Metrics on non-commutative spaces

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Chapter 1

Introduction

1.1 The general question.

The main purpose of this thesis is to explore the possibility of constructing Dirac operators which will induce the weak*-topology on the state space of a general C*-algebra. In the Chapter 3 of this thesis we present the results we have obtained for some special classes of C*-algebras. In the course of our study of different types of C*-algebras we also have obtained some general results which we have included in the first part of the thesis. This introduction is intended to offer a brief historical account of *metrics on non-commutative spaces* combined with an outline of each chapter of the thesis. It is well known that the Gelfand transform gives a 1-1 correspondence between commutative unital C*-algebras and compact topological spaces. This correspondence motivates investigations into non-commutative C*-algebras in order to find counterparts of classical notions on topological and geometrical spaces. For instance, A. Connes, [Co1], has shown us how Riemannian metrics on non-commutative spaces (C*-algebras) can be specified by means of a spectral triple. Although in this setting there is no underlying manifold on which one can obtain an ordinary metric, A. Connes, [Co1], has constructed in a simple way an ordinary metric on the state space of the C*-algebra, generalizing the Monge-Kantorovich metric on probability measures. He proposed the following definition:

Definition 1.1.1. Let $\mathcal{A}$ be a unital C*-algebra. An *unbounded Fredholm module* (H, D) over $\mathcal{A}$ is:

(C1): a Hilbert space H which is a left $\mathcal{A}$-module, that is, a Hilbert space H and a *-

representation of $\mathcal{A}$ on H;
(C1): an unbounded, self-adjoint operator D on H such that the set $\{a \in \mathcal{A} : [D, a]$is densely defined a
is norm dense in $\mathcal{A}$;
(C3): $(1 + D^2)^{-1}$ is a compact operator (i.e. D has a compact resolvent).
The triple $(\mathcal{A}, H, D)$ with the above description is called *a spectral triple.*

Still following A. Connes, if $\{a \in \mathcal{A} : \|[D, a]\| \leq 1\}/\mathbb{C}1$ is bounded, one can then introduce a metric on the state space of $\mathcal{A}$ by the formula

$$d(\varphi, \psi) = sup\{|\varphi(a) - \psi(a)| : a \in \mathcal{A}, \|[D, a]\| \leq 1\}.$$

We mentioned above a key observation due to A. Connes, [Co1], which he used to extend the notion of a metric to the non-commutative situation. We recall in the next proposition this crucial observation.

Proposition 1.1.2. *Let M be a compact, spin, Riemannian manifold, $\mathcal{A} = C(M)$, $H = L^2(M, S)$ and D the Dirac operator. Then the geodesic distance $d(P, Q)$ between two points P and Q of M is given by:*

$$d(P, Q) = sup\{|a(P) - a(Q)| : a \in \mathcal{A}, \|[D, a]\| \leq 1\}.$$

As mentioned above the metric introduced by A. Connes gives us, in the commutative case, the Monge-Kantorovich metric. Let us recall here the Monge-Kantorovich metric. Let ρ be an ordinary metric on the compact space X. The Lipschitz semi-norm L_ρ, determined by ρ is defined on $C(X)$ by:

$$L_\rho(f) := sup\{|f(x) - f(y)|/\rho(x, y) : x \neq y\}.$$

One can recover ρ from L_ρ by the relationship

$$\rho(x, y) = sup\{|f(x) - f(y)| : L_\rho(f) \leq 1\}.$$

A slight extension of this relationship defines a metric, say $\bar{\rho}$, on the space $\mathcal{P}(X)$ of probability measures on X. Explicitly,

$$\bar{\rho}(\mu, \nu) := sup\{|\int_X f d\mu - \int_X f d\nu| : f \in C(X), L_\rho(f) \leq 1\}$$

for all $\mu, \nu \in \mathcal{P}(X)$. This is the *Monge-Kantorovich metric* or the *Hutchinson metric* in the Fractal theory. The topology which it defines on $\mathcal{P}(X)$ coincides with the weak*-topology on $\mathcal{P}(X)$ coming from viewing it as the state space of the C*-algebra $C(X)$. Note also this well-known fact: for a compact space X its topology coincides with the weak*-topology coming from viewing the points of X as linear functionals on the algebra $C(X)$. Motivated by the case of ordinary compact metric spaces, it is natural to wonder if for a spectral triple, the topology from the metric on the state space coincides with the weak*-topology. If the metric topology coincides with the weak*-topology on the state space, then the metric topology should give the state space finite diameter, since the state space is compact for the weak*-topology. An elementary characterization of when the metric is bounded on the state space and furthermore when it induces the weak*-topology on this space was given by M. A. Rieffel, [Ri1], and B. Pavlović, [Pav]. This characterization reads:

Theorem 1.1.3. *Let (H, D) be an unbounded Fredholm module over a unital C*-algebra $\mathcal{A}$, and let the metric d on $\mathcal{S}(\mathcal{A})$ be defined by the formula:*

$$d(\varphi, \psi) = sup\{|\varphi(a) - \psi(a)| : a \in \mathcal{A}, \|[D, a]\| \leq 1\}.$$

for $\varphi, \psi \in \mathcal{S}(\mathcal{A})$.
(1) d is a bounded metric on $\mathcal{S}(\mathcal{A})$ if and only if

$$\{a \in \mathcal{A} : \|[D, a]\| \leq 1\}$$

has a bounded image in the quotient space $\mathcal{A}/\mathbb{C}1$, equipped with the quotient norm.
(2) the metric topology coincides with the w-topology if and only if the set*

$$\{a \in \mathcal{A} : \|[D, a]\| \leq 1\}$$

has a precompact image in the quotient space $\mathcal{A}/\mathbb{C}1$, equipped with the quotient norm.

1.2 The results in the thesis.

As we can see from the constructions above, a metric on the state space is based on a set $\mathcal{C}$ which is given by an expression like

$$\mathcal{C} = \{a \in \mathcal{A} : \|[D, a]\| \leq 1\}.$$

It seems natural to propose a definition of a metric in the non-commutative setting in terms of an object which is related directly to the algebra, and we do show that this is possible in some cases. Let us return to the theorem above. There the agreement between the metric topology and the weak*-topology is characterized by certain properties of the set $\mathcal{C}$ so our object should have similar properties. For a general unital and separable C*-algebra $\mathcal{A}$ it seems that any precompact, balanced and convex subset of $\mathcal{A}$ which separates the states on $\mathcal{A}$ can give us a metric on the state space which agrees with the weak*-topology. Our reasons for looking at such a set are partially due to our reading of works by M. A. Rieffel, [Ri1], [Ri2], and B. Pavlović, [Pav]. The key property in the above characterization is the pre compactness property. With this in mind we searched for and found a non-commutative version of *Arzelà-Ascoli theorem* which tells us how to determine pre compactness of a subset of a unital C*-algebra. Explicitly the theorem asserts that once we have given one precompact subset of $\mathcal{A}$, say $\mathcal{K}$ which separates the states then we will know all precompact subsets because they are not far away from $\mathcal{K}$. These considerations are contained in the first section of Chapter 2 of our thesis which consists of general results.

We now return to Theorem 0.3. After this characterization was established there were several attempts to verify for known spectral triples the agreement between the induced metric topology on the state space and the weak*-topology. A natural spectral triple to investigate was the one suggested by A. Connes, [Co1], which we will describe here. Consider a discrete group G endowed with a *length function*, i.e. a map $\ell : G \to \mathbb{R}_+$ such that:

1. $\ell(gh) \leq \ell(g) + \ell(h)$ for all $g, h \in G$;
2. $\ell(g^{-1}) = l(g)$ for all $g \in G$;
3. $\ell(e) = 0$.

Then we can define a Dirac operator D on $l^2(G)$ by $(D\xi)(g) = \ell(g)\xi(g)$. A. Connes proved that if the length function ℓ is a *proper length function*, i.e. $\ell^{-1}([0,c])$ is finite for each $c \in \mathbb{R}_+$, then $(l^2(G), D)$ is an unbounded Fredholm module for $C_r^*(G)$. As above we can define a *pseudo* metric d on $\mathcal{S}(C_r^*(G))$ by

$$d(\varphi, \psi) = sup\{|\varphi(a) - \psi(a)| : a \in \mathcal{S}(C_r^*(G)), \|[D,a]\| \leq 1\},$$

and note the only reason why d is not a metric is that it may not be finite. We will now focus on two classes of discrete groups for which one can construct a metric topology on the state space of the reduced C*-algebra which coincides with the weak*-topology, namely *groups which satisfy a Haagerup-type condition* and *groups with rapid decay* (or which satisfy the Haagerup inequality). M. A. Rieffel and N. Ozawa, [ORi], introduced the general setting of a *filtered C*-algebra* which satisfies a *Haagerup-type condition*, and proved in this set up that A. Connes' metric induces the weak*-topology. Close to N. Ozawa and M. A. Rieffel's article (ideas, techniques, groups to look at...even time for writing up) is our article in collaboration with Erik Christensen, [AC]. The joint paper with Erik Christensen is the core of the first section of the Chapter 2I of the thesis. We would like to sketch now the main ideas and techniques used in this article. To verify that the metric induced by the Dirac operator gives the state space finite diameter or, even more, gives the weak*-topology on the state space, one requires estimates on the norm of an element a of the C*-algebra compared with the norm of the commutator $[D, a]$. Using an estimate of the completely bounded norm of a certain Schur multiplier and some techniques concerning free groups due to U. Haagerup, [Haa], we proved the boundedness of the metric topology for a free non Abelian group. We first thought that once the boundedness question was settled the agreement with the weak*-topology would be easy to prove. A closer analysis shows that the problems involved are much more complex and of very difficult combinatorial nature. Since we have not been able to solve these problems we have looked for alternative definitions of metrics which will induce the weak*-topology on the state space of $\mathrm{C}_r^*(G)$. The most obvious thing to do seemed to be to restrict the attention to the analysis of discrete groups of rapid decay, [Jo]. The definition of this notion has inspired us to consider a relaxation of the way the Dirac operator proposed by A. Connes is used in the construction of a metric on the state space. Let us introduce here the notion of a group of rapid decay and our proposal for a metric induced by a Dirac operator. Before we state the definition we briefly explain the notation: for a discrete group G, let $\mathbb{C}G$ denote the group algebra and $\lambda : \mathbb{C}G \to B(l^2(G))$ the left regular representation.

Definition 1.2.1. A discrete group G is said to be of rapid decay (RD) if there exist a length function ℓ on G and positive reals C , k such that

$$\forall x \in \mathbb{C}G : \|\lambda(x)\| \leq C\Big(\sum_{g\in G}(1+\ell(g))^{2k}|x_g|^2\Big)^{\frac{1}{2}}.$$

Definition 1.2.2. Let G be a discrete group with a length function $\ell : G \to \mathbb{N}_0$ such that $\ell^{-1}(0) = \{e\}$. Let D denote the corresponding Dirac operator and δ the unbounded derivation on $\mathrm{C}^*_r(G)$ given by $\delta(a) =$closure($[D, a]$), when the commutator $[D, a]$ is bounded and densely defined. For any natural number k we define a seminorm L^k_D by

$$\text{domain}(L^k_D) \;=\; \text{domain}(\delta^k) \quad \text{and} \quad L^k_D(a) = \|\delta^k(a)\|.$$

For a group of rapid decay the suggested construction leads to a lot of metrics, which all give us the weak*-topology. To summarize:

Theorem 1.2.3. *Let G be a discrete group with a length function $\ell : G \to \mathbb{N}_0$ such that $\ell^{-1}(e) = 0$, ℓ is proper and G is of rapid decay with respect to ℓ. Then there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ the metric generated by L^k_D on $\mathcal{S}(\mathrm{C}^*_r(G))$ is bounded and the topology generated by the metric equals the w*-topology.*

We remark that for these metrics it is very easy to verify the agreement with the weak*-topology. The relations between the class of groups which satisfy the Haagerup-tye condition and the class of groups of rapid decay is unclear. However it was proved that the class of *word-hyperbolic groups* is contained in both classes. The inclusion in the class of groups with rapid decay is due to Jolissaint, [Jo], and the inclusion in the class of groups which satisfy a Haagerup-type condition is due to M. A. Rieffel and N. Ozawa, [ORi]. We also remark that the free groups $\mathbb{Z}^n$ with the word-length function for the standard basis doesn't satisfy a Haagerup-type condition if $n \geq 2$, as was proved by M. A. Rieffel and N. Ozawa [ORi], but it is easy to verify the fact that they are of rapid decay. Thus for $\mathbb{Z}^n$ our proposal via L^k_D gives a metric compatible with the weak*-topology, but M. A. Rieffel has already considered this case in [Ri3] and proved that A. Connes' metric has all of the right properties. His approach, very different than those from the articles [ORi] and [AC], involves the A. Connes' cosphere algebra.

An important aspect of the investigations in this thesis is the search for an extension Connes' construction of an unbounded Fredholm module based on a length function on a discrete group. As mentioned i Connes' book [Co2] the reduced group C*-algebra comes with a natural filtration induced by the word length and it also has a canonical trace state. Hence a natural candidate for a possible generalization of an unbounded Fredholm module is a filtered C*-algebra with a faithful trace. This set up, outlined by N. Ozawa and M. A. Rieffel [ORi], comes with a natural candidate for an unbounded Fredholm module, and also extends in some sense the model of a reduced group C*-algebra. To clarify the concepts, it is of importance to remark that for any discrete group G endowed with a length function there is a natural way to see $C_r^*(G)$ as a filtered C*-algebra, [ORi]. There are several well known C*-algebras fitting into this framework, such as the finitely generated C*-algebras, the uniformly hyperfinite C*-algebras (UHF), and the approximately finite dimensional C*-algebras (AF). It seems very natural to start with UHF C*-algebras. The structure here is so rich and the results so numerous, also on unbounded derivations, that a careful investigation should be possible. The second section of Chapter 3 of the thesis consists of our suggestions regarding possible constructions of unbounded Fredholm modules on UHF algebras which induce the weak*-topology on the state space. The ideas followed are of the same type as in the group C*-algebras case; we try to obtain estimates of the norm of an element a of the UHF C*-algebra compared to the norm of the commutator $[D, a]$. Our first attempt concerned the Dirac operator induced naturally by the *-filtration. To get more information about this kind of estimate we used a numerical experiment, via the mathematical software Maple. The statistical approach was done for the case of CAR C*-algebras and the results so far confirm very well our hopes for the relation between the norm of a and the norm of commutator $[D, a]$. Besides the Dirac operator induced naturally by the *-filtration, we were thinking of another construction which uses additional structure coming from a certain symmetry. We proved that this Dirac operator indeed induces the weak*-topology. The main idea in the construction given here is based on the simple observation that for any von Neumann algebra and an element in this algebra, the distance from the element to the center is equal to the distance from the element to the commutant of the algebra. Based on this we can introduce a *standard spectral triple* for the algebra $M_n(\mathbb{C})$ of complex $n \times n$ matrices.

Definition 1.2.4. Let $M_n(\mathbb{C})$ denote the complex $n \times n$ matrices, π_n the standard representation of $M_n(\mathbb{C})$ given as the GNS-representation of $M_n(\mathbb{C})$ with respect to the unique trace-state on the Hilbert space $H_n = L^2(M_n(\mathbb{C}), \frac{1}{n}\mathrm{tr})$ and T_n the self adjoint unitary operator on H_n which consists of transposing a matrix. Then the set (H_n, π_n, T_n) is called the standard spectral triple for $M_n(\mathbb{C})$.

Let us remark here that the idea behind the construction of this spectral triple for the algebra $M_n(\mathbb{C})$ can be extended so that the norm distance on the state space of any C*-algebra $\mathcal{A}$ can be recovered exactly. These considerations are contained in the Section 2 of Chapter 2. We have not seen a proposal for a spectral triple for the algebra $\mathcal{K}$ of compact operators on a separable Hilbert space. We will make a suggestion below based on an increasing sequence of finite dimensional subspaces of the underlying Hilbert space. The inspiration for a such suggestion came during our work on UHF C*-algebras. We have chosen a progression where the dimension is multiplied by some natural number larger than 1 in each step. The factor may vary from step to step so there are many possible choices. We have chosen this road because it relates to the natural choice which one can think of for UHF C*-algebras. There are, however, many possibilities for increasing sequences of products of natural numbers, so in some respects the proposed Dirac operators have very different properties. On the other hand any such operator, say D, will have the property that the set

$$\{a \in \mathcal{K} : \|[D, a]\| \leq 1\}$$

is a relatively norm compact subset of $\mathcal{K}$ and hence the sets are not too different as the non commutative Arzelà-Ascoli-Theorem, [AC] shows, and moreover the metric generated on the state space will be a metric generating the w*-topology on the state space. These considerations about compact operators are presented in Section 3 of Chapter 3 of the thesis.

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Chapter 2

General results

2.1 A non-commutative Arzelà -Ascoli Theorem

Given the notion that: *a noncommutative compact topological space is a unital C*-algebra*, it seems natural to propose a definition of a metric on such a noncommutative space in terms of an object which relates directly to the algebra and not only to its state space. For a general noncommutative, unital and separable C*-algebra $\mathcal{A}$ without any particularities it seems that any precompact balanced and convex subset of $\mathcal{A}$ which separates the states on $\mathcal{A}$ contains all the information needed to define a metric on the state space which will agree with the weak*-topology. We are motivated to consider such sets by both the work of Rieffel and Pavlović, and the main result of this section (Theorem 1.4).

Definition 2.1.1. Let $\mathcal{A}$ be a unital C*-algebra. A subset $\mathcal{C}$ of $\mathcal{A}$ is called a metric set if it is norm compact, balanced, convex and separates the states on $\mathcal{A}$.

A subset $\mathcal{B}$ of $\mathcal{A}$ is *balanced* if for any complex number μ such that $|\mu| \leq 1$, we have $\mu\mathcal{B} \subseteq \mathcal{B}$.

With this definition in hand one can easily construct metric sets for separable unital C*-algebras. For instance, a metric set in $C^*_r(G)$, where G is a countable group, $G = \{g_n \mid n \in \mathbb{N}\}$ could be given by the following expression, where $\overline{\text{conv}}$ denotes the closed convex hull.

$$\mathcal{C} := \overline{\text{conv}} \left(\bigcup_{n=1}^{\infty} \{\alpha\lambda_{g_n} + \beta\lambda^*_{g_n} \mid \alpha, \beta \in \mathbb{C} \quad \text{and} \quad |\alpha| + |\beta| \leq 1/n\} \right).$$

The next section will contain results which, hopefully, will justify this introduction of yet another concept.

