Fourier Analysis

Christian Berg

January 27, 2012

Contents

Preface These notes are based on handwritten lecture notes in Danish from a graduate course in 1999. Parts of the Danish notes were written by Tage Gutmann Madsen for the second year analysis course in the 1980ies.

I am very grateful to Natalie Gilka for converting these Danish notes into a LaTeX-file in English. At the same time additional material has been added.

Copenhagen, August 2009

These notes were used in a 9 weeks course in 2011-12. I used this opportunity to correct some minor inaccuracies.

Christian Berg

Chapter 1

Fourier series

1.1 Periodic functions and their convolution

The group behind Fourier series is the circle group $\mathbb{T}$ consisting of the complex numbers with absolute value 1. This is certainly a compact commutative group under multiplication. The mapping $t \mapsto e^{\mathrm{i}t}$ is a continuous homomorphism of the additive group $(\mathbb{R},+)$ onto $(\mathbb{T},\cdot)$

$$\mathbb{T} = \left\{e^{\mathrm{i}t} \mid t \in [0,2\pi[\ \right\} = \left\{e^{\mathrm{i}t} \mid t \in \mathbb{R}\right\}.$$

Under the mapping $t \mapsto e^{\mathrm{i}t}$ the normalized Lebesgue measure on $[0,2\pi[$ is mapped onto a probability measure m on $(\mathbb{T},\mathbb{B}(\mathbb{T}))$ (any topological space X is considered as a measurable space equipped with the σ-algebra $\mathbb{B} = \mathbb{B}(X)$ of Borel sets) such that

$$\int_{\mathbb{T}} F(z)\,dm(z) = \frac{1}{2\pi}\int_0^{2\pi} F(e^{\mathrm{i}t})\,dt = \frac{1}{2\pi}\int_a^{a+2\pi} F(e^{\mathrm{i}t})\,dt, \quad a \in \mathbb{R}$$

for any continuous function $F : \mathbb{T} \to \mathbb{C}$. The set of continuous functions $F : \mathbb{T} \to \mathbb{C}$ is denoted $C(\mathbb{T})$. It becomes a Banach space under the uniform norm

$$\|F\|_\infty = \sup_{z\in\mathbb{T}} |F(z)|.$$

For any function $F : \mathbb{T} \to \mathbb{C}$, the composed function

$$f(t) = F(e^{\mathrm{i}t}), \qquad t \in \mathbb{R} \tag{1.1.1}$$

is a 2π-periodic function, i.e.,

$$f(t + 2\pi p) = f(t), \qquad t \in \mathbb{R},\ p \in \mathbb{Z},$$

and conversely, to any such function $f : \mathbb{R} \to \mathbb{C}$, there exists a unique function $F : \mathbb{T} \to \mathbb{C}$ such that (1.1.1) holds.

Therefore we can identify $C(\mathbb{T})$ with the spaces of continuous 2π-periodic functions on $\mathbb{R}$. Similarly, we can identify the Lebesgue spaces $\mathcal{L}^p(\mathbb{T}) = \mathcal{L}^p(\mathbb{T}, \mathbb{B}(\mathbb{T}), m)$, $1 \le p \le \infty$ with the space of 2π-periodic Borel functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\begin{aligned} \|f\|_p &= \left(\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^p \, dt \right)^{1/p} < \infty, \quad 1 \le p < \infty, \\ \|f\|_\infty &= \text{ess sup} \{ |f(t)| \mid t \in [0, 2\pi[\} < \infty, p = \infty. \end{aligned}$$

For 2π-periodic Borel functions $f, g : \mathbb{R} \to \mathbb{C}$ we define the <u>convolution</u> $f * g$ as the function

$$x \mapsto f * g(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x-y) g(y) \, dy = \frac{1}{2\pi} \int_a^{a+2\pi} f(x-y) g(y) \, dy, \quad (1.1.2)$$

defined for those $x \in \mathbb{R}$ for which

$$y \mapsto f(x-y)g(y)$$

is integrable over $[0, 2\pi]$ (and thereby over any interval $[a, a+2\pi]$ of length 2π). The domain of definition $D = D(f * g)$ is a possibly empty Borel set in $\mathbb{R}$ with the property

$$x \in D \Rightarrow x + 2\pi \in D, \quad f * g(x + 2\pi) = f * g(x).$$

This can be summarized by saying that $f*g$ is 2π-periodic. Furthermore, $D(f*g) = D(g * f)$ and $f * g(x) = g * f(x)$ for $x \in D(f * g)$.

In fact, $(x, y) \mapsto f(x-y)g(y)$ is a Borel function on $\mathbb{R}^2$ as a composition of the Borel function $f \otimes g(x, y) = f(x)g(y)$ on $\mathbb{R}^2$ with the homeomorphism $(x, y) \mapsto (x - y, y)$ of $\mathbb{R}^2$.

We also have

$$D(f * g) = \left\{ x \in \mathbb{R} \mid \int_0^{2\pi} |f(x-y)| \, |g(y)| \, dy < \infty \right\}$$

which is a Borel set by Tonelli's theorem. Using the substitution $y = x - t$ (x fixed), we find

$$\int_0^{2\pi} |f(x-y)| \, |g(y)| \, dy = \int_{x-2\pi}^{x} |g(x-t)| \, |f(t)| \, dt = \int_0^{2\pi} |g(x-t)| \, |f(t)| \, dt,$$

showing that $x \in D(f * g) \Leftrightarrow x \in D(g * f)$ and $f * g(x) = g * f(x)$ for $x \in D(f * g)$.

Theorem 1.1.1 *Let $f, g : \mathbb{R} \to \mathbb{C}$ be 2π-periodic Borel functions. The following assertions hold:*

1°. If f, g are continuous then $D(f*g) = \mathbb{R}$ and $f*g$ is again continuous satisfying

$$\|f * g\|_\infty \le \|f\|_\infty \|g\|_\infty. \tag{1.1.3}$$

2°. If $1 \le p, q \le \infty$ are dual exponents, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, then if $f \in \mathcal{L}^p(\mathbb{T})$, $g \in \mathcal{L}^q(\mathbb{T})$ we have $D(f*g) = \mathbb{R}$ and $f*g$ is continuous with

$$\|f * g\|_\infty \le \|f\|_p \|g\|_q. \tag{1.1.4}$$

3°. If $f, g \in \mathcal{L}^1(\mathbb{T})$ then $\mathbb{R} \setminus D(f*g)$ is a Lebesgue null set and $f*g \in \mathcal{L}^1(\mathbb{T})$ with

$$\|f * g\|_1 \le \|f\|_1 \|g\|_1. \tag{1.1.5}$$

4°. If $f \in \mathcal{L}^1(\mathbb{T})$, $g \in \mathcal{L}^p(\mathbb{T})$, $1 \le p \le \infty$ then $\mathbb{R} \setminus D(f*g)$ is a Lebesgue null set and $f*g \in \mathcal{L}^p(\mathbb{T})$ with

$$\|f * g\|_p \le \|f\|_1 \|g\|_p. \tag{1.1.6}$$

<u>Proof.</u>

1° For each $x \in \mathbb{R}$, $y \mapsto f(x-y)g(y)$ is continuous and hence integrable over $[0, 2\pi]$ so $D(f*g) = \mathbb{R}$. A continuous periodic function f is uniformly continuous, so for given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, $|h| \le \delta$

$$|f(x+h) - f(x)| \le \varepsilon.$$

Using

$$f*g(x+h) - f*g(x) = \frac{1}{2\pi}\int_0^{2\pi} \left(f(x+h-y) - f(x-y)\right) g(y)\, dy$$

we see that for $x \in \mathbb{R}$, $|h| \le \delta$

$$|f*g(x+h) - f*g(x)| \le \frac{1}{2\pi}\int_0^{2\pi} \varepsilon\, \|g\|_\infty\, dy = \varepsilon\, \|g\|_\infty,$$

showing that $f*g$ is (uniformly) continuous. The inequality (1.1.3) is easy.

2° Because of symmetry we can assume $1 \le p < \infty$ (If $p = \infty$ then $q = 1$ so $g \in \mathcal{L}^1(\mathbb{T})$). For each $x \in \mathbb{R}$ the function $y \mapsto f(x-y)$ is in $\mathcal{L}^p(\mathbb{T})$ with

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f(x-y)|^p\, dy\right)^{1/p} = \|f\|_p$$

independent of x. By Hölder's inequality $y \mapsto f(x-y)g(y)$ is integrable and

$$|f*g(x)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(x-y)|\, |g(y)|\, dy \le \|f\|_p \|g\|_q$$

showing that $D(f*g)=\mathbb{R}$ and $f*g$ is bounded. To see that $f*g$ is continuous, we get again by Hölder's inequality

$$|f*g(x+h)-f*g(x)| \le \|f(x+h-y)-f(x-y)\|_p\|g\|_q = \|f(h-y)-f(-y)\|_p\|g\|_q$$

which shows the continuity because of the following

Lemma 1.1.2 *For $1 \le p < \infty$, $f \in \mathcal{L}^p(\mathbb{T})$, $h \in \mathbb{R}$ define*

$$\tau_h f(y) = f(y-h), \quad y \in \mathbb{R}. \tag{1.1.7}$$

Then $\tau_h f \in \mathcal{L}^p(\mathbb{T})$, $\|\tau_h f\|_p = \|f\|_p$ and $\|\tau_h f - f\|_p \to 0$ for $h \to 0$.

Proof of Lemma. If f is continuous the result follows by the uniform continuity of f. To $f \in \mathcal{L}^p(\mathbb{T})$ and $\varepsilon > 0$ there exists $\varphi \in C(\mathbb{T})$ such that $\|f-\varphi\|_p \le \varepsilon$. Using the obvious fact $\|\tau_h f\|_p = \|f\|_p$ we then get

$$\|f-\tau_h f\|_p = \|f-\varphi+\varphi-\tau_h\varphi+\tau_h(\varphi-f)\|_p \le 2\|f-\varphi\|_p + \|\varphi-\tau_h\varphi\|_p,$$

and the result follows.

3° The function $(x,y) \mapsto f(x-y)g(y)$ is integrable over $[0,2\pi]\times[0,2\pi]$ because of Tonelli's theorem:

$$\begin{aligned}&\frac{1}{(2\pi)^2}\int_0^{2\pi}\left(\int_0^{2\pi}|f(x-y)\,g(y)|dx\right)dy\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(|g(y)|\frac{1}{2\pi}\int_0^{2\pi}|f(x-y)|dx\right)dy = \|g\|_1\|f\|_1 < \infty.\end{aligned}$$

By Fubini's theorem we conclude that $y \mapsto f(x-y)g(y)$ is integrable over $[0,2\pi]$ for almost all $x \in [0,2\pi]$, i.e., $f*g(x)$ exists for almost all $x \in [0,2\pi]$, hence for almost all $x \in \mathbb{R}$ by periodicity. Furthermore, $f*g(x)$ is integrable over $[0,2\pi]$, i.e., $f*g(x) \in \mathcal{L}^1(\mathbb{T})$ and

$$\begin{aligned}\|f*g\|_1 &\le \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(x-y)|\,|g(y)|\,dy\right)dx\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(|g(y)|\frac{1}{2\pi}\int_0^{2\pi}|f(x-y)|\,dx\right)dy = \|f\|_1\|g\|_1.\end{aligned}$$

4° The case $p=1$ is considered in 3°, and the case $p=\infty$ is straightforward (and it is the special case $p=1$ of 2°). We therefore assume $1<p<\infty$, and choose the dual exponent q such that $\frac{1}{p}+\frac{1}{q}=1$. We shall apply Hölder's inequality to

$y \mapsto |f(x-y)|^{1/q} \in \mathcal{L}^q(\mathbb{T})$ and $y \mapsto |f(x-y)|^{1/p}|g(y)|$, as the latter belongs to $\mathcal{L}^p(\mathbb{T})$ for almost all x by Fubini's theorem, which is applicable because

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\left(\int_0^{2\pi}|f(x-y)|\,|g(y)|^p\,dy\right)dx = \|f\|_1\|g\|_p^p < \infty.$$

We obtain

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}|f(x-y)|\,|g(y)|\,dy &= \frac{1}{2\pi}\int_0^{2\pi}|f(x-y)|^{1/q}\,|f(x-y)|^{1/p}\,|g(y)|\,dy \\ &\le \|f\|_1^{1/q}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(x-y)|\,|g(y)|^p\,dy\right)^{1/p},\end{aligned}$$

and upon subsequent integration of the p^{th} power

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(x-y)|\,|g(y)|\,dy\right)^p dx \\ &\le \|f\|_1^{p/q}\frac{1}{(2\pi)^2}\int_0^{2\pi}\left(\int_0^{2\pi}|f(x-y)|\,|g(y)|^p\,dy\right)dx \\ &= \|f\|_1^{1+\frac{p}{q}}\|g\|_p^p = \|f\|_1^p\|g\|_p^p < \infty.\end{aligned}$$

This shows that

$$\left(\frac{1}{2\pi}\int_0^{2\pi}|f(x-y)|\,|g(y)|\,dy\right)^p < \infty$$

for almost all x, i.e., $f * g(x)$ is defined for almost all x. We obtain finally that $f * g(x) \in \mathcal{L}^p(\mathbb{T})$ since

$$|f * g(x)|^p \le \left(\frac{1}{2\pi}\int_0^{2\pi}|f(x-y)|\,|g(y)|\,dy\right)^p$$

and therefore

$$\frac{1}{2\pi}\int_0^{2\pi}|f * g(x)|^p dx \le \|f\|_1^p\|g\|_p^p,$$

from which (1.1.6) follows. □

Exercises

E 1.1 For each $\alpha \in \mathbb{R}$ define $e_\alpha(x) = e^{\mathrm{i}\alpha x}$. It is clear that $e_\alpha : \mathbb{R} \to \mathbb{T}$ is a continuous homomorphism of groups, i.e.,

$$e_\alpha(x+y) = e_\alpha(x)e_\alpha(y) \quad \text{for} \quad x, y \in \mathbb{R}.$$

(i) Prove that an arbitrary continuous homomorphism $e : \mathbb{R} \to \mathbb{T}$ has the form $e(x) = e_\alpha(x)$ for a uniquely determined $\alpha \in \mathbb{R}$.

Hint: Verify the following steps:

(a) Define
$$\varphi(x) = \int_0^x e(t)\,dt$$
and prove that $\varphi(\delta) = c \neq 0$ for suitable $\delta > 0$ and that $\varphi(x+\delta) = \varphi(x)+ce(x)$.

(b) Conclude that e is a C^1-function satisfying
$$e'(x+y) = e'(x)e(y), \quad x, y \in \mathbb{R},$$
so e solves the differential equation $e'(y) = e'(0)e(y)$.

(c) Conclude that $e'(0)$ is purely imaginary.

For each $n \in \mathbb{Z}$ it is clear that $\gamma_n : \mathbb{T} \to \mathbb{T}$ defined by $\gamma_n(z) = z^n$ is a continuous group homomorphism.

(ii) Prove that an arbitrary continuous group homomorphism $\gamma : \mathbb{T} \to \mathbb{T}$ has the form $\gamma(z) = \gamma_n(z)$ for a uniquely determined $n \in \mathbb{Z}$.

Hint: Consider $e(t) = \gamma(e^{\mathrm{i}t})$ and note that e is 2π-periodic.

E 1.2 For $n \in \mathbb{Z}$ let $e_n(t) = e^{\mathrm{i}nt}$, $t \in \mathbb{R}$. Prove that $e_n * e_m = \delta_{nm}e_n$ for $n, m \in \mathbb{Z}$, where $\delta_{nm} = 1$ if $n = m$ and $\delta_{nm} = 0$ if $n \neq m$. (The symbol δ_{nm} is called Kronecker's delta.)

E 1.3 For $f \in \mathcal{L}^1(\mathbb{T})$ consider for $1 \leq p \leq \infty$ the mapping
$$T_f : \mathcal{L}^p(\mathbb{T}) \to \mathcal{L}^p(\mathbb{T}), \quad T_f(g) = f * g.$$
(i) Show that T_f is a continuous linear mapping which induces a bounded operator $\tilde{T}_f$ in the Banach space $L^p(\mathbb{T})$ of equivalence classes of functions from $\mathcal{L}^p(\mathbb{T})$ equal almost everywhere.

(ii) Show that $\|\tilde{T}_f\| \leq \|f\|_1$.

(iii) Show that the functions e_n (from **E 1.2**) are eigenfunctions of T_f (and of $\tilde{T}_f$ properly understood), and find the corresponding eigenvalues.

E 1.4 Let $f \in \mathcal{L}^1(\mathbb{T})$ and $g \in C^1(\mathbb{T})$, i.e., g is a continuously differentiable periodic function. Prove that $f * g \in C^1(\mathbb{T})$ and $(f * g)'(x) = f * g'(x)$, $x \in \mathbb{R}$. Extend the result to $g \in C^p(\mathbb{T})$ with $p \in \mathbb{N} \cup \{\infty\}$.

E 1.5 For fixed $1 \leq p \leq \infty$ prove that if $f, g \in \mathcal{L}^p(\mathbb{T})$, then $f * g \in \mathcal{L}^p(\mathbb{T})$ and
$$\|f * g\|_p \leq \|f\|_p \|g\|_p\,.$$

1.2 Pure oscillations and trigonometric series

A real-valued function

$$x \mapsto \rho\cos(\omega x - \varphi), \qquad x \in \mathbb{R}, \tag{1.2.1}$$

where ρ, ω, $\varphi \in \mathbb{R}$, $\rho \geq 0$, $\omega \geq 0$, is called a pure oscillation. It can also be written as

$$x \mapsto a\cos(\omega x) + b\sin(\omega x), \qquad x \in \mathbb{R}, \tag{1.2.2}$$

where a, $b \in \mathbb{R}$, namely with $a = \rho\cos\varphi$, $b = \rho\sin\varphi$. Any function of the form (1.2.2) with $a, b \in \mathbb{R}$ can be written as (1.2.1) with $\rho = \sqrt{a^2+b^2}$ called the amplitude. If $\rho \neq 0$ we call $\omega/2\pi$ the frequency of the pure oscillation, and φ, determined modulo 2π is called the phase constant.

In the same way, if $a, b \in \mathbb{C}, \omega \geq 0$, we will call a complex function

$$x \mapsto a\cos(\omega x) + b\sin(\omega x), \qquad x \in \mathbb{R}, \tag{1.2.3}$$

a pure oscillation, with frequency $\omega/2\pi$. The real and imaginary part of (1.2.3) are real-valued pure oscillations with frequency $\omega/2\pi$. Note that (1.2.3) can also be written as

$$x \mapsto c_+\, e^{\mathrm{i}\omega x} + c_-\, e^{-\mathrm{i}\omega x}, \qquad x \in \mathbb{R}, \tag{1.2.4}$$

with c_+, $c_- \in \mathbb{C}$. The relationship between the coefficients is

$$\begin{aligned} a &= c_+ + c_-, & c_+ &= \frac{1}{2}(a - \mathrm{i}b) \\ b &= \mathrm{i}(c_+ - c_-), & c_- &= \frac{1}{2}(a + \mathrm{i}b). \end{aligned}$$

The last form (1.2.4) turns out to have a considerable advantage compared to (1.2.3). Herein lies an incentive to work with complex functions, which we will do in the development of the following theory. Note that a pure oscillation is real-valued exactly when c_+ and c_- are complex conjugates of each other.

For simplicity we write

$$e_\alpha(x) = e^{\mathrm{i}\alpha x}, \quad x, \alpha \in \mathbb{R}. \tag{1.2.5}$$

A pure oscillation with frequency $\omega/2\pi$, where $\omega \in \mathbb{R}_+$, can thereby be written as $c_+\, e_\omega + c_-\, e_{-\omega}$.

The theory of decomposing functions as combinations of pure oscillations is called harmonic analysis.

In this section we shall consider periodic functions $f : \mathbb{R} \to \mathbb{C}$ with a given period $\tau > 0$, i.e. $f(x+\tau) = f(x), x \in \mathbb{R}$ and hence

$$f(x + p\tau) = f(x), \quad x \in \mathbb{R}, p \in \mathbb{Z}.$$

A pure oscillation with frequency $\omega/2\pi = 1/\tau$ is called a fundamental oscillation, whereas a pure oscillation with frequency $n\omega/2\pi = n/\tau$, $n = 2, 3, \ldots$ is called an overtone. According to Joseph Fourier (Sur la propagation de la chaleur, manuscript, Paris 1807), each function with period τ can be written as the sum of a fundamental oscillation, overtones and a constant term. Fourier's statement is however a very simplified picture of the real situation.

For the sake of simplicity, we choose to consider functions with period $\tau = 2\pi$. A series with a constant term, fundamental oscillation and overtones can thereby be written as

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{1.2.6}$$

$$c_0 + \sum_{n=1}^{\infty}(c_n\, e^{\mathrm{i}nx} + c_{-n}e^{-\mathrm{i}nx}), \tag{1.2.7}$$

or by using (1.2.5)

$$c_0 + \sum_{n=1}^{\infty}(c_n\, e_n + c_{-n}\, e_{-n}). \tag{1.2.8}$$

Such a series is called a trigonometric series.

For reasons of brevity, one frequently just writes $\sum_{-\infty}^{\infty} c_n\, e^{\mathrm{i}nx}$ or $\sum_{-\infty}^{\infty} c_n\, e_n$. We note the relationships

$$a_n = c_n + c_{-n}, \qquad c_n = \frac{1}{2}(a_n - \mathrm{i}b_n), \tag{1.2.9}$$

$$b_n = \mathrm{i}(c_n - c_{-n}), \qquad c_{-n} = \frac{1}{2}(a_n + \mathrm{i}b_n), \tag{1.2.10}$$

valid for $n > 0$, and for $n = 0$ if we set $b_0 = 0$.

The following results are easy but fundamental:

Theorem 1.2.1 *The functions $e_n(x) = e^{\mathrm{i}nx}, n \in \mathbb{Z}$ form an orthonormal system in the Hilbert space $L^2(\mathbb{T})$:*

$$\langle e_n, e_m \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}nx}e^{-\mathrm{i}mx}\, dx = \begin{cases} 1 & \text{for } n = m, \\ 0 & \text{for } n \neq m. \end{cases} \tag{1.2.11}$$

Theorem 1.2.2 *If a function $f : \mathbb{R} \to \mathbb{C}$ with period 2π can be written as a sum of a uniformly convergent trigonometric series, then there is only one infinite series of this type and its coefficients are given by*

$$c_n = \langle f, e_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-\mathrm{i}nx}\, dx \quad n \in \mathbb{Z} \tag{1.2.12}$$

or in real form

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx, \quad n \in \mathbb{N}_0. \tag{1.2.13}$$

Proof. Let $c_0 + \sum_1^\infty (c_m\, e^{\mathrm{i}mx} + c_{-m}\, e^{-\mathrm{i}mx})$ be uniformly convergent for $x \in \mathbb{R}$ with sum $f(x)$, which then belongs to $C(\mathbb{T})$. For each $n \in \mathbb{Z}$ we have

$$f(x)\, e^{-\mathrm{i}nx} = c_0\, e^{-\mathrm{i}nx} + \sum_{m=1}^{\infty} (c_m\, e^{\mathrm{i}(m-n)x} + c_{-m}\, e^{\mathrm{i}(-m-n)x}), \tag{1.2.14}$$

where we again have uniform convergence, since e_{-n} is a bounded function. We are then allowed to integrate (1.2.14) term by term to get

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) e^{-\mathrm{i}nx} dx &= c_0 \int_{-\pi}^{\pi} e^{-\mathrm{i}nx} dx + \sum_{m=1}^{\infty} \left(c_m \int_{-\pi}^{\pi} e^{\mathrm{i}(m-n)x} dx + c_{-m} \int_{-\pi}^{\pi} e^{\mathrm{i}(-m-n)x} dy \right) \\ &= c_n \cdot 2\pi. \end{aligned}$$

□

Exercises

E 2.1 Show the formulas ($n = 1, 2, \ldots$)

$$\cos^{2n} x = 2^{-2n} \binom{2n}{n} + 2^{-2n+1} \sum_{k=1}^{n} \binom{2n}{n-k} \cos(2kx), x \in \mathbb{R},$$

and conclude that

$$\frac{1}{\pi} \int_0^{\pi} \cos^{2n} x \cos(2kx)\, dx = \begin{cases} 2^{-2n} \binom{2n}{n-k}, & k = 0, 1, \ldots n \\ 0, & k > n, \end{cases}$$

while

$$\frac{1}{\pi} \int_0^{\pi} \cos^{2n} x \cos((2k+1)x)\, dx = 0, k = 0, 1, \ldots.$$

1.3 Fourier series for $f \in \mathcal{L}^1(\mathbb{T}) = \mathcal{L}(\mathbb{T})$

The formula (1.2.12) makes sense for any $f \in \mathcal{L}^1(\mathbb{T})$, so we can associate a trigonometric series to f

$$c_0 \quad + \quad \sum_{n=1}^{\infty}(c_n\, e^{inx} + c_{-n}e^{-inx}) \quad \text{with} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx, \tag{1.3.1}$$

or, written alternatively,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.3.2}$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx,\ n \in \mathbb{N}_0, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx. \tag{1.3.3}$$

This series is called the Fourier series for f. Strictly speaking, it is the series $c_0 + \sum_{n=1}^{\infty}(c_n\, e_n + c_{-n}\, e_{-n})$ one has in mind here. The numbers c_n, a_n and b_n are called Fourier coefficients for f. That $\sum_{-\infty}^{\infty} c_n\, e^{inx}$ is the Fourier series for f is sometimes expressed by writing

$$f \sim \sum_{-\infty}^{\infty} c_n\, e^{inx}.$$

The symbol$\sim$ stresses that we do not know if and how the series converges. A main point in the theory is to examine if the Fourier series for f converges to f in some sense.

We can now reformulate Theorem 1.2.2:

Theorem 1.3.1 *A uniformly convergent trigonometric series is the Fourier series for its sum function, which belongs to $C(\mathbb{T})$.*

Integration theory and harmonic analysis have developed with close ties. The work in which Bernhard Riemann develops his notion of integration has the title: Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe (Göttingen 1854). Lebesgue's integration theory was instantaneously brought into fruitful application in harmonic analysis and turned out to be the natural framework.

It would have been nice if the Fourier series of any $f \in C(\mathbb{T})$ was indeed uniformly or at least pointwise convergent to $f(x)$, but unfortunately it is not so. Already Paul du Bois-Reymond gave in 1873 an example of a continuous periodic function for which the Fourier series diverges at a single point x. This has been extended:

For every null set $N \subset]-\pi, \pi]$ there is a continuous periodic function for which the Fourier series is divergent in each point of N. See [7, p.55–61].

It was a sensation when Lennart Carleson in 1966 showed that the Fourier series of a function $f \in \mathcal{L}^2(\mathbb{T})$ converges for almost all x, thereby proving a conjecture going back to Lusin from 1913. In 1968 Richard A. Hunt extended the result to $\mathcal{L}^p(\mathbb{T})$ for $1 < p < 2$, and since $\mathcal{L}^r(\mathbb{T})$ decreases with larger r one obtains therefore:

For $f \in \mathcal{L}^p(\mathbb{T})$, with $1 < p \leq \infty$, the Fourier series converges to $f(x)$ for almost all x.

Nonetheless, one can find functions $f \in \mathcal{L}^1(\mathbb{T})$ for which the Fourier series is divergent everywhere (Andrej Kolmogorov 1926), see [10, Chap. 8].

It is amazing that there is this difference between $p = 1$ and $p > 1$.

It can also be mentioned that a trigonometric series can be convergent everywhere without being a Fourier series for any function $f \in \mathcal{L}(\mathbb{T})$. A concrete example is $\sum_1^\infty \sin nx/\log(n+1)$; see e.g. [4, p.2] and Example 1.9.4.

It was in connection with studies of the set of points $x \in \mathbb{R}$, in which a given trigonometric series converges, that Georg Cantor was led to a closer study of set theory.

We shall first deal with sufficient conditions for a Fourier series to be convergent at a particular point x. Afterwards we shall investigate other possibilities for the Fourier series to represent the function.

1.4 Riemann-Lebesgue's Lemma

Lemma 1.4.1 (Riemann-Lebesgue's Lemma) *For every function $f \in \mathcal{L}^1(\mathbb{R})$*

$$\int_{\mathbb{R}} f(x)\, e^{\mathrm{i}tx}\, dx \to 0 \qquad \text{for } |t| \to \infty, t \in \mathbb{R}.$$

In particular

$$\int_{\mathbb{R}} f(x) \cos tx\, dx \to 0, \quad \int_{\mathbb{R}} f(x) \sin tx\, dx \to 0 \qquad \text{for } t \to \infty.$$

Proof. We use that step functions are dense in $\mathcal{L}^1(\mathbb{R})$.

1° If f is the characteristic function of a bounded interval $]a, b]$, the assertion follows from

$$\int_a^b e^{\mathrm{i}tx}\, dx = \left[\frac{1}{\mathrm{i}t}\, e^{\mathrm{i}tx}\right]_{x=a}^{x=b} = \frac{1}{\mathrm{i}t}(e^{\mathrm{i}tb} - e^{\mathrm{i}ta}),$$

because we get

$$\left|\int_a^b e^{\mathrm{i}tx}\,dx\right| \le \frac{2}{|t|} \qquad \text{for } t \neq 0.$$

2° If f is a step function, i.e., $f = \sum_1^n c_j \cdot 1_{\mathrm{I}_j}$, where each $\mathrm{I}_j =]a_j, b_j]$ is a bounded interval, we have by 1°

$$\int_{\mathbb{R}} f(x)\,e^{\mathrm{i}tx}\,dx = \sum_1^n c_j \int_{a_j}^{b_j} e^{\mathrm{i}tx}\,dx \to 0 \qquad \text{for } |t| \to \infty.$$

3° Finally, we consider the general case $f \in \mathcal{L}^1(\mathbb{R})$. For an arbitrary $\varepsilon \in \mathbb{R}_+$ we can find a step function g such that $\|f - g\|_1 < \frac{\varepsilon}{2}$. As long as $|t|$ is sufficiently large, we have now according to 2°

$$\left|\int_{\mathbb{R}} g(x)\,e^{\mathrm{i}tx}\,dx\right| < \frac{\varepsilon}{2}$$

and thereby

$$\left|\int_{\mathbb{R}} f(x)\,e^{\mathrm{i}tx}\,dx\right| \le \left|\int_{\mathbb{R}} (f(x) - g(x))\,e^{\mathrm{i}tx}\,dx\right| + \left|\int_{\mathbb{R}} g(x)\,e^{\mathrm{i}tx}\,dx\right| < \varepsilon$$

since

$$\left|\int_{\mathbb{R}} (f(x) - g(x))\,e^{\mathrm{i}tx}\,dx\right| \le \int_{\mathbb{R}} |f(x) - g(x)|\;|e^{\mathrm{i}tx}|\,dx = \|f - g\|_1.$$

□

Corollary 1.4.2 *For $f \in \mathcal{L}^1(\mathbb{T})$ we have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{\mathrm{i}tx}\,dx \to 0 \qquad \textit{for } |t| \to \infty,\ t \in \mathbb{R}.$$

In particular, for the Fourier coefficients a_n, b_n and c_n of f we have

$$c_n \to 0 \quad \textit{for } |n| \to \infty, \qquad a_n \to 0,\ b_n \to 0 \quad \textit{for } n \to \infty.$$

Exercises

E 4.1 Let $f \in \mathcal{L}^1(\mathbb{T})$ and $n \in \mathbb{Z}, n \neq 0$. Prove that the n'th Fourier coefficient c_n of f can be determined by the formula

$$c_n = -\frac{1}{2\pi}\int_0^{2\pi} f\left(x - \pi/|n|\right) e^{-\mathrm{i}nx}\,dx$$

and deduce that $|c_n| \le \frac{1}{2}\|f - \tau_{\pi/|n|} f\|_1$.

Use this to give a new proof that $c_n \to 0$ for $|n| \to \infty$.

1.5 Convergence of Fourier series

Let s_n denote the n'th partial sum of the Fourier series $c_0 + \sum_{n=1}^{\infty}(c_n\, e_n + c_{-n}\, e_{-n})$ for a function $f \in \mathcal{L}^1(\mathbb{T})$, i.e.,

$$s_n(x) = \sum_{k=-n}^{n} c_k\, e^{ikx} \qquad \text{for } \; n \in \mathbb{N}_0,\; x \in \mathbb{R}. \tag{1.5.1}$$

Since

$$c_k\, e^{ikx} = e^{ikx}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, e^{-iky}\, dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik(x-y)}\, f(y)\, dy = e_k * f(x),$$

we have

$$s_n = \sum_{k=-n}^{n} c_k\, e_k = \sum_{k=-n}^{n} (e_k * f) = f * \sum_{k=-n}^{n} e_k = f * D_n, \tag{1.5.2}$$

where $D_n = \sum_{k=-n}^{n} e_k$ is the n'th partial sum of the series $1 + \sum_{n=1}^{\infty}(e_n + e_{-n})$, i.e.,

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx} = 1 + 2\sum_{k=1}^{n} \cos kx. \tag{1.5.3}$$

The function D_n is called the n'th Dirichlet kernel. It is an even function, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)\, dx = 1. \tag{1.5.4}$$

For $x \neq 0 \pmod{2\pi}$, we find

$$\begin{aligned} D_n(x) = e^{-inx}\,\frac{e^{i(2n+1)x} - 1}{e^{ix} - 1} \quad &= \quad \frac{e^{i(n+1)x} - e^{-inx}}{e^{ix} - 1} \\ &= \quad \frac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} = \frac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}. \end{aligned}$$

We note: *The n'th partial sum s_n of a Fourier series for a function $f \in \mathcal{L}^1(\mathbb{T})$ is equal to $f * D_n$, i.e.,*

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\, D_n(y)\, dy \qquad \textit{for } n \in \mathbb{N}_0,\; x \in \mathbb{R}. \tag{1.5.5}$$

This is the key to the study of the convergence of Fourier series. Note that we do not aim at absolute convergence at this point.

Theorem 1.5.1 (Dini's test (1880)) *A sufficient condition for a Fourier series* $c_0 + \sum_{n=1}^{\infty}(c_n\, e_n + c_{-n}\, e_{-n})$ *for a function* $f \in \mathcal{L}^1(\mathbb{T})$ *to be convergent in the point* $x \in \mathbb{R}$ *with the sum*

$$c_0 + \sum_{n=1}^{\infty}(c_n\, e^{inx} + c_{-n}\, e^{-inx}) = s$$

is the existence of $\delta > 0$ *such that*

$$\int_0^{\delta} \left| \frac{f(x+y) + f(x-y) - 2s}{y} \right| dy < \infty. \tag{1.5.6}$$

Note that the integral is finite for every $\delta > 0$, if it is fulfilled just for one value $\delta_0 > 0$.

Proof. Since D_n is an even function, we have

$$\begin{aligned} s_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{0} f(x-y)\, D_n(y)\, dy + \frac{1}{2\pi}\int_0^{\pi} f(x-y)\, D_n(y)\, dy \\ &= \frac{1}{2\pi}\int_0^{\pi} (f(x+y) + f(x-y))\, D_n(y)\, dy. \end{aligned}$$

By (1.5.4) we have furthermore

$$\begin{aligned} s_n(x) - s &= \frac{1}{2\pi}\int_0^{\pi} (f(x+y) + f(x-y) - 2s)\, D_n(y)\, dy \\ &= \frac{1}{2\pi}\int_0^{\pi} \frac{f(x+y) + f(x-y) - 2s}{y} \cdot \frac{y}{\sin\frac{1}{2}y} \cdot \sin(n+\tfrac{1}{2})y\, dy. \end{aligned}$$

Since $y/\sin\frac{1}{2}y$ is bounded on $[0, \pi]$, the last integral assumes the form

$$\int_{\mathbb{R}} g(y)\, \sin(n + \frac{1}{2})y\, dy, \qquad \text{with } g \in \mathcal{L}^1(\mathbb{R}).$$

By Riemann-Lebesgue's Lemma (Sec. 1.4), it follows that $s_n(x) - s \to 0$. $\square$

Application. The condition in Dini's test is fulfilled, with $s = f(x)$, if the function $f \in \mathcal{L}^1(\mathbb{T})$ is continuous at x as well as differentiable from the right and left at this point.

More generally, the condition is fulfilled, with $s = \frac{1}{2}(f(x+0) + f(x-0))$, if the function $f \in \mathcal{L}^1(\mathbb{T})$ has the limit $f(x+0) \in \mathbb{C}$ and $f(x-0) \in \mathbb{C}$ from the right and from the left in the point x, and if additionally

$$\frac{f(x+y) - f(x+0)}{y} \quad \text{and} \quad \frac{f(x-y) - f(x-0)}{-y}$$

have limits in $\mathbb{C}$ for $y \to 0_+$.

Under these assumptions the function

$$y \mapsto \frac{f(x+y) + f(x-y) - 2 \cdot \frac{1}{2}(f(x+0) + (f(x-0))}{y}$$

is bounded in an interval $]0, \delta]$, so (1.5.6) is satisfied.

We will hereby leave the problem of pointwise convergence of a Fourier series but nonetheless mention the following theorem which essentially is due to G. Lejeune Dirichlet (1829). Dirichlet was the first who gave a proper proof of the convergence of Fourier series.

Theorem 1.5.2 (Dirichlet-Jordan's test) *The Fourier series for a periodic function $f : \mathbb{R} \to \mathbb{C}$ of bounded variation over $[0, 2\pi]$ is convergent in every x with the sum*

$$\frac{1}{2}\left(f(x+0) + f(x-0)\right).$$

If f is furthermore continuous on a compact interval $[a, b]$, the convergence is uniform in $[a, b]$.

We recall that the variation $V_{[a,b]}(f)$ of a function $f : [a, b] \to \mathbb{C}$ is defined as the supremum of the numbers

$$V_{\mathcal{D}}(f) = \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|, \tag{1.5.7}$$

where $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ is an arbitrary partition $\mathcal{D}$ of the interval. We say that f is of bounded variation over $[a, b]$ if $V_{[a,b]}(f) < \infty$. The set of such functions is a complex vector space $V([a, b])$.

