional

$$\text{Tr} : \mathcal{L}^1(H) \rightarrow \mathbb{C} \quad \text{Tr}(S) = \sum_{n=1}^{\infty} \langle e_n, S e_n \rangle$$

called the operator trace. Thus, we can associate a complex number $\text{Tr}([T, T^*]) \in \mathbb{C}$ to the above data. But we have even more. Let us write $T = X + iY$ where X is the real part and Y is the imaginary part of T . We then have $[T, T^*] = -2i[X, Y] \in \mathcal{L}^1(H)$. The selfadjoint operators X and Y thus form what is called an almost commuting pair. Now, for any complex

polynomial in two variables

$$p = \sum_{n,m} \lambda_{n,m} x^n y^m \in \mathbb{C}[x, y]$$

we let $p(X, Y) \in \mathcal{L}(H)$ denote the operator

$$p(X, Y) = \sum_{n,m} \lambda_{n,m} X^n Y^m$$

For any two polynomials $p, q \in \mathbb{C}[x, y]$ the commutator $[p(X, Y), q(X, Y)] \in \mathcal{L}^1(H)$ is then of trace class. In particular, we get a bilinear form

$$(\cdot, \cdot) : \mathbb{C}[x, y] \times \mathbb{C}[x, y] \rightarrow \mathbb{C} \quad (p, q) = \text{Tr}([p(X, Y), q(X, Y)])$$

This bilinear form extends to $C^\infty(\mathbb{R}^2)$ and can be calculated by a result of J. W. Helton, R. E. Howe, R. W. Carey and J. D. Pincus. See for example [14, 4, 5].

Theorem 1.1. *Let $T \in \mathcal{L}(H)$ be a completely non-normal bounded operator with trace class selfcommutator, $[T, T^*] \in \mathcal{L}^1(H)$. Then there exists a summable function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that*

- (1) *For any pair of smooth functions $f_1, f_2 \in C^\infty(\mathbb{R}^2)$ the pairing $(f_1, f_2) \in \mathbb{C}$ is given by the formula*

$$(f_1, f_2) = -\frac{1}{2\pi i} \int_{\mathbb{R}^2} J(f_1, f_2) g(x, y) dx dy$$

Here $J(f_1, f_2) = \frac{\partial f_1}{\partial x} \frac{\partial f_2}{\partial y} - \frac{\partial f_2}{\partial x} \frac{\partial f_1}{\partial y} \in C^\infty(\mathbb{R}^2)$ denotes the Jacobian of f_1 and f_2 .

- (2) *For each $\lambda \in \mathbb{C} - Z$ in the complement of the essential spectrum of $T = X + iY$ we have the identity*

$$g(\text{Re}(\lambda), \text{Im}(\lambda)) = \text{Ind}(\lambda - T)$$

The function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is called the principal function of the almost commuting pair $\text{Re}(T), \text{Im}(T) \in \mathcal{L}(H)$.

The above theorem implies that traces of commutators carry all the information on the indices of the Fredholm operators $(\lambda - T)$, $\lambda \in \mathbb{C} - Z$ and possibly a lot of extra information. We could thus understand it as a refinement of the Fredholm index invariant of topological K -theory. We will adopt this point of view. It immediately rises the question: What are traces of commutators an invariant of? We could start by expressing these quantities as values of homomorphisms defined on cyclic homology. To do so, we will have to settle ourselves in a slightly different scenario. Namely, suppose that we have a short exact sequence of Banach algebras

$$X : 0 \longrightarrow \mathcal{L}^1(H) \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{B} \longrightarrow 0$$

which satisfies the following conditions:

- (1) The Banach algebra \mathcal{E} is unital and equipped with a continuous unital algebra homomorphism $\pi : \mathcal{E} \rightarrow \mathcal{L}(H)$ which restricts to the inclusion on $\mathcal{L}^1(H)$.
- (2) There is a continuous linear split $s : \mathcal{B} \rightarrow \mathcal{E}$ and the quotient, \mathcal{B} , is unital and *commutative*.

Here we could think of \mathcal{E} as some appropriate completion of the $*$ -algebra generated by $T \in \mathcal{L}(H)$ and the operators of trace class. The assumption on the commutator $[T, T^*] \in \mathcal{L}^1(H)$ then ensures us that the quotient $\mathcal{E}/\mathcal{L}^1(H) = \mathcal{B}$ is commutative. We will think of \mathcal{B} as some algebra of differentiable functions on the essential spectrum of T . The continuous linear section is then given by some ordered functional calculus extending the above formula for polynomials. Note however that the passage from the operator T with trace class selfcommutator to the (smooth) extension scenario is far from uncomplicated in general. We can then exhibit traces of commutators in the following homological fashion:

Theorem 1.2. *The operator trace determines a well defined map on the zeroth continuous relative cyclic homology group*

$$\text{Tr} : HC_0(\mathcal{L}^1(H), \mathcal{E}) \rightarrow \mathbb{C}$$

Furthermore, any pair of elements $f, g \in \mathcal{B}$ determines a class in the first continuous cyclic homology group, $g \otimes f \in HC_1(\mathcal{B})$. This class maps to the trace of the commutator $\text{Tr}([A, B]) \in \mathbb{C}$ under the composition

$$(4) \quad HC_1(\mathcal{B}) \xrightarrow{\partial_x} HC_0(\mathcal{L}^1(H), \mathcal{E}) \xrightarrow{-\text{Tr}} \mathbb{C}$$

Here $A, B \in \mathcal{E}$ are any lifts of $f, g \in \mathcal{B}$.

For a proof of this result we refer to Section 3.3. Let us spell out the connection with the work of A. Connes at this point. Thus, consider an odd 2-summable Fredholm module (F, H) over a commutative unital Banach algebra \mathcal{A} . This amounts to the following data:

- (1) A separable Hilbert space H .
- (2) A selfadjoint unitary operator $F \in \mathcal{L}(H)$.
- (3) A unital algebra homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{L}(H)$.

The different components are related by the commutator condition

$$[F, \pi(a)] \in \mathcal{L}^2(H) \quad a \in \mathcal{A}$$

Here $\mathcal{L}^2(H)$ denotes the ideal of Hilbert-Schmidt operators on H . We will assume that the Fredholm module is continuous in the sense that

$$\|\pi(a)\|_\infty \leq \|a\| \quad \text{and} \quad \|[F, \pi(a)]\|_2 \leq \|a\|$$

for all $a \in \mathcal{A}$. Here the notation $\|\cdot\|_p : \mathcal{L}^p(H) \rightarrow [0, \infty)$ refers to the norms on the Schatten ideals. Each such Fredholm module gives rise to a character on the first continuous cyclic homology group,

$$(5) \quad \tau_F : HC_1(\mathcal{A}) \rightarrow \mathbb{C} \quad a_0 \otimes a_1 \mapsto -\frac{1}{4} \text{Tr}(F[F, a_0][F, a_1])$$

This important result is due to A. Connes, see [7, I 7. Lemma 2]. We would like to relate this Chern character with traces of commutators, in particular, with the homomorphism (4) of Theorem 1.2. To this end we form the \mathbb{C} -algebra

$$\mathcal{T}_{\mathcal{A}}^1 = \{(S, a) \in \mathcal{L}(PH) \times \mathcal{A} \mid S - PaP \in \mathcal{L}^1(PH)\}$$

which, when equipped with the norm,

$$\|(S, a)\| = \|S\|_\infty + \|S - PaP\|_1 + \|a\| + \|[P, a]\|_2$$

becomes a unital Banach algebra. Here $P = \frac{F+1}{2}$ denotes the projection onto the eigenvectors for F with eigenvalue 1. We can then write down a short exact sequence of Banach algebras

$$\mathbf{T}_F : 0 \longrightarrow \mathcal{L}^1(PH) \xrightarrow{i} \mathcal{T}_{\mathcal{A}}^1 \xrightarrow{q} \mathcal{A} \longrightarrow 0$$

which satisfies the conditions (1) and (2). The result of Theorem 1.2 thus supplies us with a homomorphism

$$HC_1(\mathcal{A}) \xrightarrow{\partial_{\mathbf{T}_F}} HC_0(\mathcal{L}^1(H), \mathcal{T}_{\mathcal{A}}^1) \xrightarrow{-\text{Tr}} \mathbb{C}$$

It is possible to prove that this character agrees with the Chern character of the Fredholm module, thus

$$(6) \quad \tau_F = (-\text{Tr}) \circ \partial_{\mathbf{T}_F}$$

See [9, Théorème 5.6] and [16, Theorem 4.2]. It follows that the evaluation of the Chern character on the class $a_0 \otimes a_1 \in HC_1(\mathcal{A})$ is given by

$$\tau_F(a_0 \otimes a_1) = -\text{Tr}([Pa_0P, Pa_1P])$$

In particular, suppose that the operators $\pi(a_0), \pi(a_1) \in \mathcal{L}(H)$ are selfadjoint and that the operator $T = P\pi(a_0)P + iP\pi(a_1)P \in \mathcal{L}(PH)$ is completely non-normal. Then for any two polynomials $p, q \in \mathbb{C}[x, y]$ we get that

$$\tau_F(p(a_0, a_1) \otimes q(a_0, a_1)) = \frac{1}{2\pi i} \int_{\mathbb{R}^2} J(p, q)g(x, y)dx dy$$

Here $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the principal function associated with $T \in \mathcal{L}(PH)$. It is perhaps worth to notice that the above integral formula looks like the Hochschild character of some appropriate spectral triple constructed out of the almost commuting pair, $P\pi(a_0)P, P\pi(a_1)P \in \mathcal{L}(PH)$. We haven't yet found occasion to pursue this track very far. For some further details on these matters we refer to the books [8] and [11]. Let us return to the main discussion.

We have seen that for any suitable short exact sequence of Banach algebras

$$X : 0 \longrightarrow \mathcal{L}^1(H) \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{B} \longrightarrow 0$$

with commutative quotient, we could express the traces of commutators in \mathcal{E} in a homological fashion. Furthermore, we have related this homological expression with Chern characters of 2-summable Fredholm modules. We are not really satisfied yet. The Fredholm indices were described in terms of a homomorphism from the first topological K -group $\text{Ind} : K_1^{\text{top}}(C(Z)) \rightarrow \mathbb{Z}$. We would like to understand traces of commutators as an extension of this invariant. That is, the origin of these quantities should not only be cyclic homology but rather some appropriate K -group. It should be noted at this point that the trace class commutator property of the unital Banach algebra \mathcal{E} is not preserved if we pass to the closure of \mathcal{E} in the operator norm. This is simply due to the fact that the trace ideal is non-closed in the C^* -algebra of bounded operators. Since topological K -theory is almost insensitive to this kind of operations we will in general need a finer K -theoretic functor to explain the traces of commutators. In this respect it was noticed by L. G. Brown in [3] that the origin of Fredholm determinants of multiplicative commutators was algebraic K -theory. Let us explain this result. Departing from the exact

sequence X , L. G. Brown constructs a *determinant invariant* on the second algebraic K -group of the quotient \mathcal{B} . This invariant is given by the following composition of homomorphisms

$$d_{(X,\pi)} : K_2(\mathcal{B}) \xrightarrow{\partial_X} K_1(\mathcal{L}^1(H), \mathcal{E}) \xrightarrow{\pi_*} K_1(\mathcal{L}^1(H), \mathcal{L}(H)) \xrightarrow{\det} \mathbb{C}^*$$

Here the K -groups are all algebraic and the last map is induced by the Fredholm determinant $\det : GL(\mathcal{L}^1(H)) \rightarrow \mathbb{C}^*$. Now, for any pair of invertible elements in the quotient $f, g \in \mathcal{B}^*$ there is a Steinberg symbol $\{f, g\} \in K_2(\mathcal{B})$ in the second algebraic K -group, see for example [29]. Assuming the existence of invertible lifts

$$A, B \in \mathcal{E}^* \quad q(A) = f, q(B) = g$$

it was then proved that

$$d_{(X,\pi)}(\{f, g\}) = \det(ABA^{-1}B^{-1})$$

Notice that the commutativity condition on the quotient implies that the above Fredholm determinant makes sense. In particular, for any elements $f, g \in \mathcal{B}$ with lifts $A, B \in \mathcal{E}$ we have the identity

$$(7) \quad d_{(X,\pi)}(\{e^f, e^g\}) = \det(e^A e^B e^{-A} e^{-B}) = e^{\text{Tr}[A, B]}$$

For a proof of the last identity we refer to [2]. Thus, at least modulo the additive subgroup $2\pi i\mathbb{Z} \subseteq \mathbb{C}$ the trace of the commutator $\text{Tr}[A, B] \in \mathbb{C}$ arises as the value of an invariant on algebraic K -theory. Before we continue, we would like to remark that the existence of invertible lifts is not a necessary condition for a concrete calculation of the determinant invariant on Steinberg symbols to apply. In this respect see [6]. Furthermore, notice that the existence of invertible lifts implies the vanishing of the boundary map in topological K -theory. Or stated in another way, the elements $f, g \in \mathcal{B}$ have trivial "winding numbers". The formula (7) suggests that the determinant theory should be interpreted as a parallel theory to the trace theory. To be slightly more precise: *The determinants of multiplicative commutators are paralleled by traces of additive commutators*. One of the main attempts of the first part of this thesis, thus the sections 2, 3, 4 is to understand what lies behind this statement. These sections are also related to the first article at the end of the thesis, [16]. So, what is it we are going to do? First of all, we will express the trace of an additive commutator by means of a K -theoretic invariant. That is, out of the exact sequence of Banach algebras X we will construct a trace invariant on the second relative K -group of the quotient. It is given by the composition

$$\tau_{(X,\pi)} : K_2^{\text{rel}}(\mathcal{B}) \xrightarrow{\text{ch}^{\text{rel}}} HC_1(\mathcal{B}) \xrightarrow{\partial_X} HC_1(\mathcal{L}^1(H), \mathcal{E}) \xrightarrow{-\text{Tr}} \mathbb{C}$$

Here the homomorphism $\text{ch}^{\text{rel}} : K_2^{\text{rel}}(\mathcal{B}) \rightarrow HC_1(\mathcal{B})$ is the second relative Chern character as introduced by M. Karoubi in [20]. Let us fix two elements $f, g \in \mathcal{B}$. Furthermore, let $\alpha : [0, 1] \rightarrow [0, 1]$ be a smooth maps with $\alpha(1) = 1$ and which vanishes on a neighborhood of the origin. From this data we are then able to associate a specific element

$$\{e^{-\alpha \cdot f}, e^{-\alpha \cdot g}\}^{\text{rel}} \in K_2^{\text{rel}}(\mathcal{B})$$

in the second relative K -group of the quotient \mathcal{B} . The application of the trace invariant to this relative Steinberg symbol is then the trace of the commutator

$$(8) \quad \tau_{(X,\pi)}(\{e^{-\alpha \cdot f}, e^{-\alpha \cdot g}\}^{\text{rel}}) = \text{Tr}([A, B]) \in \mathbb{C}$$

Here $A, B \in \mathcal{E}$ are any lifts of $f, g \in \mathcal{B}$, see Theorem 3.21. Understanding the relation between traces of additive commutators and determinants of multiplicative commutators thus amounts to understanding the relation between the trace invariant and the determinant invariant. That is, we look for a general comparison result. Now, the domain of the trace invariant is relative K -theory and the domain of the determinant invariant is algebraic K -theory. The link between these two domains is given by an explicit homomorphism,

$$\theta : K_2^{\text{rel}}(\mathcal{B}) \rightarrow K_2(\mathcal{B})$$

This homomorphism essentially associates the values at the vertices to each continuous map $\sigma : \Delta^2 \rightarrow GL(\mathcal{B})$. As an example, we have the following equality

$$\theta(\{e^{-\alpha \cdot f}, e^{-\alpha \cdot g}\}^{\text{rel}}) = \{e^f, e^g\}$$

for the application of θ to the relative Steinberg symbol. The comparison result can now be stated in the form of a commutative diagram

$$\begin{array}{ccc} K_2^{\text{rel}}(\mathcal{B}) & \xrightarrow{\theta} & K_2(\mathcal{B}) \\ \tau_{(X, \pi)} \downarrow & & d_{(X, \pi)} \downarrow \\ \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^* \end{array}$$

See Theorem 3.22. This is thus a general framework in which identities in two variables between determinants and traces can be analyzed. For example, we get a new proof of the well known equality

$$(9) \quad \det(e^A e^B e^{-A} e^{-B}) = e^{\text{Tr}([A, B])} \quad A, B \in \mathcal{E}$$

for almost commuting operators. Indeed, it is a consequence of the following calculation

$$\begin{aligned} \det(e^A e^B e^{-A} e^{-B}) &= d_{(X, \pi)}(\{e^f, e^g\}) \\ &= (d_{(X, \pi)} \circ \theta)(\{e^{-\alpha \cdot f}, e^{-\alpha \cdot g}\}^{\text{rel}}) \\ &= (\exp \circ \tau_{(X, \pi)})(\{e^{-\alpha \cdot f}, e^{-\alpha \cdot g}\}^{\text{rel}}) \\ &= e^{\text{Tr}([A, B])} \end{aligned}$$

The main goal of this thesis is to find a higher dimensional analogue of the identity (9). Let us explain what we mean by this sentence. The most desirable thing to do, would be to begin with the raw data consisting of, say, $2p$ selfadjoint operators $A_1, \dots, A_{2p} \in \mathcal{L}(H)$ with additive commutators $[A_i, A_j] \in \mathcal{L}^p(H)$ in the p^{th} Schatten ideal. To this general information J. W. Helton and R. E. Howe associates a fundamental trace form

$$\langle \cdot, \dots, \cdot \rangle : \mathcal{E}^{\otimes 2p} \rightarrow \mathbb{C} \quad x_1 \otimes \dots \otimes x_{2p} \mapsto \text{Tr}(\sum_{\sigma \in \Sigma_{2p}} \text{sgn}(\sigma) x_{\sigma(1)} \cdot \dots \cdot x_{\sigma(2p)})$$

Here $\mathcal{E} \subseteq \mathcal{L}(H)$ denotes the smallest $*$ -algebra which contains the p -almost commuting operators $A_0, \dots, A_{2p-1} \in \mathcal{L}(H)$, see [14]. Note that the above sum over the permutations on $2p$ letters is of trace class since

$$\sum_{\sigma \in \Sigma_{2p}} \text{sgn}(\sigma) x_{\sigma(1)} \cdot \dots \cdot x_{\sigma(2p)} = \sum_{s \in SE_{2p}} \text{sgn}(s) [x_{s(1)}, x_{s(2)}] \cdot \dots \cdot [x_{s(2p-1)}, x_{s(2p)}]$$

Here $SE_{2p} \subseteq \Sigma_{2p}$ denotes the subset of permutations with $s(2i-1) < s(2i)$ for all $i \in \{1, \dots, p\}$. Following the above scheme, we should then construct a trace invariant which, when applied

to appropriate higher relative Steinberg symbols produces values of the above fundamental trace form. Furthermore, the trace theory which lives on the higher relative K -groups should be paralleled by a determinant theory living on the higher algebraic K -groups. At last, the application of the higher determinant invariant to appropriate higher Steinberg symbols should be calculizable and expressed in terms of the fundamental trace form. This general program might suffer from technical problems. Even in the 1-dimensional case we used a suitable short exact sequence of Banach algebras to construct our trace invariant. The passage from the raw p -almost commuting tuple (A_1, \dots, A_{2p}) to an appropriate "smooth" extension is far from uncomplicated. We could thus limit ourselves to the setup consisting of a short exact sequence of Banach algebras

$$X : 0 \longrightarrow \mathcal{L}^p(H) \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{B} \longrightarrow 0$$

with properties similar to (1) and (2). But again, we face some obstacles. Indeed, our trace invariant was defined by means of a character on the first continuous cyclic homology group of the quotient, \mathcal{B} . This character had the operator trace on the relative continuous cyclic homology group of the pair $(\mathcal{L}^1(H), \mathcal{E})$ as an important component. There is however no immediate generalization of the operator trace to the higher relative continuous cyclic homology groups of the pair $(\mathcal{L}^p(H), \mathcal{E})$. For example, the continuous cyclic homology groups do not satisfy excision in general. We should notice at this point that it is possible to construct a continuous cyclic cocycle merely from the data consisting of a continuous linear map $s : \mathcal{B} \rightarrow \mathcal{L}(H)$ which is multiplicative modulo $\mathcal{L}^p(H)$ in a continuous sense, [7, I 7. Theorem 5]. We haven't yet found occasion to pursue the implications of this observation further. Instead we have fixed ourselves in the slightly less general framework of odd finitely summable Fredholm modules. Thus, we consider the following data:

- (1) A separable Hilbert space H and a unital commutative Banach algebra \mathcal{A} .
- (2) A selfadjoint unitary operator $F \in \mathcal{L}(H)$.
- (3) A unital algebra homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{L}(H)$.

The different components are related by the commutator condition

$$[F, \pi(a)] \in \mathcal{L}^{2p}(H) \quad a \in \mathcal{A}$$

Here $\mathcal{L}^{2p}(H)$ denotes the $2p^{\text{th}}$ Schatten ideal. We will assume that the Fredholm module is continuous in the sense that

$$\|\pi(a)\|_\infty \leq \|a\| \quad \text{and} \quad \|[F, \pi(a)]\|_{2p} \leq \|a\|$$

for all $a \in \mathcal{A}$. In this context there is a character on continuous cyclic homology

$$(10) \quad \tau_F : HC_{2p-1}(\mathcal{A}) \rightarrow \mathbb{C} \quad a_0 \otimes \dots \otimes a_{2p-1} \mapsto -\frac{1}{2^{2p} \cdot (p-1)!} \text{Tr}(F[F, a_0] \cdot \dots \cdot [F, a_{2p-1}])$$

This is the higher dimensional analogue of (5) which is also due to A. Connes, [7, I 7. Lemma 2]. The fundamental trace form of J. W. Helton and R. E. Howe can be expressed by means of the above homomorphism. Indeed, it follows from the results in [7, I 7.] that

$$(11) \quad \tau_F\left(\sum_{s \in \Sigma_{2p-1}} \text{sgn}(s) a_0 \otimes a_{s(1)} \otimes \dots \otimes a_{s(2p-1)}\right) = -\frac{1}{p!} \langle P a_0 P, \dots, P a_{2p-1} P \rangle$$

See also [17, Theorem 5.3]. In the article [9] it is then shown that the above higher cyclic cocycle gives rise to a homomorphism

$$\mathcal{A}_F : K_{2p}^{\text{rel}}(\mathcal{A}) \rightarrow \mathbb{C} \quad \mathcal{A}_F := \tau_F \circ \text{ch}^{\text{rel}}$$

which we call the additive character of the Fredholm module. Here $K_{2p}^{\text{rel}}(\mathcal{A})$ denotes the higher relative K -theory as constructed by M. Karoubi in [20]. The notation ch^{rel} refers to the relative Chern character. This is a natural homomorphism of degree minus one with values in continuous cyclic homology,

$$\text{ch}^{\text{rel}} : K_{2p}^{\text{rel}}(\mathcal{A}) \rightarrow HC_{2p-1}(\mathcal{A})$$

It was originally constructed by M. Karoubi in a differential geometric context, see [20]. The additive character is a good higher dimensional replacement for our trace invariant. Indeed, it follows immediately by (6) that

$$\mathcal{A}_F = \tau_F \circ \text{ch}^{\text{rel}} = (-\text{Tr}) \circ \partial_{\mathbf{T}_F} \circ \text{ch}^{\text{rel}} = \tau_{(\mathbf{T}_F, \pi_1)} : K_2^{\text{rel}}(\mathcal{A}) \rightarrow \mathbb{C}$$

Thus, in the one dimensional case, the additive character of the Fredholm module agrees with the trace invariant of the Fredholm module. Here the notation \mathbf{T}_F refers to the extension

$$\mathbf{T}_F : 0 \longrightarrow \mathcal{L}^1(H) \xrightarrow{i} \mathcal{T}_{\mathcal{A}}^1 \xrightarrow{q} \mathcal{A} \longrightarrow 0$$

constructed from the odd continuous 2-summable Fredholm module. As mentioned above, we are interested in computing the additive character on some higher dimensional analogue of the relative Steinberg symbol. This is one of the achievements of the second article [17] which is attached at the end of this thesis. Let us briefly explain the general scheme of the calculation. A more detailed introduction is given in Section 5 and Section 6. First of all we need to find the correct replacement for the relative Steinberg symbols on higher relative K -theory. Recall that we assume our unital Banach algebra to be commutative. In the article [25] it is shown by J.-L. Loday that the Steinberg symbols in the second algebraic K -group are special instances of a general product in algebraic K -theory. Following his ideas, we construct a rather explicit interior product on relative K -theory,

$$*^{\text{rel}} : K_n^{\text{rel}}(\mathcal{A}) \times K_m^{\text{rel}}(\mathcal{A}) \rightarrow K_{n+m}^{\text{rel}}(\mathcal{A})$$

Equipped with this structure, the relative K -theory of a unital Banach algebra becomes a graded commutative ring. Furthermore, the group homomorphism

$$\theta : K_*^{\text{rel}}(\mathcal{A}) \rightarrow K_*(\mathcal{A})$$

which links the different K -theories becomes a homomorphism of graded commutative rings. These results are proved in [17, Section 3]. Finally, in analogy with the algebraic result, the relative Steinberg symbol, which is given by an explicit homology class, can be expressed in terms of the general relative product. This identification is carried out in Section 5.4. We are therefore interested in the application of the additive character

$$\mathcal{A}_F : K_{2p}^{\text{rel}}(\mathcal{A}) \rightarrow \mathbb{C} \quad \mathcal{A}_F = \tau_F \circ \text{ch}^{\text{rel}}$$

to a higher relative Steinberg symbol of the form

$$(12) \quad [\gamma] = [\gamma_0] *^{\text{rel}} \dots *^{\text{rel}} [\gamma_{2p-1}] \in K_{2p}^{\text{rel}}(\mathcal{A})$$

Here (F, H) is a continuous odd $2p$ -summable Fredholm module over the commutative unital Banach algebra \mathcal{A} . Note at this point that the classes in the first relative K -group are

represented by smooth maps $\sigma : [0, 1] \rightarrow GL(\mathcal{A})$ with $\sigma(0) = 1$. Thus each of the elements $[\gamma_i] \in K_1^{\text{rel}}(\mathcal{A})$ is given by a smooth path of invertibles which begins at the identity. Having the calculation in the 1-dimensional case in mind we will assume that our paths are of the form

$$\gamma_i : t \mapsto e^{-ta_i} \quad a_i \in \mathcal{A}$$

See 8. Now, instead of continuing directly with the computation of the additive character we will make a detour. Namely, we will explore the behaviour of the relative Chern character

$$\text{ch}^{\text{rel}} : K_*^{\text{rel}}(\mathcal{A}) \rightarrow HC_*(\mathcal{A})[1]$$

with respect to the multiplicative structure on the relative K -groups. Note that this natural homomorphism is a part of the additive character. To be more precise, there should exist a product in cyclic homology which corresponds to the relative product in K -theory. We remark that such a product must have degree one, since the relative Chern character has degree minus one. In this respect there is a natural candidate given by the shifted shuffle product,

$$* : HC_n(\mathcal{A}) \times HC_m(\mathcal{A}) \rightarrow HC_{n+m+1}(\mathcal{A}) \quad x * y = x \times (sNy)$$

When equipped with this multiplicative structure the shifted cyclic homology theory $HC_*(\mathcal{A})[1]$ becomes a graded commutative ring. See for example [26, Section 4.4]. We prove in [17, Section 4] that the relative Chern character becomes a homomorphism of graded commutative rings. Thus, we have the equality

$$\text{ch}^{\text{rel}}([\sigma] *^{\text{rel}} [\tau]) = \text{ch}^{\text{rel}}([\sigma]) * \text{ch}^{\text{rel}}([\tau])$$

for any $[\sigma] \in K_n^{\text{rel}}(\mathcal{A})$ and $[\tau] \in K_m^{\text{rel}}(\mathcal{A})$. This effectively reduces the calculation of the relative Chern character on the higher relative Steinberg symbol

$$[\gamma] = [\gamma_0] *^{\text{rel}} \dots *^{\text{rel}} [\gamma_{2p-1}] \in K_{2p}^{\text{rel}}(\mathcal{A})$$

to a calculation of the relative Chern character on the individual terms. This is by no means a complicated thing to do. Indeed, for $\gamma_i : t \mapsto e^{-ta_i}$ we have that

$$\text{ch}^{\text{rel}}([\gamma_i]) = \int_0^1 \frac{d(e^{-ta_i})}{dt} e^{ta_i} dt = -a_i \in HC_0(\mathcal{A})$$

By the multiplicative properties of the relative Chern character we deduce that

$$(13) \quad \text{ch}^{\text{rel}}([\gamma]) = \sum_{s \in \Sigma_{2p-1}} \text{sgn}(s) a_0 \otimes a_{s(1)} \otimes \dots \otimes a_{s(2p-1)} \in HC_{2p-1}(\mathcal{A})$$

We therefore arrive at a concrete formula for the additive character on higher relative Steinberg symbol. Furthermore, we obtain an expression in terms of the fundamental trace form of J. W. Helton and R. E. Howe. To be precise, from 13 and 11 we get that

$$(14) \quad \mathcal{A}_F([\gamma]) = \sum_{s \in \Sigma_{2p-1}} \text{sgn}(s) \tau_F(a_0 \otimes a_{s(1)} \otimes \dots \otimes a_{s(2p-1)}) = -\frac{1}{p!} \langle Pa_0 P, \dots, Pa_{2p-1} P \rangle$$

Some supplementary discussion of this result is given in Section 6.3. We can proceed even further. As mentioned earlier in this introduction we would like to dispose over a determinant theory which parallels the trace theory given by the additive character. This desire is inspired by the one dimensional case where we did possess a determinant invariant on the second algebraic K -group. This invariant could be calculated by means of the trace invariant. Indeed these

two characters were intimately connected. In the higher dimensional setup we will replace the determinant invariant by the odd multiplicative character of A. Connes and M. Karoubi, see [9]. This replacement is motivated by the following result:

Theorem 1.3. *Let (F, H) be an odd 2-summable Fredholm module over a unital \mathbb{C} -algebra \mathcal{A} . We then have a commutative diagram of group homomorphisms*

$$\begin{array}{ccc} K_2(\mathcal{A}) & & \\ \mathcal{M}_F \downarrow & \searrow d_F & \\ \mathbb{C}/(2\pi i)\mathbb{Z} & \xrightarrow{\exp} & \mathbb{C}^* \end{array}$$

Here $\mathcal{M}_F : K_2(\mathcal{A}) \rightarrow \mathbb{C}/(2\pi i)\mathbb{Z}$ denotes the multiplicative character of the Fredholm module. Furthermore, $d_F : K_2(\mathcal{A}) \rightarrow \mathbb{C}^*$ denotes the determinant invariant of the short exact sequence of \mathbb{C} -algebras

$$0 \longrightarrow \mathcal{L}^1(H) \xrightarrow{i} \mathcal{T}_{\mathcal{A}}^1 \xrightarrow{q} \mathcal{A} \longrightarrow 0$$

A proof can be found in [16], see also [9] and Section 4. The interpretation of Theorem 1.3 is clear: For odd 2-summable Fredholm modules, the multiplicative character coincides with the determinant invariant. The advantage of the multiplicative character is that it generalizes to the higher K -groups. Indeed, to any odd $2p$ -summable Fredholm (F, H) over any \mathbb{C} -algebra \mathcal{A} , A. Connes and M. Karoubi associate a homomorphism

$$\mathcal{M}_F : K_{2p}(\mathcal{A}) \rightarrow \mathbb{C}/(2\pi i)^p\mathbb{Z}$$

which they call the multiplicative character of the Fredholm module. Suppose now that \mathcal{A} is a unital Banach algebra and that the Fredholm module is continuous. Let $\theta(y) \in K_{2p}(\mathcal{A})$ be an element in the image of the homomorphism

$$\theta : K_{2p}^{\text{rel}}(\mathcal{A}) \rightarrow K_{2p}(\mathcal{A}) \quad \theta(\sigma) = (\sigma(\mathbf{0})\sigma(\mathbf{1})^{-1}, \dots, \sigma(\mathbf{n}-\mathbf{1})\sigma(\mathbf{n})^{-1})$$

The application of the multiplicative character to the element $\theta(y)$ is described in terms of the additive character. Indeed, by definition

$$\mathcal{M}_F(\theta(y)) = [\mathcal{A}_F(y)] \in \mathbb{C}/(2\pi i)^p\mathbb{Z}$$

Here $[\cdot] : \mathbb{C} \rightarrow \mathbb{C}/(2\pi i)^p\mathbb{Z}$ denotes the quotient map. The multiplicative character is thus intimately related to the additive character. That is, the higher dimensional determinant theory is explicable in terms of the higher dimensional trace theory. In particular, the above calculation of the additive character entails a corresponding calculation of the multiplicative character. This is the main achievement of the article [17] to be found at the end of the present thesis. Let us give some details on this result. We need the additional assumption that our unital Banach algebra \mathcal{A} is commutative. This is really required since we need the above interior Loday products in order to construct our K -theory classes. These products only exist in the commutative setup. Let us choose $2p$ elements $a_0, \dots, a_{2p-1} \in \mathcal{A}$. By an application of the exponential function we get the invertibles $e^{a_0}, \dots, e^{a_{2p-1}} \in \mathcal{A}^*$. Each of those determine a class in the first algebraic K -group and we can thus form their interior Loday product,

$$[x] = [e^{a_0}] * \dots * [e^{a_{2p-1}}] \in K_{2p}(\mathcal{A})$$

Our aim is to get a concrete formula for the application of the multiplicative character to this higher Steinberg symbol. Notice that this is possible in the one dimensional case. Indeed, as mentioned earlier, the Loday product on algebraic K -theory identifies with the algebraic Steinberg symbol. To be precise we have that

$$[e^{a_0}] * [e^{a_1}] = \{e^{a_1}, e^{a_0}\}$$

Furthermore, by Theorem 1.3 the multiplicative character agrees with the determinant invariant. Thus, we need to calculate the quantity

$$(\exp \circ \mathcal{M}_F)\{e^{a_1}, e^{a_0}\} = d_F\{e^{a_1}, e^{a_0}\}$$

By (7) this is the determinant of a multiplicative commutator, namely

$$d_F\{e^{a_1}, e^{a_0}\} = \det(e^{Pa_1P} e^{Pa_0P} e^{-Pa_1P} e^{-Pa_0P}) = e^{\text{Tr}[Pa_1P, Pa_0P]}$$

Here the last identity is taken from (9). We have thus proved that

$$\mathcal{M}_F[e^{a_0}] * [e^{a_1}] = -[\text{Tr}[Pa_0P, Pa_1P]] \in \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

Let us return to the general case. We start by showing that $[x] \in K_{2p}(\mathcal{A})$ is in the image of $\theta : K_{2p}^{\text{rel}}(\mathcal{A}) \rightarrow K_{2p}(\mathcal{A})$. This is immediate from the multiplicative properties of this homomorphism. Indeed, with $[\gamma] \in K_{2p}^{\text{rel}}(\mathcal{A})$ defined by an iterated relative Loday product as in 12 we get that

$$\theta([\gamma]) = \theta([\gamma_0]) * \dots * \theta([\gamma_{2p-1}]) = [\gamma_0(1)^{-1}] * \dots * [\gamma_{2p-1}(1)^{-1}] = [x]$$

The multiplicative character of $[x] \in K_{2p}(\mathcal{A})$ is now expressed in terms of the additive character of the lift $[\gamma] \in K_{2p}^{\text{rel}}(\mathcal{A})$. But we already calculated this quantity in 14. In particular, we get that

$$(15) \quad \mathcal{M}_F([x]) = [\mathcal{A}_F([\gamma])] = -\left[\frac{1}{p!} \langle Pa_0P, \dots, Pa_{2p-1}P \rangle\right] \in \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

We give a rather detailed discussion of this result in Section 6.3. We will thus limit ourselves to one single comment at this point: The left hand side of (15) only depends on the exponentials $e^{a_i} \in \mathcal{A}^*$ whereas the right hand side is expressed in terms of the fundamental trace form of the logarithms a_i . This supports our point of view, that the multiplicative character should be understood as a higher determinant invariant. Let us finish this general introduction by recapitulating the achievements of the thesis:

- (1) We have understood that the higher dimensional generalization of the determinant invariant should be the multiplicative character of A. Connes and M. Karoubi.
- (2) We have argued that the correspondence between the trace invariant and the determinant invariant is generalized to a relationship between a higher additive character and the multiplicative character.
- (3) We have seen that the framework given by the additive character and the multiplicative character allows us to produce identities between determinants of multiplicative commutators and traces of additive commutators. Indeed, we were able to give a concrete formula for the application of the multiplicative character to higher Steinberg symbols. This calculation was accomplished by means of a calculation of the corresponding additive character. The final result could be expressed by the fundamental trace form of

J. W. Helton and R. E. Howe,

$$\mathcal{M}_F([e^{a_0}] * \dots * [e^{a_{2p-1}}]) = -\left[\frac{1}{p!} \langle Pa_0P, \dots, Pa_{2p-1}P \rangle\right] \in \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

This result can be interpreted as a multivariate version of the classical identity

$$\det(e^{PaP} e^{PbP} e^{-PaP} e^{-PbP}) = e^{\text{Tr}([PaP, PbP])}$$

We will now briefly go through the organization of the material in the present thesis.