The classical Arzelà-Ascoli Theorem gives a characterization of precompact subsets of $\mathrm{C}(X)$ for a compact topological space X. If X is a equipped with a metric ρ generating the topology on X one can construct a convex subset $\tilde{\mathcal{C}}$ of $\mathrm{C}(X)$ by

$$\tilde{\mathcal{C}} = \{ f \in \mathrm{C}(X) \mid \forall x, y \ \in X \ \ |f(x) - f(y)| \leq \rho(x, y) \ \}$$

This set is unbounded since any constant function belongs to $\tilde{\mathcal{C}}$. If one normalizes the set by considering the subset consisting of those elements which all vanish at a certain point x_0 then the classical Arzelá-Ascoli Theorem shows that the set, say $\mathcal{C}$, given by

$$\mathcal{C} = \{ f \in \mathrm{C}(X) \mid \forall x, y \ \in X \ \ |f(x) - f(y)| \leq \rho(x, y) \text{ and } f(x_0) = 0 \}$$

will be a compact balanced convex subset of $\mathrm{C}(X)$ which separates the points in X. The Arzelà-Ascoli-Theorem measures any other subset of $\mathrm{C}(X)$ against this set in order to see whether this subset is precompact or not. In the sequel we transfer this measuring process to the noncommutative case. The methods we use are elementary functional analytic duality results. We have wondered if this sort of result is valid in a much wider generality like operator spaces [ER], [Ke]. It seems that the validity of a generalization of Lemma 2.1.2 below, to this new setting is crucial. Before we start we want to introduce some more notation. We will be considering the self adjoint part of a unital C*-algebra which is denoted $\mathcal{A}_h$ and we want to think of the elements in $\mathcal{A}$ as affine complex w*-continuous functions on the state space $\mathcal{S}$ of $\mathcal{A}$. We will let $\mathbb{A}(\mathcal{S})$ denote the space of w*-continuous affine complex functions on $\mathcal{S}$, and for an element $a \in \mathcal{A}$, $\widehat{a}$ will denote the corresponding affine function in $\mathbb{A}(\mathcal{S})$. This presentation of $\mathcal{A}$ is called Kadison's functional representation of $\mathcal{A}$. It is well known that the functional representation is isometric on $\mathcal{A}_h$, but for a general element $a \in \mathcal{A}$ we only have the estimates

$$\|a\| \geq \sup |\widehat{a}(\varphi)| = \|\widehat{a}\| \geq \frac{1}{2}\|a\|.$$

In particular this shows that a subset $\mathcal{B}$ of $\mathcal{A}$ is bounded if and only if the subset $\widehat{\mathcal{B}}$ of $\mathbb{A}(\mathcal{S})$ is bounded. We remind the reader that a subscript attached to a Banach space like Y_μ means that we consider the closed ball of radius μ in Y, and Y^* means the dual space of Y. For the pair of Banach spaces like X and X^*, we will use the duality result known as the Bipolar Theorem. Here the polar of a set $B \subseteq X$ is denoted B° and defined by

$$B^\circ = \{ \gamma \in X^* \ : \ \forall b \in B, \ |\gamma(b)| \leq 1 \ \}$$

The Bipolar Theorem with respect to this polar then states that the bipolar $B^{\circ\circ}$, which now is a subset of X, is the smallest balanced, convex and norm closed set in X which contains B. We can now start the presentation of the generalization of the Arzelà-Ascoli Theorem. Our first lemma is closely connected to some fundamental results on states of a C*-algebra, namely, that a continuous linear self adjoint functional f can be decomposed in a unique way into a difference of two positive functionals f^+ and f^- such that

$$\|f\| = \|f^+\| + \|f^-\|.$$

Lemma 2.1.2. *Let $\mathcal{A}$ be a unital C*-algebra and $\mathcal{S}$ the state space of $\mathcal{A}$, then*

$$\mathcal{S} - \mathcal{S} = (\mathcal{A}_h^*)_2 \cap \{\mathbb{C}I\}^\perp.$$

Proof. The inclusion "$\subseteq$" is obvious. To prove the remaining inclusion "$\supseteq$", let us consider an arbitrary element f in $(\mathcal{A}_h^*)_2 \cap \{\mathbb{C}I\}^\perp$. For f in $(\mathcal{A}_h^*)_2$ we decompose $f = f^+ - f^-$ as a difference of two positive functionals such that $\|f\| = \|f^+\| + \|f^-\|$. If $f = 0$ we can write f as a difference $g - g$ where g is any state on the unital algebra $\mathcal{A}$. If $f \neq 0$ the condition $f(I) = 0$ implies that $0 \neq \|f^+\| = \|f^-\| = \frac{1}{2}\|f\| \leq 1$. Based on f^+ we can then define a positive functional g of norm $\|g\| = 1 - \|f^+\|$ by

$$g = \frac{(1 - \|f^+\|)}{\|f^+\|} f^+.$$

By construction it follows that $f^+ + g$ and $f^- + g$ are both states, and from the equality

$$f = (f^+ + g) - (f^- + g)$$

we can conclude that $f \in \mathcal{S} - \mathcal{S}$. The lemma follows. □

Following the ideas of Rieffel [Ri1], we deduce the following result

Lemma 2.1.3. *Let $\mathcal{A}$ be a unital C*-algebra, $\mathcal{S}$ the state space of $\mathcal{A}$ and $\mathcal{C}$ a norm compact subset of $\mathcal{A}$ which separates the points in the state space. Then for states φ, ψ on $\mathcal{A}$ the formula*

$$d_{\mathcal{C}}(\varphi, \psi) := \sup_{k \in \mathcal{C}} |(\varphi - \psi)(k)|$$

defines a metric on the state space $\mathcal{S}$ which generates the w-topology.*

Proof. The separation property and the compactness assumption show that $d_{\mathcal{C}}$ is a bounded metric on $\mathcal{S}$. The norm compactness of $\mathcal{C}$ and the boundedness of $\mathcal{S}$ further implies that the topology induced by $d_{\mathcal{C}}$ is a Hausdorff topology weaker than the compact w*-topology on $\mathcal{S}$. A well known theorem from topology then tells that the two topologies agree, and the lemma follows. □

We can now state and prove the main result of this section.

Theorem 2.1.4. *Let $\mathcal{A}$ be a unital C*-algebra and $\mathcal{C}$ a metric subset of $\mathcal{A}$. For any subset $\mathcal{H}$ of $\mathcal{A}$ the following conditions are equivalent*

(i) The set $\mathcal{H}$ is norm precompact.

(ii) The set of affine functions $\{\widehat{h} \in \mathbb{A}(\mathcal{S}) : \; h \in \mathcal{H}\}$ is bounded and equi continuous with respect to the w-topology on $\mathcal{S}$.*

(iii) The set $\mathcal{H}$ is bounded and for every $\varepsilon > 0$ there exists a real $N > 0$ such that

$$\mathcal{H} \subseteq \mathcal{A}_\varepsilon + N\mathcal{C} + \mathbb{C}I.$$

Proof. The equivalence between (i) and (ii) follows from the classical Arzelà-Ascoli Theorem and the fact, mentioned above, that $\mathcal{H}$ is bounded if and only if $\widehat{\mathcal{H}}$ is bounded.
To prove the equivalence between (ii) and (iii) we start with $(iii) \Rightarrow (ii)$. From the boundedness of $\mathcal{H}$ it follows that the set $\{\widehat{h} : \; h \in \mathcal{H}\}$ is bounded. To prove the equi continuity of this set let us fix an $\varepsilon > 0$ and find a positive real N which fulfills the condition (iii) with respect to $\frac{\varepsilon}{4}$. Moreover let φ and ψ be two states such that

$$d_{\mathcal{C}}(\varphi, \psi) < \frac{\varepsilon}{2N}.$$

then we will show that for any $h \in \mathcal{H}$, $|\widehat{h}(\varphi) - \widehat{h}(\psi)| \leq \varepsilon$. Now let h be an arbitrary element in $\mathcal{H}$. By (iii) we can find an element $a \in \mathcal{A}_1$, an element $c \in \mathcal{C}$ and a complex number μ such that

$$h = \frac{\varepsilon}{4}a + Nc + \mu I.$$

We then obtain

$$
\begin{aligned}
|\widehat{h}(\varphi) - \widehat{h}(\psi)| &= |(\varphi - \psi)(h)| && (2.1.1) \\
&\le |(\varphi - \psi)(\frac{\varepsilon}{4}a)| + |(\varphi - \psi)(Nc)| && (2.1.2) \\
&\le \frac{\varepsilon}{2} + N d_{\mathcal{C}}(\varphi, \psi) && (2.1.3) \\
&< \varepsilon, && (2.1.4)
\end{aligned}
$$

and the equi continuity of $\widehat{\mathcal{H}}$ has been established.
To prove the implication $(ii) \Rightarrow (iii)$ we first recall that $\mathcal{H}$ is bounded. Let $\varepsilon > 0$ be given and find, via the equi continuity assumption on $\widehat{\mathcal{H}}$, a $\delta > 0$ such that

$$\forall h \in \mathcal{H}\ \forall \varphi, \psi \in \mathcal{S} : \quad d_{\mathcal{C}}(\varphi, \psi) \le \delta \Rightarrow |(\varphi - \psi)(h)| = |\widehat{h}(\varphi) - \widehat{h}(\psi)| \le \varepsilon.$$

We will now use the bipolar theorem and remark that the expression $d_{\mathcal{C}}(\varphi, \psi) \le \delta$ exactly means that $\varphi - \psi \in \delta(\mathcal{C}^\circ)$. It is clear that $\varphi - \psi \in \mathcal{S} - \mathcal{S}$ and an application of Lemma 5.1 then shows that the implication above can just as well be expressed as

$$\forall h \in \mathcal{H}\ \forall \gamma \in (\mathcal{A}_h^*)_2 \cap \{\mathbb{C}I\}^\perp \cap \delta(\mathcal{C}^\circ) : \quad |\gamma(h)| \le \varepsilon.$$

This statement is not sufficient for our computations because it involves the space $\mathcal{A}_h^*$ rather that just $\mathcal{A}^*$. Since a functional on $\mathcal{A}$ vanishes on the identity I if and only both its hermitian and its skew hermitian part vanish on I, we can change from $\mathcal{A}_h^*$ to $\mathcal{A}^*$ at the cost of a factor of 2, so we have

$$\forall h \in \mathcal{H}\ \forall \gamma \in \mathcal{A}_2^* \cap \{\mathbb{C}I\}^\perp \cap \delta(\mathcal{C}^\circ) : \quad |\gamma(h)| \le 2\varepsilon.$$

Since all the sets involved are now convex and balanced the Bipolar Theorem can be applied very easily. Moreover, $\mathcal{C}$ is norm compact so any set of the form $\mathcal{A}_\varepsilon + \mathbb{C}I + N\mathcal{C}$ is norm closed, balanced and convex. The relation just established gives immediately the first inclusion below and the rest follows by some well known "polar techniques" and an application of the Bipolar Theorem.

$$
\begin{aligned}
\mathcal{H} &\subseteq 2\varepsilon\Big(\mathcal{A}_2^* \cap \{\mathbb{C}I\}^\perp \cap \delta(\mathcal{C}^\circ)\Big)^\circ && (2.1.5) \\
&= 2\varepsilon\Big(\mathcal{A}_{\frac{1}{2}} \cup\ \mathbb{C}I \cup \frac{1}{\delta}\mathcal{C}\Big)^{\circ\circ} && (2.1.6) \\
&= 2\varepsilon\overline{\mathrm{conv}}\Big(\mathcal{A}_{\frac{1}{2}} \cup \mathbb{C}I \cup \frac{1}{\delta}\mathcal{C}\Big) && (2.1.7) \\
&\subseteq 2\varepsilon\Big(\mathcal{A}_{\frac{1}{2}} + \mathbb{C}I + \frac{1}{\delta}\mathcal{C}\Big). && (2.1.8)
\end{aligned}
$$

Thus, given ε we have found a number $N = \frac{2\varepsilon}{\delta}$ such that

$$\mathcal{H} \subseteq \mathcal{A}_\varepsilon + \mathbb{C}I + N\mathcal{C},$$

proving the desired implication. $\square$

2.2 The norm metric on the state space

Let $\mathcal{A}$ be a unital C*-algebra. In this section we are studying the metric on the state space $\mathcal{S}(\mathcal{A})$ which is induced by the norm on the dual space $\mathcal{A}^*$ of $\mathcal{A}$.
It is well known from the Gelfand transform that for a commutative unital C*-algebra $C(X)$, where X is a compact topological space, the space X can be computed as the set of pure states equipped with the weak*-topology. This is in general seen as the guiding line for non commutative generalizations in the sense that for a spectral triple, we want the metric induced on the state space to generate the weak* topology. On the other hand for finite dimensional C*-algebras the 2 topologies agree, so we tried to look for a spectral triple for the algebra $M_n(\mathbb{C})$ of complex $n \times n$ matrices. Since Connes' suggestion for a spectral triple for a the reduced C*-algebra of a discrete group with a length function is based on the left regular representation, it seemed natural to try the "left regular" representation of $M_n(\mathbb{C})$ as left multipliers on $M_n(\mathbb{C})$ where this algebra is equipped with the inner product implemented by the trace state. As a Dirac operator the transposition operator on $M_n(\mathbb{C})$ seems to be a natural suggestion. It turned out that this procedure works very well and furthermore that one can extend this idea such that it is possible, for any C*-algebra $\mathcal{A}$, to construct a representation π of $\mathcal{A}$ on a Hilbert space H such that there exists a projection $P \in B(H)$ which has the property that the norm distance on the state space is recovered exactly if this projection P plays the role of the Dirac operator.

Lemma 2.2.1. *Let $\mathcal{A}$ be a unital C*-algebra and let ρ denote a faithful representation of $\mathcal{A}$ on a Hilbert space H. Let H_1 denote the Hilbert space tensor product $H_1 = H \otimes H$, S the flip on H_1 given by $S(\xi \otimes \eta) = \eta \otimes \xi$ and P the projection $P = (I + S)/2$. Then the representation π of $\mathcal{A}$ on H_1 given by the amplification $\pi(a) = \rho(a) \otimes I$ satisfies:*

$$\forall a = a^* \in \mathcal{A} : \quad \inf_{\lambda \in \mathbb{R}} \| a - \lambda I \| \; = \; \| [P, \pi(a)] \|$$

Proof. We will first transform the commutator slightly in order to ease the computations and show that the left side dominates the right one.

$$
\begin{aligned}
&\forall\lambda \in \mathbb{R}\ \forall a = a^* \in \mathcal{A} : \\
&\qquad \|[P,\pi(a)]\| \;=\; \frac{1}{2}\|[S,\pi(a)]\| \;=\; \frac{1}{2}\|S[S,\pi(a)]\| \\
&\qquad\qquad =\; \frac{1}{2}\|\pi(a) - S\pi(a)S\| = \frac{1}{2}\|\pi(a-\lambda I) - S\pi(a-\lambda I)S\| \\
&\qquad\qquad =\; \frac{1}{2}\|\rho(a-\lambda I)\otimes I - I\otimes\rho(a-\lambda I)\| \;\le\; \|a-\lambda I\|.
\end{aligned}
$$

In order to obtain the reverse inequality, for a certain λ, we use the computations above again but we will first remark that by spectral theory it follows that for $a = a^* \in \mathcal{A}$ with spectrum contained in the smallest possible interval $[\alpha, \beta] \subseteq \mathbb{R}$ one has

$$
\inf_{\lambda\in\mathbb{R}} \| \; a - \lambda I \; \| \;=\; \| \; a - \frac{\alpha+\beta}{2} \; \| \;=\; \frac{\beta-\alpha}{2}
$$

Let $\varepsilon > 0$ and chose unit vectors $\xi, \eta \in H$ such that $(\rho(a)\xi,\xi) \ge \beta - \varepsilon$ and $(\rho(a)\eta,\eta) \le \alpha + \varepsilon$. Then, $\xi\otimes\eta$ is a unit vector in H_1 and

$$
\begin{aligned}
\frac{1}{2}\|\rho(a)\otimes I - I\otimes\rho(a)\| \;&\ge\; \frac{1}{2}((\rho(a)\otimes I - I\otimes\rho(a))\xi\otimes\eta, \xi\otimes\eta) \\
&\ge\; \frac{(\beta-\alpha)}{2} - \varepsilon \\
&=\; \inf_{\lambda\in\mathbb{R}} \| \; a - \lambda I \; \| \; - \varepsilon.
\end{aligned}
$$

The lemma follows. □

The next lemma is well known ([Ri2]) and easy to prove.

Lemma 2.2.2. *Let A be a unital C^*-algebra then for any two states φ, ψ on A*

$$
\| \; \varphi - \psi \; \| \;=\; \sup\{ | \, (\varphi-\psi)(a) \, | : \; a = a^* \in \mathcal{A} \text{ and } \inf_{\lambda\in\mathbb{R}} \| \; \mathrm{a} - \lambda \mathrm{I} \; \| \; \le 1 \, \}.
$$

We can now combine the two lemmas above into the main result of this section.

Theorem 2.2.3. *Let $\mathcal{A}$ be a C^*-algebra and ρ a faithful non-degenerate representation of $\mathcal{A}$ on a Hilbert space H. Then there exists a representation π of $\mathcal{A}$ on a Hilbert space H_1 which is an amplification of ρ, and a projection P in $B(H_1)$ such that for any pair of states φ, ψ on $\mathcal{A}$*

$$
\| \; \varphi - \psi \; \| \;=\; \sup\{ | \, (\varphi-\psi)(a) \, | : \; a = a^* \in \mathcal{A} \text{ and } \| \; [\mathrm{P}, \pi(\mathrm{a})] \; \| \; \le 1 \, \}.
$$

If H is separable and the commutant of $\rho(\mathcal{A})$ is a properly infinite von Neumann algebra then $\pi = \rho$ is possible. If $\mathcal{A} = M_n(\mathbb{C})$ and ρ is the standard representation of $\mathcal{A}$ on $L^2(M_n(\mathbb{C}), \frac{1}{n}\mathrm{tr})$ then $\pi = \rho$ is possible and the projection $P = \frac{1}{2}(I + T)$ where T is the transposition on $M_n(\mathbb{C})$ can be used.

Proof. If $\mathcal{A}$ has no unit then we add a unit in order to obtain a unital C*-algebra $\tilde{\mathcal{A}}$. It is well known that the state space of $\mathcal{A}$ embeds isometrically into the state space of $\tilde{\mathcal{A}}$. We can then deduce the result for the non-unital case from the unital one by remarking that both of the expressions

$$|\,(\varphi - \psi)(a)\,| \text{ and } \|\,[\mathrm{P}, \pi(\mathrm{a})]\,\|$$

are left unchanged if a is replaced by $(a - \lambda I)$. Let us then assume that $\mathcal{A}$ is unital. Then, by Lemma 2.2.1 we can chose to amplify ρ by the Hilbert dimension of H, but less might do just as well. It all depends on the multiplicity of the representation ρ, or rather whether the commutant of $\rho(\mathcal{A})$ contains a subfactor isomorphic to $B(H)$. In particular, this situation occurs if H is separable and the commutant is properly infinite. If $\mathcal{A} = M_n(\mathbb{C})$ and ρ is the "left regular representation" of $\mathcal{A}$ on $L^2(M_n(\mathbb{C}), \mathrm{tr})$, then this Hilbert space is naturally identified with $\mathbb{C}^n \otimes \mathbb{C}^n$ via the mapping $M_n(\mathbb{C}) \ni a \to \sum_{i=1}^n \sum_{j=1}^n a_{ij} e_j \otimes e_i$, where the elements e_i denote the elements of the standard basis for $\mathbb{C}^n$. From here it is easy to see that the flip on the Hilbert space tensor product is nothing but the transposition operator on $M_n(\mathbb{C})$. The Lemma 2.2.1 now applies directly for $\pi = \rho$ and the projection $P = \frac{1}{2}(I + T)$, where T is the transposition operator on $M_n(\mathbb{C})$. In the arguments above we have used the trace rather than the trace-state as stated in the formulation of the theorem, the reason being that the identification of $M_n(\mathbb{C})$ with $\mathbb{C}^n \otimes \mathbb{C}^n$ fits naturally with the trace. For later use in Section 4.2 (where we discuss UHF C*-algebras) the trace state is the natural object. $\square$

Chapter 3

Spectral triples for some C*-algebras

3.1 Group C*-algebras

3.1.1 Basic definitions and results

In the sequel we are interested in discrete groups G endowed with a *length function* ℓ, i.e. a map $\ell : G \to \mathbb{R}_+$ such that:

1. $\ell(gh) \leq \ell(g) + \ell(h)$ for all $g, h \in G$;

2. $\ell(g^{-1}) = \ell(g)$ for all $g \in G$;

3. $\ell(e) = 0$.

Most of our investigations deal with properties of the reduced group C*-algebra of G, $C_r^*(G)$. We now recall its definition. Consider the left regular representation λ of the discrete group G on $l^2(G)$. The C*-algebra $C_r^*(G)$ is the norm closure in $B(l^2(G))$ of the linear span of the left translation unitaries $\{\lambda(s) : s \in G\}$. We refer to chapter 6 of [KR] for the basic properties of this C*-algebra, but we will use a slightly different notation which is inspired by Connes' presentations in [Co1] and [Co2]. This means that for an $x \in \mathbb{C}G$ we will write $\lambda(x) = \sum_g x(g)\lambda_g$ for the convolution operator on $l^2(G)$, and for $g \in G$, δ_g denotes the natural basis element in $l^2(G)$. Any element $x \in C_r^*(G)$ has a unique representation in $l^2(G)$ by $x \to x\delta_e$ so in a natural way we have

$$l^1(G) \subseteq \mathrm{C}_r^*(G) \subseteq l^2(G)$$

and, for $x \in l^1(G)$,

$$\|x\|_2 \leq \|\lambda(x)\| \leq \|x\|_1.$$

There is a subclass of discrete groups of particular interest for our purposes, namely the groups of *rapid decay*. For a group of rapid decay one has a kind of inverse to the first inequality as can be seen in the next definition.