Every monotone function is of bounded variation.

If for example f is increasing in $[a, b]$, we can write (1.5.7) as

$$\sum_{j=1}^{n} (f(x_j) - f(x_{j-1})) = f(b) - f(a),$$

thus $V_{[a,b]}(f) = f(b) - f(a)$.

It holds furthermore that

$$f \in V([a, b]) \Leftrightarrow \mathrm{Re}(f), \mathrm{Im}(f) \in V([a, b]),$$

thus every function of the form

$$f = f_1 - f_2 + \mathrm{i}(f_3 - f_4), \tag{1.5.8}$$

where $f_1, \ldots, f_4$ are increasing on $[a, b]$, is of bounded variation. This terminology was introduced by C. Jordan in 1881. He showed furthermore that every function $f \in V([a, b])$ has a representation of the form (1.5.8), cf. **E 5.2** below.

Since an increasing function $f : [a, b] \to \mathbb{R}$ has a limit from the right and from the left in every $x \in [a, b]$ (albeit for $x = a$ only from the right and for $x = b$ only from the left), namely

$$f(x + 0) = \inf\{f(y)|x < y\}, \qquad f(x - 0) = \sup\{f(y)|y < x\},$$

(and of course $f(x - 0) \leq f(x) \leq f(x + 0)$), we can note the following consequence of Jordan's result:

Every function $f \in V([a, b])$ has a limit from the left and from the right in every point of the interval $[a, b]$.

This result is of course a prerequisite for the statement of the Dirichlet-Jordan's test to be meaningful. Dirichlet proved the test for continuous functions that are piecewise monotone. The representation (1.5.8) of $f \in V([a, b])$ shows that the extension to the class $V([a, b])$ is not particularly deep. We will prove Theorem 1.5.2 in Sec. 1.6.

Exercises

E 5.1 Show that $C^1([a, b]) \subseteq V([a, b])$ and that $V_{[a,b]}(f) \leq (b - a)||f'||_\infty$.

E 5.2 For a function $f : [a, b] \to \mathbb{R}$, we introduce the positive and negative variation over $[a, b]$

$$\begin{aligned} P_{[a,b]}(f) &= \sup\left\{ P_{\mathcal{D}}(f) = \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))^+ \right\}, \\ N_{[a,b]}(f) &= \sup\left\{ N_{\mathcal{D}}(f) = \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))^- \right\}, \end{aligned}$$

where the supremum is taken over all partitions $\mathcal{D} : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$, and as usual $\alpha^+ = \max(\alpha, 0)$, $\alpha^- = \max(-\alpha, 0)$ for $\alpha \in \mathbb{R}$. Show that

$$V_{[a,b]}(f) = P_{[a,b]}(f) + N_{[a,b]}(f).$$

(Hint. Exploit that if $\mathcal{D}'$ is a further partitioning of $\mathcal{D}$, then $V_{\mathcal{D}}(f) \leq V_{\mathcal{D}'}(f)$, $P_{\mathcal{D}}(f) \leq P_{\mathcal{D}'}(f)$, $N_{\mathcal{D}}(f) \leq N_{\mathcal{D}'}(f)$.)

Show that if $V_{[a,b]}(f) < \infty$, then $P_{[a,x]}(f)$ and $N_{[a,x]}(f)$ are increasing functions for $x \in [a, b]$ and

$$f(x) = f(a) + P_{[a,x]}(f) - N_{[a,x]}(f),$$

thus f is the difference of two increasing functions.

E 5.3 Show that every increasing function $f : [a, b] \to \mathbb{R}$ is a Borel function and conclude that $V([a, b]) \subseteq \mathcal{L}^\infty([a, b])$.

E 5.4 Find the Fourier series for $f \in \mathcal{L}^1(\mathbb{T})$ given by

$$f(x) = \begin{cases} \frac{\pi}{4} & \text{for } 0 < x < \pi \\ -\frac{\pi}{4} & \text{for } \pi < x < 2\pi \\ 0 & \text{for } x = 0, x = \pi, \end{cases}$$

and show that it converges pointwise to $f(x)$ for all $x \in \mathbb{R}$.

Prove Leibniz' formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - + \cdots .$$

1.6 Summability

A sequence $s_0, s_1, \ldots$ of elements in a vector space $\mathcal{V}$ with seminorm $\|\cdot\|$ is said to be <u>limitable</u> <u>with limit</u> $s \in \mathcal{V}$ if $\|\sigma_n - s\| \to 0$ for $n \to \infty$, where

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} s_k. \tag{1.6.1}$$

Lemma 1.6.1 (Cauchy) *A convergent sequence $s_0, s_1, \ldots$ with limit s is also limitable with the same limit.*

<u>Proof.</u> For arbitrary $\varepsilon \in \mathbb{R}_+$, we can choose an $M \in \mathbb{N}$ such that

$$\|s_k - s\| < \frac{\varepsilon}{2} \qquad \text{for } k > M.$$

For every $n > M$ we have now

$$\begin{aligned} \|\sigma_n - s\| = \|\frac{1}{n+1} \sum_{k=0}^{n} (s_k - s)\| &\le \frac{1}{n+1} \|\sum_{k=0}^{M} (s_k - s)\| + \frac{1}{n+1} \sum_{k=M+1}^{n} \|(s_k - s)\| \\ &< \frac{1}{n+1} \|\sum_{k=0}^{M} (s_k - s)\| + \frac{\varepsilon}{2}. \end{aligned}$$

Since $\frac{1}{n+1} \|\sum_{k=0}^{M} (s_k - s)\| \to 0$ for $n \to \infty$, we can therefore find an $N \ge M$ such that

$$\|\sigma_n - s\| < \varepsilon \qquad \text{for } n > N.$$

A sequence can of course be limitable without being convergent. An example is $2s, 0, 2s, 0, 2s, \ldots$ which is limitable with limit s, but it is not convergent if $s \neq 0$.

A infinite series $\sum_0^\infty a_n$ with elements belonging to a vector space $\mathcal{V}$ with semi-norm is said to be summable with sum $s \in \mathcal{V}$ if the sequence of partial sums $s_n = \sum_{k=0}^n a_k$, $n = 0, 1, \ldots$, ist limitable with limit s.

An example is $2s - 2s + 2s - 2s + 2s - \ldots$ which is summable with sum s, but convergent only if $||s|| = 0$.

The arithmetic means $\sigma_n = \frac{1}{n+1} \sum_{k=0}^n s_k$ of the partial sums of a series $\sum_0^\infty a_n$ can be expressed directly in terms of the elements a_n

$$\sigma_n = \sum_{k=0}^n (1 - \frac{k}{n+1}) a_k. \tag{1.6.2}$$

In fact, writing

$$\begin{array}{ccccccccc} s_0 & = & a_0 & & & & & & \\ s_1 & = & a_0 & + & a_1 & & & & \\ \vdots & & \vdots & & \vdots & & & & \\ s_n & = & a_0 & + & a_1 & + & \ldots & + & a_n \end{array}$$

we obtain

$$(n+1)\sigma_n = (n+1)a_0 + na_1 + \ldots + 2a_{n-1} + a_n = \sum_{k=0}^n (n+1-k) a_k.$$

We shall see that the theory of summability of a Fourier series $c_0 + \sum_{n=1}^\infty (c_n\, e_n + c_{-n}\, e_{-n})$ for a function $f \in \mathcal{L}^1(\mathbb{T})$ is very elegant.

Since we found that the n'th partial sum of the Fourier series is given as $s_n = f * D_n$ (Sec. 1.5), we have for the arithmetic means

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^n s_k = \frac{1}{n+1} \sum_{k=0}^n (f * D_k) = f * \left(\frac{1}{n+1} \sum_{k=0}^n D_k \right) = f * F_n, \tag{1.6.3}$$

where $F_n = \frac{1}{n+1} \sum_{k=0}^n D_k$ is called the n'th Fejér kernel.

Since $D_n = \sum_{k=-n}^n e_k$ is the n'th partial sum of the series $1 + \sum_{n=1}^\infty (e_n + e_{-n})$, F_n is by (1.6.2) given as

$$F_n(x) = \sum_{k=-n}^n (1 - \frac{|k|}{n+1}) e^{\mathrm{i}kx} = 1 + 2 \sum_{k=1}^n (1 - \frac{k}{n+1}) \cos kx. \tag{1.6.4}$$

Note that F_n is an even function, and that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(x) dx = 1.$$

The advantage in considering summability of a Fourier series rather than convergence lies in the fact that the sequence $F_0, F_1, \ldots$ has better properties than $D_0, D_1, \ldots$. The essential point is as we shall see: $F_n(x) \geq 0$ while this is not true for $D_n(x)$.

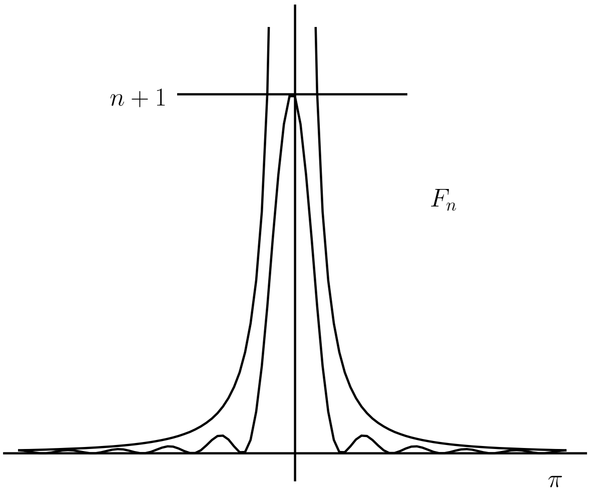

Figure 1.1: Graph of F_n and $\pi^2/(n+1)x^2$

For $x \neq 0 \pmod{2\pi}$, we have (see Sec. 1.5)

$$D_k(x) = \frac{\sin(k+\frac{1}{2})x}{\sin\frac{1}{2}x} = \frac{2\sin(k+\frac{1}{2})x\,\sin\frac{1}{2}x}{2\sin^2\frac{1}{2}x} = \frac{\cos kx - \cos(k+1)x}{2\sin^2\frac{1}{2}x}$$

and thereby

$$F_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(x) = \frac{1}{n+1}\frac{1-\cos(n+1)x}{2\sin^2\frac{1}{2}x} = \frac{1}{n+1}\left(\frac{\sin\frac{1}{2}(n+1)x}{\sin\frac{1}{2}x}\right)^2,$$

showing that $F_n \geq 0$.

Since

$$\sin\frac{1}{2}x \geq \frac{1}{\pi}x \qquad \text{for } 0 \leq x \leq \pi, \tag{1.6.5}$$

because $\sin(x/2)$ is concave, cf. Figure 1.2, it can be concluded additionally that

$$F_n(x) \leq \frac{1}{n+1}\frac{\pi^2}{x^2} \qquad \text{for } \ 0 < x \leq \pi,\ n \in \mathbb{N}_0. \tag{1.6.6}$$

This estimate shows that

$$F_n(x) \to 0 \qquad \text{for } \ n \to \infty$$

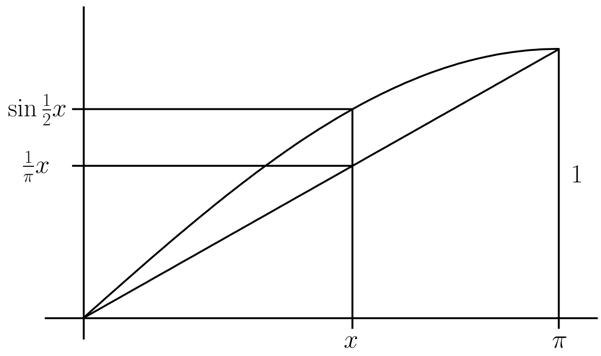

Figure 1.2: Graph of $\sin x/2$ and x/π

when $0 < |x| \leq \pi$, and that the convergence is uniform for $\delta \leq |x| \leq \pi$, when $0 < \delta < \pi$. In particular we have

$$\int_{\delta \leq |x| \leq \pi} F_n(x)dx \to 0 \qquad \text{for } n \to \infty.$$

We note that (F_n) is a Dirac sequence according to the following definition:

Definition 1.6.2 *A sequence $k_n \in \mathcal{L}^1(\mathbb{T})$ is called a Dirac sequence for $\mathbb{T}$ if it has the following properties:*

(a) $k_n \geq 0$

(b) $\displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(t)\,dt = 1$

(c) For each δ such that $0 < \delta < \pi$

$$\int_{\delta \leq |t| \leq \pi} k_n(t)\,dt \to 0 \qquad \textit{for } n \to \infty.$$

Due to (b), we can replace (c) by

(c') $\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\delta}^{\delta} k_n(t)\,dt = 1 \qquad$ for each $\delta \in \,]0, \pi[$.

We can express (a) and (b) by saying that $\frac{1}{2\pi}k_n(x)$ is a probability density on $[-\pi, \pi]$, and (c') states that the mass of this density is concentrated closer and closer around zero for increasing n. A Dirac sequence is a mathematically correct formulation of Dirac's delta function δ as a "function" which is ∞ for $x = 0$ and

0 for $x \neq 0$ and fulfills $\int \delta(x)\,dx = 1$. From a mathematical point of view such a function does not exist.

A Dirac sequence is an approximate unit with respect to convolution in the following sense: The Banach algebras $C(\mathbb{T})$ and $L^1(\mathbb{T})$ do not have a unit (i.e. a neutral element with respect to convolution) – this is shown later – but the elements k_n in a Dirac sequence satisfy

Theorem 1.6.3 *For $f \in C(\mathbb{T})$ (resp. $f \in \mathcal{L}^p(\mathbb{T})$, $1 \le p < \infty$)*

$$\lim_{n\to\infty} \|f * k_n - f\|_\infty = 0 \qquad (\textit{resp.}\ \lim_{n\to\infty} \|f * k_n - f\|_p = 0).$$

Proof. Assume first $f \in C(\mathbb{T})$. For $\varepsilon > 0$ we can find $0 < \delta < \pi$ according to uniform continuity such that

$$|f(x-t) - f(x)| \le \varepsilon \qquad \text{when } x \in \mathbb{R},\ |t| \le \delta.$$

Therefore

$$f * k_n(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x-t) - f(x))\, k_n(t)\,dt,$$

hence

$$|f * k_n(x) - f(x)| \le \varepsilon\, \frac{1}{2\pi}\int_{-\delta}^{\delta} k_n(t)\,dt + \frac{\|f\|_\infty}{\pi}\int_{\delta\le|t|\le\pi} k_n(t)\,dt.$$

Since the right-hand side is independent of x we get

$$\|f * k_n - f\|_\infty \le \varepsilon + \frac{\|f\|_\infty}{\pi}\int_{\delta\le|t|\le\pi} k_n(t)\,dt,$$

but the last term tends to 0 for $n \to \infty$ according to (c), and we can therefore find an $N \in \mathbb{N}$ such that $\|f * k_n - f\|_\infty \le 2\varepsilon$ for $n \ge N$.

As $\|f\|_p \le \|f\|_\infty$ when $f \in C(\mathbb{T})$, the second assertion follows for these f. In order to show the second assertion for arbitrary $f \in \mathcal{L}^p(\mathbb{T})$, we use that for $\varepsilon > 0$ we can find $g \in C(\mathbb{T})$ such that $\|f - g\|_p \le \varepsilon$. This leads to

$$\begin{aligned} \|f * k_n - f\|_p &\le \|(f-g) * k_n\|_p + \|g * k_n - g\|_p + \|g - f\|_p \\ &\le 2\|f-g\|_p + \|g * k_n - g\|_p \end{aligned}$$

which is $< 3\varepsilon$ for sufficiently large n. □

Remembering that the n'th average σ_n of the partial sums of a Fourier series for a function $f \in \mathcal{L}^1(\mathbb{T})$ is equal to $f * F_n$, and the Féjer kernels (F_n) form a Dirac sequence, we get by Theorem 1.6.3:

Theorem 1.6.4 (Fejér) *The Fourier series for $f \in C(\mathbb{T})$ is uniformly summable with sum f.*

The assertion here is that $\|\sigma_n - f\|_\infty \to 0$ for $n \to \infty$. This follows from $\sigma_n = f * F_n$ using Theorem 1.6.3.

Corollary 1.6.5 (The Uniqueness Theorem) *If $f, g \in \mathcal{L}^1(\mathbb{T})$ have the same Fourier series, i.e., the same Fourier coefficients, then $f = g$ almost everywhere.*

By assumption we have $f * F_n = g * F_n$ which approaches f as well as g in $\mathcal{L}^1(\mathbb{T})$, hence $f = g$ in $\mathcal{L}^1(\mathbb{T})$.

Theorem 1.6.6 *The Fourier series for an arbitrary $f \in \mathcal{L}^p(\mathbb{T})$, $1 \le p < \infty$, is summable in $\mathcal{L}^p(\mathbb{T})$ with sum f.*

Corollary 1.6.7 (Weierstrass' approximation theorem for periodic functions) *For every continuous function $f : \mathbb{R} \to \mathbb{C}$ with period 2π and every $\varepsilon > 0$ one can find a trigonometric polynomial p such that*

$$\forall\, x \in \mathbb{R} : |f(x) - p(x)| < \varepsilon.$$

For p one can use the n'th average σ_n of the partial sums of the Fourier series for f for suitably large n.

It was the Hungarian mathematician Leopold Fejér, (1880 – 1959), who first applied summability theory to Fourier series. (Untersuchungen über Fouriersche Reihen, Mathematische Annalen 58 (1904), p. 51–69). This paper was preceded by a paper in Hungarian from 1900, when he was 20 years old. Fejér also proved the following:

Theorem 1.6.8 *If $f \in \mathcal{L}^1(\mathbb{T})$ is continuous in a point $x \in \mathbb{R}$, then the Fourier series for f is summable in this point with sum $f(x)$, i.e.,*

$$\sigma_n(x) = f * F_n(x) \to f(x) \qquad \text{for } n \to \infty.$$

As in the proof of Theorem 1.6.3, we find

$$|f * F_n(x) - f(x)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-y) - f(x)|\, F_n(y)\, dy,$$

hence for $0 < \delta < \pi$

$$|f * F_n(x) - f(x)| \le \frac{1}{2\pi} \int_{-\delta}^{\delta} |f(x-y) - f(x)|\, F_n(y)\, dy +$$
$$\frac{1}{2\pi} \int_{\delta < |y| < \pi} |f(x-y)|\, F_n(y)\, dy + \frac{1}{2\pi} \int_{\delta < |y| < \pi} |f(x)|\, F_n(y)\, dy.$$

The first term can be made smaller than a given $\varepsilon > 0$ by chosing δ small–this is possible by the continuity of f at x. Once $\delta > 0$ is chosen, the third term tends to zero for $n \to \infty$ by property (c) of a Dirac sequence. Finally, using (1.6.6) the middle term can be majorized by

$$\sup\{F_n(y) \mid \delta < |y| < \pi\} \|f\|_1 \leq \frac{\pi^2}{(n+1)\delta^2} \|f\|_1,$$

which tends to 0 for $n \to \infty$. $\square$

Inspired by Fejér's results, Henri Lebesgue (Sur la convergence des séries de Fourier, Mathematische Annalen 61 (1905), p. 271–77) showed:

Theorem 1.6.9 (Fejér–Lebesgue's theorem) *The Fourier series for $f \in \mathcal{L}^1(\mathbb{T})$ is summable with sum $f(x)$ for almost all x, i.e.,*

$$\sigma_n(x) = f * F_n(x) \xrightarrow[n \to \infty]{} f(x)$$

for all $x \in \mathbb{R}$ with the exception of a Lebesgue null set.

The proof depends on Lebesgue's theorem on differentiation. For a locally integrable function $f : \mathbb{R} \to \mathbb{C}$, we define the set $\mathcal{L}(f)$ of Lebesgue points as those $x \in \mathbb{R}$ for which

$$\lim_{\delta \to 0} \frac{1}{2\delta} \int_{-\delta}^{\delta} |f(x-t) - f(x)| \, dt = 0. \tag{1.6.7}$$

Lebesgue proved that $\mathbb{R} \setminus \mathcal{L}(f)$ is a null set, i.e., that almost all x are Lebesgue points. We will not prove this result, but use it to prove the following result:

Lemma 1.6.10 *For a locally integrable function f and any $a \in \mathbb{R}$, the definite integral*

$$F(x) = \int_a^x f(t) \, dt \tag{1.6.8}$$

is differentiable with $F'(x) = f(x)$ for all $x \in \mathcal{L}(f)$.

Proof. We find

$$\frac{F(x+\delta) - F(x)}{\delta} - f(x) = \frac{1}{\delta} \int_x^{x+\delta} (f(t) - f(x)) \, dt,$$

so the right-hand side is majorized by

$$\frac{1}{\delta} \int_{-\delta}^{0} |f(x-t) - f(x)| \, dt,$$

and similarly

$$\left|\frac{F(x)-F(x-\delta)}{\delta}-f(x)\right| \le \frac{1}{\delta}\int_0^\delta |f(x-t)-f(x)|\,dt,$$

both of which approach 0 for $\delta \to 0$ by (1.6.7). A proof of Lebesgue's result can be found in [9, Chap. 7]. $\square$

We will now make use of Lemma 1.6.10 to give a proof of the Fejér-Lebesgue theorem by showing

$$f * F_n(x) \to f(x) \qquad \text{for } \ x \in \mathcal{L}(f).$$

We have

$$f * F_n(x) - f(x) = \frac{1}{2\pi}\int_0^\pi \left(f(x+t)+f(x-t)-2f(x)\right) F_n(t)\,dt,$$

and we next split the interval of integration at $t=\frac{1}{n}$. From the expression

$$F_n(x) = 1 + 2\sum_{k=1}^n \left(1-\frac{k}{n+1}\right)\cos(kx)$$

we see immediately that $F_n(x) \le F_n(0) = n+1$, and using (1.6.6) we obtain with $g(t) = f(x+t)+f(x-t)-2f(x)$

$$|f * F_n(x) - f(x)| \le \frac{n+1}{2\pi}\int_0^{\frac{1}{n}} |g(t)|\,dt + \frac{\pi}{2(n+1)}\int_{\frac{1}{n}}^\pi \frac{|g(t)|}{t^2}\,dt. \tag{1.6.9}$$

Introducing the continuous function

$$G(u) = \int_0^u |g(t)|\,dt,$$

we have for $u>0$

$$\frac{G(u)}{2u} \le \frac{1}{2u}\int_{-u}^u |f(x-t)-f(x)|\,dt,$$

which tends to zero by (1.6.7). We can evaluate the second term in (1.6.9) by partial integration, so the right-hand side of (1.6.9) can be written

$$\frac{n+1}{2\pi}G\left(\frac{1}{n}\right) + \frac{\pi}{2(n+1)}\left[\frac{G(t)}{t^2}\right]_{\frac{1}{n}}^\pi + \frac{\pi}{n+1}\int_{\frac{1}{n}}^\pi \frac{G(t)}{t^3}\,dt.$$

Since

$$0 \le n\,G\left(\frac{1}{n}\right) \le n\int_{-\frac{1}{n}}^{\frac{1}{n}} |f(x-t)-f(x)|\,dt \to 0$$

by (1.6.7), we only need to show that

$$\lim_{n\to\infty} \frac{1}{n+1} \int_{\frac{1}{n}}^{\pi} \frac{G(t)}{t^3}\, dt = 0.$$

For a given $\varepsilon > 0$ it is possible by (1.6.7) to determine $\delta > 0$ such that

$$0 \le G(t) \le t\,\varepsilon \qquad \text{for } \ 0 < t \le \delta.$$

For $n \ge \frac{1}{\delta}$ we therefore have

$$\begin{aligned}\frac{1}{n+1} \int_{\frac{1}{n}}^{\pi} \frac{G(t)}{t^3}\, dt &\le \frac{1}{n+1} \int_{\frac{1}{n}}^{\delta} \frac{\varepsilon t}{t^3}\, dt + \frac{1}{n+1} \int_{\delta}^{\pi} \frac{G(t)}{t^3}\, dt \\ &= \frac{\varepsilon}{n+1}\left(n - \frac{1}{\delta}\right) + \frac{1}{n+1} \int_{\delta}^{\pi} \frac{G(t)}{t^3}\, dt \le \varepsilon + \frac{1}{n+1} \int_{\delta}^{\pi} \frac{G(t)}{t^3}\, dt,\end{aligned}$$

which is $< 2\varepsilon$ for n sufficiently large. $\square$

We shall now apply the theory of summability to prove Dirichlet–Jordan's test. The proof depends on a "Tauberian theorem", i.e., a theorem of the form

$$\left.\begin{array}{l}\sum a_n \text{ is summable with sum } s \\ + \text{ condition on } a_n\end{array}\right\} \Rightarrow \sum a_n \text{ is convergent with sum } s.$$

The first example of this type of theorem is due to Alfred Tauber (1866 – 1942 ?).

Theorem 1.6.11 (Hardy's Tauberian Theorem (1909)) *If a series $\sum_0^\infty a_n$ is summable with sum s and the sequence of numbers $(n\,a_n)$ is bounded, then the series is convergent with sum s.*

Proof. We assume that $\lim\limits_{n\to\infty} \sigma_n = s$ and $|n\,a_n| \le A$ and shall show that $\lim\limits_{n\to\infty} s_n = s$.

Recall that $s_n = a_0 + \ldots + a_n$ and $(n+1)\sigma_n = s_0 + s_1 + \ldots + s_n$.

For $\varepsilon > 0$, there is an N such that $|s - \sigma_n| \le \varepsilon$ for $n \ge N$. For $n \ge N+1$ and $p \ge 1$ it holds that

$$(n+p)\sigma_{n+p-1} - n\,\sigma_{n-1} = s_n + s_{n+1} + \ldots + s_{n+p-1},$$

therefore

$$(n+p)(\sigma_{n+p-1} - s) - n(\sigma_{n-1} - s) = p(s_n - s) + R, \tag{1.6.10}$$

where

$$\begin{aligned} R &= (s_{n+1} - s_n) + (s_{n+2} - s_n) + \ldots + (s_{n+p-1} - s_n) \\ &= a_{n+1} + (a_{n+1} + a_{n+2}) + \ldots + (a_{n+1} + a_{n+2} + \ldots + a_{n+p-1}). \end{aligned}$$

The terms a_j above are all numerically smaller than or equal to $A/j \leq A/n$ and their number is $\frac{1}{2}p(p-1)$. Consequently,

$$|R| \leq \frac{A\,p(p-1)}{2n},$$

so from (1.6.10) we obtain

$$\begin{aligned} p|s_n - s| &\leq (n+p)\,|\sigma_{n+p-1} - s| + n\,|\sigma_{n-1} - s| + |R| \\ &\leq (2n+p)\,\varepsilon + \frac{A\,p(p-1)}{2n} \end{aligned} \tag{1.6.11}$$

or

$$|s_n - s| \leq \left(\frac{2n}{p} + 1\right)\varepsilon + \frac{A(p-1)}{2n} \qquad \text{for } \ n \geq N+1, \ p \geq 1. \tag{1.6.12}$$

We would like the right-hand side to be small, so we choose p such that it becomes as small as possible. The function

$$\varphi(p) = \frac{\alpha}{p} + \beta p + \gamma, \qquad \text{where } \ \alpha, \ \beta > 0,$$

has a minimum in $]0, \infty[$ for $p = \sqrt{\frac{\alpha}{\beta}}$. We shall therefore choose $p = 2n\sqrt{\frac{\varepsilon}{A}}$, but since p shall be an integer, we choose the integer p such that

$$2n\sqrt{\frac{\varepsilon}{A}} \leq p < 2n\sqrt{\frac{\varepsilon}{A}} + 1.$$

With this value of p, (1.6.12) yields

$$|s_n - s| \leq \left(\sqrt{\frac{A}{\varepsilon}} + 1\right)\varepsilon + A\sqrt{\frac{\varepsilon}{A}} = \varepsilon + 2\sqrt{A\varepsilon} \qquad \text{for } \ n \geq N+1, \tag{1.6.13}$$

and since $\varepsilon + 2\sqrt{A\varepsilon} \to 0$ for $\varepsilon \to 0$, it follows that $s_n \to s$ for $n \to \infty$. $\square$

<u>Proof of Dirichlet-Jordan's test (p. 18).</u>

We note first that the Fourier series is summable with sum $\frac{1}{2}(f(x+0)+f(x-0))$ for all x. To see this, we evaluate as previously:

$$\begin{aligned} &\sigma_n(x) - \frac{1}{2}(f(x+0) + f(x-0)) \\ &= \frac{1}{2\pi}\int_0^{\pi} (f(x+y) + f(x-y))\,F_n(y)\,dy - \frac{1}{2}(f(x+0) + f(x-0)) \\ &= \frac{1}{2\pi}\int_0^{\delta} + \int_{\delta}^{\pi} [f(x+y) - f(x+0) + f(x-y) - f(x-0)]\,F_n(y)\,dy. \end{aligned}$$

For a given $\varepsilon > 0$, we choose first a $\delta > 0$ such that

$$|f(x \pm y) - f(x \pm 0)| \leq \varepsilon \qquad \text{for } \ 0 < y \leq \delta,$$

and with this $\delta > 0$, the absolute value of the integral over $[0, \delta]$ can be majorized by ε. A function of bounded variation is in particular bounded, so we obtain

$$|\sigma_n(x) - \frac{1}{2}(f(x+0) + f(x-0))| \leq \varepsilon + \frac{4\|f\|_\infty}{2\pi} \int_\delta^\pi F_n(y)\, dy,$$

and the claim becomes evident.

To apply Hardy's Tauberian Theorem, it is sufficient to show that the sequence of numbers

$$n(c_n e^{inx} + c_{-n} e^{-inx})$$

is bounded for every x. It is even uniformly bounded in x, since we show

$$|n\, c_n| \leq \frac{V}{2},$$

where $V = V_{[0,2\pi]}(f)$.

For $n \neq 0$, we set $h = \pi/|n|$ and evaluate $2\pi c_n$ in the following way:

$$\begin{aligned}
2\pi c_n &= \int_0^h + \int_h^{2h} + \ldots + \int_{2\pi-h}^{2\pi} f(x)\, e^{-inx}\, dx \\
&= \int_0^h \left[f(x)\, e^{-inx} + f(x+h)\, e^{-\mathrm{i}n(x+h)} + \ldots + f(x + (2|n|-1)h)\, e^{-\mathrm{i}n(x+(2|n|-1)h)} \right] dx \\
&= \int_0^h \left[f(x) - f(x+h) + f(x+2h) - + \ldots - f(x + (2|n|-1)h) \right] e^{-inx}\, dx,
\end{aligned}$$

hence

$$2\pi |c_n| \leq \int_0^h \left[\sum_{j=0}^{|n|-1} |f(x + 2jh) - f(x + (2j+1)h)| \right] dx \leq Vh,$$

since the integrand is $\leq V$ for $x \in [0, h]$. We therefore obtain $|c_n| \leq \frac{V}{2|n|}$.

We note finally that if f in addition is continuous in $[a, b]$, then

$$|\sigma_n(x) - f(x)| \to 0 \qquad \text{uniformly for } \ x \in [a, b].$$

This follows by examination of the proof of Theorem 1.6.8. Since the boundedness condition in Hardy's Tauberian Theorem holds uniformly for $x \in \mathbb{R}$, it follows from

(1.6.13) that $s_n(x) \to f(x)$ uniformly for $x \in [a,b]$, i.e., the Fourier series converges uniformly in $[a,b]$. $\square$

Exercises

E 6.1 Consider Definition 1.6.2. Show that if (a) and (b) are fulfilled for (k_n) and (c') holds for a $\delta_1 \in]0,\pi[$, then (c') holds for every $\delta_2 \in]\delta_1,\pi[$. It is therefore sufficient to know (c') for arbitrarily small delta.

E 6.2 Let f be a periodic function of bounded variation over $[0,2\pi]$. Show that $|s_n(x)| \le \|f\|_\infty + V_{[0,2\pi]}(f)$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}$, i.e., the partial sums of the Fourier series are uniformly bounded.

Hint: $|\sigma_n(x)| \le \|f\|_\infty$ and find an estimate of $s_n(x) - \sigma_n(x)$.

E 6.3 Show that if f is periodic and of bounded variation over $[0,2\pi]$, then the Fourier series is strongly convergent in $\mathcal{L}^1(\mathbb{T})$, i.e., $\|f - s_n\|_1 \to 0$.

E 6.4 For $f \in \mathcal{L}^1(\mathbb{T})$ consider the operator $\tilde{T_f}$ from Exercise **E 1.3** as an operator on the Banach space $L^1(\mathbb{T})$. Prove that the operator norm $\|\tilde{T_f}\|$ is equal to $\|f\|_1$.

1.7 L^2-theory and Parseval's identity

The theory is probably known from an introductory course on Hilbert spaces, and we shall therefore treat it summarily and restrict ourselves to emphasizing some important points.

It is crucial that $L^2(\mathbb{T})$ is a Hilbert space and that $\{e_n(x) = e^{\mathrm{i}nx} \mid n \in \mathbb{Z}\}$ is an orthonormal basis herein. While it is elementary that $\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal system, being an orthonormal basis means in addition that if $f \in \mathcal{L}^2(\mathbb{T})$ is orthogonal to all the vectors e_n then $f = 0$, or equivalently that the space span $\{e_n \mid n \in \mathbb{Z}\}$ of trigonometric polynomials is dense in $\mathcal{L}^2(\mathbb{T})$. That this is true is non-trivial, and it follows from Weierstrass' approximation theorem, cf. Corollary 1.6.7.

The n'th partial sum $s_n(x)$ for $f \in \mathcal{L}^2(\mathbb{T})$ is the orthogonal projection on the $2n+1$ dimensional subspace $E_n = \operatorname{span}\{e^{\mathrm{i}kx} \mid -n \le k \le n\}$. The Pythagorian Theorem entails that

$$\|f\|_2^2 = \|f - s_n\|_2^2 + \|s_n\|_2^2 = \|f - s_n\|_2^2 + \sum_{k=-n}^{n} |c_k|^2. \tag{1.7.1}$$

From this equation it is completely elementary that

$$\lim_{n\to\infty} \|f - s_n\|_2 = 0 \quad \Leftrightarrow \quad \|f\|_2^2 = \sum_{-\infty}^{\infty} |c_k|^2, \tag{1.7.2}$$

i.e.,

Parseval's identity $\|f\|_2^2 = \sum_{-\infty}^{\infty} |c_k|^2$ *holds if and only if the Fourier series converges in* $\mathcal{L}^2(\mathbb{T})$.

To see that these properties are fulfilled, we proceed as follows:

The n'th average of the partial sums

$$\sigma_n = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) c_k \, e^{\mathrm{i}k\theta}$$

belongs to E_n as well as s_n, but since s_n is the best approximation of f from E_n, we have

$$\|f - s_n\|_2 \le \|f - \sigma_n\|_2.$$

The last term approaches 0 by Theorem 1.6.6 for $p = 2$, and thus also $\|f - s_n\|_2 \to 0$.

Parseval's identity goes back to 1799, but Parseval did not give a proper proof. The theory obtained a decisive rounding-off with the Lebesgue integral which made it possible to prove that $\mathcal{L}^2(\mathbb{T})$ is complete, cf. Fischer's completeness theorem. A closely related result is

Theorem 1.7.1 (Riesz-Fischer's Theorem (1909)) *To a sequence* $(c_n)_{n\in\mathbb{Z}}$ *of complex numbers satisfying* $\sum_{-\infty}^{\infty} |c_n|^2 < \infty$ *there exists a function* $f \in \mathcal{L}^2(\mathbb{T})$ *with Fourier series*

$$f \sim \sum_{k=-\infty}^{\infty} c_k \, e^{\mathrm{i}k\theta},$$

and such a function is uniquely determined almost everywhere.

The proof is simple: The sequence $s_n = \sum_{k=-n}^{n} c_k \, e^{\mathrm{i}k\theta}$ is a Cauchy sequence in $\mathcal{L}^2(\mathbb{T})$ because

$$\|s_{n+p} - s_n\|_2^2 = \sum_{k=n+1}^{n+p} \left(|c_k|^2 + |c_{-k}|^2\right),$$

and since $\mathcal{L}^2(\mathbb{T})$ is complete, $s_n \to f$ for some $f \in \mathcal{L}^2(\mathbb{T})$, which can easily be seen to have the Fourier coefficients (c_n). $\square$

Exercises

E 7.1 Find the Fourier series for $f \in \mathcal{L}^1(\mathbb{T})$ given by

$$f(x) = \begin{cases} x & \text{for } -\pi < x < \pi \\ 0 & \text{for } x = \pi, \end{cases}$$

and show that it converges pointwise to $f(x)$ for all $x \in \mathbb{R}$.

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

E 7.2 For $n \in \mathbb{N}_0$ let

$$S_n(t) = \sum_{k=-n}^{n} c_k e^{\mathrm{i}kt}$$

be a trigonometric polynomial of degree $\leq n$.

1°. Show that $\|S_n'\|_2 \leq n\|S_n\|_2$.

2°. Find all the trigonometric polynomials of degree $\leq n$ for which there is equality in the inequality in 1°.

1.8 The Fourier coefficients considered as a mapping

Considering the Fourier coefficients $(c_n) = (c_n(f))$ for $f \in \mathcal{L}^1(\mathbb{T})$, we have a mapping $C : \mathcal{L}^1(\mathbb{T}) \to C_0(\mathbb{Z})$ given by

$$C(f)(n) = c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, e^{-\mathrm{i}nx}\, dx, \qquad n \in \mathbb{Z}. \tag{1.8.1}$$

Here $C_0(\mathbb{Z})$ denotes the set of sequences $(c_n)_{n \in \mathbb{Z}}$ of complex numbers satisfying $c_n \to 0$ for $n \to \pm\infty$. (We can also view $\mathbb{Z}$ as a metric space with the discrete metric. Then $\mathbb{Z}$ becomes a locally compact Hausdorff space where each subset of $\mathbb{Z}$ is open and closed. We can thereby view $C_0(\mathbb{Z})$ as the space of continuous functions $C : \mathbb{Z} \to \mathbb{C}$ vanishing at infinity). Since two functions which are equal almost everywhere have the same Fourier coefficient, we can consider C as a mapping $C : L^1(\mathbb{T}) \to C_0(\mathbb{Z})$.