In Section 2 we give an account of the Fredholm determinant from a K -theoretic point of view. This part is quite basic and also serves to introduce the reader to some of the ideas and techniques which we apply later in the thesis.

In Section 3 we introduce the determinant invariant and the trace invariant. We calculate these homomorphisms on relative and algebraic Steinberg symbols. At last we prove our comparison result which explicits the precise relation between these two invariants. This part is still rather basic in so far that we only use homological descriptions of the involved K -groups. There is a very close connection to the first article [16] which is attached at the end of the thesis. The methods applied there are however slightly different. We feel that the exposition given in Section 3 is both more clear and more conceptual.

We continue by an examination of the relationship between determinant invariants and the multiplicative characters of A. Connes and M. Karoubi. This is related to the material in [9, Section 5] and in the article [16]. The tools which we have evolved during Section 3 allow us to prove a refinement of [9, Théorème 5.6]. Namely, in the universal case, we show that the multiplicative character factorizes through the second algebraic K -group of the Calkin algebra, $\mathcal{L}(H)/\mathcal{L}^1(H)$. The refinement is also contained in [16]. This ends the study of the low dimensional case.

In Section 5 we expose the general results on the multiplicative structure in relative K -theory. We also indicate how the relative Loday product relates to the algebraic Loday product. This part can be understood as a survey of some of the material which can be found in the second article attached at the end of the thesis, [17]. We end the section by showing that the relative Steinberg symbol is a special instance of the relative Loday product.

In Section 6 we go through the main results of the article [17]. Namely, we expose the concrete calculations of the additive and multiplicative characters of Fredholm modules. The raw description of these results is supplied by a detailed discussion of some immediate implications.

The final section is devoted to a concrete example. Namely, we consider the case of a torus of odd dimension. In this setup we show that the application of the multiplicative character to higher Steinberg symbols is given by an integral formula. To be precise, we get that

$$(16) \quad \mathcal{M}_F([e^{f_0}] * \dots * [e^{f_{2p-1}}]) = \left[C \int_{\mathbb{T}^{2p-1}} f_0 df_1 \wedge \dots \wedge df_{2p-1}\right] \in \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

where $C \in \mathbb{C}$ is an appropriate constant. In view of the above considerations, this result is by no means surprising. Indeed, our Fredholm module arises from a spectral triple and we essentially need to evaluate the character

$$\tau_F : HC_{2p-1}(C^{p+2}(\mathbb{T}^{2p-1})) \rightarrow \mathbb{C}$$

on a *Hochschild cycle*. It is well known that this quantity can be expressed in terms of an integral over the torus. By the same method we could also have found formulas similar to (16) in a more general context. For example, the odd dimensional torus could be replaced by a spin manifold of odd dimension, see [8] and [11]. For the sake of simplicity we have limited ourselves to this specific setup.

Acknowledgements: I would very much like to thank Ryszard Nest for encouraging me and listening to me patiently and continuously through the whole period of my PhD program. I remain very grateful to him. It has been highly scientifically stimulating and a general great pleasure to have him as my advisor. My thanks also go to Boris Tsygan, Jerome Kaminker and Max Karoubi for the nice talks which I have had with them on various subjects of the thesis. Finally, I would like to thank all the people at the University of Copenhagen which I have had the pleasure of knowing. Many of them has been truly great. In this respect, I should definitely mention Uuye Otgonbayar since on several occasions I have had the possibility of discussing with him some of my ideas on "smooth K -homology". The contact with him has been indeed valuable. For example, he brought my attention to the work of A. Gorokhovsky on invariance of the index cocycle in cyclic cohomology, [12]. This has helped me to greatly improve the proofs of some of the results in the article [18]. I would therefore like to thank him many times.

2. THE FREDHOLM DETERMINANT

We will give an account of the classical Fredholm determinant from the point of view of algebraic K -theory. In particular, we will give homological arguments for the main properties of the Fredholm determinant. The present approach will thus be rather different from the usual one, that is, we will only use a very limited amount of operator theory. We hope that this will serve two purposes: First of all, the reader will get an introduction to the techniques which will be used throughout the thesis, secondly, he or she might get a feeling of the power of algebraic K -theory in dealing with certain operator theoretic questions.

2.1. The Schatten ideals and the operator trace. We will briefly introduce the Schatten ideals. They are generated by selfadjoint compact operators with some extra conditions on the decay of the eigenvalues. We will then introduce the operator trace which is a generalization of the usual trace on the algebra of complex matrices. For a more detailed introduction to these matters we refer to [33].

Let H be a separable Hilbert space. Let $\mathcal{L}(H)$ denote the C^* -algebra of bounded operators on H and let $\mathcal{K}(H) \subseteq \mathcal{L}(H)$ denote the C^* -algebra of compact operators. Recall that each compact operator $T \in \mathcal{K}(H)$ has a sequence $s_0(T) \geq s_1(T) \geq s_2(T) \geq \dots$ of singular values associated to it. The entrances are the eigenvalues of the absolute value, $|T| = (T^*T)^{1/2}$, when counted with multiplicities.

Definition 2.1. Let $p \in [1, \infty)$. By the p^{th} Schatten ideal we will understand the subset of compact operators defined by

$$T \in \mathcal{L}^p(H) \Leftrightarrow (T \in \mathcal{K}(H) \text{ and } \sum_{n=0}^{\infty} |s_n(T)|^p < \infty)$$

Thus, for each operator $T \in \mathcal{L}^p(H) \subseteq \mathcal{K}(H)$, the associated sequence of singular values is p -summable.

Let $p, q \in [1, \infty)$. The Schatten ideals satisfy the following properties:

- (1) The p^{th} Schatten ideal, $\mathcal{L}^p(H)$, is an ideal in $\mathcal{L}(H)$ which is closed under adjoints.
- (2) For each $T \in \mathcal{L}^p(H)$ and $S \in \mathcal{L}^q(H)$ the product $TS \in \mathcal{L}^r(H)$ is in the r^{th} Schatten ideal, whenever $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.
- (3) The application $\|\cdot\|_p : \mathcal{L}^p(H) \rightarrow [0, \infty)$ given by $\|T\|_p = (\sum_{n=0}^{\infty} |s_n(T)|^p)^{1/p}$ defines a Banach $*$ -algebra norm on $\mathcal{L}^p(H)$. The p^{th} Schatten ideal is complete in this norm.
- (4) Whenever $T \in \mathcal{L}^p(H)$ and $S, R \in \mathcal{L}(H)$ we have the inequalities

$$\|STR\|_p \leq \|S\|_{\infty} \cdot \|T\|_p \cdot \|R\|_{\infty} \quad \text{and} \quad \|T\|_{\infty} \leq \|T\|_p$$

Here $\|\cdot\|_{\infty} : \mathcal{L}(H) \rightarrow [0, \infty)$ denotes the operator norm.

The first Schatten ideal $\mathcal{L}^1(H)$ comes equipped with an important continuous linear functional called the operator trace

$$\text{Tr} : \mathcal{L}^1(H) \rightarrow \mathbb{C} \quad \text{Tr}(T) = \sum_{n=0}^{\infty} \langle T e_n, e_n \rangle$$

Here $(e_n)_{n=0}^{\infty}$ is *any* orthonormal basis for the separable Hilbert space H . The operator trace satisfies the tracial property $\text{Tr}(TS) = \text{Tr}(ST)$ whenever the product $TS \in \mathcal{L}^1(H)$ is of trace class.

The Fredholm determinant, which we will soon define, is to be understood as a multiplicative analogue of the operator trace.

2.2. The first relative K -group of Banach algebras and their ideals. A unital Banach algebra admits two well studied K -theories, a topological K -theory and an algebraic K -theory. These two K -theories are related by a group homomorphism and the difference between them is measured by a third K -theory which is called relative K -theory. We will introduce the first relative K -group and see how it relates to its algebraic and topological counterparts. A reference for this part is certainly the book of M. Karoubi where the relative K -theory is introduced for the first time, [20]. We will then continue by defining the first relative K -group of an ideal in a Banach algebra. In fact, we will have to construct the relative K -theory when the ideal is non-closed but satisfies some milder conditions. Our main example is the Schatten ideals sitting inside the bounded operators. We shall see later that these "relative-relative" considerations can be used to define a "trace part" of the Fredholm determinant.

Let A be a unital Banach algebra. We let $GL(A)$ denote the topological group of invertible matrices over A . As a group this is the algebraic inductive limit of the general linear groups over A

$$GL(A) := \lim_{n \rightarrow \infty} GL_n(A)$$

The topology is given by considering $GL(A)$ as a subspace of the normed inductive limit of matrices over A

$$GL(A) \subseteq \lim_{n \rightarrow \infty} M_n(A)$$

Note for this, that each of the matrix algebras $M_n(A)$ becomes a Banach algebra when equipped with the norm $\|(a_{ij})\| = \sum_{i,j} \|a_{ij}\|$, for example. The inclusions $M_n(A) \rightarrow M_{n+k}(A)$ are then isometries.

Now, let $R(A)_1$ denote the group of continuous maps $\sigma : [0, 1] \rightarrow GL(A)$ with $\sigma(0) = 1$. The group structure is given by the pointwise product. We will then say that two elements $\sigma, \tau \in R(A)_1$ are equivalent if there is a continuous map $H : [0, 1] \times [0, 1] \rightarrow GL(A)$ such that

$$\begin{aligned} H(0, t) &= \sigma(t) & H(1, t) &= \tau(t) \\ H(s, 1) &= \sigma(1) & H(s, 0) &= 1 \end{aligned}$$

This determines an equivalence relation on $R(A)_1$ which we call *homotopy with fixed endpoints*, it is denoted by \sim . We let $F(A)_1$ denote the smallest subgroup of $R(A)_1$ which contains all multiplicative commutators. The image of this subgroup under the quotient map $R(A)_1 \rightarrow R(A)_1 / \sim$ will then be denoted by $F(A)_1 / \sim$. Note that $F(A)_1$ is automatically normal in $R(A)_1$.

Definition 2.2. *By the first relative K -group of the unital Banach algebra A we will understand the quotient group*

$$K_1^{\text{rel}}(A) := (R(A)_1 / \sim) / (F(A)_1 / \sim)$$

When equipped with the pointwise product this is clearly an abelian group.

Each continuous unital algebra homomorphism $\varphi : A \rightarrow B$ induces a group homomorphism $\varphi_ : K_1^{\text{rel}}(A) \rightarrow K_1^{\text{rel}}(B)$ in a functorial way.*

Let us see how the first relative K -group relates to algebraic K -theory and topological K -theory. We will recall the relevant definitions. Some good references for topological K -theory are [1, 21]. A good reference for algebraic K -theory is [32].

Definition 2.3. *By the first topological K -group of the unital Banach algebra A we will understand the quotient group*

$$K_1^{\text{top}}(A) := GL(A) / GL_0(A)$$

Here $GL_0(A)$ is the path component of the identity in the topological group of invertible matrices $GL(A)$. The first topological K -group is an abelian group.

By the second topological K -group of the unital Banach algebra A we will understand the fundamental group of the invertible matrices over A , $GL(A)$. Thus, by definition

$$K_2^{\text{top}}(A) := \pi_1(GL(A))$$

The basepoint is the unit $1 \in GL(A)$. The group structure coincides with the pointwise product of loops. This makes the second topological K -group into an abelian group.

Each continuous unital algebra homomorphism $\varphi : A \rightarrow B$ induces a group homomorphism $\varphi_ : K_*^{\text{top}}(A) \rightarrow K_*^{\text{top}}(B)$ in a functorial way.*

Definition 2.4. *By the first algebraic K -group of the unital ring R we will understand the quotient group*

$$K_1(R) := GL(R) / E(R)$$

Here $E(R) \subseteq GL(R)$ denotes the subgroup of elementary matrices in $GL(R)$. To be precise, $E(R)$ is generated by the matrices, $e_{ij}(r)$, with 1 on the diagonal and $r \in R$ in position (i, j)

with $i \neq j$. This identifies with the smallest subgroup of $GL(R)$ which contains all commutators. Note that $E(R)$ is automatically normal in $GL(R)$.

Each unital ring homomorphism $\varphi : R \rightarrow S$ induces a group homomorphism $\varphi_* : K_1(R) \rightarrow K_1(S)$ in a functorial way. The first algebraic K -group is clearly abelian.

We observe that the first algebraic K -group is defined for any unital ring R whereas the first topological K -group takes the topology of the unital Banach algebra A into account. Indeed, the topology is needed to define the path component of the identity in $GL(A)$. We note that the first algebraic K -group and the first topological K -group are non-isomorphic in general. For example, $K_1(\mathbb{C}) = \mathbb{C}^*$ and $K_1^{\text{top}}(\mathbb{C}) = \{0\}$.

Let A be a unital Banach algebra. We have the following group homomorphisms between the K -groups:

- (1) The group of elementary matrices is a subgroup of the connected component of the identity, $E(A) \subseteq GL_0(A)$, we therefore have a quotient map $q : K_1(A) = GL(A)/E(A) \rightarrow GL(A)/GL_0(A) = K_1^{\text{top}}(A)$.
- (2) There is a group homomorphism $\theta : K_1^{\text{rel}}(A) \rightarrow K_1(A)$ which on generators $\sigma \in R(A)_1$ is given by $\theta(\sigma) = \sigma(1)^{-1}$. Note that $\sigma(1)^{-1} \in E(A)$ whenever $\sigma \in F(A)_1$.
- (3) There is a group homomorphism $\partial : K_2^{\text{top}}(A) \rightarrow K_1^{\text{rel}}(A)$ which on generators $\gamma : S^1 \rightarrow GL(A)$ is given by $\partial(\gamma)(t) = \gamma(e^{2\pi it})$.

We leave it to the reader to check that these maps are well defined and that they are group homomorphisms. The relation between the K -groups can now be expressed by an exact sequence.

Theorem 2.5. *For any unital Banach algebra A there is an exact sequence of abelian groups*

$$K_2^{\text{top}}(A) \xrightarrow{\partial} K_1^{\text{rel}}(A) \xrightarrow{\theta} K_1(A) \xrightarrow{q} K_1^{\text{top}}(A) \longrightarrow 0$$

Proof. The proof is similar to the proof of Theorem 2.9 and is therefore omitted. \square

2.2.1. The first relative K -groups of Banach algebras and their ideals. We will now introduce the relative K -theory of an ideal J sitting inside a Banach algebra A . It will thus be relative in two ways. Firstly, it compares the topological and algebraic K -theory of the ideal inside the Banach algebra. Secondly, it compares the relative K -theory of the whole Banach algebra and the relative K -theory of the quotient algebra when this is defined. In order to treat the example of the Schatten ideals sitting inside the bounded operators we will need to work with non-closed ideals. This has the disadvantage that the quotient algebra A/J is not a Banach algebra. We will put some conditions on the pair (J, A) to get a sensible definition in this general situation.

Let A be a unital Banach algebra and let $J \subseteq A$ be an ideal, satisfying the following conditions:

- (1) There is a norm, $\|\cdot\|_J : J \rightarrow [0, \infty)$, which makes the ideal J into a Banach algebra.
- (2) For each $j \in J$ and $x, y \in A$ we have the inequality $\|xjy\|_J \leq \|x\|_A \|j\|_J \|y\|_A$. Here $\|\cdot\|_A : A \rightarrow [0, \infty)$ denotes the norm on A .
- (3) For each $j \in J$ we have the inequality $\|j\|_A \leq \|j\|_J$.

These properties are also satisfied when we pass to the matrix Banach algebras. Notice that we do not require the ideal J to be closed in A . However, if the ideal is closed, the pair (J, A) certainly satisfies the above conditions.

Let $J^+ = J \oplus \mathbb{C}$ denote the unitalization of J . We let $GL(J) \subseteq GL(J^+)$ denote the subgroup of invertible matrices over J^+ given by

$$g \in GL(J) \Leftrightarrow (g \in GL(J^+) \text{ and } g - 1 \in M_\infty(J))$$

Here $M_\infty(J)$ denotes the normed inductive limit of the matrices over J . The group $GL(J)$ becomes a topological group when given the topology induced by the metric $d(g, h) = \|g - h\|_{M_\infty(J)}$.

Now, let $R(J)_1$ denote the group of continuous maps $\sigma : [0, 1] \rightarrow GL(J)$ with $\sigma(0) = 1$. The group structure is given by the pointwise product. As in section 2.2 we will say that two elements $\sigma, \tau \in R(J)_1$ are equivalent if there is a continuous map $H : [0, 1] \times [0, 1] \rightarrow GL(J)$ such that

$$\begin{aligned} H(0, t) &= \sigma(t) & H(1, t) &= \tau(t) \\ H(s, 1) &= \sigma(1) & H(s, 0) &= 1 \end{aligned}$$

This determines an equivalence relation on $R(J)_1$ which we call *homotopy with fixed endpoints*, it is denoted by \sim . We let $F(J, A)_1$ denote the smallest subgroup of $R(J)_1$ such that

$$\sigma\tau\sigma^{-1}\tau^{-1} \in F(J, A)_1 \quad \text{for all } \sigma \in R(A)_1 \text{ and } \tau \in R(J)_1$$

The notation $(F(J, A)_1 / \sim) \subseteq R(J)_1 / \sim$ will then refer to the image of $F(J, A)_1$ under the quotient map. Note that $F(J, A)_1$ is automatically normal in both $R(A)_1$ and $R(J)_1$.

Definition 2.6. *By the first relative K -group of the pair (J, A) we will understand the quotient group*

$$K_1^{\text{rel}}(J, A) := (R(J)_1 / \sim) / (F(J, A)_1 / \sim)$$

Each continuous unital algebra homomorphism $\varphi : A \rightarrow B$ which restricts to a continuous algebra homomorphism $\varphi|_J : J \rightarrow I$ induces a group homomorphism $\varphi_ : K_1^{\text{rel}}(J, A) \rightarrow K_1^{\text{rel}}(I, B)$ in a functorial way. The first relative K -group of the pair (J, A) is clearly an abelian group.*

Note that the above definition depends on the enveloping Banach algebra A . We can not expect these relative K -groups to satisfy excision.

As above we would like to relate the different K -groups arising from the pair (J, A) . Let us recall the relevant definitions.

Definition 2.7. *By the first topological K -group of the ideal $J \subseteq A$ we will understand the quotient group*

$$K_1^{\text{top}}(J) = GL(J) / GL_0(J)$$

Here $GL_0(J)$ denotes the path component of the identity in $GL(J)$. The first topological K -group of the ideal is an abelian group.

By the second topological K -group of the ideal $J \subseteq A$ we will understand the fundamental group of the invertible matrices $GL(J)$, thus

$$K_2^{\text{top}}(J) = \pi_1(GL(J))$$

The basepoint is given by the identity in $1 \in GL(J)$. The group structure coincides with the pointwise product. This makes the second topological K -group of the ideal into an abelian group. Each continuous algebra homomorphism $\varphi : J \rightarrow I$ induces a group homomorphism $\varphi_* : K_*^{\text{top}}(J) \rightarrow K_*^{\text{top}}(I)$ in a functorial way.

Note that the definition of topological K -theory is independent of the unital Banach algebra A . This is due to the excision property of the topological K -groups, [1, Theorem 5.4.2].

Definition 2.8. By the first relative algebraic K -group of an ideal I in a unital ring R we will understand the quotient group

$$K_1(I, R) = GL(I)/[GL(R), GL(I)]$$

Here $[GL(R), GL(I)]$ denotes the smallest subgroup of $GL(R)$ such that

$$ghg^{-1}h^{-1} \in [GL(R), GL(I)] \quad \text{whenever } g \in GL(R) \text{ and } h \in GL(I)$$

The group $[GL(R), GL(I)]$ is automatically normal in $GL(R)$.

Clearly, the first relative algebraic K -group is an abelian group. Each unital ring homomorphism $\varphi : R \rightarrow S$ which restricts to a ring homomorphism $\varphi : I \rightarrow J$ induces a group homomorphism $\varphi_* : K_1(I, R) \rightarrow K_1(J, S)$ in a functorial way.

Note that the definition of the relative algebraic K -theory depends on the enveloping unital ring R . The algebraic K -groups do not satisfy excision in general.

We can now express the relations between the relative K -groups of the pair (J, A) by an exact sequence:

Theorem 2.9. Let A be a unital Banach algebra. For any ideal $J \subseteq A$ which satisfies the conditions (1), (2) and (3) there is an exact sequence of abelian groups

$$K_2^{\text{top}}(J) \xrightarrow{\partial} K_1^{\text{rel}}(J, A) \xrightarrow{\theta} K_1(J, A) \xrightarrow{q} K_1^{\text{top}}(J) \longrightarrow 0$$

The definition of the group homomorphisms is analogous to the non-relative case detailed out in section 2.2.

Proof. We will only prove exactness at $K_1^{\text{rel}}(J, A)$. Clearly $\theta \circ \partial = 0$. Thus, suppose that $\theta[\sigma] = 0$ for some $\sigma \in R(J)_1$. This means that $\sigma(1)^{-1} \in [GL(A), GL(J)]$. By a standard K -theoretic argument we may assume that

$$\sigma(1)^{-1} = \prod_{i=1}^n (g_i h_i g_i^{-1} h_i^{-1})$$

with $g_i \in GL_0(A)$ and $h_i \in GL_0(J)$. We can therefore choose elements $\gamma_i \in R(A)_1$ and $\tau_i \in R(J)_1$ such that

$$\gamma_i(1) = g_i \quad \text{and} \quad \tau_i(1) = h_i$$

We define the element $\beta \in F(J, A)_1$ by the product

$$\beta = \prod_{i=1}^n (\gamma_i \tau_i \gamma_i^{-1} \tau_i^{-1})$$

It follows that $[\sigma] = [\sigma \cdot \beta]$ in $K_1^{\text{rel}}(J, A)$ and that $(\sigma\beta)(1) = 1$. This proves the Theorem. \square

As stated in the beginning of this section, the first relative K -group also compares the relative K -theory of the whole Banach algebra and the relative K -theory of the quotient Banach algebra when this is defined. This statement is clarified by the next Theorem.

Theorem 2.10. *Let $J \subseteq A$ be a closed ideal in a unital Banach algebra A . Suppose that there exists a continuous linear map $s : A/J \rightarrow A$ such that $q \circ s = \text{Id}_{A/J}$. Here $q : A \rightarrow A/J$ denotes the quotient map. We then have an exact sequence of abelian groups*

$$K_1^{\text{rel}}(J, A) \xrightarrow{i_*} K_1^{\text{rel}}(A) \xrightarrow{q_*} K_1^{\text{rel}}(A/J) \longrightarrow 0$$

The group homomorphisms are induced by the inclusion $i : J \rightarrow A$ and the quotient map $q : A \rightarrow A/J$.

Proof. We will only prove exactness at $K_1^{\text{rel}}(A)$. Clearly, the composition $q_* \circ i_* = 0$. Thus, suppose that $q_*[\sigma] = 0$ for some $\sigma \in R(A)_1$. We thus have a homotopy $H : [0, 1] \times [0, 1] \rightarrow GL(A/J)$ with fixed endpoints between $q \circ \sigma$ and some commutator $\tau \in F(A/J)_1$. The existence of the continuous linear section $s : A/J \rightarrow A$ together with [1, Corollary 3.4.4] ensure us that we can find a homotopy *with fixed endpoints* $\tilde{H} : [0, 1] \times [0, 1] \rightarrow GL(A)$ such that

$$q \circ \tilde{H} = H \quad \text{and} \quad \tilde{H}(t, 0) = \sigma(t) \text{ for all } t \in [0, 1]$$

By a similar argument we can find a lift of the commutator $\tau \in F(A/J)_1$ to a commutator $\tilde{\tau} \in F(A)_1$. In particular, we can represent the class $[\sigma] \in K_1^{\text{rel}}(A)$ by the continuous map

$$t \mapsto \tilde{H}(t, 1) \cdot \tilde{\tau}^{-1}(t)$$

The proof ends by noticing that

$$q(\tilde{H}(t, 1) \cdot \tilde{\tau}^{-1}(t)) = 1 \quad \text{for all } t \in [0, 1]$$

\square

2.3. The first relative Chern character of an ideal in a Banach algebra. We will consider the setup of an ideal J in a unital Banach algebra A . The pair (J, A) is assumed to satisfy the conditions of section 2.2.1, (1), (2) and (3). We will then construct the first relative Chern character. It is going to be a homomorphism of degree minus one

$$\text{ch}^{\text{rel}} : K_1^{\text{rel}}(J, A) \rightarrow HC_0(J, A)$$

from relative K -theory to relative continuous cyclic homology. In the case where $J = \mathcal{L}^1(H) \subseteq A = \mathcal{L}(H)$ this will be an essential ingredient in our K -theoretic definition of the Fredholm determinant. In fact, as we shall see, the relative Chern character will behave as a logarithm and it will help us define the "trace part" of the Fredholm determinant.

The relative Chern character was originally introduced on relative K -theory by M. Karoubi, [20]. The formula which we use was found by A. Connes and M. Karoubi in [9]. One should also notice the work by U. Tillmann, [35]. However, all results are limited to the non-relative case.

For any two Banach algebras A and B we let $A \otimes_\pi B$ denote the projective tensorproduct of A and B . Now, let $J \subseteq A$ be an ideal which satisfies the conditions of section 2.2.1, (1), (2) and (3). We will then define a norm

$$\|\cdot\|_{(J,A)} : J \otimes_{\mathbb{C}} A + A \otimes_{\mathbb{C}} J \rightarrow [0, \infty)$$

on the vector space $J \otimes_{\mathbb{C}} A + A \otimes_{\mathbb{C}} J$. Namely, we let

$$\|x\|_{(J,A)} := \inf \left\{ \sum_{i=1}^n (\|s_i\|_J \cdot \|a_i\|_A + \|b_i\|_A \cdot \|t_i\|_J) \mid x = \sum_{i=1}^n (s_i \otimes a_i + b_i \otimes t_i) \right\}$$

We let $J \otimes_\pi A + A \otimes_\pi J$ denote the corresponding completion of $J \otimes_{\mathbb{C}} A + A \otimes_{\mathbb{C}} J$. Note that the product determines a continuous map

$$m : J \otimes_\pi A + A \otimes_\pi J \rightarrow J \quad m(s \otimes a + b \otimes t) = sa + bt$$

We are now ready to make the appropriate definitions.

Definition 2.11. *By the zeroth relative continuous cyclic homology group of the pair (J, A) we will understand the quotient space*

$$HC_0(J, A) = J/\text{Im}(b)$$

Here $b : J \otimes_\pi A + A \otimes_\pi J \rightarrow J$ extends from the Hochschild boundary

$$b : s \otimes a + b \otimes t \mapsto sa - as + bt - tb$$

Note that the topological vector space $HC_0(J, A)$ is non-Hausdorff in general. That is, the image of the Hochschild boundary is not necessarily closed in J .

Definition 2.12. *By the first relative Chern character of the pair (J, A) we will understand the homomorphism*

$$\text{ch}^{\text{rel}} : K_1^{\text{rel}}(J, A) = (R(J)_1 / \sim) / (F(J, A)_1 / \sim) \rightarrow HC_0(J, A) = J/\text{Im}(b)$$

which is induced by the application

$$(17) \quad \text{ch}^{\text{rel}} : R(J)_1 \rightarrow J \quad \sigma \mapsto \text{TR} \left(\int_0^1 \frac{d\sigma}{dt} \cdot \sigma^{-1} dt \right)$$

Here $\text{TR} : M_\infty(J) \rightarrow J$ is the generalized trace given by the sum over the diagonal.

The next lemmas will certify that the application in (17) yields a well defined homomorphism on the first relative K -group of the pair (J, A) . We will however bypass at least one technical subtlety: The definitions in Section 2.2 only concern *continuous* maps. None the less the above definition uses derivatives. This is by no means a problem. We can always assume that our classes are represented by *smooth* maps and that our homotopies are likewise *smooth*. The reader who is still feeling uncomfortable about this should consult the result [17, Lemma 3.2] which is to be found at the end of the thesis.

Lemma 2.13. *Let $\gamma_0, \gamma_1 : [0, 1] \rightarrow GL(J)$ be two smooth maps. Suppose that there exists a smooth homotopy with fixed endpoints between γ_0 and γ_1 . Then the difference $\text{ch}^{\text{rel}}(\gamma_1) - \text{ch}^{\text{rel}}(\gamma_0) \in \text{Im}(b)$ is contained in the image of $b : J \otimes_\pi A + A \otimes_\pi J \rightarrow J$. In particular we get a well defined map*

$$\text{ch}^{\text{rel}} : (R(J)_1 / \sim) \rightarrow HC_0(J, A)$$

Proof. To ease the exposition we will suppose that $\gamma_0, \gamma_1 : [0, 1] \rightarrow GL_1(J)$ are smooth maps with values in $GL_1(J)$. Likewise we will assume that the smooth homotopy with fixed endpoints $H : [0, 1] \times [0, 1] \rightarrow GL_1(J)$ takes values in $GL_1(J)$. Now, we let $L(H) \in J \otimes_\pi A + A \otimes_\pi J$ be given by the formula

$$L(H) = - \int_0^1 \int_0^1 \frac{\partial H}{\partial t} \cdot H^{-1} \otimes \frac{\partial H}{\partial s} \cdot H^{-1} dt ds$$

We claim that $b(L(H)) = \text{ch}^{\text{rel}}(\gamma_1) - \text{ch}^{\text{rel}}(\gamma_0)$.

Since $b : A \otimes_\pi J + J \otimes_\pi A \rightarrow J$ is continuous and linear we have

$$b(L(H)) = - \int_0^1 \int_0^1 \left[\frac{\partial H}{\partial t} \cdot H^{-1}, \frac{\partial H}{\partial s} \cdot H^{-1} \right] dt ds$$

Then, using some properties of the partial differential operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ we get that

$$\frac{\partial}{\partial t} \left(\frac{\partial H}{\partial s} \cdot H^{-1} \right) - \frac{\partial}{\partial s} \left(\frac{\partial H}{\partial t} \cdot H^{-1} \right) = \left[\frac{\partial H}{\partial t} \cdot H^{-1}, \frac{\partial H}{\partial s} \cdot H^{-1} \right]$$

The fundamental theorem of analysis then allows us to conclude that

$$\begin{aligned} b(L(H)) &= \int_0^1 \left(\frac{\partial H}{\partial t} \cdot H^{-1} \right)(t, 1) dt - \int_0^1 \left(\frac{\partial H}{\partial t} \cdot H^{-1} \right)(t, 0) dt \\ &\quad - \int_0^1 \left(\frac{\partial H}{\partial s} \cdot H^{-1} \right)(1, s) ds + \int_0^1 \left(\frac{\partial H}{\partial s} \cdot H^{-1} \right)(0, s) ds \end{aligned}$$

By the assumptions on H this implies the identity

$$b(L(H)) = \int_0^1 \frac{d\gamma_1}{dt} \cdot \gamma_1^{-1} dt - \int_0^1 \frac{d\gamma_0}{dt} \cdot \gamma_0^{-1} dt$$

But this is the desired result. \square

Lemma 2.14. *The relative Chern character of Definition 2.12 is a well defined homomorphism.*

Proof. We will start by proving that $\text{ch}^{\text{rel}} : (R(J)_1 / \sim) \rightarrow HC_0(J, A)$ is a homomorphism. Thus, let $\gamma_0 : [0, 1] \rightarrow GL(J)$ and $\gamma_1 : [0, 1] \rightarrow GL(J)$ be two smooth maps with vanishing derivatives at 0 and 1 and with $\gamma_0(0) = \gamma_1(0) = 1$. Let $\gamma_0 * \gamma_1 : [0, 1] \rightarrow GL(J)$ be the smooth map which is given by

$$(\gamma_0 * \gamma_1)(t) = \begin{cases} \gamma_0(2t) & \text{for } t \in [0, 1/2] \\ \gamma_0(1)\gamma_1(2t-1) & \text{for } t \in [1/2, 1] \end{cases}$$

We can then construct a smooth homotopy with fixed endpoints between the pointwise product $\gamma_0\gamma_1 : [0, 1] \rightarrow GL(J)$ and the composition product $\gamma_0 * \gamma_1 : [0, 1] \rightarrow GL(J)$. It follows that

$$\text{ch}^{\text{rel}}(\gamma_0\gamma_1) = \text{ch}^{\text{rel}}(\gamma_0 * \gamma_1)$$

by an application of Lemma 2.13. However, it is not hard to check that

$$\text{ch}^{\text{rel}}(\gamma_0 * \gamma_1) = \text{ch}^{\text{rel}}(\gamma_0) + \text{ch}^{\text{rel}}(\gamma_1)$$

This proves that the relative Chern character is a homomorphism.