Definition 3.1.1. A discrete group G is said to be of rapid decay (RD) if there exists a length function ℓ on G and positive reals $C,\ k$ such that

$$\forall x \in \mathbb{C}G : \|\lambda(x)\| \leq C\Big(\sum_{g \in G}(1+\ell(g))^{2k}|x_g|^2\Big)^{\frac{1}{2}}.$$

Such an inequality is very powerful as we shall see.
Important examples of discrete groups are of rapid decay. In the very innovative paper [Haa], Haagerup proved that the free non-Abelian groups $\mathbb{F}_n$ are of rapid decay with $C = 2$ and $k = 2$. Also, the free Abelian groups $\mathbb{Z}^k$ are of rapid decay, and one can in fact get an estimate dominating the norm $\|x\|_1$. For $k = 1$ this is obvious since for $x \in \mathbb{C}\mathbb{Z}$

$$\sum_{n \in \mathbb{Z}} |x_n| \leq \Big(\sum_{n \in \mathbb{Z}}(1+|n|)^2|x_n|^2\Big)^{\frac{1}{2}}\Big(\sum_{n \in \mathbb{Z}}(1+|n|)^{-2}\Big)^{\frac{1}{2}}.$$

The article [Jo] by Jolissaint contains many results on groups of rapid decay. Among them is a proof that a discrete group is of rapid decay if it is of polynomial growth with respect to some set of generators and the corresponding length function. Also the word hyperbolic groups of Gromov are all of rapid decay. This was proved by Jollisaint but the proof can also be found in [Co2], Theorem 5, p.241. As mentioned before, Connes defines in [Co1] a metric on a non-commutative C*-algebra via an unbounded Fredholm module. For a discrete group G with a proper length function ℓ he obtains in a very easy way an unbounded Fredholm module as follows: the Fredholm module for $C^*_r(G)$ is the Hilbert space $l^2(G)$ and the Dirac operator D on $l^2(G)$ is the selfadjoint unbounded multiplication operator which multiplies $\xi \in l^2(G)$ by ℓ pointwise.

Definition 3.1.2. Let G be a discrete group with a length function ℓ and let $\mathcal{S}$ denote the state space of $C^*_r(G)$. Then, $d_\ell : \mathcal{S} \times \mathcal{S} \to [0, \infty]$ is defined by

$$d_\ell(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : \ a \in C^*_r(G), \|[D, a]\| \leq 1\}.$$

A computation involving the properties of a length function shows that for any pair $g \in G$ and $\xi \in l^2(G)$ we have

$$\Big([D, \lambda_g]\xi\Big)(k) = \Big(\ell(k) - \ell(g^{-1}k)\Big)\xi(g^{-1}k).$$

Thus $\|[D, \lambda_g]\| \leq \ell(g)$ and we see that d_ℓ must separate the points in $\mathcal{S}$. It is not clear if $d_\ell(\varphi, \psi) < \infty$ always, but otherwise d_ℓ behaves exactly as a metric on $\mathcal{S}$, so we will call d_ℓ a possibly infinite metric on $\mathcal{S}$.

3.1.2 Schur multipliers on $B(H)$

We stressed in the introduction the importance for our investigations of certain Schur multipliers. We will now describe the Schur multipliers and the way in which they relate to the problems we are considering. We will let $M_n(\mathbb{C})$ denote the $n \times n$ complex matrices. A matrix $A \in M_n(\mathbb{C})$ induces a mapping $A_s : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by the Schur multiplication which is given by the expression, $M_n(\mathbb{C}) \ni X = (x_{ij}) \to (a_{ij}x_{ij}) = A_s(X) \in M_n(\mathbb{C})$. Here we will consider a generalization of this mapping to operators on an abstract Hilbert space H, which for some discrete group G is decomposed into a sum of orthogonal subspaces $H_g, g \in G, H = \underset{g}{\oplus} H_g$. It should be remarked that for any $g \in G$ we do permit that $H_g = 0$, and we want to emphasize that the results in this section are designed to work for the well known Abelian group $\mathbb{Z}$; the work in the in rest of the paper aims at general discrete groups of rapid decay. Despite the fact that this section really is devoted to a result on the group $\mathbb{Z}$, we present the first result in terms of a general discrete group G, a Hilbert space H and a decomposition of H indexed by G. In this setting we have a decomposition of $H = \underset{g}{\oplus} H_g$ and we get a matrix decomposition of the operators in $B(H)$ such that for any x in $B(H)$ we can write $x = (x_{s,t}), s, t \in G$ where each $x_{s,t}$ is in $B(H_t, H_s)$. It is not possible to generalize the Schur multiplication directly to this setting since $B(H_t, H_s)$ is not an algebra unless $s = t$. On the other hand, it is possible to multiply any operator $x_{s,t} \in B(H_t, H_s)$ by a complex scalar, so for an infinite scalar matrix $\Lambda = (\lambda_{s,t})$ it is possible to perform a formal Schur multiplication $B(\oplus H_g) \ni x = (x_{s,t}) \to (\lambda_{s,t}x_{s,t}) = \Lambda_s(x)$. The latter matrix may not correspond to a bounded operator, but the product is well defined as an infinite matrix and we will call it a formal Schur product. The theorem just below provides a criterion on Λ for the boundedness of the Schur product $\Lambda_s(x)$

for any x in $B(H)$. We do a little more since we do compute the so called completely bounded norm of $\Lambda_s : B(H) \to B(H)$. The theory connected to completely bounded operators is described in Paulsen's book [Pau]. In order to explain this concept in brief we consider for $n \in \mathbb{N}$ the mapping $(\Lambda_s)_n : M_n(\mathbb{C}) \otimes B(H) \to M_n(\mathbb{C}) \otimes B(H)$ given by $(\Lambda_s)_n = \mathrm{id}_{M_n(\mathbb{C})} \otimes \Lambda_s$. If the sequence of norms defined by $\|\Lambda_s\|_n := \|(\Lambda_s)_n\|$ is bounded, Λ_s is said to be completely bounded and the completely bounded norm, $\|\Lambda_s\|_{cb}$ is given as $\sup_n \|\Lambda_s\|_n$. Our criterion for $\|\Lambda_s\|_{cb}$ to be finite is based upon a generalization of a theorem by M. Bożejko and G. Fendler [BF]. In Pisier's book [Pi], he presents this result in Theorem 6.4. This theorem deals with the situation where each of the summands H_g above is one-dimensional, i. e. $H_g = \mathbb{C}$.

Theorem 3.1.3. *Let G be a discrete group, H be a Hilbert space which is decomposed into a sum of orthogonal subspaces $H_g, g \in G$, and let φ be a complex function on G. If the linear operator $T_\varphi : \lambda(\mathbb{C}G) \to \lambda(\mathbb{C}G)$ which is defined by $T_\varphi(\lambda_g) = \varphi(g)\lambda_g$ extends to a completely bounded operator on $C^*_r(G)$, then for the matrix Λ given by $\Lambda = (\lambda_{s,t})$ $s, t \in G, \lambda_{s,t} = \varphi(st^{-1})$, the mapping Λ_s is completely bounded, $\|\Lambda_s\|_{cb} \leq \|T_\varphi\|_{cb}$ and Λ_s is an ultraweakly continuous, or normal, operator on $B(H)$.*

Proof. Suppose that T_φ is completely bounded. Then, for the case where $H_g = \mathbb{C}$ for every $g \in G$, the result follows from [Pi]. The proof of the scalar case, as presented in the proof of the implication [(i) $\Rightarrow$ (iii)] of Theorem 6.4 of [Pi], shows that there exists a Hilbert space K and two functions, say ξ and η, on G with values in K such that

$$\varphi(st^{-1}) = (\xi(t), \eta(s)) \quad \forall s, t \in G$$

and

$$\|\xi(g)\| \leq \sqrt{\|T_\varphi\|_{cb}}\,, \quad \|\eta(g)\| \leq \sqrt{\|T_\varphi\|_{cb}} \quad \forall g \in G.$$

We will now turn to the operator Λ_s and show that it is completely bounded. The proof of this fact can be obtained as a modification of a part of the proof of Theorem 5.1 of [Pi]. In fact the representation of φ we have obtained above makes it possible to construct operators, say x and y, in $B(H, H \otimes K)$ such that Λ_s can be expressed as a completely bounded operator in terms of these operators. We recall that $H = \bigoplus_g H_g$ and define the operators x and y on a vector $\alpha = (\alpha_g)_{g \in G}$ by $x\alpha = (\alpha_g \otimes \xi(g))_{g \in G}$ and $y\alpha = (\alpha_g \otimes \eta(g))_{g \in G}$, and we find that both operators are of norm at most $\sqrt{\|T_\varphi\|_{cb}}$. Let π

denote the representation of $B(H)$ on $H \otimes K$ which is simply the amplification $a \to a \otimes I_K$. Then, an easy computation shows that for any pair of vectors $\alpha = (\alpha_g),\ \beta = (\beta_g)$ from $H = \underset{g}{\oplus} H_g$ and any a in $B(H)$ we have

$$(\pi(a)x\alpha, y\beta) = \sum_{s,t \in G} (a\alpha_t, \beta_s)(\xi(t), \eta(s)) \tag{3.1.1}$$

$$= \sum_{s,t \in G} (\varphi(st^{-1})a_{s,t}\alpha_t, \beta_s) \tag{3.1.2}$$

$$= (\Lambda_s(a)\alpha, \beta). \tag{3.1.3}$$

Consequently, $\Lambda_s(a) = y^*\pi(a)x$ and the cb-norm of Λ_s is at most $\|T_\varphi\|_{cb}$. The concrete description $\Lambda_s(\cdot) = y^*\pi(\cdot)x$ where π is just an amplification shows that Λ_s is ultraweakly continuous. □

Corollary 3.1.4. *If G is an Abelian discrete group and T_φ extends to a bounded operator on $C_r^*(G)$ then the mapping Λ_s is completely bounded, $\|\Lambda_s\|_{cb} \leq \|T_\varphi\|$ and Λ_s is an ultraweakly continuous, or normal, operator on $B(H)$*

Proof. We have to prove that T_φ extends to a completely bounded mapping if it extends to a bounded mapping and that the two norms on T_φ agree. In order to do so we remark that for the compact Abelian dual group $\widehat{G}$ and any natural number k we have $\mathrm{C}_r^*(G) \otimes M_k(\mathbb{C}) = \mathrm{C}(\widehat{G}, M_k(\mathbb{C}))$, the continuous $M_k(\mathbb{C})$ valued functions on $\widehat{G}$. Then for any finite sum $x = \sum \lambda_g \otimes m_g$ in $\lambda(\mathbb{C}G) \otimes M_k(\mathbb{C})$ we have

$$\|x\| = \max\{\|\sum_g \chi(g)m_g\|_{M_k(\mathbb{C})} :\ \chi \in \widehat{G}\}.$$

The norm in $M_k(\mathbb{C})$ is determined by the functionals of norm one on this algebra, so let $M_k(\mathbb{C})_1^*$ denote this unit ball and we get

$$\|x\| = \max\{|\sum_g \chi(g)\psi(m_g)| :\ \chi \in \widehat{G} \text{ and } \psi \in M_k(\mathbb{C})_1^*\}.$$

Let us now suppose that T_φ extends to a bounded operator on the group algebra of norm at most 1. For x as above and of norm at most 1 in $\mathrm{C}_r^*(G) \otimes M_k(\mathbb{C})$, we have that for any pair $\chi,\ \psi$ as above $|\sum_g \chi(g)\psi(m_g)| \leq 1$. Hence, for this fixed ψ we get the estimate $\|\sum_g \psi(m_g)\lambda_g\| \leq 1$ in $\mathrm{C}_r^*(G)$. Since $\|T_\varphi\| \leq 1$ we also get $\|\sum_g \varphi(g)\psi(m_g)\lambda_g\| \leq 1$, but this holds for any ψ so we can go back and note that T_φ is completely bounded of norm at most 1. □

The following corollary shows that a square summable function φ on a commutative discrete group G induces a completely bounded operator Λ_s. Besides these functions and the positive definite functions on G, we do not know of any other general results which can guarantee the complete boundedness of Λ_s.

Corollary 3.1.5. *Let G be an Abelian discrete group, $\varphi \in l^2(G)$ and let $\Lambda : G \times G \to \mathbb{C}$ be given by $\Lambda(s,t) = \varphi(st^{-1})$. Then Λ_s is completely bounded and $\|\Lambda_s\|_{cb} \leq \|\varphi\|_2$.*

Proof. The operator T_φ on $\mathrm{C}^*_r(G)$ can, when we look at the latter algebra as $\mathrm{C}(\widehat{G})$, be expressed as the convolution operator implemented by the Fourier transform, $\hat{\varphi} \in \mathrm{L}^2(\widehat{G})$. Here we have chosen the probability Haar measure on the compact group $\widehat{G}$ such that the Fourier transform is an isometric operator between the two Hilbert spaces. It follows directly from The Cauchy-Schwarz inequality that the norm of the convolution operator is dominated by the norm $\|\hat{\varphi}\|_2$, and the corollary follows. $\square$

The purpose of the previous corollary is actually to compute the norm of the partial inverse of certain derivations on $B(H)$. Let D be a possibly unbounded selfadjoint operator on $B(H)$ with spectrum contained in the set of integers $\mathbb{Z}$. Then, the Hilbert space H decomposes as a direct sum of the eigenspaces, say H_m, of D. Many of these spaces may vanish, but we nonetheless write $H = \underset{m\in\mathbb{Z}}{\oplus} H_m$. We will be able to use the results above concerning the norm of certain Schur multipliers together with this decomposition. We wish to give a description of a bounded operator $a \in B(H)$ which has the property that the commutator $[D, a]$ is bounded and of norm at most 1. Clearly, all bounded operators which commute with D must play a special role in this set up. This set is a von Neumann algebra and consists of the operators in the main diagonal of $B(H)$, when the latter algebra is viewed as infinite matrices with respect to the decomposition $H = \underset{m\in\mathbb{Z}}{\oplus} H_m$. We will let $\mathfrak{D}_o$ denote the commutant of D and for $k \in \mathbb{Z}$ we will define the k'th diagonal of $B(H)$ by

$$\mathfrak{D}_k = \{(x_{ij}) \in B(H) \,|\, i - j \neq k \Rightarrow x_{ij} = 0\}.$$

For $k \in \mathbb{Z}$ there is a natural projection of $B(H)$ onto $\mathfrak{D}_k$, say P_k given by the expressions

$$P_k((x_{i,j}))_{m,n} = \begin{cases} x_{m,n} & \text{if } m - n = k \;, \\ 0 & \text{if } m - n \neq k \;. \end{cases}$$

If one computes $P_k(x)^*P_k(x)$ it is easy to realize that P_k is a projection onto the k'th diagonal and of norm at most one. The problem we are facing is analogous to well known problems concerning convergence of Fourier series; it is not easy to give norm estimates of the norm of a general finite sum $\sum_{k\in C} P_k(x)$. If we disregard convergence questions for some time, it follows from elementary algebraic manipulations that for an operator $a = (a_{m,n}) \in B(H)$, the commutator $[D, a]$ must have the formal infinite matrix $c = (c_{m,n})$ given by $c_{m,n} = (m - n)a_{m,n}$. So, at least formally, we can write

$$[D, a] \;=\; \sum_{k\in\mathbb{Z}} kP_k(a),$$

and we see that this operator on $B(H)$ is in fact an unbounded Schur multiplier. Further it follows that

$$a - P_0(a) \;=\; \sum_{k\in\mathbb{Z} \text{ and } k\neq 0} \frac{1}{k}P_k([D, a]).$$

Hence, the partial inverse to the derivation $B(H) \ni a \to [D, a]$ is a Schur multiplier which according to the results above will turn out to be completely bounded. With this notation in mind we can offer norm estimates for such sums in the next theorem. The theorem is a generalization of the well known fact that the Fourier series for a differentiable 2π periodic function on $\mathbb{R}$ is uniformly convergent.

Theorem 3.1.6. *Let D be a self adjoint operator on a Hilbert space H such that the spectrum of H is contained in $\mathbb{Z}$ and let $\mathcal{C} = \{a \in B(H) : \ \|[D, a]\| \leq 1$ and $\rho_0(a) = 0\}$. Then:*

(i) every element in $\mathcal{C}$ is of norm at most $\frac{\pi}{\sqrt{3}}$;

(ii) for $a \in B(H)$ such that $\|[D, a]\| \leq 1$ the sum $\sum_{m\in\mathbb{Z}} P_m(a)$ is norm convergent and $\|\sum_{|m|>k} P_k(a)\| \leq \sqrt{\frac{2}{k}} \quad \forall k \in \mathbb{N}$.

Proof. Suppose $a \in B(H)$ satisfies $\|[D, a]\| \leq 1$ and $\rho_0(a) = 0$. Then the first statement in the theorem is, as we shall see, just a special case of the second corresponding to $k = 0$, although the estimates are slightly different. Let then $k \in \mathbb{N}_0$ be given and consider the set of Hilbert spaces $H_m, m \in \mathbb{Z}$, where the space H_m is defined as above, i.e. the eigenspace for D corresponding to the eigenvalue m. We can then apply Corollary 3.1.5

with $G = \mathbb{Z}$ and the function $\varphi_k : \mathbb{Z} \to \mathbb{R}$ given by

$$\varphi_k(m) = \begin{cases} m^{-1}, & \text{if } |m| > k \ , \\ 0 & \text{if } |m| \leq k \ . \end{cases}$$

This yields

$$\|a - \sum_{i=-k}^{k} P_i(a)\| \leq \|[D, a]\| \|\varphi_k\|_2 \leq (2 \sum_{j>k} j^{-2})^{\frac{1}{2}}.$$

After recalling the well known sum $\sum_{j \in \mathbb{N}} j^{-2} = \frac{\pi^2}{6}$ for $k = 0$ and the integral estimate $\sum_{j>k} j^{-2} < \frac{1}{k}$ for $k > 0$, the theorem follows. □

Remark 3.1.7. The proof of Theorem 3.1.6 depends on the fact that the spectrum of D is contained in the Abelian discrete group $\mathbb{Z}$. This is not likely to be a relevant condition for a result of this type and we believe that this theorem must have a more general version which is valid for an unbounded self adjoint operator whose spectrum consists of points $(s_k)_{k \in \mathbb{Z}}$ such that $|s_k| \to \infty$ for $|k| \to \infty$ and $\inf\{|s_m - s_n| \mid m, \ n \in \mathbb{Z}\} > 0$. We have already mentioned the fact that an estimate similar to the one above does exist for ordinary differentiation on C($\mathbb{T}$), but we have not found a general operator theoretic treatment of this problem.

3.1.3 Group C*-algebras as non-commutative compact metric spaces

This section is mainly devoted to the study of some metrics on the state space of a C*-algebra which is generated by the left regular representation of a discrete group of rapid decay. We will start by recalling Connes' construction [Co1] which defines a metric on the state space of a discrete group C*-algebra in terms of an unbounded Fredholm module. Let G be a discrete group with a length function $\ell : G \to \mathbb{N}_0$ such that, in the language of Section 3.1, ℓ is proper and G is of rapid decay with constants C, k with respect to ℓ. As described in [Co2] p. 241 such a length function ℓ on a discrete group G with values in $\mathbb{N}_0$ induces a decomposition of $\ell^2(G)$ into an orthogonal sum of subspaces H_m, each one being the closed linear span of the basis vectors δ_g for which $\ell(g) = m$. The Dirac operator on $\ell^2(G)$ is the self adjoint unbounded operator D which is the closure of the operator D_0 defined on the linear span of the basis vectors $\delta_g,\ g \in G$ and acts by $D_0\delta_g = \ell(g)\delta_g$. The operator D clearly has its spectrum contained in $\mathbb{Z}$, the eigenspaces all vanish for $m < 0$ and are equal to H_m for $m \geq 0$. Theorem 3.1.6 is designed to deal with this situation, but it does not work as well as expected. The theorem provides a norm estimate of $\|a - P_0(a)\|$ in terms of $\|[D, a]\|$ for an a in $\mathrm{C}^*_r(G)$, and we believed that from this it would be easy to get an estimate of $\|P_0(a)\|$, because a group is such a rigid object. Unfortunately this is not so and we can only get an estimate which also takes care of the main diagonal part P_0 if the group is a one of the free non Abelian groups $\mathbb{F}_n$.