Theorem 1.8.1 *If f, $g \in \mathcal{L}^1(\mathbb{T})$ possess the Fourier series*

$$f \sim \sum c_n\, e^{\mathrm{i}nx}, \qquad g \sim \sum d_n\, e^{\mathrm{i}nx},$$

then we have

(i) $f + g \sim \sum (c_n + d_n) e^{\mathrm{i}nx}$,

(ii) $\lambda f \sim \sum \lambda c_n e^{\mathrm{i}nx}, \quad \lambda \in \mathbb{C}$,

(iii) $f * g \sim \sum c_n \, d_n \, e^{\mathrm{i}nx}$.

Stated differently: $C : L^1(\mathbb{T}) \to C_0(\mathbb{Z})$ is an algebra homomorphism.

Proof. Only (iii) is non-trivial. By Fubini's theorem we obtain

$$\begin{aligned}
c_n(f * g) &= \frac{1}{2\pi}\int_0^{2\pi} f * g(x)\, e^{-\mathrm{i}nx}\, dx = \frac{1}{2\pi}\int_0^{2\pi} \Big(\frac{1}{2\pi}\int_0^{2\pi} f(x-y)\, g(y)\, dy\, e^{-\mathrm{i}nx}\Big) dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Big(\frac{1}{2\pi}\int_0^{2\pi} f(x-y)\, e^{-\mathrm{i}n(x-y)}\, dx\Big)\, g(y)\, e^{-\mathrm{i}ny} dy \\
&= \frac{1}{2\pi}\int_0^{2\pi} c_n(f)\, g(y)\, e^{-\mathrm{i}ny}\, dy = c_n(f)\, c_n(g).
\end{aligned}$$

□

Remarks 1.8.2 *1)* Introducing an involution $\sim$ in $\mathcal{L}^1(\mathbb{T})$ by $\tilde{f}(x) = \overline{f(-x)}$, we can see that

$$\tilde{f} \sim \sum \overline{c_n}\, e^{\mathrm{i}n\theta}, \tag{1.8.2}$$

since

$$c_n(\tilde{f}) = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(-x)}\, e^{-\mathrm{i}nx}\, dx = \overline{\frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-\mathrm{i}nx}\, dx} = \overline{c_n(f)}.$$

Complex conjugation is an involution in the algebra $C_0(\mathbb{Z})$, and (1.8.2) can be expressed that C respects the involution.

2) The Uniqueness Theorem (Theorem 1.6.5) can be expressed that C is injective.

3) Since $\sup\limits_n |c_n(f)| \leq \|f\|_1$, we have that C is norm diminishing as a mapping of the Banach space $L^1(\mathbb{T})$ into the Banach space $C_0(\mathbb{Z})$. We can add that $\|C\| \leq 1$.

4) Riesz–Fischer's and Parseval's theorems can be expressed that C maps $L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$ bijectively onto $\ell^2(\mathbb{Z}) \subseteq C_0(\mathbb{Z})$, and that the restriction of C to $L^2(\mathbb{T})$ is an isometric isomorphism onto $\ell^2(\mathbb{Z})$.

5) C maps the trigonometric polynomials onto those sequences $c \in C_0(\mathbb{Z})$ which only are $\neq 0$ for finitely many n. This subspace of $C_0(\mathbb{Z})$ can be understood as the continuous functions on $\mathbb{Z}$ with compact support.

It can be shown that $\|C\| = 1$ in 3) above, see **E 8.1**.

From 5) follows that $C(L^1(\mathbb{T}))$ is a dense subspace of $C_0(\mathbb{Z})$. It is obvious to ask if C is surjective. We shall see that it is not, cf. Remark 1.8.5. This gives rise to the question if one can say something about how quickly $c_n \to 0$ when (c_n) are Fourier coefficients for an integrable function. Here the answer is that there are Fourier coefficients which approach 0 arbitrarily slowly for $n \to \pm\infty$. The consensus among specialists is that it is impossible to give a descriptive characterization of $C(L^1(\mathbb{T}))$ as a subset of $C_0(\mathbb{Z})$.

The following sufficient condition for a sequence $c = (c_n)$ to belong to $C(L^1(\mathbb{T}))$ is often useful.

Theorem 1.8.3 *Let $c \in C_0(\mathbb{Z})$ be a sequence with the properties*

(i) $c_n \geq 0, \quad n \in \mathbb{Z}$,

(ii) $c_{-n} = c_n, \quad n \in \mathbb{Z}$ *(c is an even function on* $\mathbb{Z}$*),*

(iii) $2c_n \leq c_{n-1} + c_{n+1}$ *for* $n \geq 1$ *(c is convex on* $\mathbb{Z}_+$*).*

Then there is an $f \in \mathcal{L}^1_+(\mathbb{T})$ *with* $c_n(f) = c_n$, $n \in \mathbb{Z}$.

Proof. Condition (iii) gives $c_n - c_{n+1} \leq c_{n-1} - c_n$, i.e., $(c_n - c_{n+1})$, $n \geq 0$ is decreasing and approaches 0, the latter because $c_n \to 0$. It follows that $c_n - c_{n+1} \geq 0$ for $n \geq 0$.
For $\varepsilon > 0$ we choose an N such that $c_n \leq \varepsilon/2$ for $n \geq N$. For $p \geq 1$ we obtain now

$$\begin{aligned} c_N - c_{N+p} &= (c_N - c_{N+1}) + (c_{N+1} - c_{N+2}) + \ldots + (c_{N+p-1} - c_{N+p}) \\ &\geq p(c_{N+p-1} - c_{N+p}), \end{aligned}$$

or when $p \geq N$

$$(N+p-1)(c_{N+p-1} - c_{N+p}) \leq \frac{N+p-1}{p}(c_N - c_{N+p}) \leq \frac{2p-1}{p} c_N \leq \varepsilon.$$

We thus have

$$k(c_k - c_{k+1}) \leq \varepsilon \qquad \text{for } k \geq 2N-1,$$

which shows that $\lim\limits_{n\to\infty} n(c_n - c_{n+1}) = 0$. It follows that

$$\begin{aligned} \sum_{k=1}^{n} k(c_{k-1} + c_{k+1} - 2c_k) &= \sum_{k=1}^{n} (k(c_{k-1} - c_k) + c_k) - ((k+1)(c_k - c_{k+1}) + c_{k+1}) \\ &= c_0 - c_n - n(c_n - c_{n+1}) \end{aligned}$$

converges to c_0 for $n \to \infty$. Notice that the sum above is telescoping.

Consider now the infinite series

$$\sum_{n=1}^{\infty} n(c_{n-1}+c_{n+1}-2c_n)\,F_{n-1}(x), \tag{1.8.3}$$

where F_n is Fejér's kernel. The terms are non-negative continuous functions, and therefore the sum $f(x)$ is a non-negative Borel function, possibly infinite at certain points. (As the limit of an increasing sequence of non-negative continuous functions, f is actually lower semi-continuous.) By the monotone convergence theorem

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx &= \sum_{n=1}^{\infty} n(c_{n-1}+c_{n+1}-2c_n)\frac{1}{2\pi}\int_0^{2\pi} F_{n-1}(x)\,dx \\ &= \sum_{n=1}^{\infty} n(c_{n-1}+c_{n+1}-2c_n) = c_0,\end{aligned}$$

so $f \in \mathcal{L}^1_+(\mathbb{T})$. The partial sums s_n of (1.8.3) converge to f in $\mathcal{L}^1(\mathbb{T})$ by Lebesgue's theorem about dominated convergence.

The j'th Fourier coefficient for s_n is

$$c_j(s_n) = \sum_{k=1}^{n} k(c_{k-1}+c_{k+1}-2c_k)\frac{1}{2\pi}\int_{-\pi}^{\pi} F_{k-1}(x)\,e^{-\mathrm{i}jx}\,dx,$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_{k-1}(x)\,e^{-\mathrm{i}jx}\,dx = \begin{cases} 1-\dfrac{|j|}{k} & \text{for} \quad |j| \le k-1 \\ 0 & \text{for} \quad |j| \ge k. \end{cases}$$

For $n \ge |j|+1$ we therefore obtain

$$\begin{aligned} c_j(s_n) &= \sum_{k=|j|+1}^{n} k(c_{k-1}+c_{k+1}-2c_k)\left(1-\frac{|j|}{k}\right) \\ &= \sum_{k=|j|+1}^{n} k(c_{k-1}+c_{k+1}-2c_k) - |j|\sum_{k=|j|+1}^{n}((c_{k-1}-c_k)-(c_k-c_{k+1})) \\ &= \sum_{k=|j|+1}^{n} (k(c_{k-1}-c_k)+c_k)-((k+1)(c_k-c_{k+1})+c_{k+1}) \\ &\quad -|j|\sum_{k=|j|+1}^{n}((c_{k-1}-c_k)-(c_k-c_{k+1})) \\ &= \big((|j|+1)(c_{|j|}-c_{|j|+1})+c_{|j|+1}\big)-((n+1)(c_n-c_{n+1})+c_{n+1}) \\ &\quad -|j|\big((c_{|j|}-c_{|j|+1})-(c_n-c_{n+1})\big) \\ &= c_{|j|}+|j|(c_n-c_{n+1})-n(c_n-c_{n+1})-c_n \end{aligned}$$

by employing telescopic sums. For $n \to \infty$ this expression converges to $c_{|j|}$. Since $s_n \to f$ in $\mathcal{L}^1(\mathbb{T})$ we get $c_j(s_n) \to c_j(f)$ for $n \to \infty$ because

$$|c_j(f) - c_j(s_n)| \le \|f - s_n\|_1.$$

This gives $c_j(f) = c_{|j|} = c_j$ for $j \in \mathbb{Z}$. $\square$

Example 1.8.4 If $\varphi : [0, \infty[\to]0, \infty]$ is a decreasing convex function tending to zero at infinity, then $c_n = \varphi(|n|)$ will fulfill the conditions in Theorem 1.8.3. The theorem can be applied to

a) $\varphi(x) = \dfrac{1}{(x+1)^\alpha}, \quad \alpha > 0$

b) $\varphi(x) = \dfrac{1}{\log(a+x)} \quad$ for $a > 1$.

If φ is only defined on the open interval $]0, \infty[$, like $\varphi(x) = 1/\log(1+x)$, but still has the same properties, we can put $c_n = \varphi(|n|)$ for $n \neq 0$, and we then have to choose $c_0 \ge 0$ such that $c_0 + c_2 \ge 2c_1$. In particular, we obtain that the trigonometric series with

$$c_{-n} = c_n = \frac{1}{\log(n+1)}, \qquad n \ge 1, \qquad c_0 \ge \frac{2}{\log 2} - \frac{1}{\log 3} \;(= 1.97...),$$

i.e.,

$$c_0 + 2\sum_{n=1}^{\infty} \frac{\cos(nx)}{\log(n+1)} \tag{1.8.4}$$

is a Fourier series for a function $f \in \mathcal{L}^1_+(\mathbb{T})$. Since $\sum c_n^2 = \infty$, we know furthermore that $f \notin \mathcal{L}^2(\mathbb{T})$. The series (1.8.4) is clearly divergent for $x = 0$ and convergent (alternating series) for $x = \pi$. We shall see later that the function f is continuous on $]0, 2\pi[$, cf. Sec. 1.9.

What can we say about the corresponding sine-series?

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{\log(n+1)}. \tag{1.8.5}$$

We shall see later (Sec. 1.9) that (1.8.5) converges for all x and the sum function is continuous on $]0, 2\pi[$, but it is not integrable over $[0, 2\pi]$. This follows from Theorem 1.9.3 below because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \log(n+1)} = \infty$ (cf. the integral criterion).

Remark 1.8.5 The properties of the function (1.8.5) show that the sequence

$$c_n = \begin{cases} 0, & n = 0 \\ \dfrac{\operatorname{sgn}(n)}{\log(n+1)}, & n \neq 0 \end{cases}$$

belongs to $C_0(\mathbb{Z}) \setminus C(\mathcal{L}^1(\mathbb{T}))$, i.e., C is not surjective.

Exercises

E 8.1 Show that $C : \mathcal{L}^1(\mathbb{T}) \to C_0(\mathbb{Z})$ has norm $\|C\| = 1$, cf. Remarks 1.8.2 no. 3).

E 8.2 Show that $\varphi(x) = 1/\log^{\circ n}(a_n + x)$ is a positive, decreasing and convex function on $[0, \infty[$ with $\varphi(x) \to 0$ for $x \to \infty$, provided $a_n > \exp^{\circ n}(0)$. Here is $\log^{\circ 2}(x) = \log(\log x)$, and in general $\log^{\circ n}(x) = \log(\log^{\circ (n-1)}(x))$. Correspondingly, $\exp^{\circ n}(x) = \exp(\exp^{\circ (n-1)}(x))$. [In this way, we can construct decreasing convex functions which approach 0 very slowly, and by Theorem 1.8.3 this yields functions in $\mathcal{L}^1(\mathbb{T})$ for which the Fourier coefficients approach 0 very slowly].

1.9 Some simple trigonometric series

We shall consider pointwise convergence of a trigonometric series of the form

$$\sum_{n=1}^{\infty} \lambda_n e^{inx}, \tag{1.9.1}$$

where $(\lambda_n)_{n \geq 1}$ is a decreasing sequence of positive numbers approaching 0. This is equivalent to considering the two real-valued series

$$\sum_{n=1}^{\infty} \lambda_n \cos(nx), \qquad \sum_{n=1}^{\infty} \lambda_n \sin(nx).$$

Theorem 1.9.1 *Let (λ_n) be a decreasing sequence of positive numbers satisfying $\lim \lambda_n = 0$. Then the trigonometric series (1.9.1) converges for $x \neq 2p\pi$, and it converges uniformly on $[\delta, 2\pi - \delta]$ for every $0 < \delta < \pi$.*

Proof. Defining

$$A_n(x) = \sum_{k=1}^{n} e^{ikx} = \frac{e^{inx} - 1}{1 - e^{-ix}}, \qquad x \neq 2p\pi,$$

we have

$$|A_n(x)| \le \frac{2}{|1-e^{-\mathrm{i}x}|} = \frac{1}{\sin\frac{x}{2}} \le \frac{1}{\sin\frac{\delta}{2}} \qquad \text{for } x \in [\delta, 2\pi-\delta].$$

We next use a technique for infinite series which is analogous to integration by parts:

$$\sum_{k=n+1}^{n+p} \lambda_k\, e^{\mathrm{i}kx} = e^{\mathrm{i}nx}\left(\lambda_{n+1}A_1(x) + \sum_{k=2}^{p} \lambda_{n+k}(A_k(x) - A_{k-1}(x))\right)$$

$$= e^{\mathrm{i}nx}\left(\sum_{k=1}^{p-1} A_k(x)(\lambda_{n+k} - \lambda_{n+k+1}) + \lambda_{n+p}A_p(x)\right),$$

therefore for $x \in [\delta, 2\pi-\delta]$

$$\left|\sum_{k=n+1}^{n+p} \lambda_k\, e^{\mathrm{i}kx}\right| \le \frac{1}{\sin\frac{x}{2}}\left(\sum_{k=1}^{p-1}(\lambda_{n+k} - \lambda_{n+k+1}) + \lambda_{n+p}\right) = \frac{\lambda_{n+1}}{\sin\frac{x}{2}}, \tag{1.9.2}$$

where we have used that $\lambda_{n+p} \ge 0$, $\lambda_{n+k} - \lambda_{n+k+1} \ge 0$. From this estimate it follows that the series is uniformly convergent in the interval $[\delta, 2\pi-\delta]$. □

The theorem shows that the sum of (1.9.1)

$$f(x) = \sum_{n=1}^{\infty} \lambda_n\, e^{\mathrm{i}nx} \tag{1.9.3}$$

is a continuous function on $\mathbb{R} \setminus 2\pi\mathbb{Z}$. The real part

$$\sum_{n=1}^{\infty} \lambda_n \cos(nx)$$

converges for $x = 2p\pi$ precisely when $\sum_1^\infty \lambda_n < \infty$, but the imaginary part

$$\sum_{n=1}^{\infty} \lambda_n \sin(nx)$$

converges trivially with sum 0 for $x = 2p\pi$.

We shall now give a sufficient condition on the sequence (λ_n) which ensures that the function (1.9.3) is integrable.

Theorem 1.9.2 *Let (λ_n) be a decreasing sequence of positive numbers with $\lim \lambda_n = 0$. If $\sum_1^\infty \lambda_n/n < \infty$, then f defined by (1.9.3) belongs to $\mathcal{L}^1(\mathbb{T})$ and*

$$\|f\|_1 \le 2\sum_{n=1}^{\infty} \frac{\lambda_n}{n}. \tag{1.9.4}$$

The Fourier series of f is $\sum_{n=1}^\infty \lambda_n e^{\mathrm{i}nx}$.

Proof. Since $f(-x) = \overline{f(x)}$, it is sufficient to estimate $\int_0^\pi |f(x)|\,dx$. We let $\Lambda_k = \lambda_1 + \ldots + \lambda_k$ and find

$$\sum_{k=1}^\infty \frac{\Lambda_k}{k(k+1)} = \sum_{k=1}^\infty \frac{1}{k(k+1)} \sum_{n=1}^k \lambda_n = \sum_{n=1}^\infty \lambda_n \sum_{k=n}^\infty \left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{n=1}^\infty \frac{\lambda_n}{n}.$$

For $\frac{\pi}{k+1} \le x < \frac{\pi}{k}$, $k = 1, 2, \ldots$, we have

$$f(x) = \sum_{n=1}^k \lambda_n\, e^{\mathrm{i}nx} + \sum_{n=k+1}^\infty \lambda_n\, e^{\mathrm{i}nx},$$

hence by (1.9.2) (since we can let $p \to \infty$)

$$|f(x)| \le \Lambda_k + \frac{1}{\sin\frac{x}{2}} \lambda_{k+1}.$$

Using $\sin\frac{x}{2} \ge \frac{x}{\pi}$ for $x \in [0, \pi]$, see (1.6.5), we get the estimate

$$|f(x)| \le \Lambda_k + \frac{\pi}{x}\lambda_{k+1} \le \Lambda_k + \lambda_{k+1}(k+1),$$

where the last inequality is because $x \ge \frac{\pi}{k+1}$. We finally get

$$\begin{aligned}
\int_0^\pi |f(x)|\,dx &= \sum_{k=1}^\infty \int_{\pi/(k+1)}^{\pi/k} |f(x)|\,dx \le \sum_{k=1}^\infty \pi \left(\frac{1}{k} - \frac{1}{k+1}\right)(\Lambda_k + \lambda_{k+1}(k+1)) \\
&= \pi \sum_{k=1}^\infty \frac{\Lambda_k}{k(k+1)} + \pi \sum_{k=1}^\infty \frac{\lambda_{k+1}}{k} \le 2\pi \sum_{n=1}^\infty \frac{\lambda_n}{n}.
\end{aligned}$$

To show the last assertion, it is enough to prove that $s_n(x) = \sum_{k=1}^n \lambda_k e^{\mathrm{i}kx}$ converges to f in $\mathcal{L}^1(\mathbb{T})$. We find

$$\begin{aligned}
\|f - s_n\|_1 &= \|\sum_{k=n+1}^\infty \lambda_k e^{\mathrm{i}kx}\|_1 \\
&= \|\sum_{j=1}^\infty \lambda_{n+j} e^{\mathrm{i}jx}\|_1 \le 2 \sum_{j=1}^\infty \frac{\lambda_{n+j}}{j}
\end{aligned}$$

by (1.9.4) applied to the sequence $\lambda_{n+j}, j = 1, 2, \ldots$. We claim that

$$\sum_{j=1}^\infty \frac{\lambda_{n+j}}{j} \to 0 \quad \text{for} \quad n \to \infty$$

because of dominated convergence for sequences. In fact, $\lambda_{n+j}/j \to 0$ for $n \to \infty$ and $\lambda_{n+j}/j \le \lambda_j/j$. □

Theorem 1.9.2 has a partial converse concerning sine-series.

Theorem 1.9.3 *Let $(\lambda_n)_{n\geq 1}$ be a decreasing sequence of positive numbers such that* $\lim \lambda_n = 0$. *Then the sine series*

$$\sum_{n=1}^{\infty} \lambda_n \sin(nx) \tag{1.9.5}$$

converges to a function $S \in \mathcal{L}^1(\mathbb{T})$ if and only if

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n} < \infty. \tag{1.9.6}$$

If (1.9.6) *holds, then* (1.9.5) *is the Fourier series of S.*

Proof. We know that S is the imaginary part of the function f from Theorem 1.9.2, hence integrable if (1.9.6) holds, and (1.9.5) is the Fourier series of S.

Assume next that the sum S of the series (1.9.5) is integrable. Since S is an odd function, its Fourier series is a sine-series.

For fixed $m \in \mathbb{N}$ the series $\sum_{n=1}^{\infty} \lambda_n \sin(nx)\sin(mx)$ converges uniformly to $S(x)\sin(mx)$ for $0 < x < \pi$. In fact, for any $n, p \in \mathbb{N}$ and $0 < x < \pi$ we find

$$\begin{aligned} |\sum_{k=n+1}^{n+p} \lambda_k \sin(kx)\sin(mx)| &\leq |\sin(mx)|\,|\mathrm{Im}\left(\sum_{k=n+1}^{n+p} \lambda_k e^{\mathrm{i}kx}\right)| \\ &\leq mx\frac{\lambda_{n+1}}{\sin(x/2)} \leq \lambda_{n+1} m\pi, \end{aligned}$$

where we have used (1.9.2) and (1.6.5). From the uniform convergence we get

$$\frac{2}{\pi}\int_0^{\pi} S(x)\sin(mx)\,dx = \sum_{n=1}^{\infty} \lambda_n \frac{2}{\pi}\int_0^{\pi} \sin(nx)\sin(mx)\,dx = \lambda_m,$$

which shows that (1.9.5) is the Fourier series of S.

The definite integral $F(x) = \int_0^x S(t)\,dt$ is continuous, and it is periodic because

$$F(x+2\pi) - F(x) = \int_x^{x+2\pi} S(t)\,dt = \int_{-\pi}^{\pi} S(t)\,dt = 0.$$

It is also easy to see that F is even, so its Fourier series is a cosine series

$$F(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx).$$

For $n > 0$ we find by Fubini's theorem

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi (\int_0^x S(t)\,dt)\cos(nx)\,dx = \frac{2}{\pi}\int_0^\pi (\int_t^\pi \cos(nx)\,dx)\,S(t)\,dt \\ &= -\frac{2}{\pi}\int_0^\pi S(t)\sin(nt)/n\,dt = -\frac{\lambda_n}{n}. \end{aligned}$$

According to Fejér's theorem 1.6.8, the Fourier series for F is summable for $t = 0$ with sum 0, but since $|na_n| = \lambda_n$ is bounded (it approaches 0 for $n \to \infty$), Hardy's theorem 1.6.11 shows that the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n$$

is convergent with sum 0. In particular,

$$0 \le \sum_{n=1}^{\infty} \frac{\lambda_n}{n} = \frac{a_0}{2} = \frac{1}{\pi}\int_0^\pi F(x)\,dx < \infty.$$

□

Example 1.9.4 $\lambda_n = \dfrac{1}{\log(n+1)}, \quad n \ge 1.$

The function

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{inx}}{\log(n+1)} \tag{1.9.7}$$

is continuous on $\mathbb{R} \setminus 2\pi\mathbb{Z}$; it is not integrable because $\mathrm{Im}(f) \notin \mathcal{L}^1(\mathbb{T})$ by Theorem 1.9.3, but $\mathrm{Re}(f)(x)$ is integrable, see (1.8.4).

Example 1.9.5 $\lambda_n = \dfrac{1}{n^\alpha}, \quad \alpha > 0.$

The function

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{inx}}{n^\alpha} \tag{1.9.8}$$

is continuous on $\mathbb{R} \setminus 2\pi\mathbb{Z}$ and belongs to $\mathcal{L}^1(\mathbb{T})$, since $\sum \frac{1}{n^{1+\alpha}} < \infty$. The series is <u>not</u> absolutely convergent for $0 < \alpha \le 1$.

Example 1.9.6

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{1}{2}(\pi - x), \qquad 0 < x < 2\pi \tag{1.9.9}$$

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n} = -\ln\left(2\sin\frac{x}{2}\right), \qquad 0 \le x \le 2\pi. \tag{1.9.10}$$

To see these formulas we use the principal logarithm Log which is holomorphic in the cut plane $\mathbb{C}\backslash]-\infty,0]$ and we know that

$$-\operatorname{Log}(1-z)=\sum_{n=1}^{\infty}\frac{z^n}{n}, \qquad |z|<1.$$

Setting $z=re^{\mathrm{i}\theta}$, $0<r<1$, we have

$$-\operatorname{Log}\left(1-re^{\mathrm{i}\theta}\right)=\sum_{n=1}^{\infty}\frac{r^n}{n}e^{\mathrm{i}\theta},$$

and this series converges uniformly in θ when $r<1$. In particular,

$$\begin{aligned}
-\frac{1}{2}\ln(1+r^2-2r\cos\theta) &= \sum_{n=1}^{\infty}\frac{r^n}{n}\cos(n\theta),\\
-\operatorname{Arg}\left(1-re^{\mathrm{i}\theta}\right) &= \sum_{n=1}^{\infty}\frac{r^n}{n}\sin(n\theta),
\end{aligned}$$

where Arg is the principle argument taking values in $]-\pi,\pi[$. For $r\to 1^-$ we find

$$-\frac{1}{2}\ln(1+r^2-2r\cos\theta)\to -\ln(2\sin\frac{\theta}{2}), \quad -\operatorname{Arg}\left(1-re^{\mathrm{i}\theta}\right)\to\frac{1}{2}(\pi-\theta),$$

both for $0<\theta<2\pi$. In addition to pointwise convergence, we have convergence in $\mathcal{L}^1(\mathbb{T})$. This follows from Lebesgue's theorem on dominated convergence, and thereby $\frac{1}{2}(\pi-x)$ and $-\ln(2\sin\frac{x}{2})$ have the given Fourier series.

Exercises

E 9.1 Show that the dominated convergence theorem can be used above.

1.10 Absolutely convergent Fourier series

If $f\in\mathcal{L}^1(\mathbb{T})$ has an absolutely convergent Fourier series, i.e., if $\sum|c_n|<\infty$, then the series $\sum c_n\, e^{inx}$ converges uniformly for $\theta\in\mathbb{R}$ to a continuous periodic function g by Weierstrass' M-test. Moreover, $c_n(g)=c_n(f)$ for all n, thus by the Uniqueness Theorem 1.6.5 we have $f=g$ a. e. We can therefore say that f is equal to a continuous function a. e., or, in other words, we can change f on a null set such that it becomes continuous.

We define now

$$A(\mathbb{T})=\left\{f\in C(\mathbb{T})\,\middle|\,\sum|c_n(f)|<\infty\right\}, \tag{1.10.1}$$

which clearly is a subspace of $C(\mathbb{T})$, stable under complex conjugation. Furthermore, $A(\mathbb{T})$ contains all trigonometric polynomials. We see below in Corollary 1.10.4 that $C^1(\mathbb{T}) \subset A(\mathbb{T})$.

We can also write

$$A(\mathbb{T}) = \left\{ f \in \mathcal{L}^1(\mathbb{T}) \,|\, C(f) \in \ell^1(\mathbb{Z}) \right\}.$$

We note now that $\ell^1(\mathbb{Z})$ has a convolution structure. If $a, b \in \ell^1(\mathbb{Z})$, i.e., if $\sum |a_n| < \infty$, $\sum |b_n| < \infty$, then

$$c_n = \sum_{k\in\mathbb{Z}} a_{n-k}\, b_k, \qquad n \in \mathbb{Z} \tag{1.10.2}$$

defines a new sequence in $\ell^1(\mathbb{Z})$. In fact, the series (1.10.2) is absolutely convergent for every n because

$$\sum_{k\in\mathbb{Z}} |a_{n-k}\, b_k| \le (\sup_{n\in\mathbb{Z}} |a_n|) \sum_{k\in\mathbb{Z}} |b_k| = \|a\|_\infty \|b\|_1 < \infty,$$

and

$$\sum_{n\in\mathbb{Z}} |c_n| \le \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |a_{n-k}\, b_k| \right) = \sum_{k\in\mathbb{Z}} |b_k| \sum_{n\in\mathbb{Z}} |a_{n-k}| = \sum_{k\in\mathbb{Z}} |b_k| \sum_{n\in\mathbb{Z}} |a_n| = \|b\|_1 \|a\|_1 < \infty.$$

In other words: (1.10.2) defines a composition rule $*$ which is called a convolution in $\ell^1(\mathbb{Z})$, and it holds that

$$(a * b)_n = \sum_{k\in\mathbb{Z}} a_{n-k}\, b_k. \tag{1.10.3}$$

We leave to the reader to verify that $\ell^1(\mathbb{Z})$ is a commutative Banach algebra.

Note the analogy to $L^1(\mathbb{T})$ with respect to convolution. The crucial aspect for the construction is the group structure on $\mathbb{Z}$ resp. $\mathbb{T}$ and the translation invariant measure on $\mathbb{Z}$ and $\mathbb{T}$, namely the counting measure and Lebesgue measure. The special sequence δ_0 given by $(\delta_0)_n = 1$ for $n = 0$ and $= 0$ for $n \neq 0$ is a unit element with respect to $*$.

Theorem 1.10.1 *For $f, g \in A(\mathbb{T})$, we have $fg \in A(\mathbb{T})$ and $C(fg) = C(f) * C(g)$.*

Proof. We calculate

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \left(C(f) * C(g)\right)_n e^{\mathrm{i}nx} &= \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} c_{n-k}(f)\, c_k(g)\, e^{\mathrm{i}nx} \right) \\
&= \sum_{k\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} c_{n-k}(f)\, e^{\mathrm{i}(n-k)x} \right) e^{\mathrm{i}kx} c_k(g) = \sum_{k\in\mathbb{Z}} f(x)\, c_k(g)\, e^{\mathrm{i}kx} = f(x)g(x).
\end{aligned}$$

$\square$

Remark 1.10.2 There exist functions $f \in C(\mathbb{T})$ such that $f \notin A(\mathbb{T})$, since there are $f \in C(\mathbb{T})$ for which the Fourier series is divergent in certain points, cf. Theorem 1.11.3.

Lemma 1.10.3 *If f is periodic and differentiable with $f' \in \mathcal{L}^1(\mathbb{T})$, then $c_n(f') = \mathrm{i}n\, c_n(f)$, i.e., the Fourier series for f' is found by termwise differentiation of the Fourier series for f.*

Proof. This is a simple consequence of partial integration because

$$\begin{aligned} c_n(f') &= \frac{1}{2\pi}\int_0^{2\pi} f'(t)\, e^{-\mathrm{i}nt}\, dt = \frac{1}{2\pi}\left[f(t)\, e^{-\mathrm{i}nt}\right]_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} f(t)\,(-\mathrm{i}n)\, e^{-\mathrm{i}nt}\, dt \\ &= \mathrm{i}n\, c_n(f). \end{aligned}$$

□

Corollary 1.10.4 *If f is periodic and differentiable with $f' \in \mathcal{L}^2(\mathbb{T})$, then $f \in A(\mathbb{T})$. In particular, $C^1(\mathbb{T}) \subset A(\mathbb{T})$.*

Proof. Since $\mathcal{L}^2(\mathbb{T}) \subset \mathcal{L}^1(\mathbb{T})$, we know by Lemma 1.10.3 that $c_n(f') = \mathrm{i}n\, c_n(f)$, so by Parseval's equation we find

$$\|f'\|_2^2 = \sum_{-\infty}^{\infty} |\mathrm{i}n\, c_n(f)|^2 < \infty.$$

An application of the Cauchy-Schwarz inequality yields

$$\sum_{n\neq 0} |c_n| = \sum_{n\neq 0} |nc_n| \frac{1}{|n|} \leq \left(\sum_{n\neq 0} n^2 |c_n|^2\right)^{1/2} \left(\sum_{n\neq 0} \frac{1}{n^2}\right)^{1/2} < \infty.$$

□

Another consequence of Lemma 1.10.3 is that if $f \in C^\infty(\mathbb{T})$, then $c_n(f^{(k)}) = (\mathrm{i}n)^k c_n(f)$ and hence by the Riemann–Lebesgue lemma

$$n^k\, c_n(f) \to 0 \qquad \text{for } |n| \to \infty \qquad \text{for all } k \in \mathbb{N}. \tag{1.10.4}$$

The Fourier coefficients for $f \in C^\infty(\mathbb{T})$ therefore approach 0 faster for $|n| \to \infty$ than n^{-k} for every k. Sequences with the property (1.10.4) are called rapidly decaying sequences. They are the discrete counterpart to the Schwartz functions, which will be discussed in Chapter 2. Conversely, we shall now see that these sequences are precisely the Fourier coefficients for functions in $C^\infty(\mathbb{T})$.

Theorem 1.10.5 *Every rapidly decaying sequence $(c_n)_{n\in\mathbb{Z}}$ is the Fourier cofficients of a function $f \in C^\infty(\mathbb{T})$.*

Proof. We note first that $(n^k c_n) \in \ell^1(\mathbb{Z})$ for every $k \in \mathbb{N}_0$. In fact, since $n^{k+2} c_n \to 0$ for $|n| \to \infty$ it is bounded and therefore $|n^k c_n| \le A/n^2$ for $n \neq 0$ and some $A > 0$. We can therefore find $f_k \in A(\mathbb{T})$ such that

$$f_k(x) = \sum_{n\in\mathbb{Z}} (\mathrm{i}n)^k c_n e^{\mathrm{i}nx}, \qquad x \in \mathbb{R},\ k \in \mathbb{N}_0,$$

and each of these series are uniformly convergent on $\mathbb{R}$. By a well-known result of analysis, we see that each f_k is a C^1-function with $f_k' = f_{k+1}$, but this shows that $f = f_0 \in C^\infty(\mathbb{T})$.

□

Let us now consider a function $f \in C^\infty(\mathbb{T})$ which can be extended to a holomorphic function in a band

$$S = S_R = \left\{ z = x + \mathrm{i}y \,\middle|\, |y| < R \right\}$$

around the real axis. The functions $e^{\mathrm{i}nx}$, $n \in \mathbb{Z}$ are examples of this. They can even be extended to the entire $\mathbb{C}$.

We note first that $f : S \to \mathbb{C}$ becomes periodic in S, i.e.,

$$f(z + 2\pi) = f(z) \qquad \text{for } z \in S, \tag{1.10.5}$$

since the function $f(z+2\pi) - f(z)$ is holomorphic in S and identically 0 on $\mathbb{R}$, hence identically 0 in S. For every $y_0 \in \mathbb{R}$ with $|y_0| < R$ the function $x \mapsto f(x + \mathrm{i}y_0)$ is therefore a periodic C^∞ function. We will find its Fourier series. We apply Cauchy's integral theorem to a rectangle $\mathcal{R}$ with sides $y = 0$, $y = y_0$, $x = 0$, $x = 2\pi$.

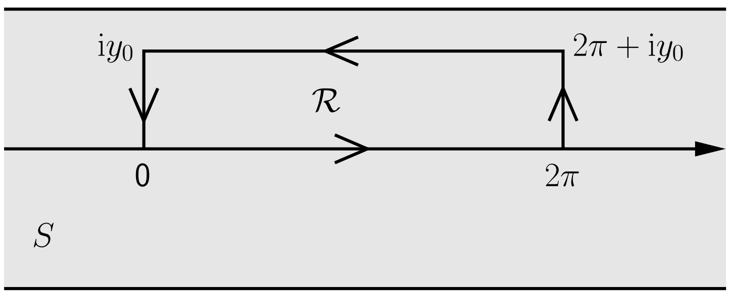

Figure 1.3: The figure shows $y_0 > 0$, but it is possible that $-R < y_0 < 0$
.

We obtain

$$\int_{\partial\mathcal{R}} f(z)\, e^{-\mathrm{i}nz}\, dz = 0,$$

or

$$\begin{aligned}
&\int_0^{2\pi} f(x)\, e^{-\mathrm{i}nx}\, dx + \mathrm{i}\int_0^{y_0} f(2\pi + \mathrm{i}t)\, e^{-\mathrm{i}n(2\pi+\mathrm{i}t)}\, dt \\
&= \int_0^{2\pi} f(x + \mathrm{i}y_0)\, e^{-\mathrm{i}n(x+\mathrm{i}y_0)}\, dx + \mathrm{i}\int_0^{y_0} f(\mathrm{i}t)\, e^{-\mathrm{i}n(\mathrm{i}t)}\, dt.
\end{aligned}$$

From (1.10.5) it follows that two of the terms are equal, and we find

$$c_n\left(f(x)\right) = e^{ny_0}\, c_n\left(f(x + \mathrm{i}y_0)\right). \tag{1.10.6}$$

This shows that

$$f(x + \mathrm{i}y_0) = \sum_{n=-\infty}^{\infty} c_n(f)\, e^{-ny_0}\, e^{\mathrm{i}nx} = \sum_{n=-\infty}^{\infty} c_n(f)\; e^{\mathrm{i}n(x+\mathrm{i}y_0)},$$

i.e., the Fourier series for $f(x + \mathrm{i}y_0)$ can be derived from the Fourier series of f by formally replacing x by $x + \mathrm{i}y_0$. We have now

Theorem 1.10.6 *Let $f : S \to \mathbb{C}$ be holomorphic in the band S and periodic with period 2π. Then the series*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n\, e^{\mathrm{i}nz}, \qquad c_n = c_n(f) \tag{1.10.7}$$

converges uniformly over compact subsets of S and for every r, $0 < r < R$, there exists a constant K_r such that

$$|c_n| \le K_r\, e^{-r|n|}, \qquad n \in \mathbb{Z}. \tag{1.10.8}$$

Proof. For an arbitrary compact set $L \subseteq S$, there exist $0 < r < R$ and $\ell > \pi$ such that $L \subseteq \tilde{L} := \left\{x + \mathrm{i}y \,\middle|\, |x| \le \ell,\, |y| \le r\right\}$. From (1.10.6) we obtain for $n > 0$

$$\begin{aligned}
|c_n| &= e^{-nr}\left|c_n\left(f(x - \mathrm{i}r)\right)\right| \le e^{-nr}\max_{z\in\tilde{L}} |f(z)|, \\
|c_{-n}| &= e^{-nr}\left|c_n\left(f(x + \mathrm{i}r)\right)\right| \le e^{-nr}\max_{z\in\tilde{L}} |f(z)|,
\end{aligned}$$

which shows (1.10.8). If we choose $\varepsilon > 0$ so small that $r + \varepsilon < R$, we have for $z \in L$

$$|c_n\, e^{\mathrm{i}nz}| \le |c_n|\, e^{|n|\,|y|} \le e^{-|n|\varepsilon}\, |c_n|\, e^{|n|(r+\varepsilon)} \le e^{-|n|\varepsilon}\, K_{r+\varepsilon},$$

and this shows that Weierstrass' M-test can be applied to (1.10.7). □

Exercises

E 10.1 For $c = (c_n) \in \ell^1(\mathbb{Z})$ define $\tilde{c}_n = \overline{c_{-n}}$ for $n \in \mathbb{Z}$. Show that $c \to \tilde{c}$ makes $\ell^1(\mathbb{Z})$ to a Banach algebra with involution.