Now, let $\sigma \in R(A)_1$ and let $\tau \in R(J)_1$. We can then construct a smooth homotopy with fixed endpoints between the elements $\sigma\tau\sigma^{-1} \in R(J)_1$ and $\sigma(1)\tau\sigma(1)^{-1} \in R(J)_1$. It follows that

$$\text{ch}^{\text{rel}}(\sigma\tau\sigma^{-1}) = \text{ch}^{\text{rel}}(\sigma(1)\tau\sigma(1)^{-1})$$

by an application of Lemma 2.13. Furthermore, we clearly have

$$b(\sigma(1)\text{ch}^{\text{rel}}(\tau) \otimes \sigma^{-1}(1)) = \text{ch}^{\text{rel}}(\sigma(1)\tau\sigma(1)^{-1}) - \text{ch}^{\text{rel}}(\tau)$$

We can thus conclude that $\text{ch}^{\text{rel}}(\sigma\tau\sigma^{-1}) = \text{ch}^{\text{rel}}(\tau)$. It now follows easily that $\text{ch}^{\text{rel}}(\beta) = 0$ for each $\beta \in F(J, A)_1 / \sim$. This proves the desired result. \square

Remark 2.15. *In the case of a unital Banach algebra A , the relative Chern character is given by the same formula. Indeed, we have*

$$\text{ch}^{\text{rel}} : K_1^{\text{rel}}(A) \rightarrow HC_0(A) \quad \text{ch}^{\text{rel}}(\sigma) = \text{TR}(\int_0^1 \frac{d\sigma}{dt} \cdot \sigma^{-1} dt)$$

We will not consider this homomorphism in detail at this place, however, we shall find occasion to introduce it in a more general fashion in Section 6.1.

2.4. A K -theoretic definition of the Fredholm determinant. At this point we have introduced a sufficient amount of general structure to give our K -theoretic definition of the Fredholm determinant. We will see that the main properties of the determinant function will follow immediately from the corresponding properties of the K -groups. This is of course an advantage. However, as we shall see later, the main concern of this thesis is to find a concrete higher dimensional analogue of the Fredholm determinant. We therefore hope that the reader will notice one important thing: The Fredholm determinant is defined by means of an *additive character of a lift in relative K -theory*. We will interpret the Fredholm determinant as a *multiplicative analogue* of this additive character. Instead of trying to generalize the Fredholm determinant directly, one might try to find a generalization via a generalized additive character. In some cases this turns out to be possible. But how does one find a concrete formula for such a map? We will leave this question open for a while and continue the exposition.

Let us consider the case where $A = \mathcal{L}(H)$ is the unital Banach algebra of operators on a separable Hilbert space and $J = \mathcal{L}^1(H) \subseteq A = \mathcal{L}(H)$ is the operators of trace class. By the considerations in section 2.1 this ideal satisfies the conditions in section 2.2.1 (1), (2) and (3). Furthermore, recall that the operator trace $\text{Tr} : \mathcal{L}^1(H) \rightarrow \mathbb{C}$ is continuous and linear and satisfies the tracial property $\text{Tr}(TS) = \text{Tr}(ST)$ whenever $TS \in \mathcal{L}^1(H)$. In particular the composition

$$\text{Tr} \circ b : \mathcal{L}^1(H) \otimes_{\pi} \mathcal{L}(H) + \mathcal{L}(H) \otimes_{\pi} \mathcal{L}^1(H) \rightarrow \mathbb{C}$$

vanishes identically. It follows that the operator trace descends to a homomorphism

$$\text{Tr} : HC_0(\mathcal{L}^1(H), \mathcal{L}(H)) \rightarrow \mathbb{C} \quad \text{Tr}[T] = \text{Tr}(T)$$

on the zeroth continuous cyclic homology of the ideal $\mathcal{L}^1(H) \subseteq \mathcal{L}(H)$.

Definition 2.16. *By the additive character on the first relative K -group $K_1^{\text{rel}}(\mathcal{L}^1(H), \mathcal{L}(H))$ we will understand the homomorphism given by the composition*

$$\tau := (-\text{Tr}) \circ \text{ch}^{\text{rel}} : K_1^{\text{rel}}(\mathcal{L}^1(H), \mathcal{L}(H)) \rightarrow \mathbb{C}$$

of the relative Chern character and minus the operator trace on the zeroth continuous cyclic homology of the ideal $\mathcal{L}^1(H) \subseteq \mathcal{L}(H)$.

As mentioned above, this is the main ingredient in our definition of the Fredholm determinant. To continue we recall some results on the topological K -theory of the Banach algebra $\mathcal{L}^1(H)$.

Theorem 2.17. *We have the following isomorphisms in topological K -theory*

$$K_1^{\text{top}}(\mathcal{L}^1(H)) \cong \{0\} \quad K_2^{\text{top}}(\mathcal{L}^1(H)) \cong \mathbb{Z}$$

The isomorphism $\mathbb{Z} \rightarrow K_2^{\text{top}}(\mathcal{L}^1(H))$ is given by

$$n \mapsto \begin{cases} z \mapsto zp_n + 1 - p_n & \text{for } n \geq 0 \\ z \mapsto z^{-1}p_n + 1 - p_n & \text{for } n < 0 \end{cases}$$

Here, for each $n \in \mathbb{Z}$, the element $p_n \in \mathcal{L}^1(H)$ is a (finite rank) projection with trace $\text{Tr}(p_n) = |n|$. This isomorphism uses Bott-periodicity in topological K -theory. See for example [21, III Theorem 1.11]

Lemma 2.18. *We have the following commutative diagram of abelian groups*

$$\begin{array}{ccc} K_2^{\text{top}}(\mathcal{L}^1(H)) & \xrightarrow{\partial} & K_1^{\text{rel}}(\mathcal{L}^1(H), \mathcal{L}(H)) \\ \downarrow \cong & & \downarrow \tau \\ \mathbb{Z} & \xrightarrow{-2\pi i \cdot} & \mathbb{C} \end{array}$$

The boundary map ∂ is defined in section 2.2.1.

Proof. Let $n \in \mathbb{N}$ and let $p_n \in \mathcal{L}^1(H)$ be a projection with $\text{Tr}(p_n) = n$. We need to check that

$$\tau(t \mapsto p_n e^{2\pi i t} + 1 - p_n) = -2\pi i \cdot n$$

However, by definition of the additive character we have

$$\tau(t \mapsto p_n e^{2\pi i t} + 1 - p_n) = -\text{Tr}\left(\int_0^1 p_n \frac{d(e^{2\pi i t})}{dt} e^{2\pi i t} dt\right) = -\text{Tr}(p_n) 2\pi i$$

This proves the desired result. \square

Combining this result with the exact sequence of Theorem 2.9 and the calculation of the topological K -group $K_1^{\text{top}}(\mathcal{L}^1(H)) \cong \{0\}$ we get the following commutative diagram of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_2^{\text{top}}(\mathcal{L}^1(H)) & \xrightarrow{\partial} & K_1^{\text{rel}}(\mathcal{L}^1(H), \mathcal{L}(H)) & \xrightarrow{\theta} & K_1(\mathcal{L}^1(H), \mathcal{L}(H)) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \tau & & \\ & & \mathbb{Z} & \xrightarrow{-2\pi i \cdot} & \mathbb{C} & & \end{array}$$

The top row is a short exact sequence. We can thus make the following definition:

Definition 2.19. *By the Fredholm determinant we will understand the group homomorphism*

$$\det : K_1(\mathcal{L}^1(H), \mathcal{L}(H)) \rightarrow \mathbb{C}/(2\pi i)\mathbb{Z} \cong \mathbb{C}^*$$

induced by the above commutative diagram. To be precise, for any $[x] \in K_1(\mathcal{L}^1(H), \mathcal{L}(H))$ we let

$$\det([x]) = (\exp \circ \tau)[\sigma]$$

where $\sigma \in R(\mathcal{L}^1(H))_1$ is any element with $\theta([\sigma]) = [x]$.

As promised, we obtain the main properties of the Fredholm determinant as easy consequences of this definition.

Corollary 2.20. *For each $g, h \in GL(\mathcal{L}^1(H))$ and each $X \in GL(\mathcal{L}(H))$ we have*

$$\det([gh]) = \det([g])\det([h]) \quad \det([XgX^{-1}]) = \det([g])$$

For each $T \in \mathcal{L}^1(H)$ we have

$$\det([e^T]) = e^{\text{Tr}(T)}$$

In particular our definition of the Fredholm determinant agrees with the usual definition. See for example [39, 8. Corollary 2].

Proof. Let us prove that $\det([e^T]) = e^{\text{Tr}(T)}$ for each $T \in \mathcal{L}^1(H)$. We let $\gamma(t) = e^{-tT}$ for all $t \in [0, 1]$. It follows that $\gamma \in R(\mathcal{L}^1(H))_1$ determines a class in the first relative K -group. Furthermore, we clearly have

$$\theta([\gamma]) = [\gamma(1)^{-1}] = [e^T]$$

By definition of the Fredholm determinant we thus get that

$$\det([e^T]) = \exp(\tau[\gamma])$$

The following calculation then proves the desired result

$$\tau[\gamma] = -\text{Tr}\left(\int_0^1 \frac{d(e^{-tT})}{dt} e^{tT} dt\right) = \text{Tr}(T)$$

□

Corollary 2.21. *Let $g \in GL(\mathcal{L}^1(H))$ then the Fredholm determinant of g can be calculated using the following formula*

$$\det(g) = (\exp \circ \text{Tr})\left(-\int_0^1 \frac{d\gamma}{dt} \cdot \gamma^{-1} dt\right)$$

where $\gamma : [0, 1] \rightarrow GL(\mathcal{L}^1(H))$ is any smooth map with $\gamma(1)^{-1} = g$ and $\gamma(0) = 1$.

3. DETERMINANTS AND TRACES OF MULTIPLICATIVE AND ADDITIVE COMMUTATORS

Let $A, B \in \mathcal{L}(H)$ be two operators and suppose that their additive commutator $[A, B] \in \mathcal{L}^1(H)$ is of trace class. We will refer to this data as an almost commuting pair of operators. Note that we do not assume our operators to be selfadjoint. Now, applying the exponential function to A and B we get two invertible elements $e^A, e^B \in \mathcal{L}(H)^*$. The multiplicative commutator

$$e^A e^B e^{-A} e^{-B} \in GL_1(\mathcal{L}^1(H))$$

of these invertibles is then of determinant class. It is well known that the Fredholm determinant of this element can be expressed by the operator trace of an additive commutator, that is

$$(18) \quad \det(e^A e^B e^{-A} e^{-B}) = e^{\text{Tr}([A, B])}$$

A possible proof runs by the Campbell-Baker-Hausdorff formula, see for example [2, §3. Lemma 2 and 3]. In this section we will give a new K -theoretic proof of this identity. The techniques which we apply are related to the material in the articles at the end of the thesis. In fact, we are going to see that the determinant of a multiplicative commutator is the value of an invariant of

the second algebraic K -group. The relevant element in algebraic K -theory is an example of a Steinberg symbol. This result was first discovered by L. G. Brown in [3]. Furthermore, we shall see that the trace of a commutator is the value of an invariant of the second relative K -group. The relevant element in relative K -theory is an example of a relative Steinberg symbol. The desired identity will then follow from a general comparison result which expresses the relation between these two invariants. The proof which we are going to present of the identity (18) is by no means easier than the usual proofs. However, it places this formula in a framework which is of far greater generality.

3.1. Noncommutative and commutative C^2 -functions in two variables. Let $A, B \in \mathcal{L}(H)$ be an almost commuting pair of operators, thus, the additive commutator $[A, B] \in \mathcal{L}^1(H)$ is of trace class. We would like to extract information out of the commutativity of the operators in the Calkin algebra, $\mathcal{L}(H)/\mathcal{L}^1(H)$, versus their noncommutativity in the algebra of bounded operators, $\mathcal{L}(H)$. Our proof of the identity (18) will rely on the different K -theories associated to a unital Banach algebra. We therefore immediately face a technical problem: The trace ideal is not closed in the operator norm, so the quotient, $\mathcal{L}(H)/\mathcal{L}^1(H)$, is not a Banach algebra. In this section we will therefore set up a formal framework which encaptures the required information, namely the difference between commutativity and noncommutativity of two variables.

Let us consider two unital \mathbb{C} -algebras $\mathbb{C}\langle X, Y \rangle$ and $\mathbb{C}[x, y]$. The first one consists of noncommutative polynomials in the formal variables X and Y with coefficients in \mathbb{C} . The second one consists of commutative polynomials in the formal variables x and y with coefficients in \mathbb{C} .

Let us think of the variables x, y, X, Y as having radius $r > 0$. We can then equip the \mathbb{C} -algebra $\mathbb{C}[x, y]$ with a C^2 -norm

$$\|\cdot\|_{2,r} : \mathbb{C}[x, y] \rightarrow [0, \infty)$$

given on monomials by

$$\|\lambda_{n,m} x^n y^m\|_{2,r} = |\lambda_{n,m}| (r^{n+m} + (n+m)r^{n+m-1} + (n+m)(n+m-1)r^{n+m-2})$$

We extend this linearly to the whole \mathbb{C} -algebra. We denote the completion by $C_r^2[x, y]$. It is the unital Banach algebra of formal *commutative* C^2 -functions in the variables x and y of radius $r > 0$.

For each $k \in \mathbb{N}_0$ we define the index set $N_k = \mathbb{N}_0 \times \underbrace{\mathbb{N} \times \dots \times \mathbb{N}}_k$. We can then associate a noncommutative monomial in $\mathbb{C}\langle X, Y \rangle$ to each $\alpha \in N_k$, namely

$$(X, Y)^\alpha = \begin{cases} X^{n_0} Y^{n_1} X^{n_2} \dots Y^{n_{2m-1}} X^{n_{2m}} & \text{for } k = 2m \\ X^{n_0} Y^{n_1} X^{n_2} \dots Y^{n_{2m-3}} X^{n_{2m-2}} Y^{n_{2m-1}} & \text{for } k = 2m - 1 \end{cases}$$

These monomials generate $\mathbb{C}\langle X, Y \rangle$ as a vector space over \mathbb{C} . By the degree of $\alpha \in N_k$ we will understand the sum $|\alpha| = \sum_{j=0}^k n_j$. We can equip the \mathbb{C} -algebra $\mathbb{C}\langle X, Y \rangle$ with a C^2 -norm

$$\|\cdot\|_{2,r} : \mathbb{C}\langle X, Y \rangle \rightarrow [0, \infty)$$

On monomials it is given by the formula

$$\|\lambda_\alpha (X, Y)^\alpha\|_{2,r} = |\lambda_\alpha| (r^{|\alpha|} + |\alpha| r^{|\alpha|-1} + |\alpha|(|\alpha| - 1) r^{|\alpha|-2})$$

We extend this linearly to the whole \mathbb{C} -algebra. The completion will be denoted by $C_r^2\langle X, Y \rangle$. It is the unital Banach algebra of formal *noncommutative* C^2 -functions in the variables X and Y of radius $R > 0$.

Lemma 3.1. *There is a short exact sequence of Banach algebra*

$$0 \longrightarrow J_r \xrightarrow{i} C_r^2\langle X, Y \rangle \xrightarrow{q} C_r^2[x, y] \longrightarrow 0$$

Here the homomorphism of unital Banach algebras $q : C_r^2\langle X, Y \rangle \rightarrow C_r^2[x, y]$ is given by $X \mapsto x$ and $Y \mapsto y$. The ideal J_r is the smallest closed ideal in $C_r^2\langle X, Y \rangle$ which contains the additive commutator $[X, Y] \in C_r^2\langle X, Y \rangle$. The short exact sequence (3.1) has a continuous linear section $s : C_r^2[x, y] \rightarrow C_r^2\langle X, Y \rangle$ given by

$$s : \lambda_{n,m} x^n y^m \mapsto \lambda_{n,m} X^n Y^m$$

Proof. We leave the proof to the reader. □

Now, let us consider two almost commuting operators $A, B \in \mathcal{L}(H)$. We can assume that their operator norm is dominated by some constant $r > 0$, thus $\|A\|_\infty, \|B\|_\infty \leq r$. We therefore have a continuous unital algebra homomorphism

$$\pi^{A,B} : C_r^2\langle X, Y \rangle \rightarrow \mathcal{L}(H) \quad \pi^{A,B}(X) = A \quad \pi^{A,B}(Y) = B$$

We want to study the behaviour of the restriction of the representation $\pi^{A,B}$ to the ideal J_r . To do so we will need the following lemma.

Lemma 3.2. *For each index $\alpha = (n_0, n_1, \dots, n_k) \in N_k$ we have the inequality*

$$\|(A, B)^\alpha - A^{|\alpha_X|} B^{|\alpha_Y|}\|_1 \leq |\alpha_X| |\alpha_Y| \cdot \|[A, B]\|_1 r^{|\alpha|-2}$$

Here $|\alpha_X| \in \mathbb{N}_0$ denotes the X -degree and $|\alpha_Y| \in \mathbb{N}_0$ denotes the Y -degree of $\alpha \in N_k$.

Proof. For simplicity we will assume that $k = 2m$ is even. We then have

$$\begin{aligned} & \|(A, B)^\alpha - A^{|\alpha_X|} B^{|\alpha_Y|}\|_1 \\ &= \left\| \sum_{i=1}^m (A, B)^{(n_0, \dots, n_{2i-2})} [B^{n_{2i-1}}, A^{\sum_{j=i}^m n_{2j}}] B^{\sum_{j=i+1}^m n_{2j-1}} \right\|_1 \\ &\leq \sum_{i=1}^m \|[B^{n_{2i-1}}, A^{\sum_{j=i}^m n_{2j}}]\|_1 r^{\sum_{j=0}^{2i-2} n_j + \sum_{j=i+1}^m n_{2j-1}} \\ &\leq \sum_{i=1}^m n_{2i-1} \left(\sum_{j=i}^m n_{2j} \right) \|[A, B]\|_1 r^{|\alpha|-2} \\ &\leq |\alpha_X| |\alpha_Y| \cdot \|[A, B]\|_1 r^{|\alpha|-2} \end{aligned}$$

This is the desired inequality. □

The above lemma certifies that the C^2 -norm on J_r controls the 1-norm on $\mathcal{L}^1(H)$. We will make this statement explicit in the next lemma.

Lemma 3.3. *The restriction of the representation $\pi^{A,B} : C_r^2\langle X, Y \rangle \rightarrow \mathcal{L}(H)$ to the closed ideal J_r factorizes through the operators of trace class. The norm of the restriction $\pi^{A,B} : J_r \rightarrow \mathcal{L}^1(H)$ is dominated by the 1-norm of the commutator, $\|\pi^{A,B}\| \leq \|[A, B]\|_1$.*

Proof. Let J^{alg} denote the kernel of the algebraic quotient map $q : \mathbb{C}\langle X, Y \rangle \rightarrow \mathbb{C}[x, y]$. The existence of the continuous linear section $s : \mathbb{C}[x, y] \rightarrow \mathbb{C}\langle X, Y \rangle$ ensures that the kernel of $q : C_r^2\langle X, Y \rangle \rightarrow C_r^2[x, y]$ is the closure of J^{alg} in $C_r^2\langle X, Y \rangle$. Since the commutator $[A, B] \in \mathcal{L}^1(H)$ is of trace class, the restriction $\pi^{A,B} : J^{\text{alg}} \rightarrow \mathcal{L}(H)$ takes values in the trace ideal $\mathcal{L}^1(H)$. Therefore we will only need to show that $\pi^{A,B} : J^{\text{alg}} \rightarrow \mathcal{L}^1(H)$ is continuous.

Let $(n, m) \in \mathbb{N}_0 \times \mathbb{N}_0$ be fixed. We will consider a noncommutative polynomial

$$p = \sum_{\alpha \in N_{n,m}} \lambda_\alpha (X, Y)^\alpha \in \mathbb{C}\langle X, Y \rangle$$

with $\sum_{\alpha \in N_{n,m}} \lambda_\alpha = 0$. Here $N_{n,m}$ denotes the set of indices α with X -degree and Y -degree equal to n and m respectively. Polynomials of the above type generates J^{alg} as a vectorspace. Now, by the assumption on $p \in J^{\text{alg}}$ we have

$$\|\pi^{A,B}(p)\|_1 = \|\pi^{A,B}(p) - \sum_{\alpha \in N_{n,m}} \lambda_\alpha A^{|\alpha_X|} B^{|\alpha_Y|}\|_1$$

By Lemma 3.2 we then get that

$$\|\pi^{A,B}(p)\|_1 \leq \sum_{\alpha \in N_{n,m}} |\lambda_\alpha| |\alpha_X| |\alpha_Y| \cdot \|[A, B]\|_1 r^{|\alpha|-2} \leq \|p\|_{2,r} \|[A, B]\|_1$$

This proves the desired result. \square

3.2. The determinant invariant and determinants of multiplicative commutators.

As mentioned in the beginning of this section we would like to understand the determinant of a multiplicative commutator in a K -theoretic framework. To do so, we will need to introduce a couple of K -theoretic concepts. First of all, we will define the second algebraic K -group of a unital ring. Next, considering an ideal in a unital ring we construct the associated boundary map $\partial : K_2(R/I) \rightarrow K_1(I, R)$. Using the interpretation of the Fredholm determinant as an invariant of relative algebraic K -theory we are then able to associate a determinant invariant to each short exact sequence satisfying certain conditions. In particular, we show that the data consisting of the almost commuting pair of operators $A, B \in \mathcal{L}(H)$ gives rise to a determinant invariant $d_{A,B} : K_2(C_r^2[x, y]) \rightarrow \mathbb{C}^*$. At last, we will construct a concrete element in the second algebraic K -group, $K_2(C_r^2[x, y])$, and calculate its determinant invariant. The result of this evaluation is the determinant of the desired multiplicative commutator

$$d_{A,B}\{e^x, e^y\} = \det(e^A e^B e^{-A} e^{-B})$$

The determinant invariant which we consider in this section is a generalization of the invariant originally introduced by L. G. Brown. We remove the assumption of commutativity of the quotient ring. Furthermore, we do not need the kernel to consist of all the operators of trace class. We replace this condition by a milder set of restrictions. We need these straight forward generalizations for the later applications which we have in mind.

3.2.1. The second algebraic K -group, boundary maps and determinant invariants. We will use a homological picture of the second algebraic K -group. Let G be any group and let us consider the following chain complex

$$0 \longleftarrow \mathbb{Z}[G] \xleftarrow{d} \dots \xleftarrow{d} \mathbb{Z}[G^{n-1}] \xleftarrow{d} \mathbb{Z}[G^n] \xleftarrow{d} \dots$$

Here the boundary $d : \mathbb{Z}[G^n] \rightarrow \mathbb{Z}[G^{n-1}]$ is given by

$$d(g_1, \dots, g_n) = (g_2, \dots, g_n) + (-1)^n(g_1, \dots, g_{n-1}) + \sum_{i=1}^{n-1} (-1)^i(g_1, \dots, g_i g_{i+1}, \dots, g_n)$$

We leave it to the reader to verify that $d^2 = 0$.

Definition 3.4. By the group homology of G with integer coefficient we will understand the homology of the above chain complex. Thus,

$$H_n(G) := Z_n(G)/B_n(G)$$

Here $Z_n(G) := \text{Ker}(d : \mathbb{Z}[G^n] \rightarrow \mathbb{Z}[G^{n-1}])$ are the cycles in degree n and $B_n(G) := \text{Im}(d : \mathbb{Z}[G^{n+1}] \rightarrow \mathbb{Z}[G^n])$ are the boundaries in degree n .

The above construction is functorial in the group G . Thus, each group homomorphism $\varphi : G \rightarrow H$ induces a homomorphism of abelian groups $\varphi_* : H_*(G) \rightarrow H_*(H)$.

Let R be a unital ring. Recall that the elementary matrices over R is the smallest subgroup of $GL(R)$ which contains all multiplicative commutators, $E(R) = [GL(R), GL(R)] \subseteq GL(R)$.

Definition 3.5. By the second algebraic K -group of the unital ring R we will understand the second homology group of the group of elementary matrices over R , thus

$$K_2(R) := H_2(E(R))$$

The second algebraic K -groups are functorial in the unital ring R .

Let $I \subseteq R$ be an ideal in a unital ring R . We can then form the quotient ring R/I which is again a unital ring. The relations between the three rings are expressed in the short exact sequence

$$0 \longrightarrow I \xrightarrow{i} R \xrightarrow{q} R/I \longrightarrow 0$$

From this data we would like to construct a boundary map

$$\partial : K_2(R/I) \rightarrow K_1(I, R)$$

on the algebraic K -groups. To do so, we note that the quotient map $q : R \rightarrow R/I$ induces a surjective group homomorphism on the level of elementary matrices, $E(q) : E(R) \rightarrow E(R/I)$. In particular, to each element $y \in \mathbb{Z}[E(R/I)^2]$ there is a lift $x \in \mathbb{Z}[E(R)^2]$ such that $E(q)_*(x) = y$. Now, suppose that $y \in Z_2(E(R/I))$ is a cycle. It follows that

$$(E(q)_*d)(x) = (d_2E(q)_*)(x) = 0$$

Using some combinatorial considerations we then get that $d(x) \in \mathbb{Z}[E(R)]$ must be of the form

$$d(x) = \sum_{i=1}^n (g_i - h_i) \quad E(q)(g_i) = E(q)(h_i)$$

By the boundary map of $y \in Z_2(E(R/I))$ we will then understand the quantity

$$\partial(y) = \prod_{i=1}^n (g_i h_i^{-1}) \in K_1(I, R) = GL(I)/[GL(I), GL(R)]$$

A priori the element $\partial(y) \in K_1(I, R)$ depends on the lift $x \in \mathbb{Z}[E(R)^2]$ and of the decomposition $d(x) = \sum_{i=1}^n (g_i - h_i)$. This ambiguity is resolved by the next lemma.

Lemma 3.6. *The homomorphism $\partial : \text{Ker}(d) \rightarrow K_1(I, R)$ is well defined and descends to a boundary homomorphism*

$$\partial : K_2(R/I) = H_2(E(R/I)) \rightarrow K_1(I, R) = GL(I)/[GL(I), GL(R)]$$

Proof. We refer to the article [16, Theorem 2.1] which is attached at the end of the thesis. See also [32, Theorem 4.3.1]. \square

Let us consider the ideal of trace class operators, $\mathcal{L}^1(H) \subseteq \mathcal{L}(H)$. We then have the short exact sequence

$$0 \longrightarrow \mathcal{L}^1(H) \xrightarrow{i} \mathcal{L}(H) \xrightarrow{q} \mathcal{L}(H)/\mathcal{L}^1(H) \longrightarrow 0$$

of rings. We have introduced the relevant concepts to make the following,

Definition 3.7. *By the universal determinant invariant we will understand the homomorphism*

$$d : K_2(\mathcal{L}(H)/\mathcal{L}^1(H)) \rightarrow \mathbb{C}^* \quad d = \det \circ \partial$$

given by the composition of the boundary map $\partial : K_2(\mathcal{L}(H)/\mathcal{L}^1(H)) \rightarrow K_1(\mathcal{L}^1(H), \mathcal{L}(H))$ and the determinant homomorphism $\det : K_1(\mathcal{L}^1(H), \mathcal{L}(H)) \rightarrow \mathbb{C}^$.*

Let us justify the word "universal" in the above definition. Namely, let $I \subseteq R$ be some ideal in a unital ring R . We thus have a short exact sequence of rings

$$X : 0 \longrightarrow I \xrightarrow{i} R \xrightarrow{q} R/I \longrightarrow 0$$

We suppose that there exists a unital ring homomorphism $\pi : R \rightarrow \mathcal{L}(H)$ such that the restriction to I factorizes through the ideal of trace class operators, $\pi : I \rightarrow \mathcal{L}^1(H)$.

Definition 3.8. *By the determinant invariant of the pair (X, π) we will understand the homomorphism*

$$d_{(X, \pi)} : K_2(R/I) \rightarrow \mathbb{C}^* \quad d_{(X, \pi)} = d \circ \pi_*$$

given by the composition of the homomorphism $\pi_ : K_2(R/I) \rightarrow K_2(\mathcal{L}(H)/\mathcal{L}^1(H))$ obtained from $\pi : R/I \rightarrow \mathcal{L}(H)/\mathcal{L}^1(H)$ by functoriality of the second algebraic K -groups and the universal determinant invariant, $d : K_2(\mathcal{L}(H)/\mathcal{L}^1(H)) \rightarrow \mathbb{C}^*$.*

In particular, to each almost commuting pair of operators $A, B \in \mathcal{L}(H)$ with $\|A\|_\infty, \|B\|_\infty \leq r$ we associate the determinant invariant

$$d_{A,B} : K_2(C_r^2[x, y]) \rightarrow \mathbb{C}^*$$

given by the short exact sequence

$$0 \longrightarrow J_r \xrightarrow{i} C_r^2\langle X, Y \rangle \xrightarrow{q} C_r^2[x, y] \longrightarrow 0$$

and the representation $\pi^{A,B} : C_r^2\langle X, Y \rangle \rightarrow \mathcal{L}(H)$.

3.2.2. The determinant invariant of a Steinberg symbol. We are interested in the application of the determinant invariant

$$d_{A,B} : K_2(C^2[x, y]) \rightarrow \mathbb{C}^* \quad d_{A,B} = \det \circ \partial \circ \pi_*^{A,B}$$

to a particular element in $K_2(C^2[x, y])$. This element is an example of a Steinberg symbol. Later on we will see that the Steinberg symbols can be described using an interior product $*$: $K_1(C^2[x, y]) \times K_1(C^2[x, y]) \rightarrow K_2(C^2[x, y])$. This discovery was made by J. -L. Loday in [25]. The point of view will be important in our attempt to generalize the determinants of multiplicative commutators. The calculations carried out in the present section can be found in a slightly different form in [3]. See also [32, Theorem 4.4.22]. We present them here for the sake of completeness.

Let R be a unital ring.

Definition 3.9. *To each pair of invertible and commuting elements $g, h \in R^*$ we associate a Steinberg symbol in the second algebraic K -group, $\{g, h\} \in K_2(R)$. It is given by the commutator*

$$\{g, h\} = (x_{13}(h), x_{12}(g)) - (x_{12}(g), x_{13}(h)) \in H_2(E(R))$$

Here $x_{12}(g), x_{13}(h) \in E(R)$ are the elementary matrices

$$x_{12}(g) = \begin{pmatrix} g & 0 & 0 \\ 0 & g^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad x_{13}(h) = \begin{pmatrix} h & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & h^{-1} \end{pmatrix}$$

Let $I \subseteq R$ be an ideal in a unital ring. Let $q : R \rightarrow R/I$ denote the quotient map. The next lemma describes the application of the boundary map $\partial : K_2(R/I) \rightarrow K_1(I, R)$ to a Steinberg symbol by an explicit formula.

Lemma 3.10. *Let $g, h \in (R/I)^*$ be a pair of invertible and commuting elements in the quotient ring R/I . Suppose that there exist invertible elements $G, H \in R^*$ with $q(G) = g$ and $q(H) = h$. The boundary map of the Steinberg symbol, $\{g, h\} \in K_2(R/I)$ is then given by the multiplicative commutator,*

$$\partial(\{g, h\}) = GHG^{-1}H^{-1} \in K_1(I, R)$$

Proof. We have that

$$E(q)_*((x_{13}(H), x_{12}(G)) - (x_{12}(G), x_{13}(H))) = (x_{13}(h), x_{12}(g)) - (x_{12}(g), x_{13}(h))$$

Furthermore, the application of the differential $d : \mathbb{Z}[E(R)^2] \rightarrow \mathbb{Z}[E(R)]$ to the lift

$$(x_{13}(H), x_{12}(G)) - (x_{12}(G), x_{13}(H)) \in \mathbb{Z}[E(R)^2]$$

yields the element

$$d((x_{13}(H), x_{12}(G)) - (x_{12}(G), x_{13}(H))) = \begin{pmatrix} GH & 0 & 0 \\ 0 & G^{-1} & 0 \\ 0 & 0 & H^{-1} \end{pmatrix} - \begin{pmatrix} HG & 0 & 0 \\ 0 & G^{-1} & 0 \\ 0 & 0 & H^{-1} \end{pmatrix}$$

Now, since $q(GH) = q(HG)$ we get that

$$\partial(\{g, h\}) = (GH)(HG)^{-1} \in K_1(I, R)$$

as desired. \square

Let us return to the particular case of an almost commuting pair of operators, $A, B \in \mathcal{L}(H)$ with $\|A\|_\infty, \|B\|_\infty \leq r \in (0, \infty)$. Applying the exponential function to the generators $x, y \in C_r^2[x, y]$ we get two invertible and commuting elements $e^x, e^y \in C_r^2[x, y]^*$. We can thus form the Steinberg symbol $\{e^x, e^y\} \in K_2(C_r^2[x, y])$. Recall that the determinant invariant of the almost commuting pair is given by the composition

$$d_{A,B} : K_2(C_r^2[x, y]) \rightarrow \mathbb{C}^* \quad d_{A,B} = \det \circ \partial \circ \pi_*^{A,B}$$

Theorem 3.11. *Let $A, B \in \mathcal{L}(H)$ be two operators with trace class commutator, $[A, B] \in \mathcal{L}^1(H)$. Let $r > 0$ be a constant with $r \geq \|A\|, \|B\|$. The determinant invariant of the Steinberg symbol $\{e^x, e^y\} \in K_2(C_r^2[x, y])$ is then given by the Fredholm determinant of a multiplicative commutator*

$$d_{A,B}\{e^x, e^y\} = \det(e^A e^B e^{-A} e^{-B})$$

Proof. From Lemma 3.10 we get that

$$\partial(\{e^x, e^y\}) = e^X e^Y e^{-X} e^{-Y} \in K_1(J, C_r^2\langle X, Y \rangle)$$

Since the homomorphism $\pi_*^{A,B} : K_1(J, C_r^2\langle X, Y \rangle) \rightarrow K_1(\mathcal{L}^1(H), \mathcal{L}(H))$ simply replaces the formal variables X and Y with the operators A and B we get that

$$(\det \circ \pi_*^{A,B})(e^X e^Y e^{-X} e^{-Y}) = \det(e^A e^B e^{-A} e^{-B})$$

This proves the desired result. \square

3.3. The trace invariant and traces of additive commutators. Our goal is now to describe the trace of an additive commutator as a special value of a K -theoretic invariant. Again we will need to introduce a couple of K -theoretic concepts. Firstly, we will define the second relative K -group of a unital Banach algebra. In analogy with our approach in the algebraic case we will use a homological picture of these abelian groups. We proceed by describing the second relative Chern character which relates the second relative K -groups with continuous cyclic homology. Having accomplished this, we associate a trace invariant to each short exact sequence satisfying certain conditions. In particular, we show that each almost commuting pair of operators $A, B \in \mathcal{L}(H)$ gives rise to a trace invariant $\tau_{A,B} : K_2^{\text{rel}}(C_r^2[x, y]) \rightarrow \mathbb{C}$. The application of this invariant to a certain relative Steinberg symbol is precisely the trace of the commutator $\text{Tr}[A, B] \in \mathbb{C}$.