Theorem 3.1.8. *Let G be a free non Abelian group on finitely many generators and ℓ the natural length function on G. Then, the metric d_ℓ on the state space $\mathcal{S}(\mathrm{C}^*_r(G))$ is bounded, and the diameter of the state space is at most 5.*

Proof. We will prove the boundedness of the metric by studying the convex and balanced subset $\mathcal{C}$ of $\mathrm{C}^*_r(G)$ given by $\mathcal{C} = \{a \in \mathrm{C}^*_r(G) :\ \|[D, a]\| \leq 1\}$. By Theorem 3.1.6 we know that for $a \in \mathcal{C}$ we will have $\|a - P_0(a)\| \leq \frac{\pi}{\sqrt{3}}$, so we only have to get an estimate of the $\|P_0(a)\|$. In order to control $\|P_0(a)\|$ we first restrict to the case where a is of finite support in G and make the extra assumption that $(a\delta_e, \delta_e) = 0$. If the second assumption does not hold we simply subtract the corresponding multiple of the unit from a. This operation has, of course, no effect on the commutator $[D, a]$. Since we know by assumption that this commutator is of norm at most 1, we get the first estimate,

$$1 \geq \|[D, a]\delta_e\|^2 = \sum_{g \in G} \ell(g)^2 |a(g)|^2.$$

Let us now pick a unit vector $\xi \in H_m$ and let F_m denote the orthogonal projection from H onto H_m. We can now try to estimate $\|P_0(a)\|$ by estimating $F_m a\xi$. Let $g \in G$ be of length m. Then g, can be expressed uniquely in terms of generators as $g = g_1 g_2 \dots g_m$. When we have to compute the value of the convolution $a * \xi(g)$ we must remember that ξ is supported on words of length m, so the sum will be an expression of the type

$$a * \xi(g) = \sum_{k=1}^{m} \sum_{\{s_1, \dots, s_k\ | s_k \neq g_k, g_{k+1}^{-1}\}} a(g_1 \dots g_k s_k^{-1} \dots s_1^{-1}) \xi(s_1 \dots s_k g_{k+1} \dots g_m).$$

We can now imitate a trick from the proof of [Haa] Lemma 1.3. In order to do so we define for a fixed $k, 1 \leq k \leq m$ (remember $a(e) = 0$ so $k > 0$) a function b_k supported on words of length k and a vector η_{m-k} supported on words of length $m-k$ by

$$b_k(g_1 \dots g_k) = (\sum_{\{s_1,\dots,s_k \,|s_k \neq g_k\}} |a(g_1 \dots g_k s_k^{-1} \dots s_1^{-1})|^2\,)^{\frac{1}{2}}, \tag{3.1.4}$$

$$b_m(g_1 \dots g_m) = |a(g_1 \dots g_m)| \tag{3.1.5}$$

$$\eta_{m-k}(g_{k+1} \dots g_m) = (\sum_{\{s_1,\dots,s_k \,|s_k \neq g_{k+1}^{-1}\}} |\xi(s_1 \dots s_k g_{k+1} \dots g_m)|^2\,)^{\frac{1}{2}} \tag{3.1.6}$$

$$\eta_0(e) = \|\xi\|_2 = 1. \tag{3.1.7}$$

With this in hand we get

$$|a * \xi(g)| \leq \sum_{k=1}^{m} b_k(g_1 \dots g_k)\eta_{m-k}(g_{k+1} \dots g_m) = (\sum_{k=1}^{m} b_k * \eta_{m-k})(g).$$

As in [Haa] we will let χ_m denote the characteristic function on the words of length m, and we will further use the statement contained in Lemma 1.3 of [Haa] which says that for functions like b_k which are supported on words of length k and the η_{m-k} which are supported on words of length $m-k$ one has

$$\|b_k * \eta_{m-k}\chi_m\|_2 \leq \|b_k\|_2 \|\eta_{m-k}\|_2.$$

A combination of the inequalities above then yields

$$\begin{aligned}
\|Q_m a\xi\|_2 &= \|a * \xi\chi_m\|_2 && (3.1.8)\\
&\leq \|\sum_{k=1}^{m} b_k * \eta_{m-k}\chi_m\|_2 && (3.1.9)\\
&\leq \sum_{k=1}^{m} \|b_k * \eta_{m-k}\chi_m\|_2 && (3.1.10)\\
&\leq \sum_{k=1}^{m} \|b_k\|_2\|\eta_{m-k}\|_2 && (3.1.11)\\
&\leq \sum_{k=1}^{m} \|b_k\|_2, \quad \text{since } \|\xi\|_2 = 1 && (3.1.12)\\
&= \sum_{k=1}^{m} k\|b_k\|_2(1/k) && (3.1.13)\\
&\leq (\sum_{k=1}^{m} k^2\|b_k\|_2^2)^{\frac{1}{2}}(\sum_{k=1}^{m} k^{-2})^{\frac{1}{2}} && (3.1.14)\\
&\leq \frac{\pi}{\sqrt{6}}(\sum_{k=1}^{m} k^2 \sum_{g,\ell(g)=2k} |a(g)|^2)^{\frac{1}{2}} && (3.1.15)\\
&\leq \frac{\pi}{\sqrt{6}}(\frac{1}{4}\sum_{g\in G} \ell(g)^2|a(g)|^2)^{\frac{1}{2}} && (3.1.16)\\
&\leq \frac{\pi}{2\sqrt{6}}. && (3.1.17)
\end{aligned}$$

The computations just above show that for an $a \in \mathcal{C}$ we have $\|P_0(a) - (a\delta_e, \delta_e)I\| \leq \frac{\pi}{2\sqrt{6}}$. From Theorem 3.1.6 we know that $\|a - P_0(a)\| \leq \frac{\pi}{\sqrt{3}}$, so we obtain $\|a - (a\delta_e, \delta_e)I\| < 2.5$. The diameter of the state space is thus at most 5. □

In [Ri2], Rieffel introduces the concept of a lower semicontinuous Lipschitz seminorm L on a C*-algebra $\mathcal{A}$. The term *Lipschitz* means that the kernel of the seminorm consists of the scalars and the term *lower semicontinuous* means that the set $\{a \in \mathcal{A} \mid L(a) \leq 1\}$ is norm closed. In our context the operator D induces several Lipschitz seminorms whose domains of definition always contain the dense subalgebra $\lambda(\mathbb{C}G)$ of $C^*_r(G)$. In order to define these seminorms we fix the setting as above. Let G be a discrete group with a length function $\ell : G \to \mathbb{N}_0$ such that $\ell^{-1}(0) = \{e\}$. Let D denote the corresponding Dirac operator and δ the unbounded derivation on $C^*_r(G)$ given by $\delta(a) = \overline{[D, a]}$, when the commutator $[D, a]$ is bounded and densely defined. For any natural number k we define a seminorm L_D^k by

$$\text{domain}(L_D^k) \;=\; \text{domain}(\delta^k) \quad \text{and} \quad L_D^k(a) = \|\delta^k(a)\|.$$

Having this notation we can state a theorem of quite general validity.

Theorem 3.1.9. *Let G be a discrete group with a length function $\ell : G \to \mathbb{N}_0$ such that $\ell^{-1}(0) = \{e\}$. For any natural number k the seminorm L_D^k on $C_r^*(G)$ is a lower semicontinuous Lipschitz seminorm.*

Proof. The condition $\ell^{-1}(0) = \{e\}$ implies that the only operators in $\mathrm{C}_r^*(G)$ which commute with D are the multiples of the unit in $\mathrm{C}_r^*(G)$. Let us define $\varphi : \mathbb{Z} \to \mathbb{R}$ by

$$\varphi(m) = \begin{cases} m^{-1}, & \text{if } m \neq 0\ , \\ 0 & \text{if } m = 0\ . \end{cases}$$

and let Λ_s denote the Schur multiplier on $B(\oplus H_m)$ implemented by the function $\lambda(m,n) = \varphi(m-n)$. Then by Theorem 3.1.6 we know that Λ_s is a completely bounded and ultraweakly continuous operator on $B(H)$. Let now $B(H)_1$ denote the unit ball in $B(H)$. Then, for any $k \in \mathbb{N}$ we have that $\Lambda_s^k(B(H)_1)$ is ultraweakly compact. We can now control most of the set

$$\{a \in C_r^*(G) \mid L_D^k(a) \leq 1\}$$

the only part missing is the main diagonal $P_0(\mathrm{C}_r^*(G))$. For $B(H)$ we have $P_0(B(H)) = \mathcal{D}_0$ i.e., the main diagonal which is clearly ultraweakly closed. The sum $\Lambda_s^k(B(H)_1) + P_0(B(H))$ is then ultraweakly closed and consequently also norm closed. The intersection below

$$[\Lambda_s^k(B(H)_1) + \rho_0(B(H))] \cap \mathrm{C}_r^*(G) = \{a \in\ \mathrm{C}_r^*(G) \mid L_D^k(a) \leq 1\}$$

is norm closed, too. □

It is rather easy to check that the metric d_ℓ introduced in Definition 3.1.2 induces a topology which is finer than the w*-topology. In fact, the norm dense group algebra $\lambda(\mathbb{C}G)$ is obviously contained in the domain of definition for the derivation δ and the metric clearly induces a topology on the state space which is finer than pointwise convergence on the operators λ_g. The question is whether the two topologies agree. At first we thought that once the boundedness question has been settled this question would be easy, because

Theorem 3.1.6 controls problems involving norms of the diagonals with large indices. A closer analysis shows that the problems involved seems to be much more complex and are, probably, of a difficult combinatorial nature. The short formulation of the problem is that for a group element, say g, such that $\ell(g)$ is "large", the unitary operator λ_g may have a lot of non vanishing diagonals with "small" indices. This makes it possible, at least in principle, for the algebra $\mathbb{C}G$ to have the property that for any natural number N, there exists an operator $x = \sum x(g)\lambda_g$ such that $x(g) \neq 0 \Rightarrow \ell(g) > N$, and $\|\sum_{|k|\leq N} P_k(x)\|$ is *big* whereas $\|\sum_{|k|>N} P_k(x)\|$ is *small.* Since we have not been able to solve these problems we have looked for alternative constructions of metrics which will induce the w*-topology on the state space of $\mathrm{C}^*_r(G)$. The most obvious thing to do, seemed to be to restrict our attention to the analysis of discrete groups of rapid decay ([Co2],[Jo]).

Before we state and prove our next result we want to recall Lemma 3.1, but in a more general version which discusses lower semicontinuity too. From Rieffel's works [Ri1] and [Ri2] and the work of Pavlović [Pav] it is known that a lower semicontinuous Lipschitz seminorm L on a unital C*-algebra $\mathcal{A}$ is bounded and induces the w*-topology on the state space of $\mathcal{A}$ if and only if the set

$$\{a \in \mathcal{A} : L(a) \leq 1\}$$

has a compact image in the quotient space $\mathcal{A}/\mathbb{C}I$, equipped with the quotient norm.

Theorem 3.1.10. *Let G be a discrete group with a length function $\ell : G \to \mathbb{N}_0$ such that $\ell^{-1}(e) = 0$, ℓ is proper and G is of rapid decay with respect to ℓ. Then there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, L^k_D is lower semicontinuous, the metric generated by L^k_D on $\mathcal{S}(\mathrm{C}^*_r(G))$ is bounded and the topology generated by the metric equals the w*-topology.*

Proof. The statement about lower semicontinuity follows from Theorem 3.1.9, and the assumption of rapid decay implies that there exist two positive reals C, s such that

$$\forall x \in \mathbb{C}G \quad \|\lambda(x)\| \leq C\Big(\sum_g \big(1+\ell(g)\big)^{2s}|\lambda(g)|^2\Big)^{\frac{1}{2}}.$$

The number k_0 is then defined by $k_0 = \lfloor s \rfloor + 1$, and given this we will fix a $k \in \mathbb{N}$ such that $k \geq k_0$. According to the statement immediately above this theorem, we have to prove that the set $\tilde{\mathcal{C}}_k$ defined by

$$\tilde{\mathcal{C}}_k := \{a \in \mathrm{C}^*_r(G) \mid L^k_D(a) \leq 1\}$$

has precompact image in $\mathrm{C}^*_r(G)/\mathbb{C}I$. We obtain this by choosing the element from each equivalence class in $\tilde{\mathcal{C}}_k$ which is of trace 0. This set is denoted $\mathcal{C}_k$, and is clearly precompact if and only if $\tilde{\mathcal{C}}_k$ has precompact image in the quotient space $\mathrm{C}^*_r(G)/\mathbb{C}I$. Consequently $\mathcal{C}_k$ is given by

$$\mathcal{C}_k = \{a \in \mathrm{C}^*_r(G) \mid L^k_D(a) \leq 1 \quad \text{and} \quad (\mathrm{a}\delta_\mathrm{e}, \delta_\mathrm{e}) = 0\}.$$

The first observation we need has already been used before, namely, that any element $a \in \mathrm{C}^*_r(G)$ can be expressed as an l^2 convergent infinite sum $\sum a(g)\delta_g$ and that $\|a\|_2 = \|a\delta_e\|$. Having this, and the fact that $D\delta_e = 0$ we have that for an $a \in \mathcal{C}_k$,

$$1 \geq L^k_D(a) = \|\delta^k(a)\| \geq \|\delta^k(a)\delta_e\| = \|\sum \ell(g)^k a(g)\delta_g\|.$$

In particular we have that

$$\sum \ell(g)^{2k}|a(g)|^2 \leq 1 \text{ for an } a \in \mathcal{C}_k.$$

The properness condition on $\ell(g)$ implies that there are only finitely many group elements of length less than any natural number n. Hence in order to prove that $\mathcal{C}_k$ is precompact it is sufficient to show that for any positive real ε there exists a natural number n such that for any $a \in \mathcal{C}_k$

$$\|\sum_{\ell(g)\geq n} a(g)\lambda_g\|_{\mathrm{C}^*_r(G)} \leq \varepsilon.$$

But this, on the other hand, is easily obtainable from the inequality at the beginning of the proof. In fact, let $n \in \mathbb{N}$ then for $g \in G$ with $\ell(g) \geq n \geq 1$. Then,

$$(1+\ell(g))^{2s} \leq 2^{2s}\ell(g)^{2s} \leq 2^{2s}n^{(2s-2k)}\ell(g)^{2k}.$$

Since $2s - 2k < 0$ there exists an $n \in \mathbb{N}$ such that $2^{2s}n^{(2s-2k)} \leq \frac{\varepsilon^2}{C^2}$. For this n we then obtain

$$\|\sum_{\ell(g)\geq n} a(g)\lambda_g\|^2_{\mathrm{C}^*_r(G)} \leq C^2 \sum_{\ell(g)\geq n} \big(1+\ell(g)\big)^{2s}|a(g)|^2 \leq C^2\frac{\varepsilon^2}{C^2}L^k_D(a)^2 \leq \varepsilon^2$$

and the theorem follows. $\square$

3.2 UHF C*-algebras

3.2.1 Introduction

We have already mentioned the construction of an unbounded Fredholm module of A. Connes for $C_r^*(G)$, where G is a discrete group endowed with a proper length function ℓ. It has been the goal of this research to investigate the possibility of constructing unbounded Fredholm modules for C*-algebras which are not group C*-algebras. In the same spirit as for group C*-algebras, we focused on those Fredholm modules which will give the weak*-topology on the state space. M. A. Riffel and N. Ozawa, [ORi], introduced in their work a very convenient set up for our goal, namely, *a filtered C*-algebra*. This is a unital C*-algebra $\mathcal{A}$ over $\mathbb{C}$ which has a *-filtration $\{A_n\}$ by finite-dimensional subspaces. This means that:

1. $A_m \subset A_n$ if $m < n$;
2. $\bigcup\limits_{n=0}^{\infty} A_n$ is norm dense in $\mathcal{A}$;
3. $A_n^* = A_n$;
4. $A_m A_n \subseteq A_{m+n}$;
5. $A_0 = \mathbb{C}1$.

One also assumes that $\mathcal{A}$ has a faithful trace, τ. For this situation there is a natural way to define an unbounded Fredholm module. Let $H = L^2(\mathcal{A}, \tau)$ and consider now the left regular representation of $\mathcal{A}$ on H as bounded operators. Identify $\mathcal{A}$ with the corresponding algebra of operators on H. Each A_n can be viewed as a finite-dimensional and hence closed subspace of H. We let P_n denote the orthogonal projection of H onto A_n. We then set $Q_0 = P_0$ and for $n > 0$ $Q_n = P_n - P_{n-1}$. The Q_n's are mutually orthogonal, and $\sum Q_n = I_{\mathcal{H}}$ in the strong operator topology. For the described situation M. A. Rieffel and N. Ozawa defined the following unbounded operator, D, on $\mathcal{H}$ by

$$D = \sum_{n=1}^{\infty} n Q_n.$$

It is quite easy to check that this representation of $\mathcal{A}$ on $L^2(\mathcal{A}, \tau)$ together with the operator D constitutes an unbounded Fredholm module. In [ORi] Ozawa and Rieffel introduced a so called *Haagerup-type condition* which we describe in Definition 4.1 below. They show for C*-algebras which satisfy the Haagerup-type condition the metric induced by D on the state space generates the weak*-topology. The Haagerup-type condition reads:

Definition 3.2.1. Let $\mathcal{A}$ and the *-filtration $\{A_n\}$ be as above, and let D be defined as above. For $k \in \mathbb{N}$ and $a \in \mathcal{A}$ the operator a_k is defined as the one which as vector in H is given by $Q_k a$. The group G is said to satisfy a *Haagerup-type condition* if there exists a positive constant C such that for any $a \in \mathcal{A}$ and any natural numbers k, m, n

$$\|P_m a_k P_n\| \leq C\|a_k\|_2$$

In the introduction we mentioned that the model of a filtered C*-algebra extends in a sense the reduced group C*-algebra setting. (We are still following N. Ozawa, M. A. Rieffel, [ORi].) For a discrete group G and any proper length function ℓ on G we obtain a *-filtration $\{A_n\}$ of the convolution algebra $\mathbb{C}G$ by setting

$$A_n = \{x \in \mathbb{C}G : x(g) = 0 \text{ if } l(g) > n\}.$$

We define a faithful trace, τ on $\mathbb{C}G$ by $\tau(x) = x(e)$. The resulting GNS Hilbert space is $l^2(G)$, and the GNS representation is the left regular representation of $\mathbb{C}(G)$. The C*-algebra generated by the left regular representation is $C^*_r(G)$. Thus we are in the setting of filtered C*-algebras. The Dirac operator corresponding to the filtration is just the operator M_ℓ of pointwise multiplication by ℓ on $l^2(G)$.
The concept of filtration gave us the inspiration to investigate spectral triples for UHF C*-algebras. A UHF C*-algebra $\mathcal{A}$ has a natural filtration since $\mathcal{A}$, by definition is the norm closure of an increasing sequence of finite dimensional full matrix algebras $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ such that $\mathcal{A}_0 = \mathbb{C}I_{\mathcal{A}}$. These algebras were studied by first by Glimm [Gl] and he proved that they can be characterized by a *super natural number* which can be computed from the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ by the following procedure. Each algebra $\mathcal{A}_n$ is isomorphic to some full matrix algebra say $\mathcal{M}_{m_n}$ and it embeds into $\mathcal{A}_{n+1} = \mathcal{M}_{m_{n+1}}$ such that the unit is preserved. This implies that m_n divides m_{n+1} and consequently the factorization of m_{n+1} into prime numbers contain the elements from the factorization of m_n. The super natural number associated to $\mathcal{A}$ is then the mapping from the primes into $\mathbb{N}_0 \cup \{\infty\}$ which counts

the total number of appearances of each prime.

We have not seen a proposal for a spectral triple for UHF C*-algebras. We will show that it is possible to construct a Dirac operator which relates to this filtration in a natural way and which will induce a metric for the w*-topology on the state space of the algebra. In our construction of a summable Fredholm module for $\mathcal{A}$, we make some choices dependent upon the concrete sequence $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ and it seems hard to compare Fredholm modules for different increasing sequences of subalgebras.