E 10.2 Show that if $f, g \in \mathcal{L}^2(\mathbb{T})$, then $f * g \in A(\mathbb{T})$.

E 10.3 Let $(c_n)_{n\in\mathbb{Z}}$ be a sequence such that

$$|c_n| \leq K\, e^{-R|n|}, \qquad n \in \mathbb{Z},$$

for appropriate K, $R > 0$. Show that there exists a holomorphic periodic function f in the band $S_R = \left\{ x + iy \,\middle|\, |y| < R \right\}$ such that the restriction of f to $\mathbb{R}$ has Fourier coefficients $(c_n)_{n\in\mathbb{Z}}$.

E 10.4 Show that the convergence in (1.10.7) is uniform over all subbands S_r of $S = S_R$ when $0 < r < R$.

1.11 Divergence of Fourier series

We begin by considering the $\mathcal{L}^1$-norm of Dirichlet's kernel D_n.

Lemma 1.11.1 *The 1-norm of the Dirichlet kernel tends to infinity with n because of the estimate*

$$\|D_n\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\, dt \geq \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k}.$$

Proof. For $x > 0$ we have $\sin\frac{x}{2} < \frac{x}{2}$, hence

$$\|D_n\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\, dt = \frac{1}{\pi}\int_0^{\pi} \left|\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\right| dt > \frac{2}{\pi}\int_0^{\pi}\frac{|\sin(n+\frac{1}{2})t|}{t}\, dt$$
$$= \frac{2}{\pi}\int_0^{(n+\frac{1}{2})\pi}\frac{|\sin u|}{u}\, du > \frac{2}{\pi}\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi}\frac{|\sin u|}{k\pi}\, du = \frac{2}{\pi}\sum_{k=1}^{n}\frac{2}{k\pi}.$$

□

Lemma 1.11.2 *For every $g \in C(\mathbb{T})$ let $L_g : C(\mathbb{T}) \to \mathbb{C}$ be defined by*

$$L_g(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, g(t)\, dt, \qquad f \in C(\mathbb{T}). \tag{1.11.1}$$

Then L_g is a continuous linear functional with norm $\|L_g\| = \|g\|_1$.

Proof. It is clear that L_g is a linear functional and since

$$|L_g(f)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(t)||g(t)|\,dt \le \|f\|_\infty \|g\|_1,$$

we get

$$\|L_g\| := \sup\left\{|L_g(f)| \,\middle|\, \|f\|_\infty \le 1\right\} \le \|g\|_1.$$

That the inequality sign holds can be seen as follows: For $\varepsilon > 0$ we set $f_\varepsilon(t) = \overline{g(t)}/(|g(t)| + \varepsilon)$ which is continuous and $\|f_\varepsilon\|_\infty < 1$. Therefore

$$\|L_g\| \ge L_g(f_\varepsilon) = \frac{1}{2\pi}\int_0^{2\pi} \frac{|g(t)|^2}{|g(t)|+\varepsilon} dt > \frac{1}{2\pi}\int_0^{2\pi} \frac{|g(t)|^2 - \varepsilon^2}{|g(t)|+\varepsilon} dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi} (|g(t)| - \varepsilon)\; dt = \|g\|_1 - \varepsilon.$$

□

Theorem 1.11.3 *For every $x_0 \in \mathbb{R}$ there exists $f \in C(\mathbb{T})$ for which the sequence of partial sums $(s_n(f)(x_0))$ of the Fourier series at x_0 is unbounded.*

In particular the Fourier series of f is divergent in the point x_0.

Proof. The proof is given by contradiction. Supposing the assertion of the theorem to be false, there exists x_0 such that for all $f \in C(\mathbb{T})$ the sequence $(s_n(f)(x_0))$ is bounded. Remember that

$$s_n(f)(x_0) = \frac{1}{2\pi}\int_0^{2\pi} f(x_0 - t) D_n(t)\,dt.$$

Since we obviously have

$$\left\{t \to f(x_0 - t) \,\middle|\, f \in C(\mathbb{T})\right\} = C(\mathbb{T}),$$

it follows that the sequence

$$L_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t) D_n(t)\,dt$$

is bounded for all $f \in C(\mathbb{T})$. This means that we have a sequence $L_n = L_{D_n}$ of continuous linear functionals on $C(\mathbb{T})$ which is pointwise bounded, i.e., $(L_n(f))$ is bounded for each f, but $\|L_n\| = \|D_n\|_1$ is unbounded and this is in contradiction with a general result from functional analysis called the Banach–Steinhaus theorem or the uniform boundedness principle:

Theorem 1.11.4 (Banach-Steinhaus) *Let E be a Banach space and $L_n : E \to \mathbb{C}$ a sequence of continuous linear functionals which is assumed to be pointwise bounded, i.e., for every $x \in E$, $(L_n(x))$ is bounded. Then $\|L_n\|$ is bounded.*

Proof. For every $p \in \mathbb{N}$ we consider

$$F_p = \left\{ x \in E \,\middle|\, \sup_n |L_n(x)| \le p \right\}.$$

Since F_p is the intersection of the closed sets

$$\left\{ x \in E \,\middle|\, |L_n(x)| \le p \right\} = L_n^{-1}\left(\left\{ z \in \mathbb{C} \,\middle|\, |z| \le p \right\} \right), \qquad n \in \mathbb{N},$$

F_p is also closed. We have furthermore $F_1 \subseteq F_2 \subseteq \ldots$, and since $\sup_n |L_n(x)| < \infty$ for every $x \in E$, we necessarily have

$$F_1 \cup F_2 \cup \ldots = E. \tag{1.11.2}$$

We conclude now (according to an idea of R. Baire) that there can be found a p such that F_p has interior points. We prove this by contradiction. If none of the sets F_p have interior points, we come to a contradiction in the following way:

Since F_1 has no interior points, we have necessarily $F_1 \ne E$, so we can choose $x_1 \in E \setminus F_1$. Since F_1 is closed, there exists a closed ball

$$B(x_1, \rho_1) = \left\{ x \in E \,\middle|\, \|x - x_1\| \le \rho_1 \right\}$$

disjoint with F_1. Since F_2 cannot contain $B(x_1, \frac{1}{2}\rho_1)$, there is an $x_2 \in B(x_1, \frac{1}{2}\rho_1) \setminus F_2$, and thereby (since F_2 is closed) there exists a closed ball $B(x_2, \rho_2)$ disjoint with F_2. Here we can of course choose $\rho_2 \le \frac{1}{2}\rho_1$, whereby we obtain $B(x_2, \rho_2) \subseteq B(x_1, \rho_1)$. By continuing in this way, we find a sequence of closed balls $B(x_p, \rho_p)$ with $\rho_{p+1} \le \frac{1}{2}\rho_p$ and $B(x_p, \rho_p) \cap F_p = \emptyset$ together with $B(x_p, \rho_p) \supseteq B(x_{p+1}, \rho_{p+1})$. Clearly $\rho_p \to 0$, and since $x_{p+1}, x_{p+2}, \ldots \in B(x_p, \rho_p)$ we see that (x_p) is a Cauchy sequence. Since E is by assumption complete, the limit $\lim_{p\to\infty} x_p = x$ exists. We have $x \in \bigcap_{p=1}^{\infty} B(x_p, \rho_p)$ and therefore $x \notin \bigcup_{p=1}^{\infty} F_p$, in contradiction with (1.11.2).

We have now shown that there exists a p_0 and a ball $B(x_0, \rho_0) \subseteq F_{p_0}$. For all y in this ball we have $|L_n(y)| \le p_0$ for every n by definition of F_{p_0}. For every $x \in E$ with $\|x\| \le 1$ we have $x_0 \pm \rho_0 x \in B(x_0, \rho_0)$, i.e.,

$$|L_n(x_0 + \rho_0 x)| \le p_0, \qquad |L_n(x_0 - \rho_0 x)| \le p_0, \qquad n \in \mathbb{N}.$$

By subtraction we obtain from this for all $n \in \mathbb{N}$

$$|2\rho_0 L_n(x)| = |L_n(x_0 + \rho_0 x) - L_n(x_0 - \rho_0 x)| \le 2p_0$$

or $|L_n(x)| \le p_0/\rho_0$. This proves that $\|L_n\| \le p_0/\rho_0$, i.e., that $\|L_n\|$ is bounded. $\square$

Exercises

E 11.1 Let X be a topological space. Show that the following two conditions are equivalent

(i) For any sequence (G_n) of open dense sets, their intersection $\cap G_n$ is dense in X.

(ii) For any sequence (F_n) of closed sets with empty interior, their union $\cup F_n$ has empty interior.

A topological space X is called a Baire space if (i) and (ii) are satisfied.

E 11.2 1° Prove that a complete metric space is a Baire space.

2° Prove that a locally compact Hausdorff space is a Baire space.

René Baire (1874-1932) proved that $X = \mathbb{R}$ is a Baire space.

1.12 Fourier coefficients for measures on $\mathbb{T}$

Let $\mathbb{M}_+(\mathbb{T})$ denote the set of positive finite measures μ defined on the Borel σ-algebra $\mathbb{B}(\mathbb{T})$ for $\mathbb{T}$, i.e., μ is a countably additive function from $\mathbb{B}(\mathbb{T})$ to $[0, \infty[$. (Note that the condition $\mu(\emptyset) = 0$ is a consequence of $\mu(\emptyset) + \mu(\emptyset) = \mu(\emptyset)$ when $\mu(\emptyset) < \infty$).

For $\mu \in \mathbb{M}_+(\mathbb{T})$, we introduce the Fourier coefficients $C(\mu) : \mathbb{Z} \to \mathbb{C}$ as

$$C(\mu)(n) = \int_{\mathbb{T}} z^{-n}\, d\mu(z), \qquad n \in \mathbb{Z}. \tag{1.12.1}$$

(If $\mathbb{T}$ is parametrized by $z = e^{\mathrm{i}\theta}$, $\theta \in [0, 2\pi[$, then $z^{-n} = e^{-\mathrm{i}n\theta}$). The Fourier coefficients $C(\mu)$ form a bounded sequence with

$$\sup_{n \in \mathbb{Z}} |C(\mu)(n)| = \mu(\mathbb{T}) < \infty, \tag{1.12.2}$$

since

$$|C(\mu)(n)| \le \int_{\mathbb{T}} |z^{-n}|\, d\mu(z) = \int_{\mathbb{T}} 1\, d\mu(z) = \mu(\mathbb{T}) = C(\mu)(0).$$

Example 1.12.1 For $\mu = \varepsilon_{e^{\mathrm{i}\theta}}$ (the point measure in $e^{\mathrm{i}\theta}$), we have $C(\mu)(n) = e^{-\mathrm{i}n\theta}$ for all n. In particular, $C(\varepsilon_1)(n) = 1$, $C(\varepsilon_{-1})(n) = (-1)^n$.

Example 1.12.2 For $\mu = f(z)\,dm(z)$, with $f \in \mathcal{L}^1_+(\mathbb{T})$, and m being the normalized Lebesgue measure on $\mathbb{T}$, we have

$$C(\mu)(n) = C(f)(n) = \int_{\mathbb{T}} z^{-n}\, f(z)\, dm(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{\mathrm{i}t})\, e^{-\mathrm{i}nt}\, dt.$$

Lemma 1.12.3 (The uniqueness theorem) *Suppose that $\mu, \nu \in \mathbb{M}_+(\mathbb{T})$ satisfy $C(\mu) = C(\nu)$, then $\mu = \nu$.*

Proof. If $C(\mu) = C(\nu)$, we obtain directly $\int p(t)\, d\mu(t) = \int p(t)\, d\nu(t)$ for all trigonometric polynomials. Since those are uniformly dense in $C(\mathbb{T})$, we have $\int f(t)\, d\mu(t) = \int f(t)\, d\nu(t)$ for all $f \in C(\mathbb{T})$. From this follows easily that $\mu = \nu$ by the usual technique: For a closed arc $B \subseteq \mathbb{T}$, there exists a decreasing sequence $f_n \in C(\mathbb{T})$ such that $f_n \to 1_B$ pointwise. Therefore $\mu(B) = \nu(B)$, and since the set of closed arcs generates $\mathbb{B}(\mathbb{T})$ and is stable under intersection, we can conclude $\mu = \nu$. (The uniqueness is also part of the Riesz representation theorem for measures.) $\square$

For $\mu, \nu \in \mathbb{M}_+(\mathbb{T})$, we introduce a convolution $\mu * \nu \in \mathbb{M}_+(\mathbb{T})$ as the image measure $p(\mu \otimes \nu)$ of $\mu \otimes \nu$ under the mapping $p : \mathbb{T}^2 \to \mathbb{T}$ given by $p(z, w) = zw$ (the multiplication in the group $\mathbb{T}$). Thus, we have for $E \in \mathbb{B}(\mathbb{T})$

$$\mu * \nu(E) = \mu \otimes \nu\left(p^{-1}(E)\right) = \mu \otimes \nu\left(\left\{(z,w) \in \mathbb{T}^2 \,\middle|\, zw \in E\right\}\right), \tag{1.12.3}$$

and we see immediately that $\mu * \nu(\mathbb{T}) = \mu \otimes \nu(\mathbb{T}^2) = \mu(\mathbb{T})\nu(\mathbb{T})$ and $\mu * \nu = \nu * \mu$. Moreover, $\mu * \varepsilon_1 = \mu$ for all $\mu \in \mathbb{M}_+(\mathbb{T})$.

Lemma 1.12.4 *Assume $\mu, \nu \in \mathbb{M}_+(\mathbb{T})$ and $f \in C(\mathbb{T})$. Then*

$$\int_{\mathbb{T}} f\, d\mu * \nu = \int_{\mathbb{T}} (\int_{\mathbb{T}} f(zw)\, d\mu(z))\, d\nu(w). \tag{1.12.4}$$

The equation holds also for positive Borel functions $f : \mathbb{T} \to [0, \infty]$ and for $f \in \mathcal{L}^1(\mu * \nu)$.

Proof. If $f = 1_E$ for $E \in \mathbb{B}(\mathbb{T})$, (1.12.4) follows directly from (1.12.3). Thereby, (1.12.4) holds also for positive simple functions. By the monotone convergence theorem we see that (1.12.4) holds for an arbitrary positive Borel function f, since there exists an increasing sequence (s_n) of simple positive Borel functions $s_n \uparrow f$. A real-valued function $f \in \mathcal{L}^1(\mu * \nu)$ can be written as a difference of two non-negative Borel functions. Therefore, the result follows for integrable f, in particular for arbitrary $f \in C(\mathbb{T})$. $\square$

Theorem 1.12.5 *$C(\mu * \nu) = C(\mu)\, C(\nu)$.*

Proof.

$$C(\mu * \nu)(n) = \int z^{-n}\, d\mu * \nu(z) = \int (\int (zw)^{-n}\, d\mu(z))\, d\nu(w)$$
$$= \int (w^{-n} \int z^{-n}\, d\mu(z))\, d\nu(w) = C(\mu)(n)\, C(\nu)(n), \qquad n \in \mathbb{Z}.$$

□

If $\mu = f\, dm$, $\nu = g\, dm$, then $\mu * \nu = (f * g)\, dm$, since for $\varphi \in C(\mathbb{T})$

$$\int \varphi\, d\mu * \nu = \int_{\mathbb{T}} (\int_{\mathbb{T}} \varphi(zw)\, f(z)\, dm(z))\, g(w)\, dm(w)$$
$$= \int_{\mathbb{T}} (\int_{\mathbb{T}} \varphi(z)\, f(zw^{-1})\, dm(z))\, g(w)\, dm(w)$$
$$= \int_{\mathbb{T}} (\int_{\mathbb{T}} f(zw^{-1})\, g(w)\, dm(w))\, \varphi(z)\, dm(z)$$
$$= \int_{\mathbb{T}} f * g(z)\, \varphi(z)\, dm(z).$$

Note that

$$f * g(z) = \int_{\mathbb{T}} f(zw^{-1})\, g(w)\, dm(w) \tag{1.12.5}$$

is the same as

$$f * g(e^{\mathrm{i}\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{\mathrm{i}(\theta - t)})\, g(e^{\mathrm{i}t})\, dt. \tag{1.12.6}$$

In the first equation, we "convolute" on the group $\mathbb{T}$, in the second, we consider functions on $\mathbb{T}$ as periodic functions on $\mathbb{R}$ by replacing $f : \mathbb{T} \to \mathbb{C}$ by $f(e^{\mathrm{i}t})$.

Thus, the convolution in $\mathbb{M}_+(\mathbb{T})$ extends the convolution of functions, when we consider a function f as the measure $f\, dm$. Thereby, we can also say that Theorem 1.12.5 extends Theorem 1.8.1(iii).

We shall give a characterization of the set of sequences $C : \mathbb{Z} \to \mathbb{C}$ which are Fourier coefficients for measures $\mu \in \mathbb{M}_+(\mathbb{T})$. By substituting the measure μ by the reflected measure $\check{\mu}$ given by

$$\check{\mu}(E) = \mu(E^{-1}) = \mu\left(\left\{\frac{1}{z} \,\middle|\, z \in E\right\}\right), \qquad E \in \mathbb{B}(\mathbb{T}),$$

we see that

$$C(\check{\mu})(n) = \int_{\mathbb{T}} z^n\, d\mu(z), \qquad n \in \mathbb{Z},$$

so the problem is the same as to characterize <u>moment sequences of measures on $\mathbb{T}$</u>.

We need an integral representation of the holomorphic functions f in the unit disc $\mathbb{D} = \left\{z \in \mathbb{C} \,\middle|\, |z| < 1\right\}$ satisfying $\mathrm{Re} f \geq 0$.

Theorem 1.12.6 (G. Herglotz, F. Riesz 1911) *The formula*

$$f(z) = \mathrm{i}\beta + \int_{\mathbb{T}} \frac{s+z}{s-z}\, d\mu(s), \qquad z \in \mathbb{D}, \tag{1.12.7}$$

gives a bijective correspondence between the set $\mathcal{H}$ of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ with $\mathrm{Re} f \geq 0$ and the set of pairs $(\beta, \mu) \in \mathbb{R} \times \mathbb{M}_+(\mathbb{T})$. For $f \in \mathcal{H}$

$$\beta = \mathrm{Im} f(0), \qquad \mu = \lim_{r \to 1} \mathrm{Re} f(rs)\, dm(s) \qquad \textit{weakly.} \tag{1.12.8}$$

Insertion on the weak topology on $\mathbb{M}_+(\mathbb{T})$.

Given a sequence (μ_n) from $\mathbb{M}_+(\mathbb{T})$, we say that $\mu_n \to \mu \in \mathbb{M}_+(\mathbb{T})$ weakly if

$$\lim_{n\to\infty} \int f\, d\mu_n = \int f\, d\mu \qquad \text{for all } \; f \in C(\mathbb{T}).$$

This corresponds to convergence in the coarsest topology on $\mathbb{M}_+(\mathbb{T})$ for which the mappings $\mu \mapsto \int f\, d\mu$ are continuous, when f is an arbitrary element in $C(\mathbb{T})$.

For $\mu \in \mathbb{M}_+(\mathbb{T})$, we define a linear functional $L_\mu : C(\mathbb{T}) \to \mathbb{C}$ by $L_\mu(f) = \int f\, d\mu$. It is positive in the sense that $f \geq 0 \Rightarrow L_\mu(f) \geq 0$. The content of Riesz' respresentation theorem (for $\mathbb{T}$) is that every positive linear functional $L : C(\mathbb{T}) \to \mathbb{C}$ has the form $L = L_\mu$ for precisely one $\mu \in \mathbb{M}_+(\mathbb{T})$ (F. Riesz 1909). Since L_μ is continuous with $\|L_\mu\| = \mu(\mathbb{T})$, we can therefore consider $\mathbb{M}_+(\mathbb{T})$ as a subset of the dual space $C(\mathbb{T})^*$, and the weak topology on $\mathbb{M}_+(\mathbb{T})$ is the restriction of the topology $\sigma(C(\mathbb{T})^*, C(\mathbb{T}))$ to $\mathbb{M}_+(\mathbb{T})$.

According to Alaoglu–Bourbaki's theorem, the unit sphere in $C(\mathbb{T})^*$ is weakly compact. This gives the following key result on compactness of measures, frequently called Helly's theorem:

Theorem 1.12.7 *For every $\alpha > 0$, the set $\left\{\mu \in \mathbb{M}_+(\mathbb{T}) \,\middle|\, \mu(\mathbb{T}) \leq \alpha\right\}$ is weakly compact, i.e., for every sequence $\mu_n \in M_+(\mathbb{T})$ with $\mu_n(\mathbb{T}) \leq \alpha$, there exists a $\mu \in \mathbb{M}_+(\mathbb{T})$ and a subsequence (μ_{n_p}) such that $\lim_{p\to\infty} \mu_{n_p} = \mu$ weakly.*

Note that since $1 \in C(\mathbb{T})$, it follows that if $\mu_n(\mathbb{T}) = \alpha$ for all n, then the accumulation point μ also has mass $\mu(\mathbb{T}) = \alpha$.

We shall now give the proof of Herglotz-Riesz' theorem.

Proof. As our starting point, we take the power series for $f \in \mathcal{H}$

$$f(z) = \sum_{n=0}^{\infty} a_n\, z^n, \qquad z \in \mathbb{D},$$

which converges uniformly over compact subsets of $\mathbb{D}$. Let $g = \mathrm{Re} f$ and $a_0 = \alpha + \mathrm{i}\beta$. For $z = re^{\mathrm{i}\theta} \in \mathbb{D}$, we have

$$g(re^{\mathrm{i}\theta}) = \alpha + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n\, r^n e^{\mathrm{i}n\theta} + \overline{a_n}\, r^n e^{-\mathrm{i}n\theta}\right),$$

which is the Fourier series for the periodic C^∞ function $g(re^{\mathrm{i}\theta})$, i.e.,

$$\alpha = \frac{1}{2\pi}\int_0^{2\pi} g(re^{\mathrm{i}\theta})\, d\theta, \qquad \frac{1}{2}a_n\, r^n = \frac{1}{2\pi}\int_0^{2\pi} g(re^{\mathrm{i}\theta})\, e^{-\mathrm{i}n\theta}\, d\theta, \qquad n \geq 1.$$

Using these formulas, we find for $z \in \mathbb{D}$, $0 < r < 1$

$$f(rz) = \sum_{n=0}^{\infty} a_n\, r^n z^n = \frac{1}{2\pi}\int_0^{2\pi} g(re^{\mathrm{i}\theta})\, d\theta + \mathrm{i}\beta + 2\sum_{n=1}^{\infty}\frac{z^n}{2\pi}\int_0^{2\pi} g(re^{\mathrm{i}\theta})e^{-\mathrm{i}n\theta}\, d\theta,$$

and interchanging $\sum$ and $\int$, which is allowed since the series

$$1 + 2\sum_{n=1}^{\infty} e^{-\mathrm{i}n\theta}\, z^n = \frac{e^{\mathrm{i}\theta} + z}{e^{\mathrm{i}\theta} - z} \tag{1.12.9}$$

converges uniformly in θ, we obtain

$$f(rz) = \mathrm{i}\beta + \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{\mathrm{i}\theta} + z}{e^{\mathrm{i}\theta} - z}\, g(re^{\mathrm{i}\theta})\, d\theta = \mathrm{i}\beta + \int_{\mathbb{T}}\frac{s+z}{s-z}\, g(rs)\, dm(s). \tag{1.12.10}$$

By assumption, we have $g \geq 0$, so $\sigma_r = g(rs)\, dm(s)$, $0 < r < 1$ is a family of measures in $\mathbb{M}_+(\mathbb{T})$ with density and they all have the same total mass (put $z = 0$ in (1.12.10))

$$\alpha = \mathrm{Re}\, f(0) = \int_{\mathbb{T}} g(rs)\, dm(s) = \sigma_r(\mathbb{T}).$$

From Helly's theorem 1.12.7 there exists $\mu \in \mathbb{M}_+(\mathbb{T})$ with $\mu(\mathbb{T}) = \alpha$ and a sequence $r_n \to 1$ such that $\sigma_{r_n} \to \mu$ weakly, and thereby (1.12.10) gives

$$f(z) = \mathrm{i}\beta + \int\frac{s+z}{s-z}\, d\mu(s), \qquad z \in \mathbb{D}.$$

We shall finally see that every function of the form (1.12.7) belongs to $\mathcal{H}$. It is holomorphic according to Morera's theorem from complex analysis, and taking the real part, we find

$$\mathrm{Re}\, f(z) = \int P(s,z)\, d\mu(s) \geq 0,$$

where

$$P(s,z) = \operatorname{Re} \frac{s+z}{s-z} = \frac{1-|z|^2}{|s-z|^2} > 0 \qquad \text{for } s \in \mathbb{T},\ z \in \mathbb{D}. \tag{1.12.11}$$

From (1.12.7) we also obtain $\beta = \operatorname{Im} f(0)$, so to finish the proof, we shall prove that

$$\lim_{r\to 1} \operatorname{Re} f(rs)\, dm(s) = \mu \qquad \text{weakly.}$$

We study in this context the function (1.12.11) a bit closer. It is called Poisson's kernel. We are therefore dealing with the function $P : \mathbb{T} \times \mathbb{D} \to \mathbb{R}$ given by (cf. (1.12.9))

$$P(s,z) = \frac{1-|z|^2}{|s-z|^2}, \qquad P(e^{\mathrm{i}\theta}, re^{\mathrm{i}\varphi}) = \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)} = 1 + 2\sum_{n=1}^{\infty} r^n \cos n(\theta-\varphi),$$

which arises from the Fourier series for the periodic function

$$P_r(\theta) = \frac{1-r^2}{1+r^2-2r\cos\theta} = 1 + 2\sum_{n=1}^{\infty} r^n \cos(n\theta) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta}, \quad 0 \le r < 1 \tag{1.12.12}$$

by substituting θ by $\theta - \varphi$.

Poisson's kernel has the following properties:

(i) $P(s,z) > 0 \qquad \text{for } s \in \mathbb{T},\ z \in \mathbb{D}$

(ii) $\displaystyle\int_{\mathbb{T}} P(s,z)\, dm(s) = 1 \qquad \text{for } z \in \mathbb{D}$

(iii) For $\delta > 0$, $s_0 \in \mathbb{T}$: $\displaystyle\lim_{z\to s_0} \int_{|s-s_0|\ge\delta} P(s,z)\, dm(s) = 0$

or equivalently

(i') $P_r(\theta) > 0$

(ii') $\displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1$

(iii') For $\delta > 0$: $\displaystyle\lim_{r\to 1} \frac{1}{2\pi}\int_{\delta\le|\theta|\le\pi} P_r(\theta)\, d\theta = 0.$

We can summarize these conditions by saying that (P_{r_n}) is a Dirac sequence for any sequence (r_n) from $]0,1[$ with $r_n \to 1$, cf. Definition 1.6.2.

The properties (i') and (ii') are straightforward from (1.12.12) and (iii') follows from the inequality

$$P_r(\theta) \le P_r(\delta) \qquad \text{when } \ \delta \le |\theta| \le \pi,$$

thus

$$\frac{1}{2\pi}\int_{\delta\le|\theta|\le\pi} P_r(\theta)\,d\theta \le \frac{\pi-\delta}{\pi}\,\frac{1-r^2}{1+r^2-2r\cos\delta} \to 0 \qquad \text{for } \ r \to 1.$$

For $z \in \mathbb{D}$, we consider the harmonic measure $\mu_z \in \mathbb{M}_+(\mathbb{T})$ defined by

$$\mu_z = P(s,z)\,dm(s).$$

This is a probability measure on $\mathbb{T}$ due to (i) and (ii). Condition (iii) implies

(iii") $\lim\limits_{z\to s_0} \mu_z = \varepsilon_{s_0}$ weakly.

If in fact $h \in C(\mathbb{T})$ and $\varepsilon > 0$ are given, then there exists $\delta > 0$ such that $|h(s) - h(s_0)| \le \varepsilon$ for $|s - s_0| \le \delta$. From this we obtain

$$\left| h(s_0) - \int h\,d\mu_z \right| \le \int_{|s-s_0|<\delta} |h(s_0) - h(s)|\ d\mu_z(s) + \int_{|s-s_0|\ge\delta} |h(s_0) - h(s)|\ d\mu_z(s)$$
$$\le \varepsilon + 2\,\|h\|_\infty \int_{|s-s_0|\ge\delta} P(s,z)\,dm(s),$$

which is $< 2\varepsilon$ according to (iii) for $|z - s_0|$ sufficiently small.

If f is holomorphic in $\mathbb{D}$, then $g = \mathrm{Re} f$ is harmonic in $\mathbb{D}$, i.e., $\Delta g = 0$ in $\mathbb{D}$. Conversely, if g is harmonic in $\mathbb{D}$, then g is the real part of a holomorphic function f and $f + \mathrm{i}\beta$, $\beta \in \mathbb{R}$ describes all holomorphic functions with real part equal to g.

We have thereby solved Dirichlet's problem for $\mathbb{D}$:

For $h \in C(\mathbb{T})$, the expression

$$H(z) = \begin{cases} h(s) & \text{for } \ z = s \in \mathbb{T} \\ \displaystyle\int h\,d\mu_z = \int h(s)\,P(s,z)\,dm(s) & \text{for } \ z \in \mathbb{D} \end{cases} \tag{1.12.13}$$

defines a continuous extension of h to $\mathbb{D}$ which is harmonic in $\mathbb{D}$. Equation (1.12.13) can for $z = re^{\mathrm{i}\varphi}$ be written as

$$H(re^{\mathrm{i}\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} P(e^{\mathrm{i}\theta}, re^{\mathrm{i}\varphi})\,h(e^{\mathrm{i}\theta})\,d\theta = P_r * h(e^{\mathrm{i}\varphi}),$$

i.e., we find the solution to the Dirichlet problem on the circle $re^{i\varphi}$ by convoluting the periodic functions P_r and h.

The following theorem is completely equivalent to the Riesz-Herglotz Theorem 1.12.6

Theorem 1.12.8 *The formula*

$$g(z) = \int_{\mathbb{T}} P(s,z)\, d\mu(s), \qquad z \in \mathbb{D} \tag{1.12.14}$$

gives a bijective correspondence between the set of positive harmonic functions g in $\mathbb{D}$ and the set of $\mu \in \mathbb{M}_+(\mathbb{T})$. For g in (1.12.14)*, we have*

$$\mu = \lim_{r\to 1} g(rs)\, dm(s) \qquad \textit{weakly.} \tag{1.12.15}$$

For the completion of the proof for Theorem 1.12.6 and Theorem 1.12.8, we shall show (1.12.15). For this we note that

$$P(s,rt) = P(t,rs) \qquad \text{for } \; s,t \in \mathbb{T}, 0 \le r \le 1, \tag{1.12.16}$$

which follows from

$$P(s,z) = \frac{1-|z|^2}{|s-z|^2}, \qquad s \in \mathbb{T}, \;\; z \in \mathbb{D},$$

since $|s-rt| = |t-rs|$, see Figure 1.4.

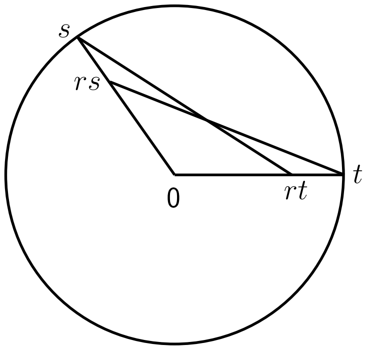

Figure 1.4: The unit disc

For $h \in C(\mathbb{T})$, we obtain from (1.12.14) and (1.12.16)

$$\begin{aligned}\int_{\mathbb{T}} h(s)\, g(rs)\, dm(s) &= \int_{\mathbb{T}} h(s) \int_{\mathbb{T}} P(t,rs)\, d\mu(t)\, dm(s) \\ &= \int_{\mathbb{T}} \Big(\int P(s,rt)\, h(s)\, dm(s)\Big)\, d\mu(t),\end{aligned}$$

and the inner integral is the solution $H(rt)$ to the Dirichlet problem with boundary values h. For $r \to 1$, this converges to $h(t)$ uniformly, and therefore the integral converges to $\int_{\mathbb{T}} h(t)\,d\mu(t)$, which proves (1.12.15).

Theorem 1.12.9 (Herglotz 1911) *For a sequence $(c_n)_{n\in\mathbb{Z}}$ of complex numbers the following two conditions are equivalent:*

(i) There exists $\mu \in \mathbb{M}_+(\mathbb{T})$ such that

$$c_n = \int_{\mathbb{T}} z^{-n}\,d\mu(z), \qquad n \in \mathbb{Z}.$$

(ii) For every $n \geq 0$, the matrices

$$T_n = (c_{j-k})_{0\leq j,k\leq n}$$

are positive semidefinite, i.e.,

$$\sum_{j,k=0}^{n} c_{j-k}\,\alpha_j\,\overline{\alpha_k} \geq 0 \qquad \forall\,(\alpha_0,\alpha_1,\ldots,\alpha_n) \in \mathbb{C}^{n+1}.$$

Proof. (i) $\Rightarrow$ (ii) is simple because

$$\sum_{j,k=0}^{n} c_{j-k}\,\alpha_j\,\overline{\alpha_k} = \int_{\mathbb{T}} \sum_{j,k=0}^{n} z^{k-j}\,\alpha_j\,\overline{\alpha_k}\,d\mu(z) = \int_{\mathbb{T}} \left|\sum_{j=0}^{n} \alpha_j\,z^{-j}\right|^2 d\mu(z) \geq 0,$$

since $\overline{z} = z^{-1}$ when $z \in \mathbb{T}$.

(ii) $\Rightarrow$ (i). In general, a positive semidefinite matrix (a_{jk}) satisfies $a_{jj} \geq 0$, $a_{jk} = \overline{a_{kj}}$ and $|a_{jk}|^2 \leq a_{jj}\,a_{kk}$. In fact, from

$$\sum_{j,k=1}^{n} a_{jk}\,\alpha_j\,\overline{\alpha_k} \geq 0 \quad \text{for all} \quad \alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{C}^n,$$

we get $a_{jj} \geq 0$ for $\alpha = e_j \in \mathbb{C}^n$. With $\alpha = e_j + t\,e_k$, $j < k$, we obtain

$$a_{jj} + |t|^2\,a_{kk} + \overline{t}\,a_{jk} + t\,a_{kj} \geq 0 \quad \text{for all} \quad t \in \mathbb{C}. \tag{1.12.17}$$

For $t = 1$ and $t = \mathrm{i}$ we get in particular

$$a_{jj} + a_{kk} + a_{jk} + a_{kj} \geq 0, \quad a_{jj} + a_{kk} + \mathrm{i}(-a_{jk} + a_{kj}) \geq 0,$$

and using $a_{jj}, a_{kk} \geq 0$, we find $a_{jk} + a_{kj},\ \mathrm{i}(-a_{jk} + a_{kj}) \in \mathbb{R}$, hence $a_{jk} = \overline{a_{kj}}$. Writing $a_{jk} = e^{\mathrm{i}\theta}|a_{jk}|$ and specializing (1.12.17) to $t = xe^{\mathrm{i}\theta}$ with $x \in \mathbb{R}$ we find

$$x^2 a_{kk} + 2x|a_{jk}| + a_{jj} \geq 0 \quad \text{for all} \quad x \in \mathbb{R},$$

so the discriminant of this second degree polynomial is ≤ 0, hence $|a_{jk}|^2 \leq a_{jj}a_{kk}$.

In our case, where $a_{jk} = c_{j-k}$, we find:

1) $c_0 \geq 0$

2) $c_{-n} = \overline{c_n}$

3) $|c_n| \leq c_0$.

We now define

$$F(z) = c_0 + 2\sum_{n=1}^{\infty} c_n\, z^n, \qquad z \in \mathbb{D},$$

and note that the series converges for $z \in \mathbb{D}$ since $|c_n| \leq c_0$. Thereby, F is holomorphic in $\mathbb{D}$. From $c_{-n} = \overline{c_n}$, $n \geq 1$ and $c_0 \geq 0$ we obtain

$$\operatorname{Re} F(z) = \sum_{n=0}^{\infty} c_n\, z^n + \sum_{n=1}^{\infty} c_{-n}\, (\overline{z})^n, \qquad z \in \mathbb{D}.$$

Multiplying this by the simple identity

$$(1-|z|^2)^{-1} = \sum_{n=0}^{\infty} z^n (\overline{z})^n, \quad z \in \mathbb{D},$$

we obtain

$$\frac{\operatorname{Re} F(z)}{1-|z|^2} = \sum_{n=0}^{\infty} c_n z^n \sum_{k=0}^{\infty} z^k \overline{z}^k + \sum_{n=1}^{\infty} c_{-n} \overline{z}^n \sum_{j=0}^{\infty} z^j \overline{z}^j.$$

In the first sum let $j = n+k$ so that $j \geq k$ and in the second sum let $k = n+j$ so that $k \geq j+1$. We then get for $z \in \mathbb{D}$

$$\begin{aligned}
\frac{\operatorname{Re} F(z)}{1-|z|^2} &= \sum_{j \geq k \geq 0} c_{j-k}\, z^j (\overline{z})^k + \sum_{j \geq 0, j+1 \leq k} c_{j-k}\, z^j (\overline{z})^k \\
&= \sum_{j,k \geq 0} c_{j-k}\, z^j (\overline{z})^k,
\end{aligned}$$

and the right-hand side is ≥ 0 since the partial sums $\sum_{j,k=0}^{n} c_{j-k}\, z^j (\overline{z})^k$ are ≥ 0 by (ii).