3.3.1. The second relative K -groups. Let A be a unital Banach algebra. For each $n \in \mathbb{N}$, let $\Delta^n \subseteq \mathbb{R}^n$ denote the standard simplex

$$\Delta^n = \{(t_1, \dots, t_n) \mid t_i \in [0, 1], \sum_{i=1}^n t_i \leq 1\}$$

The vertices, $\mathbf{0}, \dots, \mathbf{n} \in \Delta^n$ of the n^{th} standard simplex are then given by

$$\mathbf{i} = \begin{cases} (0, \dots, 0, 1, 0, \dots, 0) & \text{for } i \in \{1, \dots, n\} \\ (0, \dots, 0) & \text{for } i = 0 \end{cases}$$

Here the vertex \mathbf{i} has the number 1 in position i . Note that this model for the standard simplex does not agree precisely with the usual one. We have chosen this picture in order to emphasize the special role of the basepoint $\mathbf{0} \in \Delta^n$ and the zeroth face. We let $R(A)_n$ denote the set of continuous maps $\sigma : \Delta^n \rightarrow GL(A)$ such that $\sigma(\mathbf{0}) = 1$. The sets $R(A)_n$ can be equipped with the following *face operators*

$$d_i : R(A)_n \rightarrow R(A)_{n-1}$$

$$d_i(\sigma)(t_1, \dots, t_{n-1}) = \begin{cases} \sigma(t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}) & \text{for } i \in \{1, \dots, n\} \\ \sigma(1 - \sum_{i=1}^n t_i, t_1, \dots, t_n) \cdot \sigma(\mathbf{1})^{-1} & \text{for } i = 0 \end{cases}$$

We would like the reader to remark the (extra) factor $\sigma(\mathbf{1})^{-1}$. This ensures that the zeroth face has the value $1 \in GL(A)$ at the zeroth vertex. These face operators satisfy the simplicial relations

$$d_i d_j = d_{j-1} d_i \quad \text{for all } 0 \leq i < j \leq n$$

Now, recall that $F(A)_1$ denotes the smallest subgroup of $R(A)_1$ which contains all multiplicative commutators. For each $n \in \mathbb{N}$ we will then let $F(A)_n$ denote the subset of $R(A)_n$ defined recursively by

- (1) $F(A)_1 := F(A)_1$
- (2) $\sigma \in F(A)_{n+1} \Leftrightarrow (\sigma \in R(A)_{n+1} \text{ and } d_i(\sigma) \in F(A)_n \text{ for each } i \in \{0, 1, \dots, n+1\})$

For each $n \in \mathbb{N}$ there is a boundary operator $d : \mathbb{Z}[F(A)_{n+1}] \rightarrow \mathbb{Z}[F(A)_n]$ defined as the sum of face operators $d = \sum_{i=0}^{n+1} (-1)^i d_i$. We then have the following chain complex :

$$0 \longleftarrow \mathbb{Z}[F(A)_1] \xleftarrow{d} \dots \xleftarrow{d} \mathbb{Z}[F(A)_{n-1}] \xleftarrow{d} \mathbb{Z}[F(A)_n] \xleftarrow{d} \dots$$

We leave it to the reader to verify that $d^2 = 0$.

Definition 3.12. *By the second relative K -group of the unital Banach algebra A we will understand the second homology group of the chain complex $(\mathbb{Z}[F(A)_*], d)$. Thus, by definition*

$$K_2^{\text{rel}}(A) := H_2(F(A))$$

Remark 3.13. *In the book [20] of M. Karoubi, the second relative K -group is defined as a homotopy group of a certain space related to the simplicial set $GL(A_*)/GL(A)$ introduced by J. Milnor. We prove in [17, Corollary 3.8] that the above homological definition agrees with the original definition.*

The second relative K -group comes equipped with a homomorphism $\theta : K_2^{\text{rel}}(A) \rightarrow K_2(A)$. It is induced by the application which sends each $\sigma \in F(A)_2$ to the element $(\sigma(\mathbf{1})^{-1}, \sigma(\mathbf{1})\sigma(\mathbf{2})^{-1}) \in E(A)^2$. Note that the vertices $\sigma(\mathbf{1})$ and $\sigma(\mathbf{2})$ are elementary matrices by definition of the set $F(A)_2$. We are particularly interested in constructing lifts under this homomorphism. Indeed, recall from Section 2.4 how we proved the formula

$$\det(e^T) = e^{\text{Tr}(T)} \quad T \in \mathcal{L}^1(H)$$

for the Fredholm determinant. We found a concrete element $[\gamma] \in K_1^{\text{rel}}(\mathcal{L}^1(H), \mathcal{L}(H))$ with $\theta([\gamma]) = [e^T]$. Afterwards, we applied a trace invariant to this lift. Therefore, we might hope to calculate the determinant invariant $d_{A,B} : K_2(C_r^2[x, y]) \rightarrow \mathbb{C}^*$ of some element by finding

an appropriate lift in $K_2^{\text{rel}}(C_r^2[x, y])$ and then apply a higher trace invariant to this lift. Let us start by a discussion of the additive character which parallels the determinant invariant. The main ingredient is given by the second relative Chern character, which we will now introduce.

3.3.2. The relative Chern character. Let A be a unital Banach algebra. The second relative Chern character is a homomorphism from the second relative K -group to the first continuous cyclic homology group. Therefore, let us begin with a description of the continuous cyclic homology groups of a unital Banach algebra. The cyclic theory was discovered from different perspectives by several people, [7, 24, 36]. One should also mention the work of M. Karoubi on noncommutative de Rham homology, see for example [20]. A good general reference is certainly the book of J. -L. Loday, [26].

For each $n \in \mathbb{N}_0$ we let $C_n(A) = A \otimes_{\pi} A^{\otimes \pi n}$ denote the projective tensorproduct with $(n+1)$ factors of A . Thus, $C_n(A)$ is a Banach space. The cyclic operator on $C_n(A)$ is given by

$$t : C_n(A) \rightarrow C_n(A)$$

$$t(a_0 \otimes a_1 \otimes \dots \otimes a_n) = \begin{cases} (-1)^n a_n \otimes a_0 \otimes \dots \otimes a_{n-1} & \text{for } n \geq 1 \\ a_0 & \text{for } n = 0 \end{cases}$$

Furthermore, we define the Hochschild boundary maps

$$b : C_{n+1}(A) \rightarrow C_n(A)$$

$$b : a_0 \otimes a_1 \otimes \dots \otimes a_{n+1} \mapsto (-1)^{n+1} a_{n+1} a_0 \otimes a_1 \otimes \dots \otimes a_n + b'(a_0 \otimes a_1 \otimes \dots \otimes a_{n+1})$$

Here $b' : C_{n+1}(A) \rightarrow C_n(A)$ denotes the differential in the bar complex. It is given by the formula

$$b' : a_0 \otimes a_1 \otimes \dots \otimes a_{n+1} \mapsto \sum_{i=0}^n (-1)^i a_0 \otimes \dots \otimes a_i \cdot a_{i+1} \otimes \dots \otimes a_{n+1}$$

We then have the following,

Lemma 3.14. *The image of $(1 - t) : C_n(A) \rightarrow C_n(A)$ is closed for each $n \in \mathbb{N}_0$. The Hochschild boundaries satisfy the equality $b^2 = 0$. The Hochschild boundary preserves the image of $(1 - t) : C_*(A) \rightarrow C_*(A)$.*

Proof. We will only prove that the image of $(1 - t) : C_n(A) \rightarrow C_n(A)$ is closed. Let $n \in \mathbb{N}$. Let $N = 1 + t + \dots + t^n : C_n(A) \rightarrow C_n(A)$ denote the norm operator. We prove that $\text{Im}(1 - t) = \text{Ker}(N)$. We clearly have $N(1 - t) = 0$, thus assume that $N(y) = 0$ for some $y \in C_n(A)$. We then note that

$$\sum_{i=0}^{n-1} \frac{1}{n+1} (1 - t)(1 + t + \dots + t^i) = \sum_{j=1}^n \frac{1}{n+1} (1 - t^j)(y) = y - \frac{1}{n+1} N(y)$$

This proves the claim. We refer the reader to [26] for proofs of the other results. \square

For each $n \in \mathbb{N}_0$ we let $C_n^{\lambda}(A) = C_n(A)/\text{Im}(1 - t)$ denote the quotient Banach space. The above lemma ensures us that we have a chain complex

$$0 \longleftarrow C_0^{\lambda}(A) \xleftarrow{b} \dots \xleftarrow{b} C_n^{\lambda}(A) \xleftarrow{b} C_{n+1}^{\lambda}(A) \xleftarrow{b} \dots$$

of Banach spaces.

Definition 3.15. *By the continuous cyclic homology of the unital Banach algebra A we will understand the homology of the above chain complex. The n^{th} continuous cyclic homology group will be denoted by $HC_n(A)$. This is non-Hausdorff in general. That is, the image of the Hochschild boundary is not necessarily closed.*

For each $p \in \mathbb{N}$ and each $n \in \mathbb{N}$ there is a *generalized trace* homomorphism on continuous cyclic homology

$$\text{TR} : HC_n(M_p(A)) \rightarrow HC_n(A)$$

it is induced by the continuous linear map

$$\text{TR} : u_0 a_0 \otimes u_1 a_1 \otimes \dots \otimes u_n a_n \mapsto \text{TR}(u_0 u_1 \dots u_n) a_0 \otimes a_1 \otimes \dots \otimes a_n$$

Here $a_i \in A$ and $u_i \in M_p(\mathbb{C})$ for all $i \in \{0, 1, \dots, n\}$. For details we refer to [26, Section 1.2]. Having introduced the relevant concepts, we make the following,

Definition 3.16. *By the second relative Chern character of the unital Banach algebra A we will understand the homomorphism*

$$\text{ch}^{\text{rel}} : K_2^{\text{rel}}(A) = H_2(F(A)) \rightarrow HC_1(A)$$

which is induced by the homomorphism

$$\text{ch}^{\text{rel}} : \mathbb{Z}[F(A)_2] \rightarrow C_1^\lambda(A) \quad \sigma \mapsto \text{TR}\left(\int_0^1 \int_0^{1-t_2} \Gamma_1(\sigma) \otimes \Gamma_2(\sigma) dt_1 dt_2\right)$$

Here we have used the notation

$$\Gamma_i(\sigma) := \frac{\partial \sigma}{\partial t_i} \cdot \sigma^{-1} : \Delta^2 \rightarrow M_\infty(A)$$

This leaves some unproved claims. However, instead of considering them at this place we prefer to continue with the definition of the trace invariant. The interested reader can find the relevant arguments in the article of U. Tillmann [35]. Notice again that we pass between continuous and smooth maps at the level of homology without further notice, cf. the remark after Definition 2.12.

3.3.3. The trace invariant. Let $J \subseteq A$ be a *closed* ideal in a unital Banach algebra A . The quotient A/J is then a unital Banach algebra when equipped with the quotient norm. We thus have a short exact sequence of Banach algebras

$$X : 0 \longrightarrow J \xrightarrow{i} A \xrightarrow{q} A/J \longrightarrow 0$$

We will always assume that X comes equipped with a continuous linear section

$$s : A/J \rightarrow A \quad q \circ s = \text{Id}_{A/J}$$

Let us see how this data gives rise to a boundary map in continuous cyclic homology. By continuity and linearity of the section $s : A/J \rightarrow A$ there is an induced map

$$s_* : C_1^\lambda(A/J) \rightarrow C_1^\lambda(A) \quad s_*(x_0 \otimes x_1) = s(x_0) \otimes s(x_1)$$

We let $\partial : C_1^\lambda(A/J) \rightarrow C_0^\lambda(A)$ denote the composition $\partial = b \circ s_*$. Now, suppose that $y \in Z_1^\lambda(A/J)$ is a cyclic cycle, thus $b(y) = 0$. We then have

$$(q_* \circ \partial)(y) = (b \circ q_* \circ s_*)(y) = b(y) = 0$$

This means that $\partial(y) \in J$. In fact, we have even more,

Lemma 3.17. *The continuous linear map $\partial : C_1^\lambda(A/J) \rightarrow C_0^\lambda(A)$ descends to a boundary homomorphism in continuous cyclic homology*

$$\partial : HC_1(A/J) \rightarrow HC_0(J, A) := J/\text{Im}(b : J \otimes_\pi A + A \otimes_\pi J \rightarrow J)$$

Proof. We leave the proof to the reader. □

Now, assume that X comes equipped with a continuous algebra homomorphism $\pi : A \rightarrow \mathcal{L}(H)$ such that the restriction to the ideal J determines a continuous algebra homomorphism with trace class values, $\pi : J \rightarrow \mathcal{L}^1(H)$. We will associate a trace invariant to each such pair (X, π) . This is going to be a character on the second relative K -group of the quotient,

$$\tau_{(X, \pi)} : K_2^{\text{rel}}(A/J) \rightarrow \mathbb{C}$$

We shall see that it is intimately related to the determinant invariant which we considered in Section 3.2. Furthermore, we shall see that typical values of this trace invariant are traces of commutators.

Note that the assumption on the representation $\pi : A \rightarrow \mathcal{L}(H)$ is sufficient to get an induced homomorphism on the relative continuous cyclic homology groups

$$\pi_* : HC_0(J, A) \rightarrow HC_0(\mathcal{L}^1(H), \mathcal{L}(H))$$

Definition 3.18. *By the trace invariant of the pair (X, π) we will understand the homomorphism*

$$\tau_{(X, \pi)} : K_2^{\text{rel}}(A/J) \rightarrow \mathbb{C} \quad \tau_{(X, \pi)} = (-\text{Tr}) \circ \pi_* \circ \partial \circ \text{ch}^{\text{rel}}$$

given by the composition of the second relative Chern character, the boundary map, the representation and minus the operator trace.

In particular, let $A, B \in \mathcal{L}(H)$ be an almost commuting pair of operators with $\|A\|_\infty, \|B\|_\infty \leq r$ for some $r \in (0, \infty)$. We then have the trace invariant

$$\tau_{A, B} : K_2^{\text{rel}}(C_r^2[x, y]) \rightarrow \mathbb{C}$$

given by the short exact sequence

$$0 \longrightarrow J_r \xrightarrow{i} C_r^2\langle X, Y \rangle \xrightarrow{q} C_r^2[x, y] \longrightarrow 0$$

and the representation

$$\pi^{A, B} : C_r^2\langle X, Y \rangle \rightarrow \mathcal{L}(H) \quad \pi^{A, B} : X \mapsto A, Y \mapsto B$$

In the next section we shall see an example of a calculation of this invariant.

3.3.4. The relative Steinberg symbols and traces of commutators. We are interested in the application of the trace invariant

$$\tau_{A,B} : K_2^{\text{rel}}(C_r^2[x, y]) \rightarrow \mathbb{C}$$

to a particular element in the second relative K -group, $K_2^{\text{rel}}(C_r^2[x, y])$. This element is an example of a *relative Steinberg symbol*. Later on, we shall relate the construction to an interior product $*^{\text{rel}} : K_1^{\text{rel}}(A) \times K_1^{\text{rel}}(A) \rightarrow K_2^{\text{rel}}(A)$ on the relative K -groups. However, let us start by defining the relative Steinberg symbol.

Let A be a unital Banach algebra. For each continuous map $\sigma : [0, 1] \rightarrow GL_p(A)$ with $\sigma(0) = 1$, we let $s_0(\sigma), s_1(\sigma) : \Delta^2 \rightarrow GL_p(A)$ denote the continuous maps given by

$$s_0(\sigma)(t_1, t_2) = \sigma(t_2) \quad s_1(\sigma)(t_1, t_2) = \sigma(t_1 + t_2)$$

for each $(t_1, t_2) \in \Delta^2$. Note that $s_0(\sigma)(\mathbf{0}) = 1 = s_1(\sigma)(\mathbf{0})$. Furthermore, we let $x_{12}(\sigma), x_{13}(\sigma) : [0, 1] \rightarrow GL_{3p}(A)$ denote the continuous maps

$$x_{12}(\sigma) = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad x_{13}(\sigma) = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma^{-1} \end{pmatrix}$$

It is then possible to prove that $x_{12}(\sigma), x_{13}(\sigma) \in F(A)_1$ are products of multiplicative commutators *whenever* there exists an $\epsilon > 0$ with

$$\sigma(t) = 1 \quad \text{for all } t \in [0, \epsilon)$$

This result follows by a slight modification of the argument to be found in [32, Corollary 2.1.3], for example.

Definition 3.19. *Let A be a unital and commutative Banach algebra. Let $\sigma, \tau : [0, 1] \rightarrow A^*$ be a pair of continuous maps with values in the group of invertibles and with $\sigma(t) = \tau(t) = 1$ for all $t \in [0, \epsilon)$. By the relative Steinberg symbol of σ and τ we will then understand the element $\{\sigma, \tau\}^{\text{rel}} \in K_2^{\text{rel}}(A)$ represented by*

$$s_0(x_{12}(\sigma)) \cdot s_1(x_{13}(\tau)) - s_1(x_{12}(\sigma)) \cdot s_0(x_{13}(\tau)) \in \mathbb{Z}[F(A)_2]$$

We are particularly interested in the pair of smooth maps $\gamma_x, \gamma_y : [0, 1] \rightarrow C_r^2[x, y]^*$ given by

$$\gamma_x(t) = e^{-\alpha(t)x} \quad \gamma_y(t) = e^{-\alpha(t)y}$$

Here $\alpha : [0, 1] \rightarrow [0, 1]$ is some smooth function which vanish on a neighborhood of 0 and sends 1 to 1. Since $C_r^2[x, y]$ is a unital and commutative Banach algebra we can form the relative Steinberg symbol, $\{\gamma_x, \gamma_y\}^{\text{rel}} \in K_2^{\text{rel}}(C_r^2[x, y])$. Let us apply the second relative Chern character to this element.

Lemma 3.20. *We have the following formula*

$$\text{ch}^{\text{rel}}(\{\gamma_x, \gamma_y\}^{\text{rel}}) = y \otimes x \in HC_1(C_r^2[x, y])$$

for the application of the second relative Chern character to the relative Steinberg symbol $\{\gamma_x, \gamma_y\}^{\text{rel}} \in K_2^{\text{rel}}(C_r^2[x, y])$.

Proof. Let $\tilde{x}, \tilde{y} \in M_3(C_r^2[x, y])$ denote the matrices

$$\tilde{x} = \begin{pmatrix} x & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{y} = \begin{pmatrix} y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -y \end{pmatrix}$$

We then have the identities

$$\begin{aligned} \Gamma_1(s_0(x_{12}(\gamma_x))s_1(x_{13}(\gamma_y))) &= -s_1(\alpha') \cdot \tilde{y} \\ \Gamma_2(s_0(x_{12}(\gamma_x))s_1(x_{13}(\gamma_y))) &= -s_0(\alpha') \cdot \tilde{x} - s_1(\alpha') \cdot \tilde{y} \\ \Gamma_1(s_1(x_{12}(\gamma_x))s_0(x_{13}(\gamma_y))) &= -s_1(\alpha') \cdot \tilde{x} \\ \Gamma_2(s_1(x_{12}(\gamma_x))s_0(x_{13}(\gamma_y))) &= -s_0(\alpha') \cdot \tilde{y} - s_1(\alpha') \cdot \tilde{x} \end{aligned}$$

Here the notation $\Gamma_i(\sigma)$ refers to the smooth map

$$\Gamma_i(\sigma) = \frac{\partial \sigma}{\partial t_i} \cdot \sigma^{-1} : \Delta^2 \rightarrow M_\infty(A) \quad i = 1, 2$$

for any smooth map $\sigma : \Delta^2 \rightarrow GL(A)$.

By definition of the relative Chern character we thus have

$$\begin{aligned} \text{ch}^{\text{rel}}(\{\gamma_x, \gamma_y\}^{\text{rel}}) &= \text{TR} \left(\int_0^1 \int_0^{1-t_2} s_1(\alpha') \tilde{y} \otimes (s_1(\alpha') \tilde{y} + s_0(\alpha') \tilde{x}) dt_1 dt_2 \right) \\ &\quad - \text{TR} \left(\int_0^1 \int_0^{1-t_2} s_1(\alpha') \tilde{x} \otimes (s_1(\alpha') \tilde{x} + s_0(\alpha') \tilde{y}) dt_1 dt_2 \right) \\ &= 2 \text{TR} \left(\int_0^1 \int_0^{1-t_2} s_1(\alpha') s_0(\alpha') \tilde{y} \otimes \tilde{x} dt_1 dt_2 \right) \\ &= 2 \int_0^1 \int_0^{1-t_2} s_1(\alpha') s_0(\alpha') y \otimes x dt_1 dt_2 \end{aligned}$$

A straight forward calculation shows that

$$2 \int_0^1 \int_0^{1-t_2} s_1(\alpha') s_0(\alpha') dt_1 dt_2 = 1$$

This proves the Lemma. □

We remark that the value of the relative Chern character on the relative Steinberg symbol is independent of our choice of smooth function, $\alpha : [0, 1] \rightarrow [0, 1]$. We will see later why this is the case. Indeed, we shall see that the relative Steinberg symbol $\{\sigma, \tau\}^{\text{rel}}$ only depends on the classes of the continuous maps $\sigma, \tau : [0, 1] \rightarrow A^*$ in the first relative K -group. In particular, the relative Steinberg symbol is independent of homotopies with fixed endpoints of the terms. More generally, we shall see that the relative Steinberg symbol agrees up to a sign with an interior product on relative K -theory. Furthermore, we will describe the class $x \otimes y \in HC_1(C_R^2[x, y])$ as the result of an interior product of degree one in continuous cyclic homology. The result of Lemma 3.20 will then be a consequence of the general multiplicative properties of the relative Chern character. For now, however, let us calm down a bit and continue with the exposition.

Let $A, B \in \mathcal{L}(H)$ be an almost commuting pair of operators with $\|A\|_\infty, \|B\|_\infty \leq r$ for some $r \in (0, \infty)$. We recall that the associated trace invariant is given by the composition

$$\tau_{A,B} : K_2^{\text{rel}}(C_r^2[x, y]) \rightarrow \mathbb{C} \quad \tau_{A,B} = (-\text{Tr}) \circ \pi_*^{A,B} \circ \partial \circ \text{ch}^{\text{rel}}$$

Here $\partial : HC_1(C_r^2[x, y]) \rightarrow HC_0(J_R, C_R^2\langle X, Y \rangle)$ is the boundary map associated with the short exact sequence

$$0 \longrightarrow J_r \xrightarrow{i} C_r^2\langle X, Y \rangle \xrightarrow{q} C_r^2[x, y] \longrightarrow 0$$

Note that our continuous linear section $s : C_r^2[x, y] \rightarrow C_r^2\langle X, Y \rangle$ is given by $s(x^n y^m) = X^n Y^m$. The homomorphism $\pi_*^{A,B} : HC_0(J_r, C_r^2\langle X, Y \rangle) \rightarrow HC_0(\mathcal{L}^1(H), \mathcal{L}(H))$ is induced by the representation

$$\pi_*^{A,B} : C_r^2\langle X, Y \rangle \rightarrow \mathcal{L}(H) \quad X \mapsto A, Y \mapsto B$$

Using the above Lemma and the definitions of the involved homomorphisms, we then get that

$$\tau_{A,B}(\{\gamma_x, \gamma_y\}^{\text{rel}}) = -(\text{Tr} \circ \pi_*^{A,B} \circ \partial)(y \otimes x) = -(\text{Tr} \circ \pi_*^{A,B})(YX - XY) = \text{Tr}([A, B])$$

We have thus proved the following,

Theorem 3.21. *The image of the relative Steinberg symbol $\{\gamma_x, \gamma_y\}^{\text{rel}} \in K_2^{\text{rel}}(C_r^2[x, y])$ under the trace invariant of the almost commuting pair of operators $A, B \in \mathcal{L}(H)$ is the trace of the commutator, $\text{Tr}([A, B]) \in \mathbb{C}$. That is,*

$$\tau_{A,B}(\{\gamma_x, \gamma_y\}^{\text{rel}}) = \text{Tr}([A, B])$$

3.4. The relation between the trace invariant and the determinant invariant. In the last two sections we saw that the determinant of a multiplicative commutator and the trace of an additive commutator were both the results of K -theoretic invariants. We are thus attacking the identity

$$(19) \quad \det(e^A e^B e^{-A} e^{-B}) = e^{\text{Tr}[A, B]}$$

from a general K -theoretic point of view. Likewise, the proof which we have in mind will be the result of a more general K -theoretic consideration. Indeed, we will now find the precise relation between the trace invariant and the determinant invariant. The identity (19) will then be a consequence of this comparison result.

Let us fix a short exact sequence of Banach algebras

$$X : 0 \longrightarrow J \xrightarrow{i} A \xrightarrow{q} A/J \longrightarrow 0$$

with a continuous linear split, $s : A/J \rightarrow A$. Furthermore, we suppose that there exists a continuous unital algebra homomorphism $\pi : A \rightarrow \mathcal{L}(H)$ such that the restriction to the ideal determines a continuous algebra homomorphism $\pi : J \rightarrow \mathcal{L}^1(H)$ with values in the trace ideal.

We can then associate two invariants to the pair (X, π) . The first one is the determinant invariant

$$d_{(X, \pi)} : K_2(A/J) \rightarrow \mathbb{C}^* \quad d_{(X, \pi)} = \det \circ \partial \circ \pi_*$$

The second one is the trace invariant

$$\tau_{(X, \pi)} : K_2^{\text{rel}}(A/J) \rightarrow \mathbb{C} \quad \tau_{(X, \pi)} = (-\text{Tr}) \circ \pi_* \circ \partial \circ \text{ch}^{\text{rel}}$$

Let us recall that the link between the second relative K -group and the second algebraic K -group is established by the natural homomorphism

$$\theta : K_2^{\text{rel}}(A/J) \rightarrow K_2(A/J) \quad \theta(\sigma) = (\sigma(\mathbf{1})^{-1}, \sigma(\mathbf{1})\sigma(\mathbf{2})^{-1})$$

Here $\sigma \in F(A/J)_2$. This homomorphism allows us to compare the determinant invariant and the trace invariant. Indeed, we have the following,

Theorem 3.22. *The diagram*

$$\begin{array}{ccc} K_2^{\text{rel}}(A/J) & \xrightarrow{\theta} & K_2(A/J) \\ \tau_{(X,\pi)} \downarrow & & d_{(X,\pi)} \downarrow \\ \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^* \end{array}$$

is commutative.

Proof. We will only give a sketch of the proof. In analogy with the construction of the boundary map in algebraic K -theory it is possible to construct a boundary map in relative K -theory,

$$\partial : K_2^{\text{rel}}(A/J) \rightarrow K_1^{\text{rel}}(J, A)$$

This homomorphism makes the following diagrams commute

$$\begin{array}{ccc} K_2^{\text{rel}}(A/J) & \xrightarrow{\partial} & K_1^{\text{rel}}(J, A) \\ \theta \downarrow & & \theta \downarrow \\ K_2(A/J) & \xrightarrow{\partial} & K_1(J, A) \end{array} \quad \text{and} \quad \begin{array}{ccc} K_2^{\text{rel}}(A/J) & \xrightarrow{\partial} & K_1^{\text{rel}}(J, A) \\ \text{ch}^{\text{rel}} \downarrow & & \text{ch}^{\text{rel}} \downarrow \\ HC_1(A/J) & \xrightarrow{\partial} & HC_0(J, A) \end{array}$$

In particular, we get that

$$d_{(X,\pi)} \circ \theta = \det \circ \partial \circ \pi_* \circ \theta = \det \circ \theta \circ \pi_* \circ \partial$$

However, from Section 2.4 we know that the homomorphism

$$\det \circ \theta : K_1^{\text{rel}}(\mathcal{L}^1(H), \mathcal{L}(H)) \rightarrow \mathbb{C}^*$$

agrees with the homomorphism

$$\exp \circ (-\text{Tr}) \circ \text{ch}^{\text{rel}} : K_1^{\text{rel}}(\mathcal{L}^1(H), \mathcal{L}(H)) \rightarrow \mathbb{C}^*$$

It follows that

$$\det \circ \theta \circ \pi_* \circ \partial = \exp \circ (-\text{Tr}) \circ \pi_* \circ \partial \circ \text{ch}^{\text{rel}}$$

But this is precisely the composition of the trace invariant and the exponential function. \square

We have now finished setting up a general framework in which the identity

$$\det(e^A e^B e^{-A} e^{-B}) = e^{\text{Tr}[A,B]}$$

can be verified. Here $A, B \in \mathcal{L}(H)$ is a pair of operators with trace class commutator. Let us choose some $r > 0$ such that $\|A\|, \|B\| \leq r$ we then have the short exact sequence of Banach algebras

$$0 \longrightarrow J_r \xrightarrow{i} C_r^2\langle X, Y \rangle \xrightarrow{q} C_r^2[x, y] \longrightarrow 0$$

with representation $\pi^{A,B} : C_r^2\langle X, Y \rangle \rightarrow \mathcal{L}(H)$ sending X to A and Y to B . Furthermore, we know from Lemma 3.3 that the restriction of $\pi^{A,B}$ to the ideal J_r determines a continuous algebra homomorphism with trace class values. We can thus apply the comparison result to the invariants

$$d_{A,B} : K_2(C_r^2[x, y]) \rightarrow \mathbb{C}^* \quad \text{and} \quad \tau_{A,B} : K_2^{\text{rel}}(C_r^2[x, y]) \rightarrow \mathbb{C}$$

Theorem 3.23. *For each pair of operators $A, B \in \mathcal{L}(H)$ with trace class commutator $[A, B] \in \mathcal{L}^1(H)$ we have the formula*

$$\det(e^A e^B e^{-A} e^{-B}) = e^{\text{Tr}[A, B]}$$

for the determinant of the multiplicative commutator $e^A e^B e^{-A} e^{-B} \in \mathcal{GL}_1(\mathcal{L}^1(H))$.

Proof. From Theorem 3.21 and Theorem 3.11 we have the identities

$$\tau_{A,B}(\{\gamma_x, \gamma_y\}^{\text{rel}}) = \text{Tr}([A, B]) \quad \text{and} \quad d_{A,B}(\{e^x, e^y\}) = \det(e^A e^B e^{-A} e^{-B})$$

Furthermore, by Theorem 3.22 we have that

$$\exp \circ \tau_{A,B} = d_{A,B} \circ \theta$$

Therefore, we will only need to check the equality

$$\theta(\{\gamma_x, \gamma_y\}^{\text{rel}}) = \{e^x, e^y\}$$

However, this follows immediately since

$$\begin{aligned} (s_0(\gamma_x) s_1(\gamma_y))(\mathbf{1}) &= e^{-y} & (s_0(\gamma_x) s_1(\gamma_y))(\mathbf{2}) &= e^{-x} e^{-y} \\ (s_1(\gamma_x) s_0(\gamma_y))(\mathbf{1}) &= e^{-x} & (s_1(\gamma_x) s_0(\gamma_y))(\mathbf{2}) &= e^{-x} e^{-y} \end{aligned}$$

□

The main goal of this thesis is to find a concrete generalization of the determinant invariant

$$(e^A, e^B) \mapsto e^{\text{Tr}[A, B]}$$

As this first part of the introduction suggests this will be accomplished by the use of K -theoretic methods.

4. DETERMINANT INVARIANTS AND MULTIPLICATIVE CHARACTERS OF FREDHOLM MODULES

In their paper [9] A. Connes and M. Karoubi associate a multiplicative character on algebraic K -theory to each finitely summable Fredholm module. In particular, they show that every odd 2-summable Fredholm module (F, H) over a unital \mathbb{C} -algebra \mathcal{A} yields a homomorphism

$$\mathcal{M}_F : K_2(\mathcal{A}) \rightarrow \mathbb{C}/(2\pi i)\mathbb{Z}$$

In this section we will explore the relation between this invariant and the determinant invariant which we constructed in Section 3.2. This brings us to the core of the article [16] which can be

found at the end of the thesis. That is, we will explain our main result which is given by the commutativity of the following diagram

$$\begin{array}{ccc} K_2(\mathcal{A}) & \xrightarrow{(R_F)^*} & K_2(\mathcal{B}) \\ \mathcal{M}_F \downarrow & & \downarrow d_F \\ \mathbb{C}/(2\pi i)\mathbb{Z} & \xrightarrow{\exp} & \mathbb{C}^* \end{array}$$

Here $R_F : \mathcal{A} \rightarrow \mathcal{B}$ is a surjective unital algebra homomorphism and $d_F : K_2(\mathcal{B}) \rightarrow \mathbb{C}^*$ is a determinant invariant. They are both constructed from the odd 2-summable Fredholm module (F, H) over \mathcal{A} . Our result implies in particular that the multiplicative character factorizes through the second algebraic K -group of the Calkin algebra, $K_2(\mathcal{L}(H)/\mathcal{L}^1(H))$. Let us start with the construction of the multiplicative character.

4.1. The multiplicative character on the second algebraic K -group. Let \mathcal{H} be a fixed separable Hilbert space of infinite dimension. We let \mathcal{M}^1 denote the \mathbb{C} -algebra consisting of operators on the direct sum $\mathcal{H} \oplus \mathcal{H}$ with antidiagonal in the second Schatten ideal. Thus, the elements $x = (x_{ij}) \in \mathcal{M}^1$ are precisely the bounded operators on $\mathcal{H} \oplus \mathcal{H}$ which in their matrix representation have $x_{12}, x_{21} \in \mathcal{L}^2(\mathcal{H})$. The \mathbb{C} -algebra \mathcal{M}^1 can be equipped with the Banach algebra norm $\|\cdot\| : \mathcal{M}^1 \rightarrow [0, \infty)$ given by

$$\|x\| = \|x\|_\infty + \|x_{12}\|_2 + \|x_{21}\|_2$$

This turns \mathcal{M}^1 into a Banach algebra. The importance of \mathcal{M}^1 comes from its interpretation as a universal Fredholm algebra. Let us explain ourselves.

Definition 4.1. *Let \mathcal{A} be a unital \mathbb{C} -algebra. By an odd 2-summable Fredholm module (F, H) over \mathcal{A} we will understand a separable Hilbert space H , a unital algebra homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{L}(H)$ and a bounded operator $F \in \mathcal{L}(H)$ such that*

- (1) $F = F^*$
- (2) $F^2 = 1$
- (3) $[F, \pi(a)] \in \mathcal{L}^2(H)$ for all $a \in \mathcal{A}$.