3.2.2 A spectral triple for a UHF C*-algebra

In the sequel we will use the notation $\mathcal{M}_{m,n}$ for the complex $m \times n$ matrices. If $m = n$ then we will just write $\mathcal{M}_m$.

Recall again the construction of a spectral triple suggested by Connes [Co1] for a discrete group with a length function with values in $\mathbb{N}_0$. In this case the space of square summable functions on the group naturally decomposes into a sum of subspaces, where each subspace consists of the functions which are supported on the words of a given length. Then, the Dirac operator is the operator which has these spaces as eigenspaces with eigenvalues equal to the corresponding word lengths. In the case of a UHF C*-algebra the analogy, of the model immediately above, is, for the sake of the representation, the GNS-representation based on the trace-state. The algebra becomes a pre Hilbert space and the natural increasing sequence of finite dimensional Hilbert subspaces is then the sequence of finite dimensional C*-algebras, but now considered as subspaces of $\mathcal{A}$ with respect to the inner product induced by the trace state. This increasing sequence of finite dimensional Hilbert spaces naturally induces a sequence of pairwise orthogonal subspaces - the spaces are the differences of neighboring pairs in the increasing sequence. In analogy with the group case it seemed natural to investigate the properties of an operator which had the given sequence of subspaces as eigenspaces and with corresponding eigenvalues *somehow related* to the dimension of the C*-algebra which corresponds to the sum of all the previous eigenspaces. We tried this approach, but could not get it to work. In the next section we will report on some numerical tests, performed using MAPLE. It seems that the immediate suggestion for a Dirac operator behaves nicely, but that we just can't prove it. Instead, we tried another idea for a Dirac operator. The one we propose is based on the natural one for the algebra of complex $n \times n$ matrices. This case is treated in this section. Recall that in this finite dimensional situation the proposal for a Dirac operator is the transposition operator on the Hilbert space consisting of the entire algebra. An immediate generalization would give us the uniform metric on the state space, which is not what we want. The next guess was to take growing multiples of the transposition operator on the summands introduced before. This becomes too unbounded an operator in the sense that only the operators which are multiples of the identity will have bounded commutator with such a Dirac operator. Finally, we tried a Dirac operator where the transposition in each step is used only on the last factor. More precisely the idea is to write the increasing sequence of C*-algebras as an increasing sequence of tensor products like

$$\mathcal{A}_n = \mathcal{M}_{d_1} \otimes \cdots \otimes \mathcal{M}_{d_n} \otimes \mathbb{C} I_{\mathcal{M}_{d_{n+1}}} \subset \mathcal{M}_{d_1} \otimes \cdots \otimes \mathcal{M}_{d_{n+1}} = \mathcal{A}_{n+1}$$

and then let a multiple of the operator $I \otimes \cdots \otimes I \otimes T_{d_{n+1}}$ act on the Hilbert difference space $\mathcal{A}_{n+1} \ominus \mathcal{A}_n$. Our good friend Ryszard Nest told us that this construction seems reasonable because the differentiation operator acts on a product by summing products where just one factor in each product is being differentiated. The construction, which we introduced above, nearly has this property, but not exactly. The reason why we can not simply copy the product rule directly is that for this procedure, it has not been possible to obtain a summable Fredholm module. We will now formulate our result for a UHF C*-algebra $\mathcal{A}$

Theorem 3.2.2. *Let $\mathcal{A}$ be an infinite dimensional UHF C*-algebra and let π denote the GNS-representation induced by the trace state on a Hilbert space H. For any strictly increasing sequence $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ of finite dimensional factors such that $\mathcal{A}_0 = \mathbb{C}I_{\mathcal{A}}$ and the union of the sequence is dense in $\mathcal{A}$ and any increasing sequence of reals $(\alpha_n)_{n\in\mathbb{N}_0}$ such that $\alpha_0 = 1$ and $\sum \alpha_n^{-2} < \infty$ there exists an unbounded selfadjoint operator D on H such that*

(a) $\cup_{n\in\mathbb{N}_0}\mathcal{A}_n \subset \mathrm{dom(D)}$

(b) $(I + D^2)^{-1}$ *is of trace class.*

(c) The set $\mathcal{C} = \{a = a^* \in \mathcal{A} : \| [D, a] \| \leq 1\}$ is relatively compact in $\mathcal{A}$ and the metric which $\mathcal{C}$ induces on the state space generates the weak topology.*

Proof. We will continue to use the notation we introduced before we stated the theorem, i.e., $\mathcal{A}_n$ is isomorphic to $\mathcal{M}_{m_n}$ and $m_0 = 1$. The number m_n divides m_{n+1} and we define for $n \geq 1, d_n = m_n/m_{(n-1)}$. For simplicity in the arguments below we will always assume that all of the quotients d_n satisfy $d_n > 1$. The Hilbert space H is just the completion of pre Hilbert space $\mathcal{A}$ equipped with the inner product $(a, b) = \tau(b^*a)$, where τ is the trace state on $\mathcal{A}$. The algebras $\mathcal{A}_n$ are then closed subspaces, which we will denote (H_n). The corresponding growing sequence of projections is then denoted (P_n). As remarked earlier we want to let the Dirac operator have its eigenspaces related to the sequence of differences $H_{(n)} \ominus H_{(n-1)}$. We do therefore define a sequence (F_n) of pairwise orthogonal finite dimensional subspaces of H by $F_0 = H_0$ and $F_n = H_{(n)} \ominus H_{(n-1)}$ for $n \geq 1$. The corresponding sequence of pairwise orthogonal projections is denoted (Q_n), so $Q_0 = P_0$ and $Q_n = P_n - P_{(n-1)}$ for $n \geq 1$. We can now describe the Dirac operator's action on each of the spaces F_n. We first remark that the last factor $\mathcal{M}_{d_n}$ of $\mathcal{A}_n$ is a unital subalgebra of $\mathcal{A}$ and we can therefore write $\mathcal{A} = \mathcal{M}_{d_n} \otimes \mathcal{M}^c_{d_n}$ where $\mathcal{M}^c_{d_n}$ denotes the relative commutant of $\mathcal{M}_{d_n}$ in $\mathcal{A}$. The transposition operator on $\mathcal{M}_{d_n}$, say T_n then induces a selfadjoint unitary , a symmetry, say S_n on H by extension of the operator $T_n \otimes I_{\mathcal{M}^c_{d_n}}$ from $\mathcal{A}$ to H. It is clear that $S_n|\mathcal{A}_{(n-1)} = \text{ id } |\mathcal{A}_{(n-1)}$ and $S_n : \mathcal{A}_n \to \mathcal{A}_n$. Hence both of the subspaces $H_{(n-1)}$ and H_n are invariant for S_n, and since S_n is selfadjoint the difference space F_n is invariant, too. Finally the projection Q_n commutes with S_n. This makes it possible to define a selfadjoint operator D as the closure of the operator which on $\text{span}(\cup F_n)$ is defined by:

$$DQ_0 = Q_0 = \alpha_0 m_0 Q_0, \; n = 0$$

and

$$DQ_n = \alpha_n m_n S_n Q_n, \; n > 0.$$

Each of the finite dimensional spaces F_n consists of at most two eigenspaces for D, and it follows directly from the definition of D that the set of eigenvalues must be $\{1\} \cup \{\{-\alpha_n m_n, \alpha_n m_n\} : n \in \mathbb{N}\}$. The eigenspace corresponding to an eigenvalue x is denoted G_x and then we get

- $G_1 = F_0$ and $\dim(G_1) = 1$
- $G_{-\alpha_n m_n} \subset F_n$ and $\dim(G_{-\alpha_n m_n}) = m_n(m_n - m_{(n-1)})/2$

- $G_{\alpha_n m_n} \subset F_n$ and $\dim(G_{\alpha_n m_n}) = m_n(m_n + m_{(n-1)})/2 - m^2_{(n-1)}$
- $G_{-\alpha_n m_n} \oplus G_{\alpha_n m_n} = F_n$.

It is well known that the closure of this operator, say D, is selfadjoint and that its domain of definition $\mathrm{dom}(D)$ will be the space

$$\mathrm{dom}(D) \;=\; \{(\xi_n)_{n\in\mathbb{N}_0} \;:\; \xi_n \in F_n \text{ and } \sum_{n=0}^{\infty} \alpha_n^2 m_n^2 \|\xi_n\|^2 < \infty \;\}.$$

We have now proved the first claim in the theorem, and the considerations with respect to the dimensions of the eigenspaces yields immediately that that $(I + D^2)^{-1}$ is compact. The trace can be estimated by the following computation:

$$\mathrm{tr}((I + D^2)^{-1}) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{m_n^2 - m^2_{(n-1)}}{\alpha_n^2 m_n^2 + 1} < \sum_{n=0}^{\infty} \alpha_n^{-2} < \infty.$$

In order to prove the last claim in the theorem we remind the reader of the fact that the algebras A_n as finite dimensional C*-algebras are injective von Neumann algebras, and as such are complemented subspaces of $\mathcal{A}$ such that there exists a completely positive $\mathcal{A}_n$ bimodule projection, π_n of norm one from $\mathcal{A}$ onto $\mathcal{A}_n$. Since $\mathcal{A}$ has a unique trace state, τ, the projection π_n can be chosen such that it also has the property

$$\tau(ax_n) = \tau(\pi_n(a)x_n) \text{ for every } a \in \mathcal{A} \text{ and } x_n \in \mathcal{A}_n.$$

This property is used by several authors and for instance explained in [Ch]. The key to understand this is that for a positive a in $\mathcal{A}$ the functional $\mathcal{A}_n \ni x_n \to \tau(ax_n)$ is a positive functional on $\mathcal{A}_n$. Then this functional must have a positive Radon-Nikodym derivative with respect to the unique tracial state on $\mathcal{A}_n$ which is denoted τ_n. It is a matter of routine to show that the positivity of $\pi_n(a)$ ensures that π_n will be a projection of norm one from $\mathcal{A}$ onto $\mathcal{A}_n$ with all the desired properties. The identity above has one fundamental consequence upon which we shall build our arguments, namely, that when we consider both the operator algebra structure and the Hilbert space structure on $\mathcal{A}$ simultaneously we get

$$\pi_n(a) = P_n a \text{ for every } a \in \mathcal{A}.$$

We will now study commutators $[D, a]$ which are bounded on a dense subset of H. Since D is a diagonal operator with respect to the decomposition $H = \oplus F_n$ we will, as in Section

3.3, estimate properties of $[D,a]$ by investigating the matrix of $[D,a]$ with respect to this decomposition of H. We will use the matrix notation $a = (a_{ij})$ where $i, j \in \mathbb{N}_0$ and

$$a_{ij} = Q_i a|F_j \in B(F_j, F_i),$$

but we will also allow the use of the symbol a_{ij} with a slightly different meaning as

$$a_{ij} = Q_i a Q_j \in B(H)$$

whenever this is most convenient. With this notation we can formally write

$$[D,a]_{ij} = DQ_i a_{ij} - a_{ij} DQ_j$$

and we can define the domain of definition for the derivation, say $\delta : \mathcal{A} \to B(H)$, as the set

$$\mathrm{dom}(\delta) = \{a \in \mathcal{A} \ : \ ([D,a]_{ij}) \in B(H)\}.$$

We have that for $a \in dom(\delta)$

$$\delta(a) = \mathrm{closure}([D,a]) = ([D,a]_{ij}).$$

Let us fix an $n \in \mathbb{N}$ and let $a = a^* \in \mathcal{A}_n$. We want to study the entries in the matrix (a_{ij}) for $i > n$, and $j > n$. The inclusions $\mathcal{A}_n \subset \mathcal{A}_m$ for $m \geq n$ imply that for any such m, we have $a\mathcal{A}_m \subset \mathcal{A}_m$. When we consider the subspaces H_m of H for $m \geq n$ we see that these subspaces are all invariant for a, and since this operator is selfadjoint the projections P_m all commute with a for $m \geq n$. Then, in turn, we get that for $m > n$ the projections Q_m all commute with a. In particular this shows that

$$\forall n \in \mathbb{N}\, \forall a \in \mathcal{A}_n : \ a_{ij} = 0 \text{ for every } i, j \text{ such that } i \neq j \text{ and } i > n \text{ or } j > n.$$

The a_{mm} blocks all commute with D_m for $m > n$ since S_m by construction only acts on the m^{th} tensor factor, so we get

$$\forall m, n \in \mathbb{N}\ \forall a \in \mathcal{A}_n : \ [D,a]_{mm} = 0,$$

and all together we have obtained

$$\forall n \in \mathbb{N} \forall a \in \mathcal{A}_n : \ [D,a] \ = \ P_n[D,a]P_n.$$

We will continue the investigation of the commutator $P_n[D, a]P_n$ for some selfadjoint $a \in \mathcal{A}$. The way D is defined ensures that this commutator makes sense for any $a \in \mathcal{A}$. Let $b, c \in \mathcal{A}_n$. Then, since D commutes with P_n, we get that $x = Db \in \mathcal{A}_n$ and $y = Dc \in \mathcal{A}_n$. This yields, since D is self adjoint, the following inner product identities:

$$\begin{aligned}(P_n[D, a]P_n b, c) &= ([D, a]b, c) = (a, yb^*) - (a, cx^*) \\ &= (\pi_n(a), yb^*) - (\pi_n(a), cx^*) = ([D, \pi_n(a)]P_n b, P_n c).\end{aligned}$$

Combined with the computations above this gives us

$$\forall n \in \mathbb{N}\, \forall a \in \mathcal{A} : \; P_n[D, a]P_n \;=\; P_n[D, \pi_n(a)]P_n \;=\; [D, \pi_n(a)] \tag{3.2.1}$$

This last identity has the advantage that for an $a \in \mathcal{C} = \{a \in \mathcal{A} \,:\, \|[D, a]\| \leq 1\}$ we do not have to worry about the closure of this operator since we can get all the information from the sequence of commutators $[D, \pi_n(a)]$ and each such operator is supported on the finite dimensional Hilbert space H_n. In particular, for any $a \in \mathcal{C}$ and any $n \in \mathbb{N}$, we get $\pi_n(a) \in \mathcal{A}_n$. We will now establish the following norm estimate

$$\forall a \in \mathcal{A} : \quad \|\pi_n(a) - \pi_{(n+1)}(a)\| \;\leq\; 5\sqrt{n+1}\, 2^{-(n+1)}. \tag{3.2.2}$$

The estimate will be used to obtain the following estimates

$$\forall a \in \mathcal{C}\, \forall n \in \mathbb{N} : \; \|a - \tau(a)I\| \leq 13 \;\text{ and }\; \|a - \pi_n(a)\| \leq 20\sqrt{n}\, 2^{-n}. \tag{3.2.3}$$

Based on these inequalities and the fact that the algebra $\pi_n(\mathcal{A}) = \mathcal{A}_n$ is finite dimensional, it follows from 3.2.3 that $\mathcal{C}/\mathbb{C}I$ is a relatively norm compact set in $\mathcal{A}/\mathbb{C}I$.
The estimates above are based on detailed studies of the norm of the commutators:

$$\|[S_{n+1}, \pi_{n+1}(a)]\|. \tag{3.2.4}$$

The last commutator is studied on the Hilbert space $H_{(n+1)} = \overset{n+1}{\underset{i=0}{\oplus}} F_i$ and from the fact that $a \in \mathcal{C}$ and the equation 3.2.1 we get

$$\begin{aligned}\|Q_{n+1}[S_{n+1}, \pi_{n+1}(a)]Q_{n+1}\| &= \frac{1}{\alpha_{n+1}m_{n+1}}\|Q_{n+1}[D, \pi_{n+1}(a)]Q_{n+1}\| \\ &= \frac{1}{\alpha_{n+1}m_{n+1}}\|Q_{n+1}[D, a]Q_{n+1}\| \\ &\leq \frac{1}{\alpha_{n+1}m_{n+1}}.\end{aligned}$$

In order to make an estimate of

$$\|P_n[S_{n+1},\pi_{n+1}(a)]Q_{n+1}\|$$

we remark that

$$\|P_n[D,\pi_{n+1}(a)]Q_{n+1}\| \leq 1,$$

and consequently, for each $i \in \{0,1,\dots,n\}$, we can estimate the norms of the entries $a_{i(n+1)}$ of the infinite matrix for a by using that $a \in \mathcal{C}$, such as:

$$\|\alpha_i m_i S_i Q_i a_{i(n+1)} - \alpha_{n+1} m_{n+1} a_{i(n+1)} S_{n+1} Q_{n+1}\| \;\leq\; 1.$$

By rough estimates we then have

$$\|a_{i(n+1)}\| \;\leq\; \frac{1}{\alpha_{n+1}m_{n+1} - \alpha_n m_n} \;\leq\; \frac{2}{\alpha_{n+1}m_{n+1}}, \quad \forall i \in \{0,1,\dots,n\}.$$

Thus, by studying the operator X

$$X = P_n a Q_{n+1} + Q_{n+1} a P_n$$

we find that

$$\|X^*X\| = \|P_n a Q_{n+1}\|^2 \leq \sum_{i=0}^{n} \|a_{i(n+1)}\|^2,$$

so that

$$\|P_n a Q_{n+1} + Q_{n+1} a P_n\| \;\leq\; \frac{2\sqrt{n+1}}{\alpha_{n+1}m_{n+1}}.$$

Since the operator S_{n+1} commutes with both P_n and Q_{n+1} we can then see that

$$\begin{aligned}\|P_n[S_{n+1},a]Q_{n+1} + Q_{n+1}[S_{n+1},a]P_n\| &= \|[S_{n+1},(P_n a Q_{n+1} + Q_{n+1} a P_n)]\| \\ &\leq \frac{4\sqrt{n+1}}{\alpha_{n+1}m_{n+1}}.\end{aligned}$$

Now in order to estimate $\|[S_{n+1},a]\|$ we have only to estimate the norm $\|P_n[S_{n+1},a]P_n\|$, but again we can use that S_{n+1} commutes with P_n, and moreover that $P_n a P_n = \pi_n(a)P_n$ and S_{n+1} commutes with operators in $\mathcal{A}_n$. From this we see that

$$\|P_n[S_{n+1},a]P_n\| \;=\; \|[S_{n+1},P_n a P_n]\| \;=\; \|[S_{n+1},\pi_n(a)P_n]\| \;=\; 0.$$

Collecting the three inequalities we have

$$\|[S_{n+1},\pi_{n+1}(a)]\| \;\leq\; \frac{5\sqrt{n+1}}{\alpha_{n+1}m_{n+1}}. \tag{3.2.5}$$

In order to obtain estimates on $\|\pi_{n+1}(a) - \pi_n(a)\|$ from the inequality above we have to look into a concrete description of $\pi_n(a)$ in terms of $\pi_{n+1}(a)$. We know from the same projections P_{n+1}, P_n when considered as orthogonal projections that

$$P_{n+1} \geq P_n \text{ so } \pi_n \circ \pi_{n+1} = \pi_n.$$

The construction of π_n, given π_{n+1}, is based on the fact that $\mathcal{A}_{n+1}$ is a tensor product of full matrix algebras where one of the factors is $\mathcal{A}_n$. With a slight abuse of notation we can describe the situation as

$$\begin{aligned} \mathcal{A}_{n+1} &= \mathcal{A}_n \otimes \mathcal{M}_{d_{n+1}} \\ \mathcal{A}_n &= \mathcal{A}_n \otimes \mathbb{C} I_{\mathcal{M}_{d_{n+1}}}. \end{aligned}$$

Since the projection P_n is given with respect to the inner product implemented by the trace state

$$\tau = \tau_{\mathcal{A}_n} \otimes \tau_{d_{n+1}}, \tag{3.2.6}$$

the restriction of π_n to $\mathcal{A}_{n+1}$ can be described as a tensor product in the form

$$\pi_n : \mathcal{A}_{n+1} \to \mathcal{A}_n \text{ where } \pi_n = \mathrm{id}_{\mathcal{A}_n} \otimes \tau_{\mathrm{d}_{n+1}} \tag{3.2.7}$$

We now describe $\tau_{d_{n+1}}$ via an averaging argument. Let $\mathcal{U}$ denote the compact group of unitaries in $\mathcal{M}_{d_{n+1}}$ and let μ denote the Haar probability measure on $\mathcal{U}$. The translation invariance of the measure μ implies that

$$\forall x \in \mathcal{M}_{d_{(n+1)}} : \ \tau_{d_{n+1}}(x) I_{\mathcal{M}_{d_{(n+1)}}} \ = \int_{\mathcal{U}} u x u^* \mathrm{d}\mu(u), \tag{3.2.8}$$