From the Herglotz-Riesz Theorem 1.12.6 we know that

$$F(z) = \mathrm{i}\beta + \int_{\mathbb{T}} \frac{s+z}{s-z}\, d\mu(s)$$

and

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} g(re^{\mathrm{i}\theta})\, d\theta, \qquad c_n\, r^n = \frac{1}{2\pi} \int_0^{2\pi} g(re^{\mathrm{i}\theta})\, e^{-\mathrm{i}n\theta}\, d\theta,$$

where $g = \operatorname{Re} F$. As $\sigma_r = g(rs)\,dm(s) \to \mu$ weakly for $r \to 1$, we obtain

$$c_n = \int_{\mathbb{T}} z^{-n}\,d\mu(z), \qquad n \geq 0,$$

and since $c_{-n} = \overline{c_n}$ we get that $c_n = \int_{\mathbb{T}} z^{-n}\,d\mu(z)$, $n \in \mathbb{Z}$. $\square$

Remark 1.12.10 Matrices $T_n = (c_{i-j})_{0\leq i,j\leq n}$, where the ij'th element depends only on the difference $i-j$, are called Toeplitz matrices after Otto Toeplitz (1881–1940).

Since every complex measure μ on $\mathbb{T}$ can be written as

$$\mu = \mu_1 - \mu_2 + \mathrm{i}(\mu_3 - \mu_4),$$

with $\mu_j \in \mathbb{M}_+(\mathbb{T})$, we can define the Fourier coefficients for a complex measure μ as

$$C(\mu) = C(\mu_1) - C(\mu_2) + \mathrm{i}(C(\mu_3) - C(\mu_4)).$$

Thus, it is clear that $C(\mu)$ is a bounded sequence of complex numbers. We mention without proof that there exists bounded sequences of numbers $c : \mathbb{Z} \to \mathbb{C}$ which cannot be written $C(\mu)$ for a complex measure μ.

A sequence $c : \mathbb{Z} \to \mathbb{C}$ is called positive definite if the equivalent conditions from Theorem 1.12.9 are fulfilled. Such a sequence is automatically bounded, and it fulfills $c_{-n} = \overline{c_n}$, $|c_n| \leq c_0$ for $n \in \mathbb{Z}$.

There exist bounded sequences $c : \mathbb{Z} \to \mathbb{C}$ which cannot be written as linear combinations of positive definite sequences.

Concluding remarks about Abel summability.

We have previously mentioned summability of an infinite series. This is the simplest way to ascribe a "sum" to particular divergent series. The idea is to consider the arithmetic means σ_n of the partial sum (s_n) for the infinite series $\sum_0^\infty a_n$.

This notion is called Cesàro summability of 1. order, also denoted $(C, 1)$-summability.

One can define Cesàro summability of higher order $k \in \mathbb{N}$ by taking the arithmetic means of σ_n; if those converge to s, the series is said to be summable $(C, 2)$, etc. The higher k, the more divergent series can be given a sum.

There exist other summability theories, e.g., Nørlund summability, named after the Danish mathematician Niels Erik Nørlund (1885 – 1981), and Abel summability, which we shall briefly consider. The English mathematician Hardy has written a book: *Divergent series*, which discusses all those theories.

Definition 1.12.11 An infinite series $\displaystyle\sum_0^\infty a_n$ is called *summable* (A) or *Abel summable* with sum s if

1) The power series $\sum_0^\infty a_n\, x^n$ converges for $-1 < x < 1$, with a sum $f(x)$.

2) $\lim_{x \to 1^-} f(x) = s$.

It follows by 1) that the series $\sum_0^\infty a_n\, x^n$ has a radius of convergence $\rho \geq 1$, i.e.,

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq 1.$$

Example 1.12.12 From the power series

$$\frac{1}{1+x} = \sum_0^\infty (-1)^n\, x^n, \qquad \frac{1}{(1+x)^2} = \sum_0^\infty (-1)^n\,(n+1)\, x^n,$$

we see that the divergent series $\sum (-1)^n$, $\sum (-1)^n (n+1)$ are summable (A) with sum $\frac{1}{2}$ respectively $\frac{1}{4}$. The first is summable $(C,1)$ with sum $\frac{1}{2}$, the second is <u>not</u> summable $(C,1)$, but summable $(C,2)$ with sum $\frac{1}{4}$.

The name Abel summability is motivated by the following theorem:

Theorem 1.12.13 (Abel) *If $\sum a_n$ is convergent with sum s, then it is summable (A) with sum s.*

Abel's summability method is stronger than Cesàro summability of any order in the sense that if a series is summable (C,k) with sum s for some k, then it is also summable (A) with sum s.

We shall only prove the result for $k = 1$. At the same time we also obtain Abel's theorem because of Lemma 1.6.1 of Cauchy.

Theorem 1.12.14 *Assume that the series $\sum_0^\infty a_n$ is summable $(C,1)$ with sum s. Then the series is summable (A) with sum s.*

<u>Proof.</u> It is clearly enough to prove the result for real series. Since $\lim_{n\to\infty} \sigma_n = s$, there exists K such that $|\sigma_n| \leq K$, thus

$$|s_n| = |(n+1)\,\sigma_n - n\,\sigma_{n-1}| \leq (2n+1)\,K,$$

and ultimately $|a_n| = |s_n - s_{n-1}| \leq 4n\,K$. This shows that the power series $f(x) = \sum_{n=0}^{\infty} a_n\,x^n$ converges for $|x| < 1$. We claim that

$$f(x) = (1-x)\sum_{n=0}^{\infty} s_n\,x^n = (1-x)^2\sum_{n=0}^{\infty}(n+1)\,\sigma_n\,x^n, \qquad -1 < x < 1.$$

We have

$$\begin{aligned}\sum_{n=0}^{N} a_n\,x^n &= s_0 + (s_1 - s_0)x + (s_2 - s_1)x^2 + \ldots + (s_N - s_{N-1})x^N \\ &= (1-x)\left\{s_0 + s_1 x + \ldots + s_{N-1}\,x^{N-1}\right\} + s_N\,x^N,\end{aligned}$$

and for $|x| < 1$, $N \to \infty$, we obtain $f(x) = (1-x)\sum_{0}^{\infty} s_n\,x^n$ because $|s_N\,x^N| \leq (2N+1)K|x|^N \to 0$ for $N \to \infty$.

Furthermore, we obtain

$$\begin{aligned}&\sum_{n=0}^{N} s_n\,x^n = s_0 + (2\sigma_1 - \sigma_0)x + (3\sigma_2 - 2\sigma_1)x^2 + \ldots((N+1)\,\sigma_N - N\,\sigma_{N-1})\,x^N \\ &= (1-x)\left\{\sigma_0 + 2\sigma_1\,x + 3\sigma_2\,x^2 + \ldots + N\,\sigma_{N-1}\,x^{N-1}\right\} + (N+1)\,\sigma_N\,x^N,\end{aligned}$$

so that the other equation follows as well.

We shall make use of

$$(1-x)^{-2} = \sum_{n=0}^{\infty}(n+1)\,x^n, \qquad |x| < 1.$$

For $\varepsilon > 0$, there exists N such that for $n > N$

$$\sigma_n \in [s - \varepsilon, s + \varepsilon],$$

so for $0 < x < 1$:

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)\,\sigma_n\,x^n &\geq \sum_{n=0}^{N}(n+1)\,\sigma_n\,x^n + \sum_{n=N+1}^{\infty}(n+1)\,(s-\varepsilon)\,x^n \\ &= \frac{s-\varepsilon}{(1-x)^2} + \sum_{n=0}^{N}(n+1)\,\sigma_n\,x^n - (s-\varepsilon)\sum_{n=0}^{N}(n+1)\,x^n,\end{aligned}$$

hence

$$f(x) \geq s - \varepsilon + (1-x)^2\sum_{n=0}^{N}(n+1)\,x^n\{\sigma_n - (s-\varepsilon)\}.$$

The second term on the right-hand side approaches 0 for $x \to 1^-$, so there exists $r_1 < 1$ such that $f(x) \geq s - 2\,\varepsilon$ for $r_1 \leq x < 1$. Correspondingly, we find $f(x) \leq s + 2\,\varepsilon$ for $r_2 \leq x < 1$. $\square$

Note that for $f \in \mathcal{L}^1(\mathbb{T})$ with Fourier series $f \sim \sum c_n\, e^{\mathrm{i}n\theta}$, we have

$$f * P_r(\theta) = c_0 + \sum_{n=1}^{\infty} r^n (c_n\, e^{\mathrm{i}n\theta} + c_{-n}\, e^{-\mathrm{i}n\theta}),$$

so the question about the limit of $f * P_r(\theta)$ for $r \to 1^-$ is precisely if the Fourier series is summable (A).

For any sequence $r_n \to 1^-$ we know that $P_{r_n}(\theta)$ is a Dirac sequence for $\mathbb{T}$. From Theorem 1.6.3 we therefore get:

Theorem 1.12.15 *Let f belong to one of the spaces $C(\mathbb{T})$, $f \in \mathcal{L}^p(\mathbb{T})$, $1 \leq p < \infty$. Then $P_r * f$ belongs to the same space and $P_r * f \to f$ for $r \to 1^-$ in the norm of the space.*

Theorem 1.12.14 can be combined with Fejér-Lebesgue's Theorem (Theorem 1.6.9) and we have:

Theorem 1.12.16 *Let $f \in \mathcal{L}^1(\mathbb{T})$. Then $f * P_r(\theta) \to f(\theta)$ when $r \to 1^-$ in all Lebesgue points θ for f, in particular for almost all θ.*

Remark 1.12.17 The result of the previous theorem tell us that the solution $H(z)$ to the Dirichlet problem (cf. (1.12.13)) for the unit disc with $h \in \mathcal{L}^1(\mathbb{T})$ as boundary values converges radially to h almost everywhere, i.e.,

$$H(re^{\mathrm{i}\theta}) \to h(e^{\mathrm{i}\theta}) \quad \text{for} \quad r \to 1^-$$

for almost all θ.

Exercises

E 12.1 Let $c : \mathbb{Z} \to [0, \infty[$ fulfill

(i) $c_{-n} = c_n, \qquad n \in \mathbb{Z},$

(ii) $2c_n \leq c_{n-1} + c_{n+1}, \qquad n \geq 1,$

(iii) $c_n \geq c_{n+1}, \qquad n \geq 0.$

Show that c is positive definite, and that there exists $\alpha \geq 0$ and $f \in \mathbb{L}^1_+(\mathbb{T})$ such that $C(\alpha\,\delta_0 + f\,dm) = c$.

E 12.2 (For this exercise, it is necessary to know some distribution theory). Let $\mathcal{D}(\mathbb{T}) = C^\infty(\mathbb{T})$ be the set of periodic C^∞-functions, and consider the norms

$$p_N(f) = \max_{0\leq j\leq N} \sup_{\theta\in\mathbb{R}} |f^{(j)}(\theta)|, \qquad N = 0, 1, 2, \ldots.$$

We provide $\mathcal{D}(\mathbb{T})$ with the topology determined by the family $(p_N)_{N\geq 0}$ of norms. A distribution on $\mathbb{T}$ is a linear functional $T : C^\infty(\mathbb{T}) \to \mathbb{C}$ which is continuous in this topology, i.e., there exists $K > 0$ and $N \geq 0$ such that $|T(f)| \leq K\,p_N(f)$ for all $f \in C^\infty(\mathbb{T})$ (K and N depend on T). The set of distributions is denoted $\mathcal{D}'(\mathbb{T})$.

For a distribution T on $\mathbb{T}$ let T' denote its derivative, defined by $T'(f) = -T(f')$ for $f \in \mathcal{D}(\mathbb{T})$.

For a distribution T on $\mathbb{T}$, we define the Fourier coefficients $C(T) : \mathbb{Z} \to \mathbb{C}$ by

$$C(T)(n) = T(e^{-\mathrm{i}n\theta}), \qquad n \in \mathbb{Z}.$$

Show that

1° $T_1 = T_2 \;\Leftrightarrow\; C(T_1) = C(T_2)$,

2° If $T \in \mathcal{D}'(\mathbb{T})$ then $C(T)$ has polynomial growth, i.e.,
$\exists K, \exists N : \;\; |C(T)(n)| \leq K|n|^N \qquad$ for $n \in \mathbb{Z}$,

3° $C(T')(n) = \mathrm{i}n\,C(T)(n)$.

4° Show that if $c : \mathbb{Z} \to \mathbb{C}$ has polynomial growth, then there exists $T \in \mathcal{D}'(\mathbb{T})$ with $C(T) = c$.

E 12.3 For $a > 0$ consider the sequence $c_n = \exp(-an^2)$, $n \in \mathbb{Z}$. Show that (c_n) is a positive definite sequence.

(*Hint:* Show that the matrix $(\exp(2ajk))_{j,k=0}^n$ is positive semidefinite for all $n \geq 0$ when $a > 0$, e.g. by using the power series for the exponential function.)

Let μ_a denote the positive measure on $\mathbb{T}$ with $C(\mu_a)(n) = c_n$. Show that there exists a function $f_a \in C^\infty(\mathbb{T})$ such that $\mu_a = f_a dm$ and that

$$f_a(t) = \sum_{n=-\infty}^{\infty} e^{-an^2} e^{\mathrm{i}nt}.$$

E 12.4 1°. Let a be a complex number of absolute value 1. Show that the sequence $c_n = a^n$ $n \in \mathbb{Z}$ is positive definite by verifying condition (ii) in Herglotz' theorem. Find also the measure μ such that $C(\mu)(n) = a^n$.

2°. Prove that the sequence $c_0 = 1,\ c_n = 0,\ n \neq 0$ is positive definite and find the corresponding measure from Herglotz' theorem.

E 12.5 Let $\mathcal{P}$ denote the set of positive definite sequences $c = (c_n)$.

1°. Show that if $c, d \in \mathcal{P}$ and λ is a non-negative number, then $\lambda c + d, cd \in \mathcal{P}$.

2°. Let $c_p, p = 1, 2, \ldots$ be a sequence from $\mathcal{P}$, which converges termwise to a sequence $c : \mathbb{Z} \to \mathbb{C}$, i.e., $\lim_{p\to\infty} c_p(n) = c(n)$ for each $n \in \mathbb{Z}$. Show that $c \in \mathcal{P}$ and if $\mu_p, \mu \in \mathbb{M}_+(\mathbb{T})$ are such that $C(\mu_p) = c_p, C(\mu) = c$, then $\mu_p \to \mu$ weakly.

3°. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with radius of convergence $\rho > 0$ and assume that $a_k \geq 0$ for all k. Show that if $c \in \mathcal{P}$ satisfies $c_0 < \rho$, then $f(c) \in \mathcal{P}$, where $f(c)$ denotes the sequence $n \to f(c_n),\ n \in \mathbb{Z}$.

4°. Show that $\exp(c) \in \mathcal{P}$ for $c \in \mathcal{P}$.

E 12.6 Let $c : \mathbb{Z} \to \mathbb{C}$ be a sequence with the property that for all $N = 0, 1, \ldots$

$$f_N(t) = \sum_{k=-N}^{N} c_k e^{\mathrm{i}kt} \geq 0 \text{ for all } t \in \mathbb{R}.$$

Prove that there exists $\mu \in \mathbb{M}_+(\mathbb{T})$ such that $C(\mu) = c$, i.e., $c \in \mathcal{P}$.

(*Hint:* Define $F_N \in C(\mathbb{T})$ by $F_N(e^{\mathrm{i}t}) = f_N(t)$ and consider the measures $\mu_N = F_N\, dm \in \mathbb{M}_+(\mathbb{T})$.)

Chapter 2

Fourier integrals

2.1 Introduction

This chapter is largely based on lecture notes by Tage Gutmann Madsen to the second year analysis course around 1980. The notion of a Dirac sequence and a Dirac family, see Definition 2.6.2, is due to him.

The group behind Fourier integrals is the real line $\mathbb{R}$ with addition as group operation, and it is a locally compact abelian group. Lebesgue measure m on $\mathbb{R}$ is the uniquely determined translation invariant Borel measure normalized such that $m([0,1]) = 1$. Instead of writing $dm(x)$ we just write dx.

In the theory of Fourier series a given periodic function $f : \mathbb{R} \to \mathbb{C}$ is represented as the sum of a series with a constant term, a fundamental oscillation and its overtones, see Section 1.2

If we consider a function $f : \mathbb{R} \to \mathbb{C}$ without any periodicity, it is natural to try to represent it, not by an infinite series, but by an integral

$$f(x) \sim \int_0^\infty \left(c(t)\, e^{\mathrm{i}2\pi tx} + c(-t)\, e^{-\mathrm{i}2\pi tx} \right) dt \tag{2.1.1}$$

involving all frequencies $t > 0$. Roughly speaking, (2.1.1) can be realized by defining

$$c(t) = \int_{-\infty}^\infty f(x)\, e^{-\mathrm{i}2\pi xt}\, dx, \qquad t \in \mathbb{R}. \tag{2.1.2}$$

The function $c(t)$ is called the <u>Fourier transform</u> of f. Of course we need some assumptions for (2.1.2) to make sense, and next we have to examine in which sense the representation (2.1.1) holds.

In the following we shall use the Lebesgue spaces $\mathcal{L}^p(\mathbb{R})$, $1 \le p < \infty$, of Borel functions $f : \mathbb{R} \to \mathbb{C}$ satisfying

$$\|f\|_p = \left(\int_{-\infty}^\infty |f(x)|^p\, dx \right)^{1/p} < \infty, \qquad 1 \le p < \infty \tag{2.1.3}$$

and

$$\|f\|_\infty = \text{ess sup}\left\{|f(x)| \,\middle|\, x \in \mathbb{R}\right\} < \infty, \qquad p = \infty. \tag{2.1.4}$$

Note that if f is continuous, then the essential supremum is the same as the ordinary supremum.

It is well-known that $\|\cdot\|_p$ is a semi-norm on $\mathcal{L}^p(\mathbb{R})$, and if functions are identified if they are equal a. e., then we get the Banach spaces $L^p(\mathbb{R})$ of equivalence classes of Borel functions.

In contrast to the spaces $\mathcal{L}^p(\mathbb{T})$, which decrease in size with increasing p, there are no inclusions between the spaces $\mathcal{L}^p(\mathbb{R})$, see **E 1.1** below.

For $f \in \mathcal{L}^1(\mathbb{R})$ we can rigorously define the <u>Fourier transform</u> as the function

$$\mathcal{F}f(t) = \int_{-\infty}^{\infty} f(x)\, e^{-\mathrm{i}2\pi xt}\, dx, \qquad t \in \mathbb{R} \tag{2.1.5}$$

because $x \mapsto f(x)\, e^{-\mathrm{i}2\pi xt}$ is integrable, since it has the same absolute value as $f(x)$.

Remark 2.1.1 In some books about Fourier transformation you will see $\mathcal{F}f(t)$ defined as

$$\mathcal{F}f(t) = \widehat{f}(t) = \int_{-\infty}^{\infty} f(x)\, e^{-\mathrm{i}xt}\, dx, \qquad t \in \mathbb{R} \tag{2.1.6}$$

and the formerly defined Fourier transform is just the latter composed with the scaling $t \to 2\pi t$. Sometimes the integral in (2.1.6) is divided by $\sqrt{2\pi}$. We shall later give more explanation about this difference in notation.

Theorem 2.1.2 *The Fourier transform $\mathcal{F}f$ of $f \in \mathcal{L}^1(\mathbb{R})$ is a continuous function $\mathcal{F}f : \mathbb{R} \to \mathbb{C}$ vanishing at infinity, i.e.,*

$$\mathcal{F}f(t) \to 0 \qquad \text{for } |t| \to \infty. \tag{2.1.7}$$

Furthermore, $\|\mathcal{F}f\|_\infty \le \|f\|_1$.

<u>Proof.</u> For $t_n \to t_0$ we have

$$f(x)\, e^{-\mathrm{i}2\pi x t_n} \to f(x)\, e^{-\mathrm{i}2\pi x t_0}$$

for each $x \in \mathbb{R}$, and since $|f(x)|$ is an integrable majorant, it follows by Lebesgue's theorem on dominated convergence that $\mathcal{F}f(t_n) \to \mathcal{F}f(t_0)$, i.e., $\mathcal{F}f$ is a continuous function. Clearly $|\mathcal{F}f(t)| \le \|f\|_1$ and (2.1.7) holds by Riemann-Lebesgue's lemma from Section 1.4. $\square$

The expression for $\mathcal{F}f$ can be written

$$\mathcal{F}f(t) = \int_{-\infty}^{\infty} f(x)\cos(2\pi xt)\,dx - \mathrm{i}\int_{-\infty}^{\infty} f(x)\sin(2\pi xt)\,dx, \qquad t \in \mathbb{R}.$$

For an <u>even function</u> ($f(-x) = f(x)$) the second term vanishes, so $\mathcal{F}f$ is again an even function. If f is <u>odd</u> ($f(-x) = -f(x)$), then the first term vanishes and $\mathcal{F}f$ is also odd.

A function $g \in \mathcal{L}^1([0,\infty[)$ can be extended to an even function $f : \mathbb{R} \to \mathbb{C}$ and we find

$$\frac{1}{2}\mathcal{F}f(t) = \int_0^{\infty} g(x)\,\cos(2\pi xt)\,dx, \qquad t \in \mathbb{R},$$

called the <u>cosine transform</u> of g.

A function $g \in \mathcal{L}^1(]0,\infty[)$ can also be extended to an odd function $f : \mathbb{R} \to \mathbb{C}$, (we define $f(0) = 0$), and we then find

$$\frac{\mathrm{i}}{2}\mathcal{F}f(t) = \int_0^{\infty} g(x)\,\sin(2\pi xt)\,dx, \qquad t \in \mathbb{R},$$

called the <u>sine transform</u> of g.

Exercises

E 1.1 Prove that $f_p(x) = (1+|x|)^{-p}$ belongs to $\mathcal{L}^1(\mathbb{R})$ if and only if $p > 1$, and that

$$g_p(x) = \begin{cases} x^{-p} & \text{for } 0 < x < 1 \\ 0 & \text{for } x \le 0 \text{ and } x \ge 1 \end{cases}$$

belongs to $\mathcal{L}^1(\mathbb{R})$ if and only if $p < 1$.

Let $1 \le p_1 < p_2 \le \infty$. Construct functions in $\mathcal{L}^{p_1}(\mathbb{R}) \setminus \mathcal{L}^{p_2}(\mathbb{R})$ and in $\mathcal{L}^{p_2}(\mathbb{R}) \setminus \mathcal{L}^{p_1}(\mathbb{R})$.

E 1.2 For $f \in \mathcal{L}^1(\mathbb{R})$ and $a > 0$ let f_a denote the function equal to f on $]-a,a]$ and extended to a periodic function on $\mathbb{R}$ with period $2a$.

Show that the Fourier series of f_a can be written as

$$f_a(x) \sim \sum_{n=-\infty}^{\infty} c_{a,n} e^{\mathrm{i}n\frac{\pi}{a}x}, \quad \text{where} \quad c_{a,n} = \frac{1}{2a}\int_{-a}^{a} f(x)e^{-\mathrm{i}n\frac{\pi}{a}x}\,dx$$

and as

$$f_a(x) \sim \sum_{n=-\infty}^{\infty} \frac{1}{2a} g_a(\frac{n}{2a}) e^{\mathrm{i}n\frac{\pi}{a}x}, \quad \text{where} \quad g_a(y) = \int_{-a}^{a} f(x) e^{-\mathrm{i}2\pi xy}\, dx. \tag{2.1.8}$$

Show that $g_a(y) \to \mathcal{F}f(y)$ for $a \to \infty$ for each $y \in \mathbb{R}$.

Show that the sum in (2.1.8) can be considered as an infinite Riemann sum and explain that formally it approaches

$$\int_{-\infty}^{\infty} \mathcal{F}f(y) e^{\mathrm{i}2\pi xy}\, dy,$$

so one is tempted to claim that this integral equals $f(x)$, i.e. that

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}f(y) e^{\mathrm{i}2\pi xy}\, dy. \tag{2.1.9}$$

A lot of research for 200 years has been undertaken trying to make this rigorous.

Explain that one can write the formula (2.1.9) in the following equivalent forms

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(y) e^{\mathrm{i}xy}\, dy, \quad \text{where} \quad \widehat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-\mathrm{i}xy}\, dx,$$

and

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(y) e^{\mathrm{i}xy}\, dy, \quad \text{where} \quad \widehat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-\mathrm{i}xy}\, dx.$$

E 1.3 Let $f \in \mathcal{L}^1(\mathbb{R})$ and assume that $f(x) \geq 0$ for almost all x and $\int f(x)\, dx = 1$. (In other words f is density for a probability measure). Prove that $|\mathcal{F}f(t)| < 1$ for all $t \neq 0$. (Of course $\mathcal{F}f(0) = 1$.)

2.2 Improper integrals

Let $g :]0, \infty[\to \mathbb{C}$ be a Borel function such that $\int_0^u |g(t)|\, dt < \infty$ for each $u > 0$.

If $g \in \mathcal{L}^1(\mathbb{R}_+)$, then

$$\int_0^{\infty} g(t)\, dt = \int_{\mathbb{R}_+} g(t)\, dt = \lim_{u \to \infty} \int_0^u g(t)\, dt,$$

because for every sequence $u_1, u_2, \ldots$ in $\mathbb{R}_+$ with $u_n \to \infty$, we have

$$\int_0^{u_n} g(t)\, dt = \int_{\mathbb{R}_+} g \cdot 1_{]0,u_n]}\, dm \xrightarrow[n \to \infty]{} \int_{\mathbb{R}_+} g\, dm$$

according to Lebesgue's theorem on dominated convergence. In fact, $|g| \in \mathcal{L}^1(\mathbb{R}_+)$ is a majorant for the sequence of functions $|g \cdot 1_{]0,u_n]}|$.

However, $\int_0^u g(t)\,dt$ can have a limit $c \in \mathbb{C}$ for $u \to \infty$, even if the condition $g \in \mathcal{L}^1(\mathbb{R}_+)$ does not hold, cf. Example 2.2.1 below. We still write $c = \int_0^\infty g(t)\,dt$, possibly mentioning that the integral is improper.

In analogy with the usual convention for infinite series, one writes the symbol $\int_0^\infty g(t)\,dt$ without knowing in advance whether the integral $\int_0^u g(t)\,dt$ has a limit $c \in \mathbb{C}$ for $u \to \infty$. If the limit exists, we say that the integral $\int_0^\infty g(t)\,dt$ is convergent with value c.

One can furthermore encounter the use of language that the integral $\int_0^\infty g(t)\,dt$ is absolutely convergent. Hereby is meant that $\int_0^\infty |g(t)|\,dt$ converges. However, this is equivalent to integrability of g i.e., $g \in \mathcal{L}^1(\mathbb{R}_+)$. In fact,

$$g \in \mathcal{L}^1(\mathbb{R}_+) \;\Leftrightarrow\; |g| \in \mathcal{L}^1(\mathbb{R}_+) \;\Leftrightarrow\; \int_{\mathbb{R}_+} |g|\,dm = \lim_{u\to\infty} \int_{\mathbb{R}_+} |g| \cdot 1_{]0,u]}\,dm < \infty,$$

where Lebesgue's monotonicity theorem is used.

The reader should notice the analogy with convergent and absolutely convergent series.

Example 2.2.1 *The integral* $\displaystyle\int_0^\infty \frac{\sin t}{t}\,dt$ *is convergent with value* $\frac{\pi}{2}$*, i.e.,*

$$\int_0^u \frac{\sin t}{t}\,dt \to \frac{\pi}{2} \qquad \text{for } u \to \infty,$$

but the integrand does not belong to $\mathcal{L}^1(\mathbb{R}_+)$*.*

Proof. That $t \to \dfrac{\sin t}{t}$, $t \in \mathbb{R}_+$, does not belong to $\mathcal{L}^1(\mathbb{R}_+)$ follows from

$$\int_{(p-1)\pi}^{p\pi} \frac{|\sin t|}{t}\,dt > \int_{(p-\frac{5}{6})\pi}^{(p-\frac{1}{6})\pi} \frac{1/2}{p\pi}\,dt = \frac{1}{3p},$$

whereby

$$\int_0^\infty \left|\frac{\sin t}{t}\right| dt = \sum_{p=1}^\infty \int_{(p-1)\pi}^{p\pi} \frac{|\sin t|}{t}\,dt \geq \frac{1}{3}\sum_{p=1}^\infty \frac{1}{p} = \infty.$$

That the integral $\displaystyle\int_0^\infty \frac{\sin t}{t}\,dt$ converges can be seen in the following way: Since the sign of the integrand changes in $\pi, 2\pi, \ldots$,

$$\sum_{p=1}^\infty \int_{(p-1)\pi}^{p\pi} \frac{\sin t}{t}\,dt$$

is an alternating series. The numerical value of the terms decreases to 0 because

$$\int_{(p-1)\pi}^{p\pi} \frac{|\sin t|}{t}\, dt \geq \int_{(p-1)\pi}^{p\pi} \frac{|\sin t|}{t+\pi}\, dt = \int_{p\pi}^{(p+1)\pi} \frac{|\sin t|}{t}\, dt,$$

and

$$\int_{p\pi}^{(p+1)\pi} \frac{|\sin t|}{t}\, dt < \frac{1}{p}.$$

Consequently, the infinite series converges, i.e.,

$$\int_0^{n\pi} \frac{\sin t}{t}\, dt = \sum_{p=1}^{n} \int_{(p-1)\pi}^{p\pi} \frac{\sin t}{t}\, dt$$

has a limit c for $n \to \infty$. But from this, it follows that

$$s(u) = \int_0^u \frac{\sin t}{t}\, dt \to c \qquad \text{for } \ u \to \infty,$$

since $s(u)$ lies between $s(n\pi)$ and $s((n+1)\pi)$ for $n\pi \leq u \leq (n+1)\pi$.

We find the value c of the integral $\displaystyle\int_0^{\infty} \frac{\sin t}{t}\, dt$ as $\lim_n s((n+\frac{1}{2})\pi)$, where

$$s((n+\tfrac{1}{2})\pi) = \int_0^{(n+\frac{1}{2})\pi} \frac{\sin t}{t}\, dt = \int_0^{\pi} \frac{\sin(n+\frac{1}{2})t}{(n+\frac{1}{2})t}\,(n+\tfrac{1}{2})\, dt = \int_0^{\pi} \frac{\sin(n+\frac{1}{2})t}{t}\, dt.$$

We use that $\displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\, dt = 1$, where D_n is the Dirichlet kernel, see 1.5.3

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt} = \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t} \qquad \text{for } \ t \neq 0 \ (\text{mod } \ 2\pi).$$

The difference

$$\frac{\pi}{2} - s((n+\tfrac{1}{2})\pi) = \int_0^{\pi} \frac{1}{2}\, D_n(t)\, dt - s((n+\tfrac{1}{2})\pi)$$

can be written

$$\int_0^{\pi} \left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right) \sin(n+\tfrac{1}{2})t\, dt = \int_0^{\pi/2} \left(\frac{1}{\sin t} - \frac{1}{t}\right) \sin((2n+1)t) dt,$$

which converges to 0 for $t \to \infty$ according to Riemann-Lebesgue's lemma, cf. Section 1.4, since

$$t \to \frac{1}{\sin t} - \frac{1}{t}, \qquad 0 < t \leq \frac{\pi}{2}$$

is integrable (the function is continuous in $[0, \frac{\pi}{2}]$, when we assign the value 0 for $t = 0$). $\square$

We end this section with some comments about <u>summability</u> of an integral $\int_0^\infty g(t)\,dt$.

We assume as previously that $g :]0, \infty[\to \mathbb{C}$ is a Borel function, such that $\int_0^u |g(t)|\,dt < \infty$ for each $u \in \mathbb{R}_+$, and put

$$s(u) = \int_0^u g(t)\,dt, \qquad u \geq 0. \tag{2.2.1}$$

Since $s : [0, \infty[$ is continuous, we can define the mean values

$$\sigma(v) = \frac{1}{v}\int_0^v s(u)\,du, \qquad v > 0. \tag{2.2.2}$$

The integral $\int_0^\infty g(t)\,dt$ is said to be <u>summable</u> with the value c if $\sigma(v) \to c$ for $v \to \infty$.

Lemma 2.2.2 *A convergent integral $\int_0^\infty g(t)\,dt$ with value c is also summable with the same value.*

<u>Proof.</u> For arbitrary $\varepsilon \in \mathbb{R}_+$ there exists an $H \in \mathbb{R}_+$ such that

$$|s(u) - c| < \frac{\varepsilon}{2} \qquad \text{for } \ u > H.$$

For each $v > H$ we now get

$$\begin{aligned}
|\sigma(v) - c| &= \left|\frac{1}{v}\int_0^v (s(u) - c)\,du\right| \\
&\leq \frac{1}{v}\left|\int_0^H (s(u) - c)\,du\right| + \frac{1}{v}\int_H^v |s(u) - c|\,du \\
&\leq \frac{1}{v}\left|\int_0^H (s(u) - c)\,du\right| + \frac{\varepsilon}{2}.
\end{aligned}$$

Since $\frac{1}{v}\left|\int_0^H (s(u) - c)\,du\right| \to 0$ for $v \to \infty$, there exists a $K \geq H$ such that

$$|\sigma(v) - c| < \varepsilon \qquad \text{for } \ v > K.$$

Exercises

E 2.1 Show that the integral $\int_0^\infty \cos t\,dt$ is not convergent, but that it is summable with value 0.

2.3 Convolution of functions on $\mathbb{R}$

The real line $\mathbb{R}$ is an abelian group under addition. For $h \in \mathbb{R}$ we let $\tau_h : \mathbb{R} \to \mathbb{R}$ denote the translation by h, i.e., $\tau_h(x) = x + h$. Lebesgue measure m on $\mathbb{R}$ is translation invariant, i.e., $m(B+h) = m(B)$ for $h \in \mathbb{R}$, $B \in \mathbb{B}(\mathbb{R})$. We let τ_h act on functions $f : E \to \mathbb{C}$, where $E \subseteq \mathbb{R}$, by the following rule

$$\tau_h f(x) = f(x-h) \qquad \text{for } x \in \tau_h(E) = E + h.$$

Theorem 2.3.1 *For $f \in \mathcal{L}^p(\mathbb{R})$, $1 \le p \le \infty$ and $h \in \mathbb{R}$ we have $\tau_h f \in \mathcal{L}^p(\mathbb{R})$ and $\|\tau_h f\|_p = \|f\|_p$.*

Furthermore, if $1 \le p < \infty$ and $f \in \mathcal{L}^p(\mathbb{R})$ we have

$$\|\tau_h f - f\|_p = \left(\int_{-\infty}^{\infty} |f(x-h) - f(x)|^p \, dx \right)^{1/p} \to 0 \qquad \textit{for } h \to 0. \qquad (2.3.1)$$

Proof. The first part of the theorem is a direct consequence of the translation invariance of Lebesgue measure. For the property (2.3.1) it is important that $1 \le p < \infty$. To establish it we first consider $f \in C_c(\mathbb{R})$, where the latter denotes the continuous functions with compact support. A function $f \in C_c(\mathbb{R})$ is uniformly continuous, i.e., to $\varepsilon > 0$ there exists $0 < \delta$ such that

$$|f(x-h) - f(x)| \le \varepsilon \qquad \text{for } |h| < \delta, \ x \in \mathbb{R},$$

hence $|\tau_h f - f| \le \varepsilon$ for $|h| < \delta$. By assumption $\operatorname{supp}(f) \subseteq [-R, R]$ for suitable $R > 0$, and by assuming $\delta < 1$ we then have

$$|\tau_h f - f| \le \varepsilon \, 1_{[-R-1, R+1]},$$

hence

$$\|\tau_h f - f\|_p^p \le \varepsilon^p \, (2R+2),$$

which shows (2.3.1). To prove this equation for an arbitrary $f \in \mathcal{L}^p(\mathbb{R})$ we use that $C_c(\mathbb{R})$ is dense in $\mathcal{L}^p(\mathbb{R})$, a property which does not hold for $p = \infty$.

To $f \in \mathcal{L}^p(\mathbb{R})$ and $\varepsilon > 0$ we first choose $g \in C_c(\mathbb{R})$ such that $\|f - g\|_p < \frac{\varepsilon}{3}$. For $h \in \mathbb{R}$ we then have

$$\|\tau_h f - f\|_p \le \|\tau_h f - \tau_h g\|_p + \|\tau_h g - g\|_p + \|g - f\|_p < \frac{2\varepsilon}{3} + \|\tau_h g - g\|_p,$$

but by the first part of the proof we have $\|\tau_h g - g\|_p < \frac{\varepsilon}{3}$ if $|h| < \delta$ for $\delta > 0$ sufficiently small, hence

$$\|\tau_h f - f\|_p < \varepsilon \quad \text{for} \quad |h| < \delta.$$

□

By the <u>convolution</u> $f * g$ of two complex-valued Borel functions f and g defined on $\mathbb{R}$ we understand the function

$$x \mapsto \int_{\mathbb{R}} f(x-y)\, g(y)\, dy, \tag{2.3.2}$$

defined on the set $D(f * g)$ of those $x \in \mathbb{R}$ for which $y \mapsto f(x-y)\, g(y)$ is Lebesgue integrable on $\mathbb{R}$.