When \mathcal{A} is a unital Banach algebra we will say that the Fredholm module (F, H) is continuous when the maps

$$\pi : \mathcal{A} \rightarrow \mathcal{L}(H) \quad \text{and} \quad a \mapsto [F, \pi(a)] \in \mathcal{L}^2(H)$$

are both continuous.

We will always assume that the Hilbert spaces PH and $(1-P)H$ are infinite dimensional. Here $P = (F+1)/2$ denotes the projection associated with the selfadjoint unitary operator $F \in \mathcal{L}(H)$.

We let $F_U \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ denote the selfadjoint unitary given by the matrix

$$F_U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Furthermore we have a representation $\pi : \mathcal{M}^1 \rightarrow \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ given by inclusion. The pair $(F_U, \mathcal{H} \oplus \mathcal{H})$ then becomes a continuous odd 2-summable Fredholm module over \mathcal{M}^1 . It is universal in the following sense.

Let (F, H) denote any odd 2-summable Fredholm module over the \mathbb{C} -algebra \mathcal{A} . Since the Hilbert space PH and $(1 - P)H$ are infinite dimensional we can identify both of them with the "universal" Hilbert space \mathcal{H} . In this way we obtain an algebra homomorphism

$$\pi_F : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$$

which is well defined up to conjugation by unitary operators. The condition on the commutator $[F, \pi(a)] \in \mathcal{L}^2(H)$ ensures that the representation $\pi_F : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ factorizes through the Banach algebra \mathcal{M}^1 . Furthermore, whenever the Fredholm module (F, H) is continuous the associated unital algebra homomorphism

$$\pi_F : \mathcal{A} \rightarrow \mathcal{M}^1$$

is continuous as well.

In analogy with the K -theoretic construction of the Fredholm determinant which we gave in Section 2.4 we will use an additive character

$$\tau_1 \circ \text{ch}^{\text{rel}} : K_2^{\text{rel}}(\mathcal{M}^1) \rightarrow \mathbb{C}$$

to construct the multiplicative character. As we saw in Section 3.3 the relative Chern character

$$\text{ch}^{\text{rel}} : K_2^{\text{rel}}(\mathcal{M}^1) \rightarrow HC_1(\mathcal{M}^1)$$

is a homomorphism of degree minus one from relative K -theory to continuous cyclic homology. The homomorphism

$$\tau_1 : HC_1(\mathcal{M}^1) \rightarrow \mathbb{C}$$

comes from a low dimensional case of a theorem by A. Connes.

Theorem 4.2. [7, I 7. Lemma 2] *Let (F, H) denote a continuous odd 2-summable Fredholm module over a unital Banach algebra \mathcal{A} . The bilinear map given by*

$$\tau_F : (a^0, a^1) \mapsto -\frac{1}{4} \text{Tr}(F[F, a^0][F, a^1])$$

then determines a character $\tau_F : HC_1(\mathcal{A}) \rightarrow \mathbb{C}$ on the first continuous cyclic homology group.

The rational constant $-\frac{1}{4}$ is chosen in such a way that the multiplicative character which we are going to define agrees with the one defined in [9].

We let $\tau_1 : HC_1(\mathcal{M}^1) \rightarrow \mathbb{C}$ denote the character associated with the universal odd continuous 2-summable Fredholm module $(F_U, \mathcal{H} \oplus \mathcal{H})$ over \mathcal{M}^1 . For any odd continuous 2-summable Fredholm module (F, H) over a unital Banach algebra \mathcal{A} we then get the following commutative diagram

$$\begin{array}{ccc} HC_1(\mathcal{A}) & \xrightarrow{(\pi_F)_*} & HC_1(\mathcal{M}^1) \\ \tau_F \downarrow & & \tau_1 \downarrow \\ \mathbb{C} & \xlongequal{\quad} & \mathbb{C} \end{array}$$

As announced above, these homomorphisms and the relative Chern character constitute the additive character.

Definition 4.3. By the universal additive character we will understand the homomorphism

$$\mathcal{A}_U : K_2^{\text{rel}}(\mathcal{M}^1) \rightarrow \mathbb{C} \quad \mathcal{A} = \tau_1 \circ \text{ch}^{\text{rel}}$$

given by the composition of the second relative Chern character and the character associated with the universal odd 2-summable Fredholm module.

Definition 4.4. Let (F, H) be a continuous odd 2-summable Fredholm module over a unital Banach algebra \mathcal{A} . By the additive character of the Fredholm module we will understand the homomorphism

$$\mathcal{A}_F : K_2^{\text{rel}}(\mathcal{A}) \rightarrow \mathbb{C} \quad \tau_F := \tau_U \circ (\pi_F)_*$$

given by the composition of the homomorphism induced by the continuous unital algebra homomorphism $\pi_F : \mathcal{A} \rightarrow \mathcal{M}^1$ and the universal additive character.

Note that the naturality of the relative Chern character and the commutativity of the diagram (4.1) entail the identities

$$\mathcal{A}_F = \mathcal{A}_U \circ (\pi_F)_* = \tau_1 \circ \text{ch}^{\text{rel}} \circ (\pi_F)_* = \tau_F \circ \text{ch}^{\text{rel}}$$

In the paper [9] it is proved that the homomorphism

$$\theta : K_2^{\text{rel}}(\mathcal{M}^1) \rightarrow K_2(\mathcal{M}^1)$$

is surjective. Furthermore, it is shown that there is a well defined homomorphism

$$\mathcal{M}_U : K_2(\mathcal{M}^1) \rightarrow \mathbb{C}/(2\pi i)\mathbb{Z}$$

given by

$$\mathcal{M}_U(\theta(y)) = [\mathcal{A}_U(y)]$$

Here $[\cdot] : \mathbb{C} \rightarrow \mathbb{C}/(2\pi i)\mathbb{Z}$ denotes the quotient map. This homomorphism is called the *universal multiplicative character*.

For any odd 2-summable Fredholm module (F, H) over a \mathbb{C} -algebra \mathcal{A} we then define

$$\mathcal{M}_F : K_2(\mathcal{A}) \rightarrow \mathbb{C}/(2\pi i)\mathbb{Z} \quad \mathcal{M}_F = \mathcal{M}_U \circ (\rho_F)_*$$

This is the *multiplicative character of the Fredholm module*.

4.2. Comparison of the multiplicative character and the determinant invariant. We will now give an interpretation of the multiplicative character as a determinant invariant. This brings us to the heart of the article [16] which is attached at the end of this thesis. The work is also related to the picture of the multiplicative character on the second algebraic K -group given by A. Connes and M. Karoubi in [9]. There the invariant is exhibited by means of a certain determinant central extension. In fact, one could understand part of our main result as a rephrasing of [9, Théorème 5.6]. However, the tools which we have been developing during the last sections allows us to improve and clarify this result. At last, let us mention that the proof of the main result given in [16] relies on a slightly different approach. That is, it uses a certain cokernel complex instead of the relative K -groups of pairs which we have been introducing earlier in this text. We will not give further comment on the cokernel construction here. Let us begin the exposition.

Let \mathcal{H} be a separable infinite dimensional Hilbert space. Let \mathcal{T}^1 denote the \mathbb{C} -algebra consisting of pairs $(S, x) \in \mathcal{L}(\mathcal{H}) \times \mathcal{M}^1$ such that $S - x_{11} \in \mathcal{L}^1(H)$. We can equip this \mathbb{C} -algebra with a Banach algebra norm $\|\cdot\| : \mathcal{T}^1 \rightarrow [0, \infty)$ given by

$$\|(S, x)\| = \|x\|_\infty + \|S - x_{11}\|_1$$

This turns \mathcal{T}^1 into a Banach algebra. It comes equipped with two continuous surjective algebra homomorphisms

$$\pi_1 : \mathcal{T}^1 \rightarrow \mathcal{L}(H) \quad \pi_1(S, x) = S \quad \text{and} \quad \pi_2 : \mathcal{T}^1 \rightarrow \mathcal{M}^1 \quad \pi_2(S, x) = x$$

The kernel of π_2 is the trace ideal, $\mathcal{L}^1(H)$. In particular we have a short exact sequence

$$\mathbf{T} : 0 \longrightarrow \mathcal{L}^1(H) \xrightarrow{i} \mathcal{T}^1 \xrightarrow{\pi_2} \mathcal{M}^1 \longrightarrow 0$$

It has a continuous linear split given by

$$s : \mathcal{M}^1 \rightarrow \mathcal{T}^1 \quad s(x) = (x_{11}, x)$$

The restriction to the trace ideal of the representation $\pi_1, \pi_1|_{\mathcal{L}^1(H)} : \mathcal{L}^1(H) \rightarrow \mathcal{L}^1(H)$ is the identity homomorphism. We therefore have an associated trace invariant and an associated determinant invariant

$$\tau_{(\mathbf{T}, \pi_1)} : K_2^{\text{rel}}(\mathcal{M}^1) \rightarrow \mathbb{C} \quad d_{(\mathbf{T}, \pi_1)} : K_2(\mathcal{M}^1) \rightarrow \mathbb{C}^*$$

By Theorem 3.22 the relation between these invariants is expressed by the commutative diagram

$$(20) \quad \begin{array}{ccc} K_2^{\text{rel}}(\mathcal{M}^1) & \xrightarrow{\theta} & K_2(\mathcal{M}^1) \\ \tau_{(\mathbf{T}, \pi_1)} \downarrow & & d_{(\mathbf{T}, \pi_1)} \downarrow \\ \mathbb{C} & \xrightarrow{\text{exp}} & \mathbb{C}^* \end{array}$$

To proceed we will have to relate the trace invariant of the pair (\mathbf{T}, π_1) with the universal additive character. The main link is established by the next lemma.

Lemma 4.5. *We have the identity*

$$\tau_1 = (-\text{Tr}) \circ (\pi_1)_* \circ \partial_{\mathbf{T}} : HC_1(\mathcal{M}^1) \rightarrow \mathbb{C}$$

Thus the character of the universal Fredholm module coincides with the composition of the boundary map in continuous cyclic homology of the short exact sequence \mathbf{T} , the map induced by the representation and the operator trace on the zeroth relative cyclic homology group.

Proof. A variant of this result has been carefully proved in [16, Theorem 4.2], see also the proof of [9, Théorème 5.6]. \square

Corollary 4.6. *The universal additive character coincides with the trace invariant of the pair (\mathbf{T}, π_1) . Thus,*

$$\mathcal{A}_U = \tau_{(\mathbf{T}, \pi_1)} : K_2^{\text{rel}}(\mathcal{M}^1) \rightarrow \mathbb{C}$$

Proof. We recall that the trace invariant of the pair (\mathbf{T}, π_1) is given by the composition

$$\tau_{(\mathbf{T}, \pi_1)} = (-\text{Tr}) \circ (\pi_1)_* \circ \partial_{\mathbf{T}} \circ \text{ch}^{\text{rel}} : K_2^{\text{rel}}(\mathcal{M}^1) \rightarrow \mathbb{C}$$

The desired result is now a consequence of Lemma 4.5. \square

The description of the multiplicative character as a determinant invariant now follows immediately.

Corollary 4.7. *The composition of the universal multiplicative character and the exponential function coincides with the determinant invariant of the pair (\mathbf{T}, π_1) . To be precise, we have a commutative diagram*

$$\begin{array}{ccc} K_2(\mathcal{M}^1) & \xlongequal{\quad} & K_2(\mathcal{M}^1) \\ \mathcal{M}_U \downarrow & & d_{(\mathbf{T}, \pi_1)} \downarrow \\ \mathbb{C}/(2\pi i)\mathbb{Z} & \xrightarrow{\exp} & \mathbb{C}^* \end{array}$$

Proof. This follows from Corollary 4.6 and the commutativity of the diagram (20). \square

As mentioned in the introduction to this section, we can understand the above corollary as a rephrasing of [9, Théorème 5.6]. However, the detailed exposition which we have given actually yields a refinement of Corollary 4.7. This refinement is the main result of the first article attached at the end of the thesis. Let us give some details.

We consider the algebra homomorphism

$$\alpha : \mathcal{M}^1 \rightarrow \mathcal{L}(\mathcal{H})/\mathcal{L}^1(\mathcal{H}) \quad \alpha(x) = q(x_{11})$$

Here $q : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})/\mathcal{L}^1(\mathcal{H})$ denotes the quotient map. This coincides with the algebra homomorphism induced by the representation $\pi_1 : \mathcal{T}^1 \rightarrow \mathcal{L}(H)$. By the functoriality of the second algebraic K -groups we get an induced group homomorphism

$$\alpha_* : K_2(\mathcal{M}^1) \rightarrow K_2(\mathcal{L}(\mathcal{H})/\mathcal{L}^1(\mathcal{H}))$$

By definition of the determinant invariant of the pair (\mathbf{T}, π_1) we have

$$d_{(\mathbf{T}, \pi_1)} : K_2(\mathcal{M}^1) \rightarrow \mathbb{C}^* \quad d_{(\mathbf{T}, \pi_1)} = d \circ \alpha_*$$

Here $d : K_2(\mathcal{L}(\mathcal{H})/\mathcal{L}^1(\mathcal{H})) \rightarrow \mathbb{C}^*$ denotes the universal determinant invariant.

We have thus really proved the following

Theorem 4.8. [16] *The homomorphisms $\exp \circ (\mathcal{M}_U) : K_2(\mathcal{M}^1) \rightarrow \mathbb{C}^*$ and $d \circ \alpha_* : K_2(\mathcal{M}^1) \rightarrow \mathbb{C}^*$ coincide. That is, the universal multiplicative character agrees with the universal determinant invariant up to a canonical homomorphism. In particular, the universal multiplicative character only depends on the second algebraic K -group of the Calkin algebra $\mathcal{L}(H)/\mathcal{L}^1(H)$.*

Suppose now, that we are given an odd 2-summable Fredholm module (F, H) over a \mathbb{C} -algebra \mathcal{A} . Let us fix an algebra homomorphism $\pi_F : \mathcal{A} \rightarrow \mathcal{M}^1$. Let $\mathcal{E} \subseteq \mathcal{L}(\mathcal{H})$ denote the \mathbb{C} -algebra defined by

$$\mathcal{E} = \{T + (\pi_F(a))_{11} \mid a \in \mathcal{A} \text{ and } T \in \mathcal{L}^1(\mathcal{H})\}$$

Letting \mathcal{B} denote the image of \mathcal{E} in the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{L}^1(\mathcal{H})$ we get a short exact sequence

$$X_F : 0 \longrightarrow \mathcal{L}^1(\mathcal{H}) \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{B} \longrightarrow 0$$

and a representation $\rho : \mathcal{E} \rightarrow \mathcal{L}(\mathcal{H})$ given by inclusion. We therefore have an associated determinant invariant

$$d_F : K_2(\mathcal{B}) \rightarrow \mathbb{C}^* \quad d_F = d \circ \rho_*$$

Furthermore, we have an algebra homomorphism $R_F : \mathcal{A} \rightarrow \mathcal{B}$ given by $R_F(a) = q(\pi_F(a)_{11})$.

Corollary 4.9. [16] *The multiplicative character and the determinant invariant of the odd 2-summable Fredholm module (F, H) coincides up to the homomorphism $R_F : \mathcal{A} \rightarrow \mathcal{B}$. To be precise, the diagram*

$$\begin{array}{ccc} K_2(\mathcal{A}) & \xrightarrow{(R_F)_*} & K_2(\mathcal{B}) \\ \mathcal{M}_F \downarrow & & d_F \downarrow \\ \mathbb{C}/(2\pi i)\mathbb{Z} & \xrightarrow{\exp} & \mathbb{C}^* \end{array}$$

is commutative.

Proof. This follows from Theorem 4.8 since the unital algebra homomorphisms

$$\rho \circ R_F : \mathcal{A} \rightarrow \mathcal{L}(H)/\mathcal{L}^1(H) \quad \text{and} \quad \alpha \circ \pi_F : \mathcal{A} \rightarrow \mathcal{L}(H)/\mathcal{L}^1(H)$$

coincide. □

Let us end this section by a concrete calculation of the multiplicative character on the second algebraic K -group. Let (F, H) denote a continuous odd 2-summable Fredholm module over a *commutative* unital Banach algebra \mathcal{A} . Let us choose two elements $a, b \in \mathcal{A}$ and apply the exponential function to them. In this way we obtain the invertible elements $e^a, e^b \in \mathcal{A}^*$. We can thus form their Steinberg symbol

$$\{e^a, e^b\} \in K_2(\mathcal{A})$$

and ask ourselves of the value of the multiplicative character

$$\mathcal{M}_F : K_2(\mathcal{A}) \rightarrow \mathbb{C}/(2\pi i)\mathbb{Z}$$

on this element. Let $\pi_F : \mathcal{A} \rightarrow \mathcal{M}^1$ denote the continuous unital algebra homomorphism associated with the Fredholm module. By the result of Theorem 4.8 we then have

$$\begin{aligned} (\exp \circ \mathcal{M}_F)\{e^a, e^b\} &= (d \circ \alpha_* \circ (\pi_F)_*)\{e^a, e^b\} \\ &= d\{q(e^{a_{11}}), q(e^{b_{11}})\} \end{aligned}$$

In the second line we have suppressed the homomorphism $\pi_F : \mathcal{A} \rightarrow \mathcal{M}^1$. Following the proof of Theorem 3.11 we get that

$$d\{q(e^{a_{11}}), q(e^{b_{11}})\} = \det(e^{a_{11}} e^{b_{11}} e^{-a_{11}} e^{-b_{11}})$$

The identity of Theorem 3.23 thus entails that

$$(21) \quad (\exp \circ \mathcal{M}_F)\{e^a, e^b\} = e^{\text{Tr}[a_{11}, b_{11}]} = e^{\text{Tr}[PaP, PbP]}$$

Here $P = (F + 1)/2$ denotes the projection associated with the Fredholm module (F, H) . We have suppressed the representation $\pi : \mathcal{A} \rightarrow \mathcal{L}(H)$.

In the following sections we will obtain a multivariate version of the identity (21). This is the main result of the second article attached at the end of the thesis. To expose our calculations properly we will have to make the reader acquainted with some further K -theoretic concepts. This is the primary goal of the next section.

5. RELATIVE K -THEORY AND PRODUCTS OF CONTRACTIONS

In this section we will introduce the higher algebraic K -groups of unital rings and the higher relative K -groups of unital Banach algebras. We will then express the relation between these two K -theories and topological K -theory by means of a long exact sequence. Having accomplished this we will turn to the description of the exterior and interior Loday products on relative and algebraic K -theory. We shall see at last, that the relative Steinberg symbol, which we introduced in Section 3.3, is nothing but a special instance of the interior relative Loday product. The algebraic K -theory was originally invented by D. Quillen, see [31]. The relative K -theory was constructed by M. Karoubi who also proved the existence of the long exact sequence K -groups, see [20]. The Loday product on algebraic K -theory was found by J.-L. Loday in [25]. It was later generalized by J. P. May, see [28]. We will use the original approach of Loday to construct our product on the relative K -groups. This is accomplished in details in the article [17] which is attached at the end of the thesis. It is possible that this relative product can be viewed as a special case of May's general theory as well. We have not pursued this direction further.

5.1. Preliminaries on simplicial sets. We start out with a presentation of some basic results from the theory of simplicial sets which we will make use of in the sequel. Our main reference is the monograph by J. P. May, [27].

Definition 5.1. *By a simplicial set we will understand a sequence of sets $(X_n)_{n \geq 0}$ such that for each $n \in \mathbb{N}$ there are maps*

$$\begin{aligned} d_i : X_n &\rightarrow X_{n-1} & \text{for } i &\in \{0, 1, \dots, n\} \\ s_j : X_{n-1} &\rightarrow X_n & \text{for } j &\in \{0, 1, \dots, n-1\} \end{aligned}$$

such that

$$\begin{aligned} d_i d_j &= d_{j-1} d_i & \text{for } i < j \\ s_i s_j &= s_{j+1} s_i & \text{for } i \leq j \\ d_i s_j &= \begin{cases} s_{j-1} d_i & \text{for } i < j \\ \text{id} & \text{for } i = j, j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases} \end{aligned}$$

The maps $d_i : X_n \rightarrow X_{n-1}$ are called face operators. The maps $s_j : X_{n-1} \rightarrow X_n$ are called degeneracy operators.

By a map of simplicial sets we will understand a map of degree zero $f : X \rightarrow Y$ which commutes with the face and degeneracy operators.

There exists a family of simplicial sets which behave in a particularly nice way. For example, they admit a sequence of homotopy groups which parallels the homotopy groups of a topological space.

Definition 5.2. *By a Kan complex we will understand a simplicial set $X = (X_n)_{n \geq 0}$ which satisfies the following extension property: For each n elements $x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_n \in X_{n-1}$ which satisfy the compatibility relation*

$$d_i x_j = d_{j-1} x_i \quad \text{for each } i, j \in \{0, \dots, n\} - \{k\} \text{ with } i < j$$

there exists an $x \in X_n$ such that $d_j(x) = x_j$ for all $j \in \{0, \dots, n\} - \{k\}$.

We will give two examples of Kan complexes. Let A be a unital Banach algebra and let $GL(A)$ denote the topological group of invertible matrices. Also, let R be a unital ring and let $GL(R)$ denote the group of invertible matrices. Recall that Δ^n denotes the n^{th} standard simplex

$$\Delta^n = \{(t_1, \dots, t_n) \mid t_i \in [0, 1], \sum_{i=1}^n t_i \leq 1\}$$

The zeroth standard simplex consists of a single point, $\Delta^0 = \{0\}$. The vertices of the n^{th} standard simplex are denoted by $\mathbf{0}, \dots, \mathbf{n} \in \Delta^n$. Thus,

$$\mathbf{i} = \begin{cases} (0, \dots, 0) & \text{for } i = 0 \\ (0, \dots, 1, \dots, 0) & \text{for } i \neq 0 \end{cases}$$

where the number 1 is in position i . We have chosen this model in order to emphasize the special role of the zeroth vertex and the zeroth face in the Kan complex $R(A)$ below.

- (1) Let $BGL(R) = (GL(R)^n)_{n \geq 0}$. The face and degeneracy operators are given by

$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & \text{for } i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & \text{for } i \in \{1, \dots, n-1\} \\ (g_1, \dots, g_{n-1}) & \text{for } i = n \end{cases}$$

$$s_j(g_1, \dots, g_{n-1}) = \begin{cases} (g_1, \dots, g_{j-1}, 1, g_j, \dots, g_{n-1}) & \text{for } j \in \{0, \dots, n-1\} \end{cases}$$

The identity in degree zero will play the role of a basepoint.

- (2) Let $R(A) = (R(A)_n)$. Here $R(A)_n$ denotes the set of continuous maps $\sigma : \Delta^n \rightarrow GL(A)$ with $\sigma(\mathbf{0}) = 1$. The face and degeneracy are given by

$$d_i(\sigma)(t_1, \dots, t_{n-1}) = \begin{cases} \sigma(1 - \sum_{j=1}^{n-1} t_j, t_1, \dots, t_{n-1}) \sigma(\mathbf{1})^{-1} & \text{for } i = 0 \\ \sigma(t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}) & \text{for } i \in \{1, \dots, n\} \end{cases}$$

$$s_j(\sigma)(t_1, \dots, t_n) = \begin{cases} \sigma(t_2, \dots, t_n) & \text{for } j = 0 \\ \sigma(t_1, \dots, t_j + t_{j+1}, \dots, t_n) & \text{for } j \in \{1, \dots, n-1\} \end{cases}$$

Notice the "extra factor" $\sigma(\mathbf{1})^{-1}$ in the formula for the zeroth face operator. The identity $1 \in R(A)_0$ will play the role of a basepoint. See also Section 3.3.

Note that each unital ring homomorphism $\varphi : R \rightarrow S$ induces a simplicial map $\varphi : BGL(R) \rightarrow BGL(S)$ in a functorial way. Likewise, each continuous unital algebra homomorphism $\varphi : A \rightarrow B$ induces a simplicial map $\varphi : R(A) \rightarrow R(B)$ in a functorial way.

Remark 5.3. *The simplicial set $R(A)$ is isomorphic to the simplicial set $GL(A_*)/GL(A)$ which is used in [20] for the definition of relative K -theory. The isomorphism maps the class of a continuous map $\sigma : \Delta^n \rightarrow GL(A)$ to $\sigma \cdot \sigma(\mathbf{0})^{-1} \in R(A)$.*

The two Kan complexes (1) and (2) are linked by the simplicial map

$$\theta : R(A) \rightarrow BGL(A) \quad \theta(\sigma) = (\sigma(\mathbf{0})\sigma(\mathbf{1})^{-1}, \dots, \sigma(\mathbf{n}-1)\sigma(\mathbf{n})^{-1})$$

Let us continue by a presentation of the homotopy groups of a pointed Kan complex.

Definition 5.4. Let $X = (X_n)_{n \geq 0}$ be a pointed Kan complex with basepoint $*$ $\in X_0$. By the n^{th} homotopy group of X we will understand the set

$$\pi_n(X) := \{x \in X_n \mid d_i x = * \text{ for all } i \in \{0, \dots, n\}\} / \sim$$

Here the equivalence relation \sim is defined by

$$x \sim y \Leftrightarrow \exists z \in X_{n+1} : d_n z = x, d_{n+1} z = y \text{ and } d_i z = * \quad \forall i \in \{0, 1, \dots, n-1\}$$

The word "group" in the above definition is justified by the following lemma.

Lemma 5.5. Let $X = (X_n)_{n \geq 0}$ be a pointed Kan complex. For each $n \geq 1$ there is a well defined group structure on the homotopy group $\pi_n(X)$. The composition $*$: $\pi_n(X) \times \pi_n(X) \rightarrow \pi_n(X)$ is given by

$$x * y = d_n(z)$$

Here $z \in X_{n+1}$ is any $(n+1)$ -simplex with $d_{n-1}(z) = x$, $d_{n+1}(z) = y$ and $d_i(z) = *$ for $i \in \{0, 1, \dots, n-2\}$. The homotopy groups are commutative for all $n \geq 2$.

Let us calculate the homotopy groups of the Kan complexes (1) and (2).

Lemma 5.6. The homotopy groups of $BGL(R)$ are given by

$$\pi_n(BGL(R)) = \begin{cases} GL(R) & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

The homotopy groups of $R(A)$ are given by

$$\pi_n(R(A)) = \begin{cases} R(A)_1 / \sim & \text{for } n = 1 \\ K_{n+1}^{\text{top}}(A) & \text{for } n \geq 2 \end{cases}$$

In both cases the group structure is induced by the product of invertible matrices.

To each Kan complex, X , we can also associate a sequence of abelian groups which are analogues of the singular homology groups of a topological space. To do this, we note that the simplicial relation $d_i d_j = d_{j-1} d_i$ for all $i < j$ ensures that the operators

$$d = \sum_{i=0}^n (-1)^i d_i : \mathbb{Z}[X_n] \rightarrow \mathbb{Z}[X_{n-1}]$$

satisfy the equality $d^2 = 0$. In particular, the complex

$$\mathbb{Z}[X_0] \xleftarrow{d} \dots \xleftarrow{d} \mathbb{Z}[X_{n-1}] \xleftarrow{d} \mathbb{Z}[X_n] \xleftarrow{d} \mathbb{Z}[X_{n+1}] \xleftarrow{d} \dots$$

is a chain complex.

Definition 5.7. By the homology of the simplicial set X we will understand the homology of the above chain complex.

There is a functorial construction which associates a pointed CW-complex to each pointed Kan complex. This functor is called the geometric realization and is denoted by $|\cdot|$. We will need a precise description of this space later on, so let us introduce it properly.

Definition 5.8. Let X be a pointed Kan complex. By the geometric realization of X we will understand the quotient space

$$|X| := \coprod_{n \geq 0} \Delta^n \times X_n / \sim$$

Here each X_n is equipped with the discrete topology. The equivalence relation \sim is given by

$$\begin{aligned} (\delta^i t, x) &\sim (t, d_i x) && \text{for each } t \in \Delta^{n-1}, x \in X_n \\ (\eta^j t, x) &\sim (t, s_j x) && \text{for each } t \in \Delta^{n+1}, x \in X_n \end{aligned}$$

Here $\delta_i : \Delta^{n-1} \rightarrow \Delta^n$ and $\eta^j : \Delta^{n+1} \rightarrow \Delta^n$ are dual to the face and degeneracy operators defined in (2).

Let us state some of the properties of the geometric realization.

Theorem 5.9. Let X be a pointed Kan complex and let $|X|$ denote its geometric realization. There is then a natural isomorphism between the homotopy groups of the pointed Kan complex X and the homotopy groups of the pointed space $|X|$. Furthermore, there is a natural isomorphism between the homology of the simplicial set X and the singular homology of the topological space $|X|$.

We will think of Kan complexes as combinatorial analogues of topological spaces.

5.2. The plus construction and higher K -theory. The higher algebraic and relative K -theories are defined by means of the plus construction of D. Quillen, [31]. Loosely said, this is a way to abelianize the fundamental group of certain topological spaces without changing the singular homology. In this section we will describe the main properties of this procedure. We will then introduce the higher algebraic and relative K -groups and describe the connection between them and with topological K -theory. We refer the reader to the book of J. Rosenberg for a detailed account on algebraic K -theory, [32].

Definition 5.10. A group G is said to be perfect if it equals its own commutator subgroup. That is, the smallest subgroup of G which contains all commutators coincides with G . Note that the commutator subgroup of a group is automatically normal.

Theorem 5.11. Let X be a connected CW-complex. Let N be a perfect normal subgroup of the fundamental group of X . There exists a connected CW-complex X^+ together with an injective map $i : X \rightarrow X^+$ such that

- (1) The fundamental group of X^+ equals the quotient group $\pi_1(X)/N$ and the induced map $i_* : \pi_1(X) \rightarrow \pi_1(X^+)$ is the quotient map.
- (2) The reduced singular homology of the homotopy fiber of the inclusion $i : X \rightarrow X^+$ vanishes.

The connected CW-complex X^+ is called the plus construction of X relative to $N \subseteq \pi_1(X)$.

The plus construction of a space satisfies the following universal property:

Theorem 5.12. *Let X be a connected CW-complex and let X^+ be the plus construction of X relative to some perfect normal subgroup N of $\pi_1(X)$. Let Y be a connected CW-complex. Suppose that $f : X \rightarrow Y$ is a continuous map such that the induced map on the fundamental groups $f_* : \pi_1(X) \rightarrow \pi_1(Y)$ vanishes on $N \subseteq \pi_1(X)$. Then there exists a continuous map $f^+ : X^+ \rightarrow Y$ which is unique up to homotopy and which makes the diagram*

$$\begin{array}{ccc} X & \xrightarrow{i} & X^+ \\ & \searrow f & \downarrow f^+ \\ & & Y \end{array}$$

commute up to homotopy.

We are now ready for the definition of the higher K -groups. We are going to apply the plus construction to the geometric realizations of the Kan complexes (1) and (2) of Section 5.1. In both cases the perfect normal subgroups are going to be the commutator subgroups of the fundamental groups.

Lemma 5.13. *Let R be a unital ring and let A be a unital Banach algebra. The commutator subgroups of the fundamental groups*

$$\begin{aligned} E(R) &= [GL(R), GL(R)] \subseteq GL(R) = \pi_1(BGL(R)) && \text{and} \\ (F(A)_1 / \sim) &= [R(A)_1 / \sim, R(A)_1 / \sim] \subseteq (R(A)_1 / \sim) = \pi_1(R(A)) \end{aligned}$$

are both perfect.

Proof. The perfectness of the elementary matrices is a standard result which is carefully proved in [32, Proposition 2.1.4]. The perfectness of the group $F(A)_1 / \sim$ follows by a slight modification of the same argument. \square

Definition 5.14. *Let R be a unital ring. By the algebraic K -theory of R we will understand the homotopy groups of the pointed, connected CW-complex, $|BGL(R)|^+$. The plus construction is carried out with respect to the commutator subgroup $E(R) \subseteq \pi_1(|BGL(R)|)$. Thus, by definition*

$$K_n(R) = \pi_n(|BGL(R)|^+)$$

It follows from the universal property of the plus construction that the algebraic K -groups are functorial in the unital ring R . Indeed, each unital ring homomorphism $\varphi : R \rightarrow S$ gives rise to a continuous map $|\varphi| : |BGL(R)| \rightarrow |BGL(S)|$ which coincides with the induced group homomorphism $\varphi : GL(R) \rightarrow GL(S)$ on the fundamental groups. In particular, we get a continuous map $|\varphi|^+ : |BGL(S)|^+ \rightarrow |BGL(R)|^+$ which is uniquely defined up to homotopy. The corresponding group homomorphism between the homotopy groups is the desired application.

Definition 5.15. *Let A be a unital Banach algebra. By the relative K -theory of A we will understand the homotopy groups of the pointed, connected CW-complex, $|R(A)|^+$. The plus construction is carried out with respect to the commutator subgroup $(F(A)_1 / \sim) \subseteq \pi_1(|R(A)|)$. Thus, by definition*

$$K_n^{\text{rel}}(A) = \pi_n(|R(A)|^+)$$

It follows from the universal property of the plus construction that each continuous unital algebra homomorphism induces a homomorphism between the relative K -groups. The argument is similar to the one presented above. This makes the relative K -groups into covariant functors.

The connection between the algebraic and relative K -theories is established by the simplicial map

$$\theta : R(A) \rightarrow BGL(A) \quad \theta(\sigma) = (\sigma(\mathbf{0})\sigma(\mathbf{1})^{-1}, \dots, \sigma(\mathbf{n}-\mathbf{1})\sigma(\mathbf{n})^{-1})$$

which we have already mentioned in Section 5.1. By functoriality of the geometric realization we get a continuous map $|\theta| : |R(A)| \rightarrow |BGL(A)|$ which induces the group homomorphism

$$(R(A)_1 / \sim) \rightarrow GL(A) \quad [\sigma] \mapsto \sigma(1)^{-1}$$

on the fundamental groups. From the universal property of the plus construction we thus get a continuous map

$$|\theta|^+ : |R(A)|^+ \rightarrow |BGL(A)|^+$$

which is uniquely defined up to homotopy. In particular, there is a group homomorphism

$$\theta : K_n^{\text{rel}}(A) \rightarrow K_n(A) \quad \text{for each } n \in \mathbb{N}$$

at the level of K -theory. It is then a theorem of M. Karoubi that this link is part of long exact sequence of K -groups, see [20, Proposition 6.17].

Theorem 5.16. *There is a long exact sequence of abelian groups*

$$\dots \longrightarrow K_{n+1}^{\text{top}}(A) \xrightarrow{v} K_n^{\text{rel}}(A) \xrightarrow{\theta} K_n(A) \longrightarrow K_n^{\text{top}}(A) \xrightarrow{v} \dots$$

As we shall see later, the above result provides us with an important tool for the construction of secondary invariants.