We can then combine this with the description of the action of π_n on $\mathcal{A}_{n+1}$, but first we give the unitaries in $\mathcal{U}$ a special notation when considered as elements in $\mathcal{A}_{n+1}$, so we define

$$\forall u \in \mathcal{U} : \ \tilde{u} = I_{\mathcal{A}_n} \otimes u \in \mathcal{A}_{n+1}. \tag{3.2.9}$$

We obtain by a combination of the equations 3.2.6, 3.2.7, 3.2.8 and 3.2.9 the following description of π_n as an average,

$$\forall a \in \mathcal{A}_{n+1} : \ \pi_n(a) = \int_{\mathcal{U}} \tilde{u} a \tilde{u}^* \mathrm{d}\mu(u). \tag{3.2.10}$$

We are now ready to continue the estimates of $\|\pi_n(a) - \pi_{n+1}(a)\|$ based on the inequality 3.2.5. We remind the reader that in the tensor decomposition $\mathcal{A}_{(n+1)} = \mathcal{A}_n \otimes \mathcal{M}_{d_{(n+1)}}$ we have

$$S_{n+1} = I_{\mathcal{A}_n} \otimes T_{n+1}$$

and moreover we get, just as when we considered the case with norm metric on the state space, that

$$T_{n+1}\mathcal{M}_{d_{n+1}}T_{n+1} = \mathcal{M}'_{d_{n+1}},$$

i.e., the commutant on the space $L^2(\mathcal{M}_{d_n}, \tau_{d_n})$.
This commutation property can now be lifted to $\mathcal{A}_{(n+1)}$ in a certain way. Let u be a unitary in $\mathcal{M}_{d_n}$, then

$$S_{n+1}\tilde{u}S_{n+1} = I_{\mathcal{A}_n} \otimes T_{n+1}uT_{n+1}$$

and $S_{n+1}\tilde{u}S_{n+1}$ commutes with $\mathcal{A}_{n+1}$. We have the following equalities:

$$\begin{aligned}
\| [S_{n+1},\, \pi_{n+1}(a)] \| &= \|\pi_{n+1}(a) \; - \; S_{n+1}\pi_{n+1}(a)S_{n+1}\| \\
&= \|S_{n+1}\tilde{u}S_{n+1} \, (\, \pi_{n+1}(a) \; - \; S_{n+1}\pi_{n+1}(a)S_{n+1} \,) \, S_{n+1}\tilde{u^*}S_{n+1}\| \\
&= \|\pi_{n+1}(a) \; - \; S_{n+1}\tilde{u}\pi_{n+1}(a)\tilde{u^*}S_{n+1}\|.
\end{aligned}$$

Using that

$$\forall a \in \mathcal{A} : \;\; S_{n+1}\pi_n(a)S_{n+1} = \pi_n(a)$$

we get the estimate:

$$\begin{aligned}
\|\pi_{n+1}(a) - \pi_n(a)\| &= \|\pi_{n+1}(a) - S_{n+1}\pi_n(a)S_{n+1}\| \\
&= \|\pi_{n+1}(a) - S_{n+1}(\int_{\mathcal{U}} \tilde{u}\pi_{n+1}(a)\tilde{u}^* \mathrm{d}\mu(u) \,)S_{n+1}\| \\
&= \|\pi_{n+1}(a) - \int_{\mathcal{U}} S_{n+1}\tilde{u}\pi_{n+1}(a)\tilde{u}^* S_{n+1} \mathrm{d}\mu(u)\| \\
&= \| \int_{\mathcal{U}} (\, \pi_{n+1}(a) - S_{n+1}\tilde{u}\pi_{n+1}(a)\tilde{u}^* S_{n+1} \,)\mathrm{d}\mu(u)\| \\
&\leq \int_{\mathcal{U}} \| \, \pi_{n+1}(a) - S_{n+1}\tilde{u}\pi_{n+1}(a)\tilde{u}^* S_{n+1} \, \|\mathrm{d}\mu(u).
\end{aligned}$$

By the computations just before we know, that the last integrand is constant and equal to

$$\| \, [\, S_{n+1}, \, \pi_{n+1}(a) \,] \|,$$

so we have

$$\forall a \in \mathcal{A}: \; \|\pi_{n+1}(a) - \pi_n(a)\| \leq \|\,[S_{n+1},\, \pi_{n+1}(a)]\| \leq \frac{5\sqrt{n+1}}{\alpha_{n+1}m_{n+1}}.$$

In order to simplify the following computations we remark that $m_{n+1} = m_n d_{n+1}$, $m_0 = 1$ and $d_n \geq 2$, so we have $m_n \geq 2^n$. By assumption all the $\alpha_n \geq 1$ so we get the rougher estimate

$$\forall a \in \mathcal{A} \forall n \in \mathbb{N}: \; \|\pi_{(n+1)}(a) - \pi_n(a)\| \;\leq\; 5\sqrt{n+1}\,2^{-(n+1)}$$

For $n \in \mathbb{N}$ we have

$$\sum_{k=n+1}^{\infty} \sqrt{k}2^{-k} \leq \int_n^{\infty} \sqrt{x}2^{-x} \leq 4\sqrt{n}2^{-n}.$$

Then we can get some estimates on $\|a - \pi_n(a)\|$ and $\|a - \tau(a)I\|$ for any $a \in \mathcal{C}$. First we remark that the UHF property yields that for $k \in \mathbb{N}$ and $a \in \mathcal{C}$ we have $\pi_k(a) \to a$ for $k \to \infty$. In particular this means that for any $n \in \mathbb{N}: \; \|a - \pi_n(a)\| = \lim \|\pi_k(a) - \pi_n(a)\|$ and for operators $a \in \mathcal{C}$ we can then get

$$\forall a \in \mathcal{C} \forall n \in \mathbb{N}: \; \|a - \pi_n(a)\| \leq \sum_{k=n}^{\infty} \|\pi_{(k+1)}(a) - \pi_k(a)\| \leq 20\sqrt{n}\,2^{-n}. \tag{3.2.11}$$

and

$$\begin{aligned} \forall a \in \mathcal{C}: \; \|a - \tau(a)I\| &= \|a - \pi_0(a)\| \\ &\leq \sum_{k=0}^{\infty} \|\pi_{(k+1)}(a) - \pi_k(a)| \\ &\leq \|\pi_1(a) - \pi_0(a)\| + 20\sqrt{1}\,2^{-1} \\ &\leq \frac{5}{\alpha_1 m_1} + 10 \leq 13. \end{aligned}$$

In particular we get that $\mathcal{C}/\mathbb{C}I$ is bounded in norm by 13 and for any $\varepsilon > 0$ we can approximate elements in $\mathcal{C}$ uniformly up to ε by elements from one of the finite dimensional algebras $\mathcal{A}_n$. Hence it follows that the set $\mathcal{C}/\mathbb{C}I$ is precompact $\mathcal{A}/\mathbb{C}I$. □

3.2.3 Numerical experiments

We will keep the same notation as in the previous section. Consider the sequence of finite dimensional full matrix algebras $(\mathcal{A}_k)_{k\in\mathbb{N}}$ where each algebra $\mathcal{A}_k$ is isomorphic to

$\mathcal{M}_{d_1} \otimes \cdots \otimes \mathcal{M}_{d_k}$. Define $m_k := d_1 \dots d_k$. Let $\mathcal{A}$ be the UHF C*-algebra generated by the increasing sequence $(\mathcal{A}_k)_{k \in \mathbb{N}_0}$ where $\mathcal{A}_0 = \mathbb{C}1$, and let τ be the trace state on $\mathcal{A}$. Then the GNS Hilbert space H based on the trace state is just the pre-Hilbert space $\mathcal{A}$ completed with respect to the inner product given by $(a, b) = \tau(b^* a)$. Each algebra $\mathcal{A}_k$ can be viewed as a finite dimensional, so closed, subspace of H. We will denote this subspaces H_k. The corresponding sequence of projections is then denoted (P_k). We then set $F_0 = H_0$ and for $k \geq 1$, $F_k = H_k \ominus H_{k-1}$. It is clear that the spaces F_k are pairwise orthogonal and $H = \bigoplus_{k \in \mathbb{N}_0} F_k$. Also it can be verified that $F_k = H_{k-1} \otimes \mathcal{M}_{d_k}^\circ$, where by $\mathcal{M}_{d_k}^\circ$ we mean the set $\{x \in \mathcal{M}_{d_k} : \ \tau_{d_k}(x) = 0\}$. We let Q_k denote the orthogonal projection of H onto F_k. For the above situation we define an unbounded operator D on H by

$$D \mid_{F_k} \text{ is pointwise multiplication by } d_1 \dots d_k.$$

The left regular representation of $\mathcal{A}$ on H is also in this case the GNS representation coming from the trace state, and we identify $\mathcal{A}$ with the corresponding algebra of operators on H. We let $\|\cdot\|$ denote the operator norm of $\mathcal{A}$. The notation for a as an operator on H and a as a vector in H will be the same, so from the context it should be seen which one is intended. We remark that for any $a \in \bigcup_{n=0}^{\infty} \mathcal{A}_n$ the operator $[D, a]$ is a bounded operator. We wonder if the metric induced by this operator D gives the weak*-topology on the state space of $\mathcal{A}$. The answer can be found by studying the set

$$\mathcal{C} = \{a \in \mathcal{A} : \ \|[D, a]\| \leq 1\}.$$

More precisely, we try to verify if the image of $\mathcal{C}$ is precompact in the quotient space space $\mathcal{A}/\mathbb{C}I$ equipped with the quotient norm. The idea is to try to obtain a convenient estimate of the norm of a from the norm of the commutator $[D, a]$, where $a \in \bigcup_{n=0}^{\infty} \mathcal{A}_n$. We make the extra assumption that $\tau(a) = 0$. If the latter is not the case we simply subtract the corresponding multiple of the unit from a. Thus, we can assume that a is in some F_r and also that a is selfadjoint. We will study now the commutator [D,a]. Since D is a diagonal operator with respect to the decomposition $H = \bigoplus_{k \in \mathbb{N}_0} F_k$ we will estimate properties of $[D, a]$ by investigating the matrix of $[D, a]$ with respect to this decomposition of H. We will use also the matrix notation (a_{kl}), where $k, l \in \mathbb{N}_0$ and

$$a_{kl} = Q_k a \mid F_l \in B(F_l, F_k).$$

We will also allow to use of the symbol a_{kl} in a slightly different meaning as,

$$a_{kl} = Q_k a Q_l \in B(H),$$

whenever this is most convenient. With this notation we can formally write

$$[D, a]_{kl} = (m_k - m_l)a_{kl}.$$

On the following pages we want to investigate in details a the matrix of an element $a \in \mathcal{A}$ such that $a \in F_r$ for some $r \in \mathbb{N}$. We will therefor fix $r \in \mathbb{N}$ and $a \in F_r$ and try to get as much information as possible on the element a_{kl} for any pair $k, l \in \mathbb{N}$. In the first place it turns out that many of the entries a_{kl} vanishes. For this purpose we shall consider two arbitrary elements $\xi \in F_l$ and $\eta \in F_k$, and we will compute $(a\xi, \eta)$. We have the following cases:

1. If $l > r$ then $a\xi \in F_l$. So for $k \neq l$ we have $(a\xi, \eta) = 0$.

2. If $k > r$ then $a^*\eta \in F_l$. So for $k \neq l$ we have $(a\xi, \eta) = 0$.

3. If $l < r$ and $k < r$ then $a\xi \in F_r$ and $k \neq r$ so we have $(a\xi, \eta) = 0$.

To summarize we get $a_{kl} = 0$ except in the cases

1. $k = r$ and $l < r$

2. $l = r$ and $k < r$

3. $k = l \geq r$

For the commutator $[D, a]$ it follows that for diagonal entries we have $[D, a]_{ll} = 0$. Thus for an $a \in F_r$ the commutator $[D, a]$ has a matrix of the form:

$$\begin{array}{c|ccccccccc}
 & F_0 & F_1 & F_2 & \cdots & F_{r-1} & F_r & F_{r+1} & \cdots \\
F_0 & 0 & 0 & 0 & \cdots & 0 & ? & 0 & \cdots \\
F_1 & 0 & 0 & 0 & \cdots & 0 & ? & 0 & \cdots \\
F_2 & 0 & 0 & 0 & \cdots & 0 & ? & 0 & \cdots \\
\vdots & & & & & & & & \cdots \\
F_{r-1} & 0 & 0 & 0 & \cdots & 0 & ? & 0 & \cdots \\
F_r & ? & ? & ? & \cdots & ? & 0 & 0 & \cdots \\
F_{r+1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
\vdots & & & & & & & & \cdots
\end{array}$$

The standard representation of $\mathcal{A}_r$ as left multiplication on H_r is naturally faithful, so we will have for any $a \in F_r$

$$\|a\| = \|a\mid_{H_r}\|.$$

Based on this we will restrict our attention to $a\mid_{H_r}$ and will have the following matrix:

$$\begin{array}{c|cccccc}
 & H_0 & H_1 & H_2 & \cdots & H_{r-1} & H_r \\
H_0 & 0 & 0 & 0 & \cdots & 0 & ? \\
H_1 & 0 & 0 & 0 & \cdots & 0 & ? \\
H_2 & 0 & 0 & 0 & \cdots & 0 & ? \\
\vdots & & & & & & \\
H_{r-1} & 0 & 0 & 0 & \cdots & 0 & ? \\
H_r & ? & ? & ? & \cdots & ? & ?
\end{array}$$

As we took a selfadjoint it will be enough to compute only the row r. For the particular case of a CAR C*-algebra it is possible to compute every element of the two above matrices and to do a numerical experiment. This straightforward computation could be done because of the existence of a very convenient basis in $\mathcal{M}_2$. The elements of this basis are called Pauli's matrices, [BR]. We give now the Pauli's matrices and those properties of them that we will use later on.

$$s_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad s_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad s_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli's matrices have the following properties

1. they form an orthonormal basis for $\mathcal{M}_2$ with respect to the inner product coming from the trace state.

$$(s_i, s_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

2. $(s_i)^2 = s_0 = I$

3. $(s_i)^* = s_i$.

It is easy to verify the following table:

s_is_j	s_0	s_1	s_2	s_3
s_0	I	s_1	s_2	s_3
s_1	s_1	I	$-is_3$	is_2
s_2	s_2	is_3	I	is_1
s_3	s_3	$-is_2$	$-is_1$	I

We can write

- $F_0 = \mathbb{C}I = \text{span}\{s_0\}$;
- $F_1 = \mathcal{M}_2^\circ = \text{span}\{s_i : \ 1 \leq i \leq 3\}$;
- $F_2 = \mathcal{M}_2 \otimes \mathcal{M}_2^\circ = \text{span}\{s_i \otimes s_j : 0 \leq i \leq 3, 1 \leq j \leq 3\}$;
- $$\begin{aligned} F_3 &= \mathcal{M}_2 \otimes \mathcal{M}_2 \otimes \mathcal{M}_2^\circ \\ &= \text{span}\{s_i \otimes s_j \otimes s_k : 0 \leq i \leq 3, 0 \leq j \leq 3, 1 \leq k \leq 3\}; \end{aligned}$$

$\vdots$

- $$\begin{aligned} F_r &= \underbrace{\mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_2}_{r-1} \otimes \mathcal{M}_2^\circ \\ &= \text{span}\{s_{i_1} \otimes \cdots \otimes s_{i_r} : \ 0 \leq i_1 \leq 3, \ldots, 0 \leq i_{r-1} \leq 3, 1 \leq i_r \leq 3\}. \end{aligned}$$

With this in mind we took an arbitrary selfadjoint a in F_1 and we computed the entries of the matrices (a_{kl}) and $([D, a]_{kl})$. We can write $a = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3$, where $\alpha_1, \alpha_2, \alpha_3$

are real numbers. Then we will have the following matrix for $([D,a]_{kl})$:

$$\begin{array}{c|cccc} & H_0 & & H_1 & \\ \hline H_0 & 0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ & \alpha_1 & 0 & 0 & 0 \\ H_1 & \alpha_2 & 0 & 0 & 0 \\ & \alpha_3 & 0 & 0 & 0 \end{array}$$

Let γ be a real number, β a complex number and consider

$$a = \begin{pmatrix} \gamma & \beta \\ \bar{\beta} & -\gamma \end{pmatrix} = \mathrm{Im}\beta s_1 + \mathrm{Re}\beta s_2 + \gamma s_3,$$

then this operator is a typical element of F_1. We shall compute $\|[D,a]\|^2$. Define $A := \begin{pmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \end{pmatrix}$. We obtain

$$\begin{aligned} \|[D,a]\|^2 = \left\| \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} AA^* & 0 \\ 0 & A^*A \end{pmatrix} \right\| \\ &= \max\{\|AA^*\|, \|A^*A\|\} = \|A\|^2 \\ &= \| \begin{pmatrix} -\gamma & -\mathrm{Re}\beta & -\mathrm{Im}\beta \end{pmatrix} \|^2 = \gamma^2 + |\beta|^2. \end{aligned}$$

On the other hand,

$$p_a(\lambda) = \begin{vmatrix} \gamma - \lambda & \beta \\ \bar{\beta} & -\gamma - \lambda \end{vmatrix} = \lambda^2 - \gamma^2 - |\beta|^2.$$

In conclusion,

$$\|a\|^2 = \gamma^2 + |\beta|^2 = \|[D,a]\|. \qquad (3.2.12)$$

We now consider a in F_2. We can write

$$a = \sum_{i=0}^{3}\sum_{j=1}^{3} \alpha_{ij}(s_i \otimes s_j), \text{ where } \alpha_{ij} \in \mathbb{R}.$$

We shall compute the entries of the matrix $[D,a]$. We start with $[D,a]_{20} = (m_2 - m_0)a_{20} = 3a_{20}$. We have $\mathcal{M}_{12,1} \ni a_{20} : F_0 \to F_2$. Each component of a_{20} can be computed using the following formula:

$$
\begin{aligned}
(aI, s_p \otimes s_r) &= \tau(\sum_{i=0}^{3}\sum_{j=1}^{3}\alpha_{ij}(s_i \otimes s_j)(s_p \otimes s_r)) \\
&= \sum_{i=0}^{3}\sum_{j=1}^{3}\alpha_{ij}\tau(s_i s_p)\tau(s_j s_r)) \\
&= \alpha_{pr},
\end{aligned}
$$

where $0 \le p \le 3$ and $1 \le r \le 3$. We continue with $[D, a]_{21} = (m_2 - m_1)a_{21} = 2a_{21}$. We have $\mathcal{M}_{12,3} \ni a_{21} : F_1 \to F_2$. Each component of a_{21} can be computed using the following formula, where $1 \le p \le 3$, $0 \le i \le 3$ and $1 \le j \le 3$.

$$
\begin{aligned}
(a(s_p \otimes I), s_i \otimes s_j) &= \tau(\sum_{m=0}^{3}\sum_{n=1}^{3}\alpha_{mn}(s_m s_p \otimes s_n)(s_i \otimes s_j)) \\
&= \sum_{m=0}^{3}\sum_{n=1}^{3}\alpha_{mn}\tau(s_m s_p s_i)\tau(s_n s_j)) \\
&= \sum_{m=0}^{3}\alpha_{mj}\tau(s_m s_p s_i),
\end{aligned}
$$