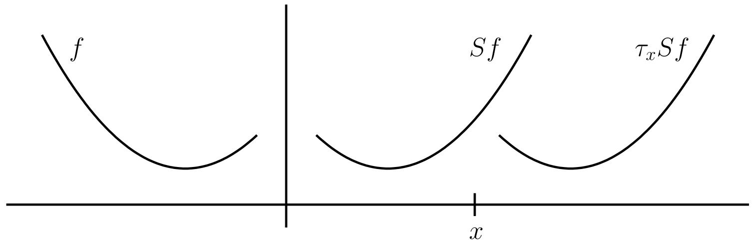

Figure 2.1: Illustration of the reflected and translated function

Denoting by Sf the function reflected in the origin, i.e., $Sf(y) = f(-y)$, we have $(\tau_x Sf)(y) = Sf(y-x) = f(x-y)$. The convolution $f * g$ is thus defined by

$$f * g(x) = \int_{\mathbb{R}} (\tau_x Sf)\, g\, dm, \tag{2.3.3}$$

for

$$x \in D(f * g) = \{x \in \mathbb{R} \mid (\tau_x Sf) g \in \mathcal{L}^1(\mathbb{R})\}. \tag{2.3.4}$$

Note that

$$D(f * g) = \{x \in \mathbb{R} \mid \int |f(x-y)g(y)|\, dy < \infty\}. \tag{2.3.5}$$

The set of definition $D = D(f * g)$ can be empty. This is true if $f = g = 1$.

Convolution is commutative like for periodic functions:

The functions $f * g$ and $g * f$ are equal on $D(f * g) = D(g * f)$.

In fact, for $x \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} |f(x-y)\, g(y)|\, dy = \int_{\mathbb{R}} |f(x+y)\, g(-y)|\, dy = \int_{\mathbb{R}} |f(y)\, g(x-y)|\, dy,$$

which shows that $x \in D(f*g) \Leftrightarrow x \in D(g*f)$. For x in this common set of definition all three integrals are finite and the equations above hold without absolute value, i.e., $f * g(x) = g * f(x)$. We have used that Lebesgue measure on $\mathbb{R}$ is invariant under reflection and translations.

Note also that if f or g is changed on a null set, this will not change the function $f * g$.

Proposition 2.3.2 *Let f, g be complex-valued Borel functions on $\mathbb{R}$. The following assertions hold:*

1° *The convolution $f * g$ is a Borel function and $D(f * g)$ is a Borel set.*

2° *Let $c \in \mathbb{C}$. If $x \in D(f * g)$ then $x \in D(f * (c\, g))$ and*

$$(f * (c\, g))(x) = c\,(f * g)(x).$$

3° *If $x \in D(f * g) \cap D(f * h)$ then $x \in D(f * (g + h))$ and*

$$(f * (g + h))(x) = (f * g)(x) + (f * h)(x).$$

4° *If $f(x) = 0$ for $x \notin A \subseteq \mathbb{R}$ and $g(x) = 0$ for $x \notin B \subseteq \mathbb{R}$, then $f * g(x)$ is defined and equal to 0 for $x \notin A + B = \left\{ a + b \,\middle|\, a \in A,\, b \in B \right\}$.*

Proof.

1° Since $f \otimes g$, i.e., the function $(x, y) \mapsto f(x)\, g(y)$ is a Borel function on $\mathbb{R}^2$, this holds also for the function obtained by composition with $(x, y) \mapsto (x - y, y)$, which is the function

$$(x, y) \mapsto f(x - y)\, g(y), \qquad x,\, y \in \mathbb{R}.$$

It follows from the proof of Fubini's Theorem that the set of points x for which $y \to f(x - y)g(y)$ is integrable is a Borel set, i.e., $D(f * g)$ is a Borel set, and the integral with respect to y is a Borel function of x, i.e., $f * g$ is a Borel function.

2°, 3° The assertions follow because an integrable function multiplied by a constant and the sum of two integrable functions are again integrable. Moreover, the integral is a linear functional.

4° In fact,

$$f(x - y)\, g(y) \neq 0 \;\Rightarrow\; x - y \in A,\; y \in B \;\Rightarrow\; x = (x - y) + y \in A + B.$$

For $x \notin A + B$ we conclude that the function $y \mapsto f(x - y)\, g(y)$ is identically 0. $\square$

Theorem 2.3.3 *Let $f, g : \mathbb{R} \to \mathbb{C}$ be Borel functions. The following assertions hold:*

1°. If $f, g \in C_c(\mathbb{R})$, the continuous functions with compact support, then $f * g \in C_c(\mathbb{R})$ and $\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f) + \operatorname{supp}(g)$.

2°. If $1 \le p, q \le \infty$ are dual exponents, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, then if $f \in \mathcal{L}^p(\mathbb{R})$, $g \in \mathcal{L}^q(\mathbb{R})$, we have $D(f * g) = \mathbb{R}$ and $f * g$ is uniformly continuous and bounded with

$$\|f * g\|_\infty \le \|f\|_p \|g\|_q. \tag{2.3.6}$$

3°. If $f, g \in \mathcal{L}^1(\mathbb{R})$ then $\mathbb{R} \setminus D(f * g)$ is a Lebesgue null set and $f * g \in \mathcal{L}^1(\mathbb{R})$ with

$$\|f * g\|_1 \le \|f\|_1 \|g\|_1. \tag{2.3.7}$$

4°. If $f \in \mathcal{L}^1(\mathbb{R})$, $g \in \mathcal{L}^p(\mathbb{R})$, $1 \le p \le \infty$ then $\mathbb{R} \setminus D(f * g)$ is a Lebesgue null set and $f * g \in \mathcal{L}^p(\mathbb{R})$ with

$$\|f * g\|_p \le \|f\|_1 \|g\|_p. \tag{2.3.8}$$

Proof. 1° Since $f(x) = 0$ for $x \notin \operatorname{supp}(f)$ and similarly with g it follows by Proposition 2.3.2 4° that $f*g(x) = 0$ for x outside the compact set $\operatorname{supp}(f)+\operatorname{supp}(g)$. This shows the assertion about the supports. The continuity of $f * g$ is an easy consequence of the uniform continuity of f, but using that $C_c(\mathbb{R}) \subset \mathcal{L}^p(\mathbb{R})$ for any $p \in [1, \infty]$ the continuity is also a consequence of 2°.

2° By the invariance properties of Lebesgue measure we know that $\tau_x S f \in \mathcal{L}^p(\mathbb{R})$, so by Hölder's inequality we get $(\tau_x S f)\, g \in \mathcal{L}^1(\mathbb{R})$ for each $x \in \mathbb{R}$, hence $D(f*g) = \mathbb{R}$ and

$$|f * g(x)| = \left| \int_{\mathbb{R}} (\tau_x S f)\, g\, dm \right| \le \|\tau_x S f\|_p \|g\|_q = \|f\|_p \|g\|_q,$$

which shows (2.3.6).

That $f * g$ is uniformly continuous can be seen from the estimate

$$\begin{aligned} |f * g(x+h) - f * g(x)| &= \left| \int_{\mathbb{R}} (\tau_{x+h} S f - \tau_x S f)\, g\, dm \right| \\ &\le \|\tau_{x+h} S f - \tau_x S f\|_p \|g\|_q = \|\tau_h S f - S f\|_p \|g\|_q, \end{aligned}$$

because $\|\tau_h S f - S f\|_p \to 0$ for $h \to 0$ by (2.3.1), provided $1 \le p < \infty$. For $p = \infty$ we have $q = 1$ and we consider $g * f$ instead.

3° From the proof of Proposition 2.3.2 1° we know that

$$(x, y) \mapsto f(x - y)\, g(y), \qquad x,\ y \in \mathbb{R}.$$

is a Borel function, and it belongs to $\mathcal{L}^1(\mathbb{R}^2)$ because by Tonelli's theorem and the translation invariance of Lebesgue measure in $\mathbb{R}$ (m_2 denotes Lebesgue measure on $\mathbb{R}^2$):

$$\begin{aligned} \int_{\mathbb{R}^2} |f(x-y)\, g(y)|\, dm_2(x, y) = &= \int_{\mathbb{R}} \left(|g(y)| \int_{\mathbb{R}} |f(x-y)|\, dx \right) dy \\ &= \int_{\mathbb{R}} \left(|g(y)| \int_{\mathbb{R}} |f(x)|\, dx \right) dy = \|f\|_1 \|g\|_1 < \infty. \end{aligned}$$

From Fubini's theorem it now follows that $y \mapsto f(x-y)\,g(y)$ is integrable for almost all $x \in \mathbb{R}$, and the almost everywhere defined function

$$x \mapsto \int_{\mathbb{R}} f(x-y)\,g(y)\,dy,$$

which is precisely $f * g$, belongs to $\mathcal{L}^1(\mathbb{R})$. Since

$$|f * g(x)| \le \int_{\mathbb{R}} |f(x-y)\,g(y)|\,dy,$$

we finally find

$$\|f * g\|_1 \le \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)\,g(y)|\,dy\,dx = \int_{\mathbb{R}^2} |f(x-y)\,g(y)|\,dm_2(x,y) = \|f\|_1 \|g\|_1,$$

which proves (2.3.7).

4° The cases $p = \infty$ and $p = 1$ are treated in 2° and 3° respectively, so we assume that $1 < p < \infty$ and determine q such that $\frac{1}{p} + \frac{1}{q} = 1$. The proof is exactly as the proof of Theorem 1.1.1 4°, but we give it a little twist by proving $D(|g|^p * |f|) \subseteq D(g * f)$. By 3° we know that $\mathbb{R} \setminus D(|g|^p * |f|)$ is a null set and therefore $g * f = f * g$ is defined for almost all x.

The inclusion follows by Hölder's inequality:

$$\begin{aligned}
&\int_{\mathbb{R}} |g(x-y)\,f(y)|\,dy = \int_{\mathbb{R}} |g(x-y)|\,|f(y)|^{1/p}\,|f(y)|^{1/q}\,dy \\
&\le \left(\int_{\mathbb{R}} |g(x-y)|^p\,|f(y)|\,dy\right)^{1/p} \left(\int_{\mathbb{R}} |f(y)|\,dy\right)^{1/q},
\end{aligned}$$

which is finite for $x \in D(|g|^p * |f|)$.

For $x \in D(g * f)$ we have

$$|g * f(x)| \le \int_{\mathbb{R}} |g(x-y)\,f(y)|\,dy,$$

hence

$$|g * f(x)|^p \le \int_{\mathbb{R}} |g(x-y)|^p\,|f(y)|\,dy \left(\int_{\mathbb{R}} |f(y)|\,dy\right)^{p/q} = |g|^p * |f|(x)\;\|f\|_1^{p-1}.$$

By 3° we know that $|g|^p * |f|$ belongs to $\mathcal{L}^1(\mathbb{R})$ with

$$\int_{\mathbb{R}} |g|^p * |f|\,dm = \|\,|g|^p * |f|\,\|_1 \le \|\,|g|^p\|_1 \|f\|_1 = \|f\|_1 \|g\|_p^p,$$

and we conclude that

$$\int_{\mathbb{R}} |g * f|^p \, dm \leq \|f\|_1^p \|g\|_p^p < \infty,$$

i.e., $g * f$ belongs to $\mathcal{L}^p(\mathbb{R})$ and

$$\|f * g\|_p \quad \leq \quad \|f\|_1 \|g\|_p,$$

which proves (2.3.8). $\square$

One should note that in particular the convolution of a function $f \in \mathcal{L}^1(\mathbb{R})$ with a continuous and bounded function g is again a continuous and bounded function.

Roughly speaking one can say that the convolution of two functions is always as "nice" as the "nicest" of the given functions. The following theorem is another illustration of this philosophy.

Theorem 2.3.4 *Let $f \in \mathcal{L}^1(\mathbb{R})$. Is $g : \mathbb{R} \to \mathbb{C}$ bounded and differentiable with a bounded derivative, then the same holds for $f * g$, and*

$$(f * g)' = f * g'.$$

Proof. The convolution $f * g = g * f$ is defined by an integral, where the integrand depends on a real-valued parameter x,

$$g * f(x) = \int_{\mathbb{R}} g(x - y) \, f(y) \, dy.$$

For fixed $x \in \mathbb{R}$, the integrand is $(\tau_x S g) \, f \in \mathcal{L}^1(\mathbb{R})$, and for fixed y, the integrand is a differentiable function of x, and

$$\left| \frac{\partial g(x - y) \, f(y)}{\partial x} \right| = |g'(x - y) f(y)| \leq \|g'\|_\infty |f(y)|,$$

so $\|g'\|_\infty |f|$ is an integrable majorant independent of x. It follows by a theorem about differentiation under the integral sign that $f * g$ is differentiable with

$$(f * g)'(x) = \int_{\mathbb{R}} \frac{\partial g(x - y) \, f(y)}{\partial x} \, dy = \int_{\mathbb{R}} g'(x - y) \, f(y) \, dy = f * g'(x).$$

Both $f * g$ and $(f * g)' = f * g'$ are bounded and uniformly continuous, since $g, g' \in \mathcal{L}^\infty(\mathbb{R})$. That g' is a Borel function follows from $(\tau_{-h_n} g - g)/h_n \to g'$ for any sequence h_n tending to zero. $\square$

Exercises

E 3.1 Let $f \in \mathcal{L}^1(\mathbb{R})$ and $h \in \mathbb{R}$. Prove that

$$\mathcal{F}(f(x)e^{2\pi i h x})(t) = \mathcal{F}(f)(t-h), \quad \mathcal{F}(\tau_h f)(t) = \mathcal{F}(f)(t)e^{-2\pi i h t}$$

E 3.2 From complex analysis it is known that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} e^{-\frac{1}{2}x^2}\, dx = e^{-\frac{1}{2}t^2}, \quad t \in \mathbb{R}.$$

Show that $e^{-\pi x^2}$ is a fixed point for $\mathcal{F}$.

E 3.3 Define $f(x) = 1/\cosh(\pi x),\ x \in \mathbb{R}$.

1°. Prove that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

2°. Prove that

$$f * f(x) = \frac{2x}{\sinh(\pi x)},$$

where the right-hand side for $x = 0$ is to be understood as the limit for $x \to 0$.

3°. Prove that $\mathcal{F}(f)(t) = 1/\cosh(\pi t),\ t \in \mathbb{R}$, i.e., that $f(x) = 1/\cosh(\pi x)$ is a fixed point for $\mathcal{F}$ (or an eigenvector for $\mathcal{F}$ corresponding to the eigenvalue 1 e.g. in the space $\mathcal{S}$ to be introduced in section 8).

Hint: Use the residue theorem for the function

$$F(z) = \frac{e^{-2\pi i t z}}{\cosh(\pi z)},$$

which is meromorphic in $\mathbb{C}$. Integrate F along the sides of the rectangle with vertices $\pm R, \pm R + i$, where $R > 0$ is fixed and let $R \to \infty$.

2.4 Convergence of Fourier integrals

The Fourier transform $\mathcal{F}f$ of a function $f \in \mathcal{L}^1(\mathbb{R})$ does not belong in general to $\mathcal{L}^1(\mathbb{R})$, ($f = 1_{[-1,1]}$, $\mathcal{F}f(x) = \frac{\sin 2\pi x}{\pi x}$). Therefore, we consider the <u>Fourier integral</u>

$$\int_0^{\infty} \left(\mathcal{F}f(t)\, e^{2\pi i t x} + \mathcal{F}f(-t)\, e^{-2\pi i t x}\right) dt$$

for $f \in \mathcal{L}^1(\mathbb{R})$ as an improper integral, i.e., we investigate the partial Fourier integrals

$$s_u(x) = \int_0^{u} \left(\mathcal{F}f(t)\, e^{2\pi i t x} + \mathcal{F}f(-t)\, e^{-2\pi i t x}\right) dt = \int_{-u}^{u} \mathcal{F}f(t)\, e^{2\pi i t x}\, dt$$

for $u \to \infty$. It turns out that the conditions of convergence correspond exactly to those known from Fourier series.

For the sake of brevity, we frequently write the Fourier integral as $\int_{-\infty}^{\infty} \mathcal{F}f(t)\, e^{2\pi itx}\, dt$.

The partial Fourier integral $s_u(x)$ can be expressed

$$s_u(x) = \int_{-u}^{u} \mathcal{F}f(t)\, e^{2\pi itx}\, dt \;=\; \int_{-u}^{u} \left(e^{2\pi \mathrm{i}tx} \int_{\mathbb{R}} f(y)\, e^{-2\pi \mathrm{i}ty}\, dy \right) dt$$
$$= \int_{-u}^{u} \int_{\mathbb{R}} f(y)\, e^{2\pi \mathrm{i}t(x-y)}\, dy\, dt.$$

Since $f \otimes 1_{[-u,u]}$ belongs to $\mathcal{L}^1(\mathbb{R} \times \mathbb{R})$, and $(y,t) \mapsto e^{2\pi \mathrm{i}t(x-y)}$ is a bounded continuous function,

$$(y,t) \mapsto f(y)\, e^{2\pi \mathrm{i}t(x-y)}, \qquad y \in \mathbb{R},\ -u \le t \le u$$

belongs to $\mathcal{L}^1(\mathbb{R} \times [-u,u])$. We can therefore apply Fubini's theorem and get

$$s_u(x) = \int_{\mathbb{R}} \int_{-u}^{u} f(y)\, e^{2\pi \mathrm{i}t(x-y)}\, dt\, dy = \int_{\mathbb{R}} f(y) \int_{-u}^{u} e^{2\pi \mathrm{i}t(x-y)}\, dt\, dy,$$

which is summarized in

The partial Fourier integral for a function $f \in \mathcal{L}^1(\mathbb{R})$ is given by

$$s_u(x) = f * \mathcal{D}_u(x), \qquad u > 0, \tag{2.4.1}$$

where $\mathcal{D}_u(x) = \int_{-u}^{u} e^{2\pi \mathrm{i}tx}\, dt$ is the partial Fourier integral of $\int_0^\infty (e^{2\pi itx} + e^{-2\pi \mathrm{i}tx})\, dt$.

The functions $\mathcal{D}_u$ play an analogous role in the theory of Fourier integrals to Dirichlet's kernel $D_n(x)$ in the theory of Fourier series, see (1.5.5).

For $x \neq 0$, we find

$$\mathcal{D}_u(x) = \frac{\sin(2\pi ux)}{\pi x} = u\, \mathcal{D}_1(u\, x). \tag{2.4.2}$$

The function $\mathcal{D}_u$ is even. It does not belong to $\mathcal{L}^1(\mathbb{R})$, but the integral $\int_{-\infty}^{\infty} \mathcal{D}_u(x)\, dx = 2\int_0^\infty \mathcal{D}_u(x)\, dx$ is convergent with the value 1.

The last assertion follows trivially from the results on $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx$ in Sec. 2.2.

Note incidentally that $\mathcal{D}_u$ is a Fourier transform, namely of the indicator function $1_{[-u,u]}$.

We now formulate a result about pointwise convergence of the Fourier integral. It is analogous to Theorem 1.5.1.

Theorem 2.4.1 (Dini's test) *A sufficient condition for the Fourier integral of a function $f \in \mathcal{L}^1(\mathbb{R})$ to converge to s in the point $x \in \mathbb{R}$, i.e.,*

$$\int_{-\infty}^{\infty} \mathcal{F}f(t)\, e^{2\pi \mathrm{i} tx}\, dt = s,$$

is that

$$\int_0^\delta \left| \frac{f(x+y)+f(x-y)-2s}{y} \right| dy < \infty \qquad \textit{for a } \ \delta > 0. \tag{2.4.3}$$

Note that the condition is fulfilled for each $\delta > 0$ if it is fulfilled just for one value $\delta_0 > 0$.

Proof. Since $\mathcal{D}_u$ is an even function, we have

$$s_u(x) = \int_{\mathbb{R}} f(x-y)\, \mathcal{D}_u(y)\, dy = \int_0^\infty (f(x+y)+f(x-y))\, \mathcal{D}_u(y)\, dy.$$

The last integral is split as $\int_0^1 + \int_1^\infty$ and for the last of these we obtain

$$\frac{1}{\pi} \int_1^\infty \frac{f(x+y)+f(x-y)}{y} \sin(2\pi u y)\, dy \to 0 \qquad \text{for } \ u \to \infty,$$

by the Riemann-Lebesgue lemma because

$$y \to \begin{cases} \frac{f(x+y)+f(x-y)}{y} & \text{for } \ y \geq 1 \\ 0 & \text{for } \ y < 1 \end{cases}$$

belongs to $\mathcal{L}^1(\mathbb{R})$. The first integral can be written

$$\frac{1}{\pi} \int_0^1 \frac{f(x+y)+f(x-y)-2s}{y} \sin(2\pi u y)\, dy + \frac{2s}{\pi} \int_0^1 \frac{\sin(2\pi u y)}{y}\, dy,$$

and the first term tends to 0 for $u \to \infty$–now we use the assumption (2.4.3) with $\delta = 1$– while for the last term, we find

$$\frac{2s}{\pi} \int_0^{2\pi u} \frac{\sin y}{y}\, dy \to s \qquad \text{for } \ u \to \infty.$$

□

Application. The condition in Dini's test is fulfilled, with $s = f(x)$, if the function $f \in \mathcal{L}^1(\mathbb{R})$ is continuous at x as well as differentiable from the right and left at this point.

More generally, the condition is fulfilled, with $s = \frac{1}{2}(f(x+0)+f(x-0))$, if the function $f \in \mathcal{L}^1(\mathbb{R})$ has the limit $f(x+0) \in \mathbb{C}$ and $f(x-0) \in \mathbb{C}$ from the right and from the left in the point x, and if additionally

$$\frac{f(x+y)-f(x+0)}{y} \quad \text{and} \quad \frac{f(x-y)-f(x-0)}{-y}$$

have limits in $\mathbb{C}$ for $y \to 0_+$.

Under these assumptions the function

$$y \mapsto \frac{f(x+y) + f(x-y) - 2 \cdot \frac{1}{2}(f(x+0) + (f(x-0))}{y}$$

is bounded in an interval $]0, \delta]$, so (2.4.3) is satisfied.

Example 2.4.2 For $f = 1_{[-1,1]}$ we find $\mathcal{F}f(t) = \mathcal{D}_1(t)$. Dini's test can be applied and we get

$$\int_0^\infty \mathcal{D}_1(t) \left(e^{\mathrm{i}2\pi tx} + e^{-\mathrm{i}2\pi tx}\right) dt = \frac{2}{\pi} \int_0^\infty \frac{\sin(2\pi t)}{t} \cos(2\pi tx)\, dt$$
$$= \frac{2}{\pi} \int_0^\infty \frac{\sin t}{t} \cos(tx)\, dt = \begin{cases} 1 & \text{for} \quad |x| < 1 \\ \frac{1}{2} & \text{for} \quad |x| = 1 \\ 0 & \text{for} \quad |x| > 1. \end{cases}$$

2.5 The group algebra $L^1(\mathbb{R})$

By Theorem 2.3.3 1° it follows that convolution is a composition law in $C_c(\mathbb{R})$, and it is easy to see that $C_c(\mathbb{R})$ equipped with the compositions $+, *$ becomes a commutative ring. Because of the structure as a vector space it is in fact a commutative algebra. However, this space does not have a norm which makes it complete, so we should rather take the completion under the norm $\|\cdot\|_1$ leading to $\mathcal{L}^1(\mathbb{R})$. This latter space is also stable under convolution but with a little defect: The convolution of two functions from $\mathcal{L}^1(\mathbb{R})$ is only defined almost everywhere, but if we extend it to the null set $\mathbb{R} \setminus D(f * g)$ by the value 0, or by any other values making it a Borel function, it belongs to $\mathcal{L}^1(\mathbb{R})$, and its norm depends only on the values on the set $D(f * g)$. This small inconvenience will disappear when we go to the Banach space $L^1(\mathbb{R})$ of equivalence classes of functions defined almost everywhere and equal almost everywhere.

Although it is not surprising, we prove that convolution in $\mathcal{L}^1(\mathbb{R})$ satisfies the associative law.

Proposition 2.5.1 *If f, g, $h \in \mathcal{L}^1(\mathbb{R})$, then*

$$f * (g * h) = (f * g) * h$$

almost everywhere in $\mathbb{R}$.

Proof. For $x \in D(f * (g * h))$ we have

$$\begin{aligned}
(f * (g * h))(x) &= \int_{\mathbb{R}} f(x-y) \cdot (g * h)(y)\, dy \\
&= \int_{D(g*h)} \left(f(x-y) \int_{\mathbb{R}} g(y-z)\, h(z)\, dz \right) dy \\
&= \int_{D(g*h)} \int_{\mathbb{R}} f(x-y)\, g(y-z)\, h(z)\, dz\, dy.
\end{aligned}$$

In the same manner, for $x \in D((f * g) * h)$ we have

$$\begin{aligned}
((f * g) * h)(x) &= \int_{\mathbb{R}} (f * g)(x-z) \cdot h(z)\, dz \\
&= \int_{x-D(f*g)} (f * g)(x-z) \cdot h(z)\, dz \\
&= \int_{x-D(f*g)} \left(\int_{\mathbb{R}} f(x-z-y)\, g(y)\, dy \cdot h(z) \right) dz \\
&= \int_{x-D(f*g)} \left(\int_{\mathbb{R}} f(x-y)\, g(y-z)\, dy \cdot h(z) \right) dz \\
&= \int_{x-D(f*g)} \int_{\mathbb{R}} f(x-y)\, g(y-z)\, h(z)\, dy\, dz.
\end{aligned}$$

Note that $\mathbb{R} \setminus D(g * h)$ and $\mathbb{R} \setminus (x - D(f * g))$ have Lebesgue measure 0.

Therefore, by Fubini's theorem,

$$(f * (g * h))(x) = ((f * g) * h)(x) = \int_{\mathbb{R}^2} f(x-y)\, g(y-z)\, h(z)\, d(y,z),$$

for each $x \in D(f * (g * h)) \cap D((f * g) * h)$ for which

$$\int_{\mathbb{R}^2} |f(x-y)\, g(y-z)\, h(z)|\, d(y,z) < \infty,$$

i.e., for almost all $x \in \mathbb{R}$, since by Tonelli's theorem

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}^2} |f(x-y)\, g(y-z)\, h(z)|\, d(y,z)\, dx \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}} |f(x-y)\, g(y-z)\, h(z)|\, dx\, d(y,z) \\
&= \int_{\mathbb{R}^2} \left(|g(y-z)\, h(z)| \int_{\mathbb{R}} |f(x-y)|\, dx \right) d(y,z) \\
&= \|f\|_1 \int_{\mathbb{R}^2} |g(y-z)\, h(z)|\, d(y,z) = \|f\|_1 \|g\|_1 \|h\|_1 < \infty.
\end{aligned}$$

We have used that

$$(x, y, z) \mapsto f(x-y)\, g(y-z)\, h(z)$$

is a Borel function and the following simple assertion: If D_n is a finite or countable family of Borel sets such that $N_n = \mathbb{R} \setminus D_n$ are null sets then $\mathbb{R} \setminus \bigcap D_n$ is a null set. (We use it in fact for a family of 3 sets). $\square$

Let us summarize what we have obtained:

Theorem 2.5.2 *The Banach space $L^1(\mathbb{R})$ equipped with convolution is a commutative Banach algebra called the group algebra for $\mathbb{R}$.*

Let us point out that this Banach algebra does not have a unit element. In fact, assume that $e \in \mathcal{L}^1(\mathbb{R})$ is a representative of a unit element in $L^1(\mathbb{R})$, then $f * e = f$ almost everywhere for any function $f \in \mathcal{L}^1(\mathbb{R})$. If we take f to be the characteristic function of the interval $[0, 1]$, the convolution $f * e$ will be continuous by Theorem 2.3.3 2° because f is bounded, but f cannot be equal to a continuous function almost everywhere, and we get a contradiction.

2.6 Approximate units in $L^1(\mathbb{R})$

If A is a commutative Banach algebra without a unit element, like $L^1(\mathbb{R})$, it is useful to consider what is called an approximate unit.

Definition 2.6.1 Let A be a commutative Banach algebra with multiplication $\cdot$. A sequence of elements (k_n) (resp. a family of elements $(k_t)_{t>0}$) from A is called an approximate unit for A if

$$\lim_{n\to\infty} \|f \cdot k_n - f\| = 0 \quad (\text{resp.} \lim_{t\to\infty} \|f \cdot k_t - f\| = 0)$$

for all elements $f \in A$.

Definition 2.6.2 A sequence of functions (k_n) on $\mathbb{R}$ is called a *Dirac sequence*, if it has the following properties

(i) $\forall\, n \in \mathbb{N} : k_n \geq 0$,

(ii) $\forall\, n \in \mathbb{N} : \displaystyle\int_{\mathbb{R}} k_n(x)\, dx = 1$,

(iii) $\forall\, \delta > 0 : \displaystyle\int_{|x|>\delta} k_n(x)\, dx \to 0 \qquad \text{for } n \to \infty$

A family of functions $(k_t)_{t>0}$ is called a *Dirac family* if it has the analogous properties where n is replaced by t and $n \to \infty$ by $t \to \infty$.

Example 2.6.3 Let $k \in \mathcal{L}^1(\mathbb{R})$ satisfy $k \geq 0$ and $\displaystyle\int_{\mathbb{R}} k(x)\,dx = 1$. Then the sequence (k_n) defined by $k_n(x) = n\,k(nx)$ is a Dirac sequence. To see (iii) let $\delta > 0$ be given. Then

$$\int_{|x|>\delta} k(nx)\,n\,dx = \int_{|x|>n\delta} k(x)\,dx,$$

and $k \cdot 1_{\{x\,|\,|x|>n\delta\}} \to 0$ for $n \to \infty$, majorized by $|k| \in \mathcal{L}^1(\mathbb{R})$, so Lebesgue's theorem can be applied.

The same reasoning shows that $(k_t)_{t>0}$, with $k_t(x) = tk(tx)$ is a Dirac family.

Theorem 2.6.4 *Every Dirac sequence (k_n) and every Dirac family $(k_t)_{t>0}$ is an approximate unit for $L^1(\mathbb{R})$.*

<u>Proof.</u> Let $f \in \mathcal{L}^1(\mathbb{R})$ be given. In each point $x \in D(f * k_n)$, i.e., for almost all x, we have

$$\begin{aligned} f * k_n(x) - f(x) &= \int_{\mathbb{R}} f(x-y)\,k_n(y)\,dy - f(x)\int_{\mathbb{R}} k_n(y)\,dy \\ &= \int_{\mathbb{R}} (f(x-y) - f(x))\,k_n(y)\,dy, \end{aligned}$$

and thereby

$$|f * k_n(x) - f(x)| \leq \int_{\mathbb{R}} |f(x-y) - f(x)|\,k_n(y)\,dy.$$

Since $(x,y) \mapsto |f(x-y) - f(x)|\,k_n(y)$ is a Borel function in $\mathbb{R}^2$, we find using Tonelli's theorem

$$\begin{aligned} \|f * k_n - f\|_1 &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y) - f(x)|\,k_n(y)\,dy\,dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y) - f(x)|\,k_n(y)\,dx\,dy \\ &= \int_{\mathbb{R}} k_n(y)\|\tau_y f - f\|_1\,dy. \end{aligned}$$

We next use Theorem 2.3.1. To $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\tau_y f - f\|_1 < \frac{\varepsilon}{2} \qquad \text{for } |y| \leq \delta.$$

For every n we then have

$$\begin{aligned}
\int_{|y|\le\delta} k_n(y)\,\|\tau_y f - f\|_1\,dy &\le \int_{|y|\le\delta} \frac{\varepsilon}{2}\,k_n(y)\,dy \le \frac{\varepsilon}{2}\\
\int_{|y|>\delta} k_n(y)\,\|\tau_y f - f\|_1\,dy &\le \int_{|y|>\delta} 2\,\|f\|_1\,k_n(y)\,dy.
\end{aligned}$$

Choosing now $N \in \mathbb{N}$ such that the last integral is less than $\frac{\varepsilon}{2}$ for $n > N$, we have

$$\|f * k_n - f\|_1 < \varepsilon \qquad \text{for } \; n > N.$$

The proof for Dirac families is similar. $\square$

Theorem 2.6.4 can be extended:

Theorem 2.6.5 *Let (k_n) be a Dirac sequence on $\mathbb{R}$ and let $1 \le p < \infty$. Then*

$$\forall\, f \in \mathcal{L}^p(\mathbb{R}) : \|f * k_n - f\|_p \to 0 \qquad \textit{for } \; n \to \infty.$$

Proof. The case $p = 1$ has already been proved, so we can assume $1 < p < \infty$. Let q be determined by $\frac{1}{p} + \frac{1}{q} = 1$. Note that $f * k_n \in \mathcal{L}^p(\mathbb{R})$ by Theorem 2.3.3 4°

In each point $x \in D(f * k_n)$, i.e., for almost all x we have

$$\begin{aligned}
|f * k_n(x) - f(x)| &= \left| \int_{\mathbb{R}} (f(x-y) - f(x))\,k_n(y)\,dy \right|\\
&= \left| \int_{\mathbb{R}} (f(x-y) - f(x))\,(k_n(y))^{1/p} \cdot (k_n(y))^{1/q}\,dy \right|\\
&\le \left(\int_{\mathbb{R}} |f(x-y) - f(x)|^p\,k_n(y)\,dy \right)^{1/p} \left(\int_{\mathbb{R}} k_n(y)\,dy \right)^{1/q},
\end{aligned}$$

where the last inequality follows from Hölder's inequality. The last factor is 1, and we therefore get by integration with respect to x and using Tonelli's theorem

$$\begin{aligned}
\int_{\mathbb{R}} |f * k_n(x) - f(x)|^p\,dx &\le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x-y) - f(x)|^p\,k_n(y)\,dy \right) dx\\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y) - f(x)|^p\,k_n(y)\,dx\,dy\\
&= \int_{\mathbb{R}} k_n(y)\,\|\tau_y f - f\|_p^p\,dy.
\end{aligned}$$

The proof is completed similarly to the case $p = 1$. $\square$

Clearly, a similar result holds for Dirac families.

Theorem 2.6.5 cannot be extended to $p = \infty$. Instead, we have:

Theorem 2.6.6 *Let (k_n) be a Dirac sequence for $\mathbb{R}$. If $f : \mathbb{R} \to \mathbb{C}$ is bounded and uniformly continuous, then*

$$\|f * k_n - f\|_\infty \to 0 \qquad \text{for } n \to \infty.$$

Note that $f * k_n$ in the theorem is uniformly continuous and bounded so the norm $\|f * k_n - f\|_\infty$ is the same as the uniform norm. The same result holds for Dirac families and the proof of the theorem is left as Exercise **E 6.1**.

Exercises

E 6.1 Prove Theorem 2.6.6.

2.7 Summability of Fourier integrals

The partial Fourier integral

$$s_u(x) = \int_{-u}^{u} \mathcal{F}f(t)\, e^{2\pi \mathrm{i} tx}\, dt$$

for a function $f \in \mathcal{L}^1(\mathbb{R})$, is given by the convolution

$$s_u(x) = f * \mathcal{D}_u(x) = \int_{\mathbb{R}} f(x-y)\, \mathcal{D}_u(y)\, dy, \tag{2.7.1}$$

with

$$\mathcal{D}_u(y) = \int_{-u}^{u} e^{2\pi \mathrm{i} ty}\, dt = \int_0^u 2\cos(2\pi ty)\, dt, \tag{2.7.2}$$

cf. (2.4.1). The mean value of the partial Fourier integrals is defined by

$$\sigma_v(x) = \frac{1}{v}\int_0^v s_u(x)\, du = \frac{1}{v}\int_0^v \int_{\mathbb{R}} f(x-y)\, \mathcal{D}_u(y)\, dy\, du. \tag{2.7.3}$$

For fixed x, $(y,u) \mapsto f(x-y)\,\mathcal{D}_u(y)$ is integrable over $\mathbb{R} \times]0,v]$, as the product of the integrable function $(\tau_x Sf) \otimes 1_{]0,v]}$ and $\mathcal{D}_u(y) = u\,\mathcal{D}_1(u\,y)$, which is continuous and bounded in $\mathbb{R} \times]0,v]$ because

$$\mathcal{D}_1(z) = \begin{cases} \frac{\sin(2\pi z)}{\pi z} & \text{for} \quad z \neq 0 \\ 2 & \text{for} \quad z = 0. \end{cases}$$

By Fubini's theorem we then get

$$\sigma_v(x) = \frac{1}{v}\int_{\mathbb{R}} \int_0^v f(x-y)\, \mathcal{D}_u(y)\, du\, dy = \int_{\mathbb{R}} \left(f(x-y) \cdot \frac{1}{v}\int_0^v \mathcal{D}_u(y)\, du \right) dy,$$

which is summarized in

The mean value of the partial Fourier integrals for a function $f \in \mathcal{L}^1(\mathbb{R})$ is given by

$$\sigma_v(x) = f * F_v(x), \qquad x \in \mathbb{R},\ v > 0, \tag{2.7.4}$$

where

$$F_v(y) = \frac{1}{v}\int_0^v \mathcal{D}_u(y)\,du \tag{2.7.5}$$

is the mean value of the integrals $\displaystyle\int_0^u \left(e^{2\pi i t y} + e^{-2\pi i t y}\right) dt.$

The functions F_v play a role in the theory of Fourier integrals which corresponds to the role of Fejér's kernels in the theory of Fourier series.

The advantage in considering the summability of Fourier integrals instead of the convergence lies in the fact that the family $(F_v)_{v>0}$ has nicer properties than $(\mathcal{D}_u)_{u>0}$.

Proposition 2.7.1 *The family $(F_v)_{v>0}$ given by (2.7.5) is a Dirac family in the sense of Definition 2.6.2.*

Proof. Using (2.4.2) we find

$$F_v(y) = \frac{1}{v}\int_0^v \mathcal{D}_u(y)\,du = \frac{1}{v}\,\frac{1-\cos(2\pi v y)}{2\pi^2 y^2},$$

i.e., $F_v(y) = v\,F(vy)$ with

$$F(y) = F_1(y) = \frac{1-\cos(2\pi y)}{2\pi^2 y^2}.$$

Note that $y = 0$ is a removable singularity for F and $\lim_{y\to 0} F(y) = 1$. Clearly, $F \geq 0$ and $F \in \mathcal{L}^1(\mathbb{R})$, so all we need in order to apply the result of Example 2.6.3 is to prove that the integral of F is 1.