5.3. The Loday product in algebraic and relative K -theory. The concern of this section is the algebraic structure on the different K -groups. We will thus describe the Loday product on algebraic K -theory and then show how similar constructions apply to the relative K -groups. As expected, the homomorphism $\theta : K_n^{\text{rel}}(A) \rightarrow K_n(A)$ preserves these multiplicative structures. The main references are the article of J.-L. Loday, [25], and the second article attached at the end of the thesis, [17].

Let us start by describing the addition on the different K -theories in terms of an additive structure at the level of spaces. To be more precise we will indicate how the connected CW -complexes $|BGL(R)|^+$ and $|R(A)|^+$ can be equipped with a commutative H -group structure. Thus, these spaces admit a composition which satisfy all the usual properties of an abelian group up to homotopy. For details on H -groups we refer to [34], for example.

Let R be a unital ring. Let us consider the "direct sum" group homomorphism, $\oplus : GL(R) \times GL(R) \rightarrow GL(R)$ given by

$$(g \oplus h)_{ij} = \begin{cases} g_{kl} & \text{for } i = 2k - 1, j = 2l - 1 \\ h_{kl} & \text{for } i = 2k, j = 2l \\ 0 & \text{elsewhere} \end{cases}$$

This homomorphism gives rise to a continuous map

$$\oplus^+ : |BGL(R) \times BGL(R)|^+ \rightarrow |BGL(R)|^+$$

Now, by [25, Proposition 1.1.4] the projections onto each factor in $BGL(R) \times BGL(R)$ yield a homotopy equivalence

$$k : |BGL(R) \times BGL(R)|^+ \rightarrow |BGL(R)|^+ \times |BGL(R)|^+$$

Let us choose an inverse up to homotopy of this map

$$k^{-1} : |BGL(R)|^+ \times |BGL(R)|^+ \rightarrow |BGL(R) \times BGL(R)|^+$$

The addition on $|BGL(R)|^+$ is then given by the composition

$$+ = \oplus^+ \circ k^{-1} : |BGL(R)|^+ \times |BGL(R)|^+ \rightarrow |BGL(R)|^+$$

The following result can then be found as [25, Théorème 1.2.6]. See also [37, Proposition 1.2].

Theorem 5.17. *The application $+: |BGL(R)|^+ \times |BGL(R)|^+ \rightarrow |BGL(R)|^+$ and the basepoint $1 \in |BGL(R)|^+$ determines a commutative H -group structure on the space $|BGL(R)|^+$.*

It is a general result from the theory of H -groups (or even H -spaces) that the composition on the homotopy groups arising from the space level composition agrees with the usual composition. In particular, we get that the addition in $K_n(R)$ can be identified as

$$[f] + [g] = [+ \circ (f, g)] \quad f, g : S^n \rightarrow |BGL(R)|^+$$

Now, let A be a unital Banach algebra. We define the direct sum of continuous maps $\sigma, \tau : \Delta^n \rightarrow GL(A)$ as a pointwise direct sum. Namely, we let

$$\oplus : R(A) \times R(A) \rightarrow R(A)$$

be the simplicial map given by

$$(\sigma \oplus \tau)(t) = \sigma(t) \oplus \tau(t) \quad \text{for each } \sigma, \tau \in R(A)_n \text{ and } t \in \Delta^n$$

The composition on $|R(A)|$ is then given by

$$+^{\text{rel}} : |R(A)|^+ \times |R(A)|^+ \rightarrow |R(A)|^+ \quad +^{\text{rel}} = \oplus^+ \circ k^{-1}$$

Here the map $k^{-1} : |R(A)|^+ \times |R(A)|^+ \rightarrow |R(A) \times R(A)|^+$ has the same description as above. The following result can then be found as [17, Theorem 3.12 and Theorem 3.13].

Theorem 5.18. *The application $+^{\text{rel}} : |R(A)|^+ \times |R(A)|^+ \rightarrow |R(A)|^+$ and the basepoint $1 \in |R(A)|^+$ determines a commutative H -group structure on the space $|R(A)|^+$. The continuous map $|\theta|^+ : |R(A)|^+ \rightarrow |BGL(A)|^+$ becomes an H -map, that is, it respects the compositions up to homotopy.*

Again the above H -group composition induces the composition on the relative K -groups. To be precise,

$$[f] + [g] = [+^{\text{rel}} \circ (f, g)] \quad f, g : S^n \rightarrow |R(A)|^+$$

Let us turn to the description of the Loday product in algebraic and relative K -theory. We begin with the algebraic case. Thus, let R and S be two unital rings. For each $p, q \in \mathbb{N}$ let us choose an isomorphism

$$\varphi : R^p \otimes_{\mathbb{Z}} S^q \rightarrow (R \otimes_{\mathbb{Z}} S)^{pq}$$

of $(R \otimes_{\mathbb{Z}} S)$ -bimodules. We then have associated group homomorphisms

$$\otimes_{\varphi} : GL_p(R) \times GL_q(S) \rightarrow GL_{pq}(R \otimes_{\mathbb{Z}} S) \subseteq GL(R \otimes_{\mathbb{Z}} S)$$

which depend on the choices made above.

Now, for each $p, q \geq 3$ we let

$$\otimes_{p,q}^+ : |BGL_p(R)|^+ \times |BGL_q(S)|^+ \rightarrow |BGL(R \otimes_{\mathbb{Z}} S)|^+ \quad \otimes_{p,q}^+ = \otimes_{\varphi}^+ \circ k^{-1}$$

denote the composition of the homotopy equivalence

$$k^{-1} : |BGL_p(R)|^+ \times |BGL_q(S)|^+ \rightarrow |BGL_p(R) \times BGL_q(S)|^+$$

and the continuous map which is functorially induced by

$$\otimes_{\varphi} : GL_p(R) \times GL_q(S) \rightarrow GL(R \otimes_{\mathbb{Z}} S)$$

The maps $\otimes_{p,q}^+$ only depend on the choices of $(R \otimes_{\mathbb{Z}} S)$ -bimodule isomorphisms up to homotopy. See [25, Section 2.1.1].

The raw tensor products above do not behave well with respect to the identity elements so we need to change them slightly. To be precise, we define the modified tensor products

$$\gamma_{p,q} : |BGL_p(R)|^+ \times |BGL_q(S)|^+ \rightarrow |BGL(R \otimes_{\mathbb{Z}} S)|^+$$

by the formulas

$$\gamma_{p,q}(x, y) = \otimes_{p,q}^+(x, y) - \otimes_{p,q}^+(1_p, y) - \otimes_{p,q}^+(x, 1_q)$$

Here the subtraction comes from the H -group structure on $|BGL(R \otimes_{\mathbb{Z}} S)|^+$. See Theorem 5.17. The above modifications allow us to glue the tensor products together to obtain a continuous map

$$\gamma : |BGL(R)|^+ \times |BGL(S)|^+ \rightarrow |BGL(R \otimes_{\mathbb{Z}} S)|^+$$

However, the choices involved in the construction of this extension imply that we only get a map which is well defined up to weak homotopies. Here we say that two continuous maps $f, g : X \rightarrow Y$ are weakly homotopic if the induced maps $f_* = g_* : [K, X] \rightarrow [K, Y]$ agree for any compact space K . The same construction can be carried out with the cartesian product replaced by the smash product. We summarize the obtained results in the next theorem which is due to J.-L. Loday, see [25, Proposition 2.1.8]. Note that the algebraic properties are really consequences of the corresponding properties for the tensor product of invertible matrices. We use the notation

$$i : |BGL_p(R)|^+ \times |BGL_q(S)|^+ \rightarrow |BGL(R)|^+ \times |BGL(S)|^+ \quad \text{and}$$

$$p : |BGL(R)|^+ \times |BGL(S)|^+ \rightarrow |BGL(R)|^+ \wedge |BGL(S)|^+$$

for the inclusion and the quotient map.

Theorem 5.19. *There exists a map $\hat{\gamma} : |BGL(R)|^+ \wedge |BGL(S)|^+ \rightarrow |BGL(R \otimes_{\mathbb{Z}} S)|^+$ which is unique up to weak homotopy and which makes the diagram*

$$\begin{array}{ccc} |BGL_p(R)|^+ \times |BGL_q(S)|^+ & \xrightarrow{poi} & |BGL(R)|^+ \wedge |BGL(S)|^+ \\ \gamma_{p,q} \downarrow & & \downarrow \hat{\gamma} \\ |BGL(R \otimes_{\mathbb{Z}} S)|^+ & \xlongequal{\quad} & |BGL(R \otimes_{\mathbb{Z}} S)|^+ \end{array}$$

commute up to homotopy. Furthermore, it is natural, bilinear, associative and commutative up to weak homotopies. We refer to [25] for details on these properties.

The above application allows us to define the exterior Loday product on the algebraic K -groups. Note that it is independent of the choices made for the construction of $\hat{\gamma}$ by compactness of the spheres.

Definition 5.20. *By the exterior Loday product we will understand the map*

$$* : K_n(R) \times K_m(S) \rightarrow K_{n+m}(R \otimes_{\mathbb{Z}} S)$$

given by the formula

$$f * g = \hat{\gamma} \circ (f \wedge g) : S^n \wedge S^m \cong S^{n+m} \rightarrow |BGL(R \otimes_{\mathbb{Z}} S)|^+$$

on representatives $f : S^n \rightarrow |BGL(R)|^+$ and $g : S^m \rightarrow |BGL(S)|^+$.

The algebraic properties of the map $\hat{\gamma}$ imply the corresponding properties of the Loday product.

Theorem 5.21. *The exterior Loday product is natural in the unital rings R and S , bilinear and associative. It is also graded commutative in the following sense: For each $x \in K_n(R)$ and $y \in K_m(S)$ we have*

$$(-1)^{nm} x * y = T_*(y * x)$$

Here $T_ : K_{n+m}(R \otimes_{\mathbb{Z}} S) \rightarrow K_{n+m}(S \otimes_{\mathbb{Z}} R)$ is the homomorphism induced by the flip $T : R \otimes_{\mathbb{Z}} S \rightarrow S \otimes_{\mathbb{Z}} R$.*

In the case where the ring R is commutative the multiplication $m : R \otimes_{\mathbb{Z}} R \rightarrow R$ is a ring homomorphism. There is thus an induced map on algebraic K -theory, $m_* : K_n(R \otimes_{\mathbb{Z}} R) \rightarrow K_n(R)$.

Definition 5.22. *By the interior Loday product we will understand the composition*

$$m_* \circ * : K_n(R) \times K_m(R) \rightarrow K_{n+m}(R)$$

This product will also be denoted by $$.*

The following theorem is then a consequence of Theorem 5.21.

Theorem 5.23. *For each commutative unital ring R the addition and the interior Loday product equip the algebraic K -theory with the structure of a graded commutative ring.*

Let us consider the case of relative K -theory. Thus, let A and B be two unital Banach algebras. For each $p, q \in \mathbb{N}$ let us choose an isomorphism

$$\varphi : A^p \otimes_{\mathbb{Z}} B^q \rightarrow (A \otimes_{\mathbb{Z}} B)^{pq}$$

of $(A \otimes_{\mathbb{Z}} B)$ -bimodules. As above we get the corresponding group homomorphisms

$$(22) \quad \otimes_{\varphi} : GL_p(A) \times GL_q(B) \rightarrow GL_{pq}(A \otimes_{\mathbb{Z}} B)$$

Now, let $A \otimes_{\pi} B$ denote the projective tensor product of A and B . We then have the "identity" homomorphism

$$\iota : A \otimes_{\mathbb{Z}} B \rightarrow A \otimes_{\pi} B$$

We will extend the group homomorphisms (22) to a pointwise version.

Definition 5.24. Let $\sigma : \Delta^n \rightarrow GL_p(A)$ and $\tau : \Delta^n \rightarrow GL_q(B)$ be continuous maps. By the pointwise tensor product of σ and τ we will understand the continuous map

$$\sigma \otimes_\varphi \tau : \Delta^n \rightarrow GL(A \otimes_\pi B)$$

given by

$$(\sigma \otimes_\varphi \tau)(t) = \iota(\sigma(t) \otimes_\varphi \tau(t))$$

The pointwise tensor products of continuous maps induce simplicial maps

$$\otimes_\varphi : R_p(A) \times R_q(B) \rightarrow R_{pq}(A \otimes_\pi B) \subseteq R(A \otimes_\pi B)$$

These maps are at the core of the relative Loday product.

Now, let $p, q \geq 3$. In analogy with the algebraic case, we define the map

$$\otimes_{p,q}^+ : |R_p(A)|^+ \times |R_q(B)|^+ \rightarrow |R(A \otimes_\pi B)|^+ \quad \otimes_{p,q}^+ = \otimes_\varphi^+ \circ k^{-1}$$

Here, as usually, $k^{-1} : |R_p(A)|^+ \times |R_q(B)|^+ \rightarrow |R_p(A) \times R_q(B)|^+$ denotes some inverse of the homotopy equivalence given by the projection onto each factor. By [17, Section 3.3] this space level tensor product only depends on the choice of bimodule isomorphism up to homotopy.

We modify the above tensor products by subtracting the base point problem. Thus we define the maps

$$\gamma_{p,q} : |R_p(A)|^+ \times |R_q(B)|^+ \rightarrow |R(A \otimes_\pi B)|^+$$

by the formulas

$$\gamma_{p,q}(x, y) = \otimes_{p,q}^+(x, y) - \otimes_{p,q}^+(1_p, y) - \otimes_{p,q}^+(x, 1_q)$$

Here the subtraction comes from the H -group structure on $|R(A \otimes_\pi B)|^+$. See Theorem 5.18. The modified tensor products behave well with respect to stabilization and smash products. To be precise:

Theorem 5.25. *There exists a map $\hat{\gamma}^{\text{rel}} : |R(A)|^+ \wedge |R(B)|^+ \rightarrow |R(A \otimes_\pi B)|^+$ which is well defined up to weak homotopy and which makes the following diagram commute up to homotopy*

$$\begin{array}{ccc} |R_p(A)|^+ \times |R_q(B)|^+ & \xrightarrow{poi} & |R(A)|^+ \wedge |R(B)|^+ \\ \gamma_{p,q} \downarrow & & \hat{\gamma}^{\text{rel}} \downarrow \\ |R(A \otimes_\pi B)|^+ & \xlongequal{\quad} & |R(A \otimes_\pi B)|^+ \end{array}$$

Furthermore, this map is natural, bilinear, associative and commutative up to weak homotopies.

For a proof of the above result we refer to [17, Section 3.3].

We can now make a sensible definition of an exterior product on the relative K -groups.

Definition 5.26. *By the exterior relative Loday product we will understand the map*

$$*^{\text{rel}} : K_n^{\text{rel}}(A) \times K_m^{\text{rel}}(B) \rightarrow K_{n+m}^{\text{rel}}(A \otimes_\pi B)$$

given by the formula

$$f *^{\text{rel}} g = \hat{\gamma}^{\text{rel}} \circ (f \wedge g) : S^n \wedge S^m \cong S^{n+m} \rightarrow |R(A \otimes_\pi B)|^+$$

on representatives $f : S^n \rightarrow |R(A)|^+$ and $g : S^m \rightarrow |R(B)|^+$.

The algebraic properties of the map $\widehat{\gamma}^{\text{rel}}$ entail corresponding properties for the exterior relative Loday product.

Theorem 5.27. *The exterior relative Loday product is natural in A and B , associative and bilinear. It is also graded commutative in the following sense: For each $x \in K_n^{\text{rel}}(A)$ and $y \in K_m^{\text{rel}}(B)$ we have*

$$(-1)^{nm}(x *^{\text{rel}} y) = T_*(y *^{\text{rel}} x)$$

Here $T : A \otimes_{\pi} B \rightarrow B \otimes_{\pi} A$ denotes the continuous unital algebra homomorphism which interchanges the factors.

Assume that the unital Banach algebra A is commutative. The product then define a continuous unital algebra homomorphism $m : A \otimes_{\pi} A \rightarrow A$. This allows us to make the following definition:

Definition 5.28. *By the interior relative Loday product we will understand the composition*

$$m_* \circ *^{\text{rel}} : K_n^{\text{rel}}(A) \times K_m^{\text{rel}}(A) \rightarrow K_{n+m}^{\text{rel}}(A)$$

of the exterior relative Loday product and the homomorphism induced by the product. The interior relative Loday product is also denoted by $^{\text{rel}}$.*

The next result is then a consequence of Theorem 5.27.

Theorem 5.29. *Let A be a commutative unital Banach algebra. The relative K -theory becomes a graded commutative ring when equipped with the interior relative Loday product and the usual addition.*

Let us investigate the relation between the two Loday products. In order to do this properly we need a modification of the usual Loday product on algebraic K -theory. This variant takes the topology of the unital Banach algebras into account.

Definition 5.30. *By the completed exterior Loday product we will understand the composition*

$$\hat{*} : K_n(A) \times K_m(B) \rightarrow K_{n+m}(A \otimes_{\pi} B) \quad \hat{*} = \iota_* \circ *$$

of the exterior Loday product and the homomorphism induced by the "identity" homomorphism $\iota : A \otimes_{\mathbb{Z}} B \rightarrow A \otimes_{\pi} B$.

As expected, the map $\theta : K_n^{\text{rel}}(A) \rightarrow K_n(A)$ respects the exterior product structures. This statement is made precise in the following,

Theorem 5.31. *We have the identity*

$$\theta(x *^{\text{rel}} y) = \theta(x) \hat{*} \theta(y)$$

for each $x \in K_n^{\text{rel}}(A)$ and $y \in K_m^{\text{rel}}(B)$. In fact, the diagram

$$\begin{array}{ccc} |R(A)|^+ \wedge |R(B)|^+ & \xrightarrow{\theta \wedge \theta} & |BGL(A)|^+ \wedge |BGL(B)|^+ \\ \widehat{\gamma}^{\text{rel}} \downarrow & & |\iota|^+ \circ \widehat{\gamma} \downarrow \\ |R(A \otimes_{\pi} B)|^+ & \xrightarrow{\theta} & |BGL(A \otimes_{\pi} B)|^+ \end{array}$$

is commutative up to weak homotopy.

We refer to [17, Theorem 3.17] for a proof of the above result.

Returning to the commutative setup we can combine the results of Theorem 5.27 and Theorem 5.31.

Theorem 5.32. *Let A be a commutative, unital Banach algebra. The map $\theta : K_*^{\text{rel}}(A) \rightarrow K_*(A)$ is a homomorphism of graded commutative rings.*

5.4. The relative Loday product and the relative Steinberg symbol. We are now going to see that the relative Steinberg symbol can be identified as a special instance of the relative Loday product. We will start by explaining the homological description of the second relative K -group which we used in Section 3.3. The identification of the relative Steinberg symbol as a relative Loday product should be compared with the identification of the Steinberg symbol as a Loday product, see [25, Proposition 2.2.3]. Furthermore, the homological description of the second relative K -group should be compared with the homological description of the second algebraic K -group, see for example [25, Proposition 1.2.1].

Let A be a unital Banach algebra. Recall from Section 3.3 that $F(A)_n$ is defined recursively by

- (1) $F(A)_1 := [R(A), R(A)]$
- (2) $\sigma \in F(A)_{n+1} \Leftrightarrow (\sigma \in R(A)_{n+1} \text{ and } d_i \sigma \in F(A)_n \text{ } i \in \{0, \dots, n+1\})$

These sets form a Kan complex $F(A)$ when equipped with the face and degeneracy operators given by restriction of the face and degeneracy operators on $R(A)$. Let $|F(A)|$ denote the geometric realization of this Kan complex. The fundamental group of $|F(A)|$ is then given by the image of $F(A)_1$ in $R(A)_1 / \sim$ under the quotient map. This group was denoted by $F(A)_1 / \sim$ in Section 2.2. It is a perfect group by Lemma 5.13 and we can thus apply the plus construction to $|F(A)|$ with respect to its fundamental group. We obtain a simply connected CW -complex $|F(A)|^+$. Furthermore, the inclusion $i : F(A) \rightarrow R(A)$ induces a continuous map $|i|^+ : |F(A)|^+ \rightarrow |R(A)|^+$. The following result can then be found as [17, Corollary 3.7].

Theorem 5.33. *The map $|i|^+ : |F(A)|^+ \rightarrow |R(A)|^+$ induces an isomorphism on the homotopy groups of dimension greater than 2. To be precise,*

$$i_*^+ : \pi_n(|F(A)|^+) \rightarrow \pi_n(|R(A)|^+) = K_n^{\text{rel}}(A)$$

is an isomorphism for all $n \geq 2$.

The next theorem is then a consequence of the Hurewicz isomorphism theorem, see for example [13, Theorem 4.3.2].

Theorem 5.34. *The Hurewicz homomorphism $h_2 : \pi_2(|F(A)|^+) \rightarrow H_2(|F(A)|^+) \cong H_2(F(A))$ is an isomorphism. In particular, the second relative K -group is isomorphic to the second homology group of the Kan complex $F(A)$.*

This shows that the different definitions of relative K -theory given in Definition 3.12 and Definition 5.15 agree up to isomorphism.

Assume that A is a commutative and unital Banach algebra. Let us fix two continuous maps $\sigma, \tau : [0, 1] \rightarrow A^*$ with values in the group of invertibles and with $\sigma(t) = 1 = \tau(t)$ for all $t \in [0, \epsilon)$.

Each of the two elements will then define a class in the first relative K -group, $[\sigma], [\tau] \in K_1^{\text{rel}}(A)$ and we can thus form their interior relative Loday product,

$$[\sigma] *^{\text{rel}} [\tau] \in K_2^{\text{rel}}(A)$$

However, we can also form the relative Steinberg symbol of the continuous maps,

$$\{\sigma, \tau\}^{\text{rel}} \in H_2(F(A))$$

Our aim is to show that these two elements coincide up to a sign under the isomorphism $K_2^{\text{rel}}(A) \cong H_2(F(A))$ which we have just described.

Let X be a pointed space. Let us start by recalling the definition of the Hurewicz homomorphism, $h_2 : \pi_2(X) \rightarrow H_2(X)$. We know that the singular homology of the sphere is given by the integers $H_2(S^2) \cong \mathbb{Z}$ and we can thus fix a generator $[1_{\mathbb{Z}}] \in H_2(S^2)$. Now, since singular homology is homotopy invariant we can define the Hurewicz homomorphism by the formula

$$h_2[f] = f_*[1_{\mathbb{Z}}] \quad [f] \in \pi_2(X)$$

We can choose an explicit cycle for the generator $[1_{\mathbb{Z}}] \in H_2(S^2)$, namely, we let

$$1_{\mathbb{Z}} = \eta_1 \wedge \eta_0 - \eta_0 \wedge \eta_1 \in \mathbb{Z}[S_2(S^2)]$$

Here $\eta_0(t_1, t_2) = e^{2\pi i t_2} \in S^1$ and $\eta_1(t_1, t_2) = e^{2\pi i(t_1+t_2)} \in S^1$.

Next, note that the elements $[\sigma], [\tau] \in K_1^{\text{rel}}(A) \cong \pi_1(|R(A)|^+)$ can be described as the following loops

$$\gamma_{\sigma}(e^{2\pi i t}) = (t, \sigma) \in |R(A)| \subseteq |R(A)|^+ \quad \text{and} \quad \gamma_{\tau}(e^{2\pi i t}) = (t, \tau) \in |R(A)| \subseteq |R(A)|^+$$

Here $t \in [0, 1]$.

Now, the geometric realization of the Kan complex $B\mathbb{Z}$ is homotopy equivalent to the circle. The homotopy equivalence is given by the continuous map

$$\gamma_1 : S^1 \rightarrow |B\mathbb{Z}| \quad \alpha(e^{2\pi i t}) = (t, 1)$$

Let us use this identification to describe the maps

$$\gamma_{\sigma} \text{ and } \gamma_{\tau} : S^1 \cong |B\mathbb{Z}| \rightarrow |R(A)| \subseteq |R(A)|^+$$

by means of simplicial maps. Namely, we let

$$\alpha_{\sigma} \text{ and } \alpha_{\tau} : B\mathbb{Z} \rightarrow R(A)$$

be the simplicial maps given by

$$\begin{aligned} \alpha_{\sigma}(i_1, \dots, i_n) &= \prod_{j=1}^n (s_{n-1} \dots \widehat{s}_{j-1} \dots s_0)(\sigma^{i_j}) \\ \alpha_{\tau}(i_1, \dots, i_n) &= \prod_{j=1}^n (s_{n-1} \dots \widehat{s}_{j-1} \dots s_0)(\tau^{i_j}) \end{aligned}$$

Here the $\widehat{}$ signifies that the term is omitted. We then have the identities

$$\gamma_{\sigma} = |\alpha_{\sigma}| \circ \gamma_1 \quad \text{and} \quad \gamma_{\tau} = |\alpha_{\tau}| \circ \gamma_1 : S^1 \rightarrow |R(A)|^+$$

Next, the torus $S^1 \times S^1$ is homotopy equivalent to the classifying space of $\mathbb{Z} \times \mathbb{Z}$ through the homotopy equivalences

$$\gamma_1 \times \gamma_1 : S^1 \times S^1 \rightarrow |B\mathbb{Z}| \times |B\mathbb{Z}| \cong |B(\mathbb{Z} \times \mathbb{Z})|$$

Here the last homotopy equivalence is given by the inverse of the projections onto the factors. Up to homotopy the continuous map

$$\gamma_\sigma \times \gamma_\tau : S^1 \times S^1 \rightarrow |R(A)|^+ \times |R(A)|^+ \cong |R(A) \times R(A)|^+$$

is thus induced by the map of simplicial sets

$$\alpha_\sigma \times \alpha_\tau : B\mathbb{Z} \times B\mathbb{Z} \rightarrow R(A) \times R(A)$$

Finally, let $x_{12}(\sigma), x_{13}(\tau) \in F(A)_1$ denote the diagonal matrices

$$x_{12}(\sigma) = d(\sigma, \sigma^{-1}, 1) \quad \text{and} \quad x_{13}(\tau) = d(\tau, 1, \tau^{-1})$$

We then get a well defined map

$$|d_{\sigma, \tau}| : S^1 \times S^1 \cong |B\mathbb{Z} \times B\mathbb{Z}| \rightarrow |F(A)| \subseteq |F(A)|^+$$

induced by the simplicial map

$$d_{\sigma, \tau} : B\mathbb{Z} \times B\mathbb{Z} \rightarrow F(A) \quad (\mathbf{i}, \mathbf{j}) \mapsto \alpha_{x_{12}(\sigma)}(\mathbf{i}) \cdot \alpha_{x_{13}(\tau)}(\mathbf{j})$$

At this point the commutativity assumption on the unital Banach algebra A is important.

We let $m : A \otimes_\pi A \rightarrow A$ denote the continuous unital algebra homomorphism given by multiplication.

Lemma 5.35. *For any $p \geq 3$ the applications*

$$|m|^+ \circ \gamma_{p,p} \circ (\gamma_\sigma \times \gamma_\tau) \quad \text{and} \quad i^+ \circ |d_{\sigma, \tau}| : S^1 \times S^1 \rightarrow |R(A)|^+$$

are homotopic. Here we are thinking of $\gamma_\sigma \times \gamma_\tau$ as taking values in the product $|R_p(A)|^+ \times |R_p(A)|^+$. Recall that $\gamma_{p,p}$ denotes the modified tensor product introduced in Section 5.3.

Proof. The proof is analogous to [25, Proposition 2.2.3] and will not be repeated here. \square

We are now in position to identify the relative interior Loday product with the relative Steinberg symbol.

Theorem 5.36. *The interior relative Loday product*

$$[\sigma] *^{\text{rel}} [\tau] \in K_2^{\text{rel}}(A)$$

coincides with the relative Steinberg symbol

$$\{\tau, \sigma\}^{\text{rel}} \in H_2(F(A))$$

under the identification $H_2(F(A)) \cong K_2^{\text{rel}}(A)$. Note that the order of the factors is reversed.

Proof. From Lemma 5.35 and the definition of the interior relative Loday product we get that the diagram

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{p} & S^1 \wedge S^1 \\ i^+ \circ |d_{\sigma \cdot \tau}| \downarrow & & \downarrow [\sigma] *^{\text{rel}} [\tau] \\ |R(A)|^+ & \xlongequal{\quad} & |R(A)|^+ \end{array}$$

commutes up to homotopy. In particular, letting $(i^+)^{-1} : |R(A)|^+ \rightarrow |F(A)|^+$ denote some inverse of the homotopy equivalence $i^+ : |F(A)|^+ \rightarrow |R(A)|^+$ we get that the diagram

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{p} & S^1 \wedge S^1 \\ |d_{\sigma \cdot \tau}| \downarrow & & \downarrow (i^+)^{-1} \circ ([\sigma] *^{\text{rel}} [\tau]) \\ |F(A)|^+ & \xlongequal{\quad} & |F(A)|^+ \end{array}$$

commutes up to homotopy.

By homotopy invariance of singular homology it follows that the Hurewicz homomorphism of the element $(i^+)^{-1} \circ ([\sigma] *^{\text{rel}} [\tau])$ coincides with the class

$$|d_{\sigma \cdot \tau}| \circ (\eta_1, \eta_0) - |d_{\sigma \cdot \tau}| \circ (\eta_0, \eta_1) \in H_2(|F(A)|^+)$$

It is not hard to see that this element coincides with the desired relative Steinberg symbol under the identification $H_2(F(A)) \cong H_2(|F(A)|^+)$. Indeed, the class

$$(\eta_1, \eta_0) - (\eta_0, \eta_1) \in H_2(S^1 \times S^1)$$

coincides with the class

$$((1, 0), (0, 1)) - ((0, 1), (1, 0)) \in H_2(B\mathbb{Z} \times B\mathbb{Z})$$

under the identification $H_2(S^1 \times S^1) \cong H_2(|B\mathbb{Z} \times B\mathbb{Z}|) \cong H_2(B\mathbb{Z} \times B\mathbb{Z})$. But this last class is mapped to the class

$$s_1(x_{12}(\sigma)) \cdot s_0(x_{13}(\tau)) - s_0(x_{12}(\sigma)) \cdot s_1(x_{13}(\tau)) \in H_2(F(A))$$

under the simplicial map which represents the continuous map

$$|d_{\sigma \cdot \tau}| : S^1 \times S^1 \rightarrow |F(A)|^+$$

This proves the desired result. □

6. THE HIGHER DETERMINANT OF A FINITELY SUMMABLE FREDHOLM MODULE

The explorations of the multiplicative structures in K -theory which we carried out in the last section shed new light on the calculation of the determinant invariant by means of the trace invariant. Let us explain ourselves. Our calculation of the determinant invariant followed a certain scheme: We found a lift in relative K -theory of the Steinberg symbol and applied the trace invariant to this lift. By the comparison result for the determinant invariant and the trace invariant, Theorem 3.22, we were then able to obtain the identity

$$(23) \quad \det(e^A e^B e^{-A} e^{-B}) = e^{\text{Tr}[A, B]}$$

Here $A, B \in \mathcal{L}(H)$ was an almost commuting pair of operators. In the last section, we identified our lift, that is the relative Steinberg symbol, as a special instance of a general relative Loday product on relative K -theory, see Theorem 5.36. The Steinberg symbol in algebraic K -theory also identifies with the algebraic Loday product, see [25, Proposition 2.2.3]. There is therefore nothing mysterious in our choice of lift. What we did was simply to lift each of the invertibles $[e^x], [e^y] \in K_1(C^2[x, y])$ to the contractions $[\gamma_x], [\gamma_y] \in K_1^{\text{rel}}(C^2[x, y])$. The relative Loday product

$$[\gamma_y] *^{\text{rel}} [\gamma_x] = \{\gamma_x, \gamma_y\}^{\text{rel}} \in K_2^{\text{rel}}(C^2[x, y])$$

should then be a lift of the algebraic Loday product

$$[e^y] * [e^x] = \{e^x, e^y\} \in K_2(C^2[x, y])$$

This is indeed true since the map

$$\theta : K_*^{\text{rel}}(C^2[x, y]) \rightarrow K_*(C^2[x, y])$$

is a homomorphism of graded rings, see Theorem 5.32. In this section, we are going to apply this idea to obtain a higher dimensional analogue of the identity (23). We will of course need a higher dimensional version of the determinant invariant. However, the multiplicative character of A. Connes and M. Karoubi which we examined in Section 4 extends to the higher K -groups, see [9]. This seems to be an appropriate generalization since the two invariants coincide on the second algebraic K -group. We have explained this result in details in Section 4. See also the paper [16] at the end of the thesis. Let us start by introducing the multiplicative character on the higher K -groups.

6.1. The multiplicative character of a finitely summable Fredholm module. In general, the multiplicative character is a homomorphism

$$\mathcal{M}_F : K_n(A) \rightarrow \mathbb{C}/(2\pi i)^{\lfloor \frac{n}{2} \rfloor} \mathbb{Z}$$

associated with an n -summable Fredholm module over a \mathbb{C} -algebra A . It was originally defined by A. Connes and M. Karoubi in [9]. We will briefly review their construction in the case where $n = 2p$ is even. Notice that there are other ways of introducing this invariant. For example, one could, I believe, use the multiplicative K -groups of M. Karoubi together with the homomorphisms in multiplicative K -theory of D. Perrot induced by finitely summable Fredholm modules. In this respect, see [22] and [30].

Let \mathcal{H} be a fixed separable infinite dimensional Hilbert space. We let \mathcal{M}^{2p-1} denote the \mathbb{C} -algebra consisting of operators on the direct sum $\mathcal{H} \oplus \mathcal{H}$ with antidiagonal in the $2p^{\text{th}}$ Schatten ideal. Thus, the elements $x = (x_{ij}) \in \mathcal{M}^1$ are precisely the bounded operators on $\mathcal{H} \oplus \mathcal{H}$ which in their matrix representation have $x_{12}, x_{21} \in \mathcal{L}^{2p}(\mathcal{H})$. The \mathbb{C} -algebra \mathcal{M}^{2p-1} can be equipped with the Banach algebra norm $\|\cdot\| : \mathcal{M}^{2p-1} \rightarrow [0, \infty)$ given by

$$\|x\| = \|x\|_{\infty} + \|x_{12}\|_{2p} + \|x_{21}\|_{2p}$$

This turns \mathcal{M}^{2p-1} into a Banach algebra. We can interpret \mathcal{M}^{2p-1} as a universal odd $2p$ -summable Fredholm algebra. Let us explain ourselves.

Definition 6.1. *Let A be a unital \mathbb{C} -algebra. By an odd $2p$ -summable Fredholm module (F, H) over A we will understand a separable Hilbert space H , a unital algebra homomorphism $\pi : A \rightarrow \mathcal{L}(H)$ and a bounded operator $F \in \mathcal{L}(H)$ such that*

- (1) $F = F^*$
- (2) $F^2 = 1$
- (3) $[F, \pi(a)] \in \mathcal{L}^{2p}(H)$ for all $a \in A$.

When A is a unital Banach algebra we will say that the Fredholm module (F, H) is continuous when the maps

$$\pi : A \rightarrow \mathcal{L}(H) \quad \text{and} \quad a \mapsto [F, \pi(a)] \in \mathcal{L}^{2p}(H)$$

are both continuous.