We can now write the matrix $([D, a])_{kl}$:

$$
\begin{array}{c|cccccccccccccccc}
 & F_0 & & F_1 & & & & & & & F_2 & & & & & & \\
\hline
 & 0 & 0 & 0 & 0 & -3\alpha_{01} & -3\alpha_{02} & -3\alpha_{03} & -3\alpha_{11} & -3\alpha_{12} & -3\alpha_{13} & -3\alpha_{21} & -3\alpha_{22} & -3\alpha_{23} & -3\alpha_{31} & -3\alpha_{32} & -3\alpha_{33} \\
 & 0 & 0 & 0 & 0 & -2\alpha_{11} & -2\alpha_{12} & -2\alpha_{13} & -2\alpha_{01} & -2\alpha_{02} & -2\alpha_{03} & 2i\alpha_{31} & 2i\alpha_{32} & 2i\alpha_{33} & -2i\alpha_{21} & -2i\alpha_{22} & -2i\alpha_{23} \\
F_1 & 0 & 0 & 0 & 0 & -2\alpha_{21} & -2\alpha_{22} & -2\alpha_{23} & -2i\alpha_{31} & -2i\alpha_{32} & -2i\alpha_{33} & -2\alpha_{01} & -2\alpha_{02} & -2\alpha_{03} & -2i\alpha_{11} & -2i\alpha_{12} & -2i\alpha_{13} \\
 & 0 & 0 & 0 & 0 & -2\alpha_{31} & -2\alpha_{32} & -2\alpha_{33} & 2i\alpha_{21} & 2i\alpha_{22} & 2i\alpha_{23} & 2i\alpha_{11} & 2i\alpha_{12} & 2i\alpha_{13} & -2\alpha_{01} & -2\alpha_{02} & -2\alpha_{03} \\
 & 3\alpha_{01} & 2\alpha_{11} & 2\alpha_{21} & 2\alpha_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 3\alpha_{02} & 2\alpha_{12} & 2\alpha_{22} & 2\alpha_{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 3\alpha_{03} & 2\alpha_{13} & 2\alpha_{23} & 2\alpha_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 3\alpha_{11} & 2\alpha_{01} & 2i\alpha_{31} & -2i\alpha_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_2 & 3\alpha_{12} & 2\alpha_{02} & 2i\alpha_{32} & -2i\alpha_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 3\alpha_{13} & 2\alpha_{03} & 2i\alpha_{33} & -2i\alpha_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 3\alpha_{21} & -2i\alpha_{31} & 2\alpha_{01} & -2i\alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 3\alpha_{22} & -2i\alpha_{32} & 2\alpha_{02} & -2i\alpha_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 3\alpha_{23} & -2i\alpha_{33} & 2\alpha_{03} & -2i\alpha_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 3\alpha_{31} & 2i\alpha_{21} & 2i\alpha_{11} & 2\alpha_{01} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 3\alpha_{32} & 2i\alpha_{22} & 2i\alpha_{12} & 2\alpha_{02} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 3\alpha_{33} & 2i\alpha_{23} & 2i\alpha_{13} & 2\alpha_{03} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

The only component left to be computed is a_{22}. We have $\mathcal{M}_{12,12} \ni a_{22} : F_2 \to F_2$. Each component of a_{22} can be computed using the following formula, where $0 \le i \le 3$, $1 \le j \le 3$, $0 \le k \le 3$ and $1 \le l \le 3$.

$$
\begin{aligned}
(a(s_i \otimes s_j), s_k \otimes s_l) &= \tau(\sum_{m=0}^{3}\sum_{n=1}^{3} \alpha_{mn}(s_m s_i \otimes s_n s_j)(s_k \otimes s_l)) \\
&= \sum_{m=0}^{3}\sum_{n=1}^{3} \alpha_{mn}\tau(s_m s_i s_k)\tau(s_n s_j s_l)),
\end{aligned}
$$

We can now write the matrix (a_{kl}):

$$
\begin{array}{ccccccccccccccccc}
 & F_0 & & F_1 & & & & & & & F_2 & & & & & & \\
 & 0 & 0 & 0 & 0 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{31} & \alpha_{32} & \alpha_{33} \\
 & 0 & 0 & 0 & 0 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{01} & \alpha_{02} & \alpha_{03} & -i\alpha_{31} & -i\alpha_{32} & -i\alpha_{33} & i\alpha_{21} & i\alpha_{22} & i\alpha_{23} \\
F_1 & 0 & 0 & 0 & 0 & \alpha_{21} & \alpha_{22} & \alpha_{23} & i\alpha_{31} & i\alpha_{32} & i\alpha_{33} & \alpha_{01} & \alpha_{02} & \alpha_{03} & i\alpha_{11} & i\alpha_{12} & i\alpha_{13} \\
 & 0 & 0 & 0 & 0 & \alpha_{31} & \alpha_{32} & \alpha_{33} & -i\alpha_{21} & -i\alpha_{22} & -i\alpha_{23} & -i\alpha_{11} & -i\alpha_{12} & -i\alpha_{13} & \alpha_{01} & \alpha_{02} & \alpha_{03} \\
 & \alpha_{01} & \alpha_{11} & \alpha_{21} & \alpha_{31} & 0 & i\alpha_{03} & -i\alpha_{02} & 0 & i\alpha_{13} & -i\alpha_{12} & 0 & i\alpha_{23} & -i\alpha_{22} & 0 & i\alpha 33 & -i\alpha_{32} \\
 & \alpha_{02} & \alpha_{12} & \alpha_{22} & \alpha_{32} & -i\alpha_{03} & 0 & -i\alpha 01 & -i\alpha_{13} & 0 & -i\alpha_{11} & -i\alpha_{23} & 0 & -i\alpha_{21} & -i\alpha_{33} & 0 & -i\alpha_{31} \\
 & \alpha_{03} & \alpha_{13} & \alpha_{23} & \alpha_{33} & i\alpha_{02} & i\alpha_{01} & 0 & i\alpha_{12} & i\alpha_{11} & 0 & i\alpha_{22} & i\alpha_{21} & 0 & i\alpha_{32} & i\alpha_{31} & 0 \\
 & \alpha_{11} & \alpha_{01} & i\alpha_{31} & -i\alpha_{21} & 0 & i\alpha_{13} & -i\alpha_{12} & 0 & i\alpha_{03} & -i\alpha_{02} & 0 & -\alpha_{33} & \alpha_{32} & 0 & \alpha_{23} & -\alpha_{22} \\
F_2 & \alpha_{12} & \alpha_{02} & i\alpha_{32} & -i\alpha_{22} & -i\alpha_{13} & 0 & -i\alpha_{11} & -i\alpha_{03} & 0 & -i\alpha_{01} & \alpha_{33} & 0 & \alpha_{31} & -\alpha_{23} & 0 & -\alpha_{21} \\
 & \alpha_{13} & \alpha_{03} & i\alpha_{33} & -i\alpha_{23} & i\alpha_{12} & i\alpha_{11} & 0 & i\alpha_{02} & i\alpha_{01} & 0 & -\alpha_{32} & -\alpha_{31} & 0 & \alpha_{22} & \alpha_{21} & 0 \\
 & \alpha_{21} & -i\alpha_{31} & \alpha_{01} & -i\alpha_{11} & 0 & i\alpha_{23} & -i\alpha_{22} & 0 & \alpha_{33} & -\alpha_{32} & 0 & i\alpha_{03} & -i\alpha_{02} & 0 & \alpha_{13} & -\alpha_{12} \\
 & \alpha_{22} & -i\alpha_{32} & \alpha_{02} & -i\alpha_{12} & -i\alpha_{23} & 0 & -i\alpha_{21} & -\alpha_{33} & 0 & -\alpha_{31} & -i\alpha_{03} & 0 & -i\alpha_{01} & -\alpha_{13} & 0 & -\alpha_{11} \\
 & \alpha_{23} & -i\alpha_{33} & \alpha_{03} & -i\alpha_{13} & i\alpha_{22} & i\alpha_{21} & 0 & \alpha_{32} & \alpha_{31} & 0 & i\alpha_{02} & i\alpha_{01} & 0 & \alpha_{12} & \alpha_{11} & 0 \\
 & \alpha_{31} & i\alpha_{21} & i\alpha_{11} & \alpha_{01} & 0 & i\alpha_{33} & -i\alpha_{32} & 0 & -\alpha_{23} & \alpha_{22} & 0 & -\alpha_{13} & \alpha_{12} & 0 & i\alpha_{03} & -i\alpha_{02} \\
 & \alpha_{32} & i\alpha_{22} & i\alpha_{12} & \alpha_{02} & -i\alpha_{33} & 0 & -i\alpha_{31} & \alpha_{23} & 0 & \alpha_{21} & \alpha_{13} & 0 & \alpha_{11} & -i\alpha_{03} & 0 & -i\alpha_{01} \\
 & \alpha_{33} & i\alpha_{23} & i\alpha_{13} & \alpha_{03} & i\alpha_{32} & i\alpha_{31} & 0 & -\alpha_{22} & -\alpha_{21} & 0 & -\alpha_{12} & -\alpha_{11} & 0 & i\alpha_{02} & i\alpha_{01} & 0
\end{array}
$$

We made a numerical experiment, via the software MAPLE, to obtain information about the relation between $\|a\|$ and $\|[D,a]\|$. In this experiment we took random values (normal distribution) for α_{ij} (in the matrix A) and computed the norms of $[D,a]$ (called DR) and a. We repeated the experiment 10 times and plotted the pairs $(x,y) = (\|[D,\alpha_i]\|, \|\alpha_i\|)$ in $\mathbb{R}^2$. The plot is printed bellow. The results tell us that it seems that $\|a\| \le \frac{3}{4}\|[D,a]\|$ or may be even with a smaller factor. The reason why we did choose the factor $\frac{3}{4}$ is that our following experiments for F_3 indicate that here a facot of $\frac{3}{8}$ might work.

```
> randentry := proc() stats[random, normald](1) end proc;
```

$$\mathit{randentry} := \mathbf{proc}()\ \mathit{stats}_{\mathit{random},\ \mathit{normald}}(1)\ \mathbf{end\ proc}$$

```
> norms := proc(n)
> A := randmatrix(4,3, entries = randentry):
> DR:=array(1..16,1..16,[[0,0,0,0,seq(seq(-3*A[i,j],j=1..3),i=1..4)],[0,
> 0,0,0,-2*A[2,1],-2*A[2,2],-2*A[2,3],-2*A[1,1],-2*A[1,2],-2*A[1,3],2*I*
> A[4,1],2*I*A[4,2],2*I*A[4,3],-2*I*A[3,1],-2*I*A[3,2],-2*I*A[3,3]],[0,0
> ,0,0,-2*A[3,1],-2*A[3,2],-2*A[3,3],-2*I*A[4,1],-2*I*A[4,2],-2*I*A[4,3]
> ,-2*A[1,1],-2*A[1,2],-2*A[1,3],-2*I*A[2,1],-2*I*A[2,2],-2*I*A[2,3]],[0
> ,0,0,0,-2*A[4,1],-2*A[4,2],-2*A[4,3],2*I*A[3,1],2*I*A[3,2],2*I*A[3,3],
> 2*I*A[2,1],2*I*A[2,2],2*I*A[2,3],-2*A[1,1],-2*A[1,2],-2*A[1,3]],[3*A[1
> ,1],2*A[2,1],2*A[3,1],2*A[4,1],0,0,0,0,0,0,0,0,0,0,0,0],[3*A[1,2],2*A[
> 2,2],2*A[3,2],2*A[4,2],0,0,0,0,0,0,0,0,0,0,0,0],[3*A[1,3],2*A[2,3],2*A
> [3,3],2*A[4,3],0,0,0,0,0,0,0,0,0,0,0,0],[3*A[2,1],2*A[1,1],2*I*A[4,1],
> -2*I*A[3,1],0,0,0,0,0,0,0,0,0,0,0,0],[3*A[2,2],2*A[1,2],2*I*A[4,2],-2*
> I*A[3,2],0,0,0,0,0,0,0,0,0,0,0,0],[3*A[2,3],2*A[1,3],2*I*A[4,3],-2*I*A
> [3,3],0,0,0,0,0,0,0,0,0,0,0,0],[3*A[3,1],-2*I*A[4,1],2*A[1,1],-2*I*A[2
> ,1],0,0,0,0,0,0,0,0,0,0,0,0],[3*A[3,2],-2*I*A[4,2],2*A[1,2],-2*I*A[2,2
> ],0,0,0,0,0,0,0,0,0,0,0,0],[3*A[3,3],-2*I*A[4,3],2*A[1,3],-2*I*A[2,3],
> 0,0,0,0,0,0,0,0,0,0,0,0],[3*A[4,1],2*I*A[3,1],2*I*A[2,1],2*A[1,1],0,0,
> 0,0,0,0,0,0,0,0,0,0],[3*A[4,2],2*I*A[3,2],2*I*A[2,2],2*A[1,2],0,0,0,0,
> 0,0,0,0,0,0,0,0],[3*A[4,3],2*I*A[3,3],2*I*A[2,3],2*A[1,3],0,0,0,0,0,0,
> 0,0,0,0,0,0]]):
> AA :=
> array(1..16,1..16,[[0,0,0,0,seq(seq(A[i,j],j=1..3),i=1..4)],[0,0,0,0,A
> [2,1],A[2,2],A[2,3],A[1,1],A[1,2],A[1,3],-I*A[4,1],-I*A[4,2],-I*A[4,3]
> ,I*A[3,1],I*A[3,2],I*A[3,3]],[0,0,0,0,A[3,1],A[3,2],A[3,3],I*A[4,1],I*
> A[4,2],I*A[4,3],A[1,1],A[1,2],A[1,3],I*A[2,1],I*A[2,2],I*A[2,3]],[0,0,
> 0,0,A[4,1],A[4,2],A[4,3],-I*A[3,1],-I*A[3,2],-I*A[3,3],-I*A[2,1],-I*A[
> 2,2],-I*A[2,3],A[1,1],A[1,2],A[1,3]],[A[1,1],A[2,1],A[3,1],A[4,1],0,I*
> A[1,3],-I*A[1,2],0,I*A[2,3],-I*A[2,2],0,I*A[3,3],-I*A[3,2],0,I*A[4,3],
> -I*A[4,2]],[A[1,2],A[2,2],A[3,2],A[4,2],-I*A[1,3],0,-I*A[1,1],-I*A[2,3
> ],0,-I*A[2,1],-I*A[3,3],0,-I*A[3,1],-I*A[4,3],0,-I*A[4,1]],[A[1,3],A[2
> ,3],A[3,3],A[4,3],I*A[1,2],I*A[1,1],0,I*A[2,2],I*A[2,1],0,I*A[3,2],I*A
> [3,1],0,I*A[4,2],I*A[4,1],0],[A[2,1],A[1,1],I*A[4,1],-I*A[3,1],0,I*A[2
> ,3],-I*A[2,2],0,I*A[1,3],-I*A[1,2],0,-A[4,3],A[4,2],0,A[3,3],-A[3,2]],
> [A[2,2],A[1,2],I*A[4,2],-I*A[3,2],-I*A[2,3],0,-I*A[2,1],-I*A[1,3],0,-I
> *A[1,1],A[4,3],0,A[4,1],-A[3,3],0,-A[3,1]],[A[2,3],A[1,3],I*A[4,3],-I*
> A[3,3],I*A[2,2],I*A[2,1],0,I*A[1,2],I*A[1,1],0,-A[4,2],-A[4,1],0,A[3,2
> ],A[3,1],0],[A[3,1],-I*A[4,1],A[1,1],-I*A[2,1],0,I*A[3,3],-I*A[3,2],0,
> A[4,3],-A[4,2],0,I*A[1,3],-I*A[1,2],0,A[2,3],-A[2,2]],[A[3,2],-I*A[4,2
> ],A[1,2],-I*A[2,2],-I*A[3,3],0,-I*A[3,1],-A[4,3],0,-A[4,1],-I*A[1,3],0
> ,-I*A[1,1],-A[2,3],0,-A[2,1]],[A[3,3],-I*A[4,3],A[1,3],-I*A[2,3],I*A[3
> ,2],I*A[3,1],0,A[4,2],A[4,1],0,I*A[1,2],I*A[1,1],0,A[2,2],A[2,1],0],[A
> [4,1],I*A[3,1],I*A[2,1],A[1,1],0,I*A[4,3],-I*A[4,2],0,-A[3,3],A[3,2],0
> ,-A[2,3],A[2,2],0,I*A[1,3],-I*A[1,2]],[A[4,2],I*A[3,2],I*A[2,2],A[1,2]
> ,-I*A[4,3],0,-I*A[4,1],A[3,3],0,A[3,1],A[2,3],0,A[2,1],-I*A[1,3],0,-I*
> A[1,1]],[A[4,3],I*A[3,3],I*A[2,3],A[1,3],I*A[4,2],I*A[4,1],0,-A[3,2],-
> A[3,1],0,-A[2,2],-A[2,1],0,I*A[1,2],I*A[1,1],0]]):
> norm(DR,2),
> norm(AA,2);
> end proc:
```

```
> l := seq([norms(n)],n = 1..10);
```

$$l := [13.25161031, 7.859616443], [13.31668019, 6.766711514],$$
$$[8.484770570, 4.621121660], [14.07692584, 7.881939684],$$
$$[7.484945620, 4.303144303], [11.62830758, 6.581594766],$$
$$[11.37170281, 6.218004504], [9.531035922, 4.935525881],$$
$$[17.29650030, 9.423200684], [8.159897084, 4.564419262]$$

Based on these computations we computed the corresponding quotients $\frac{\|a_i\|}{\|[D,a_i]\|}$ and obtained the following list of results:

$$\frac{\|a_i\|}{\|[D,a_i]\|} = (0.59,\ 0.51,\ 0.54,\ 0.56,\ 0.57,\ 0.57,\ 0.55,\ 0.52,\ 0.54,\ 0.56).$$

From here it seems that

$$\sup\{\frac{\|a\|}{\|[D,a]\|}\ :\ a \in F_2 \text{ and } a \neq 0\} \leq \frac{3}{4}.$$

```
> plot([l], x=0..15, y=0..15, style=point);
```

Plot: dirac201.eps

We proceed with a in F_3. To make our computations easier we make the following notation

$$b_i := s_k \otimes s_l, \text{ with } i = k + 4l, \text{ where } 0 \leq k \leq 3, 0 \leq l \leq 3.$$

We can write

$$a = \sum_{i=0}^{15}\sum_{j=1}^{3}\alpha_{ij}(b_i \otimes s_j), \text{ where } \alpha_{ij} \in \mathbb{R}.$$

We shall compute now the entries of the matrix $[D,a]$. We start with $[D,a]_{3,0} = (m_3 - m_0)a_{3,0} = 7a_{2,0}$. We have $\mathcal{M}_{48,1} \ni a_{3,0} : F_0 \to F_3$. Each component of $a_{3,0}$ can be computed using the following formula:

$$\begin{aligned}(aI, b_i \otimes s_j) &= \tau(\sum_{m=0}^{15}\sum_{n=1}^{3}\alpha_{mn}(b_m \otimes s_n)(b_i \otimes s_j)) \\ &= \sum_{m=0}^{15}\sum_{n=1}^{3}\alpha_{mn}\tau(b_m b_i)\tau(s_n s_j)) \\ &= \alpha_{ij},\end{aligned}$$

where $0 \leq i \leq 15$ and $1 \leq j \leq 3$. We continue with $[D, a]_{3,1} = (m_3 - m_1)a_{3,1} = 6a_{3,1}$. We have $\mathcal{M}_{48,3} \ni a_{3,1} : F_1 \to F_3$. With the introduction of b_j we have a basis for the space F_1 given by the set $\{b_1, b_2, b_3\}$. The components of $a_{3,1}$ can, for $1 \leq p \leq 3$, $0 \leq k \leq 15$ and $1 \leq l \leq 3$, be computed by

$$\begin{aligned}(a(b_p \otimes I), b_k \otimes s_l) &= \tau(\sum_{i=0}^{15}\sum_{j=1}^{3}\alpha_{ij}(b_i b_p \otimes s_j)(b_k \otimes s_l)) \\ &= \sum_{i=0}^{15}\sum_{j=1}^{3}\alpha_{ij}\tau(b_i b_p b_k)\tau(s_j s_l)) \\ &= \sum_{i=0}^{15}\alpha_{il}\tau(b_i b_p b_k).\end{aligned}$$