For $0 < w$, we obtain by partial integration

$$\begin{aligned}\int_0^w F(y)\,dy &= \int_0^w \frac{1-\cos(2\pi y)}{2\pi^2}\cdot\frac{1}{y^2}\,dy \\ &= \left[\frac{1-\cos(2\pi y)}{2\pi^2}\cdot\left(-\frac{1}{y}\right)\right]_{y=0}^{y=w} - \int_0^w \frac{\sin(2\pi y)}{\pi}\cdot\left(-\frac{1}{y}\right)dy.\end{aligned}$$

This yields

$$\int_0^w F(y)\,dy = -\frac{1-\cos(2\pi w)}{2\pi^2 w} + \frac{1}{\pi}\int_0^w \frac{\sin(2\pi y)}{y}\,dy,$$

hence for $w \to \infty$

$$\int_0^\infty F(y)\,dy = 0 + \lim_{w\to\infty} \frac{1}{\pi}\int_0^{2\pi w} \frac{\sin t}{t}\,dt = \frac{1}{2}$$

by Example 2.2.1. Since F is even we see that F has integral 1. $\square$

Theorem 2.7.2 *For every function $f \in \mathcal{L}^1(\mathbb{R})$, the Fourier integral*

$$\int_0^\infty \left(\mathcal{F}f(t)\,e^{2\pi \mathrm{i} tx} + \mathcal{F}f(-t)\,e^{-2\pi \mathrm{i} tx}\right) dt$$

is summable in $\mathcal{L}^1(\mathbb{R})$ with value f.

Proof. The claim is that $\|\sigma_v - f\|_1 \to 0$ for $v \to \infty$, where σ_v is the mean value of the truncated Fourier integral. This follows from $\sigma_v = f * F_v$, because $(F_v)_{v\in\mathbb{R}_+}$ is a Dirac family for $\mathbb{R}$, see Theorem 2.6.4. $\square$

Corollary 2.7.3 (Uniqueness theorem) *If $f, g \in \mathcal{L}^1(\mathbb{R})$ have the same Fourier transform, $\mathcal{F}f = \mathcal{F}g$, then $f = g$ almost everywhere.*

Proof. We know that $f * F_v = g * F_v$ for all $v > 0$ because it is the mean value of the partial Fourier integrals of the same function $\mathcal{F}f = \mathcal{F}g$. By the previous theorem $f * F_v$ converges to f in $\mathcal{L}^1(\mathbb{R})$ and similarly with g, hence $f = g$ almost everywhere. $\square$

Corollary 2.7.4 (Inversion theorem) *If the Fourier transform $\mathcal{F}f$ of a function $f \in \mathcal{L}^1(\mathbb{R})$ again belongs to $\mathcal{L}^1(\mathbb{R})$, i.e., if $\int_{\mathbb{R}} |\mathcal{F}f(t)|\,dt < \infty$, then*

$$f(x) = \int_{\mathbb{R}} \mathcal{F}f(t)\,e^{2\pi \mathrm{i} tx}\,dt$$

for almost all x. The equation holds for all x, if in addition f is continuous.

Proof. If $\mathcal{F}f \in \mathcal{L}^1(\mathbb{R})$, then the Fourier integral for f is convergent by Section 2.2 for each $x \in \mathbb{R}$ with sum

$$\int_{\mathbb{R}} \mathcal{F}f(t)\,e^{2\pi \mathrm{i} tx}\,dt = \mathcal{F}\mathcal{F}f(-x).$$

Being convergent, the Fourier integral is also summable with the same value, but we also know that $\|\sigma_v - f\|_1 \to 0$. We now use the following result from measure theory: *If a sequence g_n of integrable functions converge to an integrable function g in 1-norm, then a suitable subsequence of g_n converges to g almost everywhere.*

There exists therefore a sequence $v_n \to \infty$ such that $\sigma_{v_n}(x) \to f(x)$ for almost all x, hence

$$f(x) = \mathcal{F}\mathcal{F}f(-x)$$

for almost all x. The right-hand side is continuous and if f is also continuous, then the set of points where they disagree is an open null set, hence empty. $\square$

Example 2.7.5 The following formulas hold:

$$f(x) = e^{-|x|}, \qquad \mathcal{F}f(t) = \frac{2}{1+4\pi^2 t^2} \in \mathcal{L}^1(\mathbb{R}),$$

hence

$$e^{-|x|} = \int_{-\infty}^{\infty} \frac{2}{1+4\pi^2 t^2}\, e^{2\pi \mathrm{i} tx}\, dt,$$

or equivalently

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, e^{-\mathrm{i}ux}\, dx = e^{-|u|}.$$

Proof. From

$$\int_0^{\infty} e^{-xz}\, dx = \frac{1}{z} \qquad \text{for } \operatorname{Re} z > 0,$$

we obtain

$$\int_{-\infty}^{\infty} e^{-|x|}\, e^{-2\pi \mathrm{i} xt}\, dx = 2\int_0^{\infty} e^{-x}\cos(2\pi xt)\, dx = 2\operatorname{Re}\int_0^{\infty} e^{-x(1-2\pi \mathrm{i} t)}\, dx = \frac{2}{1+4\pi^2 t^2}.$$

Exercises

E 7.1 Let $f \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^{\infty}(\mathbb{R})$.

(i) Show that the Fourier integral (FI) for f is summable in x with value $f(x)$ if f is continuous in x.

(ii) Show that if f is continuous for all x in the interval $[a,b]$, then the FI is uniformly summable on $[a,b]$ with value $f(x)$, i.e.,

$$\sup_{x\in[a,b]} |\sigma_v(x) - f(x)| \to 0 \qquad \text{for } v \to \infty.$$

(iii) Show that if f is uniformly continuous on all of $\mathbb{R}$, then the FI is uniformly summable on $\mathbb{R}$ with value $f(x)$, i.e.,

$$\|\sigma_v - f\|_{\infty} \to 0 \qquad \text{for } v \to \infty.$$

E 7.2 Prove that for $v > 0$

$$\int_{-v}^{v} \left(1 - \frac{|x|}{v}\right) e^{-2\pi \mathrm{i} xt}\, dx = F_v(t) = \frac{1}{v}\,\frac{1 - \cos(2\pi vt)}{2\pi^2 t^2} = v\left(\frac{\sin(\pi vt)}{\pi vt}\right)^2,$$

i.e., that F_v is the Fourier transform of the “triangle function” in Figure 2.2, and that

$$\sigma_v(x) = f * F_v(x) = \int_{-v}^{v} \left(1 - \frac{|t|}{v}\right) \mathcal{F}f(t) e^{2\pi \mathrm{i} xt}\, dt$$

for $f \in \mathcal{L}^1(\mathbb{R})$.

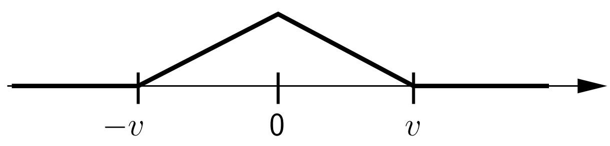

Figure 2.2: The triangle function, which is symmetric and linear from $(0, 1)$ to $(v, 0)$

2.8 Fourier transformation

The Fourier transformation $\mathcal{F}$, which transforms a function $f \in \mathcal{L}^1(\mathbb{R})$ into its Fourier transform $\mathcal{F}f$, is a mapping into the set $C_0(\mathbb{R})$ of continuous functions $g : \mathbb{R} \to \mathbb{C}$ with $g(t) \to 0$ for $|t| \to \infty$, cf. Theorem 2.1.2. Note that $C_0(\mathbb{R})$ is a Banach space under the uniform norm $\|g\|_\infty$. Under pointwise product it is also a commutative Banach algebra.

Theorem 2.8.1 *The Fourier transformation $\mathcal{F} : \mathcal{L}^1(\mathbb{R}) \to C_0(\mathbb{R})$ is linear, and*

$$\|\mathcal{F}f\|_\infty = \sup_t |\mathcal{F}f(t)| \le \|f\|_1 \qquad \text{for } f \in \mathcal{L}^1(\mathbb{R}).$$

More interestingly,

$$\mathcal{F}(f * g) = \mathcal{F}f \cdot \mathcal{F}g \qquad \text{for } f, g \in \mathcal{L}^1(\mathbb{R}). \tag{2.8.1}$$

Proof. Only (2.8.1) requires a proof. We have

$$\mathcal{F}(f * g)(t) = \int_{\mathbb{R}} f * g(x)\, e^{-2\pi \mathrm{i} xt}\, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x - y)\, g(y)\, e^{-2\pi \mathrm{i} xt}\, dy\, dx.$$

For fixed t, $(x, y) \mapsto f(x-y)\, g(y)\, e^{-2\pi i x t}$ is integrable over $\mathbb{R} \times \mathbb{R}$ because $f(x-y)\, g(y)$ is so and $e^{-2\pi i x t}$ is continuous and bounded. By Fubini's theorem we then have

$$\mathcal{F}(f * g)(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y)\, g(y)\, e^{-2\pi i x t}\, dx\, dy.$$

In the innermost integration, we substitute x by $x + y$ and obtain

$$\mathcal{F}(f * g)(t) = \int_{\mathbb{R}} \left(g(y)\, e^{-2\pi i y t} \int_{\mathbb{R}} f(x)\, e^{-2\pi i x t}\, dx \right) dy = \mathcal{F}f(t) \cdot \mathcal{F}g(t).$$

$\square$

Since $\mathcal{F}f = \mathcal{F}g$ when $f = g$ almost everywhere, we see that $\mathcal{F} : \mathcal{L}^1(\mathbb{R}) \to C_0(\mathbb{R})$ gives rise to a mapping $L^1(\mathbb{R}) \to C_0(\mathbb{R})$, which is also denoted $\mathcal{F}$. In other words, if $[f] \in L^1(\mathbb{R})$ denotes the equivalence class containing $f \in \mathcal{L}^1(\mathbb{R})$, we define $\mathcal{F}[f] = \mathcal{F}f$.

The Fourier transformation $\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is injective by the uniqueness theorem 2.7.3. The result of Theorem 2.8.1 can therefore be stated that $\mathcal{F}$ is an algebra isomorphism of the group algebra $L^1(\mathbb{R})$ onto a subalgebra $\mathcal{A}$ of $C_0(\mathbb{R})$.

We stress in particular that convolution of functions is transformed into ordinary product of funtions.

The Fourier transformation diminishes norm and distance:

$$\|\mathcal{F}(f)\|_\infty \le \|f\|_1, \quad \|\mathcal{F}(f) - \mathcal{F}(g)\|_\infty \le \|f - g\|_1 \quad \text{for } \ f, g \in L^1(\mathbb{R}).$$

Similar to Fourier series, we have $\|\mathcal{F}\| = 1$, because there exists $f \in \mathcal{L}^1(\mathbb{R})$ with $\|f\|_1 = 1$, $\|\mathcal{F}f\|_\infty = 1$, namely $f = (1/2)1_{[-1,1]}$.

At the end of Section 2.5 we have already pointed out that $L^1(\mathbb{R})$ does not have a unit element. We can also see this using Fourier transformation. In fact, if we assume that there exists an element $e \in \mathcal{L}^1(\mathbb{R})$ such that $e * f = f$ for all f, we get $\mathcal{F}e\mathcal{F}f = \mathcal{F}f$, and using $f(x) = e^{-|x|}$ from Example 2.7.5, we see that $\mathcal{F}e(x) = 1$ for all $x \in \mathbb{R}$ because $\mathcal{F}f$ does not vanish. On the other hand we know that $\mathcal{F}e$ tends to zero at infinity, and we get a contradiction.

The algebra $\mathcal{A} = \mathcal{F}(\mathcal{L}^1(\mathbb{R}))$ is dense in $C_0(\mathbb{R})$. This is a consequence of a version of the Stone-Weierstrass theorem, but we will not give any details of proof, because we see later that the Schwartz space $\mathcal{S}$ is contained in $\mathcal{A}$. Like for Fourier series, $\mathcal{A} \neq C_0(\mathbb{R})$, and there are functions in $\mathcal{A}$ which tend to zero at infinity arbitrarily slowly.

It does not seem to be possible to find a descriptive characterization of $\mathcal{A}$ as a subset of $C_0(\mathbb{R})$.

Defining $\tilde{f}(x) = \overline{f(-x)}$ for $f \in \mathcal{L}^1(\mathbb{R})$, we see that $\tilde{\ }$ is an involution in $\mathcal{L}^1(\mathbb{R})$ and $\mathcal{F}(\tilde{f}) = \overline{\mathcal{F}f}$. This can be expressed that the Fourier transformation respects (or commutes with) the involutions in $L^1(\mathbb{R})$ and $C_0(\mathbb{R})$.

2.8.1 Fourier transformation and differentiation

Roughly speaking, the behaviour of $f \in \mathcal{L}^1(\mathbb{R})$ at infinity is reflected in differentiability properties of the Fourier transform $\mathcal{F}f$: The quicker f tends to zero at infinity (meaning $\pm\infty$) the smoother is $\mathcal{F}f$. This is made precise in the next result.

Theorem 2.8.2 *Let $n \in \mathbb{N}$ and suppose that $f : \mathbb{R} \to \mathbb{C}$ and $x \mapsto x^n f(x)$ belongs to $\mathcal{L}^1(\mathbb{R})$. Then $\mathcal{F}f \in C^n(\mathbb{R})$, i.e., $\mathcal{F}f$ is n times differentiable with continuous n'th derivative $D^n(\mathcal{F}f)$, and*

$$D^j(\mathcal{F}f)(t) = (-2\pi\mathrm{i})^j \int_{\mathbb{R}} x^j\, f(x)\, e^{-2\pi\mathrm{i}xt}\, dx, \tag{2.8.2}$$

i.e.,

$$D^j(\mathcal{F}f) = (-2\pi\mathrm{i})^j \mathcal{F}(x^j f(x)), \quad j = 1, \ldots, n. \tag{2.8.3}$$

Proof. If $f : \mathbb{R} \to \mathbb{C}$ and $x \mapsto x\, f(x)$ belongs to $\mathcal{L}^1(\mathbb{R})$, then the Fourier transform $\mathcal{F}f$ is differentiable with

$$D(\mathcal{F}f)(t) = -2\pi\mathrm{i} \int_{\mathbb{R}} x\, f(x)\, e^{-2\pi\mathrm{i}xt}\, dx = -2\pi\mathrm{i}\, \mathcal{F}(xf(x))(t)$$

because

$$|D_t\left(f(x)e^{-2\pi\mathrm{i}xt}\right)| \le 2\pi|xf(x)|,$$

so we can apply a theorem about differentiation under the integral sign. The derivative $D(\mathcal{F}f)$ is continuous according to Theorem 2.1.2.

We next remark that $x \to x^j f(x)$ is integrable for each $j = 0, 1, \ldots, n$ because $|x^j f(x)| \le (1 + |x|^n)|f(x)|$, so we can differentiate the integral n times. $\square$

Theorem 2.8.3 *Let F be an indefinite integral of a function $f \in \mathcal{L}^1(\mathbb{R})$ and assume that also F is Lebesgue integrable. Then*

$$\mathcal{F}f(t) = 2\pi\mathrm{i}t\, \mathcal{F}F(t), \quad t \in \mathbb{R}.$$

Proof. By assumption

$$F(x) = c + \int_a^x f(y)\, dy,$$

hence for $x \to \infty$

$$F(x) - F(0) = \int_0^x f(y)\, dy \to \int_{\mathbb{R}_+} f(y)\, dy.$$

This shows that $F(x)$ has a limit for $x \to \infty$, and it has to be 0, since $\lim\limits_{x\to\infty} |F(x)| > 0$ will imply that $\int_0^\infty |F(x)|\,dx = \infty$ in contradiction with the integrability of F. Similarly, we have $F(x) \to 0$ for $x \to -\infty$.

By partial integration

$$\int_a^b f(x)\,e^{-2\pi\mathrm{i}xt}\,dx = \left[F(x)\,e^{-2\pi\mathrm{i}xt}\right]_{x=a}^{x=b} - \int_a^b F(x)\left(-2\pi\mathrm{i}t\,e^{-2\pi\mathrm{i}xt}\right)dx,$$

so for $a \to -\infty,\ b \to \infty$ we get

$$\mathcal{F}f(t) = 2\pi\mathrm{i}t\,\mathcal{F}F(t).$$

□

Theorem 2.8.4 *Let $n \in \mathbb{N}$ and assume that $f \in C^n(\mathbb{R})$ and $f, Df, \ldots, D^n f$ all belong to $\mathcal{L}^1(\mathbb{R})$. Then*

$$\mathcal{F}(D^j f)(t) = (2\pi\mathrm{i}\,t)^j\,\mathcal{F}f(t), \quad j = 1, \ldots, n.$$

Proof. We note that $D^{n-1}f$ is an integrable indefinite integral of $D^n f$, hence

$$\mathcal{F}(D^n f)(t) = 2\pi\mathrm{i}\,t\,\mathcal{F}(D^{n-1}f)(t),$$

and by repeated application of this we get the formula above. □

2.8.2 The Fourier transformation in the Schwartz space

For the sake of brevity, we will say about a function $\varphi : \mathbb{R} \to \mathbb{C}$ that $\varphi(x)$ tends rapidly to 0 for $|x| \to \infty$ if

$$\forall\, m \in \mathbb{N}: \quad x^m\,\varphi(x) \to 0 \qquad \text{for } |x| \to \infty.$$

The Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R})$ is defined as the set of functions $\varphi \in C^\infty(\mathbb{R})$ where

$$\forall\, n \in \mathbb{N}_0 \quad \forall\, m \in \mathbb{N}: \quad x^m\,D^n\varphi(x) \to 0 \qquad \text{for } |x| \to \infty.$$

The condition can be rephrased that φ as well as all its derivatives tend rapidly to 0.

The function space $\mathcal{S}$ is named after the French mathematician Laurent Schwartz (1915-2002), the creator of the theory of distributions. (Not to be mixed up with H.A. Schwarz (1843-1921) from the Cauchy-Schwarz inequality.)

For example, $x \mapsto e^{-x^2}$, $x \in \mathbb{R}$, will belong to $\mathcal{S}$, as well as $x \mapsto p(x)\, e^{-x^2}$, where p is a polynomial.

It is evident that $\mathcal{S}(\mathbb{R}) \subseteq \mathcal{L}^p(\mathbb{R})$ for every p, $1 \le p < \infty$, since already $x^2\, \varphi(x) \to 0$ for $|x| \to \infty$ implies that $(x^2)^p |\varphi(x)|^p$ is bounded, i.e., $|\varphi(x)|^p \le M(x^2)^{-p} \le M\, x^{-2}$ for $|x| \ge 1$.

It is also clear that $\mathcal{S}(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq \mathcal{L}^\infty(\mathbb{R})$.

We leave as an exercise to prove that $\mathcal{S}$ is a vector space and an algebra with respect to ordinary multiplication and with respect to convolution, see **E 8.2**.

We note: *If $\varphi \in \mathcal{S}$, then $D\varphi$ as well as $x\,\varphi(x)$ belong again to $\mathcal{S}$.*

In fact, $D^n(D\varphi) = D^{n+1}\varphi$ and $D^n(x\,\varphi(x)) = x\, D^n\varphi(x) + n\, D^{n-1}\varphi(x)$ tend rapidly to 0.

Using this and the results about differentiation and Fourier transformation we find the following key result:

Theorem 2.8.5 *The Fourier transform $\mathcal{F}\varphi$ of a function $\varphi \in \mathcal{S}$ belongs again to the Schwartz space $\mathcal{S}$.*

Proof.

1° Since $x \mapsto x^n\, \varphi(x)$ belongs to $\mathcal{S}$ and thereby to $\mathcal{L}^1(\mathbb{R})$ for $n = 0, 1, 2, \ldots$, we conclude from Theorem 2.8.2 that $\psi = \mathcal{F}\varphi$ belongs to $C^\infty(\mathbb{R})$ with

$$D^n\psi(t) = (-2\pi \mathrm{i})^n \int_{\mathbb{R}} x^n\, \varphi(x)\, e^{-2\pi \mathrm{i} xt}\, dx, \quad n = 0, 1, 2, \ldots.$$

In particular, we have $D^n\psi \in \mathcal{F}(\mathcal{S})$, namely $D^n\psi = \mathcal{F}\varphi_n$ with

$$\varphi_n(x) = (-2\pi \mathrm{i})^n\, x^n\, \varphi(x).$$

2° Since φ and thereby $D\varphi$, $D^2\varphi$, $\ldots$ belong to $\mathcal{S}$, i.e., in particular $\varphi \in C^\infty$ and φ, $D\varphi$, $D^2\varphi$, $\ldots \in \mathcal{L}^1(\mathbb{R})$, we conclude by Theorem 2.8.4 that

$$(2\pi \mathrm{i})^m\, t^m\, \mathcal{F}\varphi(t) = \mathcal{F}D^m\varphi(t), \quad m = 0, 1, 2, \ldots.$$

In particular, $t \mapsto t^m\, \mathcal{F}\varphi(t) \in \mathcal{F}(\mathcal{S})$.

3° For arbitrary n, m, we obtain by application of 2° on φ_n instead of φ that $t^m\, D^n\psi(t)$ belongs to $\mathcal{F}(\mathcal{S})$ and thereby to $\mathcal{F}(\mathcal{L}^1)$. Thus, by the Riemann-Lebesgue Lemma

$$t^m\, \mathcal{D}^n\psi(t) \to 0 \qquad \text{for } \ |t| \to \infty,$$

but this is precisely the requirement for $\psi = \mathcal{F}\varphi$ to belong to the Schwartz space. $\Box$

In addition to the Fourier transformation $\mathcal{F} : \mathcal{L}^1(\mathbb{R}) \to C_0(\mathbb{R})$ it is convenient to work with the co-Fourier transformation $\mathcal{F}^*$, defined for $f \in \mathcal{L}^1(\mathbb{R})$ by

$$\mathcal{F}^* f(t) = \int_{\mathbb{R}} f(x)\, e^{2\pi i x t}\, dx, \quad t \in \mathbb{R}. \tag{2.8.4}$$

Using the reflection $Sf(x) = f(-x)$, we clearly have

$$S(\mathcal{F}f) = \mathcal{F}^* f = \mathcal{F}(Sf).$$

Theorem 2.8.6 *The restriction of the Fourier transformation $\mathcal{F} : \mathcal{L}^1(\mathbb{R}) \to C_0(\mathbb{R})$ to the Schwartz space $\mathcal{S}$ is a bijective mapping onto $\mathcal{S}$. The inverse mapping is the restriction of the co-Fourier transformation $\mathcal{F}^*$ to $\mathcal{S}$.*

Proof. According to Theorem 2.8.5 both restrictions are mappings into $\mathcal{S}$. We shall show that they are inverse of each other, i.e., that

$$\mathcal{F}^*(\mathcal{F}\varphi) = \varphi \qquad \text{and} \qquad \mathcal{F}(\mathcal{F}^*\varphi) = \varphi \qquad \text{for } \varphi \in \mathcal{S}.$$

The first equation is a direct application of the Inversion Theorem 2.7.4, and the second is obtained from the first, since

$$\mathcal{F}(\mathcal{F}^*\varphi) = \mathcal{F}(\mathcal{S}(\mathcal{F}\varphi)) = \mathcal{F}^*(\mathcal{F}\varphi).$$

$\Box$

Since $\mathcal{S} \subseteq \mathcal{L}^2(\mathbb{R})$, we have in the Schwartz space $\mathcal{S}$ the usual scalar product and norm inherited from $\mathcal{L}^2$

$$\langle \varphi, \psi \rangle = \int_{\mathbb{R}} \varphi(x)\, \overline{\psi(x)}\, dx, \quad \|\varphi\|_2 = \langle \varphi, \varphi \rangle^{1/2} = \left(\int_{\mathbb{R}} |\varphi(x)|^2\, dx \right)^{1/2}.$$

Theorem 2.8.7 *The Fourier transformation $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a unitary mapping, i.e., linear and bijective with*

$$\langle \mathcal{F}\varphi, \mathcal{F}\psi \rangle = \langle \varphi, \psi \rangle \quad \textit{for } \varphi,\, \psi \in \mathcal{S}, \tag{2.8.5}$$

in particular,

$$\|\mathcal{F}\varphi\|_2 = \|\varphi\|_2 \quad \textit{for } \varphi \in \mathcal{S}. \tag{2.8.6}$$

Furthermore,

$$\langle \mathcal{F}\varphi, \psi \rangle = \langle \varphi, \mathcal{F}^*\psi \rangle \quad \textit{for } \varphi,\, \psi \in \mathcal{S}. \tag{2.8.7}$$

The identity (2.8.6) is sometimes also called Parseval's identity because of the analogy to Parseval's identity for Fourier series:

$$\frac{1}{2\pi}\int_0^{2\pi} |\varphi(t)|^2\,dt = \sum_{-\infty}^{\infty} |c_n|^2$$

Proof. We begin with the last equation:

$$\langle \mathcal{F}\varphi, \psi\rangle = \int_{\mathbb{R}} \mathcal{F}\varphi(t)\,\overline{\psi(t)}\,dt = \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x)\,\overline{\psi(t)}\,e^{-2\pi \mathrm{i}xt}\,dx\,dt.$$

Since $\varphi\otimes\overline{\psi}$ and thereby $(x,t)\mapsto \varphi(x)\,\overline{\psi(t)}\,e^{-2\pi\mathrm{i}xt}$ belong to $\mathcal{L}^1(\mathbb{R}\times\mathbb{R})$, we have

$$\langle \mathcal{F}\varphi, \psi\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x)\,\overline{\psi(t)\,e^{2\pi\mathrm{i}xt}}\,dt\,dx = \int_{\mathbb{R}} \varphi(x)\,\overline{\mathcal{F}^*\,\psi(x)}\,dx = \langle \varphi, \mathcal{F}^*\psi\rangle.$$

Using this we find

$$\langle \mathcal{F}\varphi, \mathcal{F}\psi\rangle = \langle \varphi, \mathcal{F}^*\mathcal{F}\psi\rangle = \langle\varphi,\psi\rangle.$$

□

We note that $\mathcal{F}^* : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is of course also unitary.

2.8.3 The Fourier-Plancherel transformation

For $f\in\mathcal{L}^1(\mathbb{R})$ the Fourier transform $\mathcal{F}f$ is given by

$$\mathcal{F}f(t) = \int_{\mathbb{R}} f(x)\,e^{-2\pi\mathrm{i}xt}\,dx, \qquad t\in\mathbb{R}.$$

For $f\in\mathcal{L}^2(\mathbb{R})$, this definition cannot be applied in general, since $\mathcal{L}^2(\mathbb{R})\not\subseteq\mathcal{L}^1(\mathbb{R})$. We shall introduce a Fourier transformation on $\mathcal{L}^2(\mathbb{R})$, also called the Fourier-Plancherel transformation, and which is an extension of the ordinary Fourier transformation defined on $\mathcal{L}^1(\mathbb{R})\cap\mathcal{L}^2(\mathbb{R})$. The Fourier transformation will be an isometry of the Hilbert space $L^2(\mathbb{R})$. This result is due to Michel Plancherel, a Swiss mathematician (1885 – 1967) (Rendiconti del Circolo Matematico di Palermo 30 (1910)).

We build on the results in Subsection 2.8.2 since we will introduce the Fourier-Plancherel transformation by extending the Fourier transformation in the Schwartz space by continuity.

First, some general lemmas.

Lemma 2.8.8 *A linear mapping T of a (semi)normed vector space $\mathcal{V}$ into a (semi)-normed vector space $\mathcal{W}$ is continuous, if and only if there exists a constant $M>0$ such that*

$$\|T\,x\| \le M\,\|x\| \text{ for all } x\in\mathcal{V} \tag{2.8.8}$$

Proof. It is evident that the condition even implies uniform continuity, since

$$\|Tx - Ty\| = \|T(x-y)\| \leq M\,\|x-y\|.$$

Conversely, if T is continuous at 0, then for $\varepsilon = 1$ there exists a $\delta > 0$ such that

$$\|Tx - 0\| \leq 1 \qquad \text{for } \|x\| < \delta.$$

We claim that (2.8.8) holds for $M = 1/\delta$. In fact, for any $x \in \mathcal{V}$ and any $n \in \mathbb{N}$ we have

$$\|(\delta/(\|x\| + 1/n))x\| = \frac{\|x\|}{\|x\| + 1/n}\delta < \delta,$$

hence

$$\left\| T\left(\frac{\delta}{\|x\| + 1/n}x\right)\right\| \leq 1,$$

or

$$\|Tx\| \leq \frac{\|x\| + 1/n}{\delta}.$$

Letting $n \to \infty$ we get the result. $\square$

Lemma 2.8.9 *Let $\mathcal{V}$ be a (semi)normed vector space, and let $S : \mathcal{U} \to \mathcal{W}$ be a continuous, linear mapping of a subspace $\mathcal{U} \subseteq \mathcal{V}$, that is dense in $\mathcal{V}$, into a Banach space $\mathcal{W}$. Then there exists one and only one extension of S to a continuous mapping $T : \mathcal{V} \to \mathcal{W}$, and it is linear.*

Proof. There is at most one extension of S to a continuous mapping $T : \mathcal{V} \to \mathcal{W}$, because for each $x \in \mathcal{V}$ there exists a sequence $u_1, u_2, \ldots$ from $\mathcal{U}$ with $u_n \to x$, and thus necessarily $Tx = \lim_n Tu_n = \lim_n Su_n$.

We split the proof of the existence into a series of steps.

1° *If $u_1, u_2, \ldots \in \mathcal{U}$ converges in $\mathcal{V}$, then $Su_1, Su_2, \ldots$ converges in $\mathcal{W}$.*

In fact, by Lemma 2.8.8 applied to S the exists a constant M such that

$$\|Su_n - Su_m\| \leq M\|u_n - u_m\|$$

showing that $Su_1, Su_2, \ldots$ is a Cauchy sequence, hence convergent because $\mathcal{W}$ is assumed to be complete.

2° *If $u_1, u_2, \ldots \in \mathcal{U}$ and $v_1, v_2, \ldots \in \mathcal{U}$ converge to the same element $x \in \mathcal{V}$, then $\lim_n Su_n = \lim_n Sv_n$.*

In fact, according to 1°, the mixed sequence $Su_1, Sv_1, Su_2, Sv_2, \ldots$ is convergent and therefore the subsequences (Su_n) and (Sv_n) have the same limit.

3° *The mapping $T : \mathcal{V} \to \mathcal{W}$ is well-defined by*

$$Tx = \lim_n Su_n \quad \text{if } \ u_1, u_2, \ldots \to x, \quad x \in \mathcal{V}, \ u_n \in \mathcal{U}.$$

Here we used that $\mathcal{U}$ is dense in $\mathcal{V}$, so for any $x \in \mathcal{V}$ we can choose a sequence (u_n) from $\mathcal{U}$ converging to x, and then $Tx = \lim_n Su_n$ is independent of the choice of (u_n).

4° *T is an extension of S.*

In fact, with $u_n = u \in \mathcal{U}$, we have $Tu = \lim\limits_n Su_n = Su$.

5° *T is linear.*

To show for example that $T(x+y) = Tx+Ty$ for x, $y \in \mathcal{V}$, we choose u_n, $v_n \in \mathcal{U}$ such that $u_n \to x$ and $v_n \to y$. Then $u_n + v_n \to x + y$, and consequently

$$T(x+y) = \lim_n S(u_n + v_n) = \lim_n (Su_n + Sv_n) = Tx + Ty.$$

6° *T is continuous.*

For this, we use Lemma 2.8.8. If

$$\|Su\| \le M\,\|u\| \qquad \text{for } \ u \in \mathcal{U},$$

then

$$\|Tx\| \le M\,\|x\| \qquad \text{for } \ x \in \mathcal{V};$$

because if $u_n \to x$, $u_n \in \mathcal{U}$, we have $Tx = \lim\limits_n Su_n$ and finally

$$\|Tx\| = \lim_n \|Su_n\| \le \lim_n M\,\|u_n\| = M\,\|x\|,$$

where we have used the continuity of the (semi)norm. $\square$

The Schwartz space $\mathcal{S}(\mathbb{R})$ is dense in $\mathcal{L}^2(\mathbb{R})$, since already the functions $\varphi \in C^\infty(\mathbb{R})$ with compact support are dense in $\mathcal{L}^p(\mathbb{R})$ for every p, $1 \le p < \infty$.

We now apply Lemma 2.8.9 to the "Fourier transformation" $f \to [\mathcal{F}f]$ defined on the dense subspace $\mathcal{S}$ of $\mathcal{L}^2(\mathbb{R})$ with values in the Hilbert space $L^2(\mathbb{R})$, where $[\mathcal{F}f]$ denotes the equivalence class containing the Schwartz function $\mathcal{F}f$. This mapping is continuous by (2.8.6).

The continuous linear extension $\mathcal{F}_P : \mathcal{L}^2(\mathbb{R}) \to L^2(\mathbb{R})$ is called the Fourier-Plancherel transformation. By continuity of the norm, it is an isometry like $\mathcal{F}$:

$$\|\mathcal{F}_P f\|_2 = \|f\|_2, \ \text{for } f \in \mathcal{L}^2(\mathbb{R}).$$

In particular $\mathcal{F}_P f = \mathcal{F}_P g$ if f, $g \in \mathcal{L}_2(\mathbb{R})$ satisfy $\|f-g\|_2 = 0$. Therefore $\mathcal{F}_P$: $\mathcal{L}^2(\mathbb{R}) \to L^2(\mathbb{R})$ gives rise to a mapping $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, which is also called the Fourier-Plancherel transformation and denoted $\mathcal{F}_P$. We thus have $\mathcal{F}_P[f] = \mathcal{F}_P f$.

The co-Fourier-Plancherel transformation $\mathcal{F}_P^*$ is defined in a corresponding way based on $\mathcal{F}^* : \mathcal{S} \to L^2(\mathbb{R})$.

Theorem 2.8.10 (The Fourier-Plancherel transformation) *The Fourier-Plancherel transformation $\mathcal{F}_P : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a unitary mapping of $L^2(\mathbb{R})$ onto itself, i.e., it is linear, bijective and preserves the scalar product,*

$$\langle \mathcal{F}_P f, \mathcal{F}_P g\rangle = \langle f, g\rangle \qquad \text{for } f,\, g \in L^2(\mathbb{R}). \tag{2.8.9}$$

In particular, Plancherel's equation holds:

$$\|\mathcal{F}_P f\|_2 = \|f\|_2 \qquad \text{for } f \in L^2(\mathbb{R}). \tag{2.8.10}$$

The inverse mapping is the co-Fourier-Plancherel transformation

$$\mathcal{F}_P^* : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Proof. Let us verify (2.8.9). If $\varphi_n \to f$ and $\psi_n \to g$ in 2-mean, φ_n, $\psi_n \in \mathcal{S}$, we have

$$[\mathcal{F}\varphi_n] \to \mathcal{F}_P f, \qquad [\mathcal{F}\psi_n] \to \mathcal{F}_P g \qquad \text{in } L^2(\mathbb{R}),$$

and thereby

$$\langle \mathcal{F}_P f, \mathcal{F}_P g\rangle = \lim \langle \mathcal{F}\varphi_n, \mathcal{F}\psi_n\rangle = \lim \langle \varphi_n, \psi_n\rangle = \langle f, g\rangle.$$

The mappings $\mathcal{F}_P : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $\mathcal{F}_P^* : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are the inverse of each other, i.e.,

$$\mathcal{F}_P^*\mathcal{F}_P f = f \qquad \text{and} \qquad \mathcal{F}_P\mathcal{F}_P^* f = f \qquad \text{for } f \in L^2(\mathbb{R}).$$

This is clear because if two continuous mappings agree on a dense set they are identical. □

In the definition of the Fourier-Plancherel transformation $\mathcal{F}_P : \mathcal{L}_2(\mathbb{R}) \to L_2(\mathbb{R})$, we required only agreement,

$$\mathcal{F}_P f = [\mathcal{F} f],$$

with the Fourier transformation $\mathcal{F} : \mathcal{L}^1(\mathbb{R}) \to C_0(\mathbb{R})$ for functions f belonging to the Schwartz space. However, we shall see that the equation is fulfilled merely when both sides have a meaning, i.e., for every function $f \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$.

Theorem 2.8.11 *For every function $f \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$, we have $\mathcal{F}_P f = [\mathcal{F} f]$.*

Proof.

1° First, we consider functions $f \in \mathcal{L}^2(\mathbb{R})$ with compact support, i.e., where the closure of $\left\{x \,\middle|\, f(x) \neq 0\right\}$ is compact.

Let (k_n) be a Dirac sequence for $\mathbb{R}$ of functions $k_n \in C^\infty(\mathbb{R})$ with compact support. We can for example take $k_n(x) = n\, k(nx)$, where $k \geq 0$ is a function belonging to $C^\infty(\mathbb{R})$, with compact support and $\int_{\mathbb{R}} k(x)\, dx = 1$.

Then $f * k_n$ will belong to $C^\infty(\mathbb{R})$ by Theorem 2.3.4 and will have compact support, cf. Proposition 2.3.2 4°. In particular, $f * k_n \in \mathcal{S}(\mathbb{R})$ and thereby

$$\mathcal{F}_P(f * k_n) = [\mathcal{F}(f * k_n)].$$

However, by Theorem 2.6.5 for $p = 1$ we have $\|f * k_n - f\|_1 \to 0$, and thereby

$$\|\mathcal{F}(f * k_n) - \mathcal{F} f\|_\infty \to 0.$$

On the other hand by Theorem 2.6.5 for $p = 2$ we have $\|f * k_n - f\|_2 \to 0$, hence

$$\|\mathcal{F}_P(f * k_n) - \mathcal{F}_P f\|_2 \to 0.$$

If g is a representative for $\mathcal{F}_P f$, then $\mathcal{F}(f * k_n)$ will thus converge pointwise (even uniformly) to $\mathcal{F} f$ and in the 2-mean to g. From this follows $g = \mathcal{F} f$ almost everywhere, i.e., $\mathcal{F}_P f = [g] = [\mathcal{F} f]$.