We will always assume that the Hilbert spaces PH and $(1-P)H$ are infinite dimensional. Here $P = (F+1)/2$ denotes the projection associated with the selfadjoint unitary operator $F \in \mathcal{L}(H)$.

We let $F_U \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ denote the selfadjoint unitary given by the matrix

$$F_U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Furthermore, we have a representation $\pi : \mathcal{M}^{2p-1} \rightarrow \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ given by inclusion. The pair $(F_U, \mathcal{H} \oplus \mathcal{H})$ then becomes a continuous odd $2p$ -summable Fredholm module over \mathcal{M}^{2p-1} . It is universal in the following sense.

Let (F, H) denote any odd $2p$ -summable Fredholm module over the \mathbb{C} -algebra A . Since the Hilbert space PH and $(1-P)H$ are infinite dimensional we can identify both of them with the "universal" Hilbert space \mathcal{H} . In this way we obtain an algebra homomorphism

$$\pi_F : A \rightarrow \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$$

which is well defined up to conjugation by unitary operators. The condition on the commutator $[F, \pi(a)] \in \mathcal{L}^{2p}(H)$ ensures that the representation $\pi_F : A \rightarrow \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ factorizes through the Banach algebra \mathcal{M}^{2p-1} . Furthermore, whenever the Fredholm module (F, H) is continuous the associated algebra homomorphism

$$\pi_F : A \rightarrow \mathcal{M}^{2p-1}$$

is continuous as well.

By a theorem of A. Connes each continuous odd $2p$ -summable Fredholm module gives rise to a homomorphism on continuous cyclic homology.

Theorem 6.2. [7, I 7. Lemma 2] *Let (F, H) denote a continuous odd $2p$ -summable Fredholm module over a unital Banach algebra A . The multilinear map given by*

$$\tau_F : (a^0, a^1, \dots, a^{2p-1}) \mapsto -\frac{1}{2^{2p} \cdot (p-1)!} \text{Tr}(F[F, a^0][F, a^1] \cdot \dots \cdot [F, a^{2p-1}])$$

then determines a character $\tau_F : HC_{2p-1}(A) \rightarrow \mathbb{C}$ on continuous cyclic homology.

The rational constant $-\frac{1}{2^{2p} \cdot (p-1)!}$ is chosen in such a way that the multiplicative character which we are going to define agrees with the one defined in [9]. The extra factor $\frac{1}{(2p-1)!}$ is due to the different definitions of the relative Chern character, compare [35, Section 3] and Definition 6.3.

We let $\tau_{2p-1} : HC_{2p-1}(\mathcal{M}^{2p-1}) \rightarrow \mathbb{C}$ denote the character associated with the universal odd continuous $2p$ -summable Fredholm module $(F_U, \mathcal{H} \oplus \mathcal{H})$ over \mathcal{M}^{2p-1} . For any odd continuous

$2p$ -summable Fredholm module (F, H) over a unital Banach algebra A we then get the following commutative diagram

$$(24) \quad \begin{array}{ccc} HC_{2p-1}(A) & \xrightarrow{(\pi_F)_*} & HC_{2p-1}(\mathcal{M}^{2p-1}) \\ \tau_F \downarrow & & \tau_{2p-1} \downarrow \\ \mathbb{C} & \xlongequal{\quad} & \mathbb{C} \end{array}$$

The above characters are important to us because of their connection with K -theory. The link between K -theory and cyclic homology is established by Chern characters. We will concentrate on the relative Chern character

$$\text{ch}^{\text{rel}} : K_n^{\text{rel}}(A) \rightarrow HC_{n-1}(A)$$

This natural homomorphism was originally introduced by M. Karoubi in a differential geometric context, see [21]. We will use the simplicial version of the relative Chern character which can be found in [9]. We should mention that there exists at least one more equivalent definition of this homomorphism due to M. Karoubi, see [22, Section 4.2]. In this respect one should also mention the article [40] by C. Weibel.

Let us fix a unital Banach algebra A . Recall that the relative K -theory is the homotopy groups of the pointed, connected CW -complex $|R(A)|^+$. We thus have a Hurewicz homomorphism

$$h_n : K_n^{\text{rel}}(A) = \pi_n(|R(A)|^+) \rightarrow H_n(|R(A)|^+) \cong H_n(R(A))$$

with values in the homology of the simplicial set $R(A)$.

The connection between the homology of the simplicial set $R(A)$ and cyclic homology is given by a chain map of degree minus one,

$$\begin{aligned} L : R_m(A)_n &\rightarrow C_{n-1}(M_m(A)) \\ L(\sigma) &= \frac{1}{n} \sum_{s \in \Sigma_n} \text{sgn}(s) \int_{\Delta^n} \Gamma_{s(1)}(\sigma) \otimes \dots \otimes \Gamma_{s(n)}(\sigma) dt_1 \dots dt_n \end{aligned}$$

See [35]. Here we have use the notation

$$\Gamma_i(\sigma) = \frac{\partial \sigma}{\partial t_i} \cdot \sigma^{-1} : \Delta^n \rightarrow M_m(A)$$

for any smooth map $\sigma : \Delta^n \rightarrow GL_m(A)$. Note that the above chain map only makes sense for smooth elements in $R_m(A)_n$. This is not a problem, since at the level of homology we can choose to work only with smooth maps. See [17, Lemma 3.2]. By the *logarithm* we will refer to the induced homomorphism

$$L : H_n(R(A)) \rightarrow \lim_{m \rightarrow \infty} HC_{n-1}(M_m(A))$$

The direct limit is taken over the continuous algebra homomorphisms $T \mapsto T \oplus 0$.

The generalized trace brings us from the continuous cyclic homology of the matrices to the continuous cyclic homology of the unital Banach algebra,

$$\text{TR} : \lim_{m \rightarrow \infty} HC_{n-1}(M_m(A)) \rightarrow HC_{n-1}(A)$$

This homomorphism is given by the formula

$$\mathrm{TR} : u_0 a_0 \otimes \dots \otimes u_{n-1} a_{n-1} \mapsto \mathrm{Tr}(u_0 \cdot \dots \cdot u_{n-1}) a_0 \otimes \dots \otimes a_{n-1}$$

for any $a_0, \dots, a_{n-1} \in A$ and $u_0, \dots, u_{n-1} \in M_n(\mathbb{C})$. See for example, [26, Section 1.2].

We can thus make the following definition.

Definition 6.3. *Let A be a unital Banach algebra. By the relative Chern character we will understand the homomorphism*

$$\mathrm{ch}^{\mathrm{rel}} : K_n^{\mathrm{rel}}(A) \rightarrow HC_{n-1}(A) \quad \mathrm{ch}^{\mathrm{rel}} = \mathrm{TR} \circ L \circ h_n$$

given by the composition of the Hurewicz homomorphism, the logarithm and the generalized trace.

Disposing over a continuous odd $2p$ -summable Fredholm module (F, H) over the unital Banach algebra A we can form an invariant of relative K -theory.

Definition 6.4. *By the additive character of the continuous odd $2p$ -summable Fredholm module (F, H) over the unital Banach algebra A we will understand the homomorphism*

$$\mathcal{A}_F : K_{2p}^{\mathrm{rel}}(A) \rightarrow \mathbb{C} \quad \mathcal{A}_F = \tau_F \circ \mathrm{ch}^{\mathrm{rel}}$$

By naturality of the relative Chern character and the commutativity of the diagram (24) we get the commutative diagram

$$\begin{array}{ccc} K_{2p}^{\mathrm{rel}}(A) & \xrightarrow{(\pi_F)_*} & K_{2p}^{\mathrm{rel}}(\mathcal{M}^{2p-1}) \\ \mathcal{A}_F \downarrow & & \mathcal{A}_{2p-1} \downarrow \\ \mathbb{C} & \xlongequal{\quad} & \mathbb{C} \end{array}$$

Here $\mathcal{A}_{2p-1} : K_{2p}^{\mathrm{rel}}(\mathcal{M}^{2p-1}) \rightarrow \mathbb{C}$ denotes the additive character associated with the universal Fredholm module $(F_U, \mathcal{L}(\mathcal{H} \oplus \mathcal{H}))$ over \mathcal{M}^{2p-1} .

In [9, Théorème 2.8] the topological K -groups of the unital Banach algebra \mathcal{M}^{2p-1} are computed.

Lemma 6.5. *For any $p \in \mathbb{N}$ the topological K -theory of \mathcal{M}^{2p-1} is given by*

$$K_0^{\mathrm{top}}(\mathcal{M}^{2p-1}) = \{0\} \quad K_1^{\mathrm{top}}(\mathcal{M}^{2p-1}) = \mathbb{Z}$$

In the case of the unital Banach algebra \mathcal{M}^{2p-1} the long exact sequence of K -groups from Theorem 5.16 then reads

$$\dots \xrightarrow{\theta} K_{2p+1}(\mathcal{M}^{2p-1}) \longrightarrow \mathbb{Z} \xrightarrow{v} K_{2p}^{\mathrm{rel}}(\mathcal{M}^{2p-1}) \xrightarrow{\theta} K_{2p}(\mathcal{M}^{2p-1}) \longrightarrow 0$$

In particular, the homomorphism $\theta : K_{2p}^{\mathrm{rel}}(\mathcal{M}^{2p-1}) \rightarrow K_{2p}(\mathcal{M}^{2p-1})$ is surjective. Furthermore, since both the boundary map and the universal additive character are homomorphisms we get that

$$\mathrm{Im}(\mathcal{A}_{2p-1} \circ v) = c\mathbb{Z} \subseteq \mathbb{C}$$

where $c \in \mathbb{C}$ is some constant. In [9] this constant is calculated to be $c = (2\pi i)^p$. These results allows us to define the multiplicative character.

Definition 6.6. *By the universal odd $2p$ -summable multiplicative character we will understand the homomorphism*

$$\mathcal{M}_{2p-1} : K_{2p}(\mathcal{M}^{2p-1}) \rightarrow \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

given by

$$\mathcal{M}_{2p-1}(\theta(y)) = [\mathcal{A}_{2p-1}(y)]$$

Here $[\cdot] : \mathbb{C} \rightarrow \mathbb{C}/(2\pi i)^p \mathbb{Z}$ denotes the quotient map.

In particular, let (F, H) be an odd $2p$ -summable Fredholm module over a \mathbb{C} -algebra A . By the multiplicative character of the Fredholm module we will understand the homomorphism

$$\mathcal{M}_F : K_{2p}(A) \rightarrow \mathbb{C}/(2\pi i)^p \mathbb{Z} \quad \mathcal{M}_F = \mathcal{M}_{2p-1} \circ (\pi_F)_*$$

Here $(\pi_F)_ : K_{2p}(A) \rightarrow K_{2p}(\mathcal{M}^{2p-1})$ denotes the homomorphism on algebraic K -theory induced by the representation $\pi_F : A \rightarrow \mathcal{M}^{2p-1}$.*

In the following we will address the question of calculating this homomorphism on the higher K -groups. This brings us to the heart of the second article attached at the end of the thesis.

6.2. The multiplicative properties of the relative Chern character. Let A and B be unital Banach algebras. We saw in Section 5.3 that the relative K -theory could be equipped with an exterior Loday product

$$*_n^{\text{rel}} : K_n^{\text{rel}}(A) \times K_m^{\text{rel}}(B) \rightarrow K_{n+m}^{\text{rel}}(A \otimes_{\pi} B)$$

Furthermore, we saw in the last section that the relative K -theory was related to continuous cyclic homology by means of a relative Chern character

$$\text{ch}_n^{\text{rel}} : K_n^{\text{rel}}(A) \rightarrow HC_{n-1}(A)$$

It is natural to ask for an exterior product on continuous cyclic homology which corresponds to the exterior relative Loday product. Since the relative Chern character has degree minus one such a product should have degree plus one. Such a product does indeed exist and it is the correct analogue of the relative Loday product. The main concern of this section is to expose this multiplicative result. We start by reviewing some results concerning the product of degree one in cyclic homology. We refer to [26] for further details on these matters.

Let A be a unital Banach algebra. Recall that $C_*(A)$ denotes the completed Hochschild chains. We let

$$\times' : C_n(A) \otimes_{\mathbb{C}} C_m(A) \rightarrow C_{n+m}(A)$$

denote the map given by

$$(a_0 \otimes a_1 \otimes \dots \otimes a_n) \times' (b_0 \otimes a_{n+1} \otimes \dots \otimes a_{n+m}) = \sum_{s \in \Sigma_{n,m}} \text{sgn}(s) a_0 b_0 \otimes a_{s(1)} \otimes \dots \otimes a_{s(n+m)}$$

Here $\Sigma_{(n,m)}$ denotes the set of (n, m) -shuffles. To be precise, $s \in \Sigma_{n+m}$ is an (n, m) -shuffle precisely when the associated map $s : \{1, \dots, n+m\} \rightarrow \{1, \dots, n+m\}$ is strictly increasing on each of the subsets $\{1, \dots, n\}$ and $\{n+1, \dots, n+m\}$.

Now, let B denote another unital Banach algebra. We then have the chain maps

$$1_A \otimes \cdot : C_*(B) \rightarrow C_*(A \otimes_{\pi} B) \quad \text{and} \quad \cdot \otimes 1_B : C_*(A) \rightarrow C_*(A \otimes_{\pi} B)$$

given by the formulas

$$\begin{aligned} 1_A \otimes (b_0 \otimes \dots \otimes b_n) &= (1_A \otimes b_0) \otimes \dots \otimes (1_A \otimes b_n) \quad \text{and} \\ (a_0 \otimes \dots \otimes a_n) \otimes 1_B &= (a_0 \otimes 1_B) \otimes \dots \otimes (a_n \otimes 1_B) \end{aligned}$$

Definition 6.7. *Let A and B be unital Banach algebras. By the completed exterior shuffle product we will understand the application*

$$\times : C_n(A) \otimes_{\mathbb{C}} C_m(B) \rightarrow C_{n+m}(A \otimes_{\pi} B) \quad \times = \times' \circ ((\cdot \otimes 1_B) \otimes (1_A \otimes \cdot))$$

The Hochschild boundary is a graded derivation with respect to the exterior shuffle product.

Lemma 6.8. [26, Proposition 4.2.2] *We have the identity*

$$b(x \times y) = b(x) \times y + (-1)^n x \times b(y)$$

for any $x \in C_n(A)$ and $y \in C_m(B)$.

This exterior shuffle product does not behave nicely with respect to the cyclic operator. Therefore, it will not descend to the level of cyclic homology. We can however modify the exterior shuffle product in the following way. We will make use of the following operators

$$N : C_*(A) \rightarrow C_*(A) \quad s : C_*(A) \rightarrow C_{*+1}(A)$$

called the *norm operator* and the *extra degeneracy* respectively. In degree $n \in \mathbb{N} \cup \{0\}$ they are given by the formulas

$$N = 1 + t + \dots + t^n \quad s(a_0 \otimes \dots \otimes a_n) = 1 \otimes a_0 \otimes \dots \otimes a_n$$

Here $t : C_*(A) \rightarrow C_*(A)$ denotes the cyclic operator. See Section 3.3.2.

Definition 6.9. *By the completed exterior product of degree one on cyclic homology we will understand the bilinear map*

$$* : C_n^{\lambda}(A) \otimes_{\mathbb{C}} C_m^{\lambda}(B) \rightarrow C_{n+m+1}^{\lambda}(A \otimes_{\pi} B)$$

given by

$$x * y = x \times (sNy)$$

for each $x \in C_n^{\lambda}(A)$ and $y \in C_m^{\lambda}(B)$.

The exterior product of degree one is a well defined map by [26, Theorem 4.4.2], see also [17, Section 2.3]. Let us investigate its behaviour with respect to the Hochschild boundary.

Lemma 6.10. *For each $x \in C_n^{\lambda}(A)$ and each $y \in C_m^{\lambda}(B)$ we have the identity*

$$b(x * y) = b(x) * y + (-1)^{n+1} x * b(y)$$

The Hochschild boundary is thus a (shifted) graded derivation with respect to the product of degree one.

Proof. We start by noting the identities

$$bs = 1 - t - sb' \quad \text{and} \quad b'N = Nb$$

Here $b' : C_n(A) \rightarrow C_{n-1}(A)$ denotes the differential in the continuous bar complex, see Section 3.3.2. It now follows from Lemma 6.8 that

$$b(x * y) = b(x \times (sNy)) = b(x) * y + (-1)^n x \times b(sNy) = b(x) * y + (-1)^{n+1} x * b(y)$$

This proves the Lemma. \square

In particular, we get an induced map on cyclic homology

$$* : HC_n(A) \otimes_{\mathbb{C}} HC_m(B) \rightarrow HC_{n+m+1}(A \otimes_{\pi} B)$$

This is the multiplicative structure which we are interested in. Before we proceed, we should note that the exterior product of degree one is natural in A and B , bilinear, associative and graded commutative *at the level of complexes*. See [26, Section 4.4.5] and [17, Section 2.3]. We can now state the main result of this section.

Theorem 6.11. *The relative Chern character respects the exterior products. That is, we have the identity*

$$\text{ch}^{\text{rel}}(x *^{\text{rel}} y) = \text{ch}^{\text{rel}}(x) * \text{ch}^{\text{rel}}(y)$$

for each $x \in K_n^{\text{rel}}(A)$ and each $y \in K_m^{\text{rel}}(B)$.

Proof. This result is carefully proved in [17, Section 4]. \square

Suppose that we have a *commutative* unital Banach algebra, A . In this case the product yields a continuous unital algebra homomorphism $m : A \otimes_{\pi} A \rightarrow A$. We can thus form the *interior product of degree one*

$$m_* \circ * : HC_n(A) \otimes_{\mathbb{C}} HC_m(A) \rightarrow HC_{n+m+1}(A)$$

The interior product of degree one is also denoted by $*$.

Corollary 6.12. *Let A be a commutative unital Banach algebra. The relative Chern character is a homomorphism of graded commutative rings,*

$$\text{ch}^{\text{rel}} : K_*^{\text{rel}}(A) \rightarrow HC_*(A)[1]$$

Proof. This follows by naturality of the relative Chern character and Theorem 6.11. \square

6.3. A calculation of the multiplicative character. We are interested in the following scenario. Assume that (F, H) is a continuous odd $2p$ -summable Fredholm module (F, H) over a *commutative* unital Banach algebra A . Let us take $2p$ elements in the path component of the identity, $g_0, g_1, \dots, g_{2p-1} \in GL_0(A)$. We will then give an explicit formula for the application of the multiplicative character

$$\mathcal{M}_F : K_{2p}(A) \rightarrow \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

to the interior Loday product of the invertibles, $[x] = [g_0] * \dots * [g_{2p-1}] \in K_{2p}(A)$. This is possible for the following reasons:

- (1) We can find an explicit element $[y] \in K_{2p}^{\text{rel}}(A)$ such that $\theta[y] = [x] \in K_{2p}(A)$. Namely, by Theorem 5.32 we can choose the lift as an interior relative Loday product of lifts of the invertibles $g_0, \dots, g_{2p-1} \in GL_0(A)$. Each of the individual lifts is given by a path $\gamma_i : [0, 1] \rightarrow GL(A)$ with $\gamma_i(0) = 1$ and $\gamma_i(1) = g_i^{-1}$. Thus, we let $[y] = [\gamma_0] *^{\text{rel}} \dots *^{\text{rel}} [\gamma_{2p-1}]$. It follows that

$$\theta[y] = \theta[\gamma_0] * \dots * \theta[\gamma_{2p-1}] = [g_0] * \dots * [g_{2p-1}] = [x]$$

- (2) We can calculate the relative Chern character of the lift $[y] \in K_n^{\text{rel}}(A)$. Namely, by Theorem 6.12 this is a question of applying the relative Chern character to each of the contractions $\gamma_i : [0, 1] \rightarrow GL(A)$. The desirable cycle in cyclic homology is then found by taking interior products at the level of cyclic homology.

What is left is then only to apply the homomorphism $[\tau_F] : HC_{n-1}(A) \rightarrow \mathbb{C}/(2\pi i)^p \mathbb{Z}$ to the relevant cycle. Let us see what we get...

We start by defining the form

$$\langle \cdot, \dots, \cdot \rangle_F : M_\infty(A) \wedge \dots \wedge M_\infty(A) \rightarrow \mathbb{C}$$

by the assignment

$$\begin{aligned} \langle a_0, \dots, a_{2p-1} \rangle_F &= -\frac{1}{(p-1)!} \sum_{s \in S_{2p-1}} \text{sgn}(s) \\ &\quad \text{Tr}([P\text{TR}(a_0)P, P\text{TR}(a_{s(1)})P] \cdot \dots \cdot [P\text{TR}(a_{s(2p-2)})P, P\text{TR}(a_{s(2p-1)})P]) \end{aligned}$$

Here $P = \frac{F+1}{2}$ denotes the projection associated with the Fredholm module. Furthermore, S_{2p-1} denotes the subset of permutations $s \in \Sigma_{2p-1}$ which satisfy $s(2i) < s(2i+1)$ for all $i \in \{1, \dots, p-1\}$. We would like the reader to notice that the commutator

$$[P\text{TR}(a)P, P\text{TR}(b)P] \in \mathcal{L}^p(H)$$

is in the p^{th} Schatten ideal for any $a, b \in M_\infty(A)$. This follows from the commutator condition on the Fredholm module (F, H) and the commutativity assumption on A . Indeed,

$$\begin{aligned} [P\text{TR}(a)P, P\text{TR}(b)P] &= P\text{TR}(a)[P, \text{TR}(b)]P + P\text{TR}(b)[\text{TR}(a), P]P \\ &= [P, \text{TR}(a)][P, \text{TR}(b)]P - [P, \text{TR}(b)][P, \text{TR}(a)]P \\ &\in \mathcal{L}^p(H) \end{aligned}$$

Now, for each smooth path $\gamma : [0, 1] \rightarrow GL(A)$ with $\gamma(0) = 1$ we let $l(\gamma) \in M_\infty(A)$ denote the matrix

$$l(\gamma) = \int_0^1 \frac{d\gamma}{dt} \cdot \gamma^{-1} dt$$

Furthermore, for each $a \in M_\infty(A)$ we let $\gamma_a \in [0, 1] \rightarrow GL(A)$ denote the smooth path

$$\gamma_a(t) = e^{-ta}$$

We then have the important identity

$$l(\gamma_a) = -a$$

Recall that the additive character of the continuous Fredholm module is given by

$$\mathcal{A}_F : K_{2p}^{\text{rel}}(A) \rightarrow \mathbb{C} \quad \mathcal{A}_F = \tau_F \circ \text{ch}^{\text{rel}}$$

We can calculate this invariant explicitly on relative Loday products. Indeed, the following theorem is a consequence of [17, Theorem 5.1] and [17, Theorem 5.3].

Theorem 6.13. *Let A be a commutative unital Banach algebra and suppose that (F, H) is a continuous odd $2p$ -summable Fredholm module over A . Let $\gamma_0, \dots, \gamma_{2p-1} : [0, 1] \rightarrow GL(A)$ be smooth paths with $\gamma_i(0) = 1$. We then have the formula*

$$\mathcal{A}_F([\gamma_0] *^{\text{rel}} \dots *^{\text{rel}} [\gamma_{2p-1}]) = \langle l(\gamma_0), \dots, l(\gamma_{2p-1}) \rangle_F$$

for the application of the additive character to the interior relative Loday product of the classes $[\gamma_0], \dots, [\gamma_{2p-1}] \in K_1^{\text{rel}}(A)$. In particular, for $a_0, \dots, a_{2p-1} \in M_\infty(A)$ we get that

$$\mathcal{A}_F([\gamma_{a_0}] *^{\text{rel}} \dots *^{\text{rel}} [\gamma_{a_{2p-1}}]) = \langle a_0, \dots, a_{2p-1} \rangle_F$$

Using the above theorem we get a couple of desirable properties for the form

$$\langle \cdot, \dots, \cdot \rangle_F : M_\infty(A) \wedge \dots \wedge M_\infty(A) \rightarrow \mathbb{C}$$

directly from the corresponding properties of the relative Loday product and the relative K -groups.

Corollary 6.14. *Let $\gamma_0, \dots, \gamma_{2p-1}, \tau : [0, 1] \rightarrow GL(A)$ be smooth paths with $\gamma_i(0) = 1 = \tau(0)$ and let $s \in \Sigma_{2p}$ be a permutation. We then have the identities*

$$\begin{aligned} & \langle l(\gamma_{s(0)}), \dots, l(\gamma_{s(2p-1)}) \rangle_F \\ &= \text{sgn}(s) \langle l(\gamma_0), \dots, l(\gamma_{2p-1}) \rangle_F \\ & \langle l(\gamma_0), \dots, l(\gamma_i \cdot \tau), \dots, l(\gamma_{2p-1}) \rangle_F \\ &= \langle l(\gamma_0), \dots, l(\gamma_i), \dots, l(\gamma_{2p-1}) \rangle_F + \langle l(\gamma_0), \dots, l(\tau), \dots, l(\gamma_{2p-1}) \rangle_F \end{aligned}$$

in the complex numbers.

Proof. This follows directly from the multilinearity and graded commutativity of the relative Loday product. See Theorem 5.29. \square

Corollary 6.15. *Let $\gamma_0, \dots, \gamma_{2p-1}, \tau : [0, 1] \rightarrow GL(A)$ be smooth paths with $\gamma_i(0) = 1 = \tau(0)$. Suppose that there exists an element $\sigma \in F(A) = [R(A), R(A)]$ such that $\gamma_i \cdot \sigma \sim \tau$. Here \sim denotes the equivalence relation given by homotopies with fixed endpoints. We then have the identity*

$$\langle l(\gamma_0), \dots, l(\gamma_i), \dots, l(\gamma_{2p-1}) \rangle_F = \langle l(\gamma_0), \dots, l(\tau), \dots, l(\gamma_{2p-1}) \rangle_F$$

in the complex numbers.

Proof. This follows immediately since the quantity

$$\langle l(\gamma_0), \dots, l(\gamma_i), \dots, l(\gamma_{2p-1}) \rangle_F$$

only depends on the classes of $\gamma_0, \dots, \gamma_{2p-1}$ in the first relative K -group. \square

Recall that the multiplicative character is given by the formula

$$\mathcal{M}_F(\theta[y]) = [\mathcal{A}_F[y]] \in \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

on the image of $\theta : K_{2p}^{\text{rel}}(A) \rightarrow K_{2p}(A)$. Here $[\cdot] : \mathbb{C} \rightarrow \mathbb{C}/(2\pi i)^p \mathbb{Z}$ denotes the quotient map. We can calculate this invariant explicitly on Loday products.

Theorem 6.16. [17, Theorem 5.4] *Let A be a commutative unital Banach algebra and suppose that (F, H) is a continuous odd $2p$ -summable Fredholm module over A . Let $g_0, \dots, g_{2p-1} \in GL_0(A)$ be invertibles in the path component of the identity. Let $\gamma_0, \dots, \gamma_{2p-1} : [0, 1] \rightarrow GL(A)$ denote some smooth paths with $\gamma_i(0) = 1$ and $\gamma_i(1) = g_i^{-1}$. We then have the formula*

$$\mathcal{M}_F([g_0] * \dots * [g_{2p-1}]) = [\langle l(\gamma_0), \dots, l(\gamma_{2p-1}) \rangle_F]$$

for the application of the multiplicative character to the interior Loday product of the classes $[g_0], \dots, [g_{2p-1}] \in K_1(A)$. In particular, for $a_0, \dots, a_{2p-1} \in M_\infty(A)$ we get that

$$(25) \quad \mathcal{M}_F([e^{a_0}] * \dots * [e^{a_{2p-1}}]) = [\langle a_0, \dots, a_{2p-1} \rangle_F]$$

Let us stop for a moment and discuss the above result. We would like to mention a couple of things which we find interesting. Let us begin by underlining the similarity with the identity

$$(26) \quad \det(e^A e^B e^{-A} e^{-B}) = e^{\text{Tr}[A, B]}$$

Here $A, B \in \mathcal{L}(H)$ are operators with trace class commutator. In Section 3.2 we identified the left hand side of (26) as the application of the determinant invariant $d_{A, B} : K_2(C^2[x, y]) \rightarrow \mathbb{C}^*$ to the Steinberg symbol $\{e^x, e^y\} \in K_2(C^2[x, y])$. That is

$$d_{A, B}\{e^x, e^y\} = \det(e^A e^B e^{-A} e^{-B})$$

Furthermore, we identified the trace of the commutator as the value of a trace invariant $\tau_{A, B} : K_2^{\text{rel}}(C^2[x, y]) \rightarrow \mathbb{C}$ on a relative Steinberg symbol, $\{\gamma_x, \gamma_y\}^{\text{rel}} \in K_2^{\text{rel}}(C^2[x, y])$. That is

$$\tau_{A, B}\{\gamma_x, \gamma_y\}^{\text{rel}} = \text{Tr}[A, B]$$

By the comparison result, Theorem 3.22, and the fact that $\theta\{\gamma_x, \gamma_y\}^{\text{rel}} = \{e^x, e^y\}$ we could then obtain the identity (26). In Section 4 we were then establishing the link between the determinant invariant and the multiplicative character. That is, we were able to associate a determinant invariant $d_F : K_2(\mathcal{B}) \rightarrow \mathbb{C}^*$ and a surjective algebra homomorphism $R_F : \mathcal{A} \rightarrow \mathcal{B}$ to each odd 2-summable Fredholm module over the \mathbb{C} -algebra \mathcal{A} . We could then prove the commutativity of the diagram

$$\begin{array}{ccc} K_2(\mathcal{A}) & \xrightarrow{(R_F)_*} & K_2(\mathcal{B}) \\ \mathcal{M}_F \downarrow & & \downarrow d_F \\ \mathbb{C}/(2\pi i)\mathbb{Z} & \xrightarrow{\exp} & \mathbb{C}^* \end{array}$$

See Corollary 4.9. In company with the comparison theorem this was essentially done by showing the coincidence of a universal trace invariant and a universal additive character of

2-summable Fredholm modules. In particular, assuming that (F, H) is a continuous odd 2-summable Fredholm module over a commutative Banach algebra A we get the identity

$$\begin{aligned} (\exp \circ \mathcal{M}_F)\{e^a, e^b\} &= (d_F \circ (R_F)_*)\{e^a, e^b\} \\ &= d_F\{q(P\pi(e^a)P), q(P\pi(e^b)P)\} \\ &= \exp(\operatorname{Tr}[P\pi(a)P, P\pi(b)P]) \end{aligned}$$

Or equivalently, $\mathcal{M}_F\{e^a, e^b\} = \operatorname{Tr}[P\pi(a)P, P\pi(b)P] \in \mathbb{C}/(2\pi i)\mathbb{Z}$. Finally, since the Steinberg symbol is a special instance of the interior Loday product we get the identity

$$\mathcal{M}_F([e^b] * [e^a]) = \operatorname{Tr}[P\pi(a)P, P\pi(b)P] \in \mathbb{C}/(2\pi i)\mathbb{Z}$$

This is precisely the result of Theorem 6.16 when $p = 1$. This seems to justify the interpretation of the formula (25) as a higher dimensional analogue of the formula (26). Furthermore, the odd multiplicative character could be seen as a generalization of the determinant invariant to the higher algebraic K -groups.

Let us continue our investigation of the result in Theorem 6.16. Remark that the left hand side of equation (25) only depends on the exponentials $e^{a_0}, \dots, e^{a_{2p-1}} \in GL_0(A)$. However, the right hand side only depends on the logarithms $a_0, \dots, a_{2p-1} \in M_\infty(A)$. Therefore, suppose that $b_i \in M_\infty(A)$ is another matrix over A with $e^{b_i} = e^{a_i}$. We then get that the difference

$$\langle a_0, \dots, a_i, \dots, a_{2p-1} \rangle_F - \langle a_0, \dots, b_i, \dots, a_{2p-1} \rangle_F \in (2\pi i)^p \mathbb{Z}$$

is contained in the additive subgroup $(2\pi i)^p \mathbb{Z} \subseteq \mathbb{C}$ of the complex numbers. Actually we have an even stronger invariance results.

Corollary 6.17. *Let $\gamma_0, \dots, \gamma_{2p-1} : [0, 1] \rightarrow GL(A)$ be smooth paths with $\gamma_i(0) = 1$. Suppose that $\tau_i : [0, 1] \rightarrow GL(A)$ is a smooth path with $\tau_i(0) = 1$ and $\tau_i(1) = \gamma_i(1)h$ where $h \in E(A)$ is some elementary matrix. The difference*

$$\langle l(\gamma_0), \dots, l(\gamma_i), \dots, l(\gamma_{2p-1}) \rangle - \langle l(\gamma_0), \dots, l(\tau_i), \dots, l(\gamma_{2p-1}) \rangle \in (2\pi i)^p \mathbb{Z}$$

is then contained in the additive subgroup $(2\pi i)^p \mathbb{Z} \subseteq \mathbb{C}$ of the complex numbers.

Proof. This follows from Theorem 6.16, and the fact that the multiplicative character only depends on the classes of the endpoints $\gamma_i(1)^{-1} \in GL_0(A)$ in the first algebraic K -group. \square

We feel that we have justified the following definition.

Definition 6.18. *Let (F, H) be an odd $2p$ -summable Fredholm module over some commutative unital \mathbb{C} -algebra A . By the higher determinant of (F, H) we will understand the map*

$$\det_F : GL(A)^{2p} \rightarrow \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

given by

$$\det_F(g_0, \dots, g_{2p-1}) = \mathcal{M}_F([g_0] * \dots * [g_{2p-1}])$$

We then get a couple of desirable properties for the higher determinant straight from the corresponding properties of the Loday product and the first algebraic K -group.

Corollary 6.19. *Let $g_0, \dots, g_{2p-1}, h \in GL(A)$, let $s \in \Sigma_{2p}$ be a permutation and let $e \in E(A)$ be some elementary matrix. We then have the identities*

$$\begin{aligned} \det_F(g_{s(0)}, \dots, g_{s(2p-1)}) &= \text{sgn}(s) \det_F(g_0, \dots, g_{2p-1}) \\ \det_F(g_0, \dots, g_i \cdot h, \dots, g_{2p-1}) &= \det_F(g_0, \dots, g_i, \dots, g_{2p-1}) + \det_F(g_0, \dots, h, \dots, g_{2p-1}) \\ \det_F(g_0, \dots, g_i e, \dots, g_{2p-1}) &= \det_F(g_0, \dots, g_{2p-1}) \end{aligned}$$

Now, let A be a commutative unital Banach algebra and let (F, H) be a continuous $2p$ -summable Fredholm module over A . It follows from the last corollary and Theorem 6.16 that the higher determinant is calculizable on the path component of the identity. This is due to the fact that each element in $GL_0(A)$ can be obtained as a product of exponentials, see [1, Proposition 3.4.3]. In the next section we will consider a concrete example and thus make the explicit formula for the determinant even more tangible.