We continue with $[D, a]_{3,2} = (m_3 - m_2)a_{3,2} = 4a_{3,2}$. We have $\mathcal{M}_{48,12} \ni a_{3,2} : F_2 \to F_3$. A basis for the space F_2 is given by the set $\{b_4, \ldots, b_{15}\}$ so each component of $a_{3,2}$ can be computed by

$$\begin{aligned}(a(b_p, b_k \otimes s_l) &= \tau(\sum_{i=0}^{15}\sum_{j=1}^{3}\alpha_{ij}((b_i b_p b_k) \otimes (s_j s_l))) \\ &= \sum_{i=0}^{15}\sum_{j=1}^{3}\alpha_{ij}\tau(b_i b_p b_k)\tau(s_j s_l)) \\ &= \sum_{i=0}^{15}\alpha_{il}\tau(b_i b_p b_k),\end{aligned}$$

where $4 \leq p \leq 15$, $0 \leq k \leq 15$ and $1 \leq l \leq 3$. The only component left to be computed is $a_{3,3}$. We have $\mathcal{M}_{48,48} \ni a_{3,3} : F_3 \to F_3$. Each component of $a_{2,2}$ can be computed using the following formula:

$$\begin{aligned}(a(b_p \otimes s_l), b_m \otimes s_n) &= \tau(\sum_{i=0}^{15}\sum_{j=1}^{3}\alpha_{ij}(b_i b_p \otimes s_j s_l)(b_m \otimes s_n)) \\ &= \sum_{i=0}^{15}\sum_{j=1}^{3}\alpha_{ij}\tau(b_i b_p b_m)\tau(s_j s_l s_n)),\end{aligned}$$

where $0 \leq p \leq 15$, $1 \leq l \leq 3$, $0 \leq m \leq 15$ and $1 \leq n \leq 3$. We made the same numerical experiment, via the software MAPLE, to obtain information about the relationship between $\|a\|$ and $\|[D,a]\|$. We repeated the experiment 5 times and we collected the values of $\|[D,a]\|$, respectively $\|a\|$. Then, we plotted the pairs $(x,y) = (\|a_i\|, \|[D,a_i]\|)$ and from here it seems that for $a \in F_3$ we have $\|a\| \leq \frac{3}{8}\|[D,a]\|$. The printouts from MAPLE is shown on the next couple of pages.

```
> randentry:=proc() stats[random,normald](1) end proc:
```

```
> norms:=proc(a)
> local A, AA, B, S, H1, H2, innerm1, innerm2, produ, P, outerm, H3, M1,
> M2, M3, M4, M5, DR, N1, N2, N3, AAA:
> A:=randmatrix(16,3,entries=randentry):
> AA:=matrix(48,1,[A[1,1],A[1,2],A[1,3],A[2,1],A[2,2],A[2,3],A[3,1],A[3,
> 2],A[3,3],A[4,1],A[4,2],A[4,3],A[5,1],A[5,2],A[5,3],A[6,1],A[6,2],A[6,
> 3],A[7,1],A[7,2],A[7,3],A[8,1],A[8,2],A[8,3],A[9,1],A[9,2],A[9,3],A[10
> ,1],A[10,2],A[10,3],A[11,1],A[11,2],A[11,3],A[12,1],A[12,2],A[12,3],A[
> 13,1],A[13,2],A[13,3],A[14,1],A[14,2],A[14,3],A[15,1],A[15,2],A[15,3],
> A[16,1],A[16,2],A[16,3]]):
> B[1]:=matrix(4,4,[1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1]):
> B[2]:=matrix(4,4,[0,-I,0,0,I,0,0,0,0,0,0,-I,0,0,I,0]):
> B[3]:=matrix(4,4,[0,1,0,0,1,0,0,0,0,0,0,1,0,0,1,0]):
> B[4]:=matrix(4,4,[1,0,0,0,0,-1,0,0,0,0,1,0,0,0,0,-1]):
> B[5]:=matrix(4,4,[0,0,-I,0,0,0,0,-I,I,0,0,0,0,I,0,0]):
> B[6]:=matrix(4,4,[0,0,0,-1,0,0,1,0,0,1,0,0,-1,0,0,0]):
> B[7] :=matrix(4,4,[0,0,0,-I,0,0,I,0,0,-I,0,0,I,0,0,0]):
> B[8] :=matrix(4,4,[0,0,-I,0,0,0,0,I,I,0,0,0,0,-I,0,0]):
> B[9] :=matrix(4,4,[0,0,1,0,0,0,0,1,1,0,0,0,0,1,0,0]):
> B[10]:=matrix(4,4,[0,0,0,-I,0,0,I,0,0,-I,0,0,I,0,0,0]):
> B[11]:=matrix(4,4,[0,0,0,1,0,0,1,0,0,1,0,0,1,0,0,0]):
> B[12]:=matrix(4,4,[0,0,1,0,0,0,0,-1,1,0,0,0,0,-1,0,0]):
> B[13]:=matrix(4,4,[1,0,0,0,0,1,0,0,0,0,-1,0,0,0,0,-1]):
> B[14]:=matrix(4,4,[0,-I,0,0,I,0,0,0,0,0,0,I,0,0,-I,0]):
> B[15]:=matrix(4,4,[0,1,0,0,1,0,0,0,0,0,0,-1,0,0,-1,0]):
> B[16]:=matrix(4,4,[1,0,0,0,0,-1,0,0,0,0,-1,0,0,0,0,1]):
> S[1] := matrix(4,4,[0,-I,0,0,I,0,0,0,0,0,0,-I,0,0,I,0]):
> S[2]:=matrix(4,4,[0,1,0,0,1,0,0,0,0,0,0,1,0,0,1,0]):
> S[3]:=matrix(4,4,[1,0,0,0,0,-1,0,0,0,0,1,0,0,0,0,-1]):
> H1 := matrix(48,3, (m,n) -> (1/4)*trace( sum(
> A[j,(floor((m-1)/16)+1)]* B[j] &* S[n] &* B[((m-1)mod 16)+1]
> ,j=1..16) )    ):
> H2 := matrix(48,12, (m,n) -> (1/4)*trace( sum(
> A[i,(floor((m-1)/16)+1)]* B[i] &* B[n+4] &* B[((m-1)mod 16)+1]
> ,i=1..16) )    ):
> innerm1:=(i,m,n)->1/4*trace(B[i]&*B[(n-1)mod 16+1]&*B[(m-1)mod 16+1]):
> innerm2:=(j,m,n)->1/4*trace(S[j]&*S[floor((n-1)/16)+1]&*S[floor((m-1)/
> 16)+1]):
> produ:=(i,j,m,n)->A[i,j]*innerm1(i,m,n)*innerm2(j,m,n):
> P:=[seq([seq([seq([seq(produ(i,j,m,n),i=1..16)],j=1..3)],m=1..48)],n=1
> ..48)]:
> outerm:=(m,n)->sum(sum(P[n][m][j][i],j=1..3),i=1..16):
> H3:=matrix(48,48,(m,n)->outerm(m,n)):
> M1:=matrix(16,16,0):
> M2:=matrix(48,48,0):
> M3:=blockmatrix(3,1,[-7*transpose(AA),-6*transpose(H1),-4*transpose(H2
> )]):
> M4:=blockmatrix(1,4,[7*AA,6*H1,4*H2,M2]):
> M5:=blockmatrix(1,2,[M1,M3]):
> DR:=blockmatrix(2,1,[M5,M4]):
> N1:=blockmatrix(3,1,[transpose(AA),transpose(H1),transpose(H2)]):
> N2:=blockmatrix(1,2,[M1,N1]):
> N3:=blockmatrix(1,4,[AA,H1,H2,H3]):
> AAA:=blockmatrix(2,1,[N2,N3]):
> norm(DR,2), norm(AAA,2)
> end proc:
```

```
> l:=seq([norms(a)],a=1..5);
```

$$l := [49.23519920, 12.54076608], [44.04473093, 10.66248587],$$
$$[50.23041560, 12.00635712], [55.84860974, 14.96886859],$$
$$[60.74300472, 15.09481739]$$

The corresponding quotients of the norms was then computed and they are listed in the following list.

$$\frac{\|a_i\|}{\|[D,a_i]\|} = (0.26,\ 0.24,\ 0.24,\ 0.27,\ 0.25).$$

It seems that the maximal quotient in F_3 might not be bigger than $\frac{3}{8}$.

```
> plot([l],x=10..70,y=10..20,style=point);
```

Plot: CrErik201.eps

Based on the computations and experiments for F_1, F_2 and F_3 we make the following conjecture:

$$\sup\{\frac{\|a\|}{\|[D,a]\|} \ : \ a \in F_n, a \neq 0\} = O(2^{-n}).$$

3.3 Compact operators on $l^2(\mathbb{N})$

In this section we are considering the C*-algebra of compact operators on a separable Hilbert space H. Our construction is based on a concrete orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ for H so we will identify H with $l^2(\mathbb{N})$ in the sequel. When seeking a spectral triple for the compact operators, $\mathcal{K} = \mathcal{K}(l^2(\mathbb{N}))$, it seemed natural to look for one where $\mathcal{K}$ was represented as multiplication operators on the Hilbert space consisting of Hilbert-Schmidt operators. Although we tried several natural candidates we were always forced to use an argument which was based on the irreducible representation of $\mathcal{K}$ on $l^2(\mathbb{N})$. It turns out that a quadruple of this representation will serve as a possibility for the representation of a spectral triple. The following arguments are based on computations with infinite matrices. Many of the arguments are similar to, or inspired by, the arguments used in [ORi] and [AC] for discrete groups. We will rely heavily on techniques taken from [AC].

3.3.1 A spectral triple for $\mathcal{K}(l^2(\mathbb{N}))$

Theorem 3.3.1. *Let $\mathcal{K}$ denote the compact operators on $l^2(\mathbb{N})$ and let π denote the 4-fold amplification-representation of $\mathcal{K}$ on the Hilbert space $H = \overset{4}{\underset{1}{\oplus}} l^2(\mathbb{N})$. Let S denote the unilateral shift on $l^2(\mathbb{N})$ which acts on the standard orthonormal basis by*

$$\forall n \in \mathbb{N} : Se_n = e_{n+1},$$

and let M denote the unbounded self-adjoint operator on $l^2(\mathbb{N})$ which is defined by

$$\forall n \in \mathbb{N} : \ Me_n = n^2 e_n,$$

and

$$\mathrm{dom(M) = \{x \in l^2(\mathbb{N}) : \sum n^4 |x_n|^2 < \infty\}.}$$

An operator D is defined on H with respect to matrix decomposition of $B\left(\overset{4}{\underset{1}{\oplus}} l^2(\mathbb{N})\right)$ as

$$D := \begin{pmatrix} 0 & 0 & M & 0 \\ 0 & 0 & 0 & MS^* \\ M & 0 & 0 & 0 \\ 0 & SM & 0 & 0 \end{pmatrix}$$

and

$$\mathrm{dom(D) = \{(x_1, x_2, x_3, x_4) \in H : x_i \in dom(M), 1 \leq i \leq 4\}.}$$

The operator D has the following properties:

1. *D self-adjoint;*

2. *the spectrum of D equals $\{\pm n^2 : n \in \mathbb{N}_0\}$;*

3. *the operator $(I + D^2)^{-1}$ is compact of trace class;*

4. *the sum over the nonzero eigenvalues $\sum_{n\in\mathbb{N}} |\lambda_n|^{-s} = 4\zeta(2s)$, (Riemann's zeta function);*

5. *the set $\mathcal{C} = \{k \in \mathcal{K} : \|[D, \pi(k)]\| \leq 1\}$ is relatively compact and has the property that* $\mathrm{span}(\mathcal{C})$ *is dense in $\mathcal{K}$.*

Proof. To see that (1) is true we first remark that $S^*\mathrm{dom}(M) \subseteq M$, so it is possible to define D as postulated, as a densely defined operator, which has an adjoint D^*. The operator D has the property that in the matrix for D, each row and each column has only one non-zero element, hence D^* must be given by

$$D^* = \begin{pmatrix} 0 & 0 & M & 0 \\ 0 & 0 & 0 & (SM)^* \\ M & 0 & 0 & 0 \\ 0 & (MS^*)^* & 0 & 0 \end{pmatrix}$$

Since S is an isometry we get

$$(SM)^* = MS^*$$

with

$$\mathrm{dom}(MS^*) = S\ \mathrm{dom}(M)\ \oplus\ \ker(S^*).$$

Since $\frac{n+1}{n} \to 1$ for $n \to \infty$, $\mathrm{dom}(MS^*) = \mathrm{dom}(M)$ and D is self-adjoint. The spectrum of D is clearly the union of the spectra of the operators

$$\begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & MS^* \\ SM & 0 \end{pmatrix}$$

on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$. The first operator clearly has the eigenvalues $\{\pm n^2 : n \in \mathbb{N}\}$, each counted with multiplicity one. The second operator has its polar decomposition given as

$$\begin{pmatrix} 0 & MS^* \\ SM & 0 \end{pmatrix} = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & SMS^* \end{pmatrix}.$$

Hence its eigenvalues are $\{\pm n^2 : n \in \mathbb{N}_0\}$, where the multiplicity also is one for each eigenvalue. It is now clear that $(I+D^2)^{-1}$ is compact and that the sum over the numerical values of the nonzero eigenvalues $\sum_{n\in\mathbb{N}} |\lambda_n|^{-s} = 4\zeta(2s)$. In order to prove the last part of the statement let us first remark that for elements k in $\mathcal{K}$ with finite matrices, the commutator $[D, \pi(k)]$ is clearly bounded, so the set $\mathcal{C}$ has the property that $\mathrm{span}(\mathcal{C})$ is dense in $\mathcal{K}$. Further, we will make the convention that whenever a $k \in \mathcal{K}$ has the property that $[D, \pi(k)]$ is bounded, we let $[D, \pi(k)]$ denote the closure of this bounded commutator. With this in mind we extend the convention to commutators of the form $[M, k]$, $[MS^*, k]$ and $[SM, k]$. It is clear that $[D, \pi(k)]$ is bounded if and only if all three of the commutators above are bounded, so for a $k \in \mathcal{C}$ we will try to exploit the fact that

$$\|[M, k]\| \leq 1, \ \|[MS^*, k]\| \leq 1, \ \|[SM, k]\| \leq 1.$$

The matrix for $[M, k] = ([M, k]_{ij})$ is given as

$$[M, k]_{ij} = (i^2 - j^2)k_{ij}.$$

Now $(i^2 - j^2) = (i - j)(i + j)$, and we can apply our methods from the proof of Theorem 3.6 (or the one used by N. Ozawa and M. A. Rieffel in [ORi]) to prove that the operator x with the matrix given by

$$x_{i,j} = \begin{cases} 0 & \text{if } i = j \\ \frac{1}{i-j}[M, k]_{i,j} & \text{if } i \neq j \end{cases} = \begin{cases} 0 & \text{if } i = j \\ (i + j)k_{i,j} & \text{if } i \neq j \end{cases}$$

has the property that $\|x\| \leq \frac{\pi}{\sqrt{3}}$.

This inequality does not give any information about the main diagonal of k, which we will denote k_d, but we can approximate the operator $k_{off} = k - k_d$ by an operator of finite rank by establishing norm estimates for the operators $k_{off,n}, n \in \mathbb{N}$ which are defined by

$$(k_{off,n})_{ij} = \begin{cases} 0 & \text{if } i = j \\ 0 & \text{if } i + j \leq n \\ k_{ij} & \text{if } i \neq j \text{ and } i + j > n \ . \end{cases}$$

This means that $k_{off,n}$ is the operator one gets from k_{off} when all its SW-NE diagonals up to and inclusive, number $n-1$, all are removed. In order to get an estimate of $k_{off,n}$ we use a technique similar to the one used when showing that $\|x\| \leq \frac{\pi}{\sqrt{3}}$. For $z \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ let $u(z)$ denote the unitary operator on $l^2(\mathbb{N})$ with infinite matrix given by the following expression,

$$u(z)_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ z^i & \text{if } i = j \ . \end{cases}$$

Then $\|x\| = \|u(z)xu(z)\|$ and

$$(u(z)xu(z))_{ij} = \begin{cases} 0 & \text{if } i = j \\ (i+j)z^{(i+j)}k_{ij} & \text{if } i \neq j \ . \end{cases}$$

Let now $n \in \mathbb{N}$ and define $g_n \in L^2(\mathbb{T}), \mathbb{C})$ by

$$g_n(z) = \sum_{l=n+1}^{\infty} l^{-1} z^l .$$

Then,

$$\|g_n\|_2^2 = \sum_{l=n+1}^{\infty} l^{-2} \leq \frac{1}{n},$$

so for each vector $\xi, \eta \in l^2(\mathbb{N})$ the function $h(z)$ defined by

$$h(z) = (u(z)xu(z)\xi, \eta)$$

will satisfy

$$\|(h * g_n)\|_\infty \leq \|x\|\|\xi\|\|\eta\|\|g_n\|_2 \leq \frac{1}{\sqrt{n}}\|x\|\|\xi\|\|\eta\|.$$

This means that the operator $u(z)k_{off,n}u(z)$ whose matrix is given by

$$(u(z)k_{off}, u(z))_{ij} = \begin{cases} 0 & \text{if } i = j \\ 0 & \text{if } i+j \leq n \\ z^{(i+j)}k_{ij} & \text{if } i+j > n \end{cases}$$

has norm at most $\frac{\pi}{\sqrt{3n}}$, and the same is true for $k_{off,n}$. We conclude that

$$\forall k \in \mathcal{C} : \ \|k_{off,n}\| \leq \frac{\pi}{\sqrt{3n}}. \tag{3.3.1}$$

Remark that for $n = 1$ $k_{off,n} = k_{off}$ so $\|k_{off}\| \leq \frac{\pi}{\sqrt{3}}$. The analysis above yields no information on the diagonal k_d for $k \in \mathcal{C}$. Here the relation $\|[SM, k]\| \leq 1$ can be used, especially if we want to estimate the part of the diagonal k_d for k corresponding to the entries larger than n. Such a result will enable us to get a complete norm estimate of k_n, meaning the operator obtained from k by removing the first n SW-NE diagonals. Since the unilateral shift S is supported by the first NW-SE diagonal below the main diagonal we will have to consider this diagonal of the commutator $[SM, k]$ in order to get information on the main diagonal of k. Let d_1 denote the first NW-SE diagonal of $[SM, k]$ below the main diagonal. Then $\|d_1\| \leq 1$ and

$$(d_1)_{ij} = \begin{cases} 0 & \text{if } i \neq j+1 \\ j^2(k_{jj} - k_{(j+1)(j+1)}) & \text{if } i = j+1\ . \end{cases}$$

In particular we get

$$\forall j \in \mathbb{N}: \ |k_{jj} - k_{(j+1)(j+1)}| \leq j^{-2}.$$

Consequently the sum

$$\sum_{j=m}^{\infty} (k_{jj} - k_{(j+1)(j+1)})$$

is absolutely converging with sum

$$k_{mm} - \lim_{j \to \infty} k_{(j+1)(j+1)} = k_{mm}.$$

So we have

$$|k_{11}| \leq \frac{\pi^2}{6}$$

and for $m > 1$

$$|k_{mm}| \leq \frac{1}{m-1}.$$

Altogether we have

$$\forall k \in \mathcal{C}: \ \|\text{main diag}(k)\| \leq \frac{\pi^2}{6} \tag{3.3.2}$$

and

$$\forall m \in \mathbb{N},\, m > 1\, \forall k \in \mathcal{C}: \ \sup_{n \geq m} |k_{nn}| \leq \frac{1}{m-1}. \tag{3.3.3}$$

A continuation using the results from (3.3.1), (3.3.2) and (3.3.3) shows that for all $k \in \mathcal{C}$ we have

$$\|k\| \leq \frac{\pi}{\sqrt{3}} + \frac{\pi^2}{6} \leq 4 \tag{3.3.4}$$

and

$$\forall n \geq 9: \ \|k_n\| \leq \frac{\pi}{\sqrt{3n}} + \frac{1}{n-1} \leq \frac{\pi}{\sqrt{3n}} + \frac{1}{\sqrt{n}} \leq \frac{4}{\sqrt{n}}. \tag{3.3.5}$$

For an $\varepsilon > 0$ we may choose n such that

$$\frac{4}{\sqrt{n}} < \frac{\varepsilon}{2} \text{ i.e. } n > \frac{64}{\varepsilon^2}.$$

Then for each k in $\mathcal{C}$ we have $\|k_n\| < \frac{\varepsilon}{2}$ and $k - k_n$ is supported on the first n SW-NE diagonals only. This is a finite dimensional space, actually of dimension at most

$$1 + 2 + ... + n - 1 = \frac{n(n-1)}{2} \text{ and } \|k - k_n\| \leq 4 + \frac{\varepsilon}{2},$$

so it can be covered by a finite number of balls of radius ε. We conclude that $\mathcal{C}$ is pre compact. ∎

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