2° For arbitrary $f \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$, we set $f_n = f \cdot 1_{[-n,n]}$. Since $f_n \to f$, numerically majorized by $|f|$, we have $\|f_n - f\|_1 \to 0$ as well as $\|f_n - f\|_2 \to 0$, and thereby

$$\|\mathcal{F} f_n - \mathcal{F} f\|_\infty \to 0 \qquad \text{and} \qquad \|\mathcal{F}_P f_n - \mathcal{F}_P f\|_2 \to 0.$$

By 1° we have $\mathcal{F}_P f_n = [\mathcal{F} f_n]$. If g is a representative for $\mathcal{F}_P f$, then $\mathcal{F} f_n$ will converge pointwise to $\mathcal{F} f$ and in the 2-mean to g, and it follows that $g = \mathcal{F} f$ almost everywhere, i.e., $\mathcal{F}_P f = [\mathcal{F} f]$. $\square$

Remark 2.8.12 As a byproduct of the proof we can give a more explicit characterization of $\mathcal{F}_P f$ for $f \in \mathcal{L}^2(\mathbb{R})$, namely

$$\mathcal{F}_P f = \lim_n\, [\mathcal{F} f_n] \quad \text{in } L^2(\mathbb{R}),$$

where $f_n = f \cdot 1_{[-n,n]}$, i.e.,

$$\mathcal{F} f_n(t) = \int_{-n}^{n} f(x)\, e^{-2\pi i t x}\, dx,$$

which belongs to $C_0(\mathbb{R})$ because f_n is integrable. By the isometric property we have

$$\|\mathcal{F}_P f - [\mathcal{F} f_n]\,\|_2 = \|\mathcal{F}_P f - \mathcal{F}_P f_n\|_2 = \|f - f_n\|_2 \to 0.$$

Exercises

E 8.1 Prove that the Schwartz space $\mathcal{S}$ is a vector space stable under product and convolution.

E 8.2 Let $f \in L^2(\mathbb{R})$ and $a > 0$. Show that

$$\int_{-a}^{a} f(x+y)\,dy = \int_{-\infty}^{\infty} \mathcal{F}f(t)\,e^{2\pi \mathrm{i}tx}\,\frac{\sin(2\pi ta)}{\pi t}\,dt, \qquad x \in \mathbb{R}.$$

E 8.3 Prove the following theorem about metric spaces, where we denote the metric of a space X by d_X, i.e., $d_X(a,b)$ is the distance between $a, b \in X$.

Theorem. *Let $f : A \to (Y, d_Y)$ be a mapping of a subset A of a metric space (X, d_X) and assume that (Y, d_Y) is complete and that f is uniformly continuous on the subspace (A, d_X). Then f can be uniquely extended to a continuous mapping $\tilde{f} : \overline{A} \to (Y, d_Y)$, and the extension $\tilde{f}$ is again uniformly continuous.*

2.9 Fourier transformation of measures

In the following, we will go back to the definition of the Fourier transform via the formula

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(x)\,e^{-\mathrm{i}xt}\,dx, \qquad \text{for } f \in \mathcal{L}^1(\mathbb{R}), \tag{2.9.1}$$

cf. Remark 2.1.1. With this definition the inversion theorem and Plancherel's theorem can be formulated

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(t)\,e^{\mathrm{i}tx}\,dt, \quad \text{if } \widehat{f} \in \mathcal{L}^1(\mathbb{R}) \tag{2.9.2}$$

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(t)|^2\,dt \quad \text{if } f \in \mathcal{L}^2(\mathbb{R}). \tag{2.9.3}$$

By $\mathbb{M}_+(\mathbb{R})$ we understand the set of positive Radon measures on $\mathbb{R}$. We recall that a positive Borel measure μ on $\mathbb{R}$, i.e., a positive measure on the Borel sigma-algebra $\mathbb{B}(\mathbb{R})$ is called a Radon measure if it is finite on compact sets or equivalently finite on bounded Borel subsets of $\mathbb{R}$. Lebesgue measure is a Radon measure, but the measure $\mu = |x|^{-1}\,dx$ is a Borel measure, which is not a Radon measure, because $\mu([-1,1]) = \infty$.

By $\mathbb{M}_+^b(\mathbb{R})$, we understand the set of positive Borel measures μ of finite total mass. Such measures μ are in particular finite on bounded sets, hence Radon measures. In formulas:

$$\mathbb{M}_+^b(\mathbb{R}) \subset \mathbb{M}_+(\mathbb{R}).$$

For $\mu \in \mathbb{M}^b_+(\mathbb{R})$ we introduce the Fourier transform $\widehat{\mu} : \mathbb{R} \to \mathbb{C}$ by the formula

$$\widehat{\mu}(t) = \int e^{-\mathrm{i}xt}\, d\mu(x), \tag{2.9.4}$$

(note that this formula requires μ to be of finite total mass) and we see that $\widehat{\mu}$ is a uniformly continuous bounded function with

$$|\widehat{\mu}(t)| \le \widehat{\mu}(0) = \mu(\mathbb{R}), \qquad t \in \mathbb{R}. \tag{2.9.5}$$

Concerning uniform continuity:

$$\widehat{\mu}(t+h) - \widehat{\mu}(t) = \int e^{-\mathrm{i}xt} \left(e^{-\mathrm{i}xh} - 1\right) d\mu(x),$$

hence

$$|\widehat{\mu}(t+h) - \widehat{\mu}(t)| \le \int \left|e^{-\mathrm{i}hx} - 1\right| \, d\mu(x),$$

which approaches 0 for $h \to 0$ by Lebesgue's theorem on dominated convergence.

The function $\hat{\mu}$ is also called the Fourier-Stieltjes transform of μ or (in particular in probability theory) the characteristic function of μ.

Example 2.9.1 a) For $\mu = \varepsilon_a$ we find $\widehat{\mu}(t) = e^{-\mathrm{i}ta}$.

b) For $\mu = f(x)\, dx$, where $f \in \mathcal{L}^1_+(\mathbb{R})$, we have $\widehat{\mu}(t) = \widehat{f}(t)$, using definition (2.9.1).

Theorem 2.9.2 (Uniqueness Theorem) *If two measures μ, $\nu \in \mathbb{M}^b_+(\mathbb{R})$ have the same Fourier-Stieltjes transform $\widehat{\mu}(t) = \widehat{\nu}(t)$ for $t \in \mathbb{R}$, then $\mu = \nu$.*

Proof. For $f \in \mathcal{S}$, we have by the inversion theorem

$$\begin{aligned}\int f(x)\, d\mu(x) &= \int \left(\frac{1}{2\pi} \int \widehat{f}(t)\, e^{\mathrm{i}tx}\, dt\right) d\mu(x) = \frac{1}{2\pi} \int \widehat{f}(t)\, \overline{\widehat{\mu}(t)}\, dt \\ &= \frac{1}{2\pi} \int \widehat{f}(t)\, \overline{\widehat{\nu}(t)}\, dt = \int f(x)\, d\nu(x),\end{aligned}$$

and then it is easy to see that $\mu = \nu$. For example, for an interval $[a, b]$, we can find a sequence $(f_n) \in \mathcal{S}$ with compact support such that $f_n \searrow 1_{[a,b]}$, whence $\mu([a, b]) = \nu([a, b])$, cf. Figure 2.3.

For μ, $\nu \in \mathbb{M}^b_+(\mathbb{R})$, we introduce the convolution $\mu * \nu \in \mathbb{M}^b_+(\mathbb{R})$ as the image measure $p(\mu \otimes \nu)$ of $\mu \otimes \nu$ under $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $p(x, y) = x + y$. We have

$$\mu * \nu(E) = \mu \otimes \nu \left(\left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, x + y \in E \right\} \right). \tag{2.9.6}$$

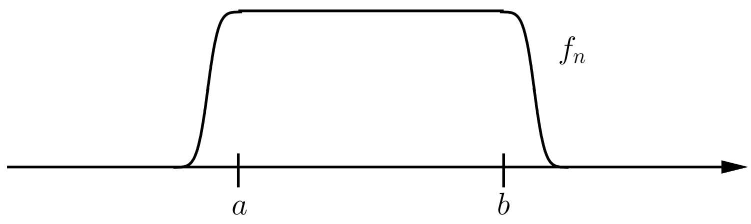

Figure 2.3: Approximation of $1_{[a,b]}$ by C^∞-functions with compact support

We see immediately that $\mu * \nu(\mathbb{R}) = \mu(\mathbb{R})\,\nu(\mathbb{R})$ and $\mu * \nu = \nu * \mu$.The measure ε_0 is a neutral element with respect to convolution: $\mu * \varepsilon_0 = \mu$.

From a theorem about integration with respect to an image measure, we obtain

$$\int_{\mathbb{R}} f\, d\mu * \nu = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y)\, d\mu(x)\, d\nu(y) \tag{2.9.7}$$

for positive Borel functions and functions $f \in \mathcal{L}^1(\mu * \nu)$.

Applied to $f(x) = e^{-\mathrm{i}tx}$, we obtain

$$\widehat{\mu * \nu}\,(t) = \widehat{\mu}(t)\,\widehat{\nu}(t), \tag{2.9.8}$$

i.e., under Fourier transformation, convolution is transformed into a product.

If $\mu = f(x)\,dx$, $\nu = g(x)\,dx$, $f,\, g \in \mathcal{L}^1_+(\mathbb{R})$, then $\mu * \nu = f * g(x)\,dx$, by the same argument as for convolution on the unit circle, i.e., for $\varphi \in C_c(\mathbb{R})$, we have

$$\int \varphi\, d\mu * \nu = \int \left(\int \varphi(x+y)\, f(x)\, dx \right) g(y)\, dy = \int \left(\int \varphi(x)\, f(x-y)\, dx \right) g(y)\, dy$$
$$= \int \left(\varphi(x) \int f(x-y)\, g(y)\, dy \right) dx = \int \varphi(x)\, f * g(x)\, dx.$$

Insertion on the vague and weak topologies on $\mathbb{M}_+(\mathbb{R})$ and $\mathbb{M}^b_+(\mathbb{R})$.

Definition 2.9.3 We say that (μ_n) from $\mathbb{M}_+(\mathbb{R})$ converges vaguely to $\mu \in \mathbb{M}_+(\mathbb{R})$ if

$$\lim_{n\to\infty} \int f\, d\mu_n = \int f\, d\mu \qquad \text{for all } \ f \in C_c(\mathbb{R}),$$

where $C_c(\mathbb{R})$ denotes the set of continuous functions $f : \mathbb{R} \to \mathbb{C}$ with compact support.

We say that (μ_n) from $\mathbb{M}^b_+(\mathbb{R})$ converges weakly to $\mu \in \mathbb{M}^b_+(\mathbb{R})$ if

$$\lim_{n\to\infty} \int f\, d\mu_n = \int f\, d\mu \qquad \text{for all } \ f \in C_b(\mathbb{R}),$$

where $C_b(\mathbb{R})$ denotes the set of bounded continuous functions $f : \mathbb{R} \to \mathbb{C}$.

Vague convergence corresponds to the initial topology on $\mathbb{M}_+(\mathbb{R})$ for the family of mappings $\mu \mapsto \int f\, d\mu$, where f is arbitrary in $C_c(\mathbb{R})$, i.e., the coarsest topology in which all these mappings are continuous.

Weak convergence corresponds to the initial topology on $\mathbb{M}_+^b(\mathbb{R})$ for the family of mappings $\mu \mapsto \int f\, d\mu$, where f is arbitrary in $C_b(\mathbb{R})$, i.e., the coarsest topology in which all these mappings are continuous.

On the set $\mathbb{M}_+^b(\mathbb{R})$ we can also consider the restriction of the vague topology. Since the vague topology on $\mathbb{M}_+^b(\mathbb{R})$ is the coarsest topology making the mappings $\mu \to \int f\, d\mu$ continuous for $f \in C_c(\mathbb{R})$ and these mappings are automatically continuous in the weak topology, we clearly have that the vague topology is coarser than the weak topology, or with other words the more obvious statement:

If $\mu_n \in \mathbb{M}_+^b(\mathbb{R})$ converges weakly to $\mu \in \mathbb{M}_+^b(\mathbb{R})$ then it automatically converges vaguely to μ.

We recall that $C_0(\mathbb{R})$ denotes the set of all continuous functions $f : \mathbb{R} \to \mathbb{C}$ vanishing at infinity, i.e.,

$$\forall \varepsilon > 0 \;\; \exists A > 0 \;\; : |f(x)| < \varepsilon \text{ for all } x \in \mathbb{R} \setminus [-A, A].$$

Lemma 2.9.4 *Let $\alpha > 0$ and let (μ_n) be a sequence from $M_+^b(\mathbb{R})$ converging vaguely to $\mu \in M_+(\mathbb{R})$. If $\mu_n(\mathbb{R}) \leq \alpha$ for all n then $\mu(\mathbb{R}) \leq \alpha$ and*

$$\lim_{n\to\infty} \int f\, d\mu_n = \int f\, d\mu \;\; \textit{for all } f \in C_0(\mathbb{R}). \tag{2.9.9}$$

Proof. For any $A > 0$ we can choose $\varphi \in C_c(\mathbb{R})$ with $0 \leq \varphi \leq 1$ and $\varphi = 1$ on $[-A, A]$, and we then get

$$\mu([-A, A]) \leq \int \varphi\, d\mu = \lim \int \varphi\, d\mu_n \leq \limsup \mu_n(\mathbb{R}) \leq \alpha,$$

and letting $A \to \infty$ we get $\mu(\mathbb{R}) \leq \alpha$.

Let now $f \in C_0(\mathbb{R})$ and $\varepsilon > 0$ be given. By definition there exists $A > 0$ such that $|f(x)| \leq \varepsilon$ for $|x| \geq A$ and let φ be as above in relation to $[-A, A]$. Since $f\varphi \in C_c(\mathbb{R})$ we have

$$\left| \int f\varphi\, d\mu_n - \int f\varphi\, d\mu \right| \leq \varepsilon$$

for $n \geq N$ where N is suitably large. For $n \geq N$ we then get

$$\begin{aligned}
&\left| \int f\, d\mu_n - \int f\, d\mu \right| \leq \\
&\quad \leq \left| \int f\, d\mu_n - \int f\varphi\, d\mu_n \right| + \left| \int f\varphi\, d\mu_n - \int f\varphi\, d\mu \right| + \left| \int f\varphi\, d\mu - \int f\, d\mu \right| \\
&\quad \leq \int |f(1-\varphi)|\, d\mu_n + \varepsilon + \int |f(1-\varphi)|\, d\mu \leq (2\alpha + 1)\varepsilon,
\end{aligned}$$

because $|f(x)(1-\varphi(x))| \leq \varepsilon$ for all $x \in \mathbb{R}$. Since ε is independent of α, we have proved (2.9.9). $\square$

If $\mu_n \to \mu$ vaguely and $\lim \mu_n(\mathbb{R}) = \alpha$, we can in general only conclude that $\mu(\mathbb{R}) \leq \alpha$, and it can happen that $\mu(\mathbb{R}) < \alpha$.

Concerning the inequality, we have $\mu_n(\mathbb{R}) \leq \alpha + \varepsilon$ for n sufficiently large, hence $\mu(\mathbb{R}) \leq \alpha + \varepsilon$ by Lemma 2.9.4. Since ε can be arbitrarily small, we get the assertion.

That we can have $\mu(\mathbb{R}) < \alpha$ is easy: $\mu_n = \varepsilon_n$ have all total mass 1 and $\lim\limits_{n\to\infty} \varepsilon_n = 0$ vaguely, but the limit has mass 0.

The following important result holds:

Lemma 2.9.5 *For a sequence (μ_n) and a measure μ from $\mathbb{M}_+^b(\mathbb{R})$ the following conditions are equivalent:*

(i) $\lim\limits_{n\to\infty} \mu_n = \mu$ *weakly*

(ii) $\lim\limits_{n\to\infty} \mu_n = \mu$ *vaguely and* $\lim\limits_{n\to\infty} \mu_n(\mathbb{R}) = \mu(\mathbb{R})$.

Proof. (i) $\Rightarrow$ (ii) is evident.

(ii) $\Rightarrow$ (i). Let $h \in C_b(\mathbb{R})$ be given.

For $\varepsilon > 0$ there exists $A > 0$ such that $\mu(\mathbb{R} \setminus [-A, A]) < \varepsilon$. Let $\varphi \in C_c(\mathbb{R})$ fulfill $0 \leq \varphi \leq 1$, $\varphi = 1$ on $[-A, A]$. By assumption (ii)

$$\lim_{n\to\infty} \int (1-\varphi)\, d\mu_n = \int (1-\varphi) d\mu \leq \mu(\mathbb{R} \setminus [-A, A]) < \varepsilon,$$

and

$$\lim_{n\to\infty} \int h\,\varphi\, d\mu_n = \int h\,\varphi\, d\mu.$$

There exists thus an N such that for $n \geq N$

$$\int (1-\varphi)\, d\mu_n < \varepsilon, \qquad \left| \int h\,\varphi\, d\mu - \int h\,\varphi\, d\mu_n \right| < \varepsilon.$$

Using this we find for $n \geq N$

$$\begin{aligned} \left| \int h\, d\mu - \int h\, d\mu_n \right| &\leq \left| \int h\, d\mu - \int h\,\varphi\, d\mu \right| + \left| \int h\,\varphi\, d\mu - \int h\,\varphi\, d\mu_n \right| \\ &\quad + \left| \int h\,\varphi\, d\mu_n - \int h\, d\mu_n \right| \\ &\leq \|h\|_\infty \int (1-\varphi)\, d\mu + \varepsilon + \|h\|_\infty \int (1-\varphi)\, d\mu_n \\ &\leq \varepsilon\,(1 + 2\,\|h\|_\infty), \end{aligned}$$

which proves the claim.

□

To $\mu \in \mathbb{M}_+^b(\mathbb{R})$ we can define a positive linear functional $L_\mu : C_0(\mathbb{R}) \to \mathbb{C}$ by $L_\mu(f) = \int f\, d\mu$.

It is easy to see that L_μ is a bounded linear functional on the Banach space $C_0(\mathbb{R})$ with $\|L_\mu\| = \mu(\mathbb{R})$. The Riesz representation theorem for finite positive Borel measures on $\mathbb{R}$ says that every bounded positive linear functional L on $C_0(\mathbb{R})$ has the form $L = L_\mu$ for precisely one $\mu \in \mathbb{M}_+^b(\mathbb{R})$.

We can therefore consider $\mathbb{M}_+^b(\mathbb{R})$ as a subset in the dual space $C_0(\mathbb{R})^*$. Alaoglu–Bourbaki's theorem states that the unit ball in $C_0(\mathbb{R})^*$ is compact in the topology $\sigma(C_0(\mathbb{R})^*, C_0(\mathbb{R}))$ and this gives the following version of Helly's theorem, cf. Theorem 1.12.7.

Theorem 2.9.6 *For every $\alpha > 0$ the set $\left\{\mu \in \mathbb{M}_+^b(\mathbb{R}) \,\middle|\, \mu(\mathbb{R}) \le \alpha\right\}$ is vaguely compact, i.e., for every sequence $\mu_n \in \mathbb{M}_+^b(\mathbb{R})$ with $\mu_n(\mathbb{R}) \le \alpha$, there exist $\mu \in \mathbb{M}_+^b(\mathbb{R})$ and a subsequence (μ_{n_p}) such that $\lim_{p\to\infty} \mu_{n_p} = \mu$ vaguely.*

Remark 2.9.7 Every bounded linear functional $L \in C_0(\mathbb{R})^*$ can be split as $L = L_1 - L_2 + i(L_3 - L_4)$, where $L_j, j = 1, \ldots, 4$ are bounded positive linear functionals. Thereby, L can be represented by a complex measure μ of the form $\mu_1 - \mu_2 + i(\mu_3 - \mu_4)$. It can be shown that $\|L\| = \|\mu\|$, where $\|\mu\|$ is the total variation of μ.

We shall now prove a counterpart to Herglotz' Theorem 1.12.9. In 1923, M. Mathias (Math. Zeitschrift 16, 103–125, 1923) introduced the following definition which corresponds to condition (ii) in Herglotz' Theorem.

A function $f : \mathbb{R} \to \mathbb{C}$ is called positive definite if for every choice of a finite set of real numbers $x_1, \ldots, x_n$, the matrix

$$(f(x_j - x_k))_{1\le j,k\le n}$$

is positive semidefinite, i.e.,

$$\sum_{j,k=1}^{n} f(x_j - x_k)\,\alpha_j\,\overline{\alpha_k} \ge 0 \qquad \forall\, \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n. \tag{2.9.10}$$

We note immediately that $f = \widehat{\mu}$, $\mu \in \mathbb{M}_+^b(\mathbb{R})$ is positive definite since

$$\begin{aligned}\sum_{j,k=1}^{n} \widehat{\mu}(x_j - x_k)\,\alpha_j\,\overline{\alpha_k} &= \int \left(\sum_{j,k=1}^{n} e^{-\mathrm{i}(x_j - x_k)x}\,\alpha_j\,\overline{\alpha_k}\right) d\mu(x)\\ &= \int \left|\sum_{j=1}^{n} e^{-\mathrm{i}x_j x}\,\alpha_j\right|^2 d\mu(x) \ge 0,\end{aligned}$$

and have thereby the easier half of

Theorem 2.9.8 (Bochner's theorem (1932).) *For a function $f : \mathbb{R} \to \mathbb{C}$ the following conditions are equivalent:*

(i) There exists $\mu \in \mathbb{M}_+^b(\mathbb{R})$ such that $f(t) = \widehat{\mu}(t) = \int e^{-itx}\, d\mu(x)$.

(ii) f is continuous and positive definite.

(Salomon Bochner (1899 – 1982), Polish–American mathematician).

Before proving Bochner's theorem, we need the following:

Lemma 2.9.9 *Let $f : \mathbb{R} \to \mathbb{C}$ be positive definite. Then*

(i) $f(-t) = \overline{f(t)}, \qquad t \in \mathbb{R}$,

(ii) $|f(t)| \le f(0), \qquad t \in \mathbb{R}$,

(iii) $|f(t) - f(s)| \le 2\, f(0)\, (f(0) - \operatorname{Re} f(s-t)), \qquad s, t \in \mathbb{R}$.

<u>Proof.</u> Choosing $x_1 = t$, $x_2 = 0$, we see that $\begin{pmatrix} f(0) & f(t) \\ f(-t) & f(0) \end{pmatrix}$ is positive semi-definite, i.e., $f(0) \ge 0$, $\overline{f(t)} = f(-t)$ together with $f(t)\, f(-t) \le f(0)^2$ or $|f(t)|^2 \le f(0)^2$. Thereby, we have proved (i) and (ii).

Choosing $x_1 = 0$, $x_2 = s$, $x_3 = t$, the matrix

$$\begin{pmatrix} f(0) & f(-s) & f(-t) \\ f(s) & f(0) & f(s-t) \\ f(t) & f(t-s) & f(0) \end{pmatrix} = \begin{pmatrix} f(0) & \overline{f(s)} & \overline{f(t)} \\ f(s) & f(0) & f(s-t) \\ f(t) & \overline{f(s-t)} & f(0) \end{pmatrix}$$

is positive semi-definite. Since in the proof of (iii), we can assume $f(t) \ne f(s)$, it makes sense for $\lambda \in \mathbb{R}$ to define

$$\alpha_1 = 1, \qquad \alpha_2 = \frac{\lambda\, |f(s) - f(t)|}{f(s) - f(t)}, \qquad \alpha_3 = -\alpha_2,$$

whereby the sum (2.9.10) becomes

$$f(0)\,(1 + 2\lambda^2) + 2\,\lambda\, |f(s) - f(t)| - 2\,\lambda^2 \operatorname{Re} f(s-t) \ge 0, \qquad \lambda \in \mathbb{R}.$$

The discriminant of this polynomial in λ of degree 2 is thus ≤ 0, i.e.,

$$|f(s) - f(t)|^2 \le 2\, f(0)\, \left(f(0) - \operatorname{Re} f(s-t)\right),$$

which gives (iii). $\square$

A positive definite function is thus bounded, and from (iii) it follows that if just $\operatorname{Re} f$ is continuous at 0, then f is uniformly continuous. A positive definite function need not be continuous, cf. **E 9.1**.

Proof of Bochner's theorem. Let f be continuous and positive definite. From the lemma it follows that it is uniformly continuous and bounded. Let now $\alpha : \mathbb{R} \to \mathbb{C}$ be a continuous integrable function. The continuous analogue of (2.9.10) is

$$\mathrm{I} := \int\int f(x-y)\,\alpha(x)\,\overline{\alpha(y)}\,dx\,dy \geq 0. \tag{2.9.11}$$

The double integral is meaningful since f is continuous and bounded, and by Lebesgue's theorem on dominated convergence, it is sufficient to show for every $A > 0$ that

$$\mathrm{I}_A = \int_{-A}^{A}\int_{-A}^{A} f(x-y)\,\alpha(x)\,\overline{\alpha(y)}\,dx\,dy \geq 0,$$

but this integral is the limit for $N \to \infty$ of the sums

$$\sum_{j,k=-N}^{N-1} f\left(\tfrac{jA-kA}{N}\right)\,\alpha\left(\tfrac{jA}{N}\right)\,\overline{\alpha\left(\tfrac{kA}{N}\right)}\,\tfrac{A^2}{N^2},$$

which are ≥ 0 by (2.9.10).

We evaluate in particular (2.9.11) for $\alpha(x) = e^{-2\varepsilon x^2}\,e^{\mathrm{i}tx}$, where $\varepsilon > 0$, $t \in \mathbb{R}$ are parameters, which gives

$$\int\int f(x-y)\,e^{-2\varepsilon x^2}\,e^{-2\varepsilon y^2}\,e^{\mathrm{i}tx}\,e^{-\mathrm{i}ty}\,dx\,dy \geq 0. \tag{2.9.12}$$

Introduce now the coordinate transformation $u = x-y$, $v = x+y$ from $\mathbb{R}^2$ onto itself, i.e., $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(v-u)$. The Jacobi determinant is

$$\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2},$$

and using $2\,x^2 + 2\,y^2 = u^2 + v^2$, (2.9.12) becomes

$$\frac{1}{2}\int\int f(u)\,e^{-\varepsilon u^2}\,e^{-\varepsilon v^2}\,e^{\mathrm{i}tu}\,du\,dv \geq 0,$$

or

$$\frac{1}{2}\int e^{-\varepsilon v^2}\,dv \int f(u)\,e^{-\varepsilon u^2}\,e^{\mathrm{i}tu}\,du \geq 0.$$

We have thus

$$\varphi_\varepsilon(t) = \frac{1}{2\pi}\int f(u)\,e^{-\varepsilon u^2}\,e^{\mathrm{i}tu}\,du \geq 0 \qquad \text{for } t \in \mathbb{R},\ \varepsilon > 0. \tag{2.9.13}$$

For $\varepsilon > 0$ consider the positive measure μ_ε with density φ_ε with respect to Lebesgue measure. It is finite with

$$\mu_\varepsilon(\mathbb{R}) = \int \varphi_\varepsilon(t)\,dt \leq f(0). \tag{2.9.14}$$

To see this we need that the density

$$g_\delta(x) = \frac{1}{\sqrt{4\pi\delta}}\,e^{-\frac{x^2}{4\delta}}, \qquad \delta > 0$$

for a normal distribution ($\int g_\delta(x)\,dx = 1$) has the Fourier transform

$$\widehat{g}_\delta(t) = e^{-\delta t^2},$$

see E 3.2.

Thus, we find for $\delta > 0$ by the inversion formula

$$\begin{aligned} 0 &\leq \int \varphi_\varepsilon(t)\,e^{-\delta t^2}\,dt = \int \varphi_\varepsilon(t)\,\widehat{g}_\delta(t)\,dt = \frac{1}{2\pi}\int\int f(u)\,e^{-\varepsilon u^2}\,e^{\mathrm{i}tu}\,du\,\widehat{g}_\delta(t)\,dt \\ &= \int f(u)\,e^{-\varepsilon u^2}\,g_\delta(u)\,du \leq \int |f(u)|\,e^{-\varepsilon u^2}\,g_\delta(u)\,du \\ &\leq f(0)\int g_\delta(u)\,du = f(0). \end{aligned}$$

Letting subsequently $\delta \to 0$, the monotone convergence theorem gives (2.9.14).

By Helly's theorem there exists $\mu \in \mathbb{M}_+^b(\mathbb{R})$ with $\mu(\mathbb{R}) \leq f(0)$ such that μ_{ε_n} converges vaguely to μ for an appropriate sequence $\varepsilon_n \to 0$.

Using the inversion theorem for the integrable function $\frac{1}{2\pi}f(u)\,e^{-\varepsilon u^2}$ which has an integrable Fourier transform according to (2.9.14), we obtain

$$f(u)\,e^{-\varepsilon u^2} = \int \varphi_\varepsilon(t)\,e^{-\mathrm{i}tu}\,dt = \widehat{\mu}_\varepsilon(u), \tag{2.9.15}$$

in particular $\widehat{\mu}_\varepsilon(0) = \mu_\varepsilon(\mathbb{R}) = f(0)$ for all $\varepsilon > 0$. The limit measure μ for μ_{ε_n} has the mass $\mu(\mathbb{R}) \leq f(0)$, and it cannot be excluded in advance that $\mu(\mathbb{R}) < f(0)$. We shall now see that $\mu(\mathbb{R}) = f(0)$, and thereby Lemma 2.9.5 shows that

$$\lim_{n\to\infty}\int h(x)\,d\mu_{\varepsilon_n}(x) = \int h(x)\,d\mu(x),$$

for all bounded continuous functions $h : \mathbb{R} \to \mathbb{C}$, in particular for $h(x) = e^{-\mathrm{i}xu}$, where $u \in \mathbb{R}$ is fixed but arbitrary, hence $\widehat{\mu_{\varepsilon_n}}(u) \to \widehat{\mu}(u)$ for $u \in \mathbb{R}$. From (2.9.15) we therefore get

$$f(u) = \int e^{-\mathrm{i}tu}\, d\mu(t), \qquad u \in \mathbb{R},$$

which was to be proved.

To see that $\mu(\mathbb{R}) = f(0)$, we integrate (2.9.15) from $-a$ to a and divide by $2a$:

$$\frac{1}{2a}\int_{-a}^{a} f(u)\, e^{-\varepsilon u^2}\, du = \int \varphi_\varepsilon(t)\, \frac{\sin(at)}{at}\, dt = \int \frac{\sin(at)}{at}\, d\mu_\varepsilon(t).$$

Putting $\varepsilon = \varepsilon_n$ and letting $n \to \infty$ we obtain

$$\frac{1}{2a}\int_{-a}^{a} f(u)\, du = \int \frac{\sin(at)}{at}\, d\mu(t).$$

Here we have used that $\sin(at)/at \in C_0(\mathbb{R})$ and next we applied Lemma 2.9.4.

Subsequently we let $a \to 0$: The left-hand side gives $f(0)$ since f is continuous, and the right-hand side gives $\mu(\mathbb{R})$ by Lebesgue's theorem on dominated convergence, i.e., $\mu(\mathbb{R}) = f(0)$. $\square$

Theorem 2.9.10 (Lévy's continuity theorem) *Assume that a sequence (μ_n) from $\mathbb{M}_+^b(\mathbb{R})$ has the properties*

(i) $\lim_{n\to\infty} \widehat{\mu_n}(t) = \varphi(t)$ *exists for all* $t \in \mathbb{R}$.

(ii) φ *is continuous for* $t = 0$.

Then there exists $\mu \in \mathbb{M}_+^b(\mathbb{R})$, *with* $\widehat{\mu} = \varphi$ *and* $\lim_{n\to\infty} \mu_n = \mu$ *weakly.*

Proof. We note first that if $\varphi_n : \mathbb{R} \to \mathbb{C}$ is a sequence of positive definite functions which converges pointwise to $\varphi : \mathbb{R} \to \mathbb{C}$, then φ is positive definite. In fact, for $x_1, \ldots, x_p \in \mathbb{R}$, $\alpha_1, \ldots, \alpha_p \in \mathbb{C}$, the inequality

$$\sum_{j,k=1}^{p} \varphi_n(x_j - x_k)\, \alpha_j\, \overline{\alpha_k} \geq 0$$

is conserved in the limit because only finitely many points are involved. Combined with (ii) and Lemma 2.9.9 we see that φ is continuous. By Bochner's theorem, $\varphi = \widehat{\mu}$ for $\mu \in \mathbb{M}_+^b(\mathbb{R})$, and since $\widehat{\mu_n}(t) \to \widehat{\mu}(t)$ pointwise, we have in particular

$\mu_n(\mathbb{R}) = \widehat{\mu_n}(0) \to \widehat{\mu}(0) = \mu(\mathbb{R})$. By Lemma 2.9.5 we thus only have to prove that $\mu_n \to \mu$ vaguely.

Let $f \in C_c(\mathbb{R})$ and $\varepsilon > 0$ be given. Since $\mathcal{F}(\mathcal{L}^1(\mathbb{R}))$ is dense in $C_0(\mathbb{R})$, cf. the discussion after Theorem 2.8.1, there is a $g \in \mathcal{L}^1(\mathbb{R})$ such that $\|\widehat{g} - f\|_\infty \le \varepsilon$. We thus have

$$\begin{aligned}
&\left|\int f\,d\mu_n - \int f\,du\right| \le \int |f - \widehat{g}|\,d\mu_n + \left|\int \widehat{g}\,d\mu_n - \int \widehat{g}\,d\mu\right| + \int |\widehat{g} - f|\,d\mu \\
&\le \varepsilon\,(\mu_n(\mathbb{R}) + \mu(\mathbb{R})) + \left|\int (\widehat{\mu}_n(x) - \widehat{\mu}(x))\,g(x)\,dx\right|.
\end{aligned}$$

The first term tends to $2\varepsilon\mu(\mathbb{R})$, and the second term tends to 0 by Lebesgue's theorem on dominated convergence, and the desired conclusion follows. □

Corollary 2.9.11 *Given μ and a sequence (μ_n) from $\mathbb{M}_+^b(\mathbb{R})$. Then*

$$\mu_n \to \mu \quad \textit{weakly} \quad \Longleftrightarrow \quad \widehat{\mu_n} \to \widehat{\mu} \quad \textit{pointwise.}$$

Exercises

E 9.1 Show that $1_{\mathbb{Q}}$ (i.e., the function $= 1$ for $x \in \mathbb{Q}$, 0 for $x \in \mathbb{R}\setminus\mathbb{Q}$) is positive definite.

Hint: Use that $\mathbb{Q}$ is a subgroup of $\mathbb{R}$.

E 9.2 Let (μ_n) be a sequence and μ a measure from $\mathbb{M}_+^b(\mathbb{R})$. Show that if $\mu_n \to \mu$ weakly, then $\widehat{\mu_n} \to \widehat{\mu}$ uniformly over compact subsets of $\mathbb{R}$.

E 9.3

1° Show that a matrix $A = (a_{jk})$ is positive semidefinite if and only if it can be written in the form $A = PP^*$, where P^* is the conjugate transpose of P, i.e., if $P = (p_{jk})$ and $P^* = (p^*_{jk})$, then $p^*_{jk} = \overline{p_{kj}}$.

2° Show that if $A = (a_{jk})$ and $B = (b_{jk})$ are positive semidefinite matrices, then $C = (a_{jk}\,b_{jk})$ is positive semidefinite (Schur).

3° Show that if f, $g : \mathbb{R} \to \mathbb{C}$ are positive definite functions, then fg is positive definite.

4° Let $F(z) = \sum_0^\infty a_n\,z^n$ be holomorphic in $|z| < R$, with $a_n \ge 0$. Let $f : \mathbb{R} \to \mathbb{C}$ be positive definite, with $f(0) < R$. Show that $F \circ f$ is positive definite.

E 9.4 Let $f \in \mathcal{L}^2(\mathbb{R})$, $\tilde{f}(x) = \overline{f(-x)}$. Show that

$$f * \tilde{f}(x) = \frac{1}{2\pi} \int e^{\mathrm{i}tx} \, |\widehat{f}(t)|^2 \, dt,$$

and conclude that $f * \tilde{f}$ is continuous and positive definite. What is the associated measure from Bochner's theorem?

E 9.5 Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and positive definite as well as integrable. Show that $\widehat{f}(t) \geq 0$ for all t. Show furthermore that $\widehat{f} \in \mathcal{L}^1(\mathbb{R})$, and thereby that

$$f(x) = \frac{1}{2\pi} \int e^{\mathrm{i}tx} \, \widehat{f}(t) \, dt.$$

Identify the measure belonging to f from Bochner's theorem.
(Hint: Show that the measure μ from Bochner's theorem fulfills

$$\int g(x) \, d\mu(x) = \frac{1}{2\pi} \int g(x) \, \widehat{f}(-x) \, dx$$

for every Schwartz function g).

Bibliography

[1] N. K. Bary, *A Treatise on Trigonometric Series I,II.* Pergamon Press, Oxford 1964.

[2] S. Bochner, *Harmonic Analysis and the Theory of Probability.* University of California Press. Berkeley 1955.

[3] H. Dym, H. P. McKean, *Fourier Series and Integrals.* Academic Press. New York 1972.

[4] R. E. Edvards, *Fourier series I.* Holt, Rinehart & Winston 1967.

[5] G. H. Hardy, W. W. Rogosinski, *Fourier Series.* Cambridge University Press. Cambridge 1950.

[6] E. Hewitt, K. A. Ross, *Abstract Harmonic Analysis I,II.* Springer. Berlin 1963, 1970.

[7] Y. Katznelson, *An introduction to harmonic analysis.* Wiley 1968.

[8] T. W. Körner, *Fourier Analysis.* Cambridge University Press, Cambridge 1988.

[9] W. Rudin, *Real and complex analysis.* Third Edition, McGraw-Hill Book Co., Singapore 1986.

[10] A. Zygmund, *Trigonometric series I,II.* 2. ed., Cambridge University Press 1959.