7. THE DETERMINANT OF AN ODD DIMENSIONAL TORUS

As mentioned above we will now consider a very concrete example, namely the case of an odd dimensional torus, \mathbb{T}^{2p-1} . We will construct a continuous odd $2p$ -summable Fredholm module (F, H) over the unital Banach algebra $C^{p+2}(\mathbb{T}^{2p-1})$ which arises from the geometry of the torus. We will then see that the evaluation of the associated character

$$\tau_F : HC_{2p-1}(C^{p+2}(\mathbb{T}^{2p-1})) \rightarrow \mathbb{C}$$

on the *Hochschild* cycle

$$x = \sum_{s \in \Sigma_{2p-1}} f_0 \otimes f_{s(1)} \otimes \dots \otimes f_{s(2p-1)} = f_0 * \dots * f_{2p-1}$$

is given by the integral

$$\tau_F(x) = C \int_{\mathbb{T}^{2p-1}} f_0 df_1 \wedge \dots \wedge df_{2p-1}$$

Here $f_0, \dots, f_{2p-1} \in C^{p+2}(\mathbb{T}^{2p-1})$ are any C^{p+2} -functions on the torus and

$$C = -\frac{\Omega_n}{n(2\pi)^n} \frac{(-i)^p (2p-1)!}{2^{p+1}(p-1)!} \in \mathbb{C} \quad n := 2p-1$$

is a constant. It follows that the higher determinant of the above Fredholm module is given by the formula

$$\det_F(e^{f_0}, \dots, e^{f_{2p-1}}) = [\tau_F(f_0 * \dots * f_{2p-1})] = [C \int_{\mathbb{T}^{2p-1}} f_0 df_1 \wedge \dots \wedge df_{2p-1}] \in \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

Here $[\cdot] : \mathbb{C} \rightarrow \mathbb{C}/(2\pi i)^p \mathbb{Z}$ denotes the quotient map. In particular the higher determinant is surjective in this case. In order to perform this calculation (in an understandable way) we will need to introduce the reader to some concepts and results from noncommutative geometry. We will use [8] and [11] as our main references. We should note that the above calculation of the Chern character over the torus is by no means new. See for example [23]. Furthermore, there should be no problems in replacing this specific manifold by any odd dimensional compact spin manifold without boundary. We would in this general case obtain a similar formula. See the

remark after [11, Theorem 10.32]. We have chosen the case of a torus to keep things as simple as possible.

7.1. Preliminaries from noncommutative geometry.

7.1.1. The Dixmier trace. Let H be a separable Hilbert space. The Dixmier trace is a trace defined on a certain ideal, $\mathcal{L}^{1+}(H)$, in the bounded operators, $\mathcal{L}(H)$. In general, the Dixmier trace of an operator will depend on the choice of a limiting procedure. In noncommutative geometry it is considered as a noncommutative integral. We will need it for our calculation of the higher determinant.

Definition 7.1. *By the Dixmier trace ideal we will understand the subset of the compact operators given by*

$$\mathcal{L}^{1+}(H) = \{T \in \mathcal{K}(H) \mid \sup_{n \geq 2} \frac{\sigma_n(T)}{\log(n)} < \infty\}$$

Here $\sigma_n(T) = s_0(T) + \dots + s_{n-1}(T)$ denotes a partial sum of the singular values of T .

We recall the well known inequalities

$$\sigma_n(T + S) \leq \sigma_n(T) + \sigma_n(S) \quad \text{and} \quad \sigma_n(ATB) \leq \|A\|_\infty \sigma_n(T) \|B\|_\infty$$

which are valid for any compact operators $T, S \in \mathcal{K}(H)$ and bounded operators $A, B \in \mathcal{L}(H)$. These inequalities ensure us that the Dixmier trace ideal is really an ideal in $\mathcal{L}(H)$. Furthermore, it becomes a Banach *-algebra when equipped with the norm

$$\|\cdot\|_{1+} : \mathcal{L}^{1+}(H) \rightarrow [0, \infty) \quad \|T\|_{1+} = \sup_{n \geq 2} \frac{\sigma_n(T)}{\log(n)}$$

Note that the Dixmier trace ideal is closed under adjoints since $T \in \mathcal{K}(H)$ and $T^* \in \mathcal{K}(H)$ have the same sequence of singular values.

In many examples we get operators which are not in the Dixmier trace ideal but in a larger ideal.

Definition 7.2. *Let $1 < p < \infty$. By the p^{th} Dixmier ideal we will understand the ideal in $\mathcal{L}(H)$ given by*

$$\mathcal{L}^{p+}(H) = \{T \in \mathcal{K}(H) \mid \sup_{n \in \mathbb{N}} \frac{\sigma_n(T)}{n^{1-1/p}} < \infty\}$$

As it is the case for the Dixmier trace ideal, the p^{th} Dixmier ideal is really an ideal in $\mathcal{L}(H)$. Furthermore, it becomes a Banach *-algebra when equipped with the norm

$$\|\cdot\|_{p+} : \mathcal{L}^{p+}(H) \rightarrow [0, \infty) \quad \|T\|_{p+} = \sup_{n \in \mathbb{N}} \frac{\sigma_n(T)}{n^{1-1/p}}$$

We then have the following result.

Lemma 7.3. [11, Lemma 7.37] *Let $A \in \mathcal{L}^{p+}(H)$ be a positive operator. The p^{th} power of A is then contained in the Dixmier ideal. That is*

$$A^p \in \mathcal{L}^{1+}(H)$$

Let us turn to the description of the Dixmier trace. Let $T \in \mathcal{K}(H)$. For each $\lambda \in [1, \infty)$ we define the number

$$\sigma_\lambda(T) = (1 - t)\sigma_n(T) + t\sigma_{n+1}(T)$$

Here $n = \lfloor \lambda \rfloor$ is the integer part of λ and $t = \lambda - n$. We can then consider the Cesàro mean of the function $\lambda \mapsto \sigma_\lambda(T)/\log(\lambda)$. That is

$$\tau_\lambda(T) = \frac{1}{\log(\lambda)} \int_3^\lambda \frac{\sigma_u(T)}{\log(u)} \frac{du}{u}$$

Clearly, whenever $T \in \mathcal{L}^{1+}(H)$ the function $\lambda \mapsto \tau_\lambda(T)$ is bounded. In general, however, nothing assures that the limit of $\tau_\lambda(T)$ exists for $\lambda \rightarrow \infty$. Instead, for positive operators in the Dixmier ideal, the value of the Dixmier trace is a generalized limit of this function. The limiting procedure will depend on the choice of a state on the quotient C^* -algebra $C_b([3, \infty))/C_0([3, \infty))$. Let us thus fix such a state

$$\omega : C_b([3, \infty))/C_0([3, \infty)) \rightarrow \mathbb{C}$$

Definition 7.4. *Let $T \in \mathcal{L}^{1+}(H)$ be a positive operator in the Dixmier ideal. By the Dixmier trace of T we will understand the positive number*

$$\mathrm{Tr}_\omega(T) = \omega(\tau_\lambda(T))$$

The Dixmier trace extends to all of $\mathcal{L}^{1+}(H)$ by noting that each $T \in \mathcal{L}^{1+}(H)$ can be written as a linear combination of positive operators. Indeed, for $T \in \mathcal{L}^{1+}(H)$ we have

$$T = \frac{T + T^*}{2} + i \frac{i(T^* - T)}{2}$$

and for each selfadjoint element $T \in \mathcal{L}^{1+}(H)$ we have

$$T = \frac{1}{2}(T + |T|) - \frac{1}{2}(|T| - T)$$

We then define the Dixmier trace using this decomposition.

Theorem 7.5. [8, IV.2. Proposition 3] *For each choice of state $\omega : C_b([3, \infty))/C_0([3, \infty)) \rightarrow \mathbb{C}$ the Dixmier trace is continuous and linear*

$$\mathrm{Tr}_\omega : \mathcal{L}^{1+}(H) \rightarrow \mathbb{C}$$

Furthermore, it satisfies the tracial property

$$\mathrm{Tr}_\omega(TS) = \mathrm{Tr}_\omega(ST)$$

for each $T \in \mathcal{L}^{1+}(H)$ and $S \in \mathcal{L}(H)$.

We say that an operator $T \in \mathcal{L}^{1+}(H)$ is *measurable* when its Dixmier trace is independent of the choice of state. For measurable operators we denote the common value of their Dixmier traces by $\mathrm{Tr}_+(T) \in \mathbb{C}$. As an example, take any positive operator of trace class, $T \in \mathcal{L}^1(H)$. In this case we get that

$$\frac{\sigma_n(T)}{\log(n)} \leq \frac{\|T\|_1}{\log(n)} \rightarrow 0$$

It follows that the Dixmier trace of T is zero no matter how we choose the state. In particular we get the following result.

Theorem 7.6. [8, IV.2. Proposition 3] *All operators of trace class are measurable and the Dixmier trace vanishes identically on $\mathcal{L}^1(H)$.*

There are however many measurable operators with nonvanishing Dixmier trace. Let us look at a very general example of this phenomena. This will also explain the interpretation of the Dixmier trace as a noncommutative integral. Namely, let M be a compact, n -dimensional, orientable Riemannian manifold without boundary. We then have the Riemannian volume form $\nu_g \in \mathcal{A}^n(M, \mathbb{R})$ and the scalar Laplacian

$$\Delta : C^\infty(M) \rightarrow C^\infty(M) \quad \Delta := -(\star d \star d)$$

Here $d : \mathcal{A}^k(M) \rightarrow \mathcal{A}^{k+1}(M)$ denotes the exterior differential and $\star : \mathcal{A}^k(M) \rightarrow \mathcal{A}^{n-k}(M)$ denotes the Hodge star operator associated with the orientation. See [38, p.149-150], for example. The scalar Laplacian is a positive elliptic differential operator of order two. In particular, we can make sense of the operator, $\Delta^{-n/2}$, as an elliptic pseudodifferential operator of order $-n$. It extends to a bounded operator on the Hilbert space, $H := L^2(M)$, of square integrable functions on M . In fact, $\Delta^{-n/2} \in \mathcal{L}^{1+}(H)$, is an element of the first Dixmier ideal. Its Dixmier trace can be computed by the following trace theorem of A. Connes. We let the continuous function on M act by multiplication on the Hilbert space $L^2(M)$.

Theorem 7.7. [11, Corollary 7.22] *The operator $f\Delta^{-n/2} \in \mathcal{L}^{1+}(H)$ is measurable for any continuous function $f \in C(M)$ and we have the identity*

$$\frac{n(2\pi)^n}{\Omega_n} \text{Tr}_+(f\Delta^{-n/2}) = \int_M f \nu_g$$

between the Dixmier trace and the integral of functions against the Riemannian volume form. Here $\Omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ is the standard volume of the sphere S^{n-1} .

7.1.2. The Hochschild character formula of A. Connes. In this section we will expose another result of A. Connes which we will use for the calculation of the higher determinant. We will consider the case where a finitely summable Fredholm module arises from a spectral triple. In this context it is possible to compute the Hochschild class of the Chern character by means of a Dixmier trace. In the case of an orientable compact Riemannian manifold it is therefore essentially given by the integral of some function. This is due to the result of A. Connes which we stated in the last section, see Theorem 7.7. We will briefly review the theory needed for the statement of the desired computational result.

Definition 7.8. *Let \mathcal{A} be a unital \mathbb{C} -algebra represented by a unital representation $\pi : \mathcal{A} \rightarrow \mathcal{L}(H)$ on a separable Hilbert space H . Let D be an unbounded selfadjoint operator on H . We say that the triple (\mathcal{A}, H, D) is an odd unital spectral triple when the conditions*

- (1) $a(\text{Dom}(D)) \subseteq \text{Dom}(D)$ and $[D, a]$ extends to a bounded operator on H .
- (2) The resolvent $(\lambda + D)^{-1}$ is compact for all $\lambda \in \mathbb{C} - \mathbb{R}$.

are satisfied for all $a \in \mathcal{A}$.

We will need some extra conditions on our spectral triples in order for the computational result to apply. We will both require a regularity condition and a stronger decay condition for the resolvent.

Definition 7.9. Let $n \in [1, \infty)$. We say that an odd unital spectral triple is $(n+)$ -summable when the resolvent $(\lambda + D)^{-1} \in \mathcal{L}^{n+}(H)$ is actually contained in the n^{th} Dixmier ideal for each $\lambda \in \mathbb{C} - \mathbb{R}$.

By the functional calculus for unbounded selfadjoint operators we can form the positive operator $|D|$. We will then consider the derivation $\delta : T \mapsto [|D|, T]$. The domain of δ consists of the bounded operators $T \in \mathcal{L}(H)$ such that

$$T(\text{Dom}(|D|)) \subseteq \text{Dom}(|D|) \quad \text{and} \quad [|D|, T] \text{ extends to a bounded operator on } H$$

We let \mathcal{A}_D denote the smallest subalgebra of $\mathcal{L}(H)$ such that

$$a \in \mathcal{A}_D \text{ and } [D, a] \in \mathcal{A}_D \quad \text{for all } a \in \mathcal{A}$$

Definition 7.10. Let $k \in \mathbb{N}$. We say that a spectral triple (\mathcal{A}, H, D) is QC^k when the algebra \mathcal{A}_D is contained in the domain of the derivation δ^m for each $1 \leq m \leq k$.

We will now outline how to construct Fredholm modules from spectral triples. So, let (\mathcal{A}, H, D) be a spectral triple. We let $P = 1_{[0, \infty)}(D)$ denote the spectral projection of D associated with the subset $[0, \infty) \subseteq \mathbb{R}$. We can then form the selfadjoint unitary $F = 2P - 1 \in \mathcal{L}(H)$. The next result is a consequence of [11, Lemma 10.18].

Theorem 7.11. Let $\epsilon > 0$ and let $n \in [1, \infty)$. If (\mathcal{A}, H, D) is an odd unital $(n+)$ -summable unital spectral triple then the pair (F, H) is an odd $(n + \epsilon)$ -summable Fredholm module over \mathcal{A} . Furthermore, if \mathcal{A} is a unital Banach algebra and both the representation and the linear map $a \mapsto [D, a]$ are continuous, then the Fredholm module (F, H) is continuous as well. See Definition 6.1.

We will need one more ingredient for the statement of the Hochschild character formula. Let \mathcal{A} be some \mathbb{C} -algebra. For each $n \in \mathbb{N}_0$ we let $C_n^{\text{alg}}(\mathcal{A}) = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}^{\otimes n}$ denote the vector space of algebraic Hochschild chains. By an *algebraic Hochschild cycle* we will then understand an element $x \in C_n^{\text{alg}}(\mathcal{A})$ with $b(x) = 0$. Here

$$b : C_n^{\text{alg}}(\mathcal{A}) \rightarrow C_{n-1}^{\text{alg}}(\mathcal{A})$$

$$b(a_0 \otimes a_1 \otimes \dots \otimes a_n) = (-1)^n a_n a_0 \otimes \dots \otimes a_{n-1} + \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n$$

denotes the Hochschild boundary. Clearly, each Hochschild cycle gives rise to an algebraic cyclic cycle by means of the quotient map

$$I : C_n^{\text{alg}}(\mathcal{A}) \rightarrow C_n^{\text{alg}}(\mathcal{A}) / \text{Im}(1 - t)$$

Now, by the formula of Theorem 6.2 each odd $2p$ -summable Fredholm module (F, H) over a unital \mathbb{C} -algebra \mathcal{A} gives rise to a homomorphism

$$\tau_F : C_{2p-1}^{\text{alg}}(\mathcal{A}) / \text{Im}(1 - t) \rightarrow \mathbb{C}$$

We could try to investigate this homomorphism on the image of the quotient map

$$I : C_{2p-1}^{\text{alg}}(\mathcal{A}) \rightarrow C_{2p-1}^{\text{alg}}(\mathcal{A}) / \text{Im}(1 - t)$$

The character formula of A. Connes deals with this question in the case where the Fredholm module comes from a spectral triple.

Let $n = 2p - 1 \in \mathbb{N}$ be an odd number. Let (\mathcal{A}, H, D) be an odd unital $(n+)$ -summable spectral triple. We can then make sense of the operator $|D|^{-1}$ on the Hilbert space $\text{Ker}(D)^\perp$. The bounded operator $|D|^{-1} \in \mathcal{L}(H)$ is then given by the matrix

$$|D|^{-1} = \begin{pmatrix} |D|^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $H = \text{Ker}(D)^\perp \oplus \text{Ker}(D)$. The $(n+)$ -summability of the spectral triple ensures us that $|D|^{-1} \in \mathcal{L}^{n+}(H)$ is in the n^{th} Dixmier ideal as well. In particular, it follows by Lemma 7.3 that $|D|^{-n} \in \mathcal{L}^{1+}(H)$ is of Dixmier trace class. We thus have a linear map

$$\varphi_D^\omega : C_{2p-1}^{\text{alg}}(\mathcal{A}) \rightarrow \mathbb{C}$$

given by

$$\varphi_D^\omega(a_0 \otimes a_1 \otimes \dots \otimes a_{2p-1}) = \text{Tr}_\omega(a_0[D, a_1] \cdot \dots \cdot [D, a_{2p-1}] |D|^{-2p+1})$$

for any choice of state $\omega : C_b([3, \infty))/C_0([3, \infty)) \rightarrow \mathbb{C}$. The character formula of A. Connes can then be stated as follows:

Theorem 7.12. [11, Theorem 10.32] *Let $n = 2p - 1 \in \mathbb{N}$ be an odd number. Let (\mathcal{A}, H, D) be an odd unital QC^2 $(n+)$ -summable spectral triple. Let (F, H) be the associated $2p$ -summable Fredholm module over \mathcal{A} . Suppose that $x \in C_{2p-1}^{\text{alg}}(\mathcal{A})$ is an algebraic Hochschild cycle. We then have the formula*

$$\tau_F(I(x)) = -\frac{1}{2^{2p} \cdot (p-1)!} \varphi_D^\omega(x)$$

for the evaluation of the Chern character on $I(x) \in C_n^{\text{alg}}(\mathcal{A})/\text{Im}(1-t)$.

7.2. An integral formula for the higher determinant of a torus. We are now ready to prove the computational result which we promised at the end of Section 6.3. We will consider the concrete example of an odd dimensional torus, \mathbb{T}^{2p-1} . In this setting there is a spectral triple $\mathbf{T}^{2p-1} = (C^{p+2}(\mathbb{T}^{2p-1}), H, D)$ which satisfies the sufficient conditions for the Hochschild character theorem to apply. The square of the selfadjoint operator D agrees with the scalar Laplacian. On Hochschild cycles we can thus express the Chern character of the associated Fredholm module by means of integration over the torus. Combining this with the calculation of the multiplicative character which we carried out in the last sections we are then able to prove the following result.

Theorem 7.13. *Let $n = 2p - 1$ be an odd number. There is a higher determinant*

$$\det_F : GL(C^{p+2}(\mathbb{T}^{2p-1}))^{2p} \rightarrow \mathbb{C}/(2\pi i)^p \mathbb{Z}$$

satisfying the properties stated in Corollary 6.19. For any $2p$ -tuple of matrices

$$f_0, \dots, f_{2p-1} \in M_k(C^{p+2}(\mathbb{T}^{2p-1}))$$

it is given by the integral

$$\det_F(e^{f_0}, \dots, e^{f_{2p-1}}) = [C \int_{\mathbb{T}^{2p-1}} \text{TR}(f_0) d\text{TR}(f_1) \wedge \dots \wedge d\text{TR}(f_{2p-1})]$$

over the torus. Here

$$C = -\frac{(-i)^p(2p-1)!}{2^{p+1} \cdot (p-1)!} \cdot \frac{\Omega_n}{n(2\pi)^n} \in \mathbb{C}$$

is a constant. Recall that $\Omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ is the standard volume of the sphere S^{n-1} .

Let us start by constructing the appropriate spectral triple, $\mathbf{T}^{2p-1} = (C^{p+2}(\mathbb{T}^{2p-1}), H, D)$. We will thus explain the terms. The unital \mathbb{C} -algebra is the unital Banach algebra of C^{p+2} -functions on the torus of dimension $(2p-1)$, $C^{p+2}(\mathbb{T}^{2p-1})$. The Hilbert space will be a direct sum of square integrable functions on the torus

$$H := \mathbb{C}^{2^{p-1}} \otimes L^2(\mathbb{T}^{2p-1})$$

The representation is the unital continuous algebra homomorphism

$$\pi : C^{p+2}(\mathbb{T}^n) \rightarrow \mathcal{L}(H) \quad \pi : f \mapsto 1_{2^{p-1}} \otimes m(f)$$

given by multiplication on the factor $L^2(\mathbb{T}^{2p-1})$. To define the unbounded operator D on H we need a bit more structure.

For each $j \in \{1, \dots, 2p-1\}$ let $\partial_j := -i \frac{\partial}{\partial \theta_j}$ denote the unbounded operator on $L^2(\mathbb{T}^{2p-1})$ given by differentiation with respect to arclength in the j^{th} copy of the circle. Furthermore, we define the two by two matrices

$$\alpha_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

For each $j \in \{1, \dots, 2p-1\}$ we then define a matrix $c_j \in M_{2^{p-1}}(\mathbb{C})$ by the formula

$$c_j = \begin{cases} \underbrace{\alpha_0 \otimes \dots \otimes \alpha_0}_{k-1} \otimes \alpha_1 \otimes 1 \otimes \dots \otimes 1 & \text{for } j = 2k-1 \\ \underbrace{\alpha_0 \otimes \dots \otimes \alpha_0}_{k-1} \otimes \alpha_2 \otimes 1 \otimes \dots \otimes 1 & \text{for } j = 2k \end{cases}$$

Here we have used an identification

$$(27) \quad \psi : M_2(\mathbb{C}) \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} M_2(\mathbb{C}) \cong M_{2^{p-1}}(\mathbb{C})$$

arising from an isomorphism of vector spaces

$$(\mathbb{C} \oplus \mathbb{C}) \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} (\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{C}^{2^{p-1}}$$

This determines a representation of the complex Clifford algebra over \mathbb{R}^{2p-1} on $\mathbb{C}^{2^{p-1}}$. Indeed, the matrices $c_j \in M_{2^{p-1}}(\mathbb{C})$ satisfy the relations

$$\begin{aligned} c_j c_k &= -c_k c_j & \text{for } 1 \leq j < k \leq 2p-1 \\ c_j c_j &= 1 & \text{for } j \in \{1, \dots, 2p-1\} \end{aligned}$$

The unbounded operator D on $H = \mathbb{C}^{2^{p-1}} \otimes L^2(\mathbb{T}^{2p-1})$ can then be written as the sum

$$D = c_1 \otimes \partial_1 + \dots + c_{2p-1} \otimes \partial_{2p-1}$$

This unbounded operator extends to a selfadjoint operator with domain equal to a direct sum of the first Sobolev space on the torus. This operator will also be denoted by D . The data which

we have just defined combines to a spectral triple with the right properties for the Hochschild character formula to apply.

Theorem 7.14. *Let $n = 2p - 1 \in \mathbb{N}$ be an odd number. The triple $\mathbf{T}^n = (C^{p+2}(\mathbb{T}^n), H, D)$ is an odd QC^2 $(n+)$ -summable unital spectral triple. Furthermore the linear map $f \mapsto [D, f]$ is continuous on $C^{p+2}(\mathbb{T}^n)$.*

Proof. We leave the proof to the reader. □

Let us fix a state

$$\omega : C_b([3, \infty))/C_0([3, \infty)) \rightarrow \mathbb{C}$$

and try to compute the quantity

$$\varphi_D^\omega(f_0 \otimes \dots \otimes f_{2p-1}) = \text{Tr}_\omega(f_0[D, f_1] \cdot \dots \cdot [D, f_{2p-1}] |D|^{-2p+1})$$

Here $f_0, \dots, f_{2p-1} \in C^{p+2}(\mathbb{T}^{2p-1})$ are C^{p+2} -functions on the torus.

We start by noting that the commutator

$$[D, f] = -i \sum_{j=1}^{2p-1} c_j \otimes \frac{\partial f}{\partial \theta_j} \quad f \in C^{p+2}(\mathbb{T}^{2p-1})$$

is given by Clifford multiplication and multiplication by the partial derivatives of f . Furthermore, we can calculate the square of the unbounded operator D

$$D^2 = 1_{2^{2p-1}} \otimes \left(\sum_{j=1}^{2p-1} \partial_j^2 \right) = 1_{2^{2p-1}} \otimes \Delta$$

Here $\Delta : C^\infty(\mathbb{T}^{2p-1}) \rightarrow C^\infty(\mathbb{T}^{2p-1})$ denotes the scalar Laplacian associated with the Riemannian volume form $\nu_g = d\theta_1 \wedge \dots \wedge d\theta_{2p-1} \in \mathcal{A}^{2p-1}(\mathbb{T}^{2p-1}, \mathbb{R})$.

To proceed we will need the following general lemma on Dixmier traces over matrices.

Lemma 7.15. *Let H be a separable Hilbert space and let $A \in M_k(\mathcal{L}^{1+}(H)) \subseteq \mathcal{L}(H^k)$ be a matrix with Dixmier trace class entrances. The Dixmier trace of A is then given by the Dixmier trace of the diagonal. Thus,*

$$\text{Tr}_\omega(A) = \sum_{j=1}^k \text{Tr}_\omega(A_{jj})$$

We also note that the trace $\text{TR} : M_{2^{2p-1}}(\mathbb{C}) \rightarrow \mathbb{C}$ can be computed on a tensor product by the formula

$$\text{TR}(\psi(A_1 \otimes \dots \otimes A_{p-1})) = \text{TR}(A_1) \cdot \dots \cdot \text{TR}(A_{p-1})$$

Here ψ denotes the identification of (27) and $A_1, \dots, A_{p-1} \in M_2(\mathbb{C})$. Combining this result with the above lemma we get that

$$\text{Tr}_\omega((c_{j_1} \cdot \dots \cdot c_{j_k}) \otimes f |D|^{-2p+1}) = 0$$

for any continuous function f and any subset $1 \leq j_1 < \dots < j_k \leq 2p-1$ with $1 \leq k < 2p-1$. Letting $n = 2p-1$ we can thus compute as follows

$$\begin{aligned}
 \varphi_D^\omega(f_0 \otimes \dots \otimes f_{2p-1}) &= (-i)^{2p-1} \text{Tr}_\omega(c_1 \cdot \dots \cdot c_{2p-1} \otimes f_0 \det(J_f) |D|^{-2p+1}) \\
 &= (-i)^p 2^{p-1} \text{Tr}_\omega(f_0 \det(J_f) |D|^{-2p+1}) \\
 &= (-i)^p 2^{p-1} \frac{\Omega_n}{n(2\pi)^n} \int_{\mathbb{T}^{2p-1}} f_0 df_1 \wedge \dots \wedge df_{2p-1}
 \end{aligned}
 \tag{28}$$

Here $J_f \in M_{2p-1}(C^{p+1}(\mathbb{T}^{2p-1}))$ denotes the Jacobian

$$(J_f)_{i,j} = \frac{\partial f_i}{\partial \theta_j}$$

For the last identity we have used the trace formula of Theorem 7.7.

We can now prove the theorem announced at the beginning of this section. We let (F, H) denote the Fredholm module over $C^{p+2}(\mathbb{T}^{2p-1})$ which we get from Theorem 7.11. We note that (F, H) is odd $2p$ -summable and continuous. In particular we have the higher determinant

$$\det_F : GL(C^{p+2}(\mathbb{T}^{2p-1}))^{2p} \rightarrow \mathbb{C}/(2\pi i)^p \mathbb{Z} \quad \det_F(g_0, \dots, g_{2p-1}) = \mathcal{M}_F([g_0] * \dots * [g_{2p-1}])$$

By the calculation of the multiplicative character given in Theorem [17, Theorem 5.4] it is given by the concrete formula

$$\det_F(e^{f_0}, \dots, e^{f_{2p-1}}) = [\tau_F(f_0 * \dots * f_{2p-1})]$$

But the product

$$f_0 * \dots * f_{2p-1} \in C_{2p-1}^{\text{alg}}(C^{p+2}(\mathbb{T}^{2p-1})) \subseteq C_{2p-1}(C^{p+2}(\mathbb{T}^{2p-1}))$$

is actually an algebraic Hochschild cycle. It therefore follows from Theorem 7.12 and the calculation in (28) that

$$\begin{aligned}
 &\tau_F(f_0 * \dots * f_{2p-1}) \\
 &= -\frac{1}{2^{2p}(p-1)!} \varphi_D^\omega(f_0 * \dots * f_{2p-1}) \\
 &= -\frac{(-i)^p (2p-1)!}{2^{p+1}(p-1)!} \frac{\Omega_n}{n(2\pi)^n} \int_{\mathbb{T}^{2p-1}} f_0 df_1 \wedge \dots \wedge df_{2p-1} \in \mathbb{C}/(2\pi i)^p \mathbb{Z}
 \end{aligned}$$

This proves Theorem 7.13.

REFERENCES

- [1] B. Blackadar, *K-Theory for Operator Algebras*, Mathematical Sciences Research Institute Publications, Springer-Verlag New York, 1986.
- [2] L. G. Brown, *The Determinant Invariant for Operators with Trace Class Self Commutators*, Proc. Conf. on Operator Theory, Lecture Notes in Math., vol. 345, Springer-Verlag, Berlin, Heidelberg, New York, 1973, 210-228.
- [3] L. G. Brown, *Operator Algebras and Algebraic K-theory*, Bull. Amer. Math. Soc., **81**, 1973, 1119-1121.
- [4] R. W. Carey and J. D. Pincus, *An Invariant for Certain Operator Algebras*, Proc. Nat. Acad. Sci. USA, vol. 71, no. 5, 1974, 1952-1956.
- [5] R. W. Carey and J. D. Pincus, *Almost Commuting Algebras*, Lecture Notes in Mathematics, vol. 575, Springer-Verlag, Berlin, Heidelberg, New York, 1977, 19-43.
- [6] R. W. Carey, J. D. Pincus, *Steinberg Symbols Modulo the Trace Class, Holonomy, and Limit Theorems for Toeplitz Determinants*, Transactions of the American Mathematical Society, vol. 358, 2004, 509-551.
- [7] A. Connes, *Non-commutative differential geometry*, Publ. Math. IHES 62, 1985, 257-360.
- [8] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.
- [9] A. Connes and M. Karoubi, *Caractère multiplicatif d'un module de Fredholm*, K-theory **2**, 1988, 431-463.
- [10] N. Dunford and J. Schwartz, *Linear Operators, Vol. II. Spectral Theory*, Interscience, 1963.
- [11] J. M. Gracia-Bondía, J. C. Várilly, H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser Advanced Texts, Birkhäuser, Boston, Basel, Berlin, 2001.
- [12] A. Gorokhovsky, *Characters of Cycles, Equivariant Characteristic Classes and Fredholm Modules*, arXiv:math/9806156v2 [math.DG], 1998.
- [13] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
- [14] J. W. Helton and R. E. Howe, *Integral Operators: Commutators, Traces, Index, and Homology*, Proc. Conf. on Operator Theory, Lecture Notes in Math., vol. 345, Springer-Verlag, Berlin, Heidelberg, New York, 1973, 141-209.
- [15] N. Higson and J. Roe, *Analytic K-homology*, Oxford Science Publications, Oxford University Press, 2000.
- [16] J. Kaad, *Comparison of secondary invariants for finitely summable Fredholm modules*, attachment.
- [17] J. Kaad, *A calculation of the multiplicative character*, attachment.
- [18] J. Kaad, *Invariance results for pairings with algebraic K-theory*, under preparation.
- [19] J. Kaminker, *Algebraic K-theory invariants for operator theory*, Contemporary Mathematics, **185**, Amer. Math. Soc., Providence, RI, 1995, 187-194.
- [20] M. Karoubi, *Homologie Cyclique et K-théorie*, Astérisque No. 149, 1987.
- [21] M. Karoubi, *K-Theory*, Grundlehren der mathematischen Wissenschaften 226, Springer-Verlag, Berlin, Heidelberg, New York, 1978.
- [22] M. Karoubi, *Sur la K-théorie multiplicative*, Proc. Conf. on Cyclic homology and Noncommutative geometry, Fields Institute Communications, Vol. 17, 1997.
- [23] E. Langmann, *Noncommutative integration calculus*, J. Math. Phys., Vol. 36, 1995, 3822-3835.
- [24] J. -L. Loday and D. Quillen, *Homologie cyclique et homologie de l'algèbre de Lie des matrices*, C. R. Acad. Sci. Paris Sér. A-B 296, 1983, 295-297.
- [25] J. -L. Loday, *K-théorie algébrique et représentations de groupes*, Ann. Scient. Ecole Norm. Sup. **9**, 1976, 309-377.
- [26] J. -L. Loday, *Cyclic Homology*, Grundlehren der mathematischen Wissenschaften, vol. 301, Springer-Verlag, Berlin, Heidelberg, New York, 1992.
- [27] J. P. May, *Simplicial Objects in Algebraic Topology*, Chicago Lectures in Mathematics, 1993.
- [28] J. P. May, *Pairings of Categories and Spectra*, Journal of Pure and Applied Algebra, Vol. 19, 1980, 299-346.
- [29] J. Milnor, *Introduction to Algebraic K-theory*, Annals of Math. Studies 7, Princeton University Press, 1974.
- [30] D. Perrot, *Secondary Invariants for Fréchet Algebras and Quasihomomorphisms*, Documenta Mathematica, Vol. 13, 2008, 275-363.

- [31] D. Quillen, *Higher Algebraic K-theory I*, Algebraic K-theory I (Battelle, 1972) (H. Bass, ed.), Lecture Notes in Math., vol. 341, Springer-Verlag, New York, Berlin, Heidelberg, 1973, 85-147.
- [32] J. Rosenberg, *Algebraic K-theory and Its Applications*, Graduate Texts in Math., vol. 147, Springer-Verlag, Berlin, Heidelberg, New York, 1994.
- [33] B. Simon, *Trace Ideals and Their Applications*, Cambridge University Press, 1979.
- [34] J. Stasheff, *H-spaces from a Homotopy Point of View*, Lecture notes in Mathematics, **161**, Springer-Verlag, Berlin, Heidelberg, New York, 1970.
- [35] U. Tillmann, *Factorization of the (Relative) Chern Character through Lie Algebra Homology*, K-theory **6**, 1992, 457-463.
- [36] B. L. Tsygan, *The homology of matrix Lie algebras over rings and the Hochschild homology*, Uspekhi Mat. Nauk 38, 1983, 217-218.
- [37] J. B. Wagoner, *Delooping Classifying Spaces in Algebraic K-theory*, Topology, **11**, 1972, 349-370.
- [38] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Graduate Texts in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 1983.
- [39] A. Wassermann, *Analysis of Operators*, Michaelmas, 2006.
- [40] C. Weibel, *Nil K-theory maps to cyclic homology*, Trans. AMS 303, 1987, 541-